Geodesic loops with non-solvable left translation groups on 3-dimensional reductive spaces

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Contents

Introduction

The first interests to study loops, which are structures with a binary multiplication having up to associativity the same properties as groups, came from the foundation of geometry, in particular from the investigation of coordinate systems of non-Desarguesian planes. Blaschke perceived that some topological questions of differential geometry, in particular the topological behavior of geodesic foliations lead to investigate loops [8].

In the last 50 years the theory of loops developed in the following three directions.

The algebraic aspect of the theory of loops was investigated by Baer [5], by Albert [2], [3], by Bruck [9], [10] and by Belousov [6]. Baer dealt with the geometry associated with loops. Bruck treated the theory of loops as a part of general algebra. First Albert saw a loop as a section in the group generated by its translations. Belousov studied loops and their associated geometry as abstract objects.

The loops play an important role in the topological algebra, in the topological geometry and in the differential geometry ([12], [13], [14], [39], [26], [1], [48]).

The theory of differentiable and analytic loops can be consider as a part of Lie theory. The aim of this research program is to generalize the first and third Lie theory for analytic loops [27] and to classify analytic loops by their tangential objects. Kuzmin [37], Kerdman [30] and Nagy [42] proved that the Lie group and Lie algebra correspondence can be extended to analytic Moufang loops and to their associated Malcev algebras. Sabinin and Miheev verified in [40] that to every Bol algebra exists a unique analytic local Bol loop (up to local isomorphisms). In contrast to the cases Lie group - Lie algebra and Moufang loop - Malcev algebra, this local Bol loop can not be embedded to a global one. Therefore the classification of global differentiable Bol loops significantly differs from the classification of local differentiable Bol loops, which is equivalent to the classification of Bol algebras. The 2-dimensional global Bol loops are determined by Nagy and Strambach [43] and the classification of the 3-dimensional global Bol loops with non-solvable left translation groups is given in [17]. In [16] we prove the Lie's fundamental theorems for very wide classes of local analytic loops for the class of geodesic loops which respect to linear connections the curvature of which is zero.

An impotant characteristic of commutative groups G is the fact that their commutators $(ba)^{-1}ab$ for all $a, b \in G$ are the identity of G. But this fact depends strongly on the associative law. For loops this behaviour changes radically. This observation leat to a broader research of loops L

in which the mapping $x \mapsto [(ba)^{-1}(a(bx))]$ is an automorphism of L (cf. [11], [6]). These loops have been called left A-loops. The investigation of differentiable left A-loops has shown that this class of loops corresponds strongly to the class of reductive spaces (cf. [32]), which are essential objects in the main stream of differential geometry (cf. [36], [23]). Moreover every almost differentiable strongly left alternative local left A-loop L is a geodesic local loop of the canonical connection ∇ of the reductive homogeneous space G/H corresponding to L (cf. [43], Proposition 5.21, p. 76).

In this thesis we investigate strongly left alternative almost differentiable connected left A-loops on 3-dimensional manifolds and describe their relations to metric space geometries.

Following the terminology of Nagy and Strambach in [43] we treat the left A-loops as images of global differentiable sharply transitive sections $\sigma : G/H \to G$ for a Lie group with the properties $\sigma(H) = 1 \in G$ and $\sigma(G/H)$ generates G such that the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H. Here G denotes the group topologically generated by the left translations $\{\lambda_x, x \in L\}$ of L and H is the stabilizer of the identity of L in G. Loops given by a differentiable section in a Lie group are called almost differentiable.

In case of a left A-loop L the tangential space $\mathbf{m} = T_1\sigma(G/H)$ of the image of the section σ at $1 \in G$ can be provided with a binary and a ternary multiplication and yields a Lie triple algebra (cf. [50], [32], Definition 7.1, p. 173) which is a generalization of a Lie triple system. As the Lie triple systems are in one-to-one correspondence to (global) simply connected affine symmetric spaces (cf. [38], [43] Section 6), the Lie triple algebras correspond to affine reductive spaces. Hence there is a strong connection between the theory of differential left A-loops and the theory of affine reductive homogeneous spaces (cf. [33]). In particular the theory of connected differentiable Bruck loops (which form a subclass of the class of left A-loops) is essentially the theory of affine symmetric spaces (cf. [43], Section 11).

The smallest connected almost differentiable non-associative left A-loops are realized on 2-dimensional manifolds. There exist precisely two isotopism classes of 2-dimensional global left A-loops. In the one class lies only the hyperbolic plane loop which is related to the hyperbolic symmetric plane (cf. [43], Section 22). This loop is even a proper Bruck loop.

The other isotopism class consists of differentiable loops, which are homeomorphic to $\mathbb{R}^2$ having the exponential solvable Lie group

$$G_{1,\alpha} = \left\{ g(u,v,w) = \begin{pmatrix} 1 & u & v \\ 0 & e^w & 0 \\ 0 & 0 & e^{\alpha w} \end{pmatrix}, \; u,v,w \in \mathbb{R} \right\},$$

with a fixed number $\alpha < 0$, as the group topologically generated by their left translations. We can choose the subgroup $H = \{g(t,t,0),\ t \in \mathbb{R}\}$ as the stabilizer of the identity of L in G. As a representative L of the other isotopism class of loops may be chosen the 2-dimensional Bruck loop which is realized on the pseudo-euclidean affine plane E such that the group topologically generated by its left translations is the connected component of the group of pseudo-euclidean motions and the elements of L are the lines of positive slope in E (cf. [43], Section 25).

Our aim in this paper is to classify the 3-dimensional connected almost differentiable strongly left alternative global left A-loops, which have a non-solvable group as the group topologically generated by their left translations. This is equivalent to the classification of all almost differentiable geodesic loops on 3-dimensional non-solvable reductive spaces.

The group G topologically generated by the left translations of a connected almost differentiable left A-loop L is a connected Lie group ([40], [43], p. 75) and the stabilizer H of $e \in L$ in G is a subgroup of G containing no non-trivial normal subgroup of G. Observing that the Lie algebra $\mathbf{g}$ of the group $G = \langle \lambda_x, x \in L \rangle$ is isomorphic to the standard enveloping Lie algebra of the tangential object for a left A-loop, we can deduce that in the case of a proper left A-loop L of dimension 3 the dimension of G is at least 4 and at most 6. We known that $L = G/H$ is a parallelizable manifold and that it can not be compact (Corollary 8 in Section 1). First we classify the 3-dimensional reductive spaces G/H, this means we determine all complements $\mathbf{m}$ of the Lie algebra $\mathbf{h}$ of H in the Lie algebra $\mathbf{g}$ of G such that $\mathbf{m}$ generates $\mathbf{g}$ and satisfies the relation $[\mathbf{h},\mathbf{m}] \subseteq \mathbf{m}$. Such a complement is called reductive. For every strongly left alternative almost differentiable left A-loops the image of the tangential space $T_H\sigma(G/H)$ under the exponential map lies in $\sigma(G/H)$. Hence the exponential image of a reductive complement $\mathbf{m}$ defines a differentiable local section in a neighbourhood U of $1 \in G$ in the sense of Lie and hence yields a local left A-loop. The classification of global almost differentiable left A-loops significantly differs from the classification of local almost differentiable left A-loops, which can be represented as local sections in non-solvable Lie groups. The question under what circumstance a local loop is embedable into a global one is difficult problem. Kuzmin, Kerdman and Nagy have proved that any local differentiable Moufang loop can be uniquely embedded in a connected simply connected global one (cf. [30], [37], [42]). But already for diassociative loops it is not more true (cf. [21]). But then we have to discuss which of these local left A-loops L can be extended to global left A-loops. For a global loop L the subset $\exp \mathbf{m}$ forms a system of representatives for the cosets $\{xH \mid x \in G\}$ in G and $\exp \mathbf{m}$ does not contain any element conjugate to an element of H.

In section 1 we collect notions and results, which we need in our later investigation.
In **1.1** we discuss connections among loops and sharply transitive sections in groups as well as relations between almost differentiable left A-loops, their tangential object and linear connections in reductive spaces.
In **1.2** we give an explicit description for the exponential map of Lie algebras $sl_2(\mathbb{R})$, $sl_2(\mathbb{C})$ and $su_2(\mathbb{C})$ following ideas of [24]. To obtain this description we need as tool the Cartan-Killing form.
In **1.3** we show which direct products having as a factor the group $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$ can occur as the group topologically generated by the left translations of an almost differentiable left A-loop. Moreover we obtain that there is no almost differentiable left A-loop homeomorphic to the 3-sphere.

In section 2 we classify all 3-dimensional almost differentiable left A-loops having a semisimple Lie group as the group topologically generated by their left translations. It is remarkable that for these loops the isotopism classes coincide with the isomorphism classes. In section 3 we show that there are (up to isotopisms) precisely 3 almost differentiable left A-loops having a 4-dimensional Lie group as the group topologically generated by their left translations, whereas in section 4 it is verified that there are no almost differentiable left A-loops with a 5-dimensional Lie group as the group topologically generated by the left translations. Finally, in section 5 we prove that there are precisely two isomorphism classes of almost differentiable left A-loops such that the group topologically generated by their left translations is a 6-dimensional non-semisimple and non-solvable Lie group, and again for these loops the isomorphism classes coincide with the isotopism classes. As an impotant tool for the proofs in section 2,3,4 and 5 we use the theory of reductive spaces.
The results of our paper can summarized in the following

Theorem *There are precisely two classes $\mathcal{C}_i$ $(i = 1, 2)$ of the connected almost differentiable simple left A-loops L having dimension 3 such that the group G generated by the left translations $\{\lambda_x; x \in L\}$ is a non-solvable Lie group.*

The class $\mathcal{C}_1$ consists of left A-loops having the simple Lie group $G = PSL_2(\mathbb{C})$ as the group topologically generated by their left translations, and the stabilizer H of $e \in L$ in G is the group $SO_3(\mathbb{R})$.
Any loop in the class $\mathcal{C}_1$ can be represented by a parameter $a \in \mathbb{R}$. For all $a \in \mathbb{R}$ the loops L_a and L_{-a} are isomorphic. These two loops form a full isotopism class. Any loop L_a with $a \geq 0$ is isomorphic to the geodesic loop of the reductive homogeneous space G/H with respect to the reductive complement $\mathbf{m}_a = T_1[\sigma_a(G/H)]$ and the corresponding canonical invariant

connection ∇_a. The hyperbolic space loop L_0, which is the unique Bruck loop in $\mathcal{C}_1$, is the geodesic loop of the hyperbolic space defined by the multiplication $x \cdot y = \tau_{e,x}(y)$, where $\tau_{e,x}$ is the hyperbolic translation moving e onto x.
The other class $\mathcal{C}_2$ of simple left A-loops consists of 3-dimensional connected differentiable left A-loops such that the group $G = PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on $\mathbb{R}^3$ is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra, is the group topologically generated by the left translations and

$$H = \left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi), x, y \in \mathbb{R} \right\}$$

is the stabilizer of $e \in L$ in G.
The loops in this class can be represented by two parameters $a, b \in \mathbb{R}$ and form precisely two isomorphism classes, which coincide with the isotopism classes. In the one isomorphism class are the Bruck loops $L_{a,0}$, $a \in \mathbb{R}$ and the pseudo-euclidean space loop $L_{0,0} = \hat{L}_0$ may be chosen as a representative of this isomorphism class. The other isomorphism class containing the loops $L_{a,b}$ with $b \neq 0$ has as a representative the loop $L_{0,1} = \hat{L}_1$. The loops $\hat{L}_0$ and $\hat{L}_1$ are realized on the pseudo-euclidean affine space $E(2,1)$ such that the group topologically generated by their left translations is the connected component of the group of pseudo-euclidean motions. The elements of the loops $\hat{L}_0$ and $\hat{L}_1$ are the planes on which the euclidean metric is induced but the sets of left translations of these loops are differ. Both loops $\hat{L}_0$ and $\hat{L}_1$ are isomorphic to the geodesic loops of the pseudo-euclidean space $G/H = E(2,1)$ with respect to the reductive complements $\mathbf{m}_{0,i} = T_1[\hat{\sigma}_i(G/H)]$ and the corresponding canonical invariant connection ∇_i, where $i = 0, 1$.
Moreover, the non-simple 3-dimensional almost differentiable left A-loops are either the direct products of a 1-dimensional Lie group with a 2-dimensional left A-loop isomorphic to the hyperbolic plane loop or the unique Scheerer extension of the Lie group $SO_2(\mathbb{R})$ by the 2-dimensional left A-loop isomorphic to the hyperbolic plane loop.

Another class of almost differentiable loops which has been thoroughly investigated is the class of differentiable Bol loops. The sections $\sigma : G/H \to G$ of Bol loops are characterized by the fact that for all $a, b \in \sigma(G/H)$ the element aba is also contained in $\sigma(G/H)$. The 3-dimensional almost differentiable Bol loops with non-solvable Lie groups have been classified in [17]; the Lie groups G topologically generated by their left translations as well as the corresponding stabilizers H are the same as in the case of 3-dimensional almost differentiable left A-loops, but the sections differ essentially. The intersection of these two classes are only the Bruck-loops and the Scheerer extension of the orthogonal group $SO_2(\mathbb{R})$ by the hyperbolic plane loop.

1 Left A-loops and geodesic loops

1.1 A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \backslash b$ and $x = b/a$. The left translation $\lambda_a : y \mapsto a \cdot y : L \to L$ is a bijection of L for any $a \in L$. Two loops $(L_1, \circ)$ and $(L_2, *)$ are called isotopic if there are three bijections $\alpha, \beta, \gamma : L_1 \to L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for any $x, y \in L_1$. An isotopism is an equivalence relation. If $\alpha = \beta = \gamma$ then the isotopic loops $(L_1, \circ)$ and $(L_2, *)$ are called isomorphic. Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops. The direct product $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$ with the multiplication $(a_1, b_1) \circ (a_2, b_2) = (a_1 \cdot a_2, b_1 * b_2)$ is again a loop, which is called the direct product of L_1 and L_2, and the loops $(L_1, \cdot)$, $(L_2, *)$ are subloops of $(L, \circ)$.
A loop is called a left A-loop if each mapping $\lambda_{x,y} = \lambda_{xy}^{-1} \lambda_x \lambda_y : L \to L$ is an automorphism of L.
If the elements of a loop L are elements of a differentiable manifold and the operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y : L \times L \to L$ are differentiable mappings then L is called an almost differentiable loop. If the right division $(x, y) \mapsto x/y$ is also differentiable then L is differentiable.
Let G be the group generated by the left translations of a loop L and let H be the stabilizer of $e \in L$ in the group G. The left translations of L form a subset of G acting on the cosets $\{xH; x \in G\}$ such that for any given cosets aH, bH there exists precisely one left translation λ_z with $\lambda_z aH = bH$.
Conversely let G be a group, H be a subgroup containing no normal non-trivial subgroup of G and $\sigma : G/H \to G$ be a section such that σ satisfies the following conditions:
1. The image $\sigma(G/H)$ forms a subset of G with $\sigma(H) = 1 \in G$.
2. $\sigma(G/H)$ generates G.
3. $\sigma(G/H)$ acts sharply transitively on the space G/H of the left cosets $\{xH, x \in G\}$, i.e. to any xH, yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$ (cf. [43], p. 18).
Then the multiplication on the factor space G/H defined by $xH * yH = \sigma(xH)yH$ yields a loop $L(\sigma)$. This loop is a left A-loop if and only if the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H.
Let L_1 be a loop defined on the factor space G_1/H_1 with respect to a section $\sigma_1 : G_1/H_1 \to G_1$ the image of which is the set $M_1 \subset G_1$ and let G_2 be a Lie group. A loop L is called a Scheerer extension of G_2 by L_1 if the loop L is defined on the factor space

$$(G_1 \times G_2)/(H_1, \varphi(H_1)), \quad \text{where} \quad \varphi : H_1 \to G_2 \quad \text{is a homomorphism,}$$

with respect to the section $\sigma : (G_1 \times G_2)/(H_1, \varphi(H_1)) \to G_1 \times G_2$ the image of which is the set $M_1 \times G_2$. Moreover the loop L contains a normal subgroup $\tilde{G}_2$ isomorphic to G_2 such that the factor loop $L/\tilde{G}_2$ is isomorphic to the loop L_1.
If G is a Lie group and σ is a differentiable section satisfying 1,2 and 3 then the loop $L(\sigma)$ is almost differentiable.
Two loops L_1 and L_2 having the same group G of the group generated by the left translations and the same stabilizer H of $e \in L_1, L_2$ are isomorphic if there is an automorphism of G leaving H invariant and mapping the section $\sigma_1(G/H)$ onto the section $\sigma_2(G/H)$. Moreover let L and L' be loops having the same group G generated by their left translations. Then L and L' are isotopic if and only if there is a loop L'' isomorphic to L' having G again as the group generated by its left translations such that there exists an inner automorphism τ of G mapping the section $\sigma''(G/H)$ belonging to L'' onto the section $\sigma(G/H)$ corresponding to L (cf. [43], Theorem 1.11. pp. 21-22).
For a differentiable manifold L the group of all autohomeomorphisms of L becomes by the compact-open topology a topological group. If L is a connected almost differentiable left A-loop, then the group G topologically generated by the left translations of L within the group of autohomeomorphisms is a connected Lie group (cf. [40]; [43], Proposition 5.20. p. 75), and we may describe L by a differentiable section.
Let L be a connected almost differentiable left A-loop. Let G be the Lie group topologically generated by the left translations of L, and let $(\mathbf{g}, [.,.])$ be the Lie algebra of G. Denote by $\mathbf{h}$ the Lie algebra of the stabilizer H of $e \in L$ in G and by $\mathbf{m} = T_1\sigma(G/H)$ the tangent space at $1 \in G$ of the image of the section $\sigma : G/H \to G$ corresponding to the left A-loop L. Then $\mathbf{m}$ generates $\mathbf{g}$ and the homogeneous space G/H is reductive, i.e. we have $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$ and $ad(\mathbf{h})\mathbf{m} \subseteq \mathbf{m}$. (cf. [44] Chapter II, p. 41; [36] Vol II, p. 190; [43], Proposition 5.20. p. 75)
Denote by ∇ the canonical connection of the homogeneous space G/H with respect to the subspace $\mathbf{m}$. The torsion tensor field T and the curvature tensor field R of ∇ are given by

$$\begin{array}{rcll} T(X,Y)_H &=& -[X,Y]_{\mathbf{m}} & \text{for all } X,Y \in \mathbf{m} \\ (R(X,Y),Z)_H &=& -\big[[X,Y]_{\mathbf{h}}, Z\big] & \text{for all } X,Y,Z \in \mathbf{m}, \end{array}$$

where $Z \mapsto Z_{\mathbf{m}} : \mathbf{g} \to \mathbf{m}$ is the projection of $\mathbf{g}$ onto $\mathbf{m}$ along the subalgebra $\mathbf{h}$ and $Z \mapsto Z_{\mathbf{h}} : \mathbf{g} \to \mathbf{h}$ is the projection of $\mathbf{g}$ onto $\mathbf{h}$ along the subalgebra $\mathbf{m}$ (cf. [33], Proposition 4. p. 6). We have $\nabla\, T = \nabla\, R = 0$.
We consider the following four identities:
Bianchi's 1st identity

$$\sigma_{X,Y,Z}\{R(X,Y)Z + T(T(X,Y),Z) - (\nabla_X T)(Y,Z)\} = 0$$

Bianchi's 2nd identity

$$\sigma_{X,Y,Z}\{R(T(X,Y),Z)W - (\nabla_X R)(Y,Z)W\} = 0$$

and

$$R(U,V)(T(X,Y)) = T(R(U,V)X,Y) + T(X,R(U,V)Y)$$
$$R(U,V)(R(X,Y)Z) = R(R(U,V)X,Y)Z+$$
$$R(X,R(U,V)Y)Z + R(X,Y)R(U,V)Z,$$

where $\sigma_{X,Y,Z}$ is the cyclic sum with respect to X, Y, Z. Evaluted these identities at the point H we obtain the following relations:

$X \cdot Y = -Y \cdot X$

$[X,Y,Z] = -[Y,X,Z]$

$\sigma_{X,Y,Z}\{[X,Y,Z] + (X \cdot Y) \cdot Z\} = 0$

$\sigma_{X,Y,Z}[(X \cdot Y), Z, W] = 0$

$[U,V,X \cdot Y] = [U,V,X] \cdot Y + X \cdot [U,V,Y]$

$[U,V,[X,Y,Z]] = [[U,V,X],Y,Z] + [X,[U,V,Y],Z] + [X,Y,[U,V,Z]],$

where $X \cdot Y := [X,Y]_{\mathbf{m}}$ and $[X,Y,Z] := [[X,Y]_{\mathbf{h}}, Z]$ (cf. [40]). These are exactly the axioms of a Lie triple algebra. The subspace $\mathbf{m}$ corresponding to a left A-loop is a Lie triple algebra with the above operations.

Any Lie triple algebra is reduced to a Lie algebra if the trilinear multiplication $[X,Y,Z]$ vanishes identically and it is reduced to a Lie triple system if the bilinear operation $X \cdot Y$ vanishes identically.

If the subspace $\mathbf{m}$ is a Lie triple system then the factor space G/H is an affine symmetric space and the corresponding loop L is a Bruck loop.

A loop L is called strongly 2-divisible if the mapping $\varphi : x \mapsto x^2 : L \to L$ is bijective.

A strongly 2-divisible differentiable loop L is called a Bruck loop if there is an involutory automorphism σ of the Lie algebra $\mathbf{g}$ of the connected Lie group G generated by the left translations of L such that the tangent space $T_e(L) = \mathbf{m}$ is the -1-eigenspace and the Lie algebra $\mathbf{h}$ of the stabilizer H of $e \in L$ in G is the $+1$-eigenspace of σ.

For any Lie triple algebra M we can define the standard enveloping Lie algebra $U(M)$ of M. It is the direct sum $U(M) = M \oplus D(M,M)$ of the Lie triple algebra M and of the set $D(M,M)$ of all inner derivations of M with the following Lie multiplication:

$[X,Y]$	$=$	$X \cdot Y + D(X,Y)$	for all	$X, Y \in M$
$[A,X]$	$=$	AX	for all	$A \in D(M,M), X \in M$
$[A,B]$	$=$	$A \circ B - B \circ A$	for all	$A, B \in D(M,M)$

(cf. [32], Theorem 7.1, p. 174).

Let L be a connected almost differentiable strongly left alternative left A-loop and let M the corresponding Lie triple algebra. Then the Lie algebra

$\mathbf{g}$ of the Lie group G topologically generated by the left translations of L is isomorphic to the standard enveloping Lie algebra $U(M)$ of M since $\mathbf{m}$ is a Lie triple algebra generating $\mathbf{g}$. If M is a n-dimensional Lie triple algebra, it is easy to see that the dimension of its standard enveloping Lie algebra $U(M)$ is at most $n + \frac{n(n-1)}{2}$.

For any connected differentiable manifold L and any fixed point e and an affine connection ∇, the local binary operation

$$x \cdot y = \exp_x \circ \tau_{e,x} \circ \exp_e^{-1}(y)$$

in some normal neighbourhood of e forms a local loop, which is called geodesic local loop at e. Here $\tau_{e,x}$ denotes the parallel translation of tangent vectors along the geodesic arc joining e to x.

According to Proposition 5.21 in [43] (p. 76) all almost differentiable strongly left alternative local left A-loop L is a local geodesic loop of the canonical connection ∇ of the homogeneous reductive space G/H with respect to $\mathbf{m} = T_H \sigma(U)$ and conversely. U is the neighbourhood of H, where the local loop L is defined.

In this thesis we investigate the strongly left alternative almost differentiable left A-loops L with dimension 3.

A connected topological (local) loop L is called strongly left alternative if it has the following properties:

(1) There is a neighbourhood U of the unit e in L which is simply covered by 1-parameter subgroups.

(2) $x^t \cdot (x^s y) = x^{t+s} \cdot y$ for all $x, y \in L$ and all (local) 1-parameter subgroups $\{x^r; r \in \mathbb{R}\}$ (cf. [43], Definition 5.3, p. 67).

If L is a strongly left alternative loop then one has $\exp[T_1 \sigma(L)] \subseteq \sigma(L)$. Then every global left A-loop contains an exponential image of a complement $\mathbf{m}$ of the Lie algebra $\mathbf{h}$ of H in the Lie algebra $\mathbf{g}$ of G, such that $\mathbf{m}$ generates $\mathbf{g}$ and satisfies the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$.

In this paper we often compute the images of subspaces $\mathbf{m}$ of the Lie algebras $sl_2(\mathbb{R})$, $sl_2(\mathbb{C})$, $su_2(\mathbb{C})$ under the exponential map.

1.2 The exponential function of the Lie algebras $sl_2(\mathbb{R})$, $sl_2(\mathbb{C})$, $su_2(\mathbb{C})$.

The exponential map $\exp : \mathbf{g} \to G$ is defined in the following way: For $X \in \mathbf{g}$ we have $\exp\ X = \gamma_X(1)$, where $\gamma_X(t)$ is the 1-parameter subgroup of G with the property $\frac{d}{dt}\big|_{t=0} \gamma_X(t) = X$. According to [24] we can choose as basis elements for a real basis of $sl_2(\mathbb{R})$ the following elements:

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Lie algebra multiplication is given by the rules:

$$[K,T]=2U,\quad [K,U]=2T,\quad [U,T]=2K.$$

The normalized Cartan-Killing form $k : sl_2(\mathbb{R})\times sl_2(\mathbb{R})\to\mathbb{R}$ of $sl_2(\mathbb{R})$ is the bilinear form defined by: $k(X,Y)=\frac{1}{8}\mathrm{trace}(\mathrm{ad}X\ \mathrm{ad}Y)$.
If $X\in sl_2(\mathbb{R})$ has the decomposition

$$X=\lambda_1\ K+\lambda_2\ T+\lambda_3\ U$$

then the Cartan-Killing form k satisfies

$$k(X)=\lambda_1^2+\lambda_2^2-\lambda_3^2.$$

Moreover the basis $\{K,T,U\}$ is orthonormal with respect to k. For any $X\in sl_2(\mathbb{R})$ we have $X^2=k(X)\cdot I$, whence all even powers of X are scalar multiples of X. Denote by $C(z)$ and $S(z)$ the following power series:

$$C(z)=1+\frac{z}{2!}+\frac{z^2}{4!}+\dots\quad\text{and}\quad S(z)=1+\frac{z}{3!}+\frac{z^2}{5!}+\dots.$$

Using these power series we obtain the formulae

1.
$$C(z^2)+z\ S(z^2)=e^z,$$

$$C(x)=\begin{cases}\cosh\sqrt{x} & \text{for}\quad 0\le x,\\ \cos\sqrt{-x} & \text{for}\quad 0>x,\end{cases}\qquad \sqrt{|x|}\ S(x)=\begin{cases}\sinh\sqrt{x} & \text{for}\quad 0\le x,\\ \sin\sqrt{-x} & \text{for}\quad 0>x,\end{cases}$$

2.
$$C(z)^2-z\ S(z)^2=1,$$

3.
$$C'(z)=\frac{1}{2}S(z).$$

Therefore we have the fundamental formula for the exponential function $\exp : sl_2(\mathbb{R})\to SL_2(\mathbb{R})$:

$$\exp\ X=C(k(X))\ I+S(k(X))\ X.$$

For $X=\lambda_1\ K+\lambda_2\ T+\lambda_3\ U$ with $\lambda_1^2+\lambda_2^2>\lambda_3^2$ we have

4.
$$\exp X=\begin{pmatrix}\cosh\sqrt{l}+\frac{\lambda_1}{\sqrt{l}}\sinh\sqrt{l} & \frac{\lambda_2+\lambda_3}{\sqrt{l}}\sinh\sqrt{l}\\ \frac{\lambda_2-\lambda_3}{\sqrt{l}}\sinh\sqrt{l} & \cosh\sqrt{l}-\frac{\lambda_1}{\sqrt{l}}\sinh\sqrt{l}\end{pmatrix},$$

where $l=\lambda_1^2+\lambda_2^2-\lambda_3^2$. For $\lambda_1^2+\lambda_2^2<\lambda_3^2$ we obtain

5.
$$\exp X=\begin{pmatrix}\cos\sqrt{l}+\frac{\lambda_1}{\sqrt{l}}\sin\sqrt{l} & \frac{\lambda_2+\lambda_3}{\sqrt{l}}\sin\sqrt{l}\\ \frac{\lambda_2-\lambda_3}{\sqrt{l}}\sin\sqrt{l} & \cos\sqrt{l}-\frac{\lambda_1}{\sqrt{l}}\sin\sqrt{l}\end{pmatrix},$$

where $l=\lambda_3^2-\lambda_1^2-\lambda_2^2$.

As a natural generalization of the fundamental formula for the exponential function of $sl_2(\mathbb{R})$ we obtain the explicite form for the exponential function of $sl_2(\mathbb{C})$. Representing the Lie algebra $\mathbf{g} = sl_2(\mathbb{C})$ as complex (2×2)-matrices we may choose as basis $\{K, T, U, \mathrm{i}K, \mathrm{i}T, \mathrm{i}U\}$, where K, T, U are the basis elements of $sl_2(\mathbb{R})$ (see in **1.2**). The Lie algebra multiplication is given by

$$[K, T] = 2U, \ [K, U] = 2T, \ [U, T] = 2K.$$

The normalized complex Cartan-Killing form $k_{\mathbb{C}} : sl_2(\mathbb{C}) \times sl_2(\mathbb{C}) \to \mathbb{C}$ of $sl_2(\mathbb{C})$ is the bilinear form defined by: $k_{\mathbb{C}}(X, Y) = \frac{1}{8}\mathrm{trace}(\mathrm{ad}X \ \mathrm{ad}Y)$. If $X \in sl_2(\mathbb{C})$ has the decomposition

$$X = \lambda_1 \ K + \lambda_2 \ T + \lambda_3 \ U + \lambda_4 \ \mathrm{i}K + \lambda_5 \ \mathrm{i}T + \lambda_6 \ \mathrm{i}U$$

then the complex Cartan-Killing form $k_{\mathbb{C}}$ satisfies

$$k_{\mathbb{C}}(X) = \lambda_1^2 + \lambda_2^2 + \lambda_6^2 - \lambda_3^2 - \lambda_4^2 - \lambda_5^2 + i \ (2 \ \lambda_1\lambda_4 + 2 \ \lambda_2\lambda_5 - 2 \ \lambda_3\lambda_6)$$

(cf. [49], Section 6.1, pp. 215-228; [22], Part II, pp. 86-100; [19], Section 1, pp. 1-3). The normalized real Cartan-Killing form $k_{\mathbb{R}} : sl_2(\mathbb{C}) \times sl_2(\mathbb{C}) \to \mathbb{R}$ is the restriction of $k_{\mathbb{C}}$ to $\mathbb{R}$ such that

$$k_{\mathbb{R}}(X) = \lambda_1^2 + \lambda_2^2 + \lambda_6^2 - \lambda_3^2 - \lambda_4^2 - \lambda_5^2$$

and the basis $\{K, T, U, \mathrm{i}K, \mathrm{i}T, \mathrm{i}U\}$ is orthonormal with respect to $k_{\mathbb{R}}$.
We consider the same power series $C(z)$ and $S(z)$ as in the above case. For $z \in \mathbb{C}$ one has $C(z) = \cosh\sqrt{z}$ and $S(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}$. The formulae 1,2,3 are satisfied too. Therefore the fundamental formula for the exponential function $\exp : sl_2(\mathbb{C}) \to SL_2(\mathbb{C})$ is

$$\exp \ X = C(k_{\mathbb{C}}(X)) \ I + S(k_{\mathbb{C}}(X)) \ X.$$

The group $SU_2(\mathbb{C})$ is the 3-dimensional compact subgroup of $SL_2(\mathbb{C})$, which can be represented by (2×2)-complex matrices having the form:

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}; a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1 \right\}.$$

Therefore the Lie algebra $\mathbf{g} = su_2(\mathbb{C})$ is the Lie algebra of matrices

$$(\lambda_1 U + \lambda_2 \ \mathrm{i}K + \lambda_3 \ \mathrm{i}T) \mapsto \begin{pmatrix} \lambda_2 i & \lambda_1 + \lambda_3 i \\ -\lambda_1 + \lambda_3 i & -\lambda_2 i \end{pmatrix}.$$

The restriction of the fundamental formula for $\exp : sl_2(\mathbb{C}) \to SL_2(\mathbb{C})$ to $su_2(\mathbb{C})$ gives the fundamental formula for $\exp : su_2(\mathbb{C}) \to SU_2(\mathbb{C})$.
For $X = \lambda_1 \ U + \lambda_2 \ \mathrm{i}K + \lambda_3 \ \mathrm{i}T$ we have $k_{\mathbb{C}}(X) = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2$ and

6. $$\exp\ X = \begin{pmatrix} \cos l + \frac{\lambda_2 i \sin l}{l} & \frac{(\lambda_1 + \lambda_3 i)\sin l}{l} \\ \frac{(-\lambda_1 + \lambda_3 i)\sin l}{l} & \cos l - \frac{\lambda_2 i \sin l}{l} \end{pmatrix},$$

where $l = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$.

1.3 Now we formulate some results which pairs (G, H) of Lie groups can be occur such that G is the group topologically generated by the left translations of a 3-dimensional almost differentiable left A-loop L and H is the stabilizer of $e \in L$ in G.
First we give some lemmata needed later and start with a well known fact from linear algebra:

Lemma 1. *If* $\mathbf{g} = \mathbf{a} \oplus \mathbf{b}$ *such that dim* $\mathbf{a} = 3$ *and the dimension of the subalgebra* $\mathbf{b}$ *is* 1 *respectively* 2 *then any* 3*-dimensional subspace* $\mathbf{m}$ *of* $\mathbf{g}$ *has an at least* 2*-dimensional respectively* 1*-dimensional intersection with the subalgebra* $\mathbf{a}$*.*

Lemma 2. *Let* L *be an almost differentiable loop and denote by* $\mathbf{m}$ *the tangent space of* L *at* $1 \in G$*. Then* $\mathbf{m}$ *does not contain any element of* $Ad_g\mathbf{h}$ *for some* $g \in G$*.*

Proof. For $g \in G$ the group $H^g = g^{-1}Hg$ is the stabilizer of the element $g^{-1}(e) \in L$ in G. □

Every subloop of a left A-loop is a left A-loop and the direct product of two left A-loops is again a left A-loop.

Proposition 3. *Let* L *be a loop and let* G *be the group generated by the left translations of* L*, and denote by* H *the stabilizer of* $e \in L$ *in* G*. If* G *and* H *are direct products* $G = G_1 \times G_2$ *and* $H = H_1 \times H_2$ *with* $H_i \subset G_i$ $(i = 1, 2)$ *then the loop* L *is the direct product of two loops* L_1 *and* L_2*, and* L_i *is isomorphic to a loop* L_i^* *having* G_i *as the group generated by the left translations of* L_i^* *and* H_i *as the corresponding stabilizer subgroup* $(i = 1, 2)$*. In particular there exists no* 3*-dimensional left A-loop* L *such that* L *is the direct product of a* 1*-dimensional and a* 2*-dimensional left A-loop and* L *has a* 5*- or* 6*-dimensional Lie group as the group topologically generated by its left translations.*

Proof. The first assertion is the Proposition 1.18. (p. 26) in [43]. The second assertion it follows from the fact that a 1-dimensional almost differentiable left A-loop is either the group $\mathbb{R}$ or the group $SO_2(\mathbb{R})$. Hence dim $G \leq 4$. □

A Lie group G is simple if the centre of G is trivial. We call a connected Lie group quasi-simple if its Lie algebra is simple. A Lie group G is called semisimple if it is the direct product of simple Lie groups. A connected Lie group G with the Lie algebra $\mathbf{g}$ is solvable if the series $\mathbf{g}_0 = \mathbf{g}$, $\mathbf{g}_i = [\mathbf{g}_{i-1}, \mathbf{g}_{i-1}]$, $i = 1, 2, \ldots$ becomes for some i zero and a connected Lie group G with the Lie algebra $\mathbf{g}$ is nilpotent if the series $\mathbf{g}_0 = \mathbf{g}$, $\mathbf{g}_i = [\mathbf{g}_{i-1}, \mathbf{g}]$, $i = 1, 2, \ldots$ becomes for some i zero.

Let L a 3-dimensional proper left A-loop. We known that the dimension of the group topologically generated by its left translations is at least 4 and at most 6.
Any 4-dimensional non-solvable Lie group is locally isomorphic to one of the following Lie groups:
1) $G = PSL_2(\mathbb{R}) \times \mathbb{R}$
2) $G = SO_3(\mathbb{R}) \times \mathbb{R}$.
Any 5-dimensional non-solvable Lie group is locally isomorphic to one of the following Lie group:
1) $PSL_2(\mathbb{R}) \times \mathbb{R}^2$
2) $PSL_2(\mathbb{R})\times$ $\mathcal{L}_2$, where $\mathcal{L}_2 \cong \{x \mapsto ax + b; a > 0, b \in \mathbb{R}\}$.
3) $SO_3(\mathbb{R}) \times \mathbb{R}^2$
4) $SO_3(\mathbb{R})\times$ $\mathcal{L}_2$
5) The semidirect product of the group $PSL_2(\mathbb{R})$ and $\mathbb{R}^2$, where $PSL_2(\mathbb{R})$ acts on the natural way on $\mathbb{R}^2$.
Any 6-dimensional semisimple Lie group is locally isomorphic to one of the following Lie groups:
1. $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$
2. $PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$
3. $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$
4. $PSL_2(\mathbb{C})$.

If a 6-dimensional Lie group G is non-solvable and non-semisimple, then it is the semidirect product of a solvable 3-dimensional Lie group G_2 and a simple 3-dimensional Lie group G_1. The simple 3-dimensional Lie groups are locally isomorphic to $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$. All solvable 3-dimensional Lie groups G_2 are explicitly known (cf. [29] pp. 12-14, [46] p. 255).
Any 6-dimensional non-semisimple and non-solvable Lie group is locally isomorphic to one of the following Lie groups:
5) $PSL_2(\mathbb{R}) \times \mathbb{R}^3$
6) $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$,where the action of $PSL_2(\mathbb{R})$ on $\mathbb{R}^3$ is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra.
7) $PSL_2(\mathbb{R}) \times \tilde{G_1}$, where $\tilde{G_1}$ is the 3-dimensional simply connected non-commutative nilpotent Lie group, which is represented on $\mathbb{R}^3$ by the multi-

plication

$$g(u_1,v_1,z_1)g(u_2,v_2,z_2)=g(u_1+u_2,v_1+v_2,z_1+z_2+\frac{1}{2}(u_1v_2-u_2v_1)).$$

8) $PSL_2(\mathbb{R})\times\tilde{G_2}$, where $\tilde{G_2}$ is the 3-dimensional Lie group $\mathcal{L}_2\times\mathbb{R}$.
9) $PSL_2(\mathbb{R})\times\tilde{G_3}$, where $\tilde{G_3}$ is the 3-dimensional solvable Lie group with trivial centre having precisely one 1-dimensional normal subgroup. This group is represented by the group of matrices

$$g(u,v,w)=\begin{pmatrix}1 & u & v\\ 0 & e^w & we^w\\ 0 & 0 & e^w\end{pmatrix},\ u,v,w\in\mathbb{R}.$$

10) $PSL_2(\mathbb{R})\times\tilde{G_4}$, where $\tilde{G_4}$ is the 3-dimensional solvable connected Lie group having precisely two 1-dimensional normal subgroups. This Lie group consists of the matrices

$$g(u,v,w)=\begin{pmatrix}1 & u & v\\ 0 & e^{aw} & 0\\ 0 & 0 & e^{bw}\end{pmatrix},u,v,w\in\mathbb{R}$$

with fixed but different numbers $a,b\in\mathbb{R}\backslash\{0\}$.
11) $PSL_2(\mathbb{R})\times\tilde{G_5}$, where $\tilde{G_5}$ is the 3-dimensional solvable connected Lie group with infinitely many 1-dimensional normal subgroups. This group which is almost abelian can be represented by the group of matrices

$$\left\{g(u,v,w)=\begin{pmatrix}1 & u & v\\ 0 & e^{aw} & 0\\ 0 & 0 & e^{aw}\end{pmatrix};u,v,w\in\mathbb{R}\right\}$$

with a fixed number $a\in\mathbb{R}\backslash\{0\}$.
12) $PSL_2(\mathbb{R})\times\tilde{G_6}$, where $\tilde{G_6}$ is the 3-dimensional solvable connected Lie group having no 1-dimensional normal subgroup. We can identify $\tilde{G_6}$ with the linear group of matrices

$$\left\{g(t,u,v)=\begin{pmatrix}1 & u & v & 0\\ 0 & \cos t & \sin t & 0\\ 0 & -\sin t & \cos t & 0\\ 0 & 0 & 0 & e^t\end{pmatrix};t,u,v\in\mathbb{R}\right\}.$$

13) G is the connected component of the group of affinities of $\mathbb{R}^2$. This group

we can identify with the group of matrices

$$\left\{ g(u,v,a,b,c,d) = \begin{pmatrix} 1 & u & v \\ 0 & a & b \\ 0 & c & d \end{pmatrix}; u,v,a,b,c,d \in \mathbb{R},\ ad - bc > 0 \right\}.$$

14) $SO_3(\mathbb{R}) \times \mathbb{R}^3$,
15) $SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, which is the connected component of the euclidean motions of $\mathbb{R}^3$.
16)-22) $SO_3(\mathbb{R}) \times \tilde{G}_i$, where $i = 1, \dots, 6$.

Proposition 4. *If L is a connected almost differentiable 3-dimensional left A-loop, then the pairs (G,H) for the factor space G/H must be different from the following cases:*
a) $G = SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$, $H \in \{U_0,\ U_1\}$ respectively $H \in \{U_0/\mathbb{Z}_2,\ U_1/\mathbb{Z}_2\}$ with $U_r = \left\{ \begin{pmatrix} z & (r-1)w \\ -(r+1)\bar{w} & \bar{z} \end{pmatrix}; |z|^2 + (r^2-1)|w|^2 = 1 \right\}$.
b) G is locally isomorphic to $SO_3(\mathbb{R}) \times \mathbb{R}$ and H is any 1-dimensional subgroup of G.
c) The group $G = SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$ is the connected component of the euclidean motion group and H is a semidirect product of a 2-dimensional translation group $\mathbb{R}^2$ by a 1-dimensional rotation group $SO_2(\mathbb{R})$.
d) $G = SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$, and H is any 3-dimensional subgroup of G.
e) $G = SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ and $H = W_r$ respectively $W_r\mathbb{Z}_2/\mathbb{Z}_2$, where $W_r = \left\{ \begin{pmatrix} \exp((ri-1)x) & 0 \\ z & \exp(-(ri-1)x) \end{pmatrix}; x \in \mathbb{R}, z \in \mathbb{C} \right\}$ for $r \in \mathbb{R}$.

Proof. In the case b) every 1-dimensional subgroup H of G containing no non-trivial normal subgroup of G has one of the following shapes:

$$H_1 = \{K \times \{0\}\}, \quad \text{or} \quad H_2 = \{K \times \varphi(K)\},$$

where K is isomorphic to $SO_2(\mathbb{R})$ and φ is a non-trivial homomorphism.
In the case d) every 3-dimensional subgroup H of G, which does not contain normal subgroup $\neq \{1\}$ of G is conjugate to $\{(a,a) \mid a \in SO_3(\mathbb{R})\}$.
In the cases a), c) and for H_1 in the case b) the factor space G/H is a topological product of spaces having as a factor one of the following non-parallelizable manifolds: S^2 or P^2. But the space $L = G/H$ must be parallelizable, since the tangent maps $(\lambda_x)_*\colon T_eL \to T_xL$ of the left translations λ_x $(x \in L)$ of an almost differentiable loop define a global section $x \mapsto \{((\lambda_x)_*e_1, \dots, (\lambda_x)_*e_n\}$ in the n-frame bundle over L, where $\{e_1, \dots, e_n\}$ is a basis of T_eL.
In the cases d), e) and for H_2 in the case b) the factor space G/H is either

S^3 or P^3. In the case d) the group G is compact and the assertion follows from Proposition 16.11 (cf. [43], p. 205). In the case e) we prove that the homogeneous spaces $SL_2(\mathbb{C})/W_r$ and $PSL_2(\mathbb{C})/(W_r\mathbb{Z}_2/\mathbb{Z}_2)$ is not reductive. We consider the real basis $\{K, T, U, \mathrm{i}K, \mathrm{i}T, \mathrm{i}U\}$ of $\mathbf{g} = sl_2(\mathbb{C})$ defined in **1.2**. Then the Lie algebra $\mathbf{h} = w_r$ of the stabilizer W_r has the following basis elements:

$$\{r\ i\ K - \ K, i\ T - i\ U, U - T\} \quad r \in \mathbb{R}.$$

For $r \neq 0$ a complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ contains a basis element with one of the following forms: $K + f(K)$ or $i\ K + f(i\ K)$, if $r = 0$ then a complement $\mathbf{m}$ contains a basis element having the form: $i\ K + f(i\ K)$, where $f : \mathbf{m} \to \mathbf{h}$ is a linear map. The complement $\mathbf{m}$ is reductive if the following relation is satisfied: $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. Since the element

$$[U - T, K + f(K)] = [U - T, K] + [U - T, f(K)] =$$

$$2U - 2T + [U - T, f(K)]$$

is an element of the intersection $\mathbf{h} \cap \mathbf{g} = \{0\}$, we have $[U - T, f(K)] = 2T - 2U$. This is the case precisely if $f(K) = -K$ but then $f(K)$ is not an element of $\mathbf{h}$. This is a contradiction. We obtain the same contradiction if $i\ K + f(i\ K) \in \mathbf{m}$.

In the case b) the Lie algebra $\mathbf{g}$ of G can be represented as $su_2(\mathbb{C}) \oplus \mathbb{R}$. Then as a basis of $\mathbf{g}$ may be chosen the following elements

$$i(K, 0), (U, 0), i(T, 0), (0, e_1),$$

where iK, U, iT is the real basis of $su_2(\mathbb{C})$ which is introduced in **1.2** and e_1 is the basis element of $\mathbb{R}$. Therefore in $\mathbf{g}$ one has the following multiplication:

$$[i(K, 0), i(T, 0)] = -2(U, 0), \ [i(K, 0), (U, 0)] = 2i(T, 0),$$

$$[(U, 0), i(T, 0)] = 2i(K, 0),$$

$$[i(K, 0), (0, e_1)] = [i(T, 0), (0, e_1)] = [(U, 0), (0, e_1)] = (0, 0).$$

Since every 1-dimensional subgroup $SO_3(\mathbb{R})$ is isomorphic to $SO_2(\mathbb{R})$ we may assume that the Lie algebra $\mathbf{h}$ of H is generated by the basis element

$$(U, c\ e_1), \ c \in \mathbb{R} \backslash \{0\}.$$

Since the automorphism $A : \mathbf{g} \to \mathbf{g}$ given by:

$$\begin{array}{rclrcl} A(i(K,0)) & = & i(K,0), & A(i(T,0)) & = & i(T,0), \\ A(U,0) & = & (U,0), & A(0,e_1) & = & (0,ce_1) \end{array}$$

maps $(U, c\ e_1)$ onto (U, e_1) we may assume $H = \{(x,x)|\ x \in SO_2(\mathbb{R})\}$ and $\mathbf{h} = \langle (U, e_1) \rangle$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the shape:

$$\mathbf{m} = \langle (iK + a_1\ U, a_1\ e_1), (iT + a_2\ U, a_2\ e_1), (a_3\ U, e_1 + a_3\ e_1) \rangle,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. From Lemma 1 we obtain that $\mathbf{m}$ has one of the following forms:

$$\mathbf{m_1} = \langle i(K,0), i(T,0), (a_3\ U, e_1 + a_3\ e_1) \rangle,$$

where $a_3 \in \mathbb{R} \backslash \{-1\}$,

$$\mathbf{m_2} = \langle i(K,0), (iT, a_2\ e_1), (U,0) \rangle,$$

where $a_2 \in \mathbb{R} \backslash \{0\}$,

$$\mathbf{m_3} = \langle (iK, a_1\ e_1), i(T,0), (U,0) \rangle,$$

where $a_1 \in \mathbb{R} \backslash \{0\}$.
For the complements $\mathbf{m}_i$ $i = 1,2,3$ must be satisfied $[\mathbf{h}, \mathbf{m}_i] \subseteq \mathbf{m}_i$. In case $i = 2$ the element

$$[(U, e_1), (iK, 0)] = (-2\ iT, 0)$$

is contained in $\mathbf{m}_2$ if and only if $a_2 = 0$ and for $i = 3$ the element

$$[(U, ce_1), (iT, 0)] = (2\ iK, 0)$$

is contained in $\mathbf{m}_3$ precisely if $a_1 = 0$. We see that $\mathbf{m}_i$ for $i = 2,3$ does not generate $\mathbf{g}$.
The complement $\mathbf{m}_1$ is reductive to $\mathbf{h}$ and generates $\mathbf{g}$.
For $a_3 > -\frac{1}{2}$ the basis element $(U, e_1) \in \mathbf{h}$ is conjugate to the element

$$-2kl(iT, 0) + (\frac{a_3}{1+a_3} U, e_1) \in \mathbf{m}_{a_3}$$

under the element $\left(\pm \begin{pmatrix} k+li & 0 \\ 0 & k-li \end{pmatrix}, 0 \right) \in G$, such that $k^2 - l^2 = \frac{a_3}{1+a_3}$ and $k^2 + l^2 = 1$. This is a contradiction to Lemma 2.
For $a_3 < -\frac{1}{2}$ we prove that there is no global section $\sigma : G/H \to G$ satisfying $\exp \mathbf{m}_{a_3} \subseteq \sigma(G/H)$. Clearly the stabilizer H has the form:

$$H = \left\{ \left(\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right); t \in \mathbb{R} \right\}.$$

The complement $\exp \mathbf{m}_{a_3}$ is the following:

$$\left(\exp(\lambda_1 U + \lambda_2 iK + \lambda_3 iT), \exp\left(\frac{\lambda_1(1+a_3)}{a_3} e_1 \right) \right),$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, a_3 < -\frac{1}{2}$. The first component of $\exp \mathbf{m}_{a_3}$ is given by the form 6 in **1.2**. The second component of $\exp \mathbf{m}_{a_3}$ can be written in the following form:

$$(\exp \mathbf{m}_{a_3})_2 = \begin{pmatrix} \cos \dfrac{\lambda_1(1+a_3)}{a_3} & \sin \dfrac{\lambda_1(1+a_3)}{a_3} \\ -\sin \dfrac{\lambda_1(1+a_3)}{a_3} & \cos \dfrac{\lambda_1(1+a_3)}{a_3} \end{pmatrix}.$$

If $\exp \mathbf{m}_{a_3}$ would be contained in a global section σ then each element $g \in G$ can be uniquely represented as a product $g = m \cdot h$ with $m \in \exp \mathbf{m}_{a_3}$ and $h \in H$. For $-2 < a_3 < -\frac{1}{2}$ the vectors

$$X_1 = \left(\frac{\pi}{2} U + \frac{\sqrt{15}}{2} \pi i K, \frac{\pi(1+a_3)}{2a_3} e_1 \right) \quad \text{and}$$

$$X_2 = \left(\frac{\pi(1+a_3)}{2} U, \frac{\pi(1+a_3)^2}{2a_3} e_1 \right)$$

are contained in $\mathbf{m}_{a_3}$. We have

$$\exp X_1 = \left(\pm I, \begin{pmatrix} \cos \frac{\pi(1+a_3)}{2a_3} & \sin \frac{\pi(1+a_3)}{2a_3} \\ -\sin \frac{\pi(1+a_3)}{2a_3} & \cos \frac{\pi(1+a_3)}{2a_3} \end{pmatrix} \right), \quad \exp X_2 =$$

$$\left(\pm \begin{pmatrix} \cos \frac{\pi(1+a_3)}{2} & \sin \frac{\pi(1+a_3)}{2} \\ -\sin \frac{\pi(1+a_3)}{2} & \cos \frac{\pi(1+a_3)}{2} \end{pmatrix}, \begin{pmatrix} \cos \frac{\pi(1+a_3)^2}{2a_3} & \sin \frac{\pi(1+a_3)^2}{2a_3} \\ -\sin \frac{\pi(1+a_3)^2}{2a_3} & \cos \frac{\pi(1+a_3)^2}{2a_3} \end{pmatrix} \right),$$

where $\pm I$ is the identity of $SO_3(\mathbb{R})$. The element

$$g = \left(\pm I, \begin{pmatrix} \cos \frac{\pi(1+a_3)}{2a_3} & \sin \frac{\pi(1+a_3)}{2a_3} \\ -\sin \frac{\pi(1+a_3)}{2a_3} & \cos \frac{\pi(1+a_3)}{2a_3} \end{pmatrix} \right) \in G$$

can be written in two different ways as a product $g = m \cdot h$, where $m \in \exp \mathbf{m}_{a_3}$ and $h \in H$. On the one hand one has $g = \exp X_1 \cdot 1$, where 1 is the identity of G, on the other hand $g = \exp X_2 \cdot h$ with

$$h = \left(\pm \begin{pmatrix} \cos \frac{\pi(1+a_3)}{2} & -\sin \frac{\pi(1+a_3)}{2} \\ \sin \frac{\pi(1+a_3)}{2} & \cos \frac{\pi(1+a_3)}{2} \end{pmatrix}, \begin{pmatrix} \cos \frac{\pi(1+a_3)}{2} & -\sin \frac{\pi(1+a_3)}{2} \\ \sin \frac{\pi(1+a_3)}{2} & \cos \frac{\pi(1+a_3)}{2} \end{pmatrix} \right).$$

For $a_3 < -2$ the vectors

$$Y_1 = \left(\frac{\pi}{2(1+a_3)} U + \sqrt{4\pi^2 - \frac{\pi^2}{(1+a_3)^2}} iK, \frac{\pi}{2a_3} e_1 \right) \quad \text{and}$$

$$Y_2 = \left(\frac{\pi}{2} U, \frac{\pi(1+a_3)}{2a_3} e_1 \right)$$

are elements of $\mathbf{m}_{a_3}$. The exponential images are

$$\exp Y_1 = \left(\pm I, \begin{pmatrix} \cos\frac{\pi}{2a_3} & \sin\frac{\pi}{2a_3} \\ -\sin\frac{\pi}{2a_3} & \cos\frac{\pi}{2a_3} \end{pmatrix} \right),$$

$$\exp Y_2 = \left(\pm \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}, \begin{pmatrix} \cos\frac{\pi(1+a_3)}{2a_3} & \sin\frac{\pi(1+a_3)}{2a_3} \\ -\sin\frac{\pi(1+a_3)}{2a_3} & \cos\frac{\pi(1+a_3)}{2a_3} \end{pmatrix} \right).$$

The element $g = \left(\pm I, \begin{pmatrix} \cos\frac{\pi}{2a_3} & \sin\frac{\pi}{2a_3} \\ -\sin\frac{\pi}{2a_3} & \cos\frac{\pi}{2a_3} \end{pmatrix} \right) \in G$ can be written on the one hand as the product $g = \exp Y_1 \cdot 1$, where 1 is the identity of G, on the other hand as the product $g = \exp Y_2 \cdot h$ with

$$h = \left(\pm \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}, \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \right) \in H.$$

□

Proposition 5. *Let $G = G_1 \times G_2$ be the group topologically generated by the left translations of a 3-dimensional connected almost differentiable left A-loop L such that $G_i \quad (i = 1, 2)$ are 3-dimensional quasi-simple Lie groups. Then G_i $(i = 1, 2)$ is isomorphic to $PSL_2(\mathbb{R})$ and the stabilizer H of $e \in L$ in G may be chosen either as*

(i) $$H_1 = \{(x, x) \mid x \in PSL_2(\mathbb{R})\}$$

or

(ii) $$H_2 = \left\{ \left(\begin{pmatrix} a & b_1 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & b_2 \\ 0 & a^{-1} \end{pmatrix} \right); a > 0, b_1, b_2 \in \mathbb{R} \right\}.$$

Proof. Denote by $\pi_i : G \to G_i$ the natural projection of G to G_i for $i = 1, 2$. Let $\dim \pi_1(H) \leq 1$. Since $H \leq \pi_1(H) \times \pi_2(H)$ one has $\dim \pi_2(H) \geq 2$. If $\dim \pi_2(H) = 2$ then H is the direct product of $\pi_1(H)$ with $\pi_2(H)$. Since the corresponding loop L is the direct product of a 2-dimensional and a 1-dimensional left A-loop we obtain a contradiction to Proposition 3. Let $\pi_2(H) = G_2$. If $\dim \pi_1(H) = 0$ then $H = \{1\} \times G_2$ which is also impossible (Proposition 3). If $\pi_1(H)$ is a 1-dimensional subgroup then H would be 4-dimensional since there is no non-trivial homomorphism from G_2 into $\pi_1(H)$. Let now $\dim \pi_1(H) = 2$. We may assume that $\dim \pi_2(H) \geq 2$ since interchanging the indices we would obtain the previous case. Therefore each of the factors of G is locally isomorphic to the group $PSL_2(\mathbb{R})$. Since the Lie algebra of G_2 is simple there is no non-trivial homomorphism from $\pi_2(H) = G_2$

into $\pi_1(H)$, so $\pi_2(H)$ cannot be G_2. If dim $\pi_2(H)$ = dim $\pi_1(H) = 2$ then $\pi_i(H) \cong \mathcal{L}_2 = \{ax + b \mid a > 0, b \in \mathbb{R}\}$ $(i = 1, 2)$ and there exist homomorphisms $\varphi_1 : \pi_1(H) \to \pi_2(H)$ and $\varphi_2 : \pi_2(H) \to \pi_1(H)$ with 1-dimensional kernels. Since $H \cap G_i = ker\ \varphi_i = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R} \right\}$ and (up to conjugation) $im\ \varphi_i = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 0 \right\}$ $(i = 1, 2)$ we obtain that

$$\pi_1(H) = ker\ \varphi_1\ im\ \varphi_2,\ \pi_2(H) = ker\ \varphi_2\ im\ \varphi_1$$

and H has the shape (ii).
Finally let dim $\pi_1(H) = 3$. From the previous arguments it follows that dim $\pi_2(H) = 3$. If there is no homomorphism $\varphi : G_1 \to G_2$ then there is no 3-dimensional subgroup H of $G = G_1 \times G_2$. If there exists a homomorphism $\varphi : G_1 \to G_2$ then the stabilizer H has the shape $\{(x, \varphi(x)) \mid x \in G_1\}$. Moreover φ must be an isomorphism since otherwise G would contain a discrete central subgroup of G. Hence G can be identified with $G_1 \times G_1$, where G_1 is isomorphic either to $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$, and H may be chosen as the diagonal subgroup $\{(x, x) \mid x \in G_1\}$. According to Proposition 2 d) we have $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. □

Proposition 6. *Let $G = G_1 \times G_2$ be a group topologically generated by the left translations of a 3-dimensional connected almost differentiable left A-loop L. Let G_2 be a commutative Lie group $\neq 1$ and G_1 be locally isomorphic to $PSL_2(\mathbb{R})$ or to $SO_3(\mathbb{R})$. Then one of the following cases can occur:*
1) L is the direct product of the hyperbolic plane loop with a 1-dimensional Lie group.
2) G is isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}$ and $H = \{(x, \varphi(x))\}$, where φ is a monomorphism from the 1-dimensional subgroup $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R} \right\}$ or from $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 0 \right\}$ of $PSL_2(\mathbb{R})$ onto $\mathbb{R}$.
3) $G = PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ such that $H = \{(x, \varphi(x))\}$, where φ is a monomorphism from a maximal compact subgroup of $PSL_2(\mathbb{R})$ onto $SO_2(\mathbb{R})$.
4) G is locally isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}^2$ and H has the shape:

$$H = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, a \right);\ a, b \in \mathbb{R} \right\}.$$

Proof. Denote by $\pi_i : G \to G_i$ the natural projection onto G_i, $i = 1, 2$ then $\pi_2(H)$ is a connected abelian Lie group. If $\pi_2(H) = 1$ then $H = H_1 \times \{1\}$ with $H_1 \leq G_1$ and $L = L_1 \times L_2$ is a direct product of a 2-dimensional almost differentiable left A-loop L_1 and a 1-dimensional almost differentiable left A-loop L_2. Moreover the loop L_1 is isomorphic to a 2-dimensional almost

differentiable left A-loop having the group G_1 as the group topologically generated by its left translations (Proposition 3). According to Theorem 27.1 and Theorem 18.14 in [43] the loop L_1 is isomorphic to the hyperbolic plane loop and the loop L_2 is isomorphic to the group $\mathbb{R}$ or to $SO_2(\mathbb{R})$.
Let now $\pi_2(H) \neq 1$. Since H does not contain normal subgroup $\neq 1$ it follows that $\varphi : \pi_2(H) \to \pi_1(H)$ is a monomorphism. The group $\varphi(\pi_2(H))$ is commutative hence $\dim \varphi(\pi_2(H)) = \dim \pi_2(H) = 1$.
If G is 4-dimensional, then the stabilizer H is 1-dimensional and it has the shape $H = \{(\varphi(x), x) \mid x \in G_2\}$. According to Proposition 3 the group G_1 is isomorphic to $PSL_2(\mathbb{R})$. The inverse of φ is again a monomorphism from $\varphi(\pi_2(H))$ onto $\pi_2(H)$ and there are two types of the 1-dimensional subgroups of $PSL_2(\mathbb{R})$ isomorphic to $\mathbb{R}$. This is the case 2 in the assertion. For $G_2 \cong SO_2(\mathbb{R})$ we obtain that $H = \{(x, \varphi^{-1}(x))\}$, where x are elements of a maximal compact subgroup of $PSL_2(\mathbb{R})$.
Let now $\dim G = 5$. The dimension of the stabilizer H is 2 and one has $H \le \pi_1(H) \times \pi_2(H)$ such that $\dim \pi_i(H) \le \dim H$ for $i = 1, 2$. Hence the dimension of $\pi_1(H)$ is either 1 or 2. If $\dim \pi_1(H) = 1$ then the stabilizer H is the direct product of $\pi_1(H)$ and $\pi_2(H)$ and the corresponding loop is again the direct product of a 2-dimensional and a 1-dimensional left A-loop. Then the group G cannot be 5-dimensional (Proposition 3). Let now $\dim \pi_1(H) = 2$. Therefore G_1 is locally isomorphic to $PSL_2(\mathbb{R})$ and we may assume that $\pi_1(H) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$. In this case we obtain only one conjugacy class of the 2-dimensional subgroups of G, which is represented by $H = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, a \right), a > 0, b \in \mathbb{R} \right\}$.
If $\dim G = 6$ then the stabilizer H has dimension 3 and from the previous arguments it follows that $\dim \pi_1(H) = 3$. Since there is no homomorphism φ from a quasi-simple Lie group $G_1 = \pi_1(H)$ onto $\pi_2(H)$ this case is impossible. □

Proposition 7. *Let $G = G_1 \times G_2$ be a Lie group topologically generated by the left translations of a connected 3-dimensional almost differentiable left A-loop. Let G_1 be locally isomorphic to one of the 3-dimensional simple Lie groups and let G_2 be a solvable non-abelian Lie group. Then one of the following cases can occur:*
1) $G \cong PSL_2(\mathbb{R}) \times \mathcal{L}_2$, where $\mathcal{L}_2 = \{ax + b \mid a > 0, b \in \mathbb{R}\}$ and
a) the stabilizer H of $e \in L$ in G is

$$H \cong \left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right), a > 0, b \in \mathbb{R} \right\},$$

b) $H \cong \left\{ \left(\left(\begin{array}{cc} e^z & b \\ 0 & e^{-z} \end{array} \right), \left(\begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) \right), z, b \in \mathbb{R} \right\}$,
c) $H \cong \{(\varphi(x), x) \mid x \in \mathcal{L}_2\}$, *where* φ *is a monomorphism,*
2 A) G *is locally isomorphic to* $PSL_2(\mathbb{R}) \times G_2$, *where* G_2 *is the* 3*-dimensional non-commutative nilpotent Lie group and the stabilizer* H *has the shape*

$$H = \left\{ \left(\left(\begin{array}{cc} e^a & b \\ 0 & e^{-a} \end{array} \right), g(c + k\ a, 0, l\ a) \right), a, b, c \in \mathbb{R} \right\},$$

where $k \in \mathbb{R}$, $l \in \mathbb{R} \backslash \{0\}$ *are given parameters,*
2 B) G *is locally isomorphic to* $PSL_2(\mathbb{R}) \times G_2$, *where* G_2 *is the* 3*-dimensional Lie group* $\mathcal{L}_2 \times \mathbb{R}$ *and the* 3*-dimensional stabilizer* H *has one of the following forms:*

$$H_1 = \left\{ \left(\left(\begin{array}{cc} e^a & b \\ 0 & e^{-a} \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & la \\ 0 & e^{c+ka} & 0 \\ 0 & 0 & 1 \end{array} \right) \right), a, b, c \in \mathbb{R} \right\},$$

k, l *are given real numbers, such that* $l \neq 0$, *or*

$$H_2 = \left\{ \left(\left(\begin{array}{cc} e^a & b \\ 0 & e^{-a} \end{array} \right), \left(\begin{array}{ccc} 1 & c + la & c + ka \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right), a, b, c \in \mathbb{R} \right\},$$

with given real numbers k, l,
2 C) G *is locally isomorphic to* $PSL_2(\mathbb{R}) \times G_2$, *where* G_2 *is the* 3*-dimensional solvable Lie group having precisely two* 1*-dimensional normal subgroups and*

$$H = \left\{ \left(\left(\begin{array}{cc} e^a & b \\ 0 & e^{-a} \end{array} \right), \left(\begin{array}{ccc} 1 & c + la & c + ka \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right), a, b, c \in \mathbb{R} \right\},$$

where k, l *are given real numbers,*
2 D) G *is locally isomorphic to* $PSL_2(\mathbb{R}) \times G_2$, *where* G_2 *is the* 3*-dimensional solvable Lie group with precisely one* 1*-dimensional normal subgroup. The stabilizer* H *of* $e \in L$ *in* G *is*

$$H = \left\{ \left(\left(\begin{array}{cc} e^a & b \\ 0 & e^{-a} \end{array} \right), \left(\begin{array}{ccc} 1 & c + ka & la \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right), a, b, c \in \mathbb{R} \right\}, k \in \mathbb{R}, l \in \mathbb{R} \backslash \{0\},$$

2 F) G *is locally isomorphic to* $PSL_2(\mathbb{R}) \times G_2$, *where* G_2 *is locally isomorphic to the group of orientation preserving motions of the euclidean plane and the*

3-dimensional stabilizer H can be written in the following shape

$$H = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, \begin{pmatrix} 1 & c+ka & la & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right), a, b, c \in \mathbb{R} \right\},$$

with $k \in \mathbb{R}$, $l \in \mathbb{R}\backslash\{0\}$.

Proof. Denote by $\pi_i : G \to G_i$ $(i = 1, 2)$ the natural projection from G onto G_i. Since every 1-dimensional Lie group is abelian we have dim $G_2 \geq 2$.
Let dim $\pi_2(H) = 0$. Since $\pi_2(H)$ is connected we have $H = H_1 \times \{1\}$ and the corresponding left A-loop L is the direct product of two left A-loops, such that the group topologically generated by its left translations is at least 5-dimensional. This is a contradiction to Proposition 3.
Let now dim $\pi_2(H) = 1$.
If the dimension of G is 5 then H is 2-dimensional. Moreover H is a subgroup of $\pi_1(H) \times \pi_2(H)$ such that dim $\pi_i(H) \leq$ dim H for $i = 1, 2$. Therefore $1 \leq$ dim $\pi_1(H) \leq 2$. If dim $\pi_1(H) = 1$ then H is the direct product $H = \pi_1(H) \times \pi_2(H)$ with $\pi_i(H) \subset G_i$ which leads to the same contradiction as above (Proposition 3). If dim $\pi_1(H) = 2$ then

$$\pi_1(H) = \mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}.$$

Then there is a homomorphism $\varphi : \pi_1(H) \to \pi_2(H)$ with 1-dimensional nucleus and H has one of the following shapes:

$$H \cong \left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right); a > 0, b \in \mathbb{R} \right\},$$

$$H \cong \left\{ \left(\begin{pmatrix} e^z & b \\ 0 & e^{-z} \end{pmatrix}, \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right), z, b \in \mathbb{R} \right\}.$$

If G has dimension 6 then the dimension of H is 3 and dim $\pi_1(H) \geq 2$. If dim $\pi_1(H) = 2$ then H is the direct product $H = \pi_1(H) \times \pi_2(H)$ with $\pi_i(H) \subset G_i$ which is impossible (Proposition 3). The dimension of $\pi_1(H)$ cannot be 3 otherwise we would have a homomorphism φ from a quasi-simple Lie group G_1 into $\pi_2(H)$. This is a contradiction.
Let now dim $\pi_2(H) = 2$.
If the dimension of G is 5 then

$$\pi_2(H) = G_2 = \mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}.$$

If $\pi_1(H) = \{1\}$ then the stabilizer H has the form $\{1\} \times \pi_2(H)$. The left A-loop L belonging to the pair (G, H) is the direct product of two left A-loops,

but this is a contradiction to Proposition 3.
If dim $\pi_1(H) = 1$ then there is a homomorphism $\psi : \pi_2(H) \to \pi_1(H)$ such that the dimension of the kernel of ψ is 1. Then the subgroup $(1, Ker\ \psi)$ is normal in G and therefore H contains a normal subgroup of G, which is a contradiction.
Let now dim $\pi_1(H) = 2$. In this case H has the form $(\varphi(G_2), G_2)$, where φ is a monomorphism from $\mathcal{L}_2$ onto a 2-dimensional subgroup of $PSL_2(\mathbb{R})$.
If G is 6-dimensional then the dimension of H is 3 and dim $\pi_1(H) \geq 1$. If dim $\pi_1(H) = 1$ then H is the direct product $H = \pi_1(H) \times \pi_2(H)$ with $\pi_i(H) \subset G_i$ which is impossible (Proposition 3).
If dim $\pi_1(H) = 3$ then $\pi_1(H) = G_1$ and we would have a homomorphism ϕ from a quasi-simple Lie group G_1 into $\pi_2(H)$ with a 1-dimensional nucleus. This is a contradiction.
If dim $\pi_1(H) = 2$ then $\pi_1(H) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$. If $H \cap (\{1\} \times G_2) = \{1\} \times \pi_2(H)$ then H is the direct product $H = \{1\} \times \pi_2(H)$. This is contradiction to dim $H = 3$. If would exist a monomorphism φ from $\pi_2(H)$ onto $\pi_1(H)$ then $\varphi(\pi_2(H)) = \pi_1(H)$ and H were 2-dimensional. Therefore there exist homomorphisms $\phi : \pi_1(H) \to \pi_2(H)$ and $\varphi : \pi_2(H) \to \pi_1(H)$ with 1-dimensional nucleus $S_1 \times \{1\} = H \cap (G_1 \times \{1\})$ and $\{1\} \times S_2 = H \cap (\{1\} \times G_2)$. Any 3-dimensional solvable non-abelian Lie group G_2 is introduced in cases 7 to 12 in **1.3**. Any 2-dimensional subgroup of G_2 is isomorphic either to $\mathcal{L}_2$ or to $\mathbb{R}^2$.
Nun $\pi_2(H)$ cannot be isomorphic to $\mathcal{L}_2$ since the 1-dimensional nucleus $\{1\} \times \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R} \right\}$ of φ is contained in H and normal in G. Therefore it is sufficient to investigate the 2-dimensional subgroups of G_2 isomorphic to $\mathbb{R}^2$. These subgroups can be chosen as $\pi_2(H)$. The subgroup $\pi_2(H)$ consists of the 1-dimensional subgroups $Im\ \phi$ and $Ker\ \varphi$, such that $Ker\ \varphi$ cannot be a normal subgroup of G_2.
A) Let G_2 be the 3-dimensional nilpotent Lie group. Let the Lie algebra $\mathbf{g}_2$ of G_2 be given by the basis $\{e_1, e_2, e_3\}$ with the multiplication $[e_1, e_2] = e_3$, $[e_3, e_i] = 0$ $(i = 1, 2)$. Any 2-dimensional subgroup $\pi_2(H)$ of G_2 isomorphic to $\mathbb{R}^2$ is either
$\{\exp(\lambda_1 e_1 + \lambda_3 e_3) \mid \lambda_i \in \mathbb{R}, i = 1, 3\} = \{g(\lambda_1, 0, \lambda_3) \mid \lambda_i \in \mathbb{R}, i = 1, 3\}$
or
$\{\exp(\lambda_2 e_2 + \lambda_3 e_3) \mid \lambda_i \in \mathbb{R}, i = 2, 3\} = \{g(0, \lambda_2, \lambda_3) \mid \lambda_i \in \mathbb{R}, i = 2, 3\}$.
Since both subgroups are isomorphic we may assume that
$\pi_2(H) = \{\exp(\lambda_1 e_1 + \lambda_3 e_3) \mid \lambda_i \in \mathbb{R}, i = 1, 3\}$.
The 1-dimensional subgroup of $\pi_2(H)$ has the following form:
$N = \{\exp\ t(d\ e_1 + c\ e_3) \mid t \in \mathbb{R}\}$, $c, d \in \mathbb{R}$, $c^2 + d^2 \neq 0$.

For $d \neq 0$ the automorphism of G_2 which corresponds to the automorphism ξ of $\mathbf{g}_2$ with $\xi(e_1) = \frac{1}{d}e_1 - \frac{c}{d}e_3$, $\xi(e_2) = de_2$, $\xi(e_3) = e_3$ maps N onto the subgroup $\{\exp\ te_1 \mid t \in \mathbb{R}\}$. Hence we may assume that $Ker\ \varphi$ has the form $\{\exp\ te_1 \mid t \in \mathbb{R}\}$ and the 3-dimensional stabilizer H has the shape:

$$H = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, g(c + k\ a, 0, l\ a) \right), a, b, c \in \mathbb{R} \right\},$$

where $k \in \mathbb{R}$, $l \in \mathbb{R}\backslash\{0\}$ are given parameters.
B) Let now G_2 is the 3-dimensional Lie group $\mathcal{L}_2 \times \mathbb{R}$. Denote by $\{e_1, e_2, e_3\}$ a real basis of the Lie algebra $\mathbf{g}_2$ of G_2 with the multiplication $[e_1, e_2] = -e_2$, $[e_3, e_i] = 0$ $(i = 1, 2)$. Then the Lie algebra $\mathbf{g}_2$ is isomorphic to the Lie algebra of matrices

$$(ue_1 + ve_2 + ze_3) \mapsto \begin{pmatrix} 0 & v & z \\ 0 & u & 0 \\ 0 & 0 & 0 \end{pmatrix}, u, v, z \in \mathbb{R}.$$

There are precisely two 2-dimensional subgroups $\pi_2(H)$ of G_2 isomorphic to $\mathbb{R}^2$:
1) $\{\exp(\lambda_1 e_1 + \lambda_3 e_3) \mid \lambda_i \in \mathbb{R}, i = 1, 3\}$
2) $\{\exp(\lambda_2 e_2 + \lambda_3 e_3) \mid \lambda_i \in \mathbb{R}, i = 2, 3\}$.
In the first case any 1-dimensional subgroup of $\pi_2(H)$ has the shape:
$N = \{\exp t(de_1 + ce_3) \mid t \in \mathbb{R}\}$, c, d are given real parameters.
If $d \neq 0$ using the automorphism of G_2 which corresponds to the automorphism ξ of $\mathbf{g}_2$ with $\xi(e_1) = e_1 + \frac{1}{d}e_3$, $\xi(e_2) = e_2$ and $\xi(e_3) = -\frac{1}{c}e_3$ we obtain $N^\xi = \{\exp\ te_1 \mid t \in \mathbb{R}\}$. Therefore we can write $Ker\ \varphi = \{\exp\ te_1 \mid t \in \mathbb{R}\}$ and the 3-dimensional stabilizer H_1 has the following form:

1. $$H_1 = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, \begin{pmatrix} 1 & 0 & la \\ 0 & e^{c+ka} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), a, b, c \in \mathbb{R} \right\},$$

k, l are given real numbers, such that $l \neq 0$.
In the second case for any real constants c, d with $cd \neq 0$ we obtain a 1-dimensional subgroup of $\pi_2(H)$

$$N = \{\exp\ t(de_2 + ce_3) \mid t \in \mathbb{R}\}.$$

For $dc \neq 0$ we can change N by the automorphism of G_2 belonging to the automorphism ξ of $\mathbf{g}_2$: $\xi(e_1) = e_1$, $\xi(e_2) = \frac{1}{d}e_2$ and $\xi(e_3) = \frac{1}{c}e_3$ such that N^ξ has the form: $\{\exp\ t(e_2 + e_3) \mid t \in \mathbb{R}\}$. Then the 3-dimensional subgroup H_2 of G has the form:

2. $$H_2 = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, \begin{pmatrix} 1 & c + la & c + ka \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), a, b, c \in \mathbb{R} \right\},$$

with given different real numbers k, l.
C) We consider the case that G_2 is the 3-dimensional solvable Lie group with precisely two 1-dimensional normal subgroups (see the case 10 in **1.3**). The Lie algebra $\mathbf{g}_2$ of G_2 consists of the matrices

$$(xe_1 + ye_2 + ze_3) \mapsto \begin{pmatrix} 0 & x & y \\ 0 & az & 0 \\ 0 & 0 & bz \end{pmatrix}, x, y, z \in \mathbb{R}.$$

The real basis $\{e_1, e_2, e_3\}$ of $\mathbf{g}_2$ satisfies the rules $[e_1, e_3] = ae_1$, $[e_2, e_3] = be_2$, and $[e_1, e_2] = 0$. The only 2-dimensional subgroup $\pi_2(H)$ of G_2 isomorphic to $\mathbb{R}^2$ is

$$\{\exp(\lambda_1 e_1 + \lambda_2 e_2) \mid \lambda_i \in \mathbb{R}, i = 1, 2\}.$$

Any 1-dimensional subgroup of $\pi_2(H)$ has the following form:
$N = \{\exp t(de_1 + ce_2) \mid t \in \mathbb{R}\}$, c, d are given real numbers.
If $cd \neq 0$ the automorphism of G_2 belonging to the automorphism ξ of $\mathbf{g}_2$ with

$$\xi(e_1) = \frac{1}{d}e_1,\ \xi(e_2) = \frac{1}{c}e_2 \text{ and } \xi(e_3) = e_3$$

gives that $N^\xi = \{\exp t(e_1 + e_2) \mid t \in \mathbb{R}\}$. Therefore we may assume that $Ker\ \varphi$ has the shape $\{\exp t(e_1 + e_2) \mid t \in \mathbb{R}\}$ and the stabilizer H has one of the following forms:

$$H = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, \begin{pmatrix} 1 & c + la & c + ka \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), a, b, c \in \mathbb{R} \right\},$$

with given different real numbers k, l.
D) Now we seek with the case that G_2 is the 3-dimensional solvable Lie group having only one 1-dimensional normal subgroup (see the case 9 in **1.3**). We denote by

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a basis of the Lie algebra $\mathbf{g}_2$ of G_2. For this basis the following relations hold:

$$[e_1, e_3] = e_1 + e_2,\ [e_2, e_3] = e_2, \text{ and } [e_1, e_2] = 0.$$

The 1-dimensional ideal of $\mathbf{g}_2$ is generated by e_2. There is precisely one 2-dimensional subgroup $\pi_2(H)$ of G_2 isomorphic to $\mathbb{R}^2$, this is the following

$$\{\exp(\lambda_1 e_1 + \lambda_2 e_2) \mid \lambda_i \in \mathbb{R}, i = 1, 2\}.$$

For any $c, d \in \mathbb{R}$ there is a 1-dimensional subgroup of $\pi_2(H)$

$$N = \{\exp t(de_1 + ce_2) \mid t \in \mathbb{R}\}.$$

If $d \neq 0$ the automorphism of G_2 corresponding to the automorphism ξ of $\mathbf{g}_2$ with $\xi(e_1) = \frac{1}{d}e_1 - \frac{c}{d^2}e_2$, $\xi(e_2) = \frac{1}{d}e_2$ and $\xi(e_3) = e_3$ maps N onto $\{\exp te_1 \mid t \in \mathbb{R}\}$. Therefore we may choose $Ker\ \varphi = \{\exp\ te_1 \mid t \in \mathbb{R}\}$ and we have

$$H = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, \begin{pmatrix} 1 & c+ka & la \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), a, b, c \in \mathbb{R} \right\}, k \in \mathbb{R}, l \in \mathbb{R}\backslash\{0\}.$$

E) Now we investigate the case that G_2 is the 3-dimensional solvable Lie group, which has infinitely many 1-dimensional normal subgroup. Choosing the following real basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathbf{g}_2$ of G_2

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the multiplication is given by the rules:

$$[e_1, e_3] = ae_1,\ [e_2, e_3] = ae_2, \text{ and } [e_1, e_2] = 0.$$

We see that the only 2-dimensional subgroup $\pi_2(H)$ of G_2 isomorphic to $\mathbb{R}^2$ is

$$\{\exp(\lambda_1 e_1 + \lambda_2 e_2) \mid \lambda_i \in \mathbb{R}, i = 1, 2\}.$$

Since $Ker\ \varphi$ is a 1-dimensional subgroup of $\pi_2(H)$ and every 1-dimensional subgroup of $\pi_2(H)$ is normal in G_2 we obtain that H contains the non-trivial normal subgroup $(1, Ker\ \varphi)$ of G. This is a contradiction.

F) Finally let G_2 be locally isomorphic to the group of orientation preserving motions of the euclidean plane (see the case 12 in **1.3**). This group has no 1-dimensional normal subgroup. Let $\{e_1, e_2, e_3\}$ be a basis of the Lie algebra $\mathbf{g}_2$ of G_2 such that the multiplication is defined as follows:

$$[e_1, e_3] = e_2,\ [e_3, e_2] = e_1, \text{ and } [e_1, e_2] = 0.$$

There exists precisely one 2-dimensional subgroup $\pi_2(H)$ of G_2 isomorphic to $\mathbb{R}^2$, which has the shape

$$\{\exp(\lambda_1 e_1 + \lambda_2 e_2) \mid \lambda_i \in \mathbb{R}, i = 1, 2\}.$$

For any real numbers $c, d \in \mathbb{R}$ we have a 1-dimensional subgroups of $\pi_2(H)$

$$N = \{\exp t(de_1 + ce_2) \mid t \in \mathbb{R}\}.$$

Since the 1-dimensional subspaces of $[\mathbf{g}, \mathbf{g}]$ generating by the vectors $de_1 + ce_2$ $(c, d \in \mathbb{R})$ are conjugate under the adjoint action of G_2, we know that N changes onto $\{\exp te_1 \mid t \in \mathbb{R}\}$ by the conjugation under suitable elements. Therefore we can assume that $Ker\ \varphi = \{\exp te_1 \mid t \in \mathbb{R}\}$ and the

3-dimensional stabilizer H can be written in the following shape

$$H = \left\{ \left(\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, \begin{pmatrix} 1 & c+ka & la & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right), a, b, c \in \mathbb{R} \right\},$$

with $k \in \mathbb{R}$, $l \in \mathbb{R} \backslash \{0\}$.

Finally we consider the case $\pi_2(H) = G_2$. Then H has the form $(\varphi(G_2), G_2)$, where $\varphi : G_2 \to G_1$ is a homomorphism. Since the homomorphism φ has a non-trivial kernel H contains the proper normal subgroup $(1, Ker\ \varphi)$. This is a contradiction. □

Corollary 8. *There is no global left A-loop L homeomorphic to the compact space S^3 or P^3.*

Proof. The group G topologically generated by the left translations of an almost differentiable proper left A-loop L homeomorphic to S^3 acts transitively on L. According to 96.16 in [48] any maximal compact subgroup of G acts also transitively on S^3. Since a transitive compact subgroup of G is a non-solvable subgroup of $SO_4(\mathbb{R})$ (96.20 in [48]) the group G is non-solvable. According to Proposition 16.11 in [43] and Propositions 4, 5, 6, 7 there is no almost differentiable left A-loop homeomorphic to S^3 or P^3 having a non-solvable Lie group as the group topologically generated by its left translations. □

2 Left A-loops as sections in semisimple Lie groups

In this section we classify all 3-dimensional connected strongly left alternative almost differentiable left A-loops having semisimple Lie groups as their left translation groups and describe the symmetric spaces and natural geometries associated with them.

If G is a direct product of quasi-simple Lie groups according to Proposition 5 it is sufficient to consider the following two cases:
1) $G \cong PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ and H_1 has the shape (i).
2) $G \cong PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ and H_2 has the form (ii).

Now let G be locally isomorphic to the group $PSL_2(\mathbb{C})$. According to ([4], pp. 273-278) there are 4 conjugacy classes of the 3-dimensional subgroups of $G = SL_2(\mathbb{C})$, which we denote by W_r, U_0, U_1 and $SU_2(\mathbb{C})$. Since $SU_2(\mathbb{C})$ contains central elements $\neq 1$ of $SL_2(\mathbb{C})$ it follows from Proposition 4 that no of these groups can be the stabilizer of $e \in L$ in G. Hence we have the following case
3) G is isomorphic to $PSL_2(\mathbb{C})$ and H is isomorphic to $SO_3(\mathbb{R})$.

Now we deal with the case 1). Then the Lie algebra $\mathbf{h}_1$ of H_1 has the shape

$$\mathbf{h}_1 = \{(X, X);\ X \in sl_2(\mathbb{R})\}.$$

Let K, U and T be the real basis of $sl_2(\mathbb{R})$ induced in **1.2**. We seek for all reductive complements $\mathbf{m}$ to $\mathbf{h}_1$ in $\mathbf{g}$. Let β be the Cartan-Killing form on $\mathbf{g}$. If $\mathbf{h}^\perp$ denotes the orthogonal complement to $\mathbf{h}$ with respect to the scalar product β then $\mathbf{h}^\perp$ is Ad_H invariant and hence $[\mathbf{h}, \mathbf{h}^\perp] \subseteq \mathbf{h}^\perp$. A 3-dimensional reductive complement $\mathbf{m} \subset \mathbf{g}$ has the shape $\{X + \varphi(X); X \in \mathbf{h}^\perp\}$, where the linear map $\varphi : \mathbf{h}^\perp \to \mathbf{h}$ satisfies the relation $Ad_h\varphi = \varphi Ad_h$ for all $h \in H$. The orthogonal complement of $\mathbf{h}$ in $\mathbf{g}$ with respect to the Cartan-Killing form β has the shape $\{(X, -X); X \in sl_2(\mathbb{R})\}$. The linear map $\varphi : \mathbf{h}^\perp \to \mathbf{h}$ is of the form $\varphi(X, -X) = (\alpha(X), \gamma(-X)) = (\alpha(X), -\gamma(X))$ with the linear map $\gamma = -\alpha : sl_2(\mathbb{R}) \to sl_2(\mathbb{R})$. The relation

$$Ad_{(h,h)}(\alpha(X), \alpha(X)) = (\alpha(Ad_hX), \alpha(Ad_hX)), \quad h \in PSL_2(\mathbb{R}), X \in sl_2(\mathbb{R})$$

implies $Ad_h\alpha = \alpha Ad_h$ for all $h \in PSL_2(\mathbb{R})$. Therefore the bijective linear map $\alpha : sl_2(\mathbb{R}) \to sl_2(\mathbb{R})$ is a scalar multiplication. Hence for any real constant $c \neq \pm 1$ we obtain a reductive complement

$$\mathbf{m} = \big\{(X, \lambda X); X \in sl_2(\mathbb{R})\big\}$$

with $0 \neq \lambda = \frac{c-1}{c+1}$. The image $\sigma(G/H)$ of the section $\sigma : G/H \to G$ satisfies the relation

$$\sigma(G/H) = \exp \mathbf{m} = \{(\exp X, (\exp X)^\lambda); X \in sl_2(\mathbb{R})\},$$

where $\exp X \mapsto (\exp X)^\lambda : PSL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ is a mapping. Denote by $S_X = \{\exp tX; t \in \mathbb{R}\}$ a 1-parameter subgroup of $PSL_2(\mathbb{R})$ isomorphic to $SO_2(\mathbb{R})$. For all x, y contained in S_X is satisfied $(xy)^\lambda = x^\lambda y^\lambda$. Hence the mapping $x \to x^\lambda$ is an endomorphism of the group S_X. Since $(S_X, S_X^\lambda) \cap H = \{(1, 1)\}$ the equation $x^\lambda = x$, $x \in S_X$ holds only for $x = 1$. Equivalently, $x^{\lambda-1}$ is an automorphism of S_X. The only non-trivial automorphism of S_X is the mapping $x \mapsto x^{-1}$. Therefore the automorphism $x \mapsto x^{\lambda-1}$ must be the identity map and we have $\lambda = 2$.

Any subgroup of G conjugate to H has the shape

$$H^d = \{(u, d^{-1}ud), u \in PSL_2(\mathbb{R})$$

with either a fixed $d \in PSL_2(\mathbb{R})$ or the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2(\mathbb{R})$. The equation $zg_1H = g_2H \quad (g_1, g_2 \in G)$ has two solution $z_1, z_2 \in \sigma(G/H)$ are contained in the coset $(g_2g_1^{-1})g_1Hg_1^{-1}$ of the conjugate subgroup $g_1Hg_1^{-1}$ of H in G. A computation shows that for $X_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ and $X_2 = \begin{pmatrix} \frac{1}{2} & -9 \\ 0 & 2 \end{pmatrix}$ we have $(X_i, X_i^2) = (R, 1)(U_i, D^{-1}U_iD)$, where $R = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $D = \begin{pmatrix} \frac{\sqrt{5}}{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}$, and $U_i = DX_i^2D^{-1}$. This means that the coset $(R, 1)H^D$ contains two different elements (X_i, X_i^2) of $\sigma(G/H)$ $(i = 1, 2)$ and consequently the section $\sigma : G/H \to G$ is not sharply transitive.

Now we consider the case 2). The Lie algebra $\mathbf{h}_2$ of H_2 is generated by the basis elements $(K, K), (U + T, 0), (0, U + T)$.
We seek for 3-dimensional complements $\mathbf{m}$, which generate the Lie algebra $\mathbf{g}$. Hence we can assume that the projection of $\mathbf{m}$ onto the first component of $\mathbf{g}$ is 3-dimensional and has as a basis the elements $U, U + T$, and K. Since the loop L^* corresponding to the pair $(H, \sigma(G/H)^g)$ with a $g \in G$ is isotopic to the loop L corresponding to the pair $(H, \sigma(G/H))$ we may assume that the second component of the first basis element is $t\ U$, $\pm\ (U + T)$ or $\lambda\ K$, respectively. Then $\mathbf{m}$ has one of the following forms:
a) $\mathbf{m}_1 = \langle (U, \mathrm{t}\ U), (U + T, \mathrm{a}\ U + \mathrm{b}\ T + \mathrm{c}\ K), (K, \mathrm{d}\ U + \mathrm{e}\ T + \mathrm{f}\ K) \rangle$, where $a, b, c, d, e, f \in \mathbb{R}$.
b) $\mathbf{m}_2 = \langle (U, \pm\ (U + T)), (U + T, \mathrm{a}\ U + \mathrm{b}\ T + \mathrm{c}\ K), (K, \mathrm{d}\ U + \mathrm{e}\ T + \mathrm{f}\ K) \rangle$, where a, b, c, d, e, f are real parameters.
c) $\mathbf{m}_3 = \langle (U, \lambda\ K), (U + T, \mathrm{a}\ U + \mathrm{b}\ T + \mathrm{c}\ K), (K, \mathrm{d}\ U + \mathrm{e}\ T + \mathrm{f}\ K) \rangle$, with $a, b, c, d, e, f \in \mathbb{R}$.
The complements $\mathbf{m}_i$ $(i = 1, 2, 3)$ must be satisfied $[\mathbf{h}, \mathbf{m}_i] \subseteq \mathbf{m}_i$. The complements $\mathbf{m}_i$ $(i = 1, 2, 3)$ cannot be reductive since the element

$$[(U + T, 0), (K, dU + eT + fK)] = -2(U + T, 0)$$

is not an element in $\mathbf{m}_i$ for $i = 1, 2, 3$. Therefore there is no 3-dimensional connected almost differentiable left A-loop as section in the Lie group $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.

Now let L be a 3-dimensional connected differentiable loop such that the group G generated by its left translations is isomorphic to $PSL_2(\mathbb{C})$ and the stabilizer H of $e \in L$ in G is isomorphic to $SO_3(\mathbb{R})$. Since H is a maximal compact subgroup of G and $\dim G - \dim H = 3$ (cf. [36], Chapter VI, Theorem 2.2 (iii)), the coset space G/H is a Riemannian manifold homeomorphic to $\mathbb{R}^3$. According to **1.2** let $\{K, T, U, \mathrm{i}K, \mathrm{i}T, \mathrm{i}U\}$ be a real basis of $\mathbf{g} = sl_2(\mathbb{C})$. We seek for all 3-dimensional complements $\mathbf{m}$ with the properties

$$\mathbf{g} = \mathbf{m} \oplus \mathbf{h},\ [\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m} \text{ and } \mathbf{m} \text{ generates } \mathbf{g}.$$

The Lie algebra of the stabilizer H is $\mathbf{h} = \langle U, \mathrm{i}T, \mathrm{i}K \rangle$, and one particular reductive component to $\mathbf{h}$ in $\mathbf{g}$ is $\mathbf{m}^* = \langle T, \mathrm{i}U, K \rangle$. We may assume that an arbitrary component $\mathbf{m}$ has the shape

$$\mathbf{m} = \langle T + \mathrm{f}(T), \mathrm{i}U + \mathrm{f}(\mathrm{i}U), K + \mathrm{f}(K) \rangle,$$

where $\mathrm{f} : \mathbf{m}^* \to \mathbf{h}$ is a linear map. This means that we can write $\mathbf{m}$ in the general form:

$\mathbf{m} = \langle T + \mathrm{a}U + \mathrm{b}\ \mathrm{i}T + \mathrm{c}\ \mathrm{i}K, \mathrm{i}U + \mathrm{d}U + \mathrm{e}\ \mathrm{i}T + \mathrm{f}\ \mathrm{i}K, K + \mathrm{g}U + \mathrm{h}\ \mathrm{i}T + \mathrm{k}\ \mathrm{i}K \rangle$, where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. A computation shows that

$$\mathbf{m} = \langle T + aiT, iU - aU, K + aiK \rangle,$$

for a real parameter a.

Now we determine the isomorphism classes and the isotopism classes of the loops L_a, $a \in \mathbb{R}$ corresponding to the complement $\mathbf{m}_a$. Two loops corresponding to $(G, H, \exp \mathbf{m}_a)$ and $(G, H, \exp \mathbf{m}_b)$ are isomorphic if and only if there exists an automorphism α of $\mathbf{g}$ such that $\alpha(\mathbf{m}_a) = \mathbf{m}_b$ and $\alpha(\mathbf{h}) = \mathbf{h}$. The automorphism group of $\mathbf{g}$ leaving $\mathbf{m}_0$ and $\mathbf{h}$ invariant is the semidirect product Θ of Ad_H and the group generated by the involutory map $\varphi : z \mapsto \bar{z}$. Since $\mathbf{m}$ is a reductive subspace the condition $\alpha(\mathbf{m}_a) = \mathbf{m}_b$, $\alpha \in \Theta$ is equivalent to $\varphi(\mathbf{m}_a) = \mathbf{m}_b$. This identity is satisfied if and only if $b = -a$. Therefore a full isomorphism class consists of the loops L_a and L_{-a} ($a \in \mathbb{R}$) and may be chosen as the representatives of these isomorphism classes the left A-loops $L_a,\ a \geq 0$. Since there is no $g \in G$ such that $g^{-1}\mathbf{m}_a g = \mathbf{m}_b$ for two different real numbers a, b the isotopism classes and the isomorphism classes of the left A-loops $L_a, a \in \mathbb{R}$ are the same.

The complement $\mathbf{m}_0 = \langle T, \mathrm{i}U, K \rangle$ is orthogonal to $\mathbf{h}$ with respect to the Cartan-Killing form $k_{\mathbb{R}}$ on $\mathbf{g}$ and satisfies $[\mathbf{m}_0, \mathbf{m}_0] = \mathbf{h}$, and $\mathbf{g} = \mathbf{m_0} \oplus [\mathbf{m_0}, \mathbf{m_0}]$. Moreover, the mapping $\mu : \mathbf{g} \to \mathbf{g},\ X \mapsto -X^*$ on $\mathbf{g}$, where X^* is the adjoint matrix of X in $sl_2(\mathbb{C})$, is an involutory automorphism such that $\mathbf{h}$ is the $(+1)$-eigenspace of μ and the (-1)-eigenspace $\mathbf{m}$ generates $\mathbf{g}$ as Lie algebra. Therefore $M_0 = exp\ \mathbf{m}_0$ is a 3-dimensional connected Riemannian symmetric space.

An elementary model of the loops $L_a, a \geq 0$ is given in the upper half space $\mathbb{R}^{3+} = \{(x, y, z) \in \mathbb{R}^3; z > 0\}$. The elements of the loops L_a are the points of the upper half space $\mathbb{R}^{3+}$. We can identificate the elements of $\mathbb{R}^3$ with the elements of the **J**-quaternion space. The **J**-quaternion space is the 3-dimensional subspace of the quaternion space which is orthogonal to the canonical basis element k (cf. [15], p. 3). We can give the action of the group $SL_2(\mathbb{C})$ on $\mathbb{R}^3$ by the linear rational functions:

$$\gamma(w) = (aw+b)(cw+d)^{-1},$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a,b,c,d \in \mathbb{C}, ad-bc=1,\ w = x+jy \in \mathbf{J},\ x \in \mathbb{C}, y \in \mathbb{R}$$

(cf. [15], p. 26). The restriction of this action onto the subspace $\mathbb{R}^{3+}$ defines the action of $SL_2(\mathbb{C})$ on the upper half space, and $(w, \pm\gamma) \mapsto \gamma(w)$ is the transitive action of $PSL_2(\mathbb{C})$ on the upper half space. The group of isometries of the hyperbolic space H_3 contains a subgroup G isomorphic to $PSL_2(\mathbb{C})$. Since the stabilizer subgroup $H = SO_3(\mathbb{R})$ leaves the point j fixed, we can choose this point as the identity element of the loops L_a. Using the foundamental formula of the exponential mapping of $sl_2(\mathbb{C})$ (see **1.2** in section 1) we can see that the image of the subspace $\mathbf{m}_a$ under the exponential mapping has the form:

$$\exp\ \mathbf{m}_a =$$

$$\pm \begin{pmatrix} \cosh k + \dfrac{\lambda_1(1+ia)}{k}\sinh k & \dfrac{\lambda_2(1+ia)+\lambda_3(i-a)}{k}\sinh k \\ \dfrac{\lambda_2(1+ia)+\lambda_3(a-i)}{k}\sinh k & \cosh k - \dfrac{\lambda_1(1+ia)}{k}\sinh k \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $k = \sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}(1+ia)$.

For $a = 0$ the image of the exponential map of $\mathbf{m}_a$ defines a differentiable sharply transitive global section $\sigma : G/H \to G$ such that $\exp\ \mathbf{m}_a = \sigma(G/H)$ since the image of this section belongs to the hyperbolic space loop L_0 (cf. [17]).

Denote by ∇_a the canonical connection of the reductive homogeneous space G/H belonging to the subspace $\mathbf{m}_a$. The geodesics through $e = j$ with respect to ∇_a have the form

$$(\exp tX)j = \frac{j + \dfrac{(\lambda_2+\lambda_3 i)}{2k'}(e^{2k't}-e^{-2k't})}{\dfrac{1}{2}(e^{2k't}+e^{-2k't}) - \dfrac{\lambda_1}{2k'}(e^{2k't}-e^{-2k't})},$$

where $X = \lambda_1(K + \text{a i}K) + \lambda_2\ (T + \text{ a i}T) + \lambda_3\ (\text{i}U - \text{ a}U)$ is an element of $\mathbf{m}_a$ and $k' = \sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}$ (cf. [36], Vol II, Proposition 2.4. p. 192). This means that the form of the geodesics through $e = j$ is independ from the parameter a.

We know from the hyperbolic geometry that the geodesics through $e = j$ in the hyperbolic space loop L_0 are the unique oriented lines connected $e = j$ and x, which are orthogonal to the plane spanned by $\{1, i\}$. The geodesics through $e = j$ of every loops L_a are the same. Therefore for any $x, y \in L_a$ there exist unique geodesics $\exp_H tX$ and $\exp_H tY$ with $x = \exp_H X$ and $y = \exp_H Y$. The parallel translation

$$\tau_{0,t} : T_H(G/H) \to T_{(\exp tX)H}(G/H)$$

along the geodesic $\{(\exp \ tX)H; \ t \in \mathbb{R}\}$ is given by the tangential map $(L_{\exp tX})_*$, where $L_{\exp tX} : yH \mapsto (\exp tX)yH$. This tangential map depends from the parameter a. This together with the uniquess of geodesics joining j with any other point defines for any $a \geq 0$ a loop L_a the multiplication of which is given by:

$$x * y = \exp_x \tau_{e,x} \exp_e^{-1}(y).$$

Summarizing our discussion we obtain

Theorem 9. *There is a class $\mathcal{C}$ of the 3-dimensional connected almost differentiable left A-loops L such that the group G generated by the left translations $\{\lambda_x; x \in L\}$ is a semisimple Lie group. The group G is isomorphic to $PSL_2(\mathbb{C})$ and the stabilizer H of $e \in L$ in G is isomorphic to $SO_3(\mathbb{R})$.*
Any loop in this class $\mathcal{C}$ can be represented by a real parameter a. The loops L_a and L_{-a} form a full isomorphism class, which is even a full isotopism class too. In $\mathcal{C}$ only the hyperbolic space loop L_0 is a Bruck loop. This loop L_0 is realized on the hyperbolic symmetric space by the multiplication $x \cdot y = \tau_{e,x}(y)$, where $\tau_{e,x}$ is the hyperbolic translation moving e onto x. The tangent space $T_1\Lambda$ of the set Λ of the left translations of the loop L_0 at the identity $1 \in G$ is the plane $\mathbf{m_0}$ through 0 in the Lie algebra $\mathbf{g}$ of G such that $\mathbf{m}_0$ is orthogonal to the 3-dimensional Lie algebra $\mathbf{h}$ of H with respect to the Cartan-Killing form of $\mathbf{g}$. Any loop L_a with $a \geq 0$ is isomorphic to the geodesic loop of the reductive homogeneous space G/H with respect to the reductive complement $\mathbf{m}_a = T_1[\sigma_a(G/H)]$ and the corresponding canonical invariant connection ∇_a.

3 3-dimensional left A-loops corresponding to 4-dimensional non-solvable Lie groups

In this section we give a classification of all 3-dimensional connected almost differentiable global left A-loops L having a 4-dimensional non-solvable Lie group G as the group topologically generated by their left translations. Then

the stabilizer H of $e \in L$ in G has dimension 1.
In this case we have $G = PSL_2(\mathbb{R}) \times G_2$, where G_2 is one of the 1-dimensional Lie groups, and H is one of the cases 2 and 3 in the Proposition 6.
The Lie algebra **g** of G can be represented as **g**$= sl_2(\mathbb{R}) \oplus \mathbb{R}$. Let $(K,0)$, $(T,0)$, $(U,0)$ with K,T,U defined in **1.2** be a real basis of $sl_2(\mathbb{R}) \oplus \{0\}$ and let $(0,e_1)$ be the generator of $\{0\} \oplus \mathbb{R}$. Then the multiplication in **g** is given by the following rules:

$$[(K,0),(T,0)] = 2(U,0),\ [(K,0),(U,0)] = 2(T,0),\ [(U,0),(T,0)] = 2(K,0),$$

$$[(K,0),(0,e_1)] = [(T,0),(0,e_1)] = [(U,0),(0,e_1)] = (0,0).$$

If H has the form $\{(x,\varphi(x))\}$, where φ is a monomorphism from the 1-dimensional subgroup $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 0 \right\}$ of $PSL_2(\mathbb{R})$ onto $\mathbb{R}$ then the Lie algebra **h** of H is generated by the basis element

$$(K, k\ e_1),\ k \in \mathbb{R}\backslash\{0\}.$$

Since the automorphism $A : \mathbf{g} \to \mathbf{g}$ defined by
$A(K,0) = (K,0), \quad A(T,0) = (T,0),$
$A(U,0) = (U,0), \quad A(0,e_1) = (0,ke_1)$
maps $(K,k\ e_1)$ onto (K,e_1) we may assume (up to an isotopism) that the Lie algebra **h** of the stabilizer H of $e \in L$ has the shape $\{\lambda(K,e_1); \lambda \in \mathbb{R}\}$.
We seek for 3-dimensional complements **m** of **h** in **g** with the properties $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$, $[\mathbf{h},\mathbf{m}] \subseteq \mathbf{m}$ and **m** generates **g**.
Since one particular complement of **h** is

$$\mathbf{m}^* = \langle (U,0),(T,0),(0,e_1)\rangle,$$

and **m** is a 3-dimensional subspace of **g** having at least a 2-dimensional intersection with $sl_2(\mathbb{R}) \oplus \{0\}$, an arbitrary complement **m** has one of the following forms:

$$\mathbf{m_1} = \langle (U,0),\ (T,0),\ (a_3\ K, e_1 + a_3\ e_1)\rangle, \quad \text{where } a_3 \in \mathbb{R}\backslash\{-1\},$$

$$\mathbf{m_2} = \langle (U,0),\ (T,a_2\ e_1),\ (-K,0)\rangle, \quad \text{where } a_2 \in \mathbb{R}\backslash\{0\},$$

$$\mathbf{m_3} = \langle (U,a_1\ e_1),\ (T,0),\ (-K,0)\rangle, \quad \text{where } a_1 \in \mathbb{R}\backslash\{0\}.$$

In the first case $\mathbf{m_1}$ yields for all $a_3 \in \mathbb{R}\backslash\{-1\}$ a reductive complement to **h**

in $\mathbf{g}$ such that $\mathbf{m_1}$ generates $\mathbf{g}$.
But the basis element $(K, e_1) \in \mathbf{h}$ is conjugate to the element

$$-(1+d+d^2)(U,0) + (-1+d+d^2)(T,0) + \left(\frac{a_3}{1+a_3}K, e_1\right) \in \mathbf{m}_{a_3}$$

under the element $\left(\pm\begin{pmatrix} 1+d & -1 \\ -d & 1 \end{pmatrix}, 0\right) \in G$, choosing d such that $2d = -\frac{1}{1+a_3}$. This is a contradiction to Lemma 2.

The complements $\mathbf{m}_i$ for $i = 2, 3$ are reductive if and only if $[\mathbf{h}, \mathbf{m}_i] \subseteq \mathbf{m}_i$. This means for $i = 2$ that the element

$$[(K, e_1), (U, 0)] = (2T, 0)$$

and for $i = 3$ the element

$$[(K, e_1), (T, 0)] = (2U, 0)$$

must be contained in $\mathbf{m}_2$ or $\mathbf{m}_3$ respectively. For $i = 2$ this is the case if and only if $a_2 = 0$, for $i = 3$ precisely if $a_1 = 0$. But then the complements $\mathbf{m}_i$ $i = 2, 3$ are subalgebras isomorphic to $sl_2(\mathbb{R})$. Hence there is no global left A-loop belonging to the pair

$$\left(G = PSL_2(\mathbb{R}) \times \mathbb{R}, H = \left\{(x, \varphi(x)), x \in \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 0\right\}\right).$$

Now let H be the following subgroup of G $\{(x, \varphi(x))\}$ such that φ is a monomorphism from the 1-dimensional subgroup $\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R}\right\}$ onto $\mathbb{R}$. Then this 1-dimensional Lie algebra $\mathbf{h}$ of H has

$$(U + T, c\, e_1),\ c \in \mathbb{R}\backslash\{0\}$$

as a basis element. With the automorphism $A : \mathbf{g} \to \mathbf{g}$ given by

$$\begin{array}{rclrcl} A(K,0) & = & (K,0), & A(T,0) & = & (T,0), \\ A(U,0) & = & (U,0), & A(0,e_1) & = & \left(0, \frac{c}{2}e_1\right) \end{array}$$

we may (up to an isotopism) assume that $\mathbf{h} = \langle (U + T, 2e_1)\rangle$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the shape:

$$\mathbf{m} = \langle (K + a_1(U+T), 2a_1 e_1), (U + a_2(U+T), 2a_2 e_1), (a_3(U+T), e_1 + 2a_3 e_1)\rangle,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Since $\mathbf{g} = sl_2(\mathbb{R}) \oplus \mathbb{R}$ and $\mathbf{m}$ is a 3-dimensional subspace of $\mathbf{g}$, then the intersection of $\mathbf{m}$ with $sl_2(\mathbb{R}) \oplus \{0\}$ at least 2-dimensional. Therefore $\mathbf{m}$ has one of the following forms:

$$\mathbf{m_1} = \langle (K, 0), (U, 0), (a_3 T, e_1 + 2a_3 e_1)\rangle,$$

where $a_3 \in \mathbb{R}\backslash \left\{-\frac{1}{2}\right\}$,

$$\mathbf{m_2} = \langle (K,0), (U, 2a_2e_1), (U+T,0)\rangle,$$

where $a_2 \in \mathbb{R}\backslash\{0\}$,

$$\mathbf{m_3} = \langle (K, 2a_1e_1), (U,0), (T,0)\rangle,$$

where $a_1 \in \mathbb{R}\backslash\{0\}$.

The complement $\mathbf{m}$ is reductive, if $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. This means for $i = 1$ that

$$[(U+T, 2e_1), (K,0)] = -2(U+T,0) \in \mathbf{m}_1$$

and for $i = 3$ that

$$[(U+T, 2e_1), (U,0)] = -2(K,0) \in \mathbf{m}_3.$$

This is satisfied for $i = 1$ if and only if $a_3 = -\frac{1}{2}$ and for $i = 3$ precisely if $a_1 = 0$. But then the complements $\mathbf{m}_1$ and $\mathbf{m}_3$ are Lie algebras, which do not generate $\mathbf{g}$.

The complement $\mathbf{m}_2$ is reductive to $\mathbf{h}$ and generates $\mathbf{g}$.
If $a_2 > 0$ then the element $(K+U, 2a_2e_1) \in \mathbf{m}_2$ is conjugate to the element $a_2(U+T, 2e_1) \in \mathbf{h}$ under the element $\left(\pm \begin{pmatrix} 1 & -\frac{2a_2}{\sqrt{2a_2}} \\ \frac{1}{\sqrt{2a_2}} & 0 \end{pmatrix}, 0\right) \in G$. This contradicts Lemma 2.
If $a_2 < 0$ then we prove that there is no global section $\sigma : G/H \to G$ such that $\exp \mathbf{m}_{a_2} \subseteq \sigma(G/H)$. The stabilizer H may be written in the following form:

$$\left\{\left(\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, l\right); l \in \mathbb{R}\right\},$$

moreover $\exp \mathbf{m}_{a_2}$ has the shape:

$$\{(\exp(\lambda_1 K + \lambda_2 U + \lambda_3(U+T)), \exp(2\lambda_2 a_2 e_1)); \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, a_2 < 0\}.$$

If $\mathbf{m}_{a_2}$ would be contained in a global section $\sigma : G/H \to G$ then each element $g \in G$ can be uniquely represented as a product $g = m\, h$ with $m \in \exp \mathbf{m}_{a_2}$ and $h \in H$. The subspace $\mathbf{m}_{a_2}$ contains the vectors

$$v_1 = (-3\pi a_2(U+T), 0), \quad v_2 = (\sqrt{5\pi^2}K + 3\pi U, 6\pi a_2 e_1).$$

According to **1.2** it follows that

$$m_1 = \exp v_1 = \left(\begin{pmatrix} 1 & -6\pi a_2 \\ 0 & 1 \end{pmatrix}, 0\right)$$

and

$$m_2 = \exp v_2 = (\pm I, 6\pi a_2),$$

where $\pm I$ is the identity of $PSL_2(\mathbb{R})$. The element $(\pm I, 6\pi a_2)$ may be written on the one hand as the product $m_1 \cdot h_1$, where $h_1 = \left(\begin{pmatrix} 1 & 6\pi a_2 \\ 0 & 1 \end{pmatrix}, 6\pi a_2 \right)$, on the other hand as the product $m_2 \cdot h_2$, where h_2 is the identity of G. Therefore the subspace $\mathbf{m}_{a_2}$ can not be the tangential space of a global sharply transitive section $\sigma : G/H \to G$ at $1 \in G$.

In the case 3) of Proposition 6 the Lie algebra $\mathbf{h}$ has the form

$$\mathbf{h} = \langle (U, c\ e_1) \rangle,\ c \in \mathbb{R} \backslash \{0\}.$$

Since the generator $(U, c\ e_1)$ changes onto (U, e_1) by the automorphism $A : \mathbf{g} \to \mathbf{g}$ defined as follows

$$\begin{array}{rclrcl} A(K,0) & = & (K,0), & A(T,0) & = & (T,0), \\ A(U,0) & = & (U,0), & A(0,e_1) & = & (0,ce_1) \end{array}$$

we may assume $H = \{(x,x) |\ x \in SO_2(\mathbb{R})\}$ and $\mathbf{h} = \langle (U, e_1) \rangle$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the shape:

$$\mathbf{m} = \langle (K + a_1\ U, a_1\ e_1), (T + a_2\ U, a_2\ e_1), (a_3\ U, e_1 + a_3\ e_1) \rangle,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. From Lemma 1 we obtain that $\mathbf{m}$ has one of the following forms:

$$\mathbf{m_1} = \langle (K,0), (T,0), (a_3\ U, e_1 + a_3\ e_1) \rangle,\ \text{where } a_3 \in \mathbb{R} \backslash \{-1\},$$

$$\mathbf{m_2} = \langle (K,0), (T, a_2\ e_1), (U,0) \rangle,\ \text{where } a_2 \in \mathbb{R} \backslash \{0\},$$

$$\mathbf{m_3} = \langle (K, a_1\ e_1), (T,0), (U,0) \rangle,\ \text{where } a_1 \in \mathbb{R} \backslash \{0\}.$$

Since $[\mathbf{h}, \mathbf{m_i}] \subseteq \mathbf{m_i}$ we obtain in case $i = 2$ that the element

$$[(U, e_1), (K, 0)] = (-2\ T, 0)$$

is an element of $\mathbf{m}_2$ and for $i = 3$ the element

$$[(U, e_1), (T, 0)] = (2\ K, 0)$$

is contained in $\mathbf{m}_3$. This holds for $i = 2$ if and only if $a_2 = 0$ and this is the case for $i = 3$ precisely if $a_1 = 0$. But these complements $\mathbf{m}_i$ $i = 2, 3$ do not generate $\mathbf{g}$.

The complement $\mathbf{m}_1$ is reductive to $\mathbf{h}$ and generates $\mathbf{g}$.
For $a_3 < -1$ the basis element $(U, e_1) \in \mathbf{h}$ is conjugate to the element

$$\left(\frac{1 - 2e^4}{2e^2} \right) (T, 0) + \left(\frac{a_3}{1 + a_3} U, e_1 \right) \in \mathbf{m}_{a_3}$$

under the element $\left(\pm\begin{pmatrix}\frac{1}{e} & 0\\ 0 & e\end{pmatrix}, 0\right) \in G$, choosing e such that $\dfrac{1+2e^4}{2e^2} = \dfrac{a_3}{1+a_3}$. This is a contradiction to Lemma 2.

If $a_3 > -1$ but $a_3 \neq 0$ $\exp \mathbf{m}_{a_3}$ has the shape:

$$\left\{\left(\exp\left(\lambda_1 K + \lambda_2 T + \lambda_3 U\right), \exp\left(\frac{\lambda_3(1+a_3)}{a_3} e_1\right)\right); \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\right\} =$$

$$\left(\exp\left(\lambda_1 K + \lambda_2 T + \lambda_3 U\right), \begin{pmatrix} \cos\frac{\lambda_3(1+a_3)}{a_3} & \sin\frac{\lambda_3(1+a_3)}{a_3} \\ -\sin\frac{\lambda_3(1+a_3)}{a_3} & \cos\frac{\lambda_3(1+a_3)}{a_3} \end{pmatrix}\right)$$

and clearly the stabilizer H has the form

$$H = \left\{\left(\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right); t \in \mathbb{R}\right\}.$$

For $a_3 > -1$ and $a_3 \neq 0$ we prove that the element

$$g = \left(\pm I, \begin{pmatrix} \cos\frac{k}{a_3} & \sin\frac{k}{a_3} \\ -\sin\frac{k}{a_3} & \cos\frac{k}{a_3} \end{pmatrix}\right) \in G,$$

where $k \in \mathbb{Z}$ such that $k > \sqrt{4\pi^2(1+a_3)^2}$ and $\pm I$ is the identity of $PSL_2(\mathbb{R})$, can be written in two different way as the product $g = m\,h$ with $m \in \exp \mathbf{m}_{a_3}$ and $h \in H$. The vectors

$$v_1 = \left(kU, \frac{k(1+a_3)}{a_3} e_1\right)$$

and

$$v_2 = \left(\sqrt{\left(\frac{k^2}{(1+a_3)^2} - 4\pi^2\right)} T + \frac{k}{1+a_3} U, \frac{k}{a_3} e_1\right)$$

are elements in the complement $\mathbf{m}_{a_3}$. According to **1.2** the images of v_1, v_2 under the exponential mapping have the forms:

$$m_1 = \exp v_1 =$$

$$\left(\pm\begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix}, \begin{pmatrix} \cos\frac{k(1+a_3)}{a_3} & \sin\frac{k(1+a_3)}{a_3} \\ -\sin\frac{k(1+a_3)}{a_3} & \cos\frac{k(1+a_3)}{a_3} \end{pmatrix}\right)$$

and

$$m_2 = \exp v_2 = \left(\pm I, \begin{pmatrix} \cos\frac{k}{a_3} & \sin\frac{k}{a_3} \\ -\sin\frac{k}{a_3} & \cos\frac{k}{a_3} \end{pmatrix}\right).$$

Therefore the element g can be written on the one hand as the product $m_1 \cdot h_1$, where

$$h_1 = \left(\pm \begin{pmatrix} \cos k & -\sin k \\ \sin k & \cos k \end{pmatrix}, \begin{pmatrix} \cos k & -\sin k \\ \sin k & \cos k \end{pmatrix} \right),$$

on the other hand as the product $m_2 \cdot 1$, where 1 is the identity of G.

For $a_3 = 0$ the complement $\mathbf{m}_{a_3}$ has the shape: $\mathbf{m}_0 = \langle (K,0), (T,0), (0,e_1) \rangle$. Then we have

$$\begin{aligned} \exp \mathbf{m}_0 &= \{\exp(\lambda_1(K,0) + \lambda_2(T,0)), \lambda_1, \lambda_2 \in \mathbb{R}\} \times \{\exp(\lambda_3(0,e_1)), \lambda_3 \in \mathbb{R}\} \\ &= M_1 \times G_2, \end{aligned}$$

such that M_1 is the image of the section σ_1 given by

$$\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} \frac{a^{-1}+a}{\pm\sqrt{b^2+(a^{-1}+a)^2}} & \frac{b}{\pm\sqrt{b^2+(a^{-1}+a)^2}} \\ -\frac{b}{\pm\sqrt{b^2+(a^{-1}+a)^2}} & \frac{a^{-1}+a}{\pm\sqrt{b^2+(a^{-1}+a)^2}} \end{pmatrix},$$

choosing $\operatorname{sign}(\pm\sqrt{b^2 + (a^{-1} + a)^2}) = \operatorname{sign} b$ if $b \neq 0$ and $+1$ for $b = 0$. The section σ_1 corresponds to the hyperbolic plane loop (cf. [43], pp. 283-284). Since $\big[[\mathbf{m}_0, \mathbf{m}_0], \mathbf{m}_0\big] \subseteq \mathbf{m}_0$ and each element $g \in G$ can uniquely be represented as a product $g = mh$ with $m \in \exp \mathbf{m}_0$ and $h \in H$ we have a global differentiable Bol loop L defined on the factor space G/H (cf. [31], Corollary 3.11, p. 51 and [43], Lemma 1.3, p. 17). This loop is a left A-loop, because of $[\mathbf{h}, \mathbf{m}_0] \subseteq \mathbf{m}_0$. But it is not a Bruck loop since there is no involutory automorphism $\sigma : \mathbf{g} \to \mathbf{g}$ such that $\sigma(\mathbf{m}_0) = -\mathbf{m}_0$ and $\sigma(\mathbf{h}) = \mathbf{h}$. According to Proposition 2.4. in [43] (p. 44) in this loop $L = G/H$ with

$$\sigma : G/H \to G,\ \sigma((x,y)(H_1, \varphi(H_1)) = (\sigma_1(xH_1), y\varphi(x^{-1}\sigma_1(xH_1)))$$

there is a normal subgroup $\tilde{G}_2$ isomorphic to $SO_2(\mathbb{R})$ and the factor loop $L/\tilde{G}_2$ is isomorphic to the hyperbolic plane loop. Therefore L is the unique Scheerer extension of the Lie group $SO_2(\mathbb{R})$ by the hyperbolic plane loop (cf. [43], Section 2).

Summarizing our discussion we have:

Theorem 10. *There are precisely three isotopism classes $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ of connected almost differentiable left A-loops with dimension 3 such that the group G topologically generated by their left translations is a 4-dimensional non-solvable Lie group.*
Every loop in the class $\mathcal{C}_1$, respectively $\mathcal{C}_2$ is the direct product of a 2-dimensional loop isomorphic to the hyperbolic plane loop with the Lie group $\mathbb{R}$, respectively $SO_2(\mathbb{R})$. These loops are differentiable Bruck loops. In the first class

the group G is isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}$, in the second class the group G is isomorphic to $PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and in both classes the stabilizer of the identity of these loops is isomorphic to $SO_2(\mathbb{R})$.
In the class $\mathcal{C}_3$ is contained up to isomorphisms only the Scheerer extension L of the Lie group $SO_2(\mathbb{R})$ by the hyperbolic plane loop. The group G topologically generated by the left translations of L is the direct product $PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and the stabilizer H of $e \in L$ in G is the group $H = \{(x, \varphi(x)) \mid x \in SO_2(\mathbb{R})\}$, where φ is a monomorphism from a compact subgroup of $PSL_2(\mathbb{R})$ onto $SO_2(\mathbb{R})$.

4 3-dimensional left A-loops belonging to 5-dimensional non-solvable Lie groups

Now we determine the 3-dimensional connected almost differentiable global left A-loops having a 5-dimensional non-solvable Lie group G as the group topologically generated by the left translations of L. In this case the stabilizer of $e \in L$ in G is a 2-dimensional closed subgroup of G containing no non-trivial normal subgroup of G. According to Propositions 6 and 7 we have to investigate the following cases:
1) G is locally isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}^2$ and H is locally isomorphic to

$$\left\{ \left(\left(\begin{array}{cc} e^a & b \\ 0 & e^{-a} \end{array} \right), a \right), a, b \in \mathbb{R} \right\},$$

2) $G \cong PSL_2(\mathbb{R}) \times \mathcal{L}_2$ and
a) $H \cong \left\{ \left(\left(\begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right), \left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \right), a > 0, b \in \mathbb{R} \right\}$
b) $H \cong \left\{ \left(\left(\begin{array}{cc} e^z & b \\ 0 & e^{-z} \end{array} \right), \left(\begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) \right), z, b \in \mathbb{R} \right\}$
c) $H \cong \{(\varphi(x), x) | x \in \mathcal{L}_2\}$, where φ is a monomorphism from $\mathcal{L}_2$ onto a 2-dimensional subgroup of $PSL_2(\mathbb{R})$,
3) G is locally isomorphic to the semi-direct product $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^2$, which is the connected component of the group for area preserving affinities of $\mathbb{R}^2$.

In the first case let us consider the following real basis of the Lie algebra $\mathbf{g} = sl_2(\mathbb{R}) \oplus \mathbb{R}^2$

$$\{(K, 0), (T, 0), (U, 0), (0, e_1), (0, e_2)\},$$

where K, T, U are (2×2)-matrices defined in 1.2 and e_1, e_2 are the generators of $\mathbb{R}^2$. The multiplication table in $\mathbf{g}$ is given by the rules:

$$[(K, 0), (T, 0)] = (2U, 0), \ [(K, 0), (U, 0)] = (2T, 0), \ [(U, 0), (T, 0)] = (2K, 0),$$

$$[(K, 0), (0, e_i)] = [(T, 0), (0, e_i)] = [(U, 0), (0, e_i)] = (0, 0), \text{for } i = 1, 2$$

$$[(0,e_1),(0,e_2)] = (0,0).$$

The Lie algebra $\mathbf{h}$ of H has the form

$$\mathbf{h} = \langle (K, e_1 + ke_2), (U+T, 0)\rangle, \quad k \in \mathbb{R}.$$

Since the automorphism $A : \mathbf{g} \to \mathbf{g}$ defined by

$$A(K,0) = (K,0), \quad A(T,0) = (T,0), \quad A(U,0) = (U,0),$$
$$A(0,e_1) = (0,e_1) - k(0,e_2), \quad A(0,e_2) = (0,e_2)$$

maps $\mathbf{h}$ onto $\mathbf{h}' = \langle (K,e_1),(U+T,0)\rangle$, we may (up to isotopism) assume that $\mathbf{h}$ has the shape $\langle (K,e_1),(U+T,0)\rangle$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ is generated by the following basis elements:

$f_1 = ((1+a_2)U + a_2T + a_1K, a_1e_1)$,

$f_2 = (b_1K + b_2(U+T), b_1e_1 + e_2)$,

$f_3 = (c_1K + c_2(U+T), c_1e_1 + e_1)$,

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. According to Lemma 1 $\mathbf{m}$ has one of the following shapes:

$$\mathbf{m}_1 = \langle ((1+a_2)U + a_2T, 0), (b_1K + b_2(U+T), b_1e_1 + e_2), (c_1K + c_2(U+T), c_1e_1 + e_1)\rangle,$$

with the real parameters a_2, b_1, b_2, c_1, c_2,

$$\mathbf{m}_2 = \langle ((1+a_2)U + a_2T + a_1K, a_1e_1), (b_1K + b_2(U+T), b_1e_1 + e_2), (-K + c_2(U+T), 0)\rangle,$$

where $a_1, a_2, b_1, b_2, c_2 \in \mathbb{R}$.

The complement $\mathbf{m}_1$ is reductive if $[\mathbf{h}, \mathbf{m}_1] \subseteq \mathbf{m}_1$ holds. The elements

$$[(U+T,0),(b_1K + b_2(U+T), b_1e_1 + e_2)] = -2b_1(U+T,0)$$

$$[(U+T,0),(c_1K + c_2(U+T), c_1e_1 + e_1)] = -2c_1(U+T,0)$$

are elements of $\mathbf{m}_1$ if and only if $b_1 = c_1 = 0$. But then the element

$$[(U+T,0),((1+a_2)U + a_2T, 0)] = -2(K,0)$$

is not contained in $\mathbf{m}_1$, hence $\mathbf{m}_1$ is not reductive.

The complement $\mathbf{m}_2$ is again not reductive since the element

$$[(U+T,0),(-K + c_2(U+T), 0)] = 2(U+T,0)$$

is not an element of $\mathbf{m}_2$.

In the second case the elements of $G = G_1 \times G_2$ we can represent as pairs of matrices

$$g = \left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ 0 & e^{-1} \end{pmatrix} \right),$$

where $ad - bc = 1, e > 0, f \in \mathbb{R}$. The group multiplication is the matrix multiplication in the both components. A real basis of the Lie algebra $\mathbf{g}$ of G is

$$\mathbf{g} = \langle (K,0),\ (T,0),\ (U,0),\ (0,e_1),\ (0,e_2)\rangle,$$

where K, T and U are the basis elements of $sl_2(\mathbb{R})$ (see **1.2**) and e_1, e_2 are the basis elements of $\mathcal{L}_2$. The multiplication in the Lie algebra $\mathbf{g}$ is given as follows:

$$[(K,0),(T,0)] = (2U,0),\ [(K,0),(U,0)] = (2T,0),\ [(U,0),(T,0)] = (2K,0),$$

$$[(K,0),(0,e_i)] = [(T,0),(0,e_i)] = [(U,0),(0,e_i)] = (0,0),$$
$$[(0,e_1),(0,e_2)] = -(0,e_2).$$

If H has the form as in 2 a) then the Lie algebra $\mathbf{h}$ of H is generated by the elements (K, e_1) and $(U+T, 0)$.

An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has as generators

$$\begin{aligned} f_1 &= (U + a_1K + a_2(U+T), a_1e_1),\\ f_2 &= (b_1K + b_2(U+T), e_2 + b_1e_1),\\ f_3 &= (c_1K + c_2(U+T), e_1 + c_1e_1), \end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. According to Lemma 1 the dimension of $\mathbf{m} \cap sl_2(\mathbb{R}) \oplus \{0\}$ is at least one. Hence we have two possibilities:

The complement $\mathbf{m}_1$ is generated by the elements

$$\{((1+a_2)U+a_2T,0),(b_1K+b_2(U+T),b_1e_1+e_2),(c_1K+c_2(U+T),c_1e_1+e_1)\},$$

where a_2, b_1, b_2, c_1, c_2 are real parameters,

the complement $\mathbf{m}_2$ has the following basis elements

$$\{((1+a_2)U+a_2T+a_1K,a_1e_1),(b_1K+b_2(U+T),b_1e_1+e_2),(-K+c_2(U+T),0)\},$$

with $a_1, a_2, b_1, b_2, c_2 \in \mathbb{R}$.

Let now the stabilizer H be in 2 b). The elements $(K, e_2), (U+T, 0)$ can be chosen as the basis elements of the Lie algebra $\mathbf{h}$ of H. The basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are

$$\begin{aligned} f_1 &= (U + a_1K + a_2(U+T), a_1e_2),\\ f_2 &= (b_1K + b_2(U+T), e_1 + b_1e_2),\\ f_3 &= (c_1K + c_2(U+T), e_2 + c_1e_2), \end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Since the intersection of $\mathbf{m}$ and $sl_2(\mathbb{R}) \oplus \{0\}$ is at least 1-dimensional (Lemma 1), we may assume that the complement $\mathbf{m}$ has one of the following shapes:

$$\mathbf{m}_1 =$$

$\langle((1{+}a_2)U{+}a_2T,0),(b_1K{+}b_2(U{+}T),b_1e_2{+}e_1),(c_1K{+}c_2(U{+}T),e_2{+}c_1e_2)\rangle$, where $a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$,

$\mathbf{m}_2 =$
$\langle((1{+}a_2)U{+}a_2T{+}a_1K,a_1e_1),(b_1K{+}b_2(U{+}T),b_1e_2{+}e_1),(-K{+}c_2(U{+}T),0)\rangle$, with the real parameters a_1,a_2,b_1,b_2,c_2.
If we compare the forms of the complements $\mathbf{m}_i$ $(i=1,2)$ and of the Lie algebra $\mathbf{h}$ in the cases 2 a) and 2 b) with the shapes of the complements $\mathbf{m}_i$ $(i=1,2)$ and of the Lie algebra $\mathbf{h}$ in the case 1) we see that the same computation leads to the same contradiction in the cases 2 a) and 2 b) as in the case 1).

Now we deal with the case 2 c). Every 2-dimensional subgroup of G having the form $\{(x,\varphi(x))\}$, where φ is a monomorphism from a 2-dimensional subgroup of $PSL_2(\mathbb{R})$ onto $\mathcal{L}_2$ is conjugate to the following

$$H=\left\{\left(\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix},\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}\right); a>0, b\in\mathbb{R}\right\}.$$

Then the Lie algebra $\mathbf{h}$ of H has as generators (K,e_1), $(U+T,e_2)$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ is generated by the following basis elements:
$f_1 = (U+a_1K+a_2(U+T),a_1e_1+a_2e_2)$,
$f_2 = (b_1K+b_2(U+T),e_1+b_1e_1+b_2e_2)$,
$f_3 = (c_1K+c_2(U+T),e_2+c_1e_1+c_2e_2)$,
where $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$. From Lemma 1 we obtain that $\mathbf{m}$ has one of the following shapes:

$$\mathbf{m}_1=\langle(U,0),((b_1K{+}b_2T),(1{+}b_1)e_1{+}b_2\,e_2),((c_1K{+}c_2T),(1{+}c_2)e_2{+}c_1e_1)\rangle,$$

where $b_1,b_2,c_1,c_2\in\mathbb{R}$.

$$\mathbf{m}_2=\langle(U+a_2(U+T),a_1e_1+a_2e_2),(K,0),(c_2(U+T),(1+c_2)e_2+c_1e_1)\rangle,$$

with the real parameters a_1,a_2,c_1,c_2.

$$\mathbf{m}_3=\langle(U+a_1K,a_1e_1+a_2e_2),(b_1K,(1+b_1)e_1+b_2\,e_2),(U+T,0)\rangle,$$

where a_1,a_2,b_1,b_2 are real parameters.
In the case $\mathbf{m}_1$ it follows from the property $[\mathbf{h},\mathbf{m}]\subseteq\mathbf{m}$ that the elements

$$[(K,e_1),(U,0)]=2(T,0)$$

and

$$[(U+T,e_2),(U,0)]=-2(K,0)$$

are in $\mathbf{m}_1$. Then $\mathbf{m}_1$ is a proper subalgebra of $\mathbf{g}$, and this is a contradiction.
Now we deal with the complement $\mathbf{m}_2$. The element

$$[(U+T,e_2),(K,0)] = -2(U+T,0)$$
lies in $\mathbf{m}_2$ if and only if $c_2 = -1, c_1 = 0$ but then $\mathbf{m}_2$ does not generate $\mathbf{g}$. Now we consider the complement $\mathbf{m}_3$. Since the elements
$$[(K,e_1),(b_1 K,(1+b_1)e_1 + b_2\ e_2)] = -(0, b_2 e_2)$$
and
$$[(U+T,e_2),(b_1 K,(1+b_1)e_1 + b_2\ e_2)] = (-2b_1(U+T),(1+b_1)e_2)$$
must be in $\mathbf{m}_3$ one has $b_2 = 0, b_1 = -1$. Then $\mathbf{g}$ is not generated by $\mathbf{m}_3$. This implies that there is no 3-dimensional left A-loop as section in the groups $G = PSL_2(\mathbb{R}) \times \mathbb{R}^2$, $G = PSL_2(\mathbb{R}) \times \mathcal{L}_2$.

In the third case we can represent the group G as the matrix group:
$$\left\{ \begin{pmatrix} 1 & u & v \\ 0 & a & b \\ 0 & c & d \end{pmatrix} ; ad - bc = 1,\ u, v \in \mathbb{R} \right\}.$$
The 2-dimensional subgroups H of G containing no non-trivial normal subgroup of G are one of the following:
a)
$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & b & a^{-1} \end{pmatrix} ; a > 0, b \in \mathbb{R} \right\}$$
b)
$$H = \left\{ \begin{pmatrix} 1 & u & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} ; a \in \mathbb{R}\backslash\{0\}, u \in \mathbb{R} \right\}$$
c)
$$H = \left\{ \begin{pmatrix} 1 & u+\varphi(b) & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix} ; b, u \in \mathbb{R} \right\}.$$
The Lie algebra $\mathbf{g}$ of G is the semi-direct product $sl_2(\mathbb{R}) \ltimes \mathbb{R}^2$. For the Lie algebra $\mathbf{g}$ of G we can choose the following basis elements:
$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$
$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The multiplication table is given by:

$$[K,e_1]=[T,e_2]=-[U,e_2]=-2\ e_1,\ [K,e_2]=-[U,e_1]=-[T,e_1]=2\ e_2,$$

$$[e_1,e_2]=0,\ [K,T]=2\ U,\ [K,U]=2\ T,\ [U,T]=2\ K.$$

The Lie algebra $\mathbf{h}$ of H in the case a) is given by $\mathbf{h}_1=\langle K,U-T\rangle$. For a complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ we have the general form:

$$\mathbf{m}=\langle e_1+a_1K+a_2(U-T),e_2+b_1K+b_2(U-T),U+c_1K+c_2(U-T)\rangle$$

for $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$. From the property $[\mathbf{h},\mathbf{m}]\subseteq\mathbf{m}$ we obtain that the element

$$[K,U+c_1K+c_2(U-T)]=2T-2c_2(U-T)$$

must be in $\mathbf{m}$. It is satisfied if and only if $c_1=0,c_2=-\frac{1}{2}$. But then the element

$$[(U-T),\frac{1}{2}(U+T)]=2K$$

must lay again in $\mathbf{m}$, which is a contradiction.

In the case b) the Lie algebra $\mathbf{h}$ of H has the basis elements K,e_1. A complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ we can write in the following form:

$$\mathbf{m}=\langle e_2+a_1K+a_2e_1,U+b_1K+b_2e_1,T+c_1K+c_2e_1\rangle$$

with the real parameters a_1,a_2,b_1,b_2,c_1,c_2. This complement $\mathbf{m}$ is reductive if $[\mathbf{h},\mathbf{m}]\subseteq\mathbf{m}$. Therefore the elements

$$\begin{array}{rcl}[e_2+a_1K+a_2e_1,e_1]&=&-2a_1e_1,\\ [e_2+a_1K+a_2e_1,K]&=&-2e_2+2a_2e_1,\\ [U+b_1K+b_2e_1,K]&=&-2T+2b_2e_1,\\ [T+c_1K+c_2e_1,K]&=&-2U+2c_2e_1\end{array}$$

are elements of $\mathbf{m}$. This is the case precisely if $a_1=a_2=c_1=b_1=0,c_2=-b_2$. Then a reductive complement $\mathbf{m}$ is generated by

$$\{e_2,U+b_2e_1,T-b_2e_1\},$$

where $b_2\in\mathbb{R}$. But the element $e_2\in\mathbf{m}$ is conjugated to the element $e_1\in\mathbf{h}$ under the element $g=\begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}\in G$.

In the last case the Lie algebra $\mathbf{h}$ of H has as basis elements $U-T,\ e_1$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ can be given as follows:

$$\mathbf{m}=\langle e_2+a_1(U-T)+a_2e_1,\ K+b_1(U-T)+b_2e_1,\ U+c_1(U-T)+c_2e_1\rangle,$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are real parameters. From the property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ we obtain that the element $[e_1, K+b_1(U-T)+b_2e_1] = 2e_1$ must be an element of $\mathbf{m}$. But this is impossible. Therefore there is not any 3-dimensional almost differentiable global left A-loop L having the group $G = PSL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as the group topologically generated by the left translations of L.

This consideration yields the following

Theorem 11. *There is no 3-dimensional connected almost differentiable global left A-loop L having a 5-dimensional non-solvable Lie group as the group topologically generated by its left translations.*

5 3-dimensional left A-loops with 6-dimensional non-solvable Lie groups

Now we consider all non-solvable and non-semisimple Lie groups with dimensional 6. We determine all 3-dimensional connected almost differentiable left A-loops such that the group topologically generated by their left translations is one of these Lie groups. According to the Propositions 6 and 7 we have to discuss the cases 2 A, 2 B, 2 C, 2 D, 2 F in the Proposition 7 and the cases
α) G is locally isomorphic to $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$,
β) G is the group for orientation preserving affinities of $\mathbb{R}^2$,
γ) G is locally isomorphic to $SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, which is the connected component of the euclidean motion group of $\mathbb{R}^3$.

In the cases 2 A, 2 B, 2 C, 2 D and 2 F the Lie algebras $\mathbf{g}$ of G can be represented as $\mathbf{g} = sl_2(\mathbb{R}) \oplus \mathbf{g}_2$, where $\mathbf{g}_2$ are the Lie algebras introduced in the cases A, B, C, D, F in the proof of Proposition 7. Let K, T and U be the real basis of $sl_2(\mathbb{R})$ defined in **1.2** (Section 1). As real basis of the Lie algebras $\mathbf{g}_2$ we choose the basises $\{e_1, e_2, e_3\}$ given in the cases A, B, C, D, F in the proof of Proposition 7.
In the case 2 A) the Lie algebra $\mathbf{h}$ of H is generated by the basis elements

$$(K, le_3 + ke_1),\ (U+T, 0),\ (0, e_1),$$

where k, l are given real parameters such that $l \neq 0$. As generators of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ may be chosen the following elements

$$f_1 = ((1+a_2)U + a_2T + a_1K, (a_1k + a_3)e_1 + a_1le_3),$$

$$f_2 = (b_1K + b_2(U+T), e_2 + b_1le_3 + (b_1k + b_3)e_1),$$

$$f_3 = (c_1K + c_2(U+T), (1+c_1l)e_3 + (c_1k+c_3)e_1),$$

with $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
In the case 2 B) the Lie algebras $\mathbf{h}_1$ of H_1 and $\mathbf{h}_2$ of H_2 have the shapes

$$\mathbf{h}_1 = \langle (K, le_3 + ke_1), (U+T, 0), (0, e_1) \rangle,$$

where $k \in \mathbb{R}, l \in \mathbb{R}\backslash\{0\}$ are given parameters,

$$\mathbf{h}_2 = \langle (K, le_2 + ke_3), (U+T, 0), (0, e_2 + e_3) \rangle,$$

where $k \neq l$ are given real numbers. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}_1$ in $\mathbf{g}$ has as basis elements:

$$f_1 = ((1+a_2)U + a_2T + a_1K, (a_1k + a_3)e_1 + a_1le_3),$$

$$f_2 = (b_1K + b_2(U+T), (1+b_1l)e_3 + (b_1k+b_3)e_1),$$
$$f_3 = (c_1K + c_2(U+T), e_2 + c_1le_3 + (c_1k+c_3)e_1),$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
An arbitrary complement $\mathbf{m}$ to $\mathbf{h}_2$ in $\mathbf{g}$ can be written as follows:

$$f_1 = ((1+a_2)U + a_2T + a_1K, (a_1l + a_3)e_2 + (a_1k+a_3)e_3),$$

$$f_2 = (b_1K + b_2(U+T), e_1 + (b_3 + b_1l)e_2 + (b_1k+b_3)e_3),$$
$$f_3 = (c_1K + c_2(U+T), (1 + c_1l + c_3)e_2 + (c_1k+c_3)e_3),$$

with the real numbers $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$.
Now we deal with the case 2 C). The basis elements of the Lie algebra $\mathbf{h}$ of H are the following

$$(K, le_1 + ke_2), (U+T, 0), (0, e_1 + e_2),$$

where k, l are given real different numbers. Then an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ is generated by the elements:

$$f_1 = ((1+a_2)U + a_2T + a_1K, (a_1l + a_3)e_1 + (a_1k+a_3)e_2),$$

$$f_2 = (b_1K + b_2(U+T), (1 + b_1l + b_3)e_1 + (b_1k+b_3)e_2),$$
$$f_3 = (c_1K + c_2(U+T), e_3 + (c_3 + c_1k)e_2 + (c_1l + c_3)e_1),$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
Now we consider the cases 2 D) and 2 F). In both cases as basis elements of the Lie algebra $\mathbf{h}$ of H can be chosen the following

$$(K, ke_1 + le_2), (U+T, 0), (0, e_1),$$

where $k \in \mathbb{R}, l \in \mathbb{R}\backslash\{0\}$ are given parameters. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the following basis elements in both cases:

$$f_1 = ((1+a_2)U + a_2 T + a_1 K, (a_1 k + a_3)e_1 + a_1 l e_2),$$

$$f_2 = (b_1 K + b_2(U+T), (1+b_1 l)e_2 + (b_1 k + b_3)e_1),$$

$$f_3 = (c_1 K + c_2(U+T), e_3 + (c_3 + c_1 k)e_1 + c_1 l e_2),$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real parameters.
We prove that in these cases there is no complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ with the properties $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$, $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ and $\mathbf{m}$ generates the Lie algebra $\mathbf{g}$.
First we consider the cases 2 A) and 2 B). From the property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ we obtain that the elements

$$[(U+T,0), f_2] = -2b_1(U+T,0)$$
$$[(U+T,0), f_3] = -2c_1(U+T,0)$$

must be elements of $\mathbf{m}$. Moreover in the cases 2 A) and for $\mathbf{h}_1$ in 2 B) the elements

$$[(K, le_3), f_2] = 2b_2(U+T,0)$$
$$[(K, le_3), f_3] = 2c_2(U+T,0)$$

for $\mathbf{h}_2$ in 2 B) the elements

$$[(K, (k-l)e_3), f_2] = 2b_2(U+T,0)$$
$$[(K, (k-l)e_3), f_3] = 2c_2(U+T,0)$$

must be again in $\mathbf{m}$. It holds if and only if $b_1 = b_2 = c_1 = c_2 = 0$. Therefore in these cases $\mathbf{m}$ does not generate $\mathbf{g}$.
Since $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ in the case 2 C) for all $c_2 \in \mathbb{R}$ and $l \neq k$ the element

$$[(K, \frac{(l-k)b}{b-a}e_1 + \frac{(l-k)a}{b-a}e_2), f_3] =$$

$$(2c_2(U+T), \frac{(l-k)ba}{b-a}(e_1+e_2)) \neq (0,0),$$

in the case 2 D) for all $c_2 \in \mathbb{R}$ and $l \in \mathbb{R}\backslash\{0\}$ the element

$$[(K, -le_1 + le_2), f_3] = (2c_2(U+T), -le_1) \neq (0,0),$$

and in the case 2 F) for all $c_2 \in \mathbb{R}$ and $l \in \mathbb{R}\backslash\{0\}$ the element

$$[(K, le_2), f_3] = (2c_2(U+T), -le_1) \neq (0,0)$$

must be in $\mathbf{m}$. This is not satisfied since these elements are elements in $\mathbf{h}$ and $\mathbf{h} \cap \mathbf{m} = \{(0,0)\}$. Therefore there is no reductive complement $\mathbf{m}$ in these cases. Moreover there is no 3-dimensional almost differentiable left A-loop such that the group topologically generated by its left translations is the direct product $G = G_1 \times G_2$, where $G_1 = PSL_2(\mathbb{R})$ and G_2 is one of the 3-dimensional solvable non-abelian Lie groups.

In the case α) the group multiplication in G is given by

$$(A_1, X_1) \circ (A_2, X_2) = (A_1\ A_2,\ A_2^{-1}\ X_1\ A_2 + X_2),$$

where (A_i, X_i), $i = 1, 2$ are two elements of G such that X_i $(i = 1, 2)$ are represented by (2×2) real matrices with trace 0.
The 3-dimensional subgroups H of G containing no normal non-trivial subgroup of G are locally isomorphic to the following subgroups:
a)

$$H = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} e + \varphi(a) & 0 \\ f & -e - \varphi(a) \end{pmatrix} \right); a > 0, e, f \in \mathbb{R} \right\},$$

b)

$$H = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} \varphi(a) & f \\ g & -\varphi(a) \end{pmatrix} \right); a > 0, f, g \in \mathbb{R} \right\},$$

c)

$$H = \left\{ \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e + \varphi(b) & f \\ 0 & -e - \varphi(b) \end{pmatrix} \right); b, e, f \in \mathbb{R} \right\},$$

d)

$$H = \left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \right); a > 0, b, f \in \mathbb{R} \right\},$$

e)

$$H = \left\{ \left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0 \right); a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\},$$

f)

$$H = \left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi), x, y \in \mathbb{R} \right\}.$$

The Lie algebra $\mathbf{g}$ of G is isomorphic to $sl_2(\mathbb{R}) \ltimes \mathbb{R}^3$. A basis of the Lie algebra $\mathbf{g}$ can be chosen as follows:

$$e_1 = \left(0, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), e_2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right), e_3 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right),$$

$$e_4 = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right), e_5 = \left(0, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right), e_6 = \left(0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

According to [29] (p. 17) we obtain the following multiplication table in $\mathbf{g}$:

$$[e_1, e_2] =: e_6,\ [e_1, e_3] =: e_5,\ [e_2, e_3] =: e_4,\ [e_5, e_4] = -e_6,$$

$$[e_1, e_4] = [e_1, e_5] = [e_1, e_6] = [e_2, e_5] = [e_3, e_6] = [e_6, e_5] = 0,$$

$$[e_2, e_6] = [e_3, e_5] = -e_1,\ [e_2, e_4] = e_3,\ [e_3, e_4] = -e_2,\ [e_6, e_4] = e_5.$$

Since $\langle e_1, e_5, e_6 \rangle$ is the radical of the Lie algebra $\mathbf{g}$, the Cartan-Killing form k on $\mathbf{g}$ is degenerate. If an element X of $\mathbf{g} = sl_2(\mathbb{R}) \ltimes \mathbb{R}^3$ has the decomposition

$$X = \lambda_1\ e_1 + \lambda_2\ e_2 + \lambda_3\ e_3 + \lambda_4\ e_4 + \lambda_5\ e_5 + \lambda_6\ e_6$$

then the following relation is satisfied

$$k(X, X) = \lambda_2^2 + \lambda_3^2 - \lambda_4^2.$$

The corresponding 3-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}$ are the following:
a) $\langle e_2,\ e_5,\ e_1 + e_6 \rangle$,
b) $\langle e_2 + k\ e_5,\ e_1,\ e_6 \rangle$, where $k \in \mathbb{R}$,
c) $\langle e_3 + e_4,\ e_5,\ e_1 - e_6 \rangle$,
d) $\langle e_2,\ e_3 + e_4,\ e_1 - e_6 \rangle$,
e) $\langle e_2, e_3, e_4 \rangle$,
f) $\langle e_4, e_5, e_6 \rangle$.
We prove that there is no 3-dimensional almost differentiable global left A-loop L, such that the group G topologically generated by its left translations is locally isomorphic to $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$ and the stabilizer of $e \in L$ in G is locally isomorphic to one of the 3-dimensional subgroups of G described in cases a), b), c), d), e).

In the case a) the basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are:

$$f_1 = e_1 + a_1 e_2 + a_2 e_5 + a_3(e_1 + e_6),$$
$$f_2 = e_4 + b_1 e_2 + b_2 e_5 + b_3(e_1 + e_6),$$
$$f_3 = e_3 + c_1 e_2 + c_2 e_5 + c_3(e_1 + e_6),$$

with $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$. If $\mathbf{m}$ is reductive, then $\mathbf{m}$ contains the element

$$[e_2, e_1 + a_1 e_2 + a_2 e_5 + a_3(e_1 + e_6)] = -(1 + a_3)e_6 - a_3 e_1.$$

This is the case if and only if $a_1 = a_2 = 0$ and $a_3 = -\frac{1}{2}$. Then the element $\frac{1}{2}(e_1 - e_6) \in \mathbf{m}$ is conjugate to $e_1 + e_6 \in \mathbf{h}$ under the element $g = \left(\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right) \in G$. This contradicts Lemma 2.

If $\mathbf{h} = \langle e_2 + k\ e_5,\ e_1,\ e_6 \rangle$ with $k \in \mathbb{R}$ then an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ has as basis elements:

$$f_1 = e_5 + a_1(e_2 + ke_5) + a_2e_1 + a_3e_6,$$

$$f_2 = e_3 + c_1(e_2 + ke_5) + c_2e_1 + c_3e_6,$$

$$f_3 = e_4 + b_1(e_2 + ke_5) + b_2e_1 + b_3e_6,$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real parameters. The complement $\mathbf{m}$ must be reductive hence the elements

$$\begin{aligned} [e_1, e_5 + a_1(e_2 + ke_5) + a_2e_1 + a_3e_6] &= -a_1e_6 \\ [e_2 + ke_5, e_5 + a_1(e_2 + ke_5) + a_2e_1 + a_3e_6] &= a_2e_6 + a_3e_1 \end{aligned}$$

are elements of $\mathbf{m}$. It is satisfied if and only if $a_1 = a_2 = a_3 = 0$. In this case the element $e_5 \in \mathbf{m}$ is conjugate to $e_6 \in \mathbf{h}$ under the element $g = \left(\pm \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}, 0 \right) \in G$. This is a contradiction to Lemma 2.

In the case c) we can choose as basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ the following:

$$f_1 = e_1 + a_1(e_3 + e_4) + a_2e_5 + a_3(e_1 - e_6),$$

$$f_2 = e_2 + b_1(e_3 + e_4) + b_2e_5 + b_3(e_1 - e_6),$$

$$f_3 = e_3 + c_1(e_3 + e_4) + c_2e_5 + c_3(e_1 - e_6),$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$. Since for the subspace $\mathbf{m}$ holds $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ the element

$$[e_5, f_2] = b_1(e_1 - e_6)$$

must be in $\mathbf{m}$. This is the case if and only if $b_1 = 0$. But then the element

$$[e_1 - e_6, e_2 + b_2e_5 + b_3(e_1 - e_6)] = e_6 - e_1$$

is not an element of $\mathbf{m}$. This contradiction shows that this $\mathbf{m}$ is not a reductive subspace.

In the case d) the generators of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are:

$$f_1 = e_1 + a_1e_2 + a_2(e_3 + e_4) + a_3(e_1 - e_6),$$

$$f_2 = e_5 + b_1e_2 + b_2(e_3 + e_4) + b_3(e_1 - e_6),$$

$$f_3 = e_3 + c_1e_2 + c_2(e_3 + e_4) + c_3(e_1 - e_6),$$

with the real parameters $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$. From the property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ we obtain that the element

$$[e_2, f_2] = b_2(e_4 + e_3) + b_3(e_1 - e_6)$$

lies in $\mathbf{m}$. It is satisfied precisely if $b_2 = b_3 = 0$. But the element

$$[e_3 + e_4, e_5 + b_1 e_2] = e_1 - e_6 + b_1(e_3 + e_4)$$

is not an element of $\mathbf{m}$. Hence there is no reductive complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$.

In the case e) one has $\mathbf{h} \cong sl_2(\mathbb{R})$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ is generated by:

$$f_1 = e_1 + a_1 e_2 + a_2 e_3 + a_3 e_4,$$
$$f_2 = e_6 + b_1 e_2 + b_2 e_3 + b_3 e_4,$$
$$f_3 = e_5 + c_1 e_2 + c_2 e_3 + c_3 e_4,$$

with $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$. We seek for reductive subspaces $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$. Since the elements

$$\begin{array}{rcl} [e_2, f_1] & = & -e_6 + a_2 e_4 + a_3 e_3, \\ [e_4, f_1] & = & -a_1 e_3 + a_2 e_2, \\ [e_3, f_1] & = & -e_5 - a_1 e_4 - a_3 e_2 \end{array}$$

must be in $\mathbf{m}$, we have $a_1 = a_2 = b_1 = c_2 = 0$, $a_2 = -b_3$, $a_3 = -b_2$, $c_1 = a_3$, $c_3 = a_1$. This means that $\mathbf{m}_a$ generated by

$$\{e_1 + a e_4, e_6 - a e_3, e_5 + a e_2\}, \quad a \in \mathbb{R}\backslash\{0\}$$

is the unique reductive complement to $\mathbf{h}$. Since the element

$$e_6 - a e_3 + e_1 + a e_4 \in \mathbf{m}_a$$

for all $a \in \mathbb{R}\backslash\{0\}$ is conjugate to the element $a e_4 - a e_3 \in \mathbf{h}$ under the element $g = \left(1, \begin{pmatrix} \frac{1}{2a} & 0 \\ 0 & -\frac{1}{2a} \end{pmatrix}\right) \in G$, we obtain a contradiction to Lemma 2.

Now we consider the last case. Since the group $SL_2(\mathbb{R})$ has no 3-dimensional linear representation the group G is isomorphic to the semidirect product of $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$ and H is isomorphic to the following 3-dimensional subgroup of G:

$$H = \left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right), t \in [0, 2\pi), x, y \in \mathbb{R} \right\}.$$

We known that the Lie algebra $\mathbf{h}$ of H is generated by the basis elements e_4, e_5, e_6. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the following basis elements:

$$\{e_1 + a_1 e_4 + a_2 e_5 + a_3 e_6, e_2 + b_1 e_4 + b_2 e_5 + b_3 e_6, e_3 + c_1 e_4 + c_2 e_5 + c_3 e_6\},$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
The complement $\mathbf{m}$ is reductive if and only if it has the following shape

$$\mathbf{m} = \mathbf{m}_{b_1, b_2} = \langle e_1, e_2 + b_1 e_6 + b_2 e_5, e_3 - b_2 e_6 + b_1 e_5 \rangle,$$

where $b_1, b_2 \in \mathbb{R}$.

Now we determine the isomorphism classes and the isotopism classes of the left A-loops L_{b_1,b_2} having the subspaces $\mathbf{m}_{b_1,b_2}$ $(b_1, b_2 \in \mathbb{R})$ as the tangent spaces $T_1 L_{b_1,b_2}$.
We have precisely two isomorphism classes $\mathcal{C}_i$ $(i = 1, 2)$ of the loops L_{b_1,b_2} belonging to the triples $(G, H, \exp \mathbf{m}_{b_1,b_2})$ for all $b_1, b_2 \in \mathbb{R}$.
The first class consists loops belonging to $\mathbf{m}_{b_1,b_2}$ for $b_2 = 0$. Denote by $\hat{\mathbf{m}}_{b_1}$ the complement $\mathbf{m}_{b_1,0}$ for all $b_1 \in \mathbb{R}$. The complements $\hat{\mathbf{m}}_{b_1}$ are orthogonal to $\mathbf{h}$ with respect to the Cartan-Killing form k on $\mathbf{g}$. One has $[\hat{\mathbf{m}}_{b_1}, \hat{\mathbf{m}}_{b_1}] = \mathbf{h}$ and $\mathbf{g} = \hat{\mathbf{m}}_{b_1} \oplus [\hat{\mathbf{m}}_{b_1}, \hat{\mathbf{m}}_{b_1}]$ for all $b_1 \in \mathbb{R}$. Therefore $M_{b_1} = \exp \hat{\mathbf{m}}_{b_1}$ is a 3-dimensional connected symmetric space for all $b_1 \in \mathbb{R}$. Every loop $L_{b_1,0}$ in this class is a Bruck loop and isomorphic to the loop $\hat{L}_0 = L_{0,0}$ corresponding to $\mathbf{m}_{0,0}$ under the automorphism $\varphi : \mathbf{g} \to \mathbf{g}$ defined by
$\varphi(e_1) = e_1$
$\varphi(e_6) = e_6$
$\varphi(e_5) = e_5$
$\varphi(e_4) = e_4$
$\varphi(e_2) = e_2 - b_1\ e_6$
$\varphi(e_3) = e_3 - b_1\ e_5$.
The other class $\mathcal{C}_2$ consists of loops L_{b_1,b_2} having $T_1 L_{b_1,b_2} = \mathbf{m}_{b_1,b_2}$ for $b_2 \neq 0$. In this case we consider the automorphism β of the Lie algebra $\mathbf{g}$:
$\beta\ (e_1) = \sqrt{c^2 + d^2}\ e_1,$
$\beta\ (e_6) = -d\ e_5 + c\ e_6,$
$\beta\ (e_5) = c\ e_5 + d\ e_6,$
$\beta\ (e_4) = e_4,$
$$\beta\ (e_2) = \frac{c}{\sqrt{c^2 + d^2}}\ e_2 - \frac{d}{\sqrt{c^2 + d^2}}\ e_3 - c\ b_1\ e_6 + d\ b_1\ e_5,$$
$$\beta\ (e_3) = \frac{c}{\sqrt{c^2 + d^2}}\ e_3 + \frac{d}{\sqrt{c^2 + d^2}}\ e_2 - d\ b_1\ e_6\ - c\ b_1\ e_5,$$
where $\varepsilon\sqrt{c^2 + d^2} = \dfrac{1}{b_2}$ with $\varepsilon = 1$ for $b_2 > 0$ and $\varepsilon = -1$ for $b_2 < 0$. This automorphism leaves the subalgebra $\mathbf{h}$ invariant and one has $\beta(\mathbf{m}_{b_1,b_2}) = \mathbf{m}_{0,1}$ for all $b_1 \in \mathbb{R}, b_2 \in \mathbb{R}\backslash\{0\}$. Hence we can choose the loop $\hat{L}_1 = L_{0,1}$ as the representative of the class $\mathcal{C}_2$.
To determine the isotopism classes we use as tool the conjugation of an element $X \in \mathbf{g}$ by an element $g \in G$. If $g \in G$ has the form
$$\left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) \in G$$
then the inverse of g is

$$g^{-1} = \left(\pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \begin{pmatrix} k & l \\ n & -k \end{pmatrix} \right),$$

where

$$k = (ay - bx)c - (ax + bz)d, \;\; l = (ax + bz)b - (ay - bx)a,$$
$$n = (cy - dx)c - (cx + dz)d.$$

The conjugation of the element $X = \left(\begin{pmatrix} e & f \\ g & -e \end{pmatrix}, \begin{pmatrix} h & i \\ j & -h \end{pmatrix} \right) \in \mathbf{g}$ by the element g of G is given as follows: The first component of $(g^{-1}Xg)$ is

$$(g^{-1}Xg)_1 = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} e & f \\ g & -e \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

the second component of $(g^{-1}Xg)$ is

$$(g^{-1}Xg)_2 = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} -e & -f \\ -g & e \end{pmatrix} \begin{pmatrix} k & l \\ n & -k \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} +$$
$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} k & l \\ n & -k \end{pmatrix} \begin{pmatrix} e & f \\ g & -e \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} +$$
$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} h & i \\ j & -h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The loops in the class $\mathcal{C}_2$ are isotopic to the loops in the class $\mathcal{C}_1$ precisely if there is a complement $\mathbf{m}_{b_1,b_2}$ with $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}\backslash\{0\}$ which is conjugate to a complement $\mathbf{m}_{b'_1,0}$ ($b'_1 \in \mathbb{R}$).
According to Iwasawa for the connected simple Lie group $PSL_2(\mathbb{R})$ there exists a unique decomposition $g = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right)$, where $a > 0, b \in \mathbb{R}, t \in [0, 2\pi)$ (cf. [28], p. 525). Hence each element of G can be written uniquely as

$$\left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) = \tag{1}$$
$$\left(\begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \right) \circ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} k & l \\ l & -k \end{pmatrix} \right)$$

with $a, b, c, d \in \mathbb{R}, \; ad - bc = 1, \; x, y, z \in \mathbb{R}, a_1 > 0, b_1 \in \mathbb{R}, \; t \in [0, 2\pi)$, such that $k = x, l = \frac{y+z}{2}, u = \frac{y-z}{2}$. Since $h^{-1}\mathbf{m}h \subseteq \mathbf{m}$ for all $h \in H$ it is sufficient to find an element $g \in G$ having the form

$$\left(\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \right); \; a > 0, b, u \in \mathbb{R}$$

such that the following matrix equation is satisfied

$$g^{-1}\mathbf{m}_{b_1,b_2}g = \mathbf{m}_{b_1',0}.$$

This matrix equation implies the following equations

$$2ab = 0, \quad \frac{1}{a^2} = a^2 - b^2, \quad 2u + b_1 = b_1', \quad b_2 = b_2'.$$

Since $b_2 = b_2'$ we see that the loops in the isomorphism class $\mathcal{C}_1$ cannot be isotopic to the loops in the class $\mathcal{C}_2$ and therefore the isotopism classes of the left A-loops L_{b_1,b_2} coincide with the isomorphism classes $\mathcal{C}_1$, $\mathcal{C}_2$.
Now we compute the image of $\mathbf{m}_{0,0}$ and $\mathbf{m}_{0,1}$ under the exponential map.
First we determine the exponential map $\exp : \mathbf{g} \to G$. For $X \in \mathbf{g}$ we have $\exp X = v_X(1)$, where $v_X(t)$ is the 1-parameter subgroup of G with the property $\frac{d}{dt}\big|_{t=0} v_X(t) = X$. In the 1-parameter subgroup $\alpha(t) = (\beta(t), \gamma(t))$ of G with the conditions

$$\alpha(t=0) = (1,0) \text{ and } \tfrac{d}{dt}\big|_{t=0}\alpha(t) = (X_1, X_2) = X \in \mathbf{g}$$

the first component $\beta(t)$ is the 1-parameter subgroup of $PSL_2(\mathbb{R})$, and the second component satisfies

$$\frac{d}{dt}\gamma(t) = \frac{d}{ds}\Big|_{s=0}\gamma(t+s) = -\frac{d}{ds}\Big|_{s=0}\beta(s)\gamma(t) + \gamma(t)\frac{d}{ds}\Big|_{s=0}\beta(s) + \frac{d}{ds}\Big|_{s=0}\gamma(s)$$

$$= -X_1\gamma(t) + \gamma(t)X_1 + X_2.$$

For $X_1 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $X_2 = \begin{pmatrix} k & u \\ y & -k \end{pmatrix}$ and $\gamma(t) = \begin{pmatrix} r(t) & s(t) \\ v(t) & -r(t) \end{pmatrix}$, where $a, b, c, k, u, y \in \mathbb{R}$ one has

$$\frac{d}{dt}\gamma(t) = \begin{pmatrix} \frac{d}{dt}r(t) & \frac{d}{dt}s(t) \\ \frac{d}{dt}v(t) & -\frac{d}{dt}r(t) \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & a \end{pmatrix}\begin{pmatrix} r(t) & s(t) \\ v(t) & -r(t) \end{pmatrix} +$$

$$\begin{pmatrix} r(t) & s(t) \\ v(t) & -r(t) \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} k & u \\ y & -k \end{pmatrix}$$

with the following properties:

$$r(0) = s(0) = v(0) = 0, \ \frac{d}{dt}\Big|_{t=0} r(t) = k, \ \frac{d}{dt}\Big|_{t=0} s(t) = u, \ \frac{d}{dt}\Big|_{t=0} v(t) = y,$$

$$\frac{d}{dt}r(t) = -bv(t) + cs(t) + k, \ \frac{d}{dt}s(t) = 2(br(t) + as(t)) + u,$$

$$\frac{d}{dt}v(t) = 2(av(t) - cr(t)) + y.$$

The solution of this inhomogeneous system of linear differential equations is:

$$r(t)=\frac{1}{8(a^2+bc)^{\frac{3}{2}}}(e^{2\sqrt{a^2+bc}t}-e^{-2\sqrt{a^2+bc}t})(-acu-bay+2kcb)+$$
$$\frac{1}{8(a^2+bc)}\big[(e^{\sqrt{a^2+bc}t}-e^{-\sqrt{a^2+bc}t})^2(-cu+by)+t(8ka^2+4acu+4aby)\big],$$
$$s(t)=\frac{1}{8(a^2+bc)^{\frac{3}{2}}}(e^{2\sqrt{a^2+bc}t}-e^{-2\sqrt{a^2+bc}t})(-b^2y+ubc-2bak+2a^2u)+$$
$$\frac{1}{8(a^2+bc)}\big[(e^{\sqrt{a^2+bc}t}-e^{-\sqrt{a^2+bc}t})^2(-2bk+2au)+t(4b^2y+4ubc+8kab)\big],$$
$$v(t)=\frac{1}{8(a^2+bc)^{\frac{3}{2}}}(e^{2\sqrt{a^2+bc}t}-e^{-2\sqrt{a^2+bc}t})(2ya^2-2cka+bcy-c^2u)+$$
$$\frac{1}{8(a^2+bc)}\big[(e^{\sqrt{a^2+bc}t}-e^{-\sqrt{a^2+bc}t})^2(-2ay+2ck)+t(8cka+4c^2u+4bcy)\big].$$

The subspaces $\mathbf{m}_{0,0}$ and $\mathbf{m}_{0,1}$ have the shapes

$$\mathbf{m}_{0,0}=\left\{\left(\begin{pmatrix}\lambda_2 & \lambda_3\\ \lambda_3 & -\lambda_2\end{pmatrix},\begin{pmatrix}0 & -\lambda_1\\ \lambda_1 & 0\end{pmatrix}\right);\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}\right\}$$

and

$$\mathbf{m}_{0,1}=\left\{\left(\begin{pmatrix}\lambda_2 & \lambda_3\\ \lambda_3 & -\lambda_2\end{pmatrix},\begin{pmatrix}-\lambda_2 & -\lambda_1-\lambda_3\\ \lambda_1-\lambda_3 & \lambda_2\end{pmatrix}\right);\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}\right\}.$$

According to **1.2** the first component of $\exp\mathbf{m}_{0,0}$ as well as of $\exp\mathbf{m}_{0,1}$ is

$$\left(\pm\begin{pmatrix}\cosh\sqrt{A}+\frac{\sinh\sqrt{A}}{\sqrt{A}}\lambda_2 & \frac{\sinh\sqrt{A}}{\sqrt{A}}\lambda_3\\ \frac{\sinh\sqrt{A}}{\sqrt{A}}\lambda_3 & \cosh\sqrt{A}-\frac{\sinh\sqrt{A}}{\sqrt{A}}\lambda_2\end{pmatrix}\right),$$

where $A=\lambda_2^2+\lambda_3^2$; the second component of $\exp\mathbf{m}_{0,0}$ is

$$(\exp(X_1,X_2))_2=\begin{pmatrix}r(1) & s(1)\\ v(1) & -r(1)\end{pmatrix},$$

where

$$r(1)=\frac{\lambda_3\,\lambda_1}{4(\lambda_2^2+\lambda_3^2)}\,(e^{\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2,$$

$$s(1)=\frac{-\lambda_1}{4\sqrt{\lambda_2^2+\lambda_3^2}}(e^{2\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-2\sqrt{\lambda_2^2+\lambda_3^2}})-\frac{\lambda_2\lambda_1}{4(\lambda_2^2+\lambda_3^2)}(e^{\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2,$$

$$v(1) = \frac{\lambda_1}{4\sqrt{\lambda_2^2+\lambda_3^2}}(e^{2\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-2\sqrt{\lambda_2^2+\lambda_3^2}}) - \frac{\lambda_2\lambda_1}{4(\lambda_2^2+\lambda_3^2)}(e^{\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2$$

and we have for the second component of $\exp \mathbf{m}_{0,1}$

$$(\exp(Y_1, Y_2))_2 = \begin{pmatrix} r'(1) & s'(1) \\ v'(1) & -r'(1) \end{pmatrix},$$

where

$$r'(1) = \frac{\lambda_3\ \lambda_1}{4(\lambda_2^2+\lambda_3^2)}\ (e^{\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2 - \lambda_2,$$

$$s'(1) = \frac{-\lambda_1}{4\sqrt{\lambda_2^2+\lambda_3^2}}(e^{2\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-2\sqrt{\lambda_2^2+\lambda_3^2}})$$
$$- \frac{\lambda_2\lambda_1}{4(\lambda_2^2+\lambda_3^2)}(e^{\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2 - \lambda_3,$$

$$v'(1) = \frac{\lambda_1}{4\sqrt{\lambda_2^2+\lambda_3^2}}(e^{2\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-2\sqrt{\lambda_2^2+\lambda_3^2}})$$
$$- \frac{\lambda_2\lambda_1}{4(\lambda_2^2+\lambda_3^2)}(e^{\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2 - \lambda_3.$$

We prove that the images of $\exp \mathbf{m}_{0,0}$ and $\exp \mathbf{m}_{0,1}$ determine sharply transitive global sections $\sigma_1 : G/H \to G$, $\sigma_2 : G/H \to G$ such that $\exp \mathbf{m}_{0,0} = \sigma_0(G/H)$ and $\exp \mathbf{m}_{0,1} = \sigma_1(G/H)$. To this we have to show on the one hand that each element $g \in G$ can be uniquely written as a product $g = mh$ with $m \in \exp \mathbf{m}_{0,0}$ respectively $m \in \exp \mathbf{m}_{0,1}$ and $h \in H$, on the other hand that $\exp \mathbf{m}_{0,0}$ and $\exp \mathbf{m}_{0,1}$ operate sharply transitively on G/H.
We know that each element of G has the unique decomposition (1) in section 5. Therefore it is sufficient to prove that there is to each element $g \in G$ with the shape

$$\left(\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \right); a > 0, b, u \in \mathbb{R}$$

precisely one $m \in \exp \mathbf{m}_{0,0}$ respectively $m \in \exp \mathbf{m}_{0,1}$ and $h \in H$ such that $g = m\ h$ or equivalently $m = g\ h^{-1}$.
The condition $m = g\ h^{-1}$ for $m \in \exp \mathbf{m}_{0,0}$ respectively $m \in \exp \mathbf{m}_{0,1}$ yields the following equations: Since the first components of the submanifolds $\exp \mathbf{m}_{0,0}$ and $\exp \mathbf{m}_{0,1}$ are the same we have

$$\cosh \sqrt{A} + \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 = a \cos t \qquad (2)$$

$$\lambda_3 \frac{\sinh \sqrt{A}}{\sqrt{A}} = a \sin t \qquad (3)$$

$$\lambda_3 \frac{\sinh\sqrt{A}}{\sqrt{A}} = b\cos t - \frac{1}{a}\sin t \tag{4}$$

$$\cosh\sqrt{A} - \frac{\sinh\sqrt{A}}{\sqrt{A}}\lambda_2 = b\sin t + \frac{1}{a}\cos t, \tag{5}$$

where $A = \lambda_2^2 + \lambda_3^2$. Moreover for $m \in \exp \mathbf{m}_{0,0}$ respectively $m \in \exp \mathbf{m}_{0,1}$ one has

$$k = r(1) \quad \text{respectively} \quad k = r'(1) \tag{6}$$

$$l = \frac{s(1) + v(1)}{2} \quad \text{respectively} \quad l = \frac{s'(1) + v'(1)}{2} \tag{7}$$

$$2u = s(1) - v(1) \quad \text{respectively} \quad 2u = s'(1) - v'(1), \tag{8}$$

where $r(1), s(1), v(1), r'(1), s'(1), v'(1)$ are defined in the form of $\exp \mathbf{m}_{0,0}$ respectively $\exp \mathbf{m}_{0,1}$. They are values of functions, which depend on the variables $\lambda_1, \lambda_2, \lambda_3$.
For given $a > 0, b, u \in \mathbb{R}$ we have to find unique solutions $\lambda_1, \lambda_2, \lambda_3, t, k, l \in \mathbb{R}$ of these equations. From the equations (2) and (3) we obtain

$$\cos t = \frac{1}{a}(\cosh\sqrt{A} + \frac{\sinh\sqrt{A}}{\sqrt{A}}\lambda_2) \tag{9}$$

$$\sin t = \frac{\lambda_3}{a}\frac{\sinh\sqrt{A}}{\sqrt{A}}. \tag{10}$$

If $\lambda_1, \lambda_2, \lambda_3$ are uniquely determined then it follows from the equations (5.9), (5.10), (5.6), (5.7) that the variables t, k, l are also unique. In both cases the variables $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are the solutions of the following equations:

$$\frac{\sinh\sqrt{A}}{\sqrt{A}}\left[-\frac{b}{a}\lambda_2 + \lambda_3\left(1 + \frac{1}{a^2}\right)\right] + \cosh\sqrt{A}\left(\frac{-b}{a}\right) = 0 \tag{11}$$

$$\frac{\sinh\sqrt{A}}{\sqrt{A}}\left[\lambda_2\left(-1 - \frac{1}{a^2}\right) - \frac{b}{a}\lambda_3\right] + \cosh\sqrt{A}\left(1 - \frac{1}{a^2}\right) = 0 \tag{12}$$

$$2u = \frac{-\lambda_1}{2\sqrt{\lambda_2^2 + \lambda_3^2}}(e^{2\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-2\sqrt{\lambda_2^2+\lambda_3^2}}) \tag{13}$$

If $b \neq 0$, then the equation (11) yields

$$\cosh\sqrt{A} = \frac{\sinh\sqrt{A}}{\sqrt{A}}\left[-\lambda_2 + \lambda_3\left(\frac{a^2+1}{ab}\right)\right].$$

Putting this into the equation (12) we obtain

$$\frac{\sinh\sqrt{A}}{\sqrt{A}}\left[-2\lambda_2+\lambda_3\left(-\frac{b}{a}-\frac{a^4-1}{a^3b}\right)\right]=0.$$

Since $\lim_{A\to 0}\dfrac{\sinh\sqrt{A}}{\sqrt{A}}=1$, one has $\dfrac{\sinh\sqrt{A}}{\sqrt{A}}\geq 1$ and $\cosh\sqrt{A}\geq 1$. Therefore we have

$$-2\lambda_2+\lambda_3\left(-\frac{b}{a}-\frac{a^4-1}{a^3b}\right)=0.$$

From this we obtain for λ_2

$$\lambda_2=\frac{1}{2}\frac{(-b^2a^2-1+a^4)}{a^3b}\lambda_3.$$

Denote by $F=\dfrac{1}{2}\dfrac{(-b^2a^2-1+a^4)}{a^3b}$. From the equation (11) one has

$$\cosh\sqrt{\lambda_3^2(1+F^2)}=\frac{\sinh\sqrt{\lambda_3^2(1+F^2)}}{\sqrt{\lambda_3^2(1+F^2)}}\left[\lambda_3\left(-F+\frac{a^2+1}{ab}\right)\right].$$

Since $-F+\dfrac{a^2+1}{ab}=\dfrac{a^2(b^2+1)+1}{a^3b}\neq 0$ this equation is uniquely solvable

$$\lambda_3=\frac{\varepsilon}{\sqrt{1+F^2}}\mathrm{aretanh}\frac{2a^3b\varepsilon\sqrt{1+F^2}}{a^2b^2+a^2+1}$$

with $\varepsilon=1$ for $b>0$ and $\varepsilon=-1$ for $b<0$. Therefore λ_2,λ_3 are uniquely determined and the equation (13) has a unique solution for λ_1

$$\lambda_1=\frac{-4u\sqrt{\lambda_2^2+\lambda_3^2}}{e^{2\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-2\sqrt{\lambda_2^2+\lambda_3^2}}}.$$

If $b=0$ then from the equation (11) it follows $\lambda_3=0$.
The equation (12)

$$\frac{\sinh\sqrt{\lambda_2^2}}{\sqrt{\lambda_2^2}}\left(\lambda_2\left(-1-\frac{1}{a^2}\right)\right)+\cosh\sqrt{\lambda_2^2}\left(1-\frac{1}{a^2}\right)=0$$

determines uniquely the variable λ_2. Similary the equation (13)

$$2u=\frac{-\lambda_1}{2\sqrt{\lambda_2^2}}(e^{2\sqrt{\lambda_2^2}}-e^{-2\sqrt{\lambda_2^2}})$$

has a unique solution for λ_1.
If $a = 1$ then $\lambda_2 = 0, \ \ \lambda_1 = -u$. For $a \neq 1$ one has

$$\lambda_2 = \varepsilon \text{aretanh} \frac{(a^2-1)\varepsilon}{a^2+1}$$

and

$$\lambda_1 = \frac{-4u\lambda_2\varepsilon}{e^{2\lambda_2} - e^{-2\lambda_2}}$$

with $\varepsilon = 1$ for $a > 1$ and $\varepsilon = -1$ for $a < 1$.
We see that the submanifolds $\exp \mathbf{m}_{0,0}$ and $\exp \mathbf{m}_{0,1}$ are images of global sections $\sigma_0 : G/H \to G$ respectively $\sigma_1 : G/H \to G$. This implies that the equation $p * v = r$ for given $p, r \in \hat{L}_0(\sigma_0)$ has a unique solution $v \in \hat{L}_0(\sigma_0)$. Moreover the submanifold $\exp \mathbf{m}_{0,0}$ is totally geodesic, i.e. $\big[[\mathbf{m}_{0,0}, \mathbf{m}_{0,0}], \mathbf{m}_{0,0}\big] \subseteq \mathbf{m}_{0,0}$ then the section σ_0 defines a global Bol loop $(\hat{L}_0, *)$ and according to [31], Corollary 3.11, (p. 51) and [43], Lemma 1.3, (p. 17) the equation $x * a = b$ has precisely one solution $x = a^{-1} * [(a*b) * a^{-1}]$ for all $a, b \in \hat{L}_0$. This loop $\hat{L}_0$ is called the pseudo-euclidean space loop (cf. [17]). It is equivalent to the fact that the section σ_0 is sharply transitive. We have to verify that the section σ_1 is also sharply transitive, this means that for given elements

$$\left(\begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_1 \\ -u_1 & 0 \end{pmatrix}\right) \quad \text{and} \quad \left(\begin{pmatrix} a_2 & 0 \\ b_2 & a_2^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_2 \\ -u_2 & 0 \end{pmatrix}\right),$$

where $a_1 > 0, a_2 > 0, b_1, b_2, u_1, u_2 \in \mathbb{R}$ there exists precisely one element $z \in \exp \mathbf{m}_{0,1}$ such that for some $h = \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} k & l \\ l & -k \end{pmatrix}\right) \in H$, where $t, k, l \in \mathbb{R}$ the equation

$$z\left(\begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_1 \\ -u_1 & 0 \end{pmatrix}\right) = \tag{14}$$
$$\left(\begin{pmatrix} a_2 & 0 \\ b_2 & a_2^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_2 \\ -u_2 & 0 \end{pmatrix}\right)\left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} k & l \\ l & -k \end{pmatrix}\right)$$

holds. The real variables $\lambda_1, \lambda_2, \lambda_3$ of $z \in \exp \mathbf{m}_{0,1}$ are determined by the following equations

$$\frac{\sinh \sqrt{A}}{\sqrt{A}} \left(\lambda_2 \left(a_1 + \frac{a_2^2}{a_1}\right) + \lambda_3 \left(b_1 + \frac{b_2 a_2}{a_1}\right)\right) + \tag{15}$$
$$\cosh \sqrt{A} \left(a_1 - \frac{a_2^2}{a_1}\right) = 0$$

$$\frac{\sinh\sqrt{A}}{\sqrt{A}}\left(\lambda_2\left(\frac{b_2a_2}{a_1}-b_1\right)+\lambda_3\left(\frac{a_1^2+b_2^2}{a_1}\right)\right)+ \qquad (16)$$
$$\cosh\sqrt{A}\left(b_1-\frac{b_2a_2}{a_1}\right)=0,$$

where $A=\lambda_2^2+\lambda_3^2$, and

$$2(u_2-u_1)+\lambda_3(b_1^2-a_1^2+a_1^{-2})+2a_1b_1\lambda_2=$$
$$\frac{-\lambda_1(b_1^2+a_1^2-a_1^{-2})}{4(\lambda_2^2+\lambda_3^2)}(e^{2\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-2\sqrt{\lambda_2^2+\lambda_3^2}})+ \qquad (17)$$
$$\lambda_1\left(\frac{\lambda_2(a_1^2-b_1^2-a_1^{-2})+2\lambda_3b_1a_1}{4(\lambda_2^2+\lambda_3^2)}\right)(e^{\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2.$$

If z is an element of $\mathbf{m}_{0,0}$ in the equation (14) then we obtain for the variables $\lambda_1,\lambda_2,\lambda_3$ of $z\in\exp\mathbf{m}_{0,0}$ the above equations (15), (16) and the equation

$$2(u_2-u_1)=\frac{-\lambda_1(b_1^2+a_1^2-a_1^{-2})}{4(\lambda_2^2+\lambda_3^2)}(e^{2\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-2\sqrt{\lambda_2^2+\lambda_3^2}})+ \qquad (18)$$
$$\left(\frac{\lambda_2(a_1^2-b_1^2-a_1^{-2})+2\lambda_3b_1a_1}{4(\lambda_2^2+\lambda_3^2)}\right)(e^{\sqrt{\lambda_2^2+\lambda_3^2}}-e^{-\sqrt{\lambda_2^2+\lambda_3^2}})^2.$$

The equations (15), (16), (18) have unique solutions because σ_0 is a sharply transitive section. Therefore the equations (15), (16), (17) are also uniquely solvable for the variables $\lambda_1,\lambda_2,\lambda_3$. Hence the sharply transitive global section σ_1 yields also a global loop $\hat{L}_1(\sigma_1)$, which is a proper left A-loop.

An elementary model of the loops $\hat{L}_0$ and $\hat{L}_1$ is given in the pseudo-euclidean affine space (cf. [20]). By $E(2,1)$ we denote the pseudo-euclidean space the points of which are represented by the matrices

$$(I,Y)=\left(I,\begin{pmatrix} x & k+l\\ k-l & -x\end{pmatrix}\right),$$

where $x,k,l\in\mathbb{R}$, and which has as norm $||(I,Y)||=x^2+k^2-l^2$. The group G acts on the space $E(2,1)$ in the following way: For given $(A,X)\in G$ and $(I,Y)\in E(2,1)$

$$(*) \qquad (A,X)*(I,Y)=(I,\ A^{-1}YA+X).$$

The norm is invariant under the action of G, therefore G is the connected component of the motion group of $E(2,1)$.

The 3-dimensional pseudo-euclidean geometry $E(2,1)$ has also a representation $\mathcal{R}$ on the affine space $\mathbb{R}^3$ such that the motion group consists of the affine mappings

$$(B,b) : (x,y,z) \mapsto (x,y,z)B^T + b,$$

where $B = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ with $\det B = 1$, $a_1^2 + a_2^2 = a_3^2 + 1$, $b_1^2 + b_2^2 = b_3^2 + 1$, $c_1^2 + c_2^2 = c_3^2 - 1$, and $b = (b_1', b_2', b_3')$ ([7], Kapitel 6). The mappings

$$\omega : \left(I, \begin{pmatrix} k & l+n \\ l-n & -k \end{pmatrix} \right) \mapsto (k,l,n)$$

and

$$\Omega : \left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} \right) \mapsto$$

$$\left(\begin{pmatrix} da+bc & cd-ba & cd+ba \\ bd-ca & \frac{a^2+d^2-b^2-c^2}{2} & \frac{d^2+b^2-a^2-c^2}{2} \\ bd+ca & \frac{d^2-b^2-a^2+c^2}{2} & \frac{d^2+b^2+a^2+c^2}{2} \end{pmatrix}, (x,y,z) \right)$$

establish an isometry from $E(2,1)$ onto $\mathcal{R}$ ([35], pp. 97-103).
The stabilizer H which is the image of the Lie algebra of the shape f) under the exponential map leaves in $E(2,1)$ the plane P consisting of the points $\left\{ \left(I, \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \right), x, y \in \mathbb{R} \right\}$ invariant. The points of P satisfy $x^2 + y^2 > 0$. The planes of $\mathcal{R}$ the points (x,y,z) of which satisfy $x^2 + y^2 > z^2$ are called euclidean planes. The connected component $\Omega(G)$ of the motion group of $\mathcal{R}$ acts transitively on the set Ψ of the euclidean planes and the sets $\Omega(\exp \mathbf{m}_{0,0})$, $\Omega(\exp \mathbf{m}_{0,1})$ are sharply transitive on Ψ. The planes of Ψ can be taken as the points of the pseudo-euclidean space loop $\hat{L}_0$ and the global left A-loop $\hat{L}_1$. The multiplication in the loop $\hat{L}_0$ is given by

$$(**) \qquad Q_1 * Q_2 = \tau_{P,Q_1}(Q_2), \quad \text{for all } Q_1, Q_2 \in \Psi,$$

where τ_{P,Q_1} is the unique element of $\Omega(\exp \mathbf{m}_{0,0})$ mapping the plane P, which is the identity of $\hat{L}_0$ onto Q_1. The multiplication in the loop $\hat{L}_1$ has the same form $(**)$, but in this case τ_{P,Q_1} is the unique element of $\Omega(\exp \mathbf{m}_{0,1})$ mapping the plane P, which is also the identity of $\hat{L}_1$ onto Q_1.
Denote by ∇_0 and ∇_1 the canonical connection of the reductive homogeneous space G/H belonging to the subspace $\mathbf{m}_{0,0}$ and $\mathbf{m}_{0,1}$. The 3-dimensional global left A-loops $\hat{L}_0$ and $\hat{L}_1$ are global geodesic loops corresponding to ∇_0 and ∇_1.

In the case β) we can identify the group G as the group of matrices

$$\left\{ g(u,v,a,b,c,d) = \begin{pmatrix} 1 & u & v \\ 0 & a & b \\ 0 & c & d \end{pmatrix} ; u,v,a,b,c,d \in \mathbb{R},\ ad-bc>0 \right\}.$$

We have the following conjugacy classes of the 3-dimensional subgroups of G, which does not contain any non-trivial normal subgroup of G:

a)

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix},\ ad-bc=1 \right\}$$

b)

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix},\ a>0,\ c>0,\ b \in \mathbb{R} \right\}$$

c)

$$H = \left\{ \begin{pmatrix} 1 & u & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix},\ a>0,\ c>0,\ u \in \mathbb{R} \right\}$$

d)

$$H = \left\{ \begin{pmatrix} 1 & u & 0 \\ 0 & a & 0 \\ 0 & b & a \end{pmatrix},\ a>0,\ b,u \in \mathbb{R} \right\}$$

e)

$$H = \left\{ \begin{pmatrix} 1 & u+\varphi(k) & 0 \\ 0 & 1 & 0 \\ 0 & h & k \end{pmatrix}, k>0,\ u,h \in \mathbb{R} \right\}$$

f)

$$H = \left\{ \begin{pmatrix} 1 & u & 0 \\ 0 & a & 0 \\ 0 & b & a^{-1} \end{pmatrix}, a>0,\ b,u \in \mathbb{R} \right\}.$$

A real basis of the Lie algebra $\mathbf{g}$ of $G = PGL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ is

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; e_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra multiplication is given by the following rules:

$$[e_1, e_2] = [e_2, e_4] = e_2, \; [e_1, e_3] = [e_3, e_4] = -e_3, \; [e_1, e_5] = [e_3, e_6] = -e_5,$$

$$[e_1, e_6] = [e_1, e_4] = [e_2, e_6] = [e_3, e_5] = [e_4, e_5] = [e_5, e_6] = 0,$$

$$[e_2, e_3] = e_1 - e_4, \; [e_2, e_5] = [e_4, e_6] = -e_6.$$

In the first case $\mathbf{h} \cong sl_2(\mathbb{R})$ and $\mathbf{h}$ has as generators: $e_1 - e_4, e_2, e_3$. The basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are:

$$f_1 = e_1 + a_1(e_1 - e_4) + a_2 e_2 + a_3 e_3,$$

$$f_2 = e_5 + b_1(e_1 - e_4) + b_2 e_2 + b_3 e_3,$$

$$f_3 = e_6 + c_1(e_1 - e_4) + c_2 e_2 + c_3 e_3,$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real parameters. The complement $\mathbf{m}$ must be satisfied $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. Therefore the elements

$$\begin{aligned} [e_2, f_1] &= -(1 + 2a_1)e_2 + a_3(e_1 - e_4), \\ [e_3, f_2] &= 2b_1 e_3 - b_2(e_1 - e_4), \\ [e_2, f_3] &= -2c_1 e_2 + c_3(e_1 - e_4) \end{aligned}$$

must be in $\mathbf{m}$. This is the case if and only if $a_3 = b_1 = b_2 = c_1 = c_3 = 0$, $a_1 = -\frac{1}{2}$. Since the commutators

$$\begin{aligned} [e_1 - e_4, \tfrac{1}{2}(e_1 + e_4)] &= 2a_2 e_2, \\ [e_1 - e_4, e_5 + b_3 e_3] &= -e_5 - 2b_3 e_3, \\ [e_1 - e_4, e_6 + c_2 e_2] &= e_6 + 2c_2 e_2 \end{aligned}$$

must be again elements of $\mathbf{m}$ one has $a_2 = b_3 = c_2 = 0$. Therefore the unique reductive subspace $\mathbf{m}$ is generated by: $e_1 + e_4, e_5, e_6$. But this yields Lie algebra.

In the case b) the Lie algebra $\mathbf{h}$ of H has as basis elements: e_1, e_2, e_4. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the following shape:

$$\mathbf{m} = \langle e_3 + a_1 e_1 + a_2 e_2 + a_3 e_4, \; e_5 + b_1 e_1 + b_2 e_2 + b_3 e_4,$$

$$e_6 + c_1 e_1 + c_2 e_2 + c_3 e_4 \rangle,$$

with the real numbers $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$. The complement $\mathbf{m}$ cannot be reductive since the element

$$[e_2, e_3 + a_1e_1 + a_2e_2 + a_3e_4] = e_1 - e_4 - a_1e_2 + a_3e_2$$
is not an element of $\mathbf{m}$.

In the case c) the generators of the Lie algebra $\mathbf{h}$ of H are: e_1, e_4, e_5. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has as generators:

$$\{f_1 = e_2 + a_1e_1 + a_2e_4 + a_3e_5,\ f_2 = e_3 + b_1e_1 + b_2e_4 + b_3e_5,$$
$$f_3 = e_6 + c_1e_1 + c_2e_4 + c_3e_5\},$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
The complement $\mathbf{m}$ is reductive if the following property holds $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. Since the elements
$$[e_4, f_1] = -e_2,\ [e_4, f_2] = -e_3,\ [e_4, f_3] = e_6$$
are elements of $\mathbf{m}$ the unique reductive complement $\mathbf{m}$ is generated by e_2, e_3, e_6. This $\mathbf{m}$ contains the element $e_2 + e_3$ which is conjugate to the element $e_1 - e_4 \in \mathbf{h}$ under the element $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{pmatrix} \in G$. This is a contradiction to Lemma 2.

In the case d) we have $\mathbf{h} = \langle e_1 + e_4, e_3, e_5 \rangle$. The basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are:

$$f_1 = e_1 + a_1(e_1 + e_4) + a_2e_3 + a_3e_5,$$
$$f_2 = e_2 + b_1(e_1 + e_4) + b_2e_3 + b_3e_5,$$
$$f_3 = e_6 + c_1(e_1 + e_4) + c_2e_3 + c_3e_5,$$

with the real parameters $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$. The element
$$[e_3, f_1] = e_3$$
is not an element of $\mathbf{m}$, hence the subspace $\mathbf{m}$ is not reductive.

In the case e) the corresponding Lie algebra $\mathbf{h}$ of H has as generators: e_3, e_4, e_5. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ is given by:

$$\langle e_1 + a_1e_3 + a_2e_4 + a_3e_5,\ e_2 + b_1e_3 + b_2e_4 + b_3e_5, e_6 + c_1e_3 + c_2e_4 + c_3e_5 \rangle,$$

with $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$. Since the element
$$[e_5, e_1 + a_1e_3 + a_2e_4 + a_3e_5] = -e_5$$
is not an element of $\mathbf{m}$, the subspace $\mathbf{m}$ cannot be reductive.

In the last case f) the Lie algebra $\mathbf{h}$ is generated by the following basis elements: $e_1 - e_4, e_3, e_5$. For the basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ one has:

$$f_1 = e_1 + a_1(e_1 - e_4) + a_2e_3 + a_3e_5,$$

$$f_2 = e_2 + b_1(e_1 - e_4) + b_2 e_3 + b_3 e_5,$$
$$f_3 = e_6 + c_1(e_1 - e_4) + c_2 e_3 + c_3 e_5,$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real numbers. This complement is also not reductive, since the element
$$[e_3, f_2] = -(e_1 - e_4) + 2b_1 e_3$$
is not an element of $\mathbf{m}$.
Therefore there is no 3-dimensional almost differentiable left A-loop corresponding to the group $PGL_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

Now we consider the case that G is locally isomorphic to $SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$. This group can be represented by the pairs of complex (2×2)-matrices

$$(A, X) = \left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} k & li + n \\ -li + n & -k \end{pmatrix} \right);$$

$a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1, k, l, n \in \mathbb{R}$. Here $\bar{a}$ denotes the complex conjugate of $a \in \mathbb{C}$. The group multiplication is given by the rule

$$(A_1, X_1) \circ (A_2, X_2) = (A_1\ A_2,\ A_2^{-1}\ X_1\ A_2 + X_2).$$

There exist precisely two conjugacy classes of the 3-dimensional subgroups H of G containing no non-trivial normal subgroup of G:
a) H has the shape as in the case c) of Proposition 4 in section 1. Hence there is no 3-dimensional almost differentiable left A-loop corresponding to this pair (G, H).
b)
$$H = \{(a, 0), a \in SO_3(\mathbb{R})\}.$$

Denote by X, Y, Z the generators correspond to 1-dimensional rotations and let V_3, V_2, V_1 be the axes of the rotation groups corresponding to X, Y respectively Z. We can identify the basis elements of $\mathbf{g}$ with the following matrices:

$$X = \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, 0 \right),\ Y = \left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, 0 \right),\ Z = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right),$$

$$V_1 = \left(0, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right),\ V_2 = \left(0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),\ V_3 = \left(0, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

According to [29] (p. 17) the multiplication table of $\mathbf{g} = su_2(\mathbb{C}) \ltimes \mathbb{R}^3$ is given by:

$$[X, Y] = Z,\ [Z, X] = Y,\ [Y, Z] = X,\ [X, V_1] = [Z, V_3] = -V_2,$$

$$[X,V_2]=[Y,V_3]=V_1,\ [Z,V_2]=-[Y,V_1]=V_3,$$
$$[X,V_3]=[Y,V_2]=[V_1,V_2]=[V_1,V_3]=[V_2,V_3]=[Z,V_1]=0.$$

An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the following shape:

$$\mathbf{m}=\langle V_1+aX+bY+cZ,\ V_2+dX+eY+fZ,\ V_3+gX+hY+iZ\rangle,$$

where $a,b,c,d,e,f,g,h,i\in\mathbb{R}$. We prove of which subspace of $\mathbf{m}$ satisfies the condition $[\mathbf{h},\mathbf{m}]\subseteq\mathbf{m}$. The subspace $\mathbf{m}$ satisfies the condition $[\mathbf{h},\mathbf{m}]\subseteq\mathbf{m}$ if and only if $\mathbf{m}$ has the following form:

$$\mathbf{m}_a=\langle V_1+aZ,V_2+aY,V_3-aX\rangle,\quad a\in\mathbb{R}\backslash\{0\}.$$

Using the automorphism φ of $\mathbf{g}$ having the form:

$\varphi(V_1)=-\frac{c_3}{2}\ V_1-\frac{\sqrt{3}c_3}{2}\ V_2,$
$\varphi(V_2)=-\frac{\sqrt{3}c_3}{2}\ V_1+\frac{c_3}{2}\ V_2,$
$\varphi(V_3)=c_3\ V_3,$
$\varphi(X)=-X,$
$\varphi(Y)=-\frac{1}{2}\ Y+\frac{\sqrt{3}}{2}\ Z,$
$\varphi(Z)=\frac{\sqrt{3}}{2}\ Y+\frac{1}{2}\ Z,$

and choosing $c_3=-\frac{1}{a}$, for all $a\in\mathbb{R}\backslash\{0\}$ we have $\varphi(\mathbf{h})=\mathbf{h}$ and $\varphi(\mathbf{m}_a)=\mathbf{m}_1$. Therefore the loops L_a having $T_1L_a=\mathbf{m}_a$ are isomorphic to the loop L_1 belonging to the reductive complement $\mathbf{m}_1$.

Now we compute the image of $\mathbf{m_1}$ under the exponential map.

The exponential map $\exp:\mathbf{g}\to G$ is given by the following way: For $X\in\mathbf{g}$ we have $\exp X=v_X(1)$, where $v_X(t)$ is the 1-parameter subgroup of G with the property $\frac{d}{dt}\big|_{t=0}v_X(t)=X$. In the 1-parameter subgroup $\alpha(t)=(\beta(t),\gamma(t))$ of G with the conditions

$$\alpha(t=0)=(1,0)\text{ and }\tfrac{d}{dt}\big|_{t=0}\alpha(t)=(X_1,X_2)=X\in\mathbf{g}$$

the first component $\beta(t)$ is the 1-parameter subgroup of $SO_3(\mathbb{R})$ and the second component $\gamma(t)$ satisfies

$$\frac{d}{dt}\gamma(t)=\frac{d}{ds}\Big|_{s=0}\gamma(t+s)=-\frac{d}{ds}\Big|_{s=0}\beta(s)\gamma(t)+\gamma(t)\frac{d}{ds}\Big|_{s=0}\beta(s)+\frac{d}{ds}\Big|_{s=0}\gamma(s)=$$
$$-X_1\gamma(t)+\gamma(t)X_1+X_2.$$

For $X_1=\begin{pmatrix}\lambda_1 i & \lambda_2 i-\lambda_3\\ -\lambda_2 i+\lambda_3 & -\lambda_1 i\end{pmatrix}$, $X_2=\begin{pmatrix}\lambda_5 & \lambda_4 i+\lambda_6\\ -\lambda_4 i+\lambda_6 & -\lambda_5\end{pmatrix}$ and $\gamma(t)=\begin{pmatrix}r(t) & v(t)i+s(t)\\ -v(t)i+s(t) & -r(t)\end{pmatrix}$, where $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6\in\mathbb{R}$ we obtain the following inhomogen system of linear differential equations:

$$\frac{d}{dt}\begin{pmatrix} r(t) \\ s(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & -2\lambda_1 & -2\lambda_3 \\ 2\lambda_1 & 0 & 2\lambda_2 \\ 2\lambda_3 & -2\lambda_2 & 0 \end{pmatrix}\begin{pmatrix} r(t) \\ s(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} \lambda_5 \\ \lambda_6 \\ \lambda_4 \end{pmatrix}$$

with the following initial conditions:

$$r(0) = s(0) = v(0) = 0,\ \frac{d}{dt}\Big|_{t=0} r(t) = \lambda_5,\ \frac{d}{dt}\Big|_{t=0} s(t) = \lambda_6,\ \frac{d}{dt}\Big|_{t=0} v(t) = \lambda_4.$$

The solution of this inhomogeneous system is:

$$r(t) = -\frac{i[(e^{2lit} - e^{2lit})(\lambda_5\lambda_1^2 + \lambda_5\lambda_3^2 - \lambda_6\lambda_3\lambda_2 + \lambda_4\lambda_1\lambda_2)]}{4l^3}$$

$$-\frac{\left[\left(e^{lit} - e^{-lit}\right)^2(-\lambda_6\lambda_1 - \lambda_4\lambda_3) + t(4\lambda_4\lambda_1\lambda_2 - 4\lambda_6\lambda_3\lambda_2 - 4\lambda_5\lambda_2^2)\right]}{4l^2},$$

$$s(t) = -\frac{i(e^{2lit} - e^{-2lit})(\lambda_6\lambda_1^2 + \lambda_6\lambda_2^2 - \lambda_5\lambda_3\lambda_2 + \lambda_4\lambda_1\lambda_3)}{4l^3}$$

$$-\frac{\left[\left(e^{lit} - e^{-lit}\right)^2(\lambda_4\lambda_2 + \lambda_5\lambda_1) + t(4\lambda_4\lambda_1\lambda_3 - 4\lambda_5\lambda_3\lambda_2 - 4\lambda_6\lambda_3^2)\right]}{4l^2},$$

$$v(t) = -\frac{i(e^{2lit} - e^{-2lit})(\lambda_4\lambda_3^2 + \lambda_4\lambda_2^2 + \lambda_5\lambda_1\lambda_2 + \lambda_6\lambda_1\lambda_3)}{4l^3}$$

$$-\frac{\left[\left(e^{lit} - e^{-lit}\right)^2(\lambda_5\lambda_3 - \lambda_6\lambda_2) + t(4\lambda_5\lambda_1\lambda_2 + 4\lambda_6\lambda_3\lambda_1 - 4\lambda_4\lambda_1^2)\right]}{4l^2},$$

where $l = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$. Since $\mathbf{m}_1$ has the form

$$\mathbf{m}_1 = \left\{\left(\begin{pmatrix} -ci & -a+bi \\ a+bi & ci \end{pmatrix}, \begin{pmatrix} -c & ai+b \\ -ai+b & c \end{pmatrix}\right); a,b,c \in \mathbb{R}\right\},$$

according to the form 6 in **1.2** (Section 1) the first component of $\exp\mathbf{m}_1$ is

$$(\exp\mathbf{m}_1)_1 = \begin{pmatrix} \cos\sqrt{k} - \frac{ci\sin\sqrt{k}}{\sqrt{k}} & \frac{(-a+bi)\sin\sqrt{k}}{\sqrt{k}} \\ \frac{(a+bi)\sin\sqrt{k}}{\sqrt{k}} & \cosh\sqrt{k} + \frac{ci\sin\sqrt{k}}{\sqrt{k}} \end{pmatrix},$$

where $k = a^2 + b^2 + c^2$, the second component of $\exp\mathbf{m}_1$ is

$$(\exp\mathbf{m}_1)_2 = \begin{pmatrix} r(1) & v(1)i + s(1) \\ -v(1)i + s(1) & -r(1) \end{pmatrix},$$

where

$$\begin{array}{rcl} r(1) & = & -c(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2, \\ s(1) & = & b(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2, \\ v(1) & = & a(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2. \end{array}$$

We prove that there is no global section $\sigma : G/H \to G$ such that $\exp \mathbf{m}_1 \subseteq \sigma(G/H)$. The submanifold $\exp \mathbf{m}_1$ is contained in the image of a section $\sigma : G/H \to G$ if and only if each element $g \in G$ can be uniquely represented as a product $g = m\ h$ with $m \in \exp \mathbf{m}_1$ and $h \in H$. Since every element of $G = SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$ may be written in a unique way as a product

$$g = (g_1, g_2) = (1, g_2')(g_1, 0), \text{ where } (g_1, 0) \in H \text{ and } g_2' = g_1 g_2 g_1^{-1}.$$

It is sufficient to verify that there is to each element $(1, g_2) \in G$ precisely one $m \in \mathbf{m}_1$ and $h \in SO_3(\mathbb{R})$ such that

$$(1, g_2) = ((\exp \mathbf{m}_1)_1, (\exp \mathbf{m}_1)_2)(h, 0).$$

From this equation one has $h = (\exp \mathbf{m}_1)_1^{-1}$. Moreover for given $e, f, g \in \mathbb{R}$ we have to find unique $a, b, c \in \mathbb{R}$ such that

$$\begin{pmatrix} e & fi + g \\ -fi + g & -e \end{pmatrix} =$$

$$(\exp \mathbf{m}_1)_1 \begin{pmatrix} -(e^{ki} - e^{-ki})^2 c & (e^{ki} - e^{-ki})^2 (ai + b) \\ (e^{ki} - e^{-ki})^2 (-ai + b) & (e^{ki} - e^{-ki})^2 c \end{pmatrix} (\exp \mathbf{m}_1)_1^{-1},$$

where $k = \sqrt{a^2 + b^2 + c^2}$. Then the following equations have to be satisfied:

$$\begin{array}{rcl} -c(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2 & = & e, \\ a(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2 & = & f, \\ b(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2 & = & g\ . \end{array}$$

Let now $e = g = 0$ and $f \neq 0$. We may assume that $c = b = 0$. Then we have

$$a(e^{\sqrt{a^2}i} - e^{-\sqrt{a^2}i})^2 = f \text{ or } a(\sinh \sqrt{a^2} i)^2 = -a(\sin \sqrt{a^2})^2 = \frac{f}{4}.$$

Since the function $x \mapsto -x(\sin \sqrt{x^2})^2$ is not injective, there exist different real numbers a_1, a_2 with the properties $\sin(\sqrt{a_1^2}) \neq \sin(\sqrt{a_2^2})$ but $a_1(\sin \sqrt{a_1^2})^2 = a_2(\sin \sqrt{a_2^2})^2$. Hence the element $\left(1, \begin{pmatrix} 0 & fi \\ -fi & 0 \end{pmatrix}\right)$ may be written in different way as product $((\exp \mathbf{m}_1)_1, (\exp \mathbf{m}_1)_2)(h, 0)$.
Summarizing our investigation there is no global 3-dimensional left A-loop as section in the group $SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$.

From the above discussion we obtain:

Theorem 12. *There is only one class $\mathcal{C}$ of the 3-dimensional connected almost differentiable left A-loops L such that the group G topologically generated by the left translations $\{\lambda_x; x \in L\}$ is a 6-dimensional non semisimple and non-solvable Lie group. The group G is isomorphic to the semidirect product $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on $\mathbb{R}^3$ is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra, and the stabilizer of the identity of the loops in $\mathcal{C}$ is the 3-dimensional subgroup of G*

$$\left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi), x, y \in \mathbb{R} \right\}.$$

Any loop in the class $\mathcal{C}$ can be characterized by two real parameters a, b and form precisely two isomorphism classes which are the isotopism classes too. In the one isomorphism class are the Bruck loops $L_{b_1,0}$, $b_1 \in \mathbb{R}$ and the pseudo-euclidean space loop $L_{0,0} = \hat{L}_0$ may be chosen as a representative of this isomorphism class. The other isomorphism class consists of left A-loops L_{b_1,b_2} with $b_2 \neq 0$ has as a representative the loop $L_{0,1} = \hat{L}_1$. The loops $\hat{L}_0$ and $\hat{L}_1$ are realized on the pseudo-euclidean affine space $E(2,1)$ such that the group topologically generated by their left translations is the connected component of the group of pseudo-euclidean motions. The elements of these loops are the planes on which the euclidean metric is induced. Both loops are isomorphic to the geodesic loops of the pseudo-euclidean space $G/H = E(2,1)$ with respect to the reductive complements $\mathbf{m}_{0,i} = T_1[\hat{\sigma}_i(G/H)]$ and the corresponding canonical invariant connection ∇_i for $i = 0, 1$.

Summary

An impotant characteristic of commutative groups G is the fact that their commutators $(ba)^{-1}ab$ for all $a, b \in G$ are the identity of G. But this fact depends strongly on the associative law. For loops which are structures with a binary multiplication having up to associativity the same properties as groups this behaviour changes radically. This observation leat to a broader research of loops L in which the mapping $x \mapsto [(ba)^{-1}(a(bx))]$ is an automorphism of L (cf. [11], [6]). These loops have been called left A-loops.

According to [43] we treat the almost differentiable left A-loops as images of global differentiable sharply transitive sections $\sigma : G/H \to G$ for a Lie group G such that the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H. Here G denote the group topologically generated by the left translations $\{\lambda_x, x \in L\}$ of L and H is the stabilizer of the identity of L in G.

In case of a left A-loop L the tangential space $\mathbf{m} = T_1\sigma(G/H)$ of the image of the section σ at $1 \in G$ can be provided with a binary and a ternary multiplication and yields a Lie triple algebra (cf. [50], [32], Definition 7.1, p. 173). Since the Lie triple algebras correspond to affine reductive spaces, which are essential objects in the main stream of differential geometry (cf. [36], [23]), there is a strong connection between the theory of differential left A-loops and the theory of affine reductive homogeneous spaces (cf. [33]). In particular the theory of connected differentiable Bruck loops (which form a subclass of the class of left A-loops) is essentially the theory of affine symmetric spaces (cf. [43], Section 11). Moreover every almost differentiable strongly left alternative local left A-loop L is a geodesic local loop of the canonical connection ∇ of the reductive homogeneous space G/H corresponding to L (cf. [43], Proposition 5.21, p. 76).

The smallest connected almost differentiable non-associative left A-loops are realized on 2-dimensional manifolds. There exist precisely two isotopism classes of 2-dimensional global left A-loops. In the one class lies only the hyperbolic plane loop which is related to the hyperbolic symmetric plane (cf. [43], Section 22). As a representative L of the other isotopism class may be chosen the 2-dimensional Bruck loop which is realized on the pseudo-euclidean affine plane E such that the group topologically generated by its left translations is the connected component of the group of pseudo-euclidean motions and the elements of L are the lines of positive slope in E (cf. [43], Section 25).

Our aim in this paper is to classify the 3-dimensional connected almost differentiable global left A-loops L, such that the group topologically generated by the left translations of L is a non-solvable Lie group. This is equiva-

lent to the classification of all almost differentiable geodesic loops, which are realized on 3-dimensional non-solvable reductive spaces.

In contrast to local almost differentiable left A-loop, which can be represented as local sections in non-solvable Lie groups G we will show that there are only five classes of global almost differentiable left A-loops with G as the group topologically generated by the left translations. These left A-loops are in strong relation to geometries on 3-dimensional manifolds.

Using the present theory of almost differentiable left A-loops it is not difficult to prove that G is four, five or six dimensional. First we classify the 3-dimensional reductive spaces G/H, this means we determine all complements $\mathbf{m}$ of the Lie algebra $\mathbf{h}$ of H in the Lie algebra $\mathbf{g}$ of G such that $\mathbf{m}$ generates $\mathbf{g}$ and satisfies the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. Such a complement is called reductive. For every strongly left alternative left A-loop the exponential image of the tangential space $\mathbf{m} = T_H\sigma(G/H)$ is contained in $\sigma(G/H)$. Hence the exponential image of a reductive complement $\mathbf{m}$ defines a local differentiable section $\sigma : G/H \to G$. Then we investigate which of these local sections can be extended to a global one. The submanifold $\exp \mathbf{m}$ can be extended to a global section if and only if $\exp \mathbf{m}$ forms a system of representatives for the cosets $\{xH|\ x \in G\}$ in G and $\exp \mathbf{m}$ does not contain any element conjugate to an element of H.
The results of our paper can summarized in the following

Theorem *There are precisely two classes $\mathcal{C}_i$ $(i = 1, 2)$ of the connected almost differentiable simple left A-loops L having dimension 3 such that the group G generated by the left translations $\{\lambda_x; x \in L\}$ is a non-solvable Lie group.*

The class $\mathcal{C}_1$ consists of left A-loops having the simple Lie group $G = PSL_2(\mathbb{C})$ as the group topologically generated by their left translations, and the stabilizer H of $e \in L$ in G is the group $SO_3(\mathbb{R})$.
Any loop in the class $\mathcal{C}_1$ can be represented by a real parameter $a \in \mathbb{R}$. For all $a \in \mathbb{R}$ the loops L_a and L_{-a} are isomorphic. These two loops form a full isotopism class. Any loop L_a with $a \geq 0$ is isomorphic to the geodesic loop of the reductive homogeneous space G/H with respect to the reductive complement $\mathbf{m}_a = T_1[\sigma_a(G/H)]$ and the corresponding canonical invariant connection ∇_a. The hyperbolic space loop L_0, which is the unique Bruck loop in $\mathcal{C}_1$, is the geodesic loop of the hyperbolic space defined by the multiplication $x \cdot y = \tau_{e,x}(y)$, where $\tau_{e,x}$ is the hyperbolic translation moving e onto x.
The other class $\mathcal{C}_2$ of simple left A-loops consists of 3-dimensional connected differentiable left A-loops such that the group $G = PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on $\mathbb{R}^3$ is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra, is the group topologically generated by the left translations

and $H = \left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi], x, y \in \mathbb{R} \right\}$ *is the stabilizer of* $e \in L$ *in* G.

The loops in this class can be represented by two real parameters a, b *and form precisely two isomorphism classes, which coincide with the isotopism classes. In the one isomorphism class are the Bruck loops* $L_{a,0}$, $a \in \mathbb{R}$ *and the pseudo-euclidean space loop* $L_{0,0} = \hat{L}_0$ *may be chosen as a representative of this isomorphism class. The other isomorphism class containing the loops* $L_{a,b}$ *with* $b \neq 0$ *has as a representative the loop* $L_{0,1} = \hat{L}_1$. *The loops* $\hat{L}_0$ *and* $\hat{L}_1$ *are realized on the pseudo-euclidean affine space* $E(2,1)$ *such that the group topologically generated by their left translations is the connected component of the group of pseudo-euclidean motions. The elements of the loops* $\hat{L}_0$ *and* $\hat{L}_1$ *are the planes on which the euclidean metric is induced but the sets of left translations differ. Both loops* $\hat{L}_0$ *and* $\hat{L}_1$ *are isomorphic to the geodesic loops of the pseudo-euclidean space* $G/H = E(2,1)$ *with respect to the reductive complements* $\mathbf{m}_{0,i} = T_1[\hat{\sigma}_i(G/H)]$ *and the corresponding canonical invariant connection* ∇_i, *where* $i = 0, 1$.

Moreover, the non-simple 3*-dimensional almost differentiable left A-loops are either the direct products of a* 1*-dimensional Lie group with a* 2*-dimensional left A-loop isomorphic to the hyperbolic plane loop or the unique Scheerer extension of the Lie group* $SO_2(\mathbb{R})$ *by the* 2*-dimensional left A-loop isomorphic to the hyperbolic plane loop.*

Összefoglalás

Geodetikus loopok 3-dimenziós nem feloldható reduktív homogén tereken

Századunk első évtizedeiben a geometria alapjainak megteremtése, ezen belül a nem Desargues-féle síkok koordinátázásának kérdései vezettek a loopok vizsgálatához, melyek az asszociativitástól eltekintve a csoport tulajdonságaival rendelkező struktúrák. Ezen kutatások vezetője Blaschke, akinek érdeklődését a loopok iránt a differenciálgeometria topologikus kérdései, ezen belül a geodetikus fóliázások topologikus viselkedése motiválta. Ezen vizsgálatok eredményeit a [8] monográfia foglalja össze.

A nem-asszociatív struktúrák elméletének további kiépítése az utóbbi 50 évben három elkülönült irányban folytatódott.

A loopok algebrai aspektusának vizsgálatát Baer [5], Albert [2], [3], Bruck [9], [10] és Belousov [6] kezdeményezték. Baer elsősorban a loopokhoz rendelt geometriát vizsgálta. Bruck a loopok elméletét az univerzális algebra részeként tárgyalja. Albert tekintette először a loopokat a transzlációik által generált csoport szeléseiként. Belousov a loopokat és a hozzájuk kapcsolódó geometriát absztrakt objektumokként tanulmányozza.

A loopok topologikus algebrai, topologikus geometriai és differenciálgeometriai vizsgálatának képviselői Chern [12], [13], [14], Hofmann [26], Akivis, Shelekhov [1], Salzmann [48].

A harmadik kutatási irányzat célja a Lie alaptételek általánosítása analitikus loopokra [27]. Az első vizsgálatokat Malcev [39] kezdeményezte. A globális Lie elméletnek analitikus Moufang loopokra és érintő algebrájukra az u.n. Malcev algebrákra történő általánosítását Kuzmin [37], Kerdman [30] és Nagy [42] végezte el. Az analitikus Bol loopok lokális Lie elméletét Sabinin és Mikheev dolgozták ki. A [40] dolgozatukban bebizonyították, hogy analitikus Bol loopokra és érintő algebrájukra, a Bol algebrákra teljesül Lie első és harmadik tétele: Minden Bol algebrához létezik egy egyértelmű lokális analitikus Bol loop (lokális izomorfizmusok erejéig). Ellentétben a Lie csoport - Lie algebra és a Moufang loop - Malcev algebra esetében elért eredményekkel az így kapott lokális Bol loop nem szükségképpen terjeszthető ki egy globális Bol looppá. Ennélfogva a globális Bol loopok osztályozása lényegesen különbözik a lokális Bol loopokétól, mely ekvivalens a Bol algebrák osztályozásával. A két dimenziós globális Bol loopokat Nagy és Strambach osztályozta [43] 25. fejezetében, a három dimenziós Bol loopok osztályozását pedig a [17] dolgozatunkban adjuk meg, abban az esetben, amikor a loop bal transzlációi által topologikusan generált Lie csoport nem feloldható. A [16] dolgozatunkban megfogalmazzuk és bebizonyítjuk a Lie alaptételeket az analitikus loopok igen széles osztályára, a zéró görbületű geodetikus analitikus loopokra, jellemezve érintő algebrájukat.

Egy G kommutatív csoport fontos jellemvonása, hogy az összes kommutátora $[a,b] = (ba)^{-1}ab$ $(a,b \in G)$ a csoport egységelemével egyezik meg. Ez a tulajdonság erősen támaszkodik az asszociativitás törvényére. Így loopokra ez a tény radikálisan változik. Ez az észrevétel vezette Bruckot, Paiget, Belousovot azon L loopok kutatásához, melyekben az

$$x \mapsto [(ba)^{-1}(a(bx))]$$

leképezések minden $a,b \in L$ esetén automorfizmusai L-nek ([11], [6]). Az ilyen tulajdonságú loopokat bal A-loopoknak nevezték el, utalva a balról történő szorzásra és az automorfizmus kifejezésre. A majdnem differenciálható bal A-loopok differenciálgeometriai szempontból is nagyon jelentősek.

Kikkawa megmutatta ([32]), hogy osztályuk szoros kapcsolatban áll a reduktív terek osztályával, melyek a differenciálgeometria lényeges objektumai ([36], [23]). Továbbá minden majdnem differenciálható erősen bal alternatív lokális L bal A-loop egy lokális geodetikus loop az L-hez tartozó reduktív tér kanonikus konnexiójára vonatkozóan ([43]), azaz az L differenciálható sokaságon a loopszorzás a következőképpen értelmezhető

$$x \cdot y = \exp_x \circ \tau_{e,x} \circ \exp_e^{-1}(y),$$

ahol e jelöli az L loop egységelemét, exp az exponenciális leképezést és $\tau_{e,x}$ pedig érintő vektoroknak a párhuzamos eltolását az e és x pontokat összekötő geodetikus mentén.

Követve Nagy és Strambach által [43]-ban kidolgozott terminológiát a loopok elméletét a csoportelmélet keretében tárgyaljuk. Ily módon minden L loop tekinthető egy olyan $\sigma : G/H \to G$ szelés képeként egy G csoportban, mely a következő tulajdonságokat teljesíti: a $\sigma(G/H)$ képhalmaz erősen tranzitívan hat a G/H faktortéren, generálja a G csoportot és $\sigma(H) = 1 \in G$. Ha H olyan részcsoportja G-nek, mely nem tartalmazza G triviálistól különböző normálosztóját ekkor a G csoport az L loop bal transzlációi által topologikusan generált csoport és a H részcsoport az L loop egységelemének a stabilizátora. Egy loopot majdnem differenciálhatónak nevezünk, ha előáll egy $\sigma : G/H \to G$ differenciálható szelés képeként egy G Lie csoportban. Egy ilyen loop bal A-loop, ha a $\sigma(G/H)$ részhalmaz invariáns marad a H elemeivel való konjugálással szemben.

Egy összefüggő majdnem differenciálható L bal A-loop bal transzlációi által topologikusan generált G csoportja egy összefüggő Lie csoport ([40]). Jelölje $\mathbf{g}$ illetve $\mathbf{h}$ a G csoport illetve a H részcsoport Lie algebráját. Az

L-hez tartozó σ szelés képének az $1 \in G$ pontbeli $\mathbf{m} = T_1\sigma(G/H)$ érintőtere komplementer a $\mathbf{h}$ Lie algebrához és rendelkezik a $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ tulajdonsággal. Ennélfogva az $\mathbf{m}$ vektortéren definiálható egy bináris és egy ternáris művelet, melyekkel $\mathbf{m}$ egy Lie hármas algebrát alkot, mely egy általánosítása a Lie hármas rendszernek ([50], [32]). A Lie hármas rendszerek egy-egyértelműen megfeleltethetők a (globális) egyszerűen összefüggő affin szimmetrikus tereknek ([38], [43] 6. fejezet), hasonlóképpen a Lie hármas algebrák megfeleltethetők a (lokális) affin reduktív tereknek. Ennélfogva létezik egy szoros kapcsolat a majdnem differenciálható bal A-loopok elmélete és az affin reduktív homogén terek emélete között ([33], 11. fejezet). Speciálisan az összefüggő differenciálható Bruck loopoknak az elmélete (melyek egy alosztályát alkotják a bal A-loopok osztályának) lényegében megegyezik az affin szimmetrikus terek elméletével [43].

A legkisebb összefüggő majdnem differenciálható nem-asszociatív bal A-loopok 2-dimenziós sokaságokon realizálhatók. A 2-dimenziós globális bal A-loopoknak pontosan két izotópia osztálya létezik. Az egyik osztályba egyedül egy valódi Bruck loop tartozik, mely a hiperbolikus sík geodetikus loopja ([43], 22. fejezet). A másik izotópia osztály egy reprezentálója az a két dimenziós Bruck loop, mely az E pszeudo-euklideszi affin síkon van értelmezve. A bal transzlációja által topologikusan generált csoport az összefüggő komponense a pszeudo-euklideszi mozgások csoportjának, a loop elemei pedig a pozitív meredekségű egyenesek E-ben ([43], 25. fejezet).

Ebben a dolgozatban osztályozzuk az összes olyan 3-dimenziós összefüggő majdnem differenciálható globális bal A-loopokat, melyeknek a bal transzlációi által topologikusan generált csoport nem feloldható. Ez a feladat ekvivalens az olyan összefüggő majdnem differenciálható geodetikus loopok osztályozásával, melyek nem feloldható reduktív tereken vannak értelmezve.

Mivel egy bal A-loop bal transzlációi által topologikusan generált csoport $\mathbf{g}$ Lie algebrája izomorf a bal A-loop $\mathbf{m} = T_1\sigma(G/H)$ érintő objektumának a sztandard burkoló Lie algebrájához, bebizonyíthatjuk, hogy egy 3-dimenziós valódi bal A-loop esetén a G csoport 4, 5 vagy 6-dimenziós. Tudjuk, hogy az $L = G/H$ sokaság párhuzamosítható és ebben a dolgozatban megmutatjuk, hogy nem kompakt (Corollary 8, 1. fejezet). Erősen bal alternatív majdnem differenciálható bal A-loopokat vizsgálunk, mivel ekkor az L loop tartalmazza a $T_1\sigma(G/H)$ érintőtérnek az exponenciális leképezés általi képét. Ennélfogva a következő eljárás szerint végezzük az osztályozást: Tekintjük az összes 4, 5 vagy 6-dimenziós nem feloldható Lie csoportot. Ezen G csoportoknak meghatározzuk az összes olyan H részcsoportját, mely nem tartalmazza G triviálistól különböző normálosztóját és amelyre teljesül $dim\ G - dim\ H = 3$. Osztályozzuk a G/H reduktív homogén tereket, azaz minden rögzített $(\mathbf{g}, \mathbf{h})$ pár esetén megkeressük a $\mathbf{g}$ Lie algebra összes olyan 3-dimenziós $\mathbf{m}$ részterét, mely komplementer a $\mathbf{h}$ részalgebrához, generálja a $\mathbf{g}$ Lie algebrát és teljesíti a $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ relációt. Egy ilyen $\mathbf{m}$ komplementer exponenciális képe definiál egy differenciálható lokális σ szelést a G egységelemének egy U környezetében úgy, hogy $\exp \mathbf{m} = \sigma(G/H)$. A globális majdnem differenciálható bal A-loopok osztályozása lényegesen különbözik a lokális bal A-loopok osztályozásától, melyek Lie csoportok lokális szeléseiként állíthatók elő. A kérdés, hogy egy lokális loop milyen feltételek mellett ágyazható be egy globálisba, egy nehéz probléma. Kuzmin, Kerdman és Nagy bebizonyították, hogy minden lokális differenciálható Moufang loop egyértelműen beágyazható egy egyszerűen összefüggő globálisba ([37], [30], [42]). De bal A-loopokra ez már nem igaz. Ezért az osztályozás következő lépéseként meg kell vizsgálni, hogy a kapott lokális szelések közül melyik terjeszthető ki differenciálhatóan az egész G/H faktortéren értelmezett globális szeléssé. Ahhoz, hogy egy $\mathbf{m}$ reduktív résztérnek az exponenciális képe egy globális metszet képe legyen, az $\exp \mathbf{m}$ képhalmaz nem tartalmazhat olyan elemeket, melyek konjugáltak a H részcsoport elemeihez és $\exp \mathbf{m}$-nek az $\{xH,\ x \in G\}$ bal oldali mellékosztályoknak egy reprezentáns rendszerét kell képeznie.

A disszertáció 5 fejezetre tagolódik, minden eredmény új, melyek [18]-ban lesznek publikálva. Az 1. fejezet összefoglalja mindazon fogalmakat és összefüggéseket, melyek a későbbi vizsgálathoz szükségesek.

Először **1.1**-ben leírjuk azt az új módszert, mely a loopokat csoportok erősen tranzitív szeléseiként állítja elő. Továbbá, megfogalmazzuk a kapcsolatokat majdnem differenciálható bal A-loopok, érintő objektumaik és reduktív terek között.

1.2-ben Hilgert és Hofmann eredményeit felhasználva ([24]) megadjuk az $sl_2(\mathbb{R})$, $sl_2(\mathbb{C})$, és $su_2(\mathbb{C})$ Lie algebrák exponenciális leképezéseit explicit formulával. Mivel az exponenciális leképezés ilyen jellegű leírása eszközként használja a Cartan-Killing formát, előzőleg megadjuk ezen Lie algebrák Cartan-Killing formáját.

1.3-ban felsoroljuk a 4, 5 vagy 6-dimenziós nem feloldható Lie csoportokat. Részletesen foglalkozunk azon Lie csoportokkal, melyek direkt szorzatai két Lie csoportnak, melyek közül az egyik a $PSL_2(\mathbb{R})$ vagy az $SO_3(\mathbb{R})$ egyszerű Lie csoport. Eredményeket fogalmazunk meg arra vonatkozóan, hogy ezen csoportok közül mely fordulhat elő egy 3-dimenziós majdnem differenciálható bal A-loop bal transzlációi által topologikusan generált csoportjaként. Továbbá bebizonyítjuk, hogy nem létezik a 3-dimenziós gömbhöz homeomorf erősen bal alternatív majdnem differenciálható bal A-loop.

A 2. fejezetben osztályozzuk az összes olyan 3-dimenziós majdnem differenciálható bal A-loopokat, melyeknek a bal transzlációik által topologikusan generált csoport egy egyszerű vagy féligegyszerű Lie csoport, és leírjuk ezen loopokhoz tartozó differenciálgeometriai struktúrákat. Fő eredményeink:

A 3-dimenziós összefüggő majdnem differenciálható bal A-loopoknak csak egy ilyen tulajdonságú $\mathcal{C}$ osztálya létezik. Ezen loopok bal transzlációi által topologikusan generált csoport izomorf $PSL_2(\mathbb{C})$-hez, és a loopok egységelemének a stabilizátora izomorf $SO_3(\mathbb{R})$-hez. Ennek az osztálynak minden loopja egy a valós paraméterrel jellemezhető. Az L_a és L_{-a} $(a \in \mathbb{R})$ loopok egy teljes izomorfia osztályt alkotnak, sőt ez egy teljes izotópia osztály is, mivel ezen loopok izomorfia osztályai megegyeznek az izotópia osztályokkal. Jelölje ∇_a a G/H reduktív homogén térnek az $\mathbf{m}_a = T_1\sigma_a(G/H)$ reduktív komplementerre vonatkozó kanonikus konnexióját. Az L_a loop (minden $a \in \mathbb{R}$ esetén) izomorf a ∇_a kanonikus konnexióhoz tartozó geodetikus looppal. A $\mathcal{C}$ osztály egyetlen Bruck loopja a 0 paraméterhez tartozó hiperbolikus térloop L_0. Az L_0 loop elemei a hiperbolikus szimmetrikus tér pontjai és a loopszorzás a következőképpen adható meg $x \cdot y = \tau_{e,x}(y)$, ahol $\tau_{e,x}$ az e pontot az x-be vivő hiperbolikus eltolás.

A 3. fejezetben olyan 3-dimenziós összefüggő majdnem differenciálható bal A-loopokat vizsgálunk, melyeknek a bal transzlációik által generált csoportja egy 4-dimenziós nem feloldható Lie csoport. Megmutatjuk:

Az ilyen tulajdonságú loopoknak pontosan három $\mathcal{C}_1$, $\mathcal{C}_2$ és $\mathcal{C}_3$-mal jelölt izotópia osztálya létezik. A $\mathcal{C}_1$ illetve a $\mathcal{C}_2$ osztály minden loopja előáll egy a hiperbolikus síkloophoz izomorf loopnak a valós számok additív csoportjával illetve az 1-dimenziós ortogonális csoporttal való direkt szorzataként. Ezek a loopok differenciálható Bruck loopok. A $\mathcal{C}_1$ osztály loopjainak a bal transzlációi által generált csoportja izomorf a $PSL_2(\mathbb{R}) \times \mathbb{R}$ Lie csoporthoz, míg a $\mathcal{C}_2$ osztály loopjainak a bal transzlációi által generált csoportja izomorf a $PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ Lie csoporthoz. Ezen loopok egységelemének a stabilizátora izomorf $SO_2(\mathbb{R})$-hez. A $\mathcal{C}_3$ osztályba izomorfia erejéig csak az $SO_2(\mathbb{R})$ ortogonális csoportnak a hiperbolikus síkloop általi L Scheerer kiterjesztése tartozik. Az L loop bal transzlációi által topologikusan generált csoportja a $PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ Lie csoport és az e egységelemének a stabilizátora a $H = \{(x, \varphi(x)) \mid x \in SO_2(\mathbb{R})\}$ csoport, ahol φ egy monomorfizmus.

A 4. fejezetben olyan 3-dimenziós bal A-loopokat keresünk, melyeknek a bal transzlációi által topologikusan generált csoportja egy 5-dimenziós nem feloldható Lie csoport. Vizsgálatunk azt mutatja:

Nem létezik 3-dimenziós majdnem differenciálható globális bal A-loop 5-dimenziós bal transzláció csoporttal.

Az 5. fejezetben olyan 3-dimenziós összefüggő majdnem differenciálható bal A-loopokkal foglalkozunk, melyeknek a bal transzlációi által topologikusan generált csoport egy 6-dimenziós nem féligegyszerű és nem feloldható Lie csoport. A lehetséges $\mathbf{m}$ reduktív komplementerek exponenciális képének globális beágyazhatóságát vizsgálva a következő eredményeket kapjuk:

Az ilyen tulajdonságokkal rendelkező bal A-loopoknak pontosan két izo-

morfia osztálya létezik és ezek az izomorfia osztályok megegyeznek az izotópia osztályokkal.

Továbbá ezen loopok bal transzlációi által topologikusan generált csoportja a $G = PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$ *Lie csoport, ahol a* $PSL_2(\mathbb{R})$ *csoport hatása* $\mathbb{R}^3$*-on éppen* $PSL_2(\mathbb{R})$ *adjugált hatása az* $sl_2(\mathbb{R})$ *Lie algebrán, és a loopok egységelemének a* H *stabilizátora a*

$$\left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi), x, y \in \mathbb{R} \right\}$$

részcsoport. Ezen loopok mindegyike két valós (a, b) *paraméterrel jellemezhetőek. Az egyik izomorfia osztályba a* $(b_1, 0)$, $b_1 \in \mathbb{R}$ *paraméterekhez tartozó* $L_{b_1,0}$ *Bruck loopok tartoznak és az* $L_{0,0} = \hat{L}_0$ *pszeudo-euklideszi térloop választható ezen osztályok reprezentálójaként. A másik izomorfia osztály az* L_{b_1,b_2}, $b_2 \neq 0$, *bal A-loopokból áll és reprezentálójaként az* $L_{0,1} = \hat{L}_1$ *loop választható. A* $\hat{L}_0$ *és az* $\hat{L}_1$ *loopok az* $E(2,1)$ *pszeudo-euklideszi affin téren realizálhatók, hiszen a bal transzlációik által topologikusan generált csoport az összefüggő komponense a pseudo-euklideszi mozgások csoportjának. Mindkét loop elemeiként választhatóak a pszeudo-euklideszi tér azon síkjai, melyeken az euklideszi norma van bevezetve, viszont a bal transzlációik halmazai különbözőek. Az* $\hat{L}_0$ *illetve az* $\hat{L}_1$ *loop izomorf az* $E(2,1)$ *pszeudo-euklideszi térbeli* $\mathbf{m}_{0,0} = T_1[\hat{\sigma}_0(G/H)]$ *illetve* $\mathbf{m}_{0,1} = T_1[\hat{\sigma}_1(G/H)]$ *reduktív komplementerre és a* ∇_0 *illetve a* ∇_1 *kanonikus konnexióra vonatkozó geodetikus loophoz.*

A majdnem differenciálható loopoknak egy másik széles körben vizsgált osztálya a differenciálható Bol loopok osztálya. A Bol loopokat megadó $\sigma : G/H \to G$ szelések a következő tulajdonsággal jellemezhetők: minden $a, b \in \sigma(G/H)$ esetén az aba elem ismét eleme $\sigma(G/H)$-nak. A 3-dimenziós majdnem differenciálható Bol loopok, melyek egy nem feloldható Lie csoport széleseiként állnak elő, osztályozva vannak [17]-ben. A 3-dimenziós differenciálható globális Bol loopok bal transzlációi által topologikusan generált G csoportok és a H stabilizátor részcsoportok megegyeznek a 3-dimenziós majdnem differenciálható globális bal A-loopok bal transzlációi által topologikusan generált G csoportokkal és a H stabilizátor részcsoportokkal, de ezen Bol loopok és bal A-loopok szelései lényegesen különbözőek. Csak a Bruck loopok és az $SO_2(\mathbb{R})$ ortogonális csoportnak a hiperbolikus sík loop általi Scheerer kiterjesztése tartoznak mind a bal A-loopok mind a Bol loopok osztályához.

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Geodesic loops with non-solvable left translation groups on 3-dimensional reductive spaces

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban

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	Dr.	....

A doktori szigorlat időpontja: 200.. . ...

Az értekezés bírálói:

	Dr.	....
	Dr.	....
	Dr.	....

A bírálóbizottság:

elnök:	Dr.	....
tagok:	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....

Az értekezés védésének időpontja: 200... . ...