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Regularity preservation for quasisums

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Abstract. A quasisum is a function $F : I_1 \times \cdots \times I_n \longrightarrow \mathbb{R}$ of the form

$$F(x_1, \ldots, x_n) = g\big(f_1(x_1) + \cdots + f_n(x_n)\big) \qquad (x_1 \in I_1, \ldots, x_n \in I_n)$$

where $n \geq 2$ is an integer and $f_k : I_k \longrightarrow \mathbb{R}$ is a continuous, strictly monotone function defined on a nonempty open interval of $\mathbb{R}$ (for $k = 1, \ldots, n$), moreover $g : f_1(I_1) + \cdots + f_n(I_n) \longrightarrow \mathbb{R}$ is also continuous, strictly monotone. In this paper we will show that if $p \in \mathbb{N}$ and the quasisum F is p-times continuously differentiable then each of the generator functions $g, f_1, \ldots, f_n$ are p-times continuously differentiable as well. We present applications of our results for p-times continuously differentiable semigroup operations and additively separable utility functions as well.

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1. Introduction

The notion of quasisums was introduced by G. Maksa in the paper [16] and the concept was soon extended to several variables [18].

Definition 1.1. Let $n \in \mathbb{N}$, $n \geq 2$ and suppose that $\emptyset \neq I_k \subseteq \mathbb{R}$ is an open interval for $k = 1, \ldots, n$. Moreover, let $f_k : I_k \longrightarrow \mathbb{R}$ be continuous, strictly monotone functions for $k = 1, \ldots, n$ and let $g : f_1(I_1) + \cdots + f_n(I_n) \longrightarrow \mathbb{R}$ be

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also continuous, strictly monotone. Then the function $F : I_1 \times \cdots \times I_n \longrightarrow \mathbb{R}$ defined by

$$F(x_1, \ldots, x_n) = g\big(f_1(x_1) + \cdots + f_n(x_n)\big) \qquad (x_1 \in I_1, \ldots, x_n \in I_n) \quad (1)$$

is called a *quasisum*. The functions $g, f_1, \ldots, f_n$ will be called the *generators* of F.

The concept of certain two variable quasisums appeared in the early works [1,2] of J. Aczél. According to [17] the terminology was also suggested by Aczél. The class of quasisums cover numerous notable families of multivariate functions, such as continuous group operations [3], quasi-arithmetic means, and also generalized and weighted quasi-arithmetic means [20], which have been the subjects of extensive study. We shall also mention here the economical problem of consistent aggregation [8]. This problem is equivalent to the functional equation of rectangular generalized bisymmetry which was investigated under various conditions by Aczél, Maksa and Taylor. It turned out that the continuous, strictly monotone solutions are quasisums, as it can be found in [5] and [16,18].

In our current approach a quasisum F (or at least its regularity properties) will be considered as known, while the generators are unknown. The defining equation (1) of quasisums is a so-called composite functional equation. This means that unknown functions are substituted into each other. The main motivation of our paper is that regularity theory of composite functional equations is much less developed than in the case of non-composite equations. Our purpose is to investigate how the regularity of a quasisum is preserved by its generator functions.

In general, by the regularity theory of functional equations we mean results of the following kind: some unknown functions in the equation are assumed to fulfill some weak regularity property, and then it is shown (using the structure of the equation and regularity properties of the known functions) that they have stronger regularity.

The most significant results about regularity for non-composite functional equations can be found in the monograph of A. Járai [14], based on papers such as [13,15]. In those works very general results can be found about how starting from weak regularity (such as measurability) one may improve regularity step by step and may eventually conclude very strong regularity (even infinitely many times differentiability or analyticity) about solutions of particular functional equations. In Járai's results multivariate, vector valued functions are concerned, which makes them applicable in many situations. On the other hand, due to this level of generality, sometimes the difficult part is to find out the correspondence between an investigated equation and the adequate general theorem.

Compared to non-composite equations, significantly less is known about the composite case. For example, the general results of Járai can only be used for very specific composite equations, when there are not "too many" unknown functions (we will touch this topic in Section 5). Most of the works about composite functional equations are concerned about finding the solutions of a specific equation, and do not discuss regularity theory for a wider class of functional equations. However, for instance, the paper [11] is an example of the latter approach. The interested reader should get familiar with the survey paper [21] by Páles, where many problems related to the regularity theory of composite and non-composite equations are collected.

Motivated by this shortage of general results in the regularity theory of composite functional equations, we investigate regularity preserving for quasisums. Since the generators are automatically assumed to be continuous and strictly monotone, it seems natural to consider the differentiability properties. It is obvious that if all generators are p-times continuously differentiable (here p is a positive integer) then the quasisum itself will be p-times Fréchet differentiable as well. The main achievement of our paper is the converse of this statement. In Sections 3 and 4 we will show that the (higher order) continuous differentiability of a quasisum is preserved by the generator functions. This is formulated in Theorems 3.1 and 4.1. The proofs are self-contained (does not rely on the machinery of the previously introduced regularity theory), our main tool is Lebesgue's Differentiation Theorem.

Here we have to mention that when we suppose strict monotonicity and continuity for the generators, it is not just because these are the natural assumptions in the framework of quasisums. On the contrary, in order to verify regularity preservation, it is also *necessary* to assume some kind of regularity for them. Otherwise the regularity of a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ given in the form $F(x_1, \ldots, x_n) = g\big(f_1(x_1) + \cdots + f_n(x_n)\big)$ does not imply the same regularity for the generators. Let us demonstrate this via an example, which was proposed to the author by Miklós Laczkovich.

Example 1.1. Let $n \in \mathbb{N}$ and let $\mathcal{B}$ be a Hamel basis of $\mathbb{R}$. Choose the sets $B_1, \ldots, B_n \subset \mathcal{B}$ in such a way that each of them has continuum cardinality and they give a partition of $\mathcal{B}$. Moreover let A_j be the linear hull of B_j over $\mathbb{Q}$ (for $j = 1, \ldots, n$). Then obviously every real number $y \in \mathbb{R}$ has a unique representation

$$y = a_1 + \cdots + a_n \quad \text{where } a_1 \in A_1, \ldots, a_n \in A_n .$$

Since A_j has also continuum cardinality, there exists a bijection $f_j : \mathbb{R} \longrightarrow A_j$ for all $j = 1, \ldots, n$. Therefore the mapping $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\varphi(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$$

is a bijection from $\mathbb{R}^n$ to $\mathbb{R}$. Finally, for any given function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ let us define $g : \mathbb{R} \longrightarrow \mathbb{R}$ by the formula

$$g(y) = F\left(\varphi^{-1}(y)\right) \qquad (y \in \mathbb{R}).$$

Now, in particular,

$$F(x_1, \dots, x_n) = g\big(\varphi(x_1, \dots, x_n)\big) = g\big(f_1(x_1) + \dots + f_n(x_n)\big).$$

Let us emphasize that $f_1, \dots, f_n$ are independent from F, and they are not continuous functions (one way to check this is to observe that $f_j(\mathbb{R}) = A_j$ is not connected), even if F is infinitely many times differentiable.

In the last section we present two applications. Firstly, we consider the representation theorem of continuous, associative, cancellative binary operations [7] by Craigen and Páles. We will prove that if the operation is p-times continuously differentiable, then the generator function of the representing quasisum is p-times continuously differentiable too. The second application is related to additively separable utility functions which are very common in mathematical economics. Their theoretical background and significance is summarized in the papers of Debreu [10] and Gorman [12]. We will show that if a p-times continuously differentiable utility function can be written in a quasisum form, then it is equivalent to an additively separable one, having p-times continuously differentiable subutility functions.

2. Notations, Preliminaries

We will use the notation $\mathbb{R}_+$ for the set of positive real numbers. For open intervals we will use a notation which may be slightly uncommon, but it helps us avoiding potential confusion of ordered pairs and open intervals. Thus, for arbitrary real numbers $a, b \in \mathbb{R}$, we write $]\,a, b\,[= \{x \in \mathbb{R} : a < x < b\}$.

If $m, p \in \mathbb{N}$ and $\emptyset \neq U \subseteq \mathbb{R}^m$ is an open set, then we say that the function $F : U \longrightarrow \mathbb{R}$ is *p-times continuously differentiable*, if all of the p-th order partial derivative functions

$$u \mapsto \partial_{i_1}\partial_{i_2}\dots\partial_{i_p}F(u) \qquad \big(u \in U \text{ and } i_1, i_2, \dots, i_p \in \{1, 2, \dots, m\}\big)$$

exist on the whole domain U and they are continuous on U. When we say that a function is *partially differentiable* then we mean that every first-order partial derivative exists on the whole domain of definition, but they are not necessarily continuous.

Finally, the following two statements are elementary, and they could be generalized in many ways. However, during the proofs of our theorems these auxiliary results will be used frequently, therefore we opted to formulate and verify them explicitly.

Lemma 2.1. *Let $\emptyset \neq I \subseteq \mathbb{R}$ be an open interval and $D \subseteq I$ be a subset which is dense in I. Suppose that the function $f : I \longrightarrow \mathbb{R}$ is continuous and strictly monotone. Then $f(I)$ is an open interval and $f(D)$ is dense in $f(I)$.*

The proof is elementary, it is a well-known fact that $f(I)$ is an open interval and f is a homeomorphism between I and $f(I)$.

Lemma 2.2. *Let $\emptyset \neq I_1 \subseteq \mathbb{R}$ and $\emptyset \neq I_2 \subseteq \mathbb{R}$ be open intervals. Suppose that $D_1 \subseteq I_1$ is an open subset which is dense in I_1 and let $D_2 \subseteq I_2$ be dense in I_2. Then $D_1 + D_2 = I_1 + I_2$.*

Proof. We only prove $I_1 + I_2 \subseteq D_1 + D_2$, as the reverse inclusion is obvious. Let $z \in I_1 + I_2$ be arbitrary. Then $z = x_1 + x_2$ for some $x_1 \in I_1$ and $x_2 \in I_2$. Since I_1, I_2 are open intervals and D_1 is dense in I_1, we may replace x_1 by $x_1 + h$ and x_2 by $x_2 - h$ with a suitably small $h > 0$ such that $x_1 \in D_1$ holds. Now D_1 is open, and thus $x_1 + k \in D_1$ for every $k > 0$ small enough. Since D_2 is dense in I_2, we can choose k such that $x_1 + k \in D_1$ and $x_2 - k \in D_2$, proving $z \in D_1 + D_2$. □

3. Preservation of first order (continuous) differentiability

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and suppose that $\emptyset \neq I_k \subseteq \mathbb{R}$ is an open interval for $k = 1, \ldots, n$. Moreover, let $f_k : I_k \longrightarrow \mathbb{R}$ be continuous, strictly monotone for $k = 1, \ldots, n$ and let $g : f_1(I_1) + \cdots + f_n(I_n) \longrightarrow \mathbb{R}$ be also continuous, strictly monotone. Consider the quasisum $F : I_1 \times \cdots \times I_n \longrightarrow \mathbb{R}$ defined by*

$$F(x_1, \ldots, x_n) = g\big(f_1(x_1) + \cdots + f_n(x_n)\big) \qquad (x_1 \in I_1, \ldots, x_n \in I_n).$$

If F is partially differentiable, then $f_1, \ldots, f_n, g$ are differentiable. Moreover, if F is continuously differentiable, then $f_1, \ldots, f_n, g$ are continuously differentiable as well.

Proof. First of all, let us observe that it is enough to carry out the proof for case of two variables. Indeed, if we fix an arbitrary index $k = 2, \ldots, n$ and any elements $y_j \in I_j$ for $j \in \{2, \ldots n\} \setminus \{k\}$ then we shall consider the function

$$\tilde{F}_{y_2, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n}(x_1, x_k) = g\left(f_1(x_1) + f_k(x_k) + \sum_{j \in \{2, \ldots, n\} \setminus \{k\}} f_j(y_j)\right)$$

defined for all $(x_1, x_k) \in I_1 \times I_k$. This inherits the same regularity from F, so (provided that the theorem holds for $n = 2$) the generators f_1 and f_k are (continuously) differentiable functions. Then the function g is also (continuously) differentiable on the open interval $f_1(I_1) + f_k(I_k) + \sum_{j \in \{2, \ldots, n\} \setminus \{k\}} f_j(y_j)$. As each y_j runs through the interval I_j and the index k runs through the set $\{2, \ldots n\}$ we obtain that g is (continuously) differentiable on its whole domain.

Thus, from now on, we focus on the two variable case, and firstly we assume merely that F is partially differentiable. Clearly the functions $s \mapsto F(s, x_2)$

and $t \mapsto F(x_1, t)$ are are strictly monotone, for all fixed $x_1 \in I_1$ and $x_2 \in I_2$, due to the strict monotonicity of the generators. Therefore the functions

$$s \mapsto \partial_1 F(s, x_2) \quad \text{and} \quad t \mapsto \partial_2 F(x_1, t)$$

cannot be identically zero on any open subinterval of I_1 and I_2, respectively.

Due to the Lebesgue Differentiation Theorem, the functions f_1, f_2 and g are differentiable at almost every point of their domain of definition. For the sake of simplicity we will suppose that f_1 is strictly increasing. The strictly decreasing case is analogous.

Claim 1. $g : f_1(I_1) + f_2(I_2) \longrightarrow \mathbb{R}$ is differentiable.

Proof of Claim 1. For any fixed $y \in I_2$ let us consider the following difference quotients:

$$\frac{F(x, y) - F(x_0, y)}{x - x_0} = \frac{g(f_1(x) + f_2(y)) - g(f_1(x_0) + f_2(y))}{(f_1(x) + f_2(y)) - (f_1(x_0) + f_2(y))} \cdot \frac{f_1(x) - f_1(x_0)}{x - x_0}$$

We shall point out that there is no zero denominator anywhere if $x \neq x_0$ due to the strict monotonicity of the functions. Taking the limit $x \to x_0$ the left hand side tends to $\partial_1 F(x_0, y)$. On the right hand side, the first factor tends to $g'(f_1(x_0) + f_2(y))$ while the second factor tends to $f_1'(x_0)$, provided that these derivatives exist.

Let us observe that if the derivative $f_1'(x_0) = \lim_{x \to x_0} \frac{f_1(x) - f_1(x_0)}{x - x_0}$ equals to a positive real number or it is $+\infty$, then g is differentiable at the point $f_1(x_0) + f_2(y)$. This follows by definition, utilizing that f_1 is continuous. What is more,

$$g'(f_1(x_0) + f_2(y)) = \frac{\partial_1 F(x_0, y)}{f_1'(x_0)} \quad \text{or} \quad g'(f_1(x_0) + f_2(y)) = 0$$

in the cases when $f_1'(x_0) \in \mathbb{R}_+$ and $f_1'(x_0) = +\infty$, respectively. As $y \in I_2$ was arbitrary, we have that g is differentiable on the open interval $f_1(x_0) + f_2(I_2)$ whenever $f_1'(x_0)$ is positive or $+\infty$. Let us define the set

$$K := \{x \in I_1 : f_1'(x) > 0 \text{ or } f_1'(x) = +\infty\}.$$

It is clear that $f_1(K)$ is nothing else but the set of points in $f_1(I_1)$ where f_1^{-1}, the inverse of f_1, has a finite derivative. By Lebesgue's theorem, f_1^{-1} is almost everywhere differentiable on its domain of definition, $f_1(I_1)$. Thus $f_1(K)$ is dense in $f_1(I_1)$. Therefore $f_1(K) + f_2(I_2) = f_1(I_1) + f_2(I_2)$ (due to Lemma 2.2). As we have already seen, g is differentiable on this set. □

Claim 2. The functions $f_k : I_k \longrightarrow \mathbb{R}$ (for $k = 1, 2$) are differentiable.

Proof of Claim 2. Since g is differentiable and strictly monotone, the set

$$L := \{z \in f_1(I_1) + f_2(I_2) \,:\, g'(z) \neq 0\}$$

is dense in the interval $f_1(I_1) + f_2(I_2)$. Now let $x \in I_1$ be arbitrary. Due to the continuity of f_2 and the density of L in $f_1(I_1) + f_2(I_2)$, there exists $y_0 \in I_2$

such that $f_1(x) + f_2(y_0) \in L$. But this means that f_1 is differentiable at x. What is more, we have

$$f_1'(x) = \frac{\partial_1 F\,(x, y_0)}{g'\,(f_1(x) + f_2(y_0))}\,.$$

As $x \in I_1$ was arbitrary, $f_1 : I_1 \longrightarrow \mathbb{R}$ is differentiable. With an analogous reasoning, the differentiability of f_2 follows from the existence of $\partial_2 F$. □

By this point we have shown that if F is partially differentiable then the generators are differentiable. In the last part we will show that if the quasisum $F : I_1 \times I_2 \longrightarrow \mathbb{R}$ is continuously differentiable then the derivatives of the generators are also continuous. Using the chain rule for the differentiation we obtain

$$\partial_k F\,(x_1\,, x_2) = g'\,(f_1(x_1) + f_2(x_2)) \cdot f_k'(x_k)$$

for $k = 1, 2$ and arbitrary numbers $x_1 \in I_1, x_2 \in I_2$. Now if $f_1'(x_0) \neq 0$ then

$$g'\,(f_1(x_0) + f_2(x_2)) = \frac{\partial_1 F\,(x_0\,, x_2)}{f_1'(x_0)}\,.$$

The right hand side is continuous in the variable x_2 , therefore g' is continuous on the interval $f_1(x_0) + f_2(I_2)$, for every fixed x_0 with $f_1'(x_0) \neq 0$. As before, using the fact that the set where f_1' does not vanish is dense in I_1 , applying Lemmas 2.1 and 2.2, we obtain that g' is continuous on $f_1(I_1) + f_2(I_2)$, so g is continuously differentiable.

Finally, let $x_0 \in I_1$ be arbitrary and choose $y_0 \in I_2$ so that $g'\,(f_1(x_0)+f_2(y_0)) \neq 0$ would be fulfilled (such y_0 exists as we have seen). Then

$$f_1'(x) = \frac{\partial_1 F\,(x\,, y_0)}{g'\,(f_1(x) + f_2(y_0))}$$

holds in a neighborhood of x_0 (namely, where the continuous g' is different from zero). Since the right hand side is continuous in the variable x, it follows that, in particular, f_1' is continuous at the point $x_0 \in I_1$. As x_0 was arbitrary, f_1' is a continuous function. With an analogous reasoning one may see that f_2' is also continuous.

Now that the theorem is proven for the two variable case, the general statement for quasisums of $n \geq 2$ variables follows, according to the first remark of the proof. □

4. Preservation of higher order continuous differentiability

Now we generalize Theorem 3.1, replacing first order continuous differentiability by any higher order continuous differentiability. For an arbitrary function $h : I \longrightarrow \mathbb{R}$ the set of points where h does not vanish is denoted by $\operatorname{supp} h$. That is, $\operatorname{supp} h = \{\, x \in I : h(x) \neq 0 \,\}$ which will be called the *support of h*.

Theorem 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and suppose that $\emptyset \neq I_k \subseteq \mathbb{R}$ is an open interval for $k = 1, \dots, n$. Moreover, let $f_k : I_k \longrightarrow \mathbb{R}$ be continuous, strictly monotone for $k = 1, \dots, n$ and let $g : f_1(I_1) + \dots + f_n(I_n) \longrightarrow \mathbb{R}$ be also continuous, strictly monotone. Let $p \in \mathbb{N}$ be a fixed positive integer. Consider the function $F : I_1 \times \dots \times I_n \longrightarrow \mathbb{R}$ defined as*

$$F\,(x_1\,, \dots, x_n) = g\big(f_1(x_1) + \dots + f_n(x_n)\big) \qquad (x_1 \in I_1\,, \dots, x_n \in I_n)\,.$$

If F is p-times continuously differentiable, then $f_1, \dots, f_n, g$ are p-times continuously differentiable as well.

Proof. As established during the proof of Theorem 3.1, the case of several variables reduces easily to the $n = 2$ variable setting, hence we perform the proof only for quasisums of two variables.

The case $p = 1$ is precisely the second statement of Theorem 3.1. We will proceed by induction, so let us suppose that the statement holds for a fixed $p \in \mathbb{N}$. Now the assumption is that F is $(p + 1)$-times continuously differentiable. According to the induction hypothesis, the generator functions f_1, f_2, g are p-times continuously differentiable. By Faà di Bruno's formula we have

$$\begin{aligned} \partial_1^{(p)} F\,(x_1\,, x_2) &= g^{(p)}\big(f_1(x_1) + f_2(x_2)\big) \cdot \big(f_1'(x_1)\big)^p \\ &\quad + \sum_{k=1}^{p-1} g^{(k)}\big(f_1(x_1) + f_2(x_2)\big) \cdot P_k\big(f_1'(x_1)\,, f_1''(x_1)\,, \dots\,, f_1^{(p)}(x_1)\big), \end{aligned} \tag{2}$$

where P_k (for $k = 1, \dots, p-1$) is a polynomial of p variables with integer coefficients and having degree at most $p-1$. In particular, $P_1\,(z_1, \dots, z_n) = z_n$ for all $(z_1, \dots, z_n) \in \mathbb{R}^n$. Of course, when $p = 1$ then the right hand side of (2) is simply the single term $g'\,(f_1(x_1) + f_2(x_2)) \cdot f_1'(x_1)$. In that case, an empty sum might appear in the further calculations.

Since f_1 is strictly monotone, $\mathrm{supp} f_1'$ is a dense, open subset of I_1. Now let us fix any $x_0 \in \mathrm{supp} f_1'$. Dividing by $(f_1'(x_0))^p$ in equation (2) and rearranging, we obtain the expression

$$\begin{aligned} g^{(p)}\big(f_1(x_0) + f_2(y)\big) &= \frac{1}{\big(f_1'(x_0)\big)^p} \cdot \partial_1^{(p)} F\,(x_0\,, y) \\ &\quad - \sum_{k=1}^{p-1} \frac{P_k\big(f_1'(x_0)\,, \dots\,, f_1^{(p)}(x_0)\big)}{\big(f_1'(x_0)\big)^p} \cdot g^{(k)}\big(f_1(x_0) + f_2(y)\big)\,, \end{aligned} \tag{3}$$

for all $y \in I_2$. The right hand side is continuously differentiable in the variable $y \in I_2$, since F is $(p+1)$-times continuously differentiable, and f_2, g are p-times continuously differentiable, due to the induction hypothesis. That is, the function $\psi : I_2 \longrightarrow \mathbb{R}$ defined as

$$\psi(y) := g^{(p)}\big(f_1(x_0) + f_2(y)\big) \qquad (y \in I_2)$$

is continuously differentiable.

On the other hand, f_2 is strictly monotone, thus the function $\varphi := f_2^{-1}$ exists. Moreover, if $y_0 \in \operatorname{supp} f_2'$ then the Inverse function theorem ensures that $\varphi = f_2^{-1}$ is continuously differentiable in a neighborhood of $f_2(y_0)$.

Consequently, if $x_0 \in \operatorname{supp} f_1'$ and $y_0 \in \operatorname{supp} f_2'$ are fixed numbers, then the function $\psi \circ \varphi : f_2(I_2) \longrightarrow \mathbb{R}$ is continuously differentiable in a neighborhood of $f_2(y_0)$. Now we have, for any $z \in f_2(I_2)$, that

$$(\psi \circ \varphi)(z) = g^{(p)}\big(f_1(x_0) + f_2(\varphi(z))\big) = g^{(p)}\big(f_1(x_0) + z\big).$$

Summarizing these observations we conclude that if $x_0 \in \operatorname{supp} f_1'$, $y_0 \in \operatorname{supp} f_2'$ then $g^{(p)}$ is continuously differentiable in a neighborhood of $f_1(x_0) + f_2(y_0)$. That is, $g^{(p)}$ is continuously differentiable in the set $f_1(\operatorname{supp} f_1') + f_2(\operatorname{supp} f_2')$.

Using Lemma 2.1 for the open set $\operatorname{supp} f_k'$ which is dense in I_k we get that $f_k(\operatorname{supp} f_k')$ is also dense in $f_k(I_k)$ (for $k = 1, 2$). Moreover $f_k(\operatorname{supp} f_k')$ is an open subset of $f_k(I_k)$, because f_k is a strictly monotone, continuous function (for $k = 1, 2$). Hence we can apply Lemma 2.2 for these dense open sets and obtain

$$f_1(\operatorname{supp} f_1') + f_2(\operatorname{supp} f_2') = f_1(I_1) + f_2(I_2).$$

That is, $g^{(p)}$ is continuously differentiable on its whole domain of definition.

Finally, we need to show that f_1, f_2 are $(p+1)$-times continuously differentiable. Due to the symmetry, it is enough to do the proof for f_1. Let us carry out a grouping in (2), and write

$$\partial_1^{(p)} F(x_1, x_2) = \sum_{k=2}^{p} g^{(k)}\big(f_1(x_1) + f_2(x_2)\big) \cdot P_k\big(f_1'(x_1), f_1''(x_1), \ldots, f_1^{(p-1)}(x_1)\big) + g'\big(f_1(x_1) + f_2(x_2)\big) \cdot f_1^{(p)}(x_1), \tag{4}$$

where P_k (for $k = 2, \ldots, p$) is a polynomial of $p - 1$ variables with integer coefficients and having degree at most p. Let $x_0 \in I_1$ be any fixed number. Since g is strictly monotone, $\operatorname{supp} g'$ is a dense open set. Therefore, using that $f_2(I_2)$ is an open interval, we may pick $y_0 \in I_2$ such that $f_1(x_0) + f_2(y_0) \in \operatorname{supp} g'$. Then there exists an open subinterval $K \subseteq I_1$ containing x_0 such that $g'(f_1(x) + f_2(y_0)) \neq 0$ holds for every $x \in K$, because g' and f_1 are continuous. Dividing in (4) by this nonzero number, we get

$$f_1^{(p)}(x) = \frac{\partial_1^{(p)} F(x, y_0)}{g'(f_1(x) + f_2(y_0))} - \sum_{k=2}^{p} \frac{g^{(k)}\big(f_1(x) + f_2(y_0)\big)}{g'(f_1(x) + f_2(y_0))} \cdot P_k\big(f_1'(x), f_1''(x), \ldots, f_1^{(p-1)}(x)\big), \tag{5}$$

for all $x \in K$. Now the right hand side is continuously differentiable in the variable $x \in K$, since f_1, f_2 are p-times continuously differentiable while F and g are $(p+1)$-times continuously differentiable. Thus $f_1^{(p)}$ is continuously differentiable on K. Since $x_0 \in I_1$ was arbitrary, a neighborhood such as K can be chosen around every $x \in I_1$ in a way that f_1 is $(p+1)$-times continuously differentiable there.

This means that $f_1 : I_1 \longrightarrow \mathbb{R}$ is $(p+1)$-times continuously differentiable. The same can be concluded about f_2 with an analogous reasoning, as the roles of the functions in the inner sum are interchangeable. □

5. Applications

5.1. Representation of continuous semigroup operations

Our first application is an augmented version of the theorem of R. Craigen and Z. Páles [7] on the representation of continuous, associative, cancellative binary operations.

Theorem 5.1. (Craigen–Páles, 1989) *Let $I \subseteq \mathbb{R}$ be a nontrivial interval and $F : I \times I \longrightarrow I$ be a continuous, associative, cancellative operation. Then there exists an unbounded interval $J \subseteq \mathbb{R}$ which is closed under addition, and there exists a continuous bijection $f : J \longrightarrow I$ such that*

$$F(x, y) = f\left(f^{-1}(x) + f^{-1}(y)\right) \qquad (\forall x, y \in I). \tag{6}$$

Remark 5.1. Let us clarify the properties of F appearing in the statement. The associativity of F means that, for all $x, y, z \in I$, we have $F(F(x, y), z) = F(x, F(y, z))$. The assumption that F is cancellative means that, for all $x, y, z \in I$, we have

$$F(x, y) = F(x, z) \Longrightarrow y = z \text{ and } F(y, x) = F(z, x) \Longrightarrow y = z.$$

By a nontrivial interval we mean an interval containing more than one point. Let us observe that the continuous bijection $f : J \longrightarrow I$ has to be strictly monotone. Moreover, one may easily check that the converse of the theorem also holds. Namely, the operation F defined by (6) is continuous, associative and cancellative.

Let us note that a similar result was proven by Aczél [1,4]. Moreover, it is worth to mention that a slightly weaker characterization theorem of continuous group operations is also due to Aczél. Namely, in [3] it is shown that if $I \subseteq \mathbb{R}$ is an open interval then $F : I \times I \longrightarrow I$ is a continuous group operation if, and only if, there exists a continuous, strictly monotone, surjective function $f : \mathbb{R} \longrightarrow I$ such that

$$\forall x, y \in I : F(x, y) = f\left(f^{-1}(x) + f^{-1}(y)\right).$$

Using Theorems 3.1 and 4.1 we can assert stronger regularity for the generator f if the operation itself is smooth.

Corollary 5.1. *Let $p \in \mathbb{N}$ and let $\emptyset \neq I \subseteq \mathbb{R}$ be an open interval. Suppose that $F : I \times I \longrightarrow I$ is a p-times continuously differentiable, associative, cancellative operation. Then there exists an unbounded open interval $J \subseteq \mathbb{R}$ which is closed under addition, and there exists a continuous bijection $f : J \longrightarrow I$ such that f^{-1} is p-times continuously differentiable, $f|_{J+J}$ is p-times continuously differentiable, moreover*

$$F(x, y) = f\left(f^{-1}(x) + f^{-1}(y)\right) \qquad (x, y \in I)\,. \tag{7}$$

Proof. The existence of a continuous bijection $f : J \longrightarrow I$ for the representation (7) follows from Theorem 5.1. $J = f^{-1}(I)$ is an open interval. The continuity and injectivity of f implies that f is strictly monotone, thus f^{-1} is continuous, strictly monotone as well. This means that the assumptions of Theorem 4.1 are fulfilled (with the notations of Theorem 4.1, consider $n = 2$ and choose both f_1 and f_2 to be f and g to be $f|_{J+J}$). Therefore f^{-1} and $f|_{J+J}$ are p-times continuously differentiable. □

Example 5.1. In the previous Corollary we cannot state that $f : J \longrightarrow I$ is continuously differentiable. We give an example when f is not even differentiable, yet the generated operation is continuously differentiable.

Let $J :=]\,2, +\infty\,[$ and $f(x) = \sqrt[3]{x-3}$ for all $x \in J$. Then $I = f(J) =]-1, +\infty\,[$ and $f^{-1}(y) = y^3 + 3$ for all $y \in I$. Now f is not differentiable at the point 3, but

$$F(x, y) = f\left(f^{-1}(x) + f^{-1}(y)\right) = \sqrt[3]{x^3 + y^3 + 3} \qquad (\,x, y \in]-1, +\infty\,[\,)$$

is a continuously differentiable function.

We would like to mention that the regularity properties of f^{-1} and $f|_{J+J}$ could have been concluded using different tools. The structure of equation (7), namely that only f and f^{-1} appear as generators of the quasisum, ensures that results of A. Járai [13,14] can be applied. For example, applying f^{-1} to both sides of (7) and rearranging the equation we arrive to the expression

$$f^{-1}(x) = f^{-1}\left(F(x, y)\right) - f^{-1}(y)$$

for which the assumptions of [13, Theorem 5.3] are fulfilled. This implies that f^{-1} is locally Lipschitz, and then [13, Theorem 6.1] yields that f^{-1} is continuously differentiable. Step by step, higher order differentiability of f^{-1} follows from [13, Theorem 7.1]. However, general quasisums with different generators does not seem to be manageable with a similar approach.

5.2. Additively separable utility functions

Finally we will present that if a utility function of quasisum form is (p-times) continuously differentiable then essentially it can be expressed as the sum of (p-times) continuously differentiable real functions of its single variables. Before displaying the details let us recall some well-known notions from the field of mathematical economics, which can be found in popular textbooks of the field (e. g. [19,22]). Let $n \in \mathbb{N}$, $n \geq 2$, then $\mathbb{R}_+^n$ denotes the n-times Cartesian product of $\mathbb{R}_+$ with itself:

$$\mathbb{R}_+^n = \{\, (x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1 > 0\,, \ldots, x_n > 0 \,\}.$$

Definition 5.1. Let $D \subseteq \mathbb{R}_+^n$ be a nonempty set and $u : D \longrightarrow \mathbb{R}$ be a function. Then an element of D will be called a *bundle of goods* (consisting of n goods) and u will be called a *utility function*. Furthermore, the relation $\preceq_u$ on the set D defined as

$$x \preceq_u y \Longleftrightarrow u(x) \leq u(y) \qquad\qquad (x, y \in D)$$

is the *preference relation* generated by the utility function u.

The utility function $u : D \longrightarrow \mathbb{R}$ is called *additively separable*, if there exist nonempty open intervals $I_k \subseteq \mathbb{R}_+$ and so-called *subutility functions* $u_k : I_k \longrightarrow \mathbb{R}$ which are continuous, strictly increasing for $k = 1, \ldots, n$ such that $D \subseteq I_1 \times \cdots \times I_n$ and

$$u\,(x_1\,, \ldots, x_n) = u_1(x_1) + \cdots + u_n(x_n) \qquad \big(\,(x_1\,, \ldots, x_n) \in D\,\big).$$

Remark 5.2. During our investigations we will suppose continuity for the utility functions. It is worth to mention that continuity often appears in particular representation theorems for preferences. For instance, the following statement was already verified by Debreu [9]. Let $D \in \mathbb{R}_+^n$ be a connected set and let $\preceq$ be a relation on D. Then $\preceq$ is reflexive, transitive, linear and continuous if, and only if, there exists a continuous function $u : D \longrightarrow \mathbb{R}$ such that

$$x \preceq_u y \Longleftrightarrow u(x) \leq u(y) \qquad\qquad (x, y \in D)\,.$$

Additively separable utility functions form a particularly important class of utility functions and they are used frequently by economists. This is partly due to their special form which makes them convenient to calculate with. More importantly, they incorporate a fundamental feature of the underlying preference relation. In the case when there are at least three goods, Debreu [10] proved that a continuous, complete preference relation is representable by a continuous, additively separable utility function if, and only if, the following holds: for any subset $I \subset \{1, \ldots, n\}$ the preference relation restricted to bundles containing goods only with indices in I is independent from the remaining values x_j for $j \notin I$. The topic of separability was further discussed in the paper [12] of Gorman.

It is worth to emphasize that if $u : D \longrightarrow \mathbb{R}$ is a utility function while $\varphi : u(D) \longrightarrow \mathbb{R}$ is strictly increasing, then $\preceq_{\varphi\circ u} = \preceq_u$. The reverse of this observation is also true.

Proposition 5.1. *Let $D \subseteq \mathbb{R}^n_+$ be a nonempty set and let $u : D \longrightarrow \mathbb{R}$ and $v : D \longrightarrow \mathbb{R}$ be two utility functions with $\preceq_u = \preceq_v$. Then there exists a strictly increasing function $\varphi : u(D) \longrightarrow \mathbb{R}$ such that $v = \varphi \circ u$.*

Proof. According to the definition of the preference relations, the equation $\preceq_u = \preceq_v$ means

$$u(x) \le u(y) \Longleftrightarrow v(x) \le v(y) \qquad \text{for all } x, y \in D.$$

This immediately implies that $u(x) = u(y)$ holds if, and only if, $v(x) = v(y)$, and therefore $u(x) < u(y)$ is equivalent to $v(x) < v(y)$, for all $x, y \in D$.

Now we proceed to define a function $\psi : u(D) \longrightarrow D$ as follows: for all $c \in u(D)$ choose $\psi(c)$ such that $\psi(c) \in u^{-1}(\{c\})$ would hold. After that we define $\varphi : u(D) \longrightarrow \mathbb{R}$ with the formula

$$\varphi(c) = v\left(\psi(c)\right).$$

Due to the definition of ψ we have $u(z) = u\left(\psi(u(z))\right)$ for every $z \in D$, which means $v(z) = v\left(\psi(u(z)\right) = \varphi\left(u(z)\right)$. That is, $v = \varphi \circ u$. Finally, for arbitrary $c, d \in u(D)$ such that $c < d$ we have $u\left(\psi(c)\right) < u\left(\psi(d)\right)$. But this is equivalent to $v\left(\psi(c)\right) < v\left(\psi(d)\right)$ which is nothing but $\varphi(c) < \varphi(d)$, so φ is strictly increasing which had to be proven. □

Hence we have obtained that utility functions u and v generate the same preference relation if, and only if, $v = \varphi \circ u$ holds with some strictly increasing real function φ. This being the case, we call u and v equivalent utility functions. This is the point where it becomes clear that additively separable utility functions are equivalent to special quasisums

$$u\left(x_1, \ldots, x_n\right) = \varphi\big(u_1(x_1) + \cdots + u_n(x_n)\big) \qquad \big(x_1 \in I_1, \ldots, x_n \in I_n\big)$$

where $\emptyset \neq I_k \subseteq \mathbb{R}_+$ are open intervals moreover $u_k : I_k \longrightarrow \mathbb{R}$ and $\varphi : u_1(I_1) + \cdots + u_n(I_n) \longrightarrow \mathbb{R}$ are continuous, strictly increasing functions.

Example 5.2. We demonstrate that the well-known Cobb–Douglas utility functions [6] are quasisums, so they are equivalent to particular additively separable utility functions. If $I_1, \ldots, I_n$ are nonempty open intervals of $\mathbb{R}_+$ while $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_+$ are given constants, then $u\left(x_1, \ldots, x_n\right) = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ is called a Cobb–Douglas utility function. Observe that

$$u\left(x_1, \ldots, x_n\right) = \exp\big(\alpha_1 \cdot \ln(x_1) + \ldots + \alpha_n \cdot \ln(x_n)\big) \qquad \big(x_1 \in I_1, \ldots, x_n \in I_n\big).$$

So u is a quasisum, and it is equivalent to the additively separable utility function $v\left(x_1, \ldots, x_n\right) = \alpha_1 \cdot \ln(x_1) + \cdots + \alpha_n \cdot \ln(x_n)$.

If we know that a preference relation is representable by a utility function of some quasisum form fulfilling (higher order) continuous differentiability properties, then, applying Theorem 4.1, we conclude that it is representable by an additively separable utility function having subutility functions with the same strong regularity. This is formulated in the next statement.

Theorem 5.2. *Let $n \in \mathbb{N}$, $n \geq 2$ and suppose that $\emptyset \neq I_k \subseteq \mathbb{R}$ is an open interval for $k = 1, \ldots, n$. Moreover, let $u_k : I_k \longrightarrow \mathbb{R}$ be continuous, strictly increasing for $k = 1, \ldots, n$ and let $\varphi : u_1(I_1) + \cdots + u_n(I_n) \longrightarrow \mathbb{R}$ be also continuous, strictly increasing. Let $p \in \mathbb{N}$ be a fixed positive integer. Suppose that the utility function $v : I_1 \times \cdots \times I_n \longrightarrow \mathbb{R}$ defined by*

$$v\,(x_1\,, \ldots, x_n) = \varphi\big(u_1(x_1) + \cdots + u_n(x_n)\big) \qquad (x_1 \in I_1\,, \ldots, x_n \in I_n)$$

is p-times continuously differentiable. Then the functions $u_1, \ldots, u_n$ are p-times continuously differentiable.

Moreover, for the additively separable utility function $u : I_1 \times \cdots \times I_n \longrightarrow \mathbb{R}$ defined by

$$u\,(x_1\,, \ldots, x_n) = u_1(x_1) + \cdots + u_n(x_n) \qquad (x_1 \in I_1\,, \ldots, x_n \in I_n)$$

we have $\preceq_u = \preceq_v$.

Proof. Applying Theorem 4.1 for v we immediately get that $\varphi, u_1, \ldots, u_n$ are p-times continuously differentiable. On the other hand, φ is a strictly increasing function. Using that $v = \varphi \circ u$ we conclude $\preceq_v = \preceq_{\varphi \circ u} = \preceq_u$. □

We shall mention that while at first glance the prior statement seems obvious, it is far from trivial. It would be tempting to simply apply φ^{-1} to v, obtain the sum u and immediately show the regularity of the terms $u_1, \ldots, u_n$. However, this does not work, since we cannot expect $\varphi^{-1} \circ v$ to be smooth (possibly this composition is not even partially differentiable if the derivative of φ does not exist or is zero at some point). Thus the proof heavily relies on Theorem 4.1.

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