Hozzájárulások a kétdimenziós fizikai rendszerek elméletéhez

Doktori (PhD) értekezés tézisei

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Contributions to the Theory of Two Dimensional Systems of Physics

Ph.D. Theses

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1. List of the topics of the thesis

Our dissertation presents some contributions to the theory of two dimensional systems of mathematical physics. These systems occupies a quite special place in mathematical physics. In many ways, they are often simpler than the higher dimensional system. Sometimes this allows not only a mathematically rigorous treatment, but a complete solution of the problem. Moreover, they serve as simple models of more complicated systems. In this thesis, we present several separate examples of two dimensional physics. We list here the topics of our dissertation:

- We study the deformations of the commutative algebra of functions on a cylinder. The nontrivial topology of the cylinder allows us to derive an interesting modification of von Neumann's formula for the deformed product.

- The deformed product of functions on a surface is used to describe the motion of an oriented membrane in M-theory. We extend this method to nonorientable surfaces. The nonsymplectic nature of nonorientable surfaces is circumvented by the use of Jordan algebras instead of associative ones.

- The simplest solvable two dimensional lattice model of statistical physics is two dimensional lattice gauge theory. Solvable two dimensional models often provide solutions of the Quantum Yang-Baxter Equation. However, the solution corresponding to gauge theory is not invertible. We present a modification of the model circumventing this problem.

- In lattice gauge theory the gauge group can be replaced by a semigroup if the semigroup possesses an involution. In two dimension, the solvability of the model is usually lost by this replacement. We study the conditions which makes these models solvable.

- Here we turn our attention to an other type of solvable models, to a complex variant of the Kortaweg-de Vries two dimensional partial differential equation. The KdV equation can be written as an isospectral deformation of a Schroedinger operator L. We present a scheme when L is replaced by its absolute value $|\mathrm{L}| = \sqrt{\mathrm{L}\mathrm{L}^*}$.

- We study the structure of the coadjoint orbits of some Lie algebras, which emerge in the description of the motion of an quantum mechanical particle in two dimension under the influence of a magnetic field and a periodic cosine potential. We basically present some examples of wild (Type II) groups in solid state physics.

- Conformal field theories with nonconformal boundary conditions might possess an extra central extension term in the algebra of the stress tensor at tha boundary. This extension was discovered by Goncharova. We present some argument for the absence of this term in the model of a massless scalar field with a semitransparent boundary condition.

2. Nonperturbative effects in deformation quantization on a cylinder

A famous result of von Neumann provides an explicit formula for the deformed product of the functions of $\mathbb{R}^2$:

$$(1)\quad f*_h g(r) = (f*_h g)(\mathbf{r}) = \frac{1}{h^2\pi^2}\int d^2\mathbf{r}' d^2\mathbf{r}'' f(\mathbf{r}')g(\mathbf{r}'')\exp\frac{-4i}{h}A(\mathbf{r},\mathbf{r}',\mathbf{r}''),$$

where A is the symplectic area of the triangle spanned by three points. This formula works for periodic functions, too, so it describes a deformation of the functions of a cylinder or a torus. A is the integral of the symplectic two form ω so if $\omega = d\beta$, then by Stokes theorem, it can be written as a line integral of β over the boundary of the triangle. On a cylinder one can draw the three straight line segments between the three point, in a manner so that they do not bound a triangle, so one is forced to use the line integral definition of A in von Neumann's formula. We studied the properties of this formula in this topologically nontrivial case.

. In the following we fix the value of the winding number w of the segments and compute this modified deformed product for the basis functions $e_{n,r} = \exp i(nx+rp)$, $n \in \mathbb{Z}, r \in \mathbb{R}$. This sort of modified evaluation of the product of basis functions gives

$$(2)\qquad e_{n,r} *_{h,w} e_{\tilde{n},\tilde{r}} = \exp[\frac{ih}{2}(r\tilde{n} - \tilde{r}n) + 2ih\pi w]e_{n+\tilde{n},r+\tilde{r}+4w\pi/h}.$$

Surprisingly, this multiplication rule is associative. In fact, by a redefinition of the of the basis we regain the original $w = 0$ multiplication rule of the $e_{n,r}$ basis.

However, the Cattaneo and Felder's path integral representation of the product suggest the inclusion of paths with different winding numbers, possibly weighted by some system of coefficients. So we do not assume anymore that w is fixed, and for the sequence of coefficients $c = \{\dots, c_{-1}, c_0, c_1, c_2, \dots\}$ we define the multiplication rule by

$$f *_c g = \sum_{w \in Z} c_w f *_{h,w} g. \tag{3}$$

Usually the linear combination of different associative products is no longer associative, but in our case it is, as both

$$(e_{n,r} *_c e_{\tilde{n},\tilde{r}}) *_c e_{\bar{n},\bar{r}} \qquad \text{and} \qquad e_{n,r} *_c (e_{\tilde{n},\tilde{r}} *_c e_{\bar{n},\bar{r}}) \tag{4}$$

evaluates to

(5)

$$\sum_{u,w} c_u c_w e_{n+\tilde{n}+\bar{n}, r+\tilde{r}+\bar{r}+4\pi(u+w)/h} \cdot \exp\{2\pi i h(u+w) + \frac{ih}{2}[\tilde{n}r - n\tilde{r} + \bar{n}r - n\bar{r} + \bar{n}\tilde{r} - \tilde{n}\bar{n}]\}.$$

So we conclude that on a cylinder the full nonperturbative evaluation of the path integral representation of our variant of von Neumann's expression gives an associative product.

Now one might suspect that after a suitable redefinition of the basis functions the $*_c$ product turns out to be the same as the $*_h$ product. However, we show that this is not quite the case. We demonstrate that when the only nonzero elements of the sequence c is $c_0 = c_1 = 1$, then the $*_c$ product algebra of the smooth functions on C does not possess an unit element.

Let us suppose (ad absurdum) that the unit u can be written as

$$u = \sum_{n,r} u_{n,r} e_{n,r}. \tag{6}$$

Then one obtains that for $k = 1, 2, 3 \dots$

(7)

$$u_{0,-k\cdot 4\pi/h} = u_{0,-4\pi/h}(-e^{-2ih\pi})^{k-1} \qquad \text{and} \qquad u_{0,k\cdot 4\pi/h} = u_{0,0}(-e^{2ih\pi})^k.$$

Since $u_{0,0} + u_{0,-4\pi/h} e^{2ih\pi} = 1$, either the $\{u_{0,-k\cdot 4\pi/h}, k = 1, 2, \dots\}$ or the $\{u_{0,k\cdot 4\pi/h}, k = 1, 2, \dots\}$ geometrical sequence has nonzero elements with constant absolute values. As the Fourier series

$$\sum_{n \geq 0} e^{inx} \alpha^n = \frac{1}{1 - \alpha e^{ix}}, \tag{8}$$

where $|\alpha| = 1$ does not represent a smooth function of x, we conclude that the unit u can not be represented by a smooth function on C. Let us note that as an unital algebra can not be deformed to a nonunital one (see for example [8]), this phenomena is a true nonperturbative effect.

3. Matrix theory of unoriented membranes and Jordan algebras

Let us recall that the Goldstone-Hoppe construction describes the classical motion of a membrane with the help of the Hamiltonian

$$H = \frac{\nu T}{4} \int d^2\sigma \left(\dot{X}^i \dot{X}^i + \frac{2}{\nu^2} \{X^i, X^j\}\{X^i, X^j\} \right), \tag{9}$$

which gives the following equations of motion:

$$\ddot{X}^i = \frac{4}{\nu^2} \{\{X^i, X^j\}, X^j\}, \qquad \{\dot{X}^i, X^i\} = 0. \tag{10}$$

In the regularization procedure of Goldstone and Hoppe [14] the X^i coordinate functions are replaced by finite size matrices, Poisson brackets by matrix commutators, and the integration over the membrane's surface by suitably normalized traces. The regularized Hamiltonian and the equations of motion are:

$$\begin{gathered} H = \frac{1}{2\pi l_p^3} \operatorname{Tr} \left(\frac{1}{2} \dot{\mathbb{X}}^i \dot{\mathbb{X}}^i - \frac{1}{4} [\mathbb{X}^i, \mathbb{X}^j][\mathbb{X}^i, \mathbb{X}^j] \right), \\ \ddot{\mathbb{X}}^i + [[\dot{\mathbb{X}}^i, \dot{\mathbb{X}}^j], \dot{\mathbb{X}}^j] = 0, \qquad [\dot{\mathbb{X}}^i, \mathbb{X}^i] = 0, \end{gathered} \tag{11}$$

where the $\mathbb{X}^i$ coordinates are now Hermitian matrices. This reformulation requires orientable surfaces. However, theses expressions can be almost completely rewritten with the help of the Jordan broduct of matrices: $\mathbb{X} \circ \mathbb{Y} = (\mathbb{X}\mathbb{Y} + \mathbb{Y}\mathbb{X})/2$ In term of this product the matrix Hamiltonian looks as

$$H = \frac{1}{2\pi l_p^3} \operatorname{Tr} \left(\frac{1}{2} \dot{\mathbb{X}}^i \circ \dot{\mathbb{X}}^i - \mathbb{X}^i \circ (\mathbb{X}^j \circ (\mathbb{X}^i \circ \mathbb{X}^j)) + (\mathbb{X}^i \circ \mathbb{X}^i) \circ (\mathbb{X}^j \circ \mathbb{X}^j) \right). \tag{12}$$

The accelerations $\ddot{\mathbb{X}}^i$ of the matrix coordinates are given by double commutators (35), so it can be expressed by with the help of the associator of the Jordan algebra:

$$(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) = (\mathbb{X} \circ \mathbb{Y}) \circ \mathbb{Z} - \mathbb{X} \circ (\mathbb{Y} \circ \mathbb{Z}) = \frac{1}{4}[\mathbb{Y}, [\mathbb{X}, \mathbb{Z}]], \tag{13}$$

which identity gives the following equation of motion

$$\ddot{\mathbb{X}}^i = 4(\mathbb{X}^i, \mathbb{X}^j, \mathbb{X}^j) = [\mathbb{X}^j, [\mathbb{X}^i, \mathbb{X}^j]]. \tag{14}$$

The substitution of double commutators by associators occurs in almost all papers on the Jordan algebraic reformulation of quantum mechanics. Our next task is to express the constraints $[\dot{\mathbb{X}}^i, \mathbb{X}^i] = 0$ with the Jordan product of matrices. This is obviously impossible directly. The best we can do is to require that

$$4(\dot{\mathbb{X}}^i, \mathbb{U}, \mathbb{X}^i) = [\mathbb{U}, [\dot{\mathbb{X}}^i, \mathbb{X}^i]] = 0 \tag{15}$$

for any matrix $\mathbb{U}$. Since only the multiples of the identity matrix commute with all the other matrices, this equation implies that $[\dot{\mathbb{X}}^i, \mathbb{X}^i] = c \cdot 1$, but it is well known that this is impossible for finite size matrices (by taking the trace of both sides).

We demonstrated that this variant of the Goldstone-Hoppe construction associates the Jordan algebra of Hermitian matrices to a surface with the topology of S^2. This was pretty much expected. In the case of the projective plane we obtained the Jordan algebra of real symmetric matrices. This result is in sharp contrast compared to the construction of [15], where the Lie algebra of $USp(N)$ was used, since the closest Jordan algebraic relative of $USp(N)$ is $H_N(\mathbb{H})$, i.e. the set of selfadjoint quaternionic matrices.

4. Finite groups, semigroups and the Quantum Yang-Baxter Equation

The solvability of many 2d lattice statistical models is closely connected to the Quantum Yang-Baxter equation (QYBE) [26, 27]. Solutions of the QYBE provide the weight functions of vertex models.

Probably the most simple 2d integrable system is (lattice) gauge theory. The weights of the field configurations around a plaquette (Fig.1a) satisfy the QYBE. (The gauge group is assumed to be finite.)

$$w(a,b,c,d) = R_{a,b}^{d^{-1},c^{-1}}(\lambda) = \sum_{r\in R(G)} \lambda^r \chi_r(abcd) \tag{16}$$

$$\sum_{g,h,i\in G} R_{a,b}^{g^{-1},i}(\lambda) R_{i^{-1},c}^{h,d}(\mu) R_{g,h^{-1}}^{f,e}(\nu) = \sum_{g,h,i\in G} R_{b,c}^{h^{-1},g}(\nu) R_{a,h}^{f,i^{-1}}(\mu) R_{i,g^{-1}}^{e,d}(\lambda). \tag{17}$$

($a, b, c, \ldots$ are elements of the group G, χ_r is the character of irreducible representation $r \in R(G)$. [28])

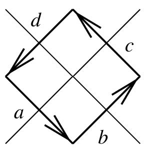

Fig. 1a

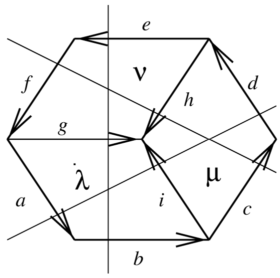

Fig. 1b

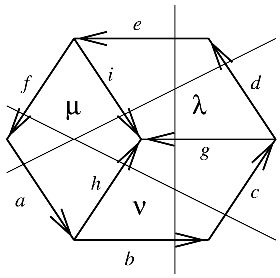

Fig. 1c

We discovered a slight modification of (37) which also satisfies QYBE:

(18)

$$w(a,b,c,d) = R_{a,b}^{d^{-1},c^{-1}}(\lambda) =$$
$$\sum_{r\in R(G)} \Big(\lambda_0^r \chi_r(abcd) + \lambda_1^r \big(\chi_r(bd) + \chi_r(ac)\big) \Big) + \sum_{t,u\in R(G)} \lambda_2^{tu} \chi_t(bd)\chi_u(ac),$$

where $\lambda_2^{tu} = \lambda_2^{ut}$.

These R matrices are usually invertible, in contrast to the ones of the original lattice gauge theory.

We attempted to extend these results for the case when the gauge group is replaced by a semigroup. We demonstrated that solvability can be more or less preserved for the fairly simple type of semigroups G_0, where G_0 is obtained from a finite group G by adjoining a zero element 0 with the multiplication rules $0g = g0 = 00 = 0$ for any $g \in G$. The involution i leaves the zero unchanged. We checked if the character indentity is satisfied for G_0. Let $w : G_0 \to R$ be a conjugacy invariant function, i.e. $w(c) = w(gcg^{-1})$ for $\forall g \in G$. Then w can be expanded as a linear combination of the characters χ_s, $s \in \hat{G}$ of the irreducible representations of G plus an additional term χ_0 for the zero element

$$w(c) = \sum_{s\in\hat{G}} \lambda_s \chi_s(c) \; + \; \lambda_0 \chi_0(c) \tag{19}$$

where $\chi_s(0)=0$, and $\chi_0(0)=1,\ \chi_0(g)=0$ for $\forall g\in G$.

Our result is that

(20)
$$\sum_{x\in G_0} w(ax)\tilde{w}(i(x)b)=\begin{cases}|G|\sum_{s\in\hat{G}}\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab)\ +\ \lambda_0\tilde{\lambda}_0, & \text{if } a,b\in G,\\ \lambda_0\tilde{\lambda}_1|G|+\lambda_0\tilde{\lambda}_0, & \text{if } a=0,b\in G,\\ (|G|+1)\lambda_0\hat{\lambda}_0. & \text{if } a=b=0.\end{cases}$$

We intend to write this expression in the form $\bar{w}(ab)$ for a conjugacy invariant function $\bar{w}$. Since in the second and the third cases $ab=0$, this is possible only if $\tilde{\lambda}_1=\tilde{\lambda}_0$ (and $\lambda_1=\lambda_0$ from the $a\in G,\ \ b=0$ case). Under this condition (39) is equal to

(21)
$$\hat{w}(ab)=\begin{cases}\sum_{s\in\hat{G},s\neq 1}|G|\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab)\ +\ (\lambda_1\tilde{\lambda}_1+\lambda_0\tilde{\lambda}_0)\chi_1(ab) & \text{if } a,b\in G,\\ (|G|+1)\lambda_0\tilde{\lambda}_0 & \text{if } ab=0.\end{cases}$$

$$(22)\quad =\sum_{s\in\hat{G},s\neq 1}|G|\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab)\ +\ (\lambda_1\tilde{\lambda}_1+\lambda_0\tilde{\lambda}_0)\chi_1(ab)+(|G|+1)\lambda_0\tilde{\lambda}_0.$$

So we see that if the statistical summation is performed over a link joining two plaquettes, then the coefficients of the character expansions of the effective weights of the joint plaquette is obtained by the rule:

(23)
$$(\lambda_0,\lambda_1,\lambda_s),\quad(\tilde{\lambda}_0,\tilde{\lambda}_1,\tilde{\lambda}_s)\longrightarrow\left((|G|+1)\lambda_0\tilde{\lambda}_0,(|G|+1)\lambda_1\tilde{\lambda}_1,|G|\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\right)$$

if $\lambda_0=\lambda_1$ and $\tilde{\lambda}_0=\tilde{\lambda}_1$. Note that this condition remains true for the coefficients of the joint plaquettes, too. This ensures that the recursive elimination of the link variables can be continued indefinitely. So we conclude that the 2d lattice semigroup gauge theory of G_0 is solvable under the derived restriction on the weight system.

We also performed some computer algebra experiments for the semigroup consisting of the following matrices:

$$(24)\quad g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad g_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad g_3=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad g_4=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$$g_5=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad g_6=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad g_7=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

The results is that this semigroups behaves as if the group $\mathbb{Z}_2$ were extended by five extra zero elements corresponding to the nonivertible ones of the semigroup.

5. Unitary deformations and complex soliton equations

Since the Lax-equation

$$\dot{L} = PL - LP$$

is the infinitesimal form of the similarity transformation $L \to e^{tP} L e^{-tP}$, such equations leave invariant the 'spectrum' of L [32], ensuring the existence of nontrivial conserved quantities. One can try to preserve the spectrums of L^*L and LL^* instead of L's one. In fact, it is fairly natural to associate the operator L^*L to L, since every $L \in B(\mathfrak{H})$ can be uniquely written as $L = U|L|$, where $|L| = (L^*L)^{1/2}$ is a positive operator, while U is a partial isometry [35].

The spectrum of L^*L is not preserved by general similarity transformations of L. However, the transformation

$$L \to e^{itP} L e^{-itQ}$$

does leave it invariant if e^{itP} and e^{-itQ} are unitary, i.e. if P and Q are self-adjoint operators. Indeed

$$LL^* \to \left(e^{itP} L e^{-itQ}\right)\left(\left(e^{-itQ}\right)^* L^* \left(e^{itP}\right)^*\right) = e^{itP} LL^* e^{-itP}.$$

These considerations suggest that it might be possible to obtain integrable equations in the following form:

$$\dot{L} = i(PL - LQ), \quad P = P^*, \quad Q = Q^*. \tag{25}$$

The preservation of the spectrum of the product of operators occured in the literature in several ways, for example in the works of Drinfeld-Sokolov, Hirota-Satsuma and Dodd-Fordy.

We applied the Gelfand-Dickey scheme for the construction of PDEs of this form. As the self-adjointness of operators has an important role, we use the self-adjoint derivation $D = i\partial$ instead of ∂. Let

$$L = D^2 + v(x)D + u(x),$$

where u and v are complex functions of x. Instead of $L^{1/2}$ (which is not self-adjoint), we would like to obtain self-adjoint pseudo-differential operators

$$A = D + a_0 + a_{-1}D^{-1} + a_{-2}D^{-2} + \dots,$$
$$B = D + b_0 + b_{-1}D^{-1} + b_{-2}D^{-2} + \dots,$$

satisfying

$$i(AL - LB) = 0.$$

So

$$AL = LB \quad \Rightarrow \quad L^{-1}AL = B = B^* = L^*AL^{-1*} \quad \Rightarrow \quad A(LL^*) = (LL^*)A,$$

which implies that

$$A = (LL^*)^{1/4} \quad \text{and} \quad B = (L^*L)^{1/4}.$$

The operator equations

$$\partial_{t_k}L = \partial_{t_k}vD + \partial_{t_k}u = i\left\{(A^k)_+L - L(B^k)_+\right\} = -i\left\{(A^k)_-L - L(B^k)_-\right\} \tag{26}$$

generate integrable equations for $u(x,t)$ and $v(x,t)$.

We[1] compute the explicit forms of the first few of the hierarchy (45). For the odd flows, we present only their constrained, $v = 0$ form, while for the even flows, v's value is constrained to be pure imaginary ($v = iw$).

(1) flow:

$$\partial_{t_1}u = u'$$

(2) flow:

$$\partial_{t_2}w = 2\Im u'$$

$$\partial_{t_2}u = \frac{i}{2}\Big(w''' - w'(2w' + 2w^2 - 4u) - w(u' - \bar{u}' - w'')\Big)$$

(3) flow:

$$\partial_{t_3}u = \frac{1}{8}\Big(u''' - 3\bar{u}''' - 6uu' + 6u'\bar{u} + 12u\bar{u}'\Big)$$

(4) flow:

$$\partial_{t_4}u = \partial_{t_4}w = 0$$

[1] Me and a symbolic algebra package.

(5) flow:

$$\partial_{t_5}u = \frac{1}{32}\Big\{ -3u^{(5)} + 5\bar{u}^{(5)} + 5u'''(3u-\bar{u}) + 5\bar{u}'''(-5u+\bar{u}) \\ + 5u'\left(-3u^2 + 6u\bar{u} + \bar{u}^2 - 3u'' - 5\bar{u}''\right) \\ + 5\bar{u}'\left(4u^2 + 4u\bar{u} - 3u'' - 3\bar{u}''\right)\Big\}$$

6. Coadjoint orbits of wild groups in solid-state physics

Lie-groups are divided into two classes (Types I and II) according to the behaviour of their representations [43]. The unitary representations of Type I (tame) groups have essentially unique decompositions into irreducible representations, while in the case of Type II (wild) groups such decomposition can be highly nonuniqe. According to a theorem of Auslander and Kostant [44], a solvable Lie-group is tame if and only if the set of its coadjoint orbits are separable and the their standard symplectic two-forms are exact. This theorem provides a fairly convenient method to prove the wildness of some solvable groups.

In Kirillov's book [45] two simple examples of wild solvable groups are given. These examples are not just mathematical curiosities, but they emerge naturally in the description of some quasi-periodic systems in solid-state physics. Kirillov's first example has the following physical realization: The functions $a\cos x$, $a\sin x$, $b\cos\alpha x$, $b\sin\alpha x$, and the derivation ∂_x form a five dimensional Lie-algebra. If α is irrational, then its Lie-group is wild. These operators are the building blocks of the Hamiltonian of an electron moving a quasi-periodic cosine potential

$$H = -\frac{1}{2}\partial_x^2 + a\cos x + b\cos\alpha x.$$

The Lie-algebra of the second example can be represented by operators which are necessary for the description of the motion of an electron in two dimension under the influence of periodic cosine potentials and uniform magnetic field. The corresponding group contains the magnetic translation group [46, 47].

The physics of quasi-periodic systems has many characteristic features like the unusual band structure, various types of (de)localisations, etc. [48]. The wildness of the groups in these examples foreshadows the appearance of such features, so the theorem of Auslander and Kostant can be used to predict the qualitative nature of physical systems connected with solvable Lie-groups.

We study what happens if the magnetic translation group is extended by generators generating fluctuations of the magnetic field. In this case the conditions of tameness in the Auslander-Kostant theorem is violated only by a single exeptional coadjoint orbit. As all the other orbits satisfy the conditions of the theorem, we expect that this physical system does not exhibit the unusual phenomenas of the quasiperiodic disordered systems. The Lie algebra of the last example is

$$\begin{array}{lll} [P_x, S_x] = C_x, & [P_y, S_y] = C_y, & [P_x, X] = I, \\ [P_x, C_x] = -S_x, & [P_y, C_y] = -S, & [P_y, Y] = I, \end{array} \tag{27}$$

$$\begin{array}{ll} [E, P_x] = -Y, & [P_x, P_y] = 2B, \\ [E, P_y] = X, & [E, B] = I. \end{array}$$

An interesting aspect of this example is that most of the coadjoint orbits do not violate the conditions of the Auslander-Kostant theorem. This might imply that despite the Type II nature of the group the behaviour of the corresponding systems of solid-state physics are similar to the behaviour of the periodic systems.

7. On the stress tensor near a nonconformal boundary

This chapter contains only preliminary results. In [59] the authors considered the case of a massless scalar field on the figure eight as the simplest example of quantum field theory on a network. They realized that the system has a nontrivial analogue of conformal symmetry only in special cases. We study only the „half" of this network, i.e. the quantum field theory of a massless scalar on a circle with a point-like impurity.

We investigate the question that what sort of modification can be expected for the usual Virasoro algebra of the stress tensor. Let us imagine that ϕ satisfies the nonconformal boundary condition at $x = 0$:

$$\phi'(0) + \alpha\phi(0) = 0. \tag{28}$$

(Actually this is not a very sensible boundary condition for a closed physical system, so we consider only it as an illustration.) This condition is invariant against the transformation $\phi(x) \to \phi(f(x))$ only if $f(x) =$

$x + f_1 x^2 + \dots$. The Lie algebra of such transformation is contained in $L_1(1)$, i.e. in the Lie algebra of formal vector fields on a line of the form

$$v = v_1 x^2 \partial + v_2 x^3 + \partial + \dots . \tag{29}$$

In general, $L_k(1)$ is the formal vector fields on the one dimensional line with the following form:

$$v = v_k x^{k+1} \partial + v_{k+1} x^{k+2} + \partial + \dots . \tag{30}$$

The (nontrivial) central extensions of $L_1(1)$ was computed by Goncaharova [60]. Since for a free scalar field ϕ the commutator $[\phi, \phi] \sim const$, the schematic form of the Lie algebra of bilinear stress tensor T must be

$$[T, T] \sim [\phi\phi, \phi\phi] \sim \phi\phi + const. \tag{31}$$

(In this condensed notation for example $[\phi, \phi] \sim const$ stands for $[\phi(x), \dot{\phi}(y)] = \delta(x-y)$.) The $\phi\phi$ term in the result has the same structure as in the classical system, while the $const$ term of the central extension must be some sum of the Virasoro (Gelfand-Fuks) and the Goncharova cocycles. We note here that Goncharova's cocycle is local in the sense that it can be calculated from the first few derivatives of the vector field v at $x = 0$. Of course we cannot guess in advance the coefficients of the various cocycles in the $const.$ central extensions. They might be even 0 or ∞ in a QFT calculation. We argue that in this case the central extension is absent.

We computed the form of a nonconformal boundary condition for a massless scalar on the interval $[0, \pi]$:

$$\begin{pmatrix} \phi(\pi) \\ \phi_x(\pi) \end{pmatrix} = R \begin{pmatrix} \phi(0) \\ \phi_x(0) \end{pmatrix}, \qquad R \in SL(2.\mathbb{R}). \tag{32}$$

This can be interpreted as a semitransparent boundary condition corresponding to an impurity at $x = 0 \equiv \pi$. We studied the scattering of plane waves at the endpoints. This enabled us to guess the form of the modification of the Green functions of the theory near the boundary. We concluded that the modification is too small to produce Goncharova's term in the central extension of the Lie algebra of the stress tensor. Unfortunately, this conclusion was at most heuristic.

8. List of publications and preprints

The results of the thesis are contained in the following papers, preprints or manuscripts:

- 1. Nonperturbative effects in deformation quantization on a cylinder.

  Submitted to the Journal of Physics A.

- 2. Unoriented membranes and Jordan algebras.

  Accepted for publication by the Journal of Mathematical Physics.

- 3. Finite groups, semigroups and the Quantum Yang-Baxter Equation.

  Lett. Math. Phys. 43 (1998), no. 4, 295–298.

- 4. On the solvability of two dimensional semigroup gauge theories.

  Submitted to the European Journal of Physics.

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- Minimax games, spin glasses, and the polynomial-time hierarchy of complexity classes.

  Phys. Rev. E (3) 57 (1998), no. 6, 6487–6492.

- Relative entropy in the Sherrington-Kirkpatrick spin glass model.

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9. A disszertáció témái

Tézisünkben azokat az eredményeket emeljük ki, amelyek a legjobb tudomásunk szerint eredetiek. Ezeket a következőekben fejezetenként tárgyaljuk.

- *2.Fejezet. A deformácios kvantálás nemperturbative effektusai.*

  Ez a fejezet a következő geometriai képen alapul. A $C^\infty(\mathbb{R}^2)$-beli sima függvények szorzásának egy deformációját Neumann formulája adja meg

  $$(f *_h g)(\mathbf{r}) = \frac{1}{h^2\pi^2}\int d^2\mathbf{r}' d^2\mathbf{r}'' f(\mathbf{r}')g(\mathbf{r}'')\exp\frac{-4i}{h}A(\mathbf{r},\mathbf{r}',\mathbf{r}''),$$

  ahol $\mathbf{r} = (x, p)$ és A a szimplektikus területe a $T = \Delta(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ háromszögnek. Mivel ez a formula értelemmel bír periodikus függvények esetében is, egyszersmind egy henger függvényei algebrájanak valamilyen deformációját is. Az $A(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ tényezőt felírhatjuk egyrészt a szimplektikus formának a $\Delta(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ háromszög feletti integráljaként, de úgy is mint egy alkalmas egy-formának a háromszög pereme fölötti integrálját. Bár a geodetikus ívek három pont között egy hengeren nem feltétlenül határolnak egy háromszöget, az A tényező második megadása ebben az esetben is működik. Így megvizsgálhatjuk a hatását ezeknek a topologikus szempontból nemtriviális konfigurációknak Neumann képletében. Ha ezek közül csak azokat vesszük figyelembe, amelyek pontosan w-szer csavarodnak a hengerre, akkor a szorzasi szabály a következő lesz

  $$e_{n,r} *_{h,w} e_{\tilde{n},\tilde{r}} = \exp[\frac{ih}{2}(r\tilde{n} - \tilde{r}n) + 2ih\pi w]e_{n+\tilde{n},r+\tilde{r}+4w\pi/h},$$

  ahol $e_{n,r} = e^{i(nx+rp)}$. Az eredeti szorzatot a $w = 0$ esetben kapjuk vissza. Tulajdonképpen a $w \neq 0$ szorzat ekvivalens a régivel egy megfelelő bázistranszformáció után. Ez azonban már nem lesz így, ha különböző csavarodási számokat is megengedünk. Ekkor a szorzási szabály

  $$f *_c g = \sum_{w\in Z} c_w f *_{h,w} g,$$

ahol a c_w számok tetszőleges együtthatók.

Megmutattuk, hogy ha a nem nulla együtthatók $c_0 = c_1 = 1$, akkor a $*_c$ szorzás algebrájának nics egysegeleme. Ez a viselkedes nem fordulhatna elő algebrai deformaciók esetében, így ez egy valódi nemperturbatív effektus. Mi több, a $*_c$ szorzás asszociatív marad, ami igazi meglepetés, mivel általában asszociatív szorzatok lineáris szuperpozíciója nem asszociatív (erre a jelenségre a legismertebb példák a Lie és Jordan algebrák). Ez a következő szorzatok kiszámításával ellenőrizhető:

$$(e_{n,r} *_c e_{\tilde{n},\tilde{r}}) *_c e_{\bar{n},\bar{r}} \qquad \text{and} \qquad e_{n,r} *_c (e_{\tilde{n},\tilde{r}} *_c e_{\bar{n},\bar{r}}) \tag{33}$$

Mindkettő eredménye

$$\sum_{u,w} c_u c_w e_{n+\tilde{n}+\bar{n}, r+\tilde{r}+\bar{r}+4\pi(u+w)/h} \cdot \exp\{2\pi i h(u+w) + \frac{ih}{2}[\tilde{n}r - n\tilde{r} + \bar{n}r - n\bar{r} + \bar{n}\tilde{r} - \tilde{n}\bar{n}]\}, \tag{34}$$

ami mutatja a szorzási szabály asszociativitását.

- *3. Fejezet. Jordan algebrák és a nemorientálható membránok mátrix elmélete.*

Ez a fejezet egy egyszerű megfigyelésen alapul. Jól ismert, hogy a egy orientálható felület relativisztikus mozgásegyenlete közelíthető a nagyméretű ermitikus matrixok nemkommutatív algebrájának a segítségével. Ez a módszer a

Poisson algebra a felületen $\longleftrightarrow$ *Mátrix kommutátorok*

összefüggést használja ki. A megfeleő Poisson algebra létezése kikényszeríti a felület orientálhatóságát. Viszont a membrán lokális mozgásegyenlete nem tartalmaz semmiféle utalást a membrán orientációjára, így szinte biztos, hogy valahogy meg lehet taláni a fenti kapcsolat megfelelőjét a nemorientálható esetben is. Mi ezt a célt a

A felület deformált algebrájának a Jordan szorzata

$$\longleftrightarrow$$

Jordán mátrix szorzat

megfeleltetéssel értük el. Mindez a membrán mozgásának Jordan algebrai leírására a következő egyszerű összefüggésen keresztül használható fel

$$\mathrm{Tr}\left(\frac{1}{4}[\mathbb{X}^i,\mathbb{X}^j][\mathbb{X}^i,\mathbb{X}^j]\right) =$$

$$\mathrm{Tr}\left(\mathbb{X}^i\circ(\mathbb{X}^j\circ(\mathbb{X}^i\circ\mathbb{X}^j))-(\mathbb{X}^i\circ\mathbb{X}^i)\circ(\mathbb{X}^j\circ\mathbb{X}^j)\right).$$

Mivel a reguralizált Goldstone-Hoppe féle Hamilton függvény és a mozgásegyenetek a következőek:

$$H=\frac{1}{2\pi l_p^3}\,\mathrm{Tr}\,\left(\frac{1}{2}\dot{\mathbb{X}}^i\dot{\mathbb{X}}^i-\frac{1}{4}[\mathbb{X}^i,\mathbb{X}^j][\mathbb{X}^i,\mathbb{X}^j]\right), \tag{35}$$
$$\ddot{\mathbb{X}}^i+[[\dot{\mathbb{X}}^i,\dot{\mathbb{X}}^j],\dot{\mathbb{X}}^j]=0, \qquad [\dot{\mathbb{X}}^i,\mathbb{X}^i]=0,$$

ahol a $\mathbb{X}^i$ koordináták ermitikus mátrixok, így mindez felírható a Jordan szorzás segítségével is.

Így ki lehet fejezni a membrán potenciális energiáját a Jordan féle szorzás segítségével. Meghatároztuk, hogy milyen Jordán algebrák szükségesek a gömb, illetve a projektív sík topológiáju membránok leírására. Ez utóbbihoz a valós szimmetrikus mátrixok $H_N(\mathbb{R})$ algebrája kellett.

- *4. Fejezet. Véges csoportok, félcsoportok és a kvantum Yang-Baxter egyenlet.*

Migdal munkássága nyomán jól ismert, hogy a kétdimenziós rács mértékelmélet egzaktul megoldható. A kétdimenziós megoldható modellek igen gyakran megadják néhány megoldását a Kvantum Yang-Baxter Egyenlet néhány megoldását. A kétdimenziós rács mértékelmélet esetében a súlyok amelyek egy négyzet körüli konfigurációhoz tartoznak kielégítik a QYBE egyenletet.

$$w(a,b,c,d)=R_{a,b}^{d^{-1},c^{-1}}(\lambda)=\sum_{r\in R(G)}\lambda^r\chi_r(abcd) \tag{36}$$

$$\sum_{g,h,i\in G}R_{a,b}^{g^{-1},i}(\lambda)R_{i^{-1},c}^{h,d}(\mu)R_{g,h^{-1}}^{f,e}(\nu)=\sum_{g,h,i\in G}R_{b,c}^{h^{-1},g}(\nu)R_{a,h}^{f,i^{-1}}(\mu)R_{i,g^{-1}}^{e,d}(\lambda). \tag{37}$$

(Itt $a,b,c,\ldots$ a G, csoport elemei, míg χ_r egy irreducibilis reprezentáció karaktere.)

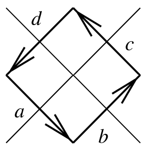

Fig. 1a

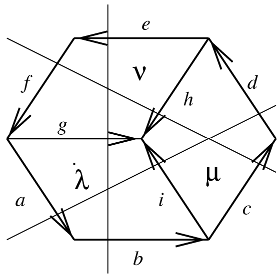

Fig. 1b

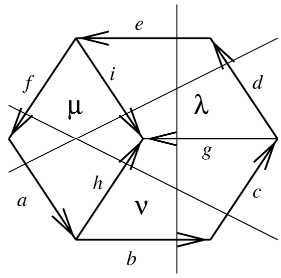

Fig. 1c

Sajnos a kétdimenziós rács mértékelméletnek megfelelő megoldás nem invertálható. Ezt a csorbát sikerült kiküszöbölni a modell egy apró módosításával

$$R_{a,b}^{d^{-1},c^{-1}}(\lambda) =$$

$$\sum_{r\in R(G)} \left(\lambda_0^r \chi_r(abcd) + \lambda_1^r \big(\chi_r(bd) + \chi_r(ac)\big) \right) + \sum_{t,u \in R(G)} \lambda_2^{tu} \chi_t(bd)\chi_u(ac).$$

(Itt az a, b, c, d csoportelemek a a címkéi annak a vektortérnek amelyen R hat, míg χ a mertékcsoport karaktere.) Ez az eredmény egy, a csoportok reprezentációelméletéből jól ismert azonosságon alapul:

$$\left(\sum_{r\in\hat{G}} \lambda_r^{I} \chi_r(ax) \right) \cdot \left(\sum_{s\in\hat{G}} \lambda_s^{II} \chi_s(x^{-1}b) \right) = |G| \sum_{r\in\hat{G}} \frac{\lambda_r^{I}\lambda_r^{II}}{\dim r} \chi_r(ab)$$

Ha be tudnánk bizonyítani ezt az összefüggést félcsoportokra is, akkor az előzőek már nem csak a csoportok esetén teljesülnének. Abban az igen egyszerű esetben, amikor a G_0 félcsoport egy véges csoport kibővítve egy zéruselemmel, a fenti reláció megfelelője igaz marad, ha $\lambda_0 = \lambda_1$. Ez a feltétel azt jelenti, hogy a csoport triviális reprezentációjának és a nulla elemhez tartozó karakternek az együtthatója megegyezik.

Ehhez tekintettünk a konjugálás invariáns $w : G_0 \to R$ függvényt. Ez kifejezhető a csoport irreducibilis kakaktereinek lineáris kombinációjaként plusz egy extra taggal, ami a nulla elemen vesz fel nemnulla értéket

(38)
$$w(c)=\sum_{s\in\hat{G}}\lambda_s\chi_s(c)\ +\ \lambda_0\chi_0(c).$$

Eredményeink szerint

(39)
$$\sum_{x\in G_0} w(ax)\tilde{w}(i(x)b)=\begin{cases}|G|\sum_{s\in\hat{G}}\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab)\ +\ \lambda_0\tilde{\lambda}_0, & \text{if } a,b\in G,\\ \lambda_0\tilde{\lambda}_1|G|+\lambda_0\tilde{\lambda}_0, & \text{if } a=0,b\in G,\\ (|G|+1)\lambda_0\hat{\lambda}_0. & \text{if } a=b=0.\end{cases}$$

Ezt abban az esetben lehet kifejteni konjugálás invariáns függvények segíségével, ha $\lambda_0=\lambda_1$. Ekkor

(40)
$$\hat{w}(ab)=\begin{cases}\sum_{s\in\hat{G},s\neq 1}|G|\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab)\ +\ (\lambda_1\tilde{\lambda}_1+\lambda_0\tilde{\lambda}_0)\chi_1(ab) & \text{if}\quad a,b\in G,\\ (|G|+1)\lambda_0\tilde{\lambda}_0 & \text{if}\quad ab=0.\end{cases}$$

(41)
$$=\sum_{s\in\hat{G},s\neq 1}|G|\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\chi_s(ab)\ +\ (\lambda_1\tilde{\lambda}_1+\lambda_0\tilde{\lambda}_0)\chi_1(ab)+(|G|+1)\lambda_0\tilde{\lambda}_0.$$

Vagyis az nettó súlya az egyesített négyzeteknek

(42)
$$(\lambda_0,\lambda_1,\lambda_s),\quad (\tilde{\lambda}_0,\tilde{\lambda}_1,\tilde{\lambda}_s)\longrightarrow\left((|G|+1)\lambda_0\tilde{\lambda}_0,(|G|+1)\lambda_1\tilde{\lambda}_1,|G|\frac{\lambda_s\tilde{\lambda}_s}{\dim s}\right)$$

lesz, ha $\lambda_0=\lambda_1$ és $\tilde{\lambda}_0=\tilde{\lambda}_1$. Ezek a feltételek igazak maradnak az egyesített négyzetre is.

Ezenfelül megvizsgáltuk ezt a kérdést egy kis méretű félcsoport esetében is egy szimbolikus algebra program segítségével:

(43)
$$g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad g_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad g_3=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad g_4=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$
$$g_5=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad g_6=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad g_7=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Ez a félcsoport olyan módon viselkedett, mintha $\mathbb{Z}_2$ lett volna kiterjesztve öt extra nulla elemmel, amelyek a nem invertálható mátrixoknak feleltek meg.

- *5. Fejezet. A Lax egyenlet egy módosítása.*

Mivel a Lax-egyenlet

$$\dot{L} = PL - LP$$

a hasonósági transzformáció $L \to e^{tP} L e^{-tP}$ infinitezimális formája, így invariánsan hagyja L „sprektrumát". Mi megpróbáltuk az L^*L és LL^* operátorok sprektrumát megőrizni. Ezt a

$$L \to e^{itP} L e^{-itQ}$$

transzformáció invariánsan hagyja, ha e^{itP} és e^{-itQ} unitérek, vagyis P és Q önadjungáltak. Valóban:

$$LL^* \to \left(e^{itP} L e^{-itQ}\right)\left(\left(e^{-itQ}\right)^* L^* \left(e^{itP}\right)^*\right) = e^{itP} LL^* e^{-itP}.$$

Ezek a megfontolások abba az irányba mutatnak, hogy lehetséges következő alakú integálható egyenleteket kapni:

$$\dot{L} = i(PL - LQ), \quad P = P^*, \quad Q = Q^*. \tag{44}$$

Itt jegyezzük meg, hogy a szorzatok spektrumának a megőrzésének a gondolata korábban is előfordult a tudományos irodalomban, például Drinfeld-Sokolov, Hirota-Satsuma és Dodd-Fordy munkáiban.

Mi a Gelfand és Dickey által kidolgozott módszert alkalmaztuk ilyen integrálható egyenletek előállítására. Legyen

$$L = D^2 + v(x)D + u(x),$$

ahol $D = i\partial$. Ekkor $L^{1/2}$ helyett szeretnénk olyan önadjungált pszeudo-differenciál operátorokat

$$A = D + a_0 + a_{-1}D^{-1} + a_{-2}D^{-2} + \ldots,$$

$$B = D + b_0 + b_{-1}D^{-1} + b_{-2}D^{-2} + \ldots,$$

konstruálni, amelyekre igaz, hogy

$$i(AL - LB) = 0.$$

Így

$$AL = LB \quad \Rightarrow \quad L^{-1}AL = B = B^* = L^*AL^{-1*} \quad \Rightarrow \quad A(LL^*) = (LL^*)A,$$

amiből azt kapjuk, hogy

$$A = (LL^*)^{1/4} \quad \text{and} \quad B = (L^*L)^{1/4}.$$

Ekkor az

$$(45) \qquad \begin{aligned} \partial_{t_k} L = \partial_{t_k} \nu D + \partial_{t_k} u &= i\left\{(A^k)_+ L - L(B^k)_+\right\} \\ &= -i\left\{(A^k)_- L - L(B^k)_-\right\} \end{aligned}$$

operátor egyenletek kommutáló folyamokat adnak meg.

Egy szimbolikus algebra program segítségével kiszámltunk néhányat ezek közűl:

(1) flow:

$$\partial_{t_1} u = u'$$

(2) flow:

$$\partial_{t_2} w = 2\Im u'$$

$$\partial_{t_2} u = \frac{i}{2}\Big(w''' - w'(2w' + 2w^2 - 4u) - w(u' - \bar{u}' - w'')\Big)$$

(3) flow:

$$\partial_{t_3} u = \frac{1}{8}\Big(u''' - 3\bar{u}''' - 6uu' + 6u'\bar{u} + 12u\bar{u}'\Big)$$

(4) flow:

$$\partial_{t_4} u = \partial_{t_4} w = 0$$

(5) flow:

$$\begin{aligned} \partial_{t_5} u = \frac{1}{32}\Big\{ &-3u^{(5)} + 5\bar{u}^{(5)} + 5u'''(3u - \bar{u}) + 5\bar{u}'''(-5u + \bar{u}) \\ &+ 5u'\left(-3u^2 + 6u\bar{u} + \bar{u}^2 - 3u'' - 5\bar{u}''\right) \\ &+ 5\bar{u}'\left(4u^2 + 4u\bar{u} - 3u'' - 3\bar{u}''\right)\Big\} \end{aligned}$$

- *6. Fejezet. Vad csoportok koadjungált obitjai a szilárdtest-fizikában*

A síkon a koszinusz függvények és a parciális deriválások operátorai egy feloldható Lie algebrát alkotnak. Mivel ezekből az operátorokból felépíthető egy kvantummechanikai részecske periodikus vagy kváziperiodikus potenciálban történő mozgását leíró Hamilton operátor, ésszerű feltételezni valamilyen kapcsolatot a Lie algebra típusa és a Hamilton operátor spektruma között. Erre a kapcsolatra valószínuleg először J.Zak mutatott rá egy állandó mágneses térben elhelyezkedő rács esetében.

A feloldható Lie csoportok típusát meg lehet határozni Auslander és Kostant egyik tétele segíségével. Mi ellenőriztük ezen tétel feltételeinek teljesülését több, a szilárdtest fizikában előforduló csoport esetében. Ehhez a Lie algebrák koadjungált orbitjait kellett megkeresnünk. Esetünkben a Lie algebra a következő volt:

$$\begin{aligned} &[P_x, S_x] = C_x, \qquad [P_y, S_y] = C_y, \qquad [P_x, X] = I, \\ &[P_x, C_x] = -S_x, \qquad [P_y, C_y] = -S, \qquad [P_y, Y] = I, \end{aligned} \tag{46}$$

$$\begin{aligned} &[E, P_x] = -Y, \qquad [P_x, P_y] = 2B, \\ &[E, P_y] = X, \qquad [E, B] = I. \end{aligned}$$

Itt egy kivételes orbit kivételével az orbitok kielégítették az Auslander-Kostant tétel feltételeit. Ez arra mutathat, hogy bár a csoport maga nem eggyes típusú, az esetunkben felmerülő szilárdtest fizikai rendszer mégis inkább a periodikus rendszerekhez hasonlóan viselkedhet.

- *7. Fejezet. Az energia-momentum algebra viselkedése nemkonformális határfeltételek mellett.*

Ebben a fejezetben megkíséreltük kiszámítani a kétdimenziós nulla tömegü skaláris tér viselkedését nemkonformális határfeltételek mellett. Ezt a problémát igen sokat tanulmányozták abban az esetben, amikor a határfeltételek konformálisak voltak. A nemkonformális esetben elvileg új jelenség is felléphetne. Ennek az a potenciális oka, hogy a $\phi(0) + \phi'(0) = 0$ típusu határfeltételeket (itt a 0 a határ térkoordinátáját jelöli) a $\phi(x) \to \phi(f(x))$ transzformáció csak akkor hagyja invariánsan, ha nem csak $f(0)$, de $f'(0)$ is eltűnik. Ezeknek a transzformációknak a (formális) Lie algebráját $L_1(1)$-nek nevezik. Mivel $H^2(L_1(1))$ kétdimenziós, így az ennek megfelelő centrális kiterjesztés elvben előfordulhatna az energia-momentum algebra szokásos Virasoro típusú kiterjesztése mellett. Mi felsorakoztatunk néhány érvet amellett, hogy esetünkben ez nem fordulhat elő.

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