Introduction

The PhD dissertation consists of three chapters. Most of the results of the dissertation have been published in our papers [52], [54] and [51]. In the introduction of the dissertation, we define relators and relator spaces.

Definition. A nonvoid family $\mathcal{R}$ of binary relations on a nonvoid set X is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space.

Moreover, if $\mathcal{R}$ is a relator on X, then the relator $\mathcal{R}^{-1} = \{R^{-1} : R \in \mathcal{R}\}$ is called the inverse of $\mathcal{R}$.

I

By establishing some intimate connections between unary operations and set-valued functions for relators, we greatly extend and supplement some of the former results of Á. Száz and J. Mala on the various refinements and modifications of relators.

A function $\square$ of the family of all relators on X into itself is called a unary operation for relators on X. And we write $\mathcal{R}^{\square} = \square(\mathcal{R})$ for every relator $\mathcal{R}$ on X. Moreover, a function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a set-valued function for relators on X. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

If $\square$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\square$ is expansive if $\mathcal{R} \subset \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X;

(2) $\square$ is idempotent if $\mathcal{R}^{\square} = \mathcal{R}^{\square\square}$ for every relator $\mathcal{R}$ on X;

(3) $\mathfrak{F}$ is increasing if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$;

(4) $\square$ is a refinement if it is expansive, idempotent and increasing.

Moreover, if $\mathfrak{F}$ is an increasing set-valued function for relators on X, then the induced unary operation $\square_{\mathfrak{F}}$ is defined by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \{S \subset X^2 : \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}\},$$

for every relator $\mathcal{R}$ on X.

Finally, an increasing set-valued function for relators on X is called regular if

$$\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$$

for every relator $\mathcal{R}$ on X. And, an increasing set-valued function for relators on X is called normal if

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}$$

for every relator $\mathcal{R}$ on X.

The most important theorems of this chapter will show, that a normal set-valued function for relators on X is, in particular, regular. On the other hand, a unary operation induced by a regular set-valued function is a refinement. Moreover, if $\mathfrak{F}$ is a

regular set-valued function for relators on X and $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\square_{\mathfrak{F}}}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$.

For instance, if $\mathcal{R}$ is a relator on X, then the relation $\mathrm{int}_{\mathcal{R}}$ defined by

$$\mathrm{int}_{\mathcal{R}}(A) = \{x \in X : \exists R \in \mathcal{R} : R(x) \subset A\}$$

for all $A \subset X$, is called the topological interior induced by $\mathcal{R}$. Moreover, the members of the families

$$\mathcal{T}_{\mathcal{R}} = \{A \subset X : A \subset \mathrm{int}_{\mathcal{R}}(A)\} \qquad \text{and} \qquad \mathcal{E}_{\mathcal{R}} = \{A \subset X : \mathrm{int}_{\mathcal{R}}(A) \neq \emptyset\}$$

are called the topologically open and the fat subsets of the relator space $X(\mathcal{R})$, respectively.

Furthermore, the unary operations

$$\wedge = \square_{\mathrm{int}} \qquad \text{and} \qquad \triangle = \square_{\mathcal{E}}$$

are called the topological and the paratopological refinements, respectively. Namely, since int and $\mathcal{E}$ are normal increasing set-valued functions for relators on X, the induced unary operations $\wedge$ and $\triangle$ are refinements.

Finally, the preorder modification of the relator $\mathcal{R}$ on X is defined by

$$\mathcal{R}^{\infty} = \{R^{\infty} : R \in \mathcal{R}\},$$

where $R^{\infty} = \Delta_X \cup \bigcup_{n=1}^{\infty} R^n$. Note, that the unary operation ∞ is not expansive, therefore it is not a refinement.

II

In this chapter, a unified treatment of some old and new well-chainedness and connectedness properties of the most basic topological structures (such as closures, proximities and uniformities, for instance) is offered in the framework of relators and their fundamental refinements.

The results obtained show that the various connectedness properties are actually particular cases of Cantor's well-chainedness property neglected by several authors. Moreover, they show that the hyperconnectedness introduced by L. A. Steen and J. A. Seebach is a particular case of our paratopological connectedness.

A relator $\mathcal{R}$ on X will be called properly well-chained or chain-connected if

$$\mathcal{R}^{\infty} = \{X^2\}.$$

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ on X will be called $\square$-well-chained if the relator $\mathcal{R}^{\square}$ is properly well-chained.

The condition $\mathcal{R}^{\infty} = \{X^2\}$, in a detailed form, means only that for every $R \in \mathcal{R}$ and $x, y \in X$, with $x \neq y$, there exists an $n \in \mathbb{N}$ such that $(x, y) \in R^n$. That is, there exists a family $(x_i)_{i=0}^n$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R$ for all $i = 1, \ldots, n$.

To characterize well-chained relators, we need the notion of the Davis–Pervin relation. For each $A \subset X$, the relation

$$R_A = A^2 \cup (X \setminus A) \times X$$

is called the Davis–Pervin relation on X generated by A.

In the dissertation, we prove that a relator $\mathcal{R}$ on X is properly well-chained if and only if $\mathcal{P}(R_A) \cap \mathcal{R} = \emptyset$ for every proper nonvoid subset A of X, or equivalently $\mathcal{P}(R) \cap \mathcal{R} = \emptyset$ for every proper preorder R on X.

Moreover, a relator $\mathcal{R}$ on X is topologically (paratopologically) well-chained if and only if $R_A \notin \mathcal{R}^\wedge$ ($R_A \notin \mathcal{R}^\vartriangle$) for every proper nonvoid subset A of X, or equivalently $R \notin \mathcal{R}^\wedge$ ($R \notin \mathcal{R}^\vartriangle$) for every proper preorder R on X.

Furthermore, a relator $\mathcal{R}$ on X is topologically well-chained if and only if $\mathcal{T}_\mathcal{R} = \{\emptyset, X\}$. And, if $\mathrm{card}(X) > 1$, then the relator $\mathcal{R}$ on X is paratopologically well-chained if and only if $\mathcal{E}_\mathcal{R} = \{X\}$, or equivalently $\mathcal{R} = \{X^2\}$.

A relator $\mathcal{R}$ on X will be called properly connected if the relator

$$\mathcal{R} \triangledown \mathcal{R}^{-1} = \{R \cup R^{-1} : R \in \mathcal{R}\}$$

is properly well-chained. Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ on X will be called $\square$-connected if the relator $\mathcal{R}^\square$ is properly connected.

To characterize connected relators, we need the symmetrization of the Davis–Pervin relation. For each $A \subset X$, the relation

$$S_A = R_A \cap R_A^{-1}$$

is called the symmetrization of the Davis–Pervin relation R_A.

In the dissertation, we prove that a relator $\mathcal{R}$ on X is properly connected if and only if $\mathcal{P}(S_A) \cap \mathcal{R} = \emptyset$ for every proper nonvoid subset A of X, or equivalently $\mathcal{P}(S) \cap \mathcal{R} = \emptyset$ for every proper equivalence S on X.

Moreover, a relator $\mathcal{R}$ on X is topologically (paratopologically) connected if and only if $S_A \notin \mathcal{R}^\wedge$ ($S_A \notin \mathcal{R}^\vartriangle$) for every proper nonvoid subset A of X, or equivalently $S \notin \mathcal{R}^\wedge$ ($S \notin \mathcal{R}^\vartriangle$) for every proper equivalence S on X.

Furthermore, a relator $\mathcal{R}$ on X is topologically connected if and only if the complement of any proper nonvoid topologically open set is not topologically open. And, if $\mathrm{card}(X) > 1$, then the relator $\mathcal{R}$ on X is paratopologically connected if and only if the intersection of any two fat sets is non empty.

At the end of this chapter, a diagram can be found which shows the main implications among the various well-chainedness and connectedness properties of relators.

III

In chapter III, some published and unpublished results of Árpád Száz, József Mala and Jenő Deák on simple and quasi-simple relators are illustrated and supplemented.

A relator $\mathcal{R}$ on X is called properly simple if it is a singleton. Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ on X is called $\square$-simple if

there exists a relation R on X such that $\mathcal{R}^{\square} = \{R\}^{\square}$. On the other hand, a relator is called quasi-$\square$-simple, if it is $\square\infty$-simple. We remark, that for instance, the topological well-chainedness is a particular case of quasi-topological simplicity.

In the dissertation, we prove that a relator $\mathcal{R}$ on X is quasi-properly simple if and only if $R^{\infty} = S^{\infty}$ for all $R, S \in \mathcal{R}$, or equivalently $\mathcal{R}^{\infty}$ is properly simple. And, if a relator is $\square$-simple, then it is also quasi-$\square$-simple.

Moreover, for instance, we prove that a relator $\mathcal{R}$ on X is (quasi-)topologically simple if and only if $\bigcap \mathcal{R} \in \mathcal{R}^{\wedge}$ ($\bigcap \mathcal{R}^{\wedge\infty} \in \mathcal{R}^{\wedge}$). And, we state that the paratopological simplicity is equivalent to the quasi-paratopological simplicity. After this, we characterize paratopological simple relators, to construct a a non-paratopologically simple, equivalence relator.

At the end of this chapter, a diagram can be found which shows the main implications among the various simplicity and quasi-simplicity properties of relators.

1 Relators and their induced basic tools

Definition 1.1. If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$ and $x, y \in X$ we write:

(1) $B \in \mathrm{Int}_{\mathcal{R}}(A)$ if $R(B) \subset A$ for some $R \in \mathcal{R}$;

(2) $B \in \mathrm{Cl}_{\mathcal{R}}(A)$ if $R(B) \cap A \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $x \in \mathrm{int}_{\mathcal{R}}(A)$ if $\{x\} \in \mathrm{Int}_{\mathcal{R}}(A)$;

(4) $x \in \mathrm{cl}_{\mathcal{R}}(A)$ if $\{x\} \in \mathrm{Cl}_{\mathcal{R}}(A)$;

(5) $y \in \sigma_{\mathcal{R}}(x)$ if $y \in \mathrm{int}_{\mathcal{R}}\big(\{x\}\big)$;

(6) $y \in \rho_{\mathcal{R}}(x)$ if $y \in \mathrm{cl}_{\mathcal{R}}\big(\{x\}\big)$;

The relations $\mathrm{Int}_{\mathcal{R}}$, $\mathrm{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called the proximal, the topological, and the infinitesimal interiors induced by $\mathcal{R}$ on X, respectively. And, the relations $\mathrm{Cl}_{\mathcal{R}}$, $\mathrm{cl}_{\mathcal{R}}$ and $\rho_{\mathcal{R}}$ are called the proximal, the topological, and the infinitesimal closures induced by $\mathcal{R}$ on X, respectively.

Definition 1.2. If $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we write:

(1) $A \in \tau_{\mathcal{R}}$ if $A \in \mathrm{Int}_{\mathcal{R}}(A)$;

(2) $A \in \tau\!\!\!-_{\mathcal{R}}$ if $X \setminus A \notin \mathrm{Cl}_{\mathcal{R}}(A)$;

(3) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \mathrm{int}_{\mathcal{R}}(A)$;

(4) $A \in \mathcal{F}_{\mathcal{R}}$ if $\mathrm{cl}_{\mathcal{R}}(A) \subset A$;

(5) $A \in \mathcal{E}_{\mathcal{R}}$ if $\mathrm{int}_{\mathcal{R}}(A) \neq \emptyset$;

(6) $A \in \mathcal{D}_{\mathcal{R}}$ if $\mathrm{cl}_{\mathcal{R}}(A) = X$.

(7) $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}}$;

(8) $D_{\mathcal{R}} = \bigcup (\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}})$.

The members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, the topologically open, and the fat subsets of $X(\mathcal{R})$, respectively. And, the members of the families $\tau\!\!\!-_{\mathcal{R}}$, $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are called the proximally closed, the topologically closed, and the dense subsets of the relator space $X(\mathcal{R})$, respectively.

Definition 1.3. If $\mathcal{R}$ is a relator on X, then the relators

(1) $\mathcal{R}^{*} = \big\{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\big\}$;

(2) $\mathcal{R}^{\#} = \big\{S \subset X^2 : \forall A \subset X : A \in \mathrm{Int}_{\mathcal{R}}\big(S(A)\big)\big\}$;

(3) $\mathcal{R}^{\wedge} = \big\{S \subset X^2 : \forall x \in X : x \in \mathrm{int}_{\mathcal{R}}\big(S(x)\big)\big\}$;

(4) $R^{\triangle} = \{S \subset X^2 : \forall x \in X : S(x) \in \mathcal{E}_{\mathcal{R}}\}$;

(5) $\mathcal{R}^{\bullet} = \{\rho_{\mathcal{R}}^{-1}\}^*$; (6) $\mathcal{R}^{\blacktriangle} = \{X \times E_{\mathcal{R}}\}^*$

are called the uniform, the proximal, the topological, the paratopological, the infinitesimal and the parainfinitesimal refinements of $\mathcal{R}$, respectively.

To provide a general framework for the investigation of the above basic tools, we have introduced in [52] several new definitions and established their most important consequences.

We define unary operations (for instance refinements, modifications) and set-valued functions for relators.

Definition 1.4. A function $\square$ of the family of all relators on X into itself is called a unary operation for relators on X. And we write $\mathcal{R}^{\square} = \square(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

Moreover, a function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a set-valued function for relators on X. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

Definition 1.5. If $\square$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\square$ is expansive if $\mathcal{R} \subset \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X;

(2) $\square$ is idempotent if $\mathcal{R}^{\square} = \mathcal{R}^{\square\square}$ for every relator $\mathcal{R}$ on X;

(3) $\mathfrak{F}$ is increasing (decreasing) if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$) for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$.

(4) $\square$ is a modification if it is idempotent and increasing;

(5) $\square$ is a refinement if it is expansive, idempotent and increasing;

(6) $\mathfrak{F}$ is $\square$-increasing ($\square$-decreasing) if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$);

(7) the relators $\mathcal{R}$ and $\mathcal{S}$ on X are $\mathfrak{F}$-equivalent if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$;

(8) the relator $\mathcal{R}$ on X is $\square$-fine if $\mathcal{R} = \mathcal{R}^{\square}$.

Theorem 1.6. *If $\square$ is a unary operation for relators on X, then the following assertions are equivalent:*

(1) $\square$ is a refinement;

(2) there exists a $\square$-increasing ($\square$-decreasing) set-valued function $\mathfrak{F}$ for relators on X.

Theorem 1.7. *If $\square$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ is $\square$-increasing ($\square$-decreasing);

(2) $\mathfrak{F}$ is increasing (decreasing) and, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}^{\square}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}^{\square}}$).

Corollary 1.8. *If $\square$ is a unary operation and $\mathfrak{F}$ is a $\square$-monotonic set-valued for relators on X, then for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.*

Definition 1.9. If $\mathfrak{F}$ is an increasing (decreasing) set-valued function for relators on X, then the operation $\square_{\mathfrak{F}}$, defined by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{S \subset X^2 : \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}\right\} \qquad \left(\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{S \subset X^2 : \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\{S\}}\right\}\right)$$

for every relator $\mathcal{R}$ on X, is called the operation induced by $\mathfrak{F}$.

Definition 1.10. A monotone set-valued function $\mathfrak{F}$ for relators on X is called regular if

$$\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$$

for every relator $\mathcal{R}$ on X.

Theorem 1.11. *If $\mathfrak{F}$ is a monotone set-valued function for relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ is regular; *(2) $\mathfrak{F}$ is $\square_{\mathfrak{F}}$-monotonic;*

(3) $\mathfrak{F}$ is $\square$-monotonic for some unary operation $\square$ for relators on X.

Definition 1.12. An increasing (decreasing) set-valued function $\mathfrak{F}$ for relators on X is called normal if, for every relator $\mathcal{R}$ on X, we have

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \qquad \left(\mathfrak{F}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}\right).$$

Note that, the above definition means only that a monotone set-valued function for relators on X is normal if and only if

$$\bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \subset \mathfrak{F}_{\mathcal{R}} \subset \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}$$

for every relator $\mathcal{R}$ on X.

Theorem 1.13. *A normal set-valued function for relators on X is, in particular, regular.*

Theorem 1.14. *$*$, Int, int, σ, τ, $\tau\!\!\!-$, $\mathcal{E}$, and D are normal increasing set-valued functions for relators on X.*

While, Cl, cl, *ρ, $\mathcal{D}$, and E are normal decreasing set-valued functions for relators on X.*

Unfortunately, the increasing set-valued functions $\mathcal{T}$ and $\mathcal{F}$, on which topology was based on, are not even regular, in general.

Therefore, if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator on X, such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$ $\left(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\square}}\right)$.

Definition 1.15. If $\diamondsuit$ and $\square$ are unary operations for relators on X, then we say that:

(1) $\square$ is $\diamondsuit$-dominating if $\mathcal{R}^{\diamondsuit} \subset \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X;

(2) $\square$ is $\diamondsuit$-invariant if $\mathcal{R}^{\square} = \mathcal{R}^{\square\diamondsuit}$ for every relator $\mathcal{R}$ on X;

(3) $\square$ is $\diamondsuit$-absorbing if $\mathcal{R}^{\square} = \mathcal{R}^{\diamondsuit\square}$ for every relator $\mathcal{R}$ on X;

(4) $\square$ is $\diamondsuit$-compatible if $\mathcal{R}^{\square\diamondsuit} = \mathcal{R}^{\diamondsuit\square}$ for every relator $\mathcal{R}$ on X.

Remark 1.16. In particular, the operation $\square$ will be called inversion compatible if $\left(\mathcal{R}^{\square}\right)^{-1} = \left(\mathcal{R}^{-1}\right)^{\square}$ for every relator $\mathcal{R}$ on X.

Now, as some useful consequences of the above definitions, we can also state the following theorems.

Theorem 1.17. *If $\diamondsuit$ is an expansive and $\square$ is a $\diamondsuit$-dominating idempotent operation for relators on X, then $\square$ is $\diamondsuit$-invariant.*

Theorem 1.18. *If $\diamondsuit$ is an expansive and $\square$ is a $\diamondsuit$-dominating modification for relators on X, then $\square$ is $\diamondsuit$-absorbing.*

Theorem 1.19. *If $\mathfrak{F}$ and $\mathfrak{G}$ are regular set-valued functions for relators on X such that $\mathfrak{F} \circ \mathfrak{G}^{-1}$ is a function, then $\square_{\mathfrak{F}}$ is $\square_{\mathfrak{G}}$-dominating.*

Corollary 1.20. *If $\mathfrak{F}$ and $\mathfrak{G}$ are regular set-valued functions for relators on X such that $\mathfrak{F} \circ \mathfrak{G}^{-1}$ and $\mathfrak{G} \circ \mathfrak{F}^{-1}$ are both functions, then $\square_{\mathfrak{F}} = \square_{\mathfrak{G}}$.*

Corollary 1.21. *If $\mathfrak{F}$ and $\mathfrak{G}$ are regular set-valued functions for relators on X such that $\mathfrak{F} \circ \mathfrak{G}^{-1}$ is a function, then $\square_{\mathfrak{F}}$ is $\square_{\mathfrak{G}}$-invariant, $\square_{\mathfrak{G}}$-absorbing and $\square_{\mathfrak{G}}$-compatible.*

Theorem 1.22. *If $\mathfrak{F}$ and $\mathfrak{G}$ are set-valued functions for relators on X such that*

$$(\mathfrak{G}, \mathfrak{F}) \in \big\{(*, \mathrm{Int}), (*, \mathrm{Cl}), (\mathrm{Int}, \mathrm{int}), (\mathrm{Cl}, \mathrm{cl}), \\ (\mathrm{int}, \mathcal{E}), (\mathrm{cl}, \mathcal{D}), (\mathrm{int}, \sigma), (\mathrm{cl}, \rho), (\mathcal{E}, E), (\mathcal{D}, D)\big\},$$

then $\mathfrak{F} \circ \mathfrak{G}^{-1}$ is a function.

Theorem 1.23. *If $\mathfrak{F}$ and $\mathfrak{G}$ are set-valued functions for relators on X such that*

$$(\mathfrak{G}, \mathfrak{F}) \in \big\{(\mathrm{Int}, \mathrm{Cl}), (\mathrm{int}, \mathrm{cl}), (\mathcal{E}, \mathcal{D}), (E, D)\big\},$$

then $\mathfrak{F} \circ \mathfrak{G}^{-1}$ $\mathfrak{G} \circ \mathfrak{F}^{-1}$ are both functions.

The appropriateness of our former definitions is already apparent from the following theorems.

Theorem 1.24.

(1) $* = \square_{*}$;

(2) $\# = \square_{\mathrm{Int}} = \square_{\mathrm{Cl}}$;

(3) $\wedge = \square_{\mathrm{int}} = \square_{\mathrm{cl}}$;

(4) $\vartriangle = \square_{\mathcal{E}} = \square_{\mathcal{D}}$;

(5) $\bullet = \square_{\rho}$;

(6) $\blacktriangle = \square_{E} = \square_{D}$.

Theorem 1.25. *The above operations are refinements for relators on X such that, for any relator $\mathcal{R}$ on X,*

(1) $\mathcal{R}^{\#}$ is the largest relator on X such that $\mathrm{Int}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^{\#}}$, or equivalently $\mathrm{Cl}_{\mathcal{R}} = \mathrm{Cl}_{\mathcal{R}^{\#}}$;

(2) $\mathcal{R}^{\wedge}$ is the largest relator on X such that $\mathrm{int}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}^{\wedge}}$, or equivalently $\mathrm{cl}_{\mathcal{R}} = \mathrm{cl}_{\mathcal{R}^{\wedge}}$;

(3) $\mathcal{R}^{\triangle}$ is the largest relator on X such that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\triangle}}$, or equivalently $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\triangle}}$;

(4) $\mathcal{R}^{\bullet}$ is the largest relator on X such that $\rho_{\mathcal{R}} = \rho_{\mathcal{R}^{\bullet}}$.

(5) $\mathcal{R}^{\blacktriangle}$ is the largest relator on X such that $E_{\mathcal{R}} = E_{\mathcal{R}^{\blacktriangle}}$, or equivalently $D_{\mathcal{R}} = D_{\mathcal{R}^{\blacktriangle}}$.

Theorem 1.26. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then*

$$\begin{array}{lllll}
(1)\ \mathcal{S} \subset \mathcal{R}^{\#} & \iff & \mathrm{Int}_{\mathcal{S}} \subset \mathrm{Int}_{\mathcal{R}} & \iff & \mathrm{Cl}_{\mathcal{R}} \subset \mathrm{Cl}_{\mathcal{S}} \\
(2)\ \mathcal{S} \subset \mathcal{R}^{\wedge} & \iff & \mathrm{int}_{\mathcal{S}} \subset \mathrm{int}_{\mathcal{R}} & \iff & \mathrm{cl}_{\mathcal{R}} \subset \mathrm{cl}_{\mathcal{S}} \\
(3)\ \mathcal{S} \subset \mathcal{R}^{\triangle} & \iff & \mathcal{E}_{\mathcal{S}} \subset \mathcal{E}_{\mathcal{R}} & \iff & \mathcal{D}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{S}} \\
(4)\ \mathcal{S} \subset \mathcal{R}^{\bullet} & \iff & \rho_{\mathcal{R}} \subset \rho_{\mathcal{S}} & & \\
(5)\ \mathcal{S} \subset \mathcal{R}^{\blacktriangle} & \iff & E_{\mathcal{R}} \subset E_{\mathcal{S}} & \iff & D_{\mathcal{S}} \subset D_{\mathcal{R}}
\end{array}$$

Theorem 1.27. *If $\mathcal{R}$ is a relator on X, then*

$$\begin{array}{ccccccccc}
\mathcal{R} \subset \mathcal{R}^{*} \subset \mathcal{R}^{\#} \subset \mathcal{R}^{\wedge} \subset \mathcal{R}^{\triangle} \\
\cap \qquad \cap \\
\mathcal{R}^{\bullet} \subset \mathcal{R}^{\blacktriangle}.
\end{array}$$

Unfortunately, the relators $\mathcal{R}^{\triangle}$ and $\mathcal{R}^{\bullet}$ are incomparable, in general.

Theorem 1.28. *If $\diamondsuit, \square \in \{*, \#, \wedge, \circ, \blacktriangle\}$, where $\circ = \triangle$ or $\bullet$, such that $\diamondsuit$ precedes $\square$ in the above list, then $\square$ is both $\diamondsuit$-invariant and $\diamondsuit$-absorbing.*

Moreover, in the dissertation we prove that under the notations $\mathcal{R}^{\vee} = \mathcal{R}^{\wedge -1}$ and $\mathcal{R}^{\nabla} = \mathcal{R}^{\triangle -1}$, we have the following

Theorem 1.29. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{R}^{\bullet} = \mathcal{R}^{\vee\vee} \qquad \text{and} \qquad \mathcal{R}^{\blacktriangle} = \mathcal{R}^{\nabla\nabla}.$$

In the dissertation, we investigate some other set-valued functions and unary operations for relators on X, for instance

Definition 1.30. If R is a relation on X, then the relation

$$R^{\infty} = \bigcup_{n=0}^{\infty},$$

where $R^n = R \circ R^{n-1}$ and $R^0 = \Delta_X$ is called the preorder hull of R.

Moreover if $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^{\infty} = \{R^{\infty} : R \in \mathcal{R}\} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \{S \subset X^2 : S^{\infty} \in \mathcal{R}^{\infty}\}$$

are called the preorder modification and the inverse preorder refinement of $\mathcal{R}$, respectively.

Simple applications of the corresponding definitions give the following

Theorem 1.31. *∞ is an inversion compatible, normal modification for relators on X such that, for every relator $\mathcal{R}$ on X, we have*

$$\mathcal{R}^{\infty} \subset \mathcal{R}^{*\infty} \subset \mathcal{R}^{\infty *} \subset \mathcal{R}^{*}.$$

Moreover, $\partial = \square_{\infty}$ is an inversion compatible, normal refinement for relators on X such that, ∞ is ∂-absorbing and ∂ is ∞-absorbing, and for every relator $\mathcal{R}$ on X, we have

$$\mathcal{R}^{\partial} \cup \mathcal{R}^{*} \subset \mathcal{R}^{\partial *} \subset \mathcal{R}^{*\partial}.$$

At the end of the first chapter of the dissertation, we investigate some binary operations for relators, and some special relators, which are interesting from the point of view of topology. Moreover, we write about continuous relations in relator spaces. For instance, we shall use the following

Definition 1.32. If $\mathcal{R} = \{R_i\}_{i \in I}$ and $\mathcal{S} = \{S_i\}_{i \in I}$ are relators on X, then by trusting to the reader's good sense to avoid confusions we define

$$\mathcal{R} \triangledown \mathcal{S} = \big\{R_i \cup S_i : i \in I\big\}.$$

Remark 1.33. Note, that if $\mathcal{R}$ is a relator on X, then by considering $\mathcal{R} = \{R\}_{R \in \mathcal{R}}$ and $\mathcal{R}^{-1} = \{R^{-1}\}_{R \in \mathcal{R}}$ we have

$$\mathcal{R} \triangledown \mathcal{R}^{-1} = \big\{R \cup R^{-1} : R \in \mathcal{R}\big\}.$$

Definition 1.34. If $\mathcal{R}$ is a relator on X and for all $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subset R(x)$, then we say that $\mathcal{R}$ is topological.

Theorem 1.35. *A relator $\mathcal{R}$ on X is topological if and only if for all $A \subset X$*

$$\operatorname{int}_{\mathcal{R}}(A) = \bigcup \big\{V \in \mathcal{T}_{\mathcal{R}} : V \subset A\big\}, \quad \textit{or equivalently} \quad \operatorname{cl}_{\mathcal{R}}(A) = \bigcap \big\{W \in \mathcal{F}_{\mathcal{R}} : A \subset W\big\}.$$

Theorem 1.36. *If $\mathcal{R}$ is a topological relator on X, then*

$$\mathcal{E}_{\mathcal{R}} = \big\{A \subset X : \exists V \in \mathcal{T}_{\mathcal{R}} : \emptyset \neq V \subset A\big\}.$$

Definition 1.37. If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square$ is a unary operation for relators, then the relation F is said to be $\square$-continuous, or more precisely mildly $\square$-continuous [80] if

$$\big(F^{-1} \circ \mathcal{S}^{\square} \circ F\big)^{\square} \subset \mathcal{R}^{\square}.$$

Thus, in particular, we have the following two theorems.

Theorem 1.38. *If f is a function on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square \in \{*, \#, \wedge, \bullet\}$, then the following assertions are equivalent:*

(1) f is $\square$-continuous; *(2) $f^{-1} \circ \mathcal{S} \circ f \subset \mathcal{R}^{\square}$.*

Theorem 1.39. *If f is a function on one relator space $X(\mathcal{R})$ onto another $Y(\mathcal{S})$ and $\square \in \{\triangle, \blacktriangle\}$, then the following assertions are equivalent:*

(1) f is $\square$-continuous; *(2) $f^{-1} \circ \mathcal{S} \circ f \subset \mathcal{R}^{\square}$.*

2 Well-chainednesses and connectednesses of relators

In this chapter we investigate well-chained and connected relators.

Analogously to the problem of finding a powerful and flexible notion of a spatial structure, the problem of finding an appropriate notion of connectedness also has a long history. The three most important definitions were suggested by K. Weierstrass, G. Cantor and C. Jordan. (See [86] and [82, p. 29].)

According to Cantor, a metric space $X(d)$ may be called well-chained or chain-connected if for every $x, y \in X$ and every $\varepsilon > 0$ there exists a finite family $(x_i)_{i=0}^n$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, \ldots, n$.

That is, there exists a natural number n such that, for the ε-sized d-surrounding $B_\varepsilon^d = \{(u, v) \in X^2 : d(u, v) < \varepsilon\}$, which is only a tolerance (reflexive and symmetric relation) on X, we have $(x, y) \in \left(B_\varepsilon^d\right)^n$, where the nth power is taken with respect to composition.

Therefore, we may define the well-chainedness properties in the following way.

Definition 2.1. A relator $\mathcal{R}$ on X will be called properly well-chained or chain-connected if $\mathcal{R}^\infty = \{X^2\}$.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ will be called $\square$-well-chained if the relator $\mathcal{R}^{\square}$ is properly well-chained.

According to the above defined refinements for relators, we may naturally call a relator $\mathcal{R}$ on X uniformly, proximally, topologically, paratopologically, infinitesimally and parainfinitesimally well-chained if it is $\square$-well-chained with $\square = *, \#, \wedge, \triangle, \bullet, \blacktriangle$, respectively.

The well-chainedness of metric or uniform spaces is usually neglected by the authors of the standard textbooks on topology. The only exceptions seem to be Berge [3, p. 96–99], Gaal [15, p. 101 and 142] and Whyburn and Duda [85, p. 34–37].

Several interesting new characterizations of well-chained metric and uniform spaces were established by Mathews [40], Mrówka and Pervin [41] and Levine [31]. Moreover, the well-chainedness of nearness spaces has also been studied by Baboolal and Ori [2].

Some of the results of Levine were extended to reflexive relators by Kurdics and Száz in [26]. The latter authors also investigated the uniform, proximal and topological well-chainednesses of reflexive relators. But, the other well-chainedness properties have been considered first in [54].

For this, we need the following definition and theorem.

Definition 2.2. For each $A \subset X$, the relation

$$R_A = A^2 \cup (X \setminus A) \times X$$

is called the Davis–Pervin relation on X generated by A.

Remark 2.3. Namely, the relations R_A were first used by Davis [9] and Pervin [55] in their uniformization procedures of topological spaces.

Theorem 2.4. *If $A, B \subset X$, then R_A is a preorder on X such that*

$$R_A(B) = \emptyset \text{ if } B = \emptyset, \qquad R_A(B) = A \text{ if } \emptyset \neq B \subset A,$$
$$R_A(B) = X \text{ if } B \not\subset A.$$

Theorem 2.5. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^*$; *(2) $A \in \tau_{\mathcal{R}}$.*

Theorem 2.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $R_A \notin \mathcal{R}^*$ for every proper nonvoid subset A of X;

(3) $R \notin \mathcal{R}^*$ for every proper preorder R on X;

(4) $R \notin \mathcal{R}^*$ for every proper nonvoid transitive relation R on X;

(5) $\tau_{\mathcal{R}} = \{\emptyset, X\}$; *(6) $\tau_{\mathcal{R}} = \{\emptyset, X\}$.*

The $*$-invariances of the refinements $*, \#, \wedge, \triangle, \bullet, \blacktriangle$ show the importance of the following

Theorem 2.7. *If $\mathcal{R}$ is a relator on X and $\square$ is an $*$-invariant operation for relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-well-chained;

(2) $R_A \notin \mathcal{R}^{\square}$ for every proper nonvoid subset A of X;

(3) $R \notin \mathcal{R}^{\square}$ for every proper preorder R on X;

(4) $R \notin \mathcal{R}^{\square}$ for every proper nonvoid transitive relation R on X.

By the above mentioned dominating properties, this theorem shows that 'paratopologically or infinitesimally well-chained' $\Longrightarrow$ 'topologically well-chained' $\Longrightarrow$ 'proximally well-chained' $\Longrightarrow$ 'uniformly well-chained' $\Longrightarrow$ 'properly well-chained'. And 'parainfinitesimally well-chained' $\Longrightarrow$ 'paratopologically and infinitesimally well-chained'.

For further characterization of the well-chainedness of refinement relators we need the following

Theorem 2.8. *If $\mathcal{R}$ is a relator on X, then*

(1) $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^*} = \tau_{\mathcal{R}^\#}$ *and* $\tau\!\!\!-_{\mathcal{R}} = \tau\!\!\!-_{\mathcal{R}^*} = \tau\!\!\!-_{\mathcal{R}^\#}$;

(2) $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$ *and* $\tau\!\!\!-_{\mathcal{R}^\wedge} = \mathcal{F}_{\mathcal{R}}$;

(3) $\tau_{\mathcal{R}^\triangle} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$ *and* $\tau\!\!\!-_{\mathcal{R}^\triangle} = (\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}) \cup \{X\}$;

(4) $\tau_{\mathcal{R}^\bullet} = \tau\!\!\!-_{\{\rho_{\mathcal{R}}\}}$ *and* $\tau\!\!\!-_{\mathcal{R}^\bullet} = \tau_{\{\rho_{\mathcal{R}}\}}$;

(5) $\tau_{\mathcal{R}^\blacktriangle} = \{A \subset X : E_{\mathcal{R}} \subset A\} \cup \{\emptyset\}$ *and* $\tau\!\!\!-_{\mathcal{R}^\blacktriangle} = \mathcal{P}(D_{\mathcal{R}}) \cup \{X\}$.

Theorem 2.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $\mathcal{R}$ is uniformly well-chained; *(3) $\mathcal{R}$ is proximally well-chained.*

Theorem 2.10. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically well-chained;

(2) $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$; *(3)* $\mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.

Theorem 2.11. *If $\mathcal{R}$ is a relator on X and* $\mathrm{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically well-chained; *(2) $\mathcal{R}$ is parainfinitesimally well-chained;*

(3) $\mathcal{E}_{\mathcal{R}} = \{X\}$; *(4)* $\mathcal{D}_{\mathcal{R}} = \mathcal{P}(X) \setminus \{\emptyset\}$;

(5) $\mathcal{R} = \{X^2\}$.

Theorem 2.12. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is infinitesimally well-chained; *(2) $\{\rho_{\mathcal{R}}\}$ is properly well-chained;*

(3) $\tau_{\{\rho_{\mathcal{R}}\}} = \{\emptyset, X\}$; *(4)* $\tau\!\!\!-_{\{\rho_{\mathcal{R}}\}} = \{\emptyset, X\}$;

(5) $\rho_{\mathcal{R}}^{\infty} = X^2$.

Now, according to the results of [23] and [27], we may define connectedness properties in the following way.

Definition 2.13. A relator $\mathcal{R}$ on X will be called properly connected if the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is properly well-chained.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ will be called $\square$-connected if the relator $\mathcal{R}^{\square}$ is properly connected.

Moreover, analogously to the corresponding well-chainedness properties, the relator $\mathcal{R}$ may be naturally called uniformly, proximally, topologically, paratopologically, infinitesimally and parainfinitesimally connected if it is $\square$-connected with $\square = *, \#, \wedge, \triangle, \bullet, \blacktriangle$, respectively.

We need the following definition and theorem

Definition 2.14. For each $A \subset X$, the relation

$$S_A = R_A \cap R_A^{-1}$$

is called the symmetrization of the Davis–Pervin relation R_A.

Theorem 2.15. *If $A, B \subset X$, then S_A is an equivalence on X such that*

$$S_A(B) = \emptyset \text{ if } B = \emptyset, \qquad S_A(B) = X \setminus A \text{ if } \emptyset \neq B \subset X \setminus A,$$
$$S_A(B) = A \text{ if } \emptyset \neq B \subset A, \qquad S_A(B) = X \text{ if } B \not\subset A \text{ and } B \not\subset X \setminus A.$$

Theorem 2.16. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\#}$; *(2) $A \in \tau_{\mathcal{R}} \cap \tau_{\mathcal{R}}$.*

Theorem 2.17. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) $S_A \notin \mathcal{R}^*$ for every proper nonvoid subset A of X;

(3) $S \notin \mathcal{R}^*$ for every proper equivalence S on X;

(4) $S \notin \mathcal{R}^*$ for every proper nonvoid symmetric and transitive relation S on X.

Theorem 2.18. *If $\mathcal{R}$ is a relator on X and $\square$ is an $*$-invariant operation for relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-connected;

(2) $S_A \notin \mathcal{R}^{\square}$ for every proper nonvoid subset A of X;

(3) $S \notin \mathcal{R}^{\square}$ for every proper equivalence S on X;

(4) $S \notin \mathcal{R}^{\square}$ for every proper nonvoid symmetric and transitive relation S on X.

By the above mentioned dominating properties, this theorem shows that 'paratopologically or infinitesimally connected' $\Longrightarrow$ 'topologically connected' $\Longrightarrow$ 'proximally connected' $\Longrightarrow$ 'uniformly connected' $\Longleftrightarrow$ 'properly connected'. And 'parainfinitesimally connected' $\Longrightarrow$ 'paratopologically and infinitesimally connected'.

Moreover, by the required assertions we have the following theorems.

Theorem 2.19. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected; *(2) $\tau_{\mathcal{R}} \cap \tau_{\mathcal{R}} = \{\emptyset, X\}$.*

Theorem 2.20. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected; *(2) $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.*

Theorem 2.21. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$; *(3) $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{E}_{\mathcal{R}}$.*

Theorem 2.22. *If $\mathcal{R}$ is a topological relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{D}_{\mathcal{R}}$; *(3) $U \cap V \neq \emptyset$ for all $U, V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.*

Theorem 2.23. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is infinitesimally connected;

(2) $\tau_{\{\rho_\mathcal{R}\}} \cap \tau_{\{\rho_\mathcal{R}\}} = \{\emptyset, X\}$; *(3) $\{\rho_\mathcal{R}\}$ is properly connected.*

Theorem 2.24. *If $\mathcal{R}$ is a relator on X and $\operatorname{card}(x) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is parainfinitesimally connected;

(2) $E_\mathcal{R} \neq \emptyset$; *(3) $D_\mathcal{R} \neq X$.*

Moreover, we investigate the properties of continuous relations and functions of a relator space into another.

Theorem 2.25. *If $\mathcal{R}$ is a relator on X, $\square \in \{*, \#, \wedge, \bullet\}$ and $\operatorname{card}(Y) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-connected;

(2) every $\square$-continuous function f of $X(\mathcal{R})$ into $Y(\Delta_Y)$ is constant.

Theorem 2.26. *If $\mathcal{R}$ is a relator on X, $\square \in \{\triangle, \blacktriangle\}$ and $\operatorname{card}(Y) = 2$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-connected;

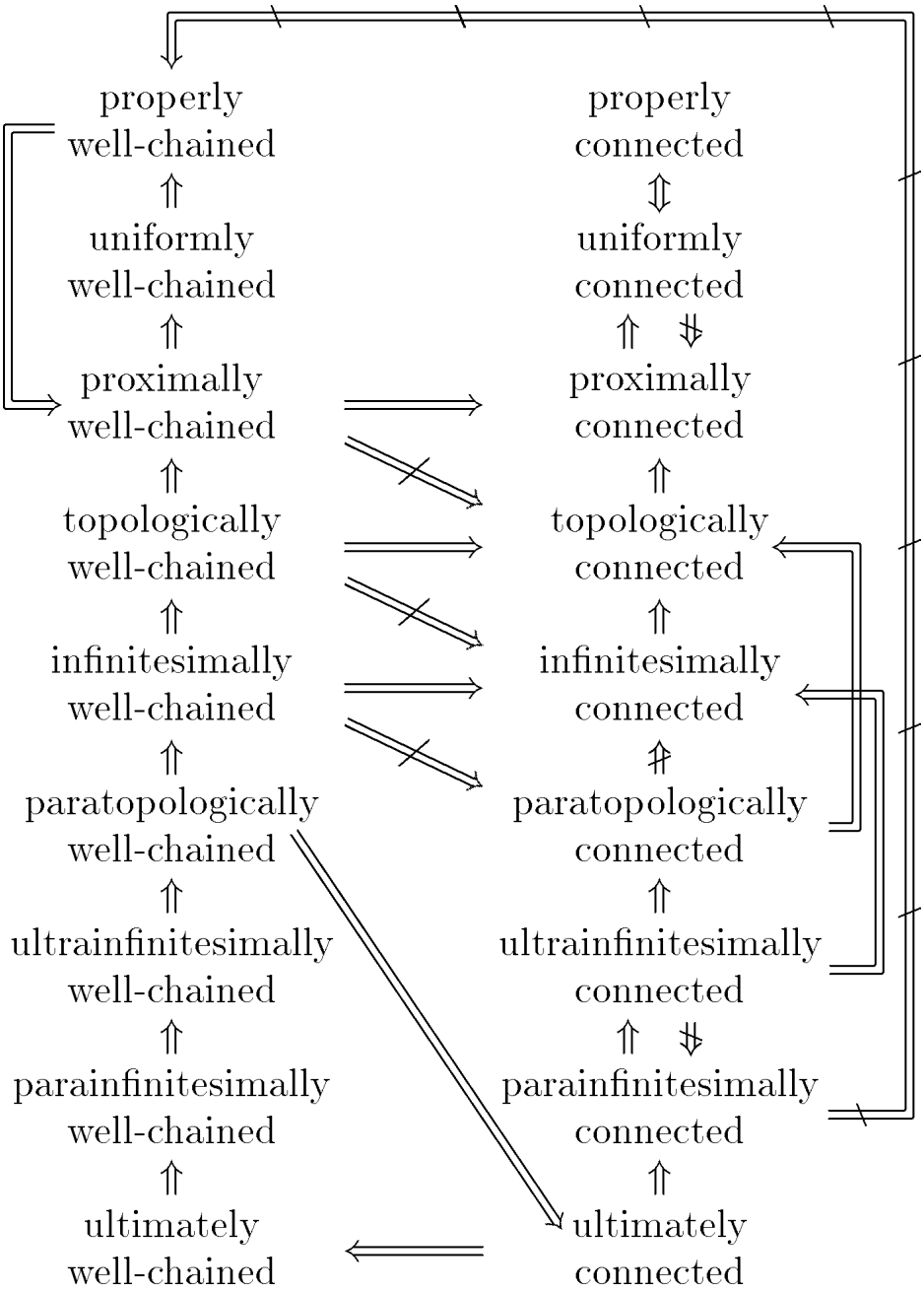
(2) every $\square$-continuous function f of $X(\mathcal{R})$ into $Y(\Delta_Y)$ is constant.

Finally, we compare the various well-chainedness and connectedness properties. For instance

Theorem 2.27. *If $\square$ is a unary operation for relators on X and $\mathcal{R}$ is a $\square$-well-chained relator on X, then $\mathcal{R}$ is, in particular, $\square$-connected.*

We show counterexamples for implications between the well-chainedness and connectedness properties which do not hold, in general.

The following diagram shows the main implications among the various well-chainedness and connectedness properties of relators.



3 Simplicity of relators

In this chapter we investigate simple and quasi-simple relators.

Simple and quasi-simple relators were mainly investigated by Árpád Száz, but several interesting problems have been left open. The most exciting ones were solved by József Mala and Jenő Deák. However, the results of the latter author have not been published because of his early and tragic death.

Therefore, the main purpose of [51] was not only to solve some of the remaining open problems of Árpád Száz, but also to present the relevant results of Jenő Deák. The latter author provided a useful characterization of paratopologically simple relators, which lead us to the investigation of two natural operations on families of sets.

Definition 3.1. A relator $\mathcal{R}$ on X is called properly simple if it is a singleton. That is, there exists a relation R on X such that $\mathcal{R} = \{R\}$.

Moreover, if $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called $\square$-simple if it is $\square$-equivalent to a properly simple relator.

Finally, if $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called quasi-$\square$-simple if it is $\square\infty$-simple.

In particular the relator $\mathcal{R}$ on X is quasi-properly simple if it is ∞-simple.

Remark 3.2. We could define the $\mathfrak{F}$-simplicity of relators on X for an arbitrary set-valued function $\mathfrak{F}$ for relators on X, but we would not get any new notion. Namely, for instance, the Int-, cl- and $\mathcal{E}$-simplicity would be equivalent to the #-, $\wedge$- and $\triangle$-simplicity, respectively. Moreover, the τ- and $\mathcal{F}$-simplicity would be equivalent to the quasi-#- and quasi-$\wedge$-simplicity, respectively.

Furthermore, we could not define the quasi-$\mathfrak{F}$-simplicity for any set-valued functions $\mathfrak{F}$ for relators on X.

We remark that, for instance, the topologically well-chainedness is a particular case of quasi-topologically simplicity.

By the above mentioned absorbing properties, we can see that 'properly simple' $\implies$ 'uniformly simple' $\implies$ 'proximally simple' $\implies$ 'topologically simple' $\implies$ 'paratopologically simple and infinitesimally simple'.

On the other hand, 'paratopologically simple or infinitesimally simple' $\implies$ 'parainfinitesimally simple'.

Moreover, it is also clear that if $\square$ is a unary operation for relators on X, then '$\square$-simple' $\implies$ 'quasi-$\square$-simple'.

Theorem 3.3. *If $\square \in \{*, \#, \wedge, \bullet\}$, then the relator $\mathcal{R}$ on X is $\square$-simple if and only if $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\square}$.*

By the above theorem, we can see that every relator is infinitesimally simple. Therefore, quasi-infinitesimally, parainfinitesimally and quasi-parainfinitesimally simple relators need not be study anymore. Moreover, 'paratopologically simple' $\implies$ 'infinitesimally simple'.

After this, we characterize quasi-simple relators.

Theorem 3.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-properly simple;

(2) $\mathcal{R}^{\infty} = \{\rho_{\mathcal{R}^{\infty}}^{-1}\}$; *(3)* $\mathcal{R}^{\infty} = \{\rho_{\mathcal{R}^{\infty}}^{-1}\}^{\infty}$.

The following three basic theorems have also been mostly established in [68].

Theorem 3.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-uniformly simple; *(2)* $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{*}$;

(3) $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{\infty}$; *(4)* $\mathcal{R}^{*\infty} = \{\rho_{\mathcal{R}^{\infty}}^{-1}\}^{*\infty}$.

Theorem 3.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-proximally simple;

(2) $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{\#}$; *(3)* $\mathcal{R}^{\#\infty} = \{\rho_{\mathcal{R}^{\infty}}^{-1}\}^{\#\infty}$.

Theorem 3.7. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically simple;

(2) $\rho_{\mathcal{R}^{\wedge\infty}}^{-1} \in \mathcal{R}^{\wedge}$; *(3)* $\mathcal{R}^{\wedge\infty} = \{\rho_{\mathcal{R}^{\wedge\infty}}^{-1}\}^{\wedge\infty}$.

Theorem 3.8. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X such that $\mathcal{S}$ is total, then the following assertions are equivalent:*

(1) $\mathcal{S}^{\Delta} \subset \mathcal{R}^{\Delta}$; *(2)* $\mathcal{S}^{\Delta\infty} \subset \mathcal{R}^{\Delta\infty}$.

Corollary 3.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically simple;

(2) $\mathcal{R}$ is quasi-paratopologically simple.

Now, we only have to study paratopologically simple relators. For this, we introduce two operations on families of sets.

Definition 3.10. If $\mathcal{A} \subset \mathcal{P}(X)$, then we write

$$\mathcal{A}^{*} = \{B \subset X : \exists A \in \mathcal{A} : A \subset B\}.$$

Note that the notion of the uniform refinement of relators is a special case of the above definition. As a useful reformulation of the above definition, we can at once state the following

Proposition 3.11. *If $\mathcal{A} \subset \mathcal{P}(X)$, then*

$$\mathcal{A}^{*} = \{B \subset X : \mathcal{A} \cap \mathcal{P}(B) \neq \emptyset\}.$$

Definition 3.12. If $\mathcal{A} \subset \mathcal{P}(X)$, then we write

$$\mathcal{A}^{\diamond} = \{B \in \mathcal{A} : A \in \mathcal{A}, A \subset B \implies A = B\}.$$

Note that thus $\mathcal{A}^\diamond$ is just the family of the minimal members of $\mathcal{A}$. As a useful reformulation of the above definition, we can at once state the following

Proposition 3.13. *If $\mathcal{A} \subset \mathcal{P}(X)$, then*

$$\mathcal{A}^\diamond = \big\{B \subset X : \mathcal{A} \cap \mathcal{P}(B) = \{B\}\big\}.$$

Definition 3.14. If $\mathcal{A}$ is an ascending family in $\mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{A}$ such that for each $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $B \subset A$, then $\mathcal{B}$ will be called a base for $\mathcal{A}$.

Theorem 3.15. *If $\mathcal{A}$ is an ascending family in $\mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{B}$ is a base for $\mathcal{A}$; *(2) $\mathcal{A} \subset \mathcal{B}^*$;* *(3) $\mathcal{A} = \mathcal{B}^*$.*

Theorem 3.16. *If $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B}$ is a base for $\mathcal{A}$, then $\mathcal{A}^\diamond = \mathcal{B}^\diamond$.*

Theorem 3.17. *If $\mathcal{A} \subset \mathcal{P}(X)$ such that each nonvoid chain contained in $\mathcal{A}$ has a minimal element, then $\mathcal{A}^* = \mathcal{A}^{\diamond *}$.*

Corollary 3.18. *If $\mathcal{A}$ is an ascending subfamily of $\mathcal{P}(X)$ such that each nonvoid chain contained in $\mathcal{A}$ has a minimal element, then $\mathcal{A}^\diamond$ is the smallest base for $\mathcal{A}$.*

Lemma 3.19. *If $A, B \subset X$ such that $B \neq \emptyset$, then* $\operatorname{card}(B) \leq \operatorname{card}(A)$ *if and only if there exists a function f of A onto B.*

Now, we can state a theorem of Jenő Deák which was originally stated in terms of the minimum of the cardinalities of the bases of the family $\mathcal{E}_\mathcal{R}$.

Theorem 3.20. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically simple;

(2) $\mathcal{E}_\mathcal{R}$ has a base $\mathcal{B}$ with $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(X)$.

Theorem 3.21. *If $\mathcal{R}$ is a relator on X such that each nonvoid chain contained in $\mathcal{E}_\mathcal{R}$ has a minimal element, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically simple;

(2) $\operatorname{card}(\mathcal{E}_\mathcal{R}^\diamond) \leq \operatorname{card}(X)$;

(3) $\operatorname{card}\big(\{R(x) : x \in X, R \in \mathcal{R}\}^\diamond\big) \leq \operatorname{card}(X)$.

The above theorem allows us to easily check the main assertion of the following

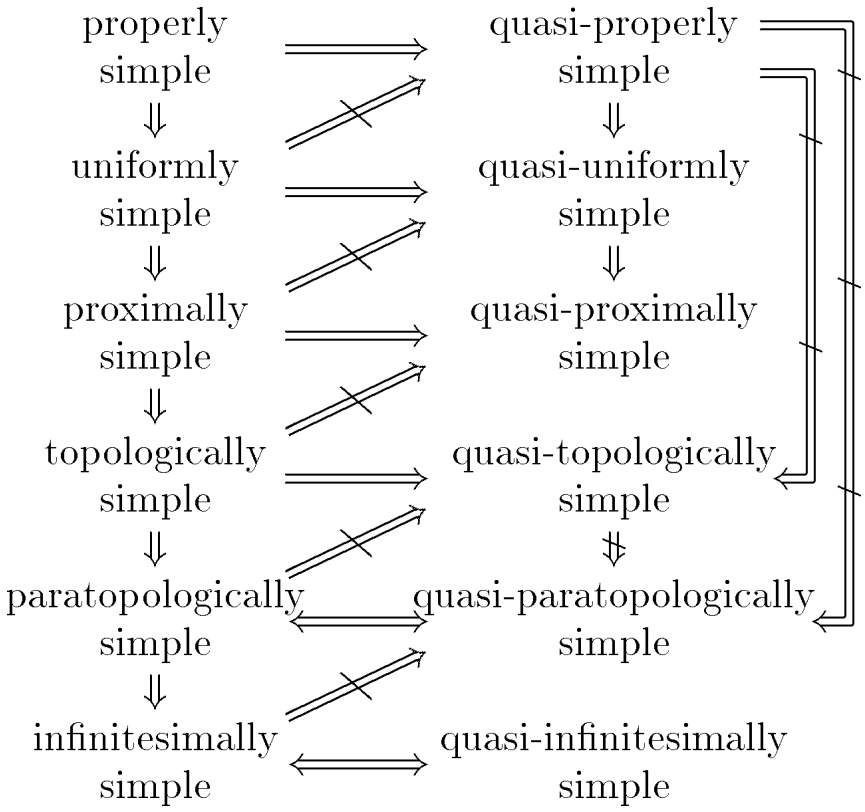
Example 3.22. If $X = \{1, 2, 3, 4\}$ and $R_i \subset X^2$ for all $i = 1, 2, 3$ such that

$$\begin{aligned} R_1(1) = R_1(2) &= \{1, 2\}, & R_1(3) = R_1(4) &= \{3, 4\}; \\ R_2(1) = R_2(3) &= \{1, 3\}, & R_2(2) = R_2(4) &= \{2, 4\}; \\ R_3(1) = R_3(4) &= \{1, 4\}, & R_3(2) = R_3(3) &= \{2, 3\}; \end{aligned}$$

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is an equivalence relator on X such that $\mathcal{R}$ is not paratopologically simple.

Remark 3.23. The above example substantially improves an example of Jenő Deák [11], which gives only a non-paratopologically simple, non-reflexive relator on a four element set.

The following diagram shows the main implications among the various simplicity properties of relators.



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