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Symmetric spectral synthesis

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Dedicated to the memory of Professor János Aczél.

Abstract. According to the famous and pioneering result of Laurent Schwartz, any closed translation invariant linear space of continuous functions on the reals is synthesizable from its exponential monomials. Due to a result of D. I. Gurevič there is no straightforward extension of this result to higher dimensions. Following Székelyhidi (Acta Math Hungar 153(1):120–142, 2017), with the aid of Gelfand pairs and K-spherical functions, K-synthesizability of K-varieties can be described. In this paper we contribute to this direction in the special case when K is the symmetric group of order d.

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1. Introduction

Spectral analysis and synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. A fundamental problem is to discover the structure of such spaces of functions, or more precisely, to find the appropriate class of basic functions, the so-called building blocks that serve as ‘typical elements’ of the space, i.e., a kind of basis. These turn out be the so-called exponential monomials. Consider the space of all complex-valued continuous functions on the real line with respect to the pointwise linear

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operations and to the topology of uniform convergence on compact sets. Suppose that a closed translation invariant subspace of this space is given. This subspace may or may not contain an exponential monomial. If it does, then we say that *spectral analysis* holds for this subspace. In such a situation, the exponential function itself is contained in this subspace. The complex number, characterizing this exponential function, can be considered as a *spectral value*. The complete description of the subspace, however, means that all the exponential monomials corresponding to the spectral exponentials and their multiplicities *characterize* the subspace: their linear hull is dense in this subspace. If so, we say that *spectral synthesis holds for this subspace*.

In his pioneering paper [5], Schwartz proved that any closed translation invariant linear space of continuous functions on the reals is synthesizable from its exponential monomials. The above construction can obviously be generalized: instead of the topological group of the reals, the set of all continuous, complex-valued functions on a locally compact Abelian group (equipped with the pointwise linear operation and with the topology of uniform convergence on compact sets) can be considered. Furthermore, exponential functions and exponential polynomials should also be defined in this setting. After this, the problems of spectral analysis and synthesis can be formulated: is it true that any nonzero closed translation invariant subspace of the above space contains an exponential (spectral analysis), and is it true that in any subspace of this kind, the linear hull of all exponential monomials is dense (spectral synthesis)?

The case of discrete Abelian groups is an interesting special case, which is the main aim of the monograph [9]. While the proof of Schwartz's result uses complex function theory (which is unavailable in the theory of topological groups), in the discrete case classical theorems on Artin and Noether rings are incorporated resulting in a method that depends on the annihilators of varieties, see [10], as well.

In the above-mentioned paper Schwartz asked whether the investigated spaces have this property when $n > 1$. In [2] Gurevič provided counterexamples of spectral analysis and synthesis for $n \geq 2$ for the spaces $\mathscr{E}^n$ (all infinitely many differentiable functions on $\mathbb{R}^n$), $\mathscr{H}$ (all entire functions over $\mathbb{C}^n$) and $\mathscr{D}'$ (distributions over $\mathbb{R}^n$). In particular, he also constructed a homogeneous system of convolution type equations such that exponential polynomial solutions of this system are not dense in the space of all their solutions.

To prove affirmative results for higher dimensional cases, in [11] Székelyhidi replaced 'translation invariance' by 'invariance with respect to some compact group of automorphisms'. For this purpose, the basic concepts and notions of spectral analysis and synthesis need to be introduced with the help of Gelfand pairs. The role of exponentials then is played by spherical functions. After elaborating the general theory, in the special case where the basic group is $\mathbb{R}^n$ and the compact group of automorphisms is $\mathrm{SO}(n)$, the main result below (see [11]) is considered as an extension.

Theorem. *Let n be a positive integer and $K = \mathrm{SO}(n)$ acting on $\mathbb{R}^n$. Then K-spectral synthesis holds on $\mathbb{R}^n$.*

2. The spaces $\mathscr{C}(G)$ and $\mathscr{M}_c(G)$

In what follows $\mathbb{C}$ denotes the set of complex numbers. If X is a locally compact Hausdorff topological space, then $\mathscr{C}(X)$ stands for the locally convex topological vector space of all continuous complex valued functions on X equipped with the pointwise linear operations and with the *compact-open topology*, or *topology of compact convergence*. The topological dual $\mathscr{C}(X)^*$ of $\mathscr{C}(X)$ is identified with the space $\mathscr{M}_c(X)$ of all compactly supported complex Borel measures on X. For the sake of simplicity elements of the space $\mathscr{C}(X)$ will be called *functions* and elements of the space $\mathscr{M}_c(X)$ will be called *measures* on X. The pairing between $\mathscr{M}_c(X)$ and $\mathscr{C}(X)$ will be denoted by

$$\langle \mu, f \rangle = \int_X f \, d\mu$$

whenever μ is a measure and f is a function.

For each subset H of $\mathscr{C}(X)$ the *orthogonal complement* of H in $\mathscr{M}_c(X)$, respectively, of a subset I of $\mathscr{M}_c(X)$ in $\mathscr{C}(X)$, is defined by

$$H^\perp = \{\mu \in \mathscr{M}_c(X) \,|\, \langle \mu, h \rangle = 0 \text{ for each}\, h \,\text{in}\, H\},$$

and

$$I^\perp = \{f \in \mathscr{C}(X) \,|\, \langle \mu, h \rangle = 0 \text{ for each } \mu \, in \, I\},$$

respectively.

We note that if X is discrete, then $\mathscr{C}(X)$ is the space of all complex valued functions on X equipped with the linear operations and with the topology of pointwise convergence, and $\mathscr{M}_c(X)$ is identified with the space of all finitely supported complex valued functions on X. For more details about $\mathscr{C}(X)$ and $\mathscr{M}_c(X)$ we shall refer to [10, 11].

3. Invariant functions and measures

Suppose that G is a locally compact topological group. In this case, $\mathscr{M}_c(G)$ is a unital complex algebra with the convolution of measures defined as

$$\langle \mu * \nu, f \rangle = \int_G \int_G f(xy) \, d\mu(x) \, d\nu(y)$$

for each of the measures μ, ν and function f. The identity of this algebra is δ_e, where e denotes the identity element in G, and, in general, δ_x is the point mass supported at the element x of G. In fact, $\mathscr{M}_c(G)$ is an involutive algebra with the *involution* $\mu \mapsto \check{\mu}$ defined by

$$\langle \check{\mu}, f \rangle = \langle \mu, \check{f} \rangle,$$

where, for each function f, $\check{f}$ is the *inversion* of f defined by $\check{f}(x) = f(x^{-1})$ for each x in G. More generally, for a subset H in $\mathscr{C}(G)$ and for a subset I in $\mathscr{M}_c(G)$ we write

$$\check{H} = \left\{ \check{f} \,|\, f \in H \right\} \quad \text{and} \quad \check{I} = \{ \check{\mu} \,|\, \mu \in I \}.$$

The function space $\mathscr{C}(G)$ is a topological module over the algebra of measures with respect to both the left and right action of $\mathscr{M}_c(G)$ defined as

$$\mu * f(x) = \int_G f(y^{-1}x)\, d\mu(y), \quad f * \mu(x) = \int_G f\left(xy^{-1}\right)\, d\mu(y)$$

for each x in G.

Let G be a locally compact topological group and $K \subseteq G$ a compact subgroup with normalized Haar measure ω_K; we recall that ω_K is left and right translation invariant and inversion invariant. We consider ω_K as a measure on G by the obvious extension

$$\langle \omega_K, f \rangle = \int_K f\, d\omega_K$$

for every function f. Then ω_K is in $\mathscr{M}_c(G)$. We call a function f *K-invariant*, if $f(kxl) = f(x)$ holds for each x in G and k, l in K. Clearly, this is equivalent to the property $f = \omega_K * f * \omega_K$. It is easy to see that all K-invariant functions form a closed subspace $\mathscr{C}_K(G)$ in $\mathscr{C}(G)$. The topological dual $\mathscr{C}_K(G)^*$ of the locally compact topological vector space $\mathscr{C}_K(G)$ plays a fundamental role in this paper. We shall identify this space as a weak*-closed subspace of $\mathscr{M}_c(G)$. The measure μ will be called *K-invariant*, if $\mu = \omega_K * \mu * \omega_K$ holds. Clearly, the space of all K-invariant measures is a weak*-closed subspace in $\mathscr{M}_c(G)$, which we denote by $\mathscr{M}_{c,K}(G)$.

Theorem 1. *The measure μ is K-invariant if and only if*

$$\int_G \left[\int_K \int_K f(kxl)\, d\omega_K(k)\, d\omega_K(l) \right] d\mu(x) = \int_K f\, d\omega_K \tag{1}$$

holds for each function f.

Proof. Clearly, the left side of (1) is

$$\langle \omega_K * \mu * \omega_K, f \rangle,$$

and the right side is $\langle \omega_K, f \rangle$. □

The function $x \mapsto \int_K \int_K f(kxl)\, d\omega_K(k)\, d\omega_K(l)$ in the square bracket in the above integral is nothing but $\omega_K * f * \omega_K$, and it is obviously continuous and K-invariant. We denote it by $f^\#$ and it is called the *K-projection* of f. Also, the measure $\omega_K * \mu * \omega_K$ is called the *K-projection* of μ whenever μ is an arbitrary measure. The following theorem is easy to prove.

Theorem 2. *The mapping $f \mapsto f^\#$ is a continuous linear mapping of $\mathscr{C}(G)$ onto $\mathscr{C}_K(G)$. For each f in $\mathscr{C}(G)$ we have $f^{\#\#} = f^\#$. The function f in $\mathscr{C}(G)$ is K-invariant if and only if $f^\# = f$.*

Now we identify the dual of $\mathscr{C}_K(G)$.

Theorem 3. (see [11, Theorem 2]) *Let G be a locally compact topological group, and $K \subseteq G$ a compact subgroup. The topological dual $\mathscr{C}_K(G)^*$ of the space of all K-invariant functions is topologically isomorphic to the space of K-invariant measures.*

Proof. Let λ be an element in $\mathscr{C}_K(G)^*$, that is, a continuous linear functional on $\mathscr{C}_K(G)$. We define $\Lambda\colon \mathscr{C}(G) \to \mathbb{C}$ as

$$\Lambda(f) = \langle \lambda, f^\# \rangle$$

for each f in $\mathscr{C}(G)$. Clearly, Λ is a continuous linear functional on $\mathscr{C}(G)$, hence there is a unique measure μ_λ in $\mathscr{M}_c(G)$ such that

$$\lambda(f) = \Lambda(f) = \langle \mu_\lambda, f \rangle$$

holds for each f in $\mathscr{C}_K(G)$. We show that μ_λ is a K-invariant measure. We have for each f in $\mathscr{C}(G)$:

$$\langle \mu_\lambda, f \rangle = \Lambda(f) = \langle \lambda, f^\# \rangle = \langle \lambda, f^{\#\#} \rangle = \Lambda(f^\#) = \langle \mu_\lambda, f^\# \rangle,$$

hence μ_λ is K-invariant, by Theorem 1. Clearly, the mapping $\lambda \mapsto \mu_\lambda$ is linear, continuous and open. Obviously, every K-invariant measure is a continuous linear functional on $\mathscr{C}_K(G)$, which proves surjectivity. To prove injectivity we assume that $\langle \mu_\lambda, f \rangle = 0$ for each f in $\mathscr{C}_K(G)$—then Λ vanishes on $\mathscr{C}_K(G)$, consequently $\lambda = 0$. The proof is complete. □

The following result is an easy consequence of Theorem 3.

Theorem 4. *The mapping $\mu \mapsto \mu^\#$ is a continuous linear mapping of $\mathscr{M}_c(G)$ onto $\mathscr{M}_{c,K}(G)$. For each μ in $\mathscr{M}_c(G)$ we have $\mu^{\#\#} = \mu^\#$. The measure μ in $\mathscr{M}_c(G)$ is K-invariant if and only if $\mu^\# = \mu$.*

It turns out that $\mathscr{M}_{c,K}(G)$ is not just a weak*-closed subspace of $\mathscr{M}_c(G)$, but it is also a subalgebra, as it is shown by the following theorem.

Theorem 5. *The space $\mathscr{M}_{c,K}(G)$ is a subalgebra of $\mathscr{M}_c(G)$.*

Proof. We need to show that the convolution of K-invariant measures is K-invariant. First we note that

$$\omega_K * \omega_K = \omega_K$$

holds, by the left and right invariance of ω_K. Applying this we can calculate as follows: for every μ, ν in $\mathscr{M}_{c,K}(G)$

$$\begin{aligned}\omega_K * (\mu * \nu) * \omega_K &= \omega_K * [(\omega_K * \mu * \omega_K) * (\omega_K * \nu * \omega_K)] * \omega_K \\ &= (\omega_K * \omega_K) * \mu * (\omega_K * \omega_K) * \nu * (\omega_K * \omega_K) \\ &= \omega_K * \mu * (\omega_K * \omega_K) * \nu * \omega_K \\ &= (\omega_K * \mu * \omega_K) * (\omega_K * \nu * \omega_K) = \mu * \nu,\end{aligned}$$

hence $\mu * \nu$ is K-invariant. □

Theorem 6. *The space $\mathscr{C}_K(G)$ is a topological vector module over the algebra $\mathscr{M}_{c,K}(G)$ under the action $(\mu, f) \mapsto \mu * f$ defined by*

$$\mu * f(x) = \int_G f\left(y^{-1}x\right)\, d\mu(y)$$

for each x in G.

Proof. The only statement we have to prove is that $\mu * f$ is a K-invariant function for each μ in $\mathscr{M}_{c,K}(G)$ and f in $\mathscr{C}_K(G)$. Clearly, if f is K-invariant, then $\check{f}$ is K-invariant, as well, and we have

$$\begin{aligned}\mu * f(kxl) = \mu * f^{\#}(kxl) &= \int_G f^{\#}\left(y^{-1}kxl\right)\, d\mu(y) \\ &= \int_G \widecheck{f^{\#}}\left(x^{-1}k^{-1}y\right)\, d\mu(y) = \int_G \left(\check{f}\right)^{\#}\left(x^{-1}k^{-1}y\right)\, d\mu(y) \\ &= \int_G\int_K\int_K \check{f}\left(x^{-1}k^{-1}l_1yl_2\right)\, d\omega_K(l_1)\, d\omega_K(l_2)\, d\mu(y) \\ &= \int_G\int_K\int_K \check{f}\left(x^{-1}kyl\right)\, d\omega_K(k)\, d\omega_K(l)\, d\mu(y) \\ &= \int_G (\check{f})^{\#}(x^{-1}y)\, d\mu(y) = \int_G \check{f}(x^{-1}y)\, d\mu(y) \\ &= \int_G f(y^{-1}x)\, d\mu(y) = \mu * f(x).\end{aligned}$$

□

Given the locally compact topological group G and the compact subgroup $K \subseteq G$ we say that (G, K) is a *Gelfand pair*, if the algebra $\mathscr{M}_{c,K}(G)$ of all K-invariant measures is commutative.

4. Spherical spectral analysis and synthesis

Let (G,K) be a Gelfand pair. For each y in G we introduce the K-*translation with increment* y as the linear mapping $\tau_y^{\#}\colon \mathscr{C}(G)\to\mathscr{C}(G)$ defined by

$$\tau_y^{\#}f(x)=\delta_{y^{-1}}^{\#}*f(x)=\int_G f\left(z^{-1}x\right)\,d\delta_{y^{-1}}^{\#}(z)$$
$$=\int_G\int_K\int_K f(kylx)\,d\omega_K(k)\,d\omega_K(l).$$

In particular, if f is a K-invariant function, then

$$\tau_y^{\#}f(x)=\int_K f(ykx)\,d\omega_K(k).$$

The following result is known (see [11, Theorem 7]).

Theorem 7. *Let G be a locally compact topological group and $K\subseteq G$ a compact subgroup. Then (G,K) is a Gelfand pair if and only if all K-translations form a commuting family of linear operators on $\mathscr{C}_K(G)$.*

A closed linear subspace of $\mathscr{C}_K(G)$ is called a K-*variety*, if it is invariant with respect to all K-translations. By Theorem 7 in [11], it turns out that K-varieties are exactly the closed submodules of the module $\mathscr{C}_K(G)$ over $\mathscr{M}_{c,K}(G)$. The intersection of all K-varieties containing the given function f in $\mathscr{C}_K(G)$ is called the K-*variety of* f, and it is denoted by $\tau_K(f)$.

The smallest nonzero K-variety is one dimensional: it is spanned by a single function. If we assume that the function s in $\mathscr{C}_K(G)$ spans a one dimensional K-variety, and s is normalized, i.e. $s(e)=1$, then s is called a K-*spherical function*. We have the following characterization of K-spherical functions:

Theorem 8. *Let (G,K) be a Gelfand pair. The nonzero K-invariant function s is a K-spherical function if and only if it is normalized and*

$$\int_K s(xky)\,d\omega_K(k)=s(x)s(y). \tag{2}$$

Proof. Equation (2) can be written in the form

$$\tau_y^{\#}s=s(y)s, \tag{3}$$

which shows that if s is nonzero, and satisfies (2), then $\tau_K(s)$ is one dimensional. On the other hand, putting $x=e$ in (2) we have $s=s(e)s$, hence $s(e)=1$.

Conversely, if s is a K-spherical function, then

$$\tau_y^{\#}s(x)=\lambda(y)s(x)$$

holds for each x,y in G with some function $\lambda\colon G\to\mathbb{C}$. Putting $x=e$ we have $\lambda=s$, hence s satisfies (2). □

Corollary 1. *Let (G, K) be a Gelfand pair. The K-spherical functions are exactly the joint normalized K-invariant eigenfunctions of all K-translations.*

The existence of K-spherical functions in K-varieties is the so-called K-spectral analysis property. More exactly, we say that *K-spectral analysis* holds for a given K-variety V, if every nonzero K-subvariety of V contains a K-spherical function.

Theorem 9. *Let (G, K) be a Gelfand pair. Then K-spectral analysis holds for every finite dimensional K-variety.*

Proof. Let V be a nonzero finite dimensional K-variety. Then V is a joint invariant subspace of all K-translations. As all K-translations form a commuting family of linear operators on V, by linear algebra, they have a joint eigenfunction in V. □

We introduce the following K-invariant measure: given the K-spherical function s for each y in G we let

$$\mathrm{D}_{s;y} = \delta^{\#}_{y^{-1}} - s(y)\delta_e,$$

which is called the *modified K-difference with increment y corresponding to s*, or simply *modified K-difference*. Iterated modified K-differences will be written as

$$\mathrm{D}_{s;y_1,y_2,\dots,y_{n+1}} = \prod_{j=1}^{n+1} \mathrm{D}_{s;y_j},$$

where Π stands for convolution product.

Let s be a K-spherical function. The K-invariant function f is called a *generalized s-monomial*, if there exists a natural number n such that

$$\mathrm{D}_{s;y_1,y_2,\dots,y_{n+1}} * f = 0 \tag{4}$$

holds for each $y_1, y_2, \dots, y_{n+1}$ in G. If f is nonzero, then s is unique, and the smallest n which satisfies (4) is called the *degree* of f. If the K-variety of a generalized s-monomial is finite dimensional, then we simply call it an *s-monomial*. The function f is called a (*generalized*) *K-monomial*, if it is a (generalized) s-monomial for some K-spherical function s, and linear combinations of (generalized) K-monomials are called (*generalized*) *K-polynomials*.

We say that a given K-variety V is *K-synthesizable*, if all K-monomials in V span a dense subspace. We say that *K-spectral synthesis* holds for a K-variety V, if every K-subvariety of V is synthesizable. It is known (see [11]) that if an s-monomial is in a K-variety, then s itself belongs to the variety, hence K-spectral synthesis for a K-variety implies its K-spectral analysis.

The case when $K = \{e\}$ is the trivial subgroup attracts special attention. In this case the spherical concepts reduce to the classical notions of spectral synthesis. Obviously, $(G, \{e\})$ is a Gelfand pair if and only if G is commutative.

In this case all functions and measures are K-invariant and we omit 'K' from the names of the above concepts. Also, we simply talk about *spectral analysis, synthesizability, spectral synthesis* without the specification 'K'. In this case the $\{e\}$-spherical functions and (generalized) $\{e\}$-monomials are traditionally called *exponentials*, resp. (*generalized*) *exponential monomials*, and if the exponential is the identically 1 function, then the exponential monomials are simply called (*generalized*) *polynomials*. In this case the mapping $\mu \longmapsto \mu^{\#}$ reduces to the identity mapping and we omit # in the notation, and we write $\Delta_{m;y}$ instead of $\mathrm{D}_{m;y}$.

The polynomials of degree 1, that is, the solutions of the functional equation

$$\Delta_{1,y,z} * f(x) = 0$$

are called *additive functions* if $f(0) = 0$. They are characterized by the equation

$$a(x + y) = a(x) + a(y) \tag{5}$$

for each x, y in G. Let d be a positive integer, a function $A\colon\ G^d \to \mathbb{C}$ is called *d-additive* on G, if it is additive in each variable, while the other variables are kept fixed.

For our later purposes a characterization theorem of exponential polynomials based on the notion of decomposability will be useful (see [3,6]). Let G be a group and $n \geq 2$ an integer. A function $F\colon\ G^n \to \mathbb{C}$ is said to be *decomposable* if it can be written as a finite sum of products $F_1 \cdot F_2 \dots F_k$, where for all $i, j = 1, \dots, k$, $i \neq j$ the function F_j does not depend on the variables of the functions F_i and also conversely.

Remark. Without loss of generality we can suppose that $k = 2$ in the above definition, that is, decomposable functions are mappings that can be written in the form

$$F\left(x_1, x_2, \dots, x_n\right) = \sum_E \sum_j A_j^E B_j^E$$

where E runs through all non-void proper subsets of $\{1, 2, \dots, n\}$ and for each E and j the function A_j^E depends only on the variables x_i with i in E, while B_j^E depends only on the variables x_i with i not in E.

Theorem 10. *Let G be a commutative topological group. A function f in $\mathscr{C}(G)$ is a generalized exponential polynomial if and only if there is a positive integer $n \geq 2$ such that the mapping*

$$\left(x_1, x_2, \dots, x_n\right) \longmapsto f\left(x_1 + x_2 + \dots + x_n\right)$$

is decomposable.

5. Semidirect products

The concept of K-spectral analysis and synthesis has been introduced in the situation where K is a compact subgroup of a locally compact topological group G, and (G, K) forms a Gelfand pair. The K-varieties are objects which are invariant with respect to K-translations and they will be analyzed and synthesized using spherical monomials as building blocks. Another important situation is if a locally compact topological group N is given and we consider objects which are invariant with respect to the action of a group of automorphisms of N. Now we show how to cover this situation using the previous ideas and the concept of semidirect products of topological groups. Here we restrict ourselves to the case which is comfortable for our purposes in this paper. For the details and proofs of the forthcoming statements the reader should consult [11].

Let N be a commutative topological group and $K \subseteq \operatorname{Aut} N$ a compact subgroup of the automorphism group of N. The group operation in N will be denoted by $+$, and the topology on $\operatorname{Aut} N$ is the compact-open topology. We consider the semidirect product of N and K: $G = N \rtimes K$, which is a locally compact topological group, the group operation is defined as

$$(x, k) \cdot (y, l) = (ky + x, k \circ l)\,,$$

the identity is $(0, \mathrm{id})$, and the inverse of (x, k) is $(-k^{-1}x, k^{-1})$. Then K is topologically isomorphic to the compact subgroup $\{(0, k)\colon k \in K\}$ of G, and they will be identified. Moreover, N is topologically isomorphic to the closed normal subgroup $\{(x, \mathrm{id})\colon x \in N\}$ of G, and they will be identified. It is easy to see that the function $f\colon G \to \mathbb{C}$ is K-invariant if and only if it has the form

$$f(x, k) = \varphi(x)$$

and

$$\varphi(kx) = \varphi(x)$$

for each (x, k) in G, hence $\mathscr{C}_K(G)$ will be identified with $\mathscr{C}_r(N)$, the subspace of all *K-radial functions* in $\mathscr{C}(N)$. (For the terminology see [11].) The topological dual of $\mathscr{C}_r(N)$ is $\mathscr{M}_{c,r}(N)$, the space of *K-radial measures*, which is the closed subspace of $\mathscr{M}_c(N)$ consisting of measures μ satisfying

$$\int_N f(kx)\, d\mu(x) = \int_N f(x)\, d\mu(x)$$

for each f in $\mathscr{C}(G)$ and k in K. Hence the topological algebra $\mathscr{M}_{c,K}(G)$ of K-invariant measures on G is identified with the algebra of K-radial measures, where the convolution coincides with the usual convolution in $\mathscr{M}_c(N)$, and also the convolution between $\mathscr{M}_{c,K}(G)$ and $\mathscr{C}_K(G)$ corresponds to the ordinary convolution between $\mathscr{M}_c(N)$ and $\mathscr{C}(N)$. Consequently, (G, K) is a Gelfand pair. It follows that in this situation we shall deal with K-radial functions and measures on N, K-spherical functions and K-monomials will be considered as functions on N, and K-translation can be expressed as

$$\tau_y^{\#} f(x) = \int_K f(x+ky)\, d\omega_K(k)$$

whenever f is a K-radial function, and x, y are in N. K-spherical functions are characterized by the conditions

$$\int_K s(x+ky)\, d\omega_K(k) = s(x)s(y), \quad s(0) = 1 \tag{6}$$

for each x, y in N.

6. Symmetric functions and measures

In this paper we have the following setting. Let N be a locally compact Abelian group, d a positive integer, and we consider the permutation group $\mathscr{P}_d$ acting on N^d according to the rule

$$\sigma(x) = \sigma\left(x_1, x_2, \dots, x_d\right) = \left(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)}\right)$$

for each element $\sigma\colon \{1, 2, \dots, d\} \to \{1, 2, \dots, d\}$ of $\mathscr{P}_d$ and for any $x = (x_1, x_2, \dots, x_d)$ in N^d. Then $\mathscr{P}_d$ is a finite, hence compact subgroup of $\operatorname{Aut} N^d$. The $\mathscr{P}_d$-invariant measures and functions will be called *symmetric*. The spaces of continuous symmetric functions, resp. symmetric measures will be denoted by $\mathscr{C}_d(N^d)$, resp. $\mathscr{M}_{c,d}(N^d)$. As N is commutative, $(N^d \rtimes \mathscr{P}_d, \mathscr{P}_d)$ is a Gelfand pair, as we established above. We shall use the notation $G = N^d \rtimes \mathscr{P}_d$. In this context we shall modify our above terminology and use the terms *symmetric translation*, *symmetric variety*, *symmetric spherical function*, *symmetric spectral synthesis*, etc. The normalized Haar measure on $\mathscr{P}_d$ will be denoted by ω_d and it has the form

$$\int_{\mathscr{P}_d} \varphi\, d\omega_d = \frac{1}{d!} \sum_{\sigma \in \mathscr{P}_d} \varphi(\sigma)$$

whenever $\varphi\colon \mathscr{P}_d \to \mathbb{C}$ is a function. Accordingly, the symmetric projection of the function $f\colon N^d \to \mathbb{C}$ has the form

$$f^{\#}(x) = \frac{1}{d!} \sum_{\sigma \in \mathscr{P}_d} f(\sigma x)$$

for each x in N^d, and the symmetric projection of the measure μ has the form

$$\langle \mu^{\#}, f \rangle = \frac{1}{d!} \sum_{\sigma \in \mathscr{P}_d} \int_{N^d} f(\sigma x)\, d\mu(x)$$

for each function $f\colon N^d \to \mathbb{C}$.

7. Description of symmetric spherical monomials

In this section we describe all symmetric monomials on N^d. We shall use the following notation: for $d \geq 2$ and $1 \leq k \leq d$ we write $[x]_k$ for the element of N^k whose components are all equal to x. Hence $[x]_1 = x$, $[x]_2 = (x, x)$, etc. If $A : N^d \to \mathbb{C}$ is a d-additive function and it is symmetric in its variables, then the function $x \mapsto A([x]_d)$ is called its *diagonalization.*

Theorem 11. [1,4] *Let N be a commutative locally compact topological group and d a positive integer. Every symmetric spherical function s on N^d has the form $s = m^{\#}$ with some exponential m on N^d. In other words,*

$$s(x) = \frac{1}{d!} \sum_{\sigma \in \mathscr{P}_d} m(\sigma x)$$

for each x in N^d.

Lemma 1. [4] *A function $\varphi\colon N^d \to \mathbb{C}$ fulfills*

$$\mathrm{D}_{1;y} * \varphi(x) = 0$$

that is,

$$\int_{\mathscr{P}_d} \varphi(x + \sigma y) d\omega_d(\sigma) = \varphi(x)$$

for each x, y in N^d if and only if for each $k = 1, 2, \ldots, d! - 1$ there exist symmetric, k-additive functions $A_k\colon (N^d)^k \to \mathbb{C}$ and a complex constant A_0 such that, for each x, y in N^d we have

$$\varphi(x) = A_0 + \sum_{k=1}^{d!-1} A_k([x]_k)$$

and

$$\int_{\mathscr{P}_d} A_k \left([x]_{k-i}, [\sigma y]_i\right) d\omega_d(\sigma) = 0.$$

Remark. We remark that using some results of [7] we immediately get that symmetric 1-monomials are symmetric generalized polynomials. Indeed, if $\varphi\colon N^d \to \mathbb{C}$ is a symmetric 1-monomial, then there exists a nonnegative integer n such that

$$D_{1;y_1,y_2,\ldots,y_{n+1}} * \varphi(x) = 0$$

holds for any $x, y_1, \ldots, y_{n+1}$ in N^d. Clearly, in this case the less restrictive equation,

$$D_{1;y}^{n+1} * \varphi(x) = 0$$

is also satisfied for each x, y in N^d. After computing the symmetric modified differences we obtain that a functional equation of the form

$$\varphi(x) + \sum_{i=1}^{k} \varphi_i\left(x + \psi_i(y)\right) = 0$$

holds for each x, y in N^d yielding that φ is *of degree* n, which, in view of Theorem 3.6 of [7], immediately implies that φ is a generalized polynomial. Here we have for each x in N^d and for $i = 1, 2, \ldots, k$:

$$\varphi_i(x) = \alpha_i \varphi(x) \quad \text{and} \quad \psi_i(x) = \beta_i \sigma_i x \qquad \left(x \in G^d\right)$$

with certain complex constants α_i, β_i.

In [4] Theorem 2.10 describes the solutions of the functional equation

$$D_{s;y} * \varphi(x) = 0,$$

where x, y are in N^d. More exactly we have the following result:

Theorem 12. *The non-zero functions $\varphi\colon N^d \to \mathbb{C}$ and $s\colon N^d \to \mathbb{C}$ satisfy the functional equation*

$$\int_{\mathscr{P}_d} \varphi(x + \sigma y) d\omega_d(\sigma) = s(y)\varphi(x)$$

for each x, y in N^d if and only if there exists an exponential $m\colon N^d \to \mathbb{C}$ such that $s = m^{\#}$ and there are complex constants A_0^σ and symmetric, k-additive functions A_k^σ such that for each x, y in N^d we have

$$\varphi(x) = \sum_{\sigma \in \mathscr{P}_d^*} m(\sigma x) \cdot \left(A_0^\sigma + \sum_{i=1}^{\left|\mathscr{P}_d^0\right| - 1} A_i^\sigma\left([x]_i\right) \right),$$

and

$$\sum_{\sigma \in \mathscr{P}_d^0} A_k^\sigma\left([x]_{k-i}, [\xi y]_i\right) = 0 \qquad \left(\xi \in \mathscr{P}_d^*, 1 \le i \le k \le \left|\mathscr{P}_d^0\right| - 1\right).$$

Here $\mathscr{P}_d^0 = \{\sigma \in \mathscr{P}_d \,|\, m \circ \sigma = m\}$, and $\mathscr{P}_d^*$ stands for the quotient group $\mathscr{P}_d / \mathscr{P}_d^0$.

Theorem 13. *Let n be a positive integer and $s\colon N^d \to \mathbb{C}$ be a symmetric spherical function. If a symmetric function $\varphi\colon N^d \to \mathbb{C}$ satisfies*

$$D_{s;y_1,y_2,\ldots,y_{n+1}} * \varphi(x) = 0 \tag{7}$$

for each $x, y_1, y_2 \ldots, y_{n+1}$ in N^d, then φ is a generalized exponential polynomial.

Proof. Let n be a positive integer, $s\colon\ N^d \to \mathbb{C}$ a symmetric spherical function, and assume that for the mapping $\varphi\colon\ N^d \to \mathbb{C}$ equation (7) holds for each $x, y_1, y_2, \ldots, y_{n+1}$ in N. In view of Theorem 11, there exists an exponential m on G^d such that $s = m^{\#}$.

On computing the symmetric modified differences, Eq. (7) implies that the function φ has to fulfill

$$\sum_{i=0}^{n+1} \sum_{1\leq j_1<\cdots<j_l\leq n+1} (-1)^{j_1+\cdots+j_l} s\left(y_{j_1}\right)\cdot s\left(y_{j_2}\right)\cdots s\left(y_{j_l}\right) \\ \cdot \sum_{\sigma_1\in\mathscr{P}_d} \cdots \sum_{\sigma_{j+1}\in\mathscr{P}_d} \varphi\left(x+\varepsilon_1\sigma_1\left(y_1\right)+\varepsilon_2\sigma_2\left(y_2\right)+\cdots+\varepsilon_{j+1}\sigma_{j+1}(y_{j+1})\right)=0 \tag{8}$$

for all $x, y_1, y_2, \ldots, y_{n+1}$ in N^d, where in each summand ε_i in $\{0, 1\}$ depends on whether i does or does not belong to the set $\{j_1, j_2, \ldots, j_l\}$. Using this equation, our point is to apply different substitutions for the variables $x, y_1, y_2, \ldots, y_{n+1}$ in N^d to arrive at the conclusion that the mapping

$$(x_1, x_2, \ldots, x_{n+2}) \longmapsto \varphi\circ\pi_1\left(x_1+x_2+\cdots+x_{n+2}\right)$$

is decomposable, and then we apply Theorem 10.

We introduce the following technical notation: if k and $1 \leq n_1 < n_2 < \cdots < n_k \leq d$ are positive integers, then $\pi_{n_1,n_2,\ldots,n_k}y$ denotes that element in N^d which may differ from y in only those coordinates which do not belong to the set $\{n_1, n_2, \ldots, n_k\}$, and those coordinates are zero. For instance $\pi_1 y = (y^{(1)}, 0, 0, \ldots, 0)$, or $\pi_{1,d} y = (y^{(1)}, 0, \ldots, 0, y^{(d)})$, etc., for each $y = (y^{(1)}, y_2, \ldots, y^{(d)})$ in N^d. In addition, let $\pi_0 y = 0$ for each y.

We have to consider the cases $n + 2 \geq d$ and $n + 2 < d$ separately. If $n + 2 \geq d$, then the substitutions

$$\pi_1 x \quad \text{for} \quad x, \quad \pi_{2,3,\ldots,d}y_1 \quad \text{for} \quad y_1, \quad \text{and} \quad \pi_0 y_j \quad \text{for} \quad y_j \quad \text{if} \quad 2\leq j\leq n+1$$

into (8) yield that the mapping φ can be expressed with the one-variable function $\varphi\circ\pi_1$ and the $(d-1)$-variable function $\varphi\circ\pi_{2,\ldots,d}$. The substitutions

$$\pi_1 x \quad \text{for} \quad x, \quad \pi_{3,4,\ldots,d}y_1 \quad \text{for} \quad y_1, \quad \text{and} \quad \pi_0 y_j \quad \text{for} \quad y_j \quad \text{if} \quad 2\leq j\leq n+1$$

into (8) show that $\varphi\circ\pi_{2,3,\ldots,d}$ can be expressed with the one-variable function $\varphi\circ\pi_1$, the $(d-2)$-variable function $\varphi\circ\pi_{3,4,\ldots,d}$, and the exponential m. Continuing this descending argument, we finally get that the mapping φ can be written with the aid of the one-variable function $\varphi\circ\pi_1$ and the exponential m.

On the other hand, if $n + 2 < d$, then the substitutions

$$\pi_1 x \quad \text{for} \quad x, \quad \pi_2 y_1 \quad \text{for} \quad y_1, \quad \ldots \quad, \pi_{n+1}y_n \quad \text{for} \quad y_n$$

and

$$\pi_{n+2,\dots,d}y_{n+1} \quad \text{for} \quad y_{n+1}$$

into (8) yield that the mapping φ can be expressed with the one-variable function $\varphi\circ\pi_1$ and the $(d-n-2)$-variable function $\varphi\circ\pi_{n+2,n+3,\dots,d}$. Further, the substitutions

$$\pi_1 x \quad \text{for} \quad x, \quad \pi_2 y_1 \quad \text{for} \quad y_1, \quad \dots \quad , \pi_{n+1}y_n \quad \text{for} \quad y_n$$

and

$$\pi_{n+2,n+3,\dots,d-1}y_{n+1} \quad \text{for} \quad y_{n+1}$$

into (8) show that $\varphi\circ\pi_{n+2,n+3,\dots,d}$ can be expressed with the one-variable function $\varphi\circ\pi_1$ and the $(d-n-3)$-variable function $\varphi\circ\pi_{n+2,n+3,\dots,d-1}$ and the exponential m. Continuing this descending argument, we finally get that the mapping φ can be expressed in terms of the one-variable function $\varphi\circ\pi_1$ and the exponential m, in this case, too.

Finally we conclude that the mapping φ can be expressed in terms of the one-variable function $\varphi\circ\pi_1$ and the exponential m, in this case, too. Writing this form back into Eq. (7) we obtain that

$$\begin{aligned}(\varphi\circ\pi_1)\left(x_1+x_2+\dots+x_{n+2}\right) = &\sum_{i=1}^{n+1}\Phi_{i,1}\left(x_i\right)\Psi_{i,n+1}\left(\sum_{j\neq i}x_j\right)\\ &+\sum_{i\neq j}\Phi_{i,j,2}\left(x_i+x_j\right)\cdot\Psi_{i,j,n}\left(\sum_{k\neq i,j}x_k\right)+\cdots\end{aligned}$$

for each $x_1,x_2,\dots,x_{n+2}$ in N^d, showing that the mapping

$$\left(x_1,x_2,\dots,x_{n+2}\right)\longmapsto\varphi\circ\pi_1\left(x_1+x_2+\dots+x_{n+2}\right)$$

is decomposable. Using Theorem 10, we obtain that $\varphi\circ\pi_1$ is a generalized exponential polynomial on N^d. Since φ is a (continuous) polynomial of $\varphi\circ\pi_1$, we get that φ is a generalized exponential polynomial, as well. The proof is complete. □

We shall use the following symmetrization operator: for each positive integer k and continuous function $F\colon (N^d)^k\to\mathbb{C}$ we write

$$\begin{aligned}&\operatorname{sym}_{u_1,u_2,\dots,u_k}F\left(u_1,u_2,\dots,u_k\right)\\ &\quad=\int_{\mathscr{P}_n}\dots\int_{\mathscr{P}_n}F\left(\sigma_1u_1,\sigma_2u_2,\dots,\sigma_ku_k\right)d\omega\left(\sigma_1\right)d\omega\left(\sigma_2\right)\dots d\omega\left(\sigma_k\right).\end{aligned}$$

The subscripts of sym indicate which variables are subjected to symmetrization. For instance, we have for each f in $\mathscr{C}(N^d)$ and exponential m on N^d:

$$\operatorname{sym}_{x,y}\left[f(x+y)-m(y)f(x)\right]$$

$$= \int_{\mathscr{P}_d}\int_{\mathscr{P}_d} \left[f\left(\sigma_1 x + \sigma_2 y\right) - m\left(\sigma_2 y\right) f\left(\sigma_1 x\right)\right] d\omega\left(\sigma_1\right)\, d\omega\left(\sigma_2\right)$$
$$= \int_{\mathscr{P}_d}\left(\int_{\mathscr{P}_d}\left[f\left(\sigma_1\left(x+\sigma_1^{-1}\sigma_2 y\right) - f\left(\sigma_1 x\right) m\left(\sigma_2 y\right)\right] d\omega\left(\sigma_1\right)\right) d\omega\left(\sigma_2\right)$$
$$= \int_{\mathscr{P}_d}\left[f^{\#}\left(x+\sigma y\right) - f^{\#}(x) m(\sigma y)\right] d\omega(\sigma)$$
$$= \int_{\mathscr{P}_d} f^{\#}\left(x+\sigma y\right) d\omega(\sigma) - f^{\#}(x) m^{\#}(y) = \mathrm{D}_{m^{\#};y} * f^{\#}(x),$$

or

$$\operatorname{sym}_{x,y}\left[\Delta_{m;y} * f(x)\right] = \mathrm{D}_{m^{\#};y} * f^{\#}(x)$$

for each x, y in N^d. More generally, the modified differences have the following remarkable property.

Proposition 1. *Let n be a natural number, $m\colon N^d \to \mathbb{C}$ an exponential and f in $\mathscr{C}(N^d)$. Then we have for each $x, y_1, y_2, \ldots, y_{n+1}$ in N*

$$\mathrm{D}_{m^{\#};y_1,y_2,\ldots,y_{n+1}} * f^{\#}(x) = \operatorname{sym}_{x,y_1,y_2,\ldots,y_{n+1}}\left[\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x)\right]. \quad (9)$$

Proof. Let $m\colon N^d \to \mathbb{C}$ be an exponential and f be in $\mathscr{C}(N^d)$. We prove the identity by induction on n. Due to the above remark, the statement holds true for $n = 1$.

Suppose now that there exists a natural number n such that the above identity holds for n, that is, we have

$$\mathrm{D}_{m^{\#};y_1,y_2,\ldots,y_n} * f^{\#}(x) = \operatorname{sym}_{x,y_1,y_2,\ldots,y_n}\left[\Delta_{m;y_1,y_2,\ldots,y_n} * f(x)\right].$$

for each $x, y_1, y_2, \ldots, y_n$ in N^d.

Let $x, y_1, y_2, \ldots, y_n, y_{n+1}$ be in N^d, then due to the induction hypothesis, we have

$$\mathrm{D}_{m^{\#};y_1,y_2,\ldots,y_n,y_{n+1}} * f^{\#}(x)$$
$$= \left(\mathrm{D}_{m^{\#};y_{n+1}} * \mathrm{D}_{m^{\#};y_1,y_2,\ldots,y_n}\right) * f^{\#}(x)$$
$$= \mathrm{D}_{m^{\#};y_{n+1}} * \left(\mathrm{D}_{m^{\#};y_1,y_2,\ldots,y_n}\right) * f^{\#}(x)$$
$$= \mathrm{D}_{m^{\#};y_{n+1}} * \left(\operatorname{sym}_{x,y_1,y_2,\ldots,y_n}\Delta_{m;y_1,y_2,\ldots,y_n} * f(x)\right)$$
$$= \left(\delta^{\#}_{-y_{n+1}} - m^{*}(y_{n+1})\delta_0\right) * \left(\operatorname{sym}_{x,y_1,y_2,\ldots,y_n}\Delta_{m;y_1,y_2,\ldots,y_n} * f(x)\right)$$
$$= \delta^{\#}_{-y_{n+1}} * \left(\int_{\mathscr{P}_n}\cdots\int_{\mathscr{P}_n}\Delta_{m;\sigma_1 y_1,\ldots,\sigma_n y_n} * f(\sigma x)\, d\omega(\sigma)\, d\omega(\sigma_1)\ldots\omega(\sigma_n)\right)$$
$$- \int_{\mathscr{P}_d} m\left(\sigma_{n+1} y_{n+1}\right) d\omega\left(\sigma_{n+1}\right)$$
$$\times\left(\int_{\mathscr{P}_n}\cdots\int_{\mathscr{P}_n}\Delta_{m;\sigma_1 y_1,\ldots,\sigma_n y_n} * f(\sigma x) d\omega(\sigma) d\omega\left(\sigma_1\right)\ldots\omega\left(\sigma_n\right)\right)$$

$$= \int_{\mathscr{P}_n} \cdots \int_{\mathscr{P}_n} \Delta_{m;\sigma_1 y_1,\ldots,\sigma_n y_n} * f\left(\sigma x + \sigma_{n+1} y_{n+1}\right) d\omega(\sigma) d\omega\left(\sigma_1\right) \ldots \omega\left(\sigma_{n+1}\right)$$
$$- \int_{\mathscr{P}_n} \cdots \int_{\mathscr{P}_n} m\left(\sigma_{n+1} y_{n+1}\right) \cdot \Delta_{m;\sigma_1 y_1,\ldots,\sigma_n y_n} * f(\sigma x) d\omega(\sigma) d\omega\left(\sigma_1\right)$$
$$\ldots \omega\left(\sigma_{n+1}\right)$$
$$= \int_{\mathscr{P}_n} \cdots \int_{\mathscr{P}_n} \Delta_{m;\sigma_1 y_1,\sigma_2 y_2,\ldots,\sigma_{n+1} y_{n+1}} * f\left(\sigma x\right) d\omega(\sigma) d\omega\left(\sigma_1\right)\, d\omega\left(\sigma_2\right)$$
$$\ldots \omega\left(\sigma_{n+1}\right) = \operatorname{sym}_{x,y_1,y_2,\ldots,y_{n+1}}\left[\Delta_{m;y_1,y_2,\ldots,y_{n+1}} * f(x)\right],$$

proving that the statement holds also for $n+1$. □

In view of Theorem 13, every symmetric monomial on N^d is a generalized exponential polynomial. Knowing this, we are able to describe symmetric monomials on N^d.

Assume therefore that a symmetric function $\varphi\colon N^d \to \mathbb{C}$ is a spherical monomial, that is, there exists a symmetric spherical function s on N^d and a positive integer n such that

$$\mathrm{D}_{s;y_1,y_2,\ldots,y_{n+1}} * \varphi(x) = 0$$

hold for each $x, y_1, y_2, \ldots, y_{n+1}$. By Theorem 13, the function φ in $\mathscr{C}(N^d)$ is a generalized exponential polynomial, so there is a positive integer k, there exist different exponentials $m_1, m_2, \ldots, m_k$ on N^d and there are non-zero generalized polynomials $P_1, P_2, \ldots, P_k$ on N^d such that for each x in N^d

$$\varphi(x) = \sum_{i=1}^{k} P_i(x) m_i(x).$$

Since φ is symmetric, the representation

$$\varphi(x) = \sum_{i=1}^{k} \left(P_i m_i\right)^{\#}(x)$$

also holds for each x in N^d. First we show that $k \leq d!$ and for each $i = 1, 2, \ldots, k$ there exists a σ_i in $\mathscr{P}_d$ such that

$$m_i(x) = m(\sigma x)$$

holds for each x in N^d, where the exponential m is determined by $s = m^{\#}$. To do so, the following lemmata are needed.

Hereinafter Lemma 4.3 from [8], that is, the statement below will be used several times.

Proposition 2. *Let G be an Abelian group and n be a positive integer. Suppose that $\sum_{k=1}^{n} P_k m_k = 0$, where $m_1, m_2, \ldots, m_n\colon G \to \mathbb{C}$ are different exponentials, and $P_1, P_2, \ldots, P_n\colon G \to \mathbb{C}$ are (generalized) polynomials. Then for each $i = 1, 2, \ldots, n$ the polynomial P_i is identically zero.*

It follows that the representation $f=\sum_{k=1}^n P_k m_k$ of the generalized exponential polynomial f with different exponentials is unique up to the order of the terms, and it is called the *canonical representation.*

Lemma 2. *Let m_1, m_2 be exponentials on N^d. If $m_1^{\#}=m_2^{\#}$, then there is a σ in $\mathscr{P}_d$ such that $m_2=\sigma m_1$.*

Proof. It is known (see e.g. [8, Theorem 6.11]) that there are exponentials $m_{1,j}, m_{2,j}: N\to\mathbb{C}$ for $j=1,2,\dots,d$ on N such that

$$m_1(x_1,x_2,\dots,x_d)=m_{1,1}(x_1)\cdot m_{1,2}(x_2)\cdots m_{1,d}(x_d),$$

and

$$m_2(x_1,x_2,\dots,x_d)=m_{2,1}(x_1)\cdot m_{2,2}(x_2)\cdots m_{2,d}(x_d),$$

holds for each $x=(x_1,x_2,\dots,x_d)$ in N^d. As $m_1^{\#}=m_2^{\#}$, substituting $\pi_1 x$ for x we have for each x in N^d

$$\sum_{i=1}^d m_{1,i}(x_1)=\sum_{i=1}^d m_{2,i}(x_1).$$

Using Proposition 2 and the uniqueness of the canonical representation we infer that the exponentials $m_{2,i}$ are the same as the exponentials $m_{1,i}$, possibly listed in a different order. But this is exactly the statement of the lemma. □

Lemma 3. *Let n be a positive integer, m_k different exponentials on N^d, and α_k nonzero complex numbers $(k=1,2,\dots,n)$. If $\sum_{k=1}^n \alpha_k m_k^{\#}=0$, then there is an exponential m on N^d such that, for each $k=1,2,\dots,n$ there exists σ_k in $\mathscr{P}_d$ for which $m_k=\sigma_k m$. In particular, $n\le d!$.*

Proof. We prove by induction on n and the case $n=1$ is clear, by the previous lemma. Suppose that we have proved the statement for some $n\ge 1$, and assume that $\sum_{k=1}^{n+1}\alpha_k m_k^{\#}=0$. We define $f=\sum_{k=1}^{n+1}\alpha_k m_k$, where the m_k's are different exponentials on N^d and the α_k's are nonzero complex numbers. Then, for each $j=1,2,\dots,n+1$ and x,y in N^d, we have

$$0=\mathrm{D}_{m_j^{\#};y} * f^{\#}(x)=\mathrm{sym}_{x,y}\left[\Delta_{m_j;y} * f(x)\right].$$

With the notation $\widetilde{\alpha_i}(y)=\alpha_i\big(m_i^{\#}(y)-m_j^{\#}(y)\big)$ $(i=1,2,\dots,n+1, i\ne j)$ this can be written as

$$\begin{aligned}g_j(x)=&\sum_{i=1,i\ne j}^{n+1}\widetilde{\alpha_i}(y)m_i^{\#}(x)\\ =&\sum_{\substack{i=1\\ i\ne j}}^{n}\alpha_i\left(\int_{\mathscr{P}_d}\left[m_i(\sigma_2 y)-m_j(\sigma_2 y)\right]d\omega_d(\sigma_2)\right)\cdot\left(\int_{\mathscr{P}_d} m_i(\sigma_1 x)d\omega_d(\sigma_1)\right)=0,\end{aligned}$$

whenever x, y is in N^d. First we suppose that for any choice of j, there is a y such that $\widetilde{\alpha}_i(y) \neq 0$ for each $i \neq j$. Then, by induction, we conclude that

$$m_i^{\#} = m_j^{\#} \quad \text{whenever} \quad i = 1, 2, \ldots, n+1, i \neq j.$$

Interchanging j with an $i \neq j$ and applying the same argument, by Lemma 2, we obtain the statement of the theorem.

Otherwise we have that, for each $j = 1, 2, \ldots, n+1$, there is $i \neq j$ such that $\widetilde{\alpha}_i = 0$. Then, for that pair i, j we have $m_i^{\#} = m_j^{\#}$, and, by Lemma 2, we infer that $m_i = \sigma m_j$. Clearly, this will lead to the same conclusion stated by the theorem.

Finally, if we assume indirectly that $n > d!$, then, by the previous step, there exists an exponential m on N^d so that for each $i = 1, 2, \ldots, n$, there is a certain $\sigma \in \mathscr{P}_d$ such that $m_i = \sigma m$. Since $|\mathscr{P}_d| = d! < n$, this would mean that the exponentials $m_1, m_2, \ldots, m_n$ cannot be different, contrary to our assumption. □

Proposition 3. *Let n be a positive integer, let $m_1, m_2, \ldots, m_n$ be different exponentials on N^d further let $P_1, P_2, \ldots, P_n\colon N^d \to \mathbb{C}$ be nonzero generalized polynomials. If $f = \sum_{k=1}^n P_k m_k$, and $f^{\#} = 0$, then there is an exponential m on N^d, and for each $i = 1, 2, \ldots, n$ there exists a σ_i in $\mathscr{P}_d$ such that $m_i = \sigma_i m$.*

Proof. The statement is obvious for $n = 1$, hence we assume that $n \geq 2$. As the projection of the function f is identically zero, we have that for each x in N^d

$$\sum_{i=1}^{n} \sum_{j=1}^{d!} P_i(\sigma_j x)\, m_i(\sigma_j x) = 0$$

holds. Since the functions $m_1, m_2, \ldots, m_n$ are exponentials for all possible values of $i = 1, 2, \ldots, n$ and $j = 1, 2 \ldots, d!$, it follows that the mappings $m_i \circ \sigma_j$ are exponentials on N^d, as well. In view of Proposition 2, the set

$$\{\sigma_j m_i \colon i = 1, 2, \ldots, n; j = 1, 2 \ldots, d!\}$$

cannot consist of *different* exponentials. Similarly as in the proof of Lemma 2, we conclude that this can happen in two different ways: either

$$\sigma_j m_i = \sigma_l m_k$$

holds for some i, k in $\{1, 2, \ldots, n\}$ with $i \neq k$ and j, l in $\{1, 2, \ldots, d!\}$ with $j \neq l$, or at least one of the coefficient polynomials is identically zero. In any of these two cases the above sum reduces to a sum with fewer terms but the same form, and, by induction, our statement follows.

The statement $n \leq d!$ can be proved as in the previous lemma. □

Theorem 14. *Let n be a positive integer, let $m_1, m_2, \ldots, m_n$ be different exponentials on N^d, further let $P_1, P_2, \ldots, P_n\colon N^d \to \mathbb{C}$ be nonzero generalized polynomials. If $s\colon N^d \to \mathbb{C}$ is a symmetric spherical function and $f =$*

$\sum_{k=1}^n P_k m_k$ is a symmetric monomial corresponding to s, then $n \leq d!$, and there exists an exponential m on N^d such that for each $i = 1, 2, \ldots, n$ there is a σ_i in $\mathscr{P}_d$ for which $m_i = \sigma m$ and $m^\# = s$.

Proof. For $n = 1$ the statement follows from the previous theorem. Let $n \geq 2$. By Theorem 11, $s = m^\#$ for some exponential m on N^d, and by assumption, there is a natural number k such that

$$\begin{aligned}
0 &= \mathrm{D}_{s;y_1,y_2,\ldots,y_{k+1}} * f(x) = \mathrm{D}_{s;y_1,y_2,\ldots,y_{k+1}} * f^\#(x) \\
&= \sum_{i=1}^n \mathrm{D}_{s;y_1,y_2,\ldots,y_{n+1}} * (P_i \cdot m_i)^\# (x) \\
&= m(x + y_1 + y_2 + \cdots + y_{k+1}) \\
&\quad \times \sum_{i=1}^n \mathrm{sym}_{x,y_1,y_2,\ldots,y_{k+1}} \left[\Delta_{y_1,y_2,\ldots,y_{k+1}} * (P_i \cdot m_i \cdot \check{m})(x)\right]
\end{aligned}$$

for each $x, y_1, y_2, \ldots, y_{k+1}$ in N^d. This means that

$$\sum_{i=1}^n \mathrm{sym}_{x,y_1,y_2,\ldots,y_{k+1}} \left[\Delta_{y_1,y_2,\ldots,y_{k+1}} * (P_i \cdot m_i \cdot \check{m})(x)\right] = 0$$

holds whenever $x, y_1, y_2, \ldots, y_{k+1}$ is in N^d. Since $m_1, m_2, \ldots, m_n$ and also $\check{m}$ are exponentials, after computing the left hand side of this identity, we can conclude that Lemma 2 can be applied to deduce that for each $i = 1, 2, \ldots, n$ there exists $\sigma_i \in \mathscr{P}_d$ with $m_i = \sigma m$. □

8. Symmetric spectral analysis and synthesis

Let N be a locally compact Abelian group and d a positive integer. In this section we study symmetric spectral analysis and synthesis for symmetric varieties on N^d. First we note that if a symmetric variety V contains a nonzero s-monomial for some symmetric spherical function s, then V contains s, as well (see [11]). Another observation is about the relation between classical spectral synthesis properties on N and symmetric spectral synthesis properties on N^d. We introduce the mapping $S: N^d \to N$ by

$$Sx = x_1 + x_2 + \cdots + x_d$$

for each $x = (x_1, x_2, \ldots, x_d)$ in N^d. We shall also use the embedding $E\colon\ N \to N^d$ defined by

$$Ex = (x, 0, \ldots, 0)$$

for each x in N. Then $T = E \circ S: N^d \to N^d$ is an idempotent endomorphism of N^d: for each $x = (x_1, x_2, \ldots, x_d)$ in N^d we have

$$Tx = (x_1 + x_2 + \cdots + x_d, 0, \ldots, 0)\,.$$

The mapping $f \mapsto f \circ T$ is a continuous linear idempotent operator of $\mathscr{C}(N^d)$, and its restriction to $\mathscr{C}_d(N^d)$ is a continuous linear idempotent operator of $\mathscr{C}_d(N^d)$.

We call a function $f\colon N^d \to \mathbb{C}$ *sum-dependent function*, if there exists a function $\psi\colon N \to \mathbb{C}$ such that $f = \psi \circ S$. This is equivalent to the property $f = f \circ T$. Indeed, if $f = f \circ T$, then

$$f = f \circ T = f \circ E \circ S = (f \circ E) \circ S,$$

hence we can take $\psi = f \circ E$. Conversely, if $f = \psi \circ S$, then

$$f \circ T = (\psi \circ S) \circ T = \psi \circ (S \circ T) = \psi \circ S = f,$$

as obviously $S \circ T = S$.

The set of all sum-dependent functions in $\mathscr{C}(N^d)$ is denoted by $\mathscr{C}_S(N^d)$. The measure μ in $\mathscr{M}_c(N^d)$ is called *sum-dependent measure*, if

$$\langle \mu, f \rangle = \langle \mu, f \circ T \rangle$$

holds for each f in $\mathscr{C}(N^d)$. The set of all sum-dependent measures in $\mathscr{M}_c(N^d)$ is denoted by $\mathscr{M}_{c,S}(N^d)$. In general, for each μ in $\mathscr{M}_c(N^d)$, the measure μ_T is defined by

$$\langle \mu_T, f \rangle = \langle \mu, f \circ T \rangle,$$

which is a sum-dependent measure, moreover, μ is sum-dependent if and only if $\mu = \mu_T$, as it is easy to see.

Theorem 15. *The space $\mathscr{C}_S(N^d)$ is a closed linear subspace in $\mathscr{C}_d(N^d)$ and its dual is topologically isomorphic to $\mathscr{M}_{c,S}(N^d)$, which is a weak*-closed subalgebra in $\mathscr{M}_{c,d}(N^d)$.*

Proof. It is obvious that $\mathscr{C}_S(N^d)$ is a closed linear subspace in $\mathscr{C}_d(N^d)$. We show that $\mathscr{C}_S(N^d)^*$ is topologically isomorphic to $\mathscr{M}_{c,S}(N^d)$. We observe that, for every f in $\mathscr{C}(N^d)$ the function $f \circ T$ is in $\mathscr{C}_S(N^d)$. Indeed, we have

$$(f \circ T) \circ T = f \circ (T \circ T) = f \circ T,$$

as T is idempotent. For each continuous linear functional λ in $\mathscr{C}_S(N^d)^*$ we define $F_\lambda\colon \mathscr{C}(N^d) \to \mathbb{C}$ by

$$F_\lambda(f) = \langle \lambda, f \circ T \rangle.$$

It is easy to see that F_λ is a continuous linear functional on $\mathscr{C}(N^d)$. Hence there is a measure μ_λ in $\mathscr{M}_c(N^d)$ such that

$$\langle \mu_\lambda, f \rangle = \langle \lambda, f \circ T \rangle$$

holds for each f in $\mathscr{C}(N^d)$. We also have

$$\langle \mu_\lambda, f \rangle = \langle \lambda, f \circ T \rangle = \langle \lambda, f \circ (T \circ T) \rangle = \langle \lambda, (f \circ T) \circ T \rangle = \langle \mu_\lambda, f \circ T \rangle,$$

hence μ_λ is a sum-dependent measure. What is left is to show that the mapping $\lambda \leftrightarrow \mu_\lambda$ is a topological isomorphism. It is clear that this mapping is linear, continuous and open. Suppose that $\mu_\lambda = 0$ for some λ; then λ vanishes on the sum-dependent functions, that is $\lambda = 0$, which proves bijectivity. For surjectivity, assume that μ is a sum-dependent measure, then clearly, $f \mapsto \langle \mu, f\rangle$ is a continuous linear functional on $\mathscr{C}_S(N^d)$, which we denote by λ. It is obvious that $\mu_\lambda = \mu$ and this proves the topological isomorphism between $\mathscr{C}_S(N^d)^*$ and $\mathscr{M}_{c,S}(N^d)$.

It is clear that $\mathscr{M}_{c,S}(N^d)$ is weak*-closed in $\mathscr{M}_{c,d}(N^d)$. We need to show only that $\mathscr{M}_{c,S}(N^d)$ is closed with respect to convolution. Let μ, ν be sum-dependent measures—we show that $\mu * \nu$ is sum-dependent. For each f in $\mathscr{C}(N^d)$ we have

$$\begin{aligned}
\langle \mu * \nu, f\circ T\rangle &= \int_{N^d}\int_{N^d} (f\circ T)(x+y)\,d\mu(x)\,d\nu(y)\\
&= \int_{N^d}\left[\int_{N^d} (f\circ (T\circ T))\,(x+y)\,d\mu(x)\right]\,d\nu(y)\\
&= \int_{N^d}\left[\int_{N^d} ((f\circ T)\circ T)\,(x+y)\,d\mu(x)\right]\,d\nu(y)\\
&= \int_{N^d}\left[\int_{N^d} (f\circ T)(Tx+y)\,d\mu(x)\right]\,d\nu(y)\\
&= \int_{N^d}\left[\int_{N^d} (f\circ T)(x+y)\,d\mu_T(x)\right]\,d\nu(y)\\
&= \int_{N^d}\left[\int_{N^d} (f\circ T)(x+y)\,d\nu(y)\right]\,d\mu_T(x)\\
&= \int_{N^d}\left[\int_{N^d} f(x+Ty)\,d\nu(y)\right]\,d\mu_T(x)\\
&= \int_{N^d}\int_{N^d} f(x+y)\,d\nu_T(y)\,d\mu_T(x)\\
&= \int_{N^d}\int_{N^d} f(x+y)\,d\nu(y)\,d\mu(x) = \langle \mu*\nu, f\rangle,
\end{aligned}$$

and the theorem is proved. □

The symmetric translation on $\mathscr{C}_S(N^d)$, resp. on $\mathscr{M}_{c,S}(N^d)$ takes the form

$$\begin{aligned}
\tau_y^{\#} f(x) &= \frac{1}{d!}\sum_{\sigma\in\mathscr{P}_d} f\left(x+\sigma y\right)\\
&= \frac{1}{d!}\sum_{\sigma\in\mathscr{P}_d} f\left(T(x+\sigma y)\right) = \frac{1}{d!}\sum_{\sigma\in\mathscr{P}_d} f\left(T(x)+T(\sigma y)\right)\\
&= \frac{1}{d!}\sum_{\sigma\in\mathscr{P}_d} f\left(T(x)+T\left(y\right)\right) = \frac{1}{d!}\sum_{\sigma\in\mathscr{P}_d} (f\circ T)(x+y)\\
&= f(x+y) = \tau_y f(x),
\end{aligned}$$

that is, it reduces to ordinary translation. Consequently, symmetric varieties in $\mathscr{C}_S(N^d)$ reduce to ordinary varieties, sum-dependent spherical functions are exponentials, and sum-dependent symmetric monomials and polynomials are exponential monomials. Symmetric spectral analysis and synthesis questions for varieties in $\mathscr{C}_S(N^d)$ reduce to problems about ordinary spectral analysis and synthesis problems. If V is a variety on N, then $V_S = \{f\colon f \in \mathscr{C}_S(N^d), f \circ E \in V\}$ is also a variety in $\mathscr{C}_S(N^d)$, as it is easy to see. On the other hand, for each symmetric variety W in $\mathscr{C}_S(N^d)$ we define

$$V = \{f \circ E\colon f \in W\}\,,$$

then V is a variety on N, and $V_S = W$, as it is easy to see. It follows that $V \leftrightarrow V_S$ is a one-to-one correspondence between the varieties on N and the symmetric varieties in $\mathscr{C}_S(N^d)$. Using this observation we can prove the following theorem.

Theorem 16. *Let N be a locally compact Abelian group, d a positive integer, and V a variety on N. Spectral analysis holds for V if and only if symmetric spectral analysis holds for V_S. The variety V is synthesizable if and only if symmetric synthesizability holds for V_S, and spectral synthesis holds for V if and only if symmetric spectral synthesis holds for V_S.*

Proof. By [11, Theorem 19], it is enough to show that the algebras

$$\mathscr{M}_c(N)/\operatorname{Ann} V \quad \text{and} \quad \mathscr{M}_{c,S}(N^d)/\operatorname{Ann} V_S$$

are topologically isomorphic. We emphasize that $\operatorname{Ann} V_S$ denotes the annihilator of V_S in the algebra $\mathscr{M}_{c,S}(N^d)$. We recall that $\mathscr{M}_c(N)/\operatorname{Ann} V$, resp. $\mathscr{M}_{c,S}(N^d)/\operatorname{Ann} V_S$ is the topological dual of the topological vector space V, resp. V_S. First we define an algebra homomorphism of $\mathscr{M}_c(N)$ into $\mathscr{M}_{c,S}(N^d)$ by

$$F(\mu)(f) = \langle \mu, f \circ E\rangle = \int_N f(t, 0, \ldots, 0)\, d\mu(t)$$

whenever μ is in $\mathscr{M}_c(N)$ and f is in $\mathscr{C}_S(N^d)$. It is a routine calculation that F is an algebra homomorphism, continuous with respect to the weak*-topologies. It is also clear, that if μ annihilates V, then $F(\mu)$ annihilates V_S. Consequently, F can be considered as an algebra homomorphism of the space $\mathscr{M}_c(N)/\operatorname{Ann} V$ into $\mathscr{M}_{c,S}(N^d)/\operatorname{Ann} V_S$. Injectivity of F is the consequence of the surjectivity of the mapping $f \mapsto f \circ E$ from $\mathscr{C}_S(N^d)$ onto $\mathscr{C}(N)$. Finally, to prove the surjectivity of F we take ν from $\mathscr{M}_{c,S}(N^d)$ and we define for φ in $\mathscr{C}(N)$:

$$\langle \mu_\nu, \varphi\rangle = \langle \nu, \varphi \circ S\rangle.$$

Clearly, $\varphi \circ S$ is in $\mathscr{C}_S(N^d)$, hence the right hand side is well-defined, and μ_ν is in $\mathscr{M}_c(N)$. We have for each f in $\mathscr{C}_S(N^d)$

$$F\left(\mu_\nu\right)(f) = \langle \mu_\nu, f \circ E\rangle = \int_N f(t, 0, \ldots, 0)\, d\mu_\nu(t)$$

$$= \int_{N^d} f\left(x_1 + \cdots + x_d, 0, \ldots, 0\right) \, d\nu(x) = \int_{N^d} f \, d\nu,$$

hence $F(\mu_\nu) = \nu$, and the proof is complete. □

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Symmetric spectral synthesis

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