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Some numerical characteristics of Sylvester and Hadamard matrices

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Abstract. We introduce numerical characteristics of *Sylvester* and *Hadamard matrices* and give their estimates and some applications.

1. Introduction

In this paper we introduce and study numerical characteristics of Hadamard and Sylvester matrices. The considered characteristics and the structure of Hadamard and Sylvester matrices play an important role in the investigation of convergence of series in Banach spaces (see, for example, [1], [2], [11], [12], [13], [14]) and have an independent interest as well. For this characteristics we give estimates (cf. Theorems 3.1, 3.7, 4.2 and 4.8). It seems to us that the proposed characteristics may have applications in other fields of mathematics.

In Section 2 some concepts, definitions and auxiliary results required for further discussion are given.

In Section 3 the numerical characteristic $\varrho^{(n)}$ of Sylvester matrices is introduced and its estimations for the case of a Banach space with a subsymmetric basis are studied. For any positive integer n we prove the following estimations

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(cf. Theorems 3.1 and 3.7)

$$\max\left\{\frac{n+2}{6}\cdot\lambda(2^n),\, 2^n\right\} \le \varrho^{(n)} \le \min\left\{\left(1+\sum_{j=1}^{n} 2^{-j}\lambda(2^{j-1})\right)\cdot 2^n, \lambda(n)\cdot 2^n\right\}.$$

In Section 4 we define the analogue characteristic for Hadamard matrices. For any positive integer n for which there exists a Hadamard matrix of order n we show the following estimations (cf. Theorems 4.2 and 4.8)

$$\max\left\{(1/\sqrt{2})\,\lambda(n)\,\sqrt{n},\, n\right\} \le \varrho_n \le \lambda([\sqrt{n}]+1)\,n,$$

where $[\sqrt{n}]$ is the integer part of $\sqrt{n}$.

As an application of the introduced notions we give a characterization for the spaces isomorphic to l_1 in terms of these characteristics (cf. Theorem 4.10).

In Section 5 we pose one open problem which has naturally arisen from our investigations.

Most of the results of this paper were announced in [5] without proofs. Here these and some new results are given with complete proofs.

2. Notation and Preliminaries

We follow the standard notation and terminology used, for example, in [7]. Notations c_0, l_p and $L_p, 1 \le p < \infty$, have their usual meaning.

A sequence (φ_i) of nonzero elements in a real Banach space X is called a (*Schauder*) *basis* of X if for every $x \in X$ there is a unique sequence of scalars (α_i) so that $x = \sum_{i=1}^{\infty} \alpha_i \varphi_i$. If (φ_i) is a basis in a Banach space X with a norm $\|\cdot\|$, then there is a constant $K \ge 1$ so that for every choice of scalars (α_i) and positive integers $n < m$, we have

$$\|\sum_{i=1}^{n} \alpha_i\varphi_i\| \le K\|\sum_{i=1}^{m} \alpha_i\varphi_i\|.$$

The smallest possible constant K in this inequality is called the *basis constant* of (φ_i). Note that in X there exists an *equivalent norm* $|||\cdot|||$ (i.e. for some positive constants C_1, C_2: $C_1\|x\| \le |||x||| \le C_2\|x\|$ for every $x \in X$) under for which the basis constant $K = 1$.

A basis (φ_i) is called *normalized* if $\|\varphi_i\| = 1$ for all i. Let (φ_i) be a basis of a Banach space X. A sequence of linear bounded functionals (φ_i^*) defined by the relation $\langle \varphi_i^*, \varphi_j \rangle = \delta_{ij}$, where δ_{ij} is the *Kronecker delta*, is called the sequence of *biorthogonal functionals* associated to the basis (φ_i). Two bases (φ_i) of X and (ψ_i) of Y are called equivalent provided a series $\sum_{i=1}^{\infty} \alpha_i \varphi_i$ converges if and only if $\sum_{i=1}^{\infty} \alpha_i \psi_i$ converges.

A basis (φ_i) of a Banach space X is *unconditional* if for any permutation $\pi : \mathbb{N} \to \mathbb{N}$ of the set N of positive integers $(\varphi_{\pi(i)})$ is a basis in X. If (φ_i) is an unconditional basis of a Banach space X. Then there is a constant $K \geq 1$ so that for any choice of scalars (α_i) for which $\sum_{i=1}^{\infty} \alpha_i \varphi_i$ converges and every choice of bounded scalars (λ_i) we have

$$\| \sum_{i=1}^{\infty} \lambda_i \alpha_i \varphi_i \| \leq K \sup_i \lambda_i \| \sum_{i=1}^{\infty} \alpha_i \varphi_i \|.$$

The smallest possible constant K in this inequality is called the *unconditional constant* of (φ_i). If (φ_i) is an unconditional basis of X, then there is an equivalent norm in X so that the unconditional constant becomes 1.

The sequence of unit vectors $e_i = (0, 0, \ldots, \overset{i}{1}, 0, \ldots), i = 1, 2, \ldots,$ is an example of an unconditional basis in c_0 and in l_p, $1 \leq p < \infty$ (the basis (e_i) is called the *natural basis* of the corresponding spaces). The *Haar system* is unconditional basis in the functional spaces $L_p(0,1)$, $1 < p < \infty$. This system is also basis in $L_1(0,1)$, but in this space does not exist an unconditional basis.

Every normalized unconditional basis in l_1, l_2 or c_0 is equivalent to the natural basis of the space. Moreover, a Banach space has, up to equivalence, a unique unconditional basis if and only if it is isomorphic to one of the following three spaces: l_1, l_2 or c_0.

Let $(X, \| \cdot \|)$ be a Banach space with a normalized basis (φ_i). Consider the expression

$$\lambda(n) = \| \sum_{i=1}^{n} \varphi_i \|, \quad n = 1, 2, \ldots.$$

For any space with unconditional basis for which the unconditional constant is equal to 1 we have that $(\lambda(n))$ is a non-decreasing sequence and $\lim_{n \to \infty} \lambda(n) = \infty$, except the case of the space c_0. More precisely, if $\sup_n \lambda(n) < \infty$, then (φ_i) is equivalent to the unit vectors of the space c_0 (see, for example, [7], p. 120).

A basis (φ_i) of a Banach space X is said to be *symmetric* if for any permutation π of the positive integers $(\varphi_{\pi(i)})$ is equivalent to (φ_i). If (φ_i) is a symmetric basis of a Banach space X, then there is a constant K so that for any choice of scalars (α_i) for which $\sum\limits_{i=1}^{\infty} \alpha_i\varphi_i$ converges, every choice of signs $\vartheta = (\vartheta_i)$ and any permutation π of the integers, we have

$$\|\sum_{i=1}^{\infty} \vartheta_i\alpha_i\varphi_{\pi(i)}\| \le K\|\sum_{i=1}^{\infty} \alpha_i\varphi_i\|.$$

The smallest possible constant K in this inequality is called the *symmetric constant* of (φ_i).

A basis (φ_i) of a Banach space X is called *subsymmetric* if it is unconditional and for every increasing sequence of integers (i_n), (φ_{i_n}) is equivalent to (φ_i). If (φ_i) is a subsymmetric basis of a Banach space X, then there is a constant K so that for any choice of scalars (α_i) for which $\sum\limits_{i=1}^{\infty} \alpha_i\varphi_i$ converges for every choice of signs $\vartheta = (\vartheta_i)$ and for every increasing sequence of integers (i_n) we have

$$\|\sum_{n=1}^{\infty} \vartheta_n\alpha_n\varphi_{i_n}\| \le K\|\sum_{i=1}^{\infty} \alpha_i\varphi_i\|.$$

The smallest possible constant K in this inequality is called the *subsymmetric constant* of (φ_i).

Every symmetric basis is subsymmetric. The converse of this assertion is not true. The unit vectors in l_p, $1 \le p < \infty$, and c_0 are examples of symmetric basis.

Proposition 2.1. *(see* [7]*, Proposition 3.a.7, p. 119). Let* $(X, \|\cdot\|)$ *be a Banach space with a symmetric basis* (φ_i) *whose symmetric constant is equal to 1. Then there exists a new norm* $\|\cdot\|_0$ *on* X *such that:*

(a). $\|x\| \le \|x\|_0 \le 2\|x\|$ *for all* $x \in X$*;*

(b). *The symmetric constant of* (φ_i) *with respect to* $\|\cdot\|_0$ *is equal to 1;*

(c). *If we put* $\lambda_0(n) = \|\sum\limits_{i=1}^{n} \varphi_i\|_0, n = 1, 2, \dots,$ *then* $\{\lambda_0(n+1) - \lambda_0(n)\}$ *is a non-increasing sequence, i.e.* $\lambda_0(\cdot)$ *is a concave function on the integers.*

The converse of the last assertion is also true in the sense that, for every concave non-decreasing sequence of positive numbers (λ_k) there exists at least one Banach space X having a symmetric basis (φ_i) with symmetric constant equal to 1 such that $\|\sum\limits_{i=1}^{n} \varphi_i\| = \lambda_n$ for every n.

Proposition 2.2. *(see* [7]*, Proposition 3.a.4, p. 116).* (**A**). *Let* X *be a Banach space with a normalized subsymmetric basis* (φ_i) *whose subsymmetric* *constant is 1. Then the following inequality is valid*

$$\|\sum_{i=1}^{n} \alpha_i \varphi_i\| \geq \frac{\sum\limits_{i=1}^{n} |\alpha_i|}{n} \lambda(n), \quad n = 1, 2, \ldots .$$

(**B**). *Moreover, if* (φ_i) *is a subsymmetric basis, then one has*

$$\|\sum_{i=1}^{n} \alpha_i \varphi_i\| \geq \frac{\sum\limits_{i=1}^{n} |\alpha_i|}{2n} \lambda(n), \quad n = 1, 2, \ldots .$$

From this it follows that if $\lim\limits_{n\to\infty} \sup \lambda(n)/n > 0$, then (φ_i) is equivalent to the unit vectors of the space l_1 (see, for example, [7], p. 120).

The *Rademacher function* $r_k : [0,1] \to \{0,1\}$, $k = 1, 2, \ldots$, is defined by the equality

$$r_k(t) = sign(\sin 2^k \pi t).$$

Let us note the well-known *Khintchine inequality*: for every $0 < p < \infty$ there exist positive constants A_p and B_p so that

$$A_p \left(\sum_{k=1}^{m} \alpha_k^2 \right)^{1/2} \leq \left(\int_0^1 \left| \sum_{k=1}^{m} \alpha_k r_k(t) \right|^p dt \right)^{1/p} \leq B_p \left(\sum_{k=1}^{m} \alpha_k^2 \right)^{1/2}, \quad m = 1, 2, \ldots,$$

for every choice of scalars $(\alpha_1, \alpha_2, \ldots, \alpha_m)$. For $p = 1$ the best constant is $A_1 = 1/\sqrt{2}$ (see [9]).

A Banach space X is said to be of *type p* if there is a constant $T_p = T_p(X) \geq 0$ such that for any finite collection of vectors $x_1, x_2, \ldots, x_n$ in X we have

$$\left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|^2 dt \right)^{1/2} \leq T_p \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}, \quad n = 1, 2, \ldots .$$

In the Khintchine's inequality the notions type p have meaning for the case $0 < p \leq 2$. Every Banach space has type p for $0 < p \leq 1$. The spaces $l_p, L_p([0,1]), 1 \leq p < \infty$, have type $\min(2, p)$.

A Hadamard matrix is a square matrix of order n with entries ± 1 such that any two columns (rows) are orthogonal (see *e.g.* [4], p. 238, [8], p. 44). A Hadamard matrix of order n will be denoted by $\mathcal{H}_n = [h_{ki}^n]$. It is easy to see that the order of a Hadamard matrix is 1 or 2 or is divisible by 4. *Hadamard* puts forward the conjecture that for any n divisible by 4 there exists a Hadamard matrix of order n. As far as we know, *Hadamard's conjecture* remains open yet. Let $\mathbb{N}_{\mathcal{H}}$ be the set of all positive integers n for which there exists a Hadamard matrix of order n.

The following property follows from the definition of Hadamard matrices. If $\mathcal{H}_n = [h_{ki}^n]$ is a Hadamard matrix, then for every n, $n \in \mathbb{N}_{\mathcal{H}}$, we have

$$\sum_{i=1}^{n} h_{ki}^n h_{mi}^n = n\,\delta_{km}, \quad \sum_{k=1}^{n} h_{ki}^n h_{kj}^n = n\,\delta_{ij}.$$

Therefore for any n, $n \in \mathbb{N}_{\mathcal{H}}$, and every sequence $(\beta_i)_{i \le n}$ of real numbers one has

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} h_{ki}^n \beta_i \right)^2 = n \sum_{i=1}^{n} \beta_i^2.$$

It is easily seen that multiplying any row or any column of a Hadamard matrix by -1 we get again a Hadamard matrix.

Let a triple $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space, where Ω is a nonempty set, $\mathfrak{A}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure on a measurable space $(\Omega, \mathfrak{A})$ (that is $\mathbb{P}$ is a non-negative measure on $(\Omega, \mathfrak{A})$ satisfying the condition $\mathbb{P}(\Omega) = 1$). Let X be a real Banach space with the topological dual space X^*. A function $\xi : \Omega \to X$ is scalarly measurable (respectively scalarly integrable) if for each $x^* \in X^*$ the scalar function $\langle x^*, \xi \rangle$ is measurable (respectively integrable, i.e. $\langle x^*, \xi \rangle \in L_1(\Omega, \mathfrak{A}, \mathbb{P})$). A scalarly integrable function $\xi : \Omega \to X$ is *Pettis integrable* (or *weak integrable*) if for each $A \in \mathfrak{A}$ there exists a vector $m_{\xi,A} \in X$ such that

$$\langle x^*, m_{\xi,A} \rangle = \int_A \langle x^*, \xi \rangle \, d\mathbb{P}$$

for every $x^* \in X^*$. For a Pettis integrable function $\xi : \Omega \to X$ the element $m_{\xi,\Omega}$ is called *Pettis integral* of ξ with respect to $\mathbb{P}$. It is also called the *mean value* of the function ξ. The Pettis integral of the function ξ we denote by the symbol $\mathbb{E}\,\xi$. If a function $\xi : \Omega \to X$ has a measurable norm and there exists $\mathbb{E}\,\xi$, then $\|\mathbb{E}\,\xi\| \le \mathbb{E}\,\|\xi\|$. For every separably valued function $\xi : \Omega \to X$ from the condition $\mathbb{E}\,\|\xi\| < \infty$ it follows the existence of the Pettis integral $\mathbb{E}\,\xi$ ("separably valued" i.e. $\xi(\Omega)$ is a separable subset of X).

All materials considered here and much more with proofs and discussions one can find in [7] and [10].

3. Sylvester matrices

The *Sylvester matrices* are special cases of Hadamard matrices. They are defined by the recursion relations [8], p. 45:

$$\mathcal{S}^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathcal{S}^{(n)} = \begin{bmatrix} \mathcal{S}^{(n-1)} & \mathcal{S}^{(n-1)} \\ \mathcal{S}^{(n-1)} & -\mathcal{S}^{(n-1)} \end{bmatrix}, \quad n = 2, 3, \ldots.$$

$\mathcal{S}^{(n)}$ is a Hadamard matrix of order 2^n and hence $2^n \in \mathbb{N}_{\mathcal{H}}$ for all $n = 1, 2, \ldots$..

If the first column of a Hadamard matrix $\mathcal{H}_n = [h^n_{ki}]$ consists of only $+1$, then one has

$$\sum_{k=1}^{n} h^n_{ki} = \begin{cases} n, & \text{for} \quad i = 1, \\ 0, & \text{for} \quad i = 2, 3, \ldots, n. \end{cases}$$

In particular, if $\mathcal{S}^{(n)} = \left[s^{(n)}_{ki}\right]$, $n = 1, 2, \ldots$, is a Sylvester matrix, then we get

$$\sum_{k=1}^{2^n} s^{(n)}_{ki} = \begin{cases} 2^n, & \text{for} \quad i = 1, \\ 0, & \text{for} \quad i = 2, 3, \ldots, 2^n \end{cases}$$

and

$$\sum_{k=1}^{2^{n-1}} s^{(n)}_{ki} = \begin{cases} 2^{n-1}, & \text{for} \quad i = 1 \quad \text{and} \quad i = 2^{n-1} + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}^{(n)} = \left[s^{(n)}_{ki}\right]$, $n = 1, 2, \ldots$, be the Sylvester matrix and X be a Banach space with a norm $\|\cdot\|$ and a normalized basis (φ_i). Consider the functional

$$\varrho^{(n)}(m) = \left\| \sum_{i=1}^{2^n} \left(\sum_{k=1}^{m} s^{(n)}_{ki} \right) \varphi_i \right\|, \; m = 1, 2, \ldots, 2^n. \tag{3.1}$$

One has $\varrho^{(n)}(1) = \lambda(2^n), \varrho^{(n)}(2) = 2 \left\| \sum\limits_{i=1}^{2^{n-1}} \varphi_{2i-1} \right\|, \varrho^{(n)}(2^n) = 2^n$, where $\lambda(2^n) = \left\| \sum\limits_{i=1}^{2^n} \varphi_i \right\|$. The expression $\varrho^{(n)}(m)$ obviously depends on X, the norm in X, and the choice of basis (φ_i). In particular, for the case of the spaces l_p, $1 \le p < \infty$, with respect to the natural basis, it has the form $\varrho^{(n)}(m) = \left(\sum\limits_{i=1}^{2^n} \left| \sum\limits_{k=1}^{m} s^{(n)}_{ki} \right|^p \right)^{1/p}$.

We set

$$\varrho^{(n)} = \max_{1 \le m \le 2^n} \varrho^{(n)}(m). \tag{3.2}$$

The functional $\varrho^{(n)}(m)$ can be expressed as follows. Let $a_k = \sum_{i=1}^{2^n} s_{ki}^{(n)} \varphi_i,\ k = 1, 2, \ldots, 2^n$. Then one has $\varrho^{(n)}(m) = \left\| \sum_{k=1}^{m} a_k \right\|$. If (φ_i) is an unconditional basis with unconditional constant 1, then, obviously, $||a_k|| = \lambda(2^n)$ for any $k = 1, 2, \ldots, 2^n$ and $\varrho^{(n)} \le \lambda(2^n)\, 2^n \le 2^{2n}$.

In $l_p,\ 1 \le p < \infty$, it was proved in [11] that $\varrho^{(n)} \le n\, 2^n$.

The following theorem gives a similar estimate of $\varrho^{(n)}$ in the case of general Banach spaces with subsymmetric basis.

Theorem 3.1. *Let X be a Banach space with normalized subsymmetric basis whose subsymmetryc constant is equal to 1. Then for $\varrho^{(n)}$ defined by (3.2) one has the following estimation*

$$\varrho^{(n)} \le \min \left\{ \left(1 + \sum_{j=1}^{n} 2^{-j} \lambda(2^{j-1}) \right) \cdot 2^n, \quad \lambda(n) \cdot 2^n \right\}, \quad n = 1, 2, \ldots. \tag{3.3}$$

Proof. First consider the inequality $\varrho^{(n)} \le \left(1 + \sum_{j=1}^{n} 2^{-j} \lambda(2^{j-1}) \right) \cdot 2^n$. We prove this inequality by induction. For $n = 1$ it is true since the left hand side of (3.3) is equal to 2 and the right hand side is equal to 3. Let now $n \ge 2$ and introduce the following notation

$$\alpha_i^{(n)}(m) = \sum_{k=1}^{m} s_{ki}^{(n)}, \quad 1 \le i, m \le 2^n. \tag{3.4}$$

Therefore we get

$$\alpha_1^{(n)}(m) = m \tag{3.5}$$

and

$$\alpha_{2^{n-1}+1}^{(n)}(m) = \begin{cases} m, & \text{for} \quad 1 \le m \le 2^{n-1}, \\ 2^n - m, & \text{for} \quad 2^{n-1} + 1 \le m \le 2^n. \end{cases} \tag{3.6}$$

Since $i \le 2^n$ we can write that $i = \varepsilon_n 2^n + \varepsilon_{n-1} 2^{n-1} + \ldots + \varepsilon_1 2 + \varepsilon_0$, where ε_j for every j is equal to 0 or to 1. Then by the definition and the properties of the Sylvester matrices we can prove by induction that for any i

$$\max_{1 \le m \le 2^n} \left| \alpha_i^{(n)}(m) \right| = 2^{f(i)}, \tag{3.7}$$

where the function $f : \{1, 2, \dots, n\} \to \{0, 1, 2, \dots, n\}$ is defined by the following way: $f(1) = n$; $f(i) = 0$ if $\varepsilon_0 = 0$ (i.e. i is an even number) and if $\varepsilon_0 = 1$ (i.e. i is an odd number), then for $f(i)$ we have: $\varepsilon_{f(i)} = 1$ and $\varepsilon_j = 0$ for every $j = 1, 2, \dots, f(i) - 1$.

For $i = 1$ and $i = 2^{n-1} + 1$ the equality (3.7) is valid since from the relations (3.5) and (3.6) it follows, that $\max\limits_{1 \le m \le 2^n} \left|\alpha_1^{(n)}(m)\right| = 2^n$ and $\max\limits_{1 \le m \le 2^n} \left|\alpha_{2^{n-1}+1}^{(n)}(m)\right| = 2^{n-1}$. To prove (3.7) for the rest indexes i, we are needed the following equalities

$$\max_{1 \le m \le 2^{n+1}} \left|\alpha_{2^n+i}^{(n+1)}(m)\right| = \max_{1 \le m \le 2^{n+1}} \left|\alpha_i^{(n+1)}(m)\right| = \max_{1 \le m \le 2^n} \left|\alpha_i^{(n)}(m)\right| \tag{3.8}$$

for any $i = 2, 3, \dots, 2^n$, which is a consequence of the definition and the properties of the Sylvester matrices. Every positive integer i, $1 \le i \le 2^{n+1}$, has the unique representation given by

$$i = \begin{cases} \varepsilon_n 2^n + \ldots + \varepsilon_1 2 + \varepsilon_0, & \text{for} \quad 1 \le i \le 2^n, \\ 2^n + \varepsilon_n 2^n + \ldots + \varepsilon_1 2 + \varepsilon_0, & \text{for} \quad 2^n + 1 \le i \le 2^{n+1}. \end{cases} \tag{3.9}$$

If i is an even number, then in (3.9) we have $\varepsilon_0 = 0$ and by (3.7) and (3.8), we obtain $\max\limits_{1 \le m \le 2^{n+1}} \left|\alpha_i^{(n+1)}(m)\right| = 1$. But if i is an odd number and, in addition, $i \ne 1$ and $i \ne 2^n + 1$, then one can write (3.9) in the following way

$$i = \begin{cases} \varepsilon_n 2^n + \ldots + \varepsilon_{j_0+1} 2^{j_0+1} + 2^{j_0} + 1, & \text{for} \quad 3 \le i \le 2^n, \\ 2^n + \varepsilon_n 2^n + \ldots + \varepsilon_{j_0+1} 2^{j_0+1} + 2^{j_0} + 1, & \text{for} \quad 2^n + 3 \le i \le 2^{n+1}, \end{cases}$$

where $j_0 = 1, 2, \dots, n-1$. Using again relations (3.7) and (3.8) we certainly have $\max\limits_{1 \le m \le 2^{n+1}} \left|\alpha_i^{(n+1)}(m)\right| = 2^{j_0}$.

Applying now a simple combinatorial calculation we get that the number of indexes i, $1 \le i \le 2^n$, for which $\max\limits_{1 \le m \le 2^n} \left|\alpha_i^{(n)}(m)\right| = 2^j$, is equal to 2^{n-j-1} for $j = 0, 1, 2, \dots, n-1$, and the equality $\max\limits_{1 \le m \le 2^n} \left|\alpha_i^{(n)}(m)\right| = 2^n$ is satisfied only in one case for $i = 1$.

As the basis (φ_i) has unit subsymmetryc constant, using (3.7), we obtain for every $m = 1, 2, \dots, 2^n$ the following relations:

$$\varrho^{(n)}(m) = \left\| \sum_{i=1}^{2^n} \left|\alpha_i^{(n)}(m)\right| \varphi_i \right\| \le \left\| \sum_{i=1}^{2^n} \max_{1 \le m \le 2^n} \left|\alpha_i^{(n)}(m)\right| \varphi_i \right\| =$$

$$= \left\| 2^n \varphi_1 + \sum_{j=1}^{n} 2^{n-j} \sum_{i=2^{j-1}}^{2^j-1} \varphi_{\pi(i+1)} \right\|, \tag{3.10}$$

where π is a permutation of a sequence of the positive integers $\{2, 3, \ldots, 2^n\}$. Applying now the triangular inequality on the right hand side of (3.10) and using subsymmetricity of (φ_i) we get that the required inequality is valid.

Now let us prove the inequality $\varrho^{(n)} \leq \lambda(n)\, 2^n$. The number of basis elements (not necessarily different) involved in the right hand side of the inequality (3.10) is equal or less than $n \cdot 2^n$ (more exactly, $(1+n/2) \cdot 2^n$). Hence we get the following equality

$$2^n \varphi_1 + \sum_{j=1}^{n} 2^{n-j} \sum_{i=2^{j-1}}^{2^j-1} \varphi_{\pi(i+1)} = \sum_{k=1}^{2^n} \sum_{i=1}^{l_k} \varphi_{k_i}, \tag{3.11}$$

where $1 \leq l_k \leq n$ for any $k = 1, 2, \ldots, 2^n$, moreover, $\varphi_{k_i} \in \{\varphi_1, \varphi_2, \ldots, \varphi_{2^n}\}$ and for any fixed k we have $\varphi_{k_i} \neq \varphi_{k_j}$ for every $i \neq j$, $i, j = 1, 2, \ldots, l_k$, and for any fixed i the elements φ_{k_i} for different indexes k can be coincide. As the basis (φ_i) is subsymmetric with unit subsymmetric constant and (3.10) and (3.11) is valid we have for every m

$$\varrho^{(n)}(m) \leq \left\| 2^n \varphi_1 + \sum_{j=1}^{n} 2^{n-j} \sum_{i=2^{j-1}}^{2^j-1} \varphi_{\pi(i+1)} \right\| = \left\| \sum_{k=1}^{2^n} \sum_{i=1}^{l_k} \varphi_{k_i} \right\| \leq$$

$$\leq \sum_{k=1}^{2^n} \left\| \sum_{i=1}^{l_k} \varphi_{k_i} \right\| = \sum_{k=1}^{2^n} \lambda(l_k) \leq \lambda(n)\, 2^n$$

and the theorem is proved. □

Remark 3.2. For the estimations proved in Theorem 3.1 with respect to the natural basis in the case $X = l_1$ we have $1 + \sum_{j=1}^{n} 2^{-j} \lambda(2^{j-1}) \leq \lambda(n)$, but for the case $X = c_0$ we have the converse relation $1 + \sum_{j=1}^{n} 2^{-j} \lambda(2^{j-1}) \geq \lambda(n)$.

Let X be a Banach space (not necessarily with basis), $x_1, x_2, \ldots, x_{2^n}$ be a sequence of elements from the unit ball of X and $\mathcal{S}^{(n)}$ be the Sylvester matrix of order 2^n, $n = 1, 2, \ldots$. By analogy with the definition of $\varrho^{(n)}$ let $\hat{\varrho}^{(n)}(m) = \left\| \sum_{i=1}^{2^n} \left(\sum_{k=1}^{m} s_{ki}^{(n)} \right) x_i \right\|$, $m = 1, 2, \ldots, 2^n$, and let $\hat{\varrho}^{(n)} = \max_{1 \leq m \leq 2^n} \hat{\varrho}^{(n)}(m)$.

Corollary 3.3. *We have* $\hat{\varrho}^{(n)} \leq n \cdot 2^n$.

Proof. Using the triangular inequality and the fact that $||x_i|| \leq 1$ for any i, we have

$$\hat{\varrho}^{(n)}(m) \leq \sum_{i=1}^{2^n} \left| \sum_{k=1}^{m} s_{ki}^{(n)} \right| .$$

The right hand side of the last relation is the expression $\varrho^{(n)}(m)$ in the space l_1 with respect to the natural basis, which is for every $m = 1, 2, \dots, 2^n$ less or equal than $n\, 2^n$ (cf. Theorem 3.1). □

Corollary 3.4. *Let X be a Banach space of type p, $p > 1$, with a normalized subsymmetric basis (φ_i) whose subsymmetric constant is 1. Then one has*

$$\varrho^{(n)} \leq c \cdot 2^n,$$

where the constant $c \geq 1$ depends only on the space X.

Proof. Since (φ_i) is a normalized subsymmetric basis, whose subsymmetric constant is 1, then $\lambda(2^{j-1}) \leq T_p(X)\, 2^{(j-1)/p}$ for every $j \geq 1$, where $T_p(X)$ is the constant involved in the definition of the space of type p. Then for the right hand side of (3.3) we get

$$1 + \sum_{j=1}^{n} 2^{-j} \lambda(2^{j-1}) \leq 1 + T_p(X) \sum_{j=1}^{n} 2^{-j+(j-1)/p} \leq 1 + T_p(X)/(2 - 2^{1/p}).$$

Taking as c the value $1 + T_p(X)/(2 - 2^{1/p})$ the proof is finished. □

It should be pointed out that in the space c_0 we have a similar estimation, namely $\varrho^{(n)} \leq 2^n$ (cf. Theorem 3.1), although c_0 is a space of type 1. As $\varrho^{(n)} \geq 2^n$, we get $\varrho^{(n)} = 2^n$ in the space c_0.

Thus, in the Banach spaces of type p, $p > 1$, (as well as in c_0), we have $\sup\limits_n \varrho^{(n)}/2^n < \infty$. But in generally this is not true. The following statement shows this fact in the space l_1.

Theorem 3.5. [6]. *For the space l_1 with the natural basis, one has*

$$\varrho^{(n)} = \max_{1 \leq m \leq 2^n} \varrho^{(n)}(m) = (3n + 7)2^n/9 + 2(-1)^n/9, \quad n \geq 1.$$

For any n the maximum is attained at the points $m_n = (2^{n+1} + (-1)^n)/3$ and $m_n^{'} = (5 \cdot 2^{n-1} + (-1)^{n-1})/3$.

Let us estimate $\varrho^{(n)}$ from below.

Theorem 3.6. *If a Banach space X satisfies the conditions of Theorem 3.1, then one has*

$$\varrho^{(n)} \geq \max\left\{\frac{n+2}{6}\,\lambda(2^n),\ 2^n\right\}, \quad n = 1, 2, \ldots .$$

Proof. By definition of the expression $\varrho^{(n)}(m)$ for any integer n we have, $\varrho^{(n)} \geq \varrho^{(n)}(2^n) = 2^n$ and the inequality $\varrho^{(n)} \geq 2^n$ is evident.

Let us prove that for any integer n the inequality $\varrho^{(n)} \geq \frac{n+2}{6}\,\lambda(2^n)$ is also true. Using inequality (1.2) in Proposition 2.2(**B**) for any integer n we have

$$\left\|\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|\varphi_i\right\| \geq \frac{\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|}{2^{n+1}}\,\lambda(2^n), \quad \text{for any} \quad m = 1, 2, \ldots, 2^n,$$

where the numbers $\alpha_i^{(n)}(m)$ are defined by (3.4). Then for any integer n we get

$$\max_{1\leq m\leq 2^n}\left\|\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|\varphi_i\right\| \geq \frac{\max_{1\leq m\leq 2^n}\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|}{2^{n+1}}\,\lambda(2^n). \tag{3.12}$$

As it is known $\left\|\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|\varphi_i\right\| = \varrho^{(n)}(m)$ and $\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right|$ is the value of the expression $\varrho^{(n)}(m)$ in the space l_1 with respect to the natural basis. Therefore, by Theorem 3.5 we have

$$\max_{1\leq m\leq 2^n}\sum_{i=1}^{2^n}\left|\alpha_i^{(n)}(m)\right| = \frac{3n+7}{9}\,2^n + (-1)^n\,\frac{2}{9}, \quad \text{for any} \quad n = 1, 2, \ldots .$$

Substituting these expressions in (3.12) an elementary calculation yields the assertion. $\square$

Remark 3.7. If a basis (φ_i) of a space X is, in addition, symmetric, then using inequality (1.1) of Proposition 2.2(**A**) by analogy with Theorem 3.6 we can prove that

$$\varrho^{(n)} \geq \max\left\{\frac{n+2}{3}\,\lambda(2^n),\ 2^n\right\}, \quad n = 1, 2, \ldots .$$

From Theorem 3.6 it follows that in the spaces of type p, $p > 1$, for sufficiently large n the lower estimation 2^n is more precisely than the estimation $\frac{n+2}{6}\,\lambda(2^n)$, because of in such spaces we have $\lambda(2^n) \leq T_p(X)\,2^{n/p}$. Hence, the lower estimation $\frac{n+2}{6}\,\lambda(2^n)$ can compete with 2^n in the spaces having type 1.

The following example shows that besides l_1 there exist Banach spaces, for which $\sup_n \varrho^{(n)}/2^n = \infty$.

Example 3.8. Consider the real function $f(t) = \frac{\sqrt{\log_2 5}}{5} \frac{t+4}{\sqrt{\log_2(t+4)}}$, $t \geq 1$. It is a concave function since its second derivative

$$f^{''}(t) = \frac{\sqrt{\log_2 5}}{10 \ln 2} \cdot \frac{-\log_2(t+4) + 3/(2\ln 2)}{(t+4)\log_2^{5/2}(t+4)} \leq 0$$

for every $t \geq 1$. Consider a sequence (λ_n) of real numbers, defined by the equations: $\lambda_n = f(n)$, $n = 1, 2, \ldots$. The sequence (λ_n) is concave (cf. Proposition 2.1(c)). Then there exists at least one Banach space X with symmetric basis (φ_i), whose symmetric constant is equal to 1 and for which $\lambda(n) = \left\| \sum\limits_{i=1}^{n} \varphi_i \right\| = \lambda_n$ for every $n = 1, 2, \ldots$ (see [7], p. 120). Hence, by Remark 3.7 for any integer n we have

$$\varrho^{(n)} \geq \frac{n+2}{3} \cdot \lambda(2^n) = \frac{n+2}{3} \cdot \frac{\sqrt{\log_2 5}}{5} \cdot \frac{2^n + 4}{\sqrt{\log_2(2^n+4)}} >$$

$$> \frac{\sqrt{\log_2 5}}{15} \cdot \frac{n+2}{\sqrt{n+2}} \cdot 2^n \geq \frac{\sqrt{\log_2 5}}{15} \cdot \sqrt{n+2} \cdot 2^n.$$

The space X does not isomorphic to l_1, as

$$\lim_{n\to\infty} \sup_n \frac{\lambda(n)}{n} = \lim_{n\to\infty} \sup_n \left(\frac{\sqrt{\log_2 5}}{5} \cdot \frac{n+4}{n\sqrt{\log_2(n+4)}} \right) = 0.$$

From the obtained estimation by Corollary 3.4 in particular it follows that the type of the space X does not exceed 1.

4. Hadamard matrices

In connection with the Hadamard matrices the following natural problem arises: are there generalizations of the above estimates for general Hadamard matrices? Let $\mathcal{H}_n^{all}$ be the set of all Hadamard matrices of order n, $n \in \mathbb{N}_{\mathcal{H}}$. For a Hadamard matrix $\mathcal{H}_n = [h_{ki}^n]$, consider the same numerical characteristic $\varrho_{\mathcal{H}_n}(m) = \left\| \sum\limits_{i=1}^{n} \left(\sum\limits_{k=1}^{m} h_{ki}^n \right) \varphi_i \right\|$, $m = 1, 2, \ldots, n$, where (φ_i) is a normalized basis of a Banach space X. Setting $a_k = \sum\limits_{i=1}^{n} h_{ki}^n \varphi_i$, we notice that

$$\varrho_{\mathcal{H}_n}(m) = \left\| \sum_{k=1}^{m} a_k \right\|. \tag{4.1}$$

If (φ_i) is an unconditional basis with unconditional constant 1, then one has $\max\limits_{1\le m\le n} \varrho_{\mathcal{H}_n}(m) \le \lambda(n)\, n \le n^2$ for any $\mathcal{H}_n \in \mathcal{H}_n^{all}$.

Finally we set $\varrho_{\mathcal{H}_n} = \max\limits_{1\le m\le n} \varrho_{\mathcal{H}_n}(m)$ and $\varrho_n = \max\limits_{\mathcal{H}_n\in\mathcal{H}_n^{all}} \varrho_{\mathcal{H}_n}$.

Remark 4.1. Note that the characteristic $\varrho_{\mathcal{H}_n} = \varrho(\mathcal{H}_n)$ can be regarded as a norm of the Hadamard matrix $\mathcal{H}_n$. Indeed, denote by $\mathbf{M}_n$ the vector space of all square matrices of the order n, $n \in \mathbb{N}_{\mathcal{H}}$, and let X be a Banach space with a basis (φ_i). Obviously $\mathcal{H}_n^{all} \subset \mathbf{M}_n$. Let $\mathcal{T}_n = [t_{ki}^n] \in \mathbf{M}_n$ be a matrix and $\varrho(\mathcal{T}_n) = \max\limits_{1\le m\le n} \left\| \sum\limits_{i=1}^{n} \left(\sum\limits_{k=1}^{m} t_{ki}^n \right) \varphi_i \right\|$. It is easy to see that ϱ is a norm in $\mathbf{M}_n$ and with respect with this norm $\mathbf{M}_n$ is a Banach space.

The following theorem gives us the lower estimate for ϱ_n.

Theorem 4.2. *Let X be a Banach space with a normalized unconditional basis whose unconditional constant is 1. Then*

$$\varrho_n \ge \max\left\{ (1/\sqrt{2})\, \lambda(n)\, \sqrt{n},\, n \right\} \quad \text{for any} \quad n \in \mathbb{N}_{\mathcal{H}}.$$

Proof. If one of the columns of a Hadamard matrix $\mathcal{H}_n$ consists of only $+1$, then we have $\varrho_{\mathcal{H}_n}(n) = n$ and the inequality $\varrho_n \ge n$ is evident.

Let now $\mathcal{H}_n = [h_{ki}^n]$ be a Hadamard matrix of order n and let $(r_k(t))_{k\le n}$ be a sequence of Rademacher functions defined on the interval $[0,1]$. The matrix $\mathcal{H}_{n,t} = [h_{ki}^n\, r_k(t)]$ for every $t \in [0,1]$ is also a Hadamard matrix, for which $\varrho_{\mathcal{H}_{n,t}} = \max\limits_{1\le m\le n} \varrho_{\mathcal{H}_{n,t}}(m) = \max\limits_{1\le m\le n} \left\| \sum\limits_{k=1}^{m} a_k\, r_k(t) \right\|$, where $a_k = \sum\limits_{i=1}^{n} h_{ki}^n \varphi_i,\ k = 1,2,\ldots,n$.

Denote $\xi(t) = \sum\limits_{i=1}^{n} \left| \sum\limits_{k=1}^{n} \langle \varphi_i^*, a_k\, r_k(t) \rangle \right| \varphi_i$. Using the fact that (φ_i) is an unconditional basis for which the unconditional constant is equal to 1, it is easy to see that

$$||\xi(t)|| = \left\| \sum_{i=1}^{n} \sum_{k=1}^{n} \langle \varphi_i^*, a_k \rangle\, r_k(t) \varphi_i \right\| =$$

$$= \left\| \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \langle \varphi_i^*, a_k \rangle\, \varphi_i \right) r_k(t) \right\| = \left\| \sum_{k=1}^{n} a_k\, r_k(t) \right\|$$

for every $t \in [0,1]$. From boundedness of the Rademacher functions it follows the integrability of $||\xi(t)||$ with respect to Lebesgue measure on $[0,1]$. Hence, there exists the Pettis integral $\mathbb{E}\,\xi$ of the measurable function ξ, and, as it is well-known, $\mathbb{E}\,||\xi|| \ge ||\mathbb{E}\,\xi||$. It is easy to see that $\mathbb{E}\,\xi = \sum\limits_{i=1}^{n} \left(\mathbb{E} \left| \sum\limits_{k=1}^{n} \langle \varphi_i^*, a_k \rangle\, r_k(t) \right| \right) \varphi_i$.

From boundedness of the Rademacher functions it follows also the integrability of the expression $\varrho_{\mathcal{H}_{n,t}}$ with respect to Lebesgue measure on $[0,1]$ and using the Khintchine inequality we have

$$\infty > \mathbb{E}\,\varrho_{\mathcal{H}_{n,t}} = \mathbb{E}\max_{1\le m\le n}\left\|\sum_{k=1}^{m} a_k\, r_k(t)\right\| \ge \mathbb{E}\,||\xi|| \ge ||\mathbb{E}\,\xi|| \ge$$

$$\ge (1/\sqrt{2})\left\|\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\langle\varphi_i^*, a_k\rangle^2\right)^{1/2}\varphi_i\right\| = (1/\sqrt{2})\lambda(n)\sqrt{n},$$

where (φ_i^*) are the biorthogonal functionals corresponding to the basis (φ_i). Then, clearly, there exists a point $t_0 \in [0,1]$, such that $\varrho_{\mathcal{H}_{n,t_0}} \ge \mathbb{E}\,\varrho_{\mathcal{H}_{n,t}}$ and, therefore, $\varrho_n \ge \varrho_{\mathcal{H}_{n,t_0}} \ge (1/\sqrt{2})\lambda(n)\sqrt{n}$. □

The immediate consequence of this theorem is the following corollary.

Corollary 4.3. *In spaces l_p, $1 \le p < 2$, with respect to the natural basis we have* $\sup\limits_{n\in\mathbb{N}_{\mathcal{H}}} \varrho_n/n = \infty$.

Among the spaces l_p the similar fact for the Sylvester matrices is satisfied only in the space l_1 (see Theorem 3.5).

Let us estimate ϱ_n from above for the case of spaces l_p, $1 \le p < \infty$.

Theorem 4.4. *In the space l_p, $1 \le p < \infty$, with respect to the natural basis for any $n \in \mathbb{N}_{\mathcal{H}}$ the following inequality is valid*

$$\varrho_n \le \max\left\{n^{(p+2)/2p},\, n\right\}.$$

Proof. Let $p \ge 2$ and $\mathcal{H}_n \in \mathcal{H}_n^{all}$ be a Hadamard matrix of order n. Using definition (4.1) and the fact $||a||_{l_p} \le ||a||_{l_2}$, one can see that

$$\varrho_{\mathcal{H}_n} \le \max_{1\le m\le n}\left\|\sum_{k=1}^{m} a_k\right\|_{l_2} = \max_{1\le m\le n}\left(\sum_{k=1}^{m} a_k, \sum_{k=1}^{m} a_k\right)^{1/2} = n,$$

where by the symbol $(\cdot,\cdot)$ we denote the inner product in the space l_2.

As $\mathcal{H}_n$ is a Hadamard matrix, it follows that in spaces l_p, $p \ge 2$, for every Hadamard matrices the estimate $\varrho_n \le n$ is valid.

Now let $1 \le p \le 2$ and $\mathcal{H}_n \in \mathcal{H}_n^{all}$ be again a Hadamard matrix. If $a = (\alpha_i) \in l_p$ is a sequence length of which is n (i.e. $\alpha_n \ne 0$ and $\alpha_i = 0$ for any $i > n$), then one has $||a||_{l_p} \le n^{(2-p)/2p}\,||a||_{l_2}$. Hence, we have

$$\varrho_{\mathcal{H}_n} \le n^{(2-p)/2p}\max_{1\le m\le n}\left\|\sum_{k=1}^{m} a_k\right\|_{l_2} = n^{(p+2)/2p},$$

and by arbitrariness of a Hadamard matrix $\mathcal{H}_n$, the proof is finished. □

For Sylvester matrices Corollary 4.3 and Theorem 4.4 yield the following corollary.

Corollary 4.5. *Let $S^{(n)}$ be a Sylvester matrix of order 2^n, $n = 1, 2, \ldots$. Then in the space l_p, $p \geq 2$, with respect to the natural basis we have*

$$\varrho^{(n)} = 2^n.$$

From Theorem 4.2 and 4.4 it follows the following result.

Corollary 4.6. *In the space l_p, $1 \leq p \leq \infty$, with respect to the natural basis for every $n \in \mathbb{N}_{\mathcal{H}}$ we have*

$$(1/\sqrt{2})\, n^{(p+2)/2p} \leq \varrho_n \leq n^{(p+2)/2p}, \quad \text{for} \quad 1 \leq p < 2,$$

$$\varrho_n = n, \quad \text{for} \quad p \geq 2.$$

Let X be a Banach space (not necessarily with a basis), let $x_1, x_2, \ldots, x_n$ be a sequence of elements from the unit ball of X and let $\mathcal{H}_n \in \mathcal{H}_n^{all}$, $n \in \mathbb{N}_{\mathcal{H}}$. Let us consider $\hat{\varrho}_{\mathcal{H}_n}(m) = \left\| \sum_{i=1}^{n} \left(\sum_{k=1}^{m} h_{ki}^n \right) x_i \right\|$, $m = 1, 2, \ldots, n$, $\hat{\varrho}_{\mathcal{H}_n} = \max_{1 \leq m \leq n} \hat{\varrho}_{\mathcal{H}_n}(m)$ and $\hat{\varrho}_n = \max_{\mathcal{H}_n \in \mathcal{H}_n^{all}} \hat{\varrho}_{\mathcal{H}_n}$.

Corollary 4.7. *For any $n \in \mathbb{N}_{\mathcal{H}}$ one has $\hat{\varrho}_n \leq n\sqrt{n}$.*

Proof. Using Corollary 4.6 for the case $p = 1$, the proof passes in the analogous way as the proof of Corollary 3.3. □

Now we prove the analogue of Theorem 3.1 for the case of the Hadamard matrices.

Theorem 4.8. *Let X be a Banach space with a normalized subsymmetric basis whose subsymmetric constant is 1. Then one has for any $n \in \mathbb{N}_{\mathcal{H}}$*

$$\varrho_n \leq \lambda([\sqrt{n}] + 1)\, n,$$

where $[\sqrt{n}]$ is the integer part of $\sqrt{n}$.

Proof. Let $\mathcal{H}_n = [h_{ki}^n]$ be a Hadamard matrix of order n. As we already noted

$$\varrho_{\mathcal{H}_n} = \max_{1 \leq m \leq n} \left\| \sum_{i=1}^{n} \left(\sum_{k=1}^{m} h_{ki}^n \right) \varphi_i \right\| \leq \max_{1 \leq m \leq n} \sum_{i=1}^{n} \left| \sum_{k=1}^{m} h_{ki}^n \right| \leq n\sqrt{n} \tag{4.2}$$

for every $\mathcal{H}_n \in \mathcal{H}_n^{all}$. For the convenience let us introduce the notation

$$\alpha_i^{(n)}(m) = \left|\sum_{k=1}^{m} h_{ki}^n\right| \quad \textit{for any} \quad i, m = 1, 2, \ldots, n. \tag{4.3}$$

From the definition of Hadamard matrices and from (4.2) it follows the following properties of the numbers $\alpha_i^{(n)}(m)$:

(a). For every indexes i and m the number $\alpha_i^{(n)}(m)$ is an integer and $0 \leq \alpha_i^{(n)}(m) \leq n$.

(b). For any m the following condition $\sum\limits_{i=1}^{n} \alpha_i^{(n)}(m) \leq n\sqrt{n}$ is satisfied.

If we denote by M a subset of X consisting of the following n points $M = \{\sum\limits_{i=1}^{n} \alpha_i^{(n)}(m)\varphi_i : m = 1, 2, \ldots, n\}$, where $\alpha_i^{(n)}(m)$ is defined by (4.3), then one has $\varrho_{\mathcal{H}_n} = \max\limits_{x \in M} ||x||$.

Let us consider in X the following subsets:

$$S = \{\sum_{i=1}^{n} t_i \varphi_i : 0 \leq t_i \leq n, i = 1, 2, \ldots, n\} \quad \text{and} \quad T = \{\sum_{i=1}^{n} t_i \varphi_i : \sum_{i=1}^{n} t_i \leq n\sqrt{n}\}.$$

Since S is a n-dimensional parallelepiped and T is a hyperplane in X the sets S, T and their intersection $S \cap T$ are convex. Moreover, one has $M \subset S \cap T$. As $S \cap T$ is a bounded set in a n-dimensional subset of X spanned on the basis vectors $\varphi_1, \varphi_2, \ldots, \varphi_n$ it is compact and by the Krein-Milman theorem (see, for example, [3], p. 104) it is a closed convex span of its extreme points. Hence we have

$$\varrho_{\mathcal{H}_n} = \max_{x \in M} ||x|| \leq \sup_{x \in S \cap T} ||x|| = \sup_{x \in E} ||x||, \tag{4.4}$$

where E is the set of the extreme points of $S \cap T$. The extreme points of the set S are the vertices of the parallelepiped S, i.e. the points of the form $\sum\limits_{i=1}^{n} \beta_i \varphi_i$, where each β_i takes the values 0 or n. As $E \subset S \cap T$ the set E contains those extreme points of S for which the condition $\sum\limits_{i=1}^{n} \beta_i \leq n\sqrt{n}$ is satisfied. If we denote by l the number of these β_i, which are different from zero, then the last condition we can write in the following way: $ln \leq n\sqrt{n}$, i.e. $l \leq \sqrt{n}$. As l is an integer we get $l \leq [\sqrt{n}]$. Since the basis (φ_i) is subsymmetric, the norm of each such element can be estimated in the following way $\left\|\sum\limits_{i=1}^{n} \beta_i \varphi_i\right\| \leq \lambda(l)\, n < \lambda([\sqrt{n}] + 1)\, n$.

It is easy to check that the set E, besides of the vertices of the parallelepiped S, also contains the points of intersection of bound of the set T with the edges of the parallelepiped S, the points of which have the form $\sum_{i=1}^{n} \beta_i \varphi_i$, where one of β_i satisfies the condition $0 \le \beta_{i_0} \le n$, and all other β_i take the values 0 or n. Denote by l the number of β_i, for which $\beta_i = n$. By virtue of the condition $\sum_{i=1}^{n} \beta_i \varphi_i \in T$, we have $\beta_{i_0} + ln \le n\sqrt{n}$. As $\beta_{i_0} \ge 0$ and l is an integer we have $l \le [\sqrt{n}]$. Since $0 \le \beta_{i_0} \le n$, using again the subsymmetry of the basis (φ_i), we obtain

$$\left\| \sum_{i=1}^{n} \beta_i \varphi_i \right\| = \left\| \beta_{i_0} \varphi_{i_0} + \sum_{i_0 \ne i=1}^{n} \beta_i \varphi_i \right\| \le \left\| n \varphi_{i_0} + \sum_{i_0 \ne i=1}^{n} \beta_i \varphi_i \right\| \le \lambda([\sqrt{n}] + 1)\, n.$$

Thus, for every point x of the set E the estimation $||x|| \le \lambda([\sqrt{n}] + 1)\, n$ is valid and using (4.4) the theorem is proved. □

Remark 4.9. We can rephrase Theorem 4.8 in such a way. Let $\mathcal{H}_n = [h_{ki}^n]$ be a Hadamard matrix of order $n \in \mathbb{N}_{\mathcal{H}}$ and let $a_k = \sum_{i=1}^{n} h_{ki}^n \varphi_i, k = 1, 2, \ldots, n$, where (φ_i) is a normalized subsymmetric basis of a Banach space X with subsymmetric constant 1. Then one has

$$\max_{1 \le m \le n} \left\| \sum_{k=1}^{m} \vartheta_k a_k \right\| \le \lambda \left([\sqrt{n}] + 1 \right) \, n$$

for any signs $\vartheta_k = \pm 1$, $k = 1, 2, \ldots, n$, any Hadamard matrices $\mathcal{H}_n \in \mathcal{H}_n^{all}$ and positive integer $n \in \mathbb{N}_{\mathcal{H}}$.

As follows from Theorem 3.1, in a Banach space with a normalized subsymmetric basis whose subsymmetric constant is equal to 1 we have $\varrho^{(n)} / \, (n \cdot 2^n) \le 1$. On the other hand, by Theorem 3.5 in the space l_1 we have $\varrho^{(n)} / \, (n \cdot 2^n) \ge 1/3$. Using Sylvester and Hadamard matrices we can characterize the spaces isomorphic to l_1 in the following way.

Theorem 4.10. *Let X be a Banach space with a normalized subsymmetric basis (φ_i) whose subsymmetric constant is 1. The following statements are equivalent:*

(i). There is a constant $\delta > 0$ such that $\varrho_n / \, (n\sqrt{n}) \ge \delta$ for every $n \in \mathbb{N}_{\mathcal{H}}$, where δ is independent of n.

(ii). X is isomorphic to l_1.

(iii). There exists a constant $\varepsilon > 0$ not depending on n such that for any $n = 1, 2, \ldots$ we have $\varrho^{(n)} / \, (n \cdot 2^n) \ge \varepsilon$.

Proof. $(i) \Rightarrow (ii)$. Using Theorem 4.8 for any $n \in \mathbb{N}_{\mathcal{H}}$ we have

$$0 < \delta \le \varrho_n/(n\sqrt{n}) \le \lambda([\sqrt{n}]+1)\, n/(n\sqrt{n}) = \lambda([\sqrt{n}]+1)/\sqrt{n}.$$

Therefore $\lambda([\sqrt{n}])/\sqrt{n} \ge \delta/2 > 0$ for infinitely many n. Now the validity of the statement (ii) follows from the fact which was mentioned in Section 2: if

$$\lim_{n\to\infty} \sup \lambda(n)/n > 0,$$

then X is isomorphic to l_1.

$(ii) \Rightarrow (iii)$. Let X be isomorphic to l_1 and denote by $T : X \to l_1$ an isomorphism between X and l_1. It is clear that $(T\varphi_i)$ is an unconditional basis in l_1. Since in l_1 all normalized unconditional bases are equivalent (see [7], p. 71), there exists a bounded linear operator $S : l_1 \to l_1$ with bounded inverse operator, such that $T\varphi_i = Se_i$ for any integer i, where (e_i) is a sequence of unit vectors in l_1. By Theorem 3.5 for every integer n we have

$$1/3 \le \max_{1\le m\le 2^n} \left\| \sum_{i=1}^{2^n} \left| \sum_{k=1}^{m} s_{ki}^{(n)} \right| e_i \right\| / \left(n\cdot 2^n\right) =$$

$$= \max_{1\le m\le 2^n} \left\| \sum_{i=1}^{2^n} \left| \sum_{k=1}^{m} s_{ki}^{(n)} \right| S^{-1}T\varphi_i \right\| / \left(n\cdot 2^n\right) \le \|S^{-1}T\| \max_{1\le m\le 2^n} \varrho^{(n)}(m)/\left(n\cdot 2^n\right).$$

Hence, denoting $\varepsilon = 1/\left(3||S^{-1}T||\right) > 0$, we get the validity of assertion (iii).

The implication $(iii) \Rightarrow (i)$ is true because $2^n \in \mathbb{N}_{\mathcal{H}}$. □

5. Unsolved problem

Let (e_i) be the natural basis of the space l_1, $\mathcal{S}^{(n)} = \left[s_{ki}^{(n)}\right]$ be a Sylvester matrix of order 2^n and $(a_k)_{k\le 2^n}$ be a sequence in l_1 defined as follows

$$a_k = \sum_{i=1}^{2^n} s_{ki}^{(n)} e_i, \quad k = 1, 2, \ldots, 2^n.$$

Let us formulate the assertion of Theorem 3.5 in the following manner

$$\varrho^{(n)} = \left\| \sum_{k=1}^{m_n} a_k \right\|_{l_1} = (3n+7)2^n/9 + 2(-1)^n/9,$$

where $m_n = (2^{n+1} + (-1)^n)/3$.

Now let us consider a permutation $\sigma : \{1, 2, \ldots, 2^n\} \to \{1, 2, \ldots, 2^n\}$ and the following expression:

$$\left\| \sum_{k=1}^{m_n} a_{\sigma(k)} \right\|_{l_1}.$$

By Corollary 4.6 for any permutation $\sigma : \{1, 2, \ldots, 2^n\} \to \{1, 2, \ldots, 2^n\}$ we have

$$\left\| \sum_{k=1}^{m_n} a_{\sigma(k)} \right\|_{l_1} \leq 2^{3n/2}.$$

The authors do not know yet the answer of the following conjecture:

Conjecture 5.1. *For any positive integer $n \geq 1$ and for any permutation of integers $\sigma : \{1, 2, \ldots, 2^n\} \to \{1, 2, \ldots, 2^n\}$ we have*

$$\left\| \sum_{k=1}^{m_n} a_{\sigma(k)} \right\|_{l_1} \geq (3n + 7)2^n/9 + 2(-1)^n/9.$$

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