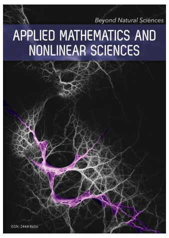

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Geometric Properties of a Linear Complex Operator on a Subclass of Meromorphic Functions: An Analysis of Hurwitz-Lerch-Zeta Functions

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Abstract

Geometric function theory (GFT) is one of the richest research disciplines in complex analysis. This discipline also deals with the extended differential inequality theory, known as the differential subordination theory. Based on these theories, this study focuses on analyzing intriguing aspects of the geometric subclass of meromorphic functions in terms of a linear complex operator and a special class of Hurwitz-Lerch-Zeta functions. Hence, several of its geometric attributes are deduced. Furthermore, the paper highlights the different fascinating advantages and applications of various new geometric subclasses in relation to the subordination and inclusion theorems.

Keywords: Meromorphic function; Hadamard product; Subordination principle; Gaussian hypergeometric function; Hurwitz-LerchZeta function.
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1 Introduction

In the field of complex analysis, the research of Geometric Function Theory (GFT) is a leading topic. It was instituted in the 19th century by introducing the Riemann mapping theorem (RMT) launched by Riemann in 1850. Then, in 1907, Koebe proposed discussing univalent regular functions, and subsequently, in light of RMT, started to study the attributes of regular and univalent functions on the complex open unit disk instead of a general complex domain that is simply connected. After the development of RMT, studies related to GFT began to develop, and numerous remarkable outcomes correlating with univalent functions were yielded. In this context, the meromorphic function, which is a univalent regular function in the punctured complex unit disk $\Delta^* = \Delta \setminus 0$ where $\Delta = z \in \mathbb{C}: 0 \leq z < 1$, is a fundamental concept in the study of GFT because of its interesting attributes. Following discussions on the usage of hypergeometric functions in proving Bieberbach's conjecture, presented by Brange in 1985, the interest in the combination of meromorphic functions and special functions has succeeded in acquiring inspiring outcomes related to a lot of aspects, such as the creation of new subclasses of these combined functions in Δ^* and their application in creating new complex operators. The private theme of Hurwitz-Lerch-Zeta functions appeared as part of this study [9], [6], [13]. This idea is investigated from numerous points of view in meromorphic function theory [11], [12], [23].

Class Σ consists of meromorphic functions $f(z)$ in Δ^* expressed as:

$$f(z) = \frac{1}{z} + \sum_{\kappa=1}^{\infty} \sigma_\kappa z^\kappa. \tag{1}$$

Consider $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)(0 \leq \eta)$ as the subclasses of Σ denoting meromophic starlike and convex functions of order η in Δ^*, respectively. Hadamard's principle corresponding to meromorphic functions $f_v(z)(v = 1; 2)$ in Σ is formulated by

$$(f_1 * f_2) = \frac{1}{z} + \sum_{\kappa=1}^{\infty} \sigma_{\kappa,1}\sigma_{\kappa,2} z^\kappa. \tag{2}$$

Let us first consider the following special function $\tilde{\psi}(\gamma, ð; z)$ by:

$$\tilde{\psi}(\gamma, ð; z) = \frac{1}{z} + \sum_{\kappa=0}^{\infty} \frac{(\gamma)_{\kappa+1}}{(ð)_{\kappa+1}} z^\kappa, \tag{3}$$

For $\gamma \in C \setminus \{0\}$, and $ð \neq 0, -1, -2, \ldots$, where $(T)_\kappa = T(T+1)_{\kappa+1}$ denotes the increasing factorial (Pochhammer symbol). Note that

$$\tilde{\psi}(\gamma, ð; z) = \frac{1}{z}\, {}_2F_1(1, \gamma, ð; z),$$

where

$${}_2F_1(\wp, \gamma, ð; z) = \sum_{\kappa=0}^{\infty} \frac{(\wp)_\kappa(\gamma)_\kappa}{(ð)_\kappa} \frac{z^\kappa}{\kappa!}$$

is a hypergeometric series, called a "Gaussian hypergeometric function".

The generic Hurwitz Lerch zeta function is symbolized by E (z,κ,μ) and is represented as [21], [22]:

$$\mathcal{E}(z, \varkappa, \mu) = \frac{1}{\mu^\varkappa} + \sum_{\kappa=1}^{\infty} \frac{z^\kappa}{(\kappa+\mu)^\varkappa}. \tag{4}$$

The function $\mathcal{E}(z,\varkappa,\mu)$ has several interesting specific instances, as such, the function of Reimann zeta $\mathcal{R}(\varkappa)=\mathcal{E}(1,\varkappa,1)$, the function of Hurwitz zeta $\mathcal{Z}(\varkappa,\mu)=\mathcal{E}(1,\varkappa,\mu)$, the function of Lerch zeta $\mathcal{L}_{\varkappa}(\mathcal{R})=\mathcal{E}\left(\exp^{2\pi i\zeta},\varkappa,1\right)$, where $\Re(\varkappa)>1,\zeta\in\mathcal{R}$, and the polylogarithm $\mathcal{L}_{\varkappa}^{i}(z)=z\mathcal{E}(z,\varkappa,\mu)$ and so on. For recent outcomes on $\mathcal{E}(z.\varkappa,\mu)$, see [23], [24], [25] and [26]. By using this standardized function, we propose a new special function as follows:

$$\begin{aligned}\mathcal{G}_{\varkappa,\mu}(z)&=(1+\mu)^{\varkappa}\left[\mathcal{E}(z,\varkappa,\mu)-\mu^{-\varkappa}+\frac{1}{z(1+\mu)^{\varkappa}}\right]\\&=\frac{1}{z}+\sum_{\kappa=1}^{\infty}\left(\frac{1+\mu}{\kappa+\mu}\right)^{\varkappa}z^{\kappa}.\end{aligned}\tag{5}$$

The Hadamard technique and $\mathcal{G}_{\varkappa,\mu}(z)$ are used to enforce a new complex linear operator $\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð)$ acting on Σ:

$$\begin{aligned}&\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð)f(z)=\psi(\gamma,ð;z)*\mathcal{G}_{\varkappa,\mu}(z)*f(z)\\&=\frac{1}{z}+\sum_{\kappa=1}^{\infty}\frac{(\gamma)_{\kappa+1}}{(ð)_{\kappa+1}}\left(\frac{1+\mu}{\kappa+\mu}\right)^{\varkappa}a_{\kappa}z^{\kappa},\quad(z\in\Delta^{*}).\end{aligned}\tag{6}$$

Many others have recently addressed the theory of meromorphic functions, which connects the class of hypergeometric functions and the class of Hurwitz Lerch zeta functions, for example [8], [5], [10], [14] and [15].

From (6), we gain

$$z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma+1,ð)f(z)\right)'=\gamma\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð)f(z)\right)-(\gamma+1)\mathcal{L}_{\mu}^{\varkappa}(\gamma+1,b)f(z),\tag{7}$$

and

$$z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð)f(z)\right)'=ð\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð+1)f(z)\right)-(ð+1)\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð)f(z).\tag{8}$$

The main idea of this paper is to develop and study a new geometric subclass of meromorphic functions in terms of Hurwitz Lerch zeta and hypergeometric functions. Moreover, their inclusion features are discussed.

Definition 1. The function $f(z)\in\Sigma$ is called in the class $\Sigma_{\mu}^{\varkappa}(\gamma,ð;\delta;v,\omega)$ if it fulfills the following situation for fixed parameters $v,\omega(-1\le\omega<v\le1)$, and $0\le\delta<1$:

$$\frac{1}{1-\delta}\left(\frac{-z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð)f(z)\right)'}{z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,ð)f(z)\right)}-\delta\right)\prec\frac{1+vz}{1+\omega z},\quad(z\in\Delta^{*}).\tag{9}$$

It is still a common practice to investigate different operators using the theory of differential superordination; some investigations contribute with results like the ones presented here [1], [2], [3]. It provides prerequisite conditions for numerous formulations of this proposed operator to establish subordination that can be identified and analyzed. The results are an extension of earlier findings on starlikeness, convexity, and close-to-convex, see [4], [7] and [20].

2 Preliminaries

We will need the following lemmas to get started with our results:

Lemma 1. *[16] For $v, \gamma, , \omega, t \in \mathbb{C}$ escorted by $t \neq 0$ and $|\omega| \leq 1$, assume that these constants meet the following conditions*

$$\Re(t(1-v)(1-\bar{\omega}) + \gamma|1-\omega|^2) > 0, \tag{10}$$

and

$$\Re(t(1-v)(1-\bar{\omega}) + \gamma|1-\omega|^2)\Re(t(1+v)(1+\bar{\omega}) + \gamma|1+\omega|^2) \\ -(\mathrm{Im}\,(t(\bar{\omega}-v) + \gamma(\bar{\omega}-\omega)))^2 \geq 0,$$

or

$$\Re(t(1+v)(1+\bar{\omega}) + \gamma|1+\omega|^2) \geq 0, \tag{11}$$

and

$$\Re(t(1-v)(1-\bar{\omega}) + \gamma|1-\omega|^2) = 0.$$

The solution to the differential equation

$$h(z) + \frac{zq'(z)}{th(z)+\gamma} = \frac{1+vz}{1+\omega z}$$

that is univalent is expressed as

$$h(z) = \begin{cases} \dfrac{z^{t+\gamma}(1+\omega z)^{\frac{t(v-\omega)}{\omega}}}{t\int_0^z t^{t+\gamma-1}(1+\omega t)^{\frac{t(v-\omega)}{\omega}}dt} - \dfrac{\gamma}{t} & \omega, t \neq 0, \\ \dfrac{z_0^{t+\gamma}e^{evz}}{t\int_0^2 t^{t+\gamma-1}e^{tvt}dt} - \dfrac{\gamma}{t} & t \neq 0. \end{cases} \tag{12}$$

Accordingly, $\psi(z) \prec h(z) \prec \frac{1+vz}{1+\omega z}$ and $q(z)$ are the best dominant if $\psi(z)$ is regular in Δ^* and fulfills

$$\psi(z) + \frac{z\psi'(z)}{t\psi(z)+\gamma} \prec \frac{1+vz}{1+\omega z}.$$

Lemma 2. [26]. *Suppose that φ is a measure that is positive over the interval $[0,1]$. Let $\lambda(z,m)$ be a complex valued function given on the interval $\Delta[0,1]$ in which $\lambda(.,m)$ is regular in Δ for each $m \in [0,1]$ and $\lambda(z,.)$ is ϕ-integrable over the interval $[0,1]$ for every $z \in \Delta$.*

Furthermore, consider that $\Re(\lambda(z,m))$ is positive, then $\lambda(-r,m) \in (-\infty,\infty)$, and

$$\Re\left(\frac{1}{\lambda(z,m)}\right) \geq \frac{1}{\lambda(-r,m)} \, (|z| \leq r < 1, m \in [0,1]).$$

If the function $\Lambda(z)$ is given as:

$$\Lambda(z) = \int_0^1 \lambda(z,m)dv(t),$$

Then

$$\Lambda(z) = \int_0^1 \lambda(z,m)dv(t)\Re\left(\frac{1}{\Lambda(z)}\right) \geq \frac{1}{\Lambda(-r)}(|z| \leq r < 1). \tag{13}$$

Lemma 3. *([27]). For real or complex numbers $\gamma, ð, \wp, (\wp \notin \mathbb{Z}_0^-)$, we gain*

$$\int_0^1 t^{ð-1}(1-t)^{\wp-ð-1}(1-tz)^{-\gamma}dt = \frac{\Gamma(ð)\Gamma(\wp-ð)}{\Gamma(\wp)}\ {}_2F_1(\gamma, ð; \wp; z), \tag{14}$$

$$(\Re(\wp) > \Re(ð) > 0, z \in \Delta);$$

where

$${}_2F_1(\gamma, ð; \wp; z) = {}_2F_1(ð, \gamma; \wp; z), \tag{15}$$

$${}_2F_1(\gamma, ð; \wp; z) = (1-z)^{-\gamma}\ {}_2F_1\left(\gamma, \wp - ð; \wp; \frac{z}{z-1}\right), \tag{16}$$

and

$$(ð+1)_2F_1(1, ð; ð+1; z) = (ð+1) + ðz_2F_1(1, ð+1; ð+2; z). \tag{17}$$

3 Geometric Outcomes

Theorem 4. *Let $f \in \Sigma_\mu^\varkappa(\gamma, ð; \delta; v, \omega)$ and the function Λ be given on Δ as*

$$\Lambda(z) = \begin{cases} \int_0^1 \left(\frac{1+\omega uz}{1+\omega z}\right)^{-\frac{(1-\delta)(v-\omega)}{\omega}} du & \omega \neq 0 \\ \int_0^1 e^{-v(1-\delta)(u-1)z} du & \omega = 0 \end{cases}, \tag{18}$$

and assuming the additional restrictions $0 < \omega < 1$ *and*

$$2 > \frac{(1-\gamma)(v-\omega)}{\omega} \tag{19}$$

are satisfied, then

$$\frac{1-|v|}{1-|\omega|} < \frac{1}{1-\delta}\left(-\Re\frac{z\left(\mathcal{L}_\mu^\varkappa(\gamma,ð)f(z)\right)'}{z\left(\mathcal{L}_\mu^\varkappa(\gamma,ð)f(z)\right)} - \delta\right) < \beta_1, \tag{20}$$

where

$$\beta_1 = \frac{1}{1-\delta}\left((2-\delta) - \frac{1}{{}_2F_1\left(1, \frac{(1-\gamma)(v-\omega)}{\omega}; 2; \frac{\omega}{(\omega-1)}\right)}\right). \tag{21}$$

The bound for β_1 *is the best that can be found.*

Proof: To prove (20), we note that the application of the subordination principle in (9) gives

$$\frac{1-|v|}{1-|\omega|} < \frac{1}{1-\delta}\left(-\Re\frac{z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,\eth)f(z)\right)'}{z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,\eth)f(z)\right)} - \delta\right),$$

which corresponds to the left-hand inequality in (20). Moreover, according to the concept of subordination

$$\frac{1}{1-\delta}\left(-\Re\frac{z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,\eth)f(z)\right)'}{z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma,\eth)f(z)\right)} - \delta\right) \leq \sup_{z\in\Delta^*}\Re\left(\lambda_1(z)\right)$$

$$= \sup_{z\in\Delta^*}\left(\frac{1}{1-\delta}\left((2-\delta) - \Re\left(\frac{1}{\Lambda(z)}\right)\right)\right)$$

$$= \frac{1}{1-\delta}\left(2-\delta-\inf_{z\in\Delta^*}\left(\Re\left(\frac{1}{\Lambda(z)}\right)\right)\right). \quad (22)$$

The remainder of the proof focuses on determining $\inf_{z\in\Delta^*}\left(\Re\left(\frac{1}{\Lambda(z)}\right)\right)$. As a result of the hypothesis $w \neq 0$, we have

$$\Lambda(z) = (1+\omega z)^{\chi}\int_0^1 (1+\omega uz)^{-\chi}du(z\in\Delta^*),$$

where $\chi = \frac{(1-\gamma)(v-\omega)}{\omega}$. Using (14)-(16) of Lemma 2.3, we obtain

$$\Lambda(z) = {}_2F_1\left(1,\frac{(1-\gamma)(v-\omega)}{\omega};2;\frac{\omega z}{(\omega z+1)}\right). \quad (23)$$

Furthermore, by applying (14) from Lemma 2.3 into (23), we obtain

$$\Lambda(z) = \int_0^1 g(z,u)dv(u),$$

where

$$g(z,u) = \frac{1+\omega z}{1+(1-u)\omega z}(0 \leq u \leq 1)$$

and

$$dv(u) = \frac{1}{\Gamma(\chi)\Gamma(2-\chi)}u^{\chi-1}(1-u)^{-\chi+1}du$$

is a positive value for $u \in [0,1]$. It is worth noting that $\Re\left(g(z,u)\right) > 0, g(-r,u)$ is real for $u \in [0,1]$ and $0 \leq r < 1$. As a result of Lemma 2.2, we have

$$\Re\left(\frac{1}{\Lambda(z)}\right) \geq \frac{1}{\Lambda(-r)}, (|z| \leq r < 1)$$

and

$$\inf_{z\in\Delta^*}\left(\frac{1}{\Lambda(z)}\right)=\inf_{-1<z<1}\frac{1}{\Lambda(-r)}=\frac{1}{\int_0^1 g(-1,u)du}=\frac{1}{\Lambda(-1)}$$

$$=\frac{1}{{}_2F_1\left(1,\frac{(1-\gamma)(v-\omega)}{\omega};2;\frac{\omega}{\omega-1}\right)}. \tag{24}$$

As a result of (22), the right-hand inequality of (20) comes from (24), which is

$$2>\frac{(1-\gamma)(v-\omega)}{\omega}.$$

The restriction of β_1 is sharp by consistence with the hypothesis of subordination. Therefore, the proof of **Theorem 4** is complete.

Theorem 5. *(i) Suppose that* $\frac{v-\omega}{1+\omega}\leq\frac{\Re(\gamma)}{1-\delta}$. *Let* $f\in\Sigma_\mu^\varkappa(\gamma,ð;\delta;v,\omega)$ *and the function* Λ *be given on* Δ *by*

$$\Lambda(z)=\begin{cases}\int_0^1 u^{\gamma-1}\left(\frac{1+\omega uz}{1+\omega z}\right)^{-\frac{(1-\delta)(v-\omega)}{\omega}}du & \omega\neq 0,\\ \int_0^1 u^{\gamma-1}e^{-v(1-\delta)(u-1)z}du & \omega=0.\end{cases} \tag{25}$$

Then,

$$\frac{1}{1-\delta}\left(\frac{-z\left(\mathcal{L}_\mu^\varkappa(\gamma+1,ð)f(z)\right)'}{z\left(\mathcal{L}_\mu^\varkappa(\gamma+1,ð)f(z)\right)}-\delta\right)\prec\frac{1}{1-\delta}\left((\gamma+1-\delta)-\frac{1}{\Lambda(z)}\right)$$

$$:=\lambda_2(z)\prec\frac{1+vz}{1+\omega z},z\in\Delta^* \tag{26}$$

and the best dominant of (26) *is* $\lambda_2(z)$.

(ii) Furthermore, if the additional requirements 0 < ω < 1 and

$$\Re(\gamma)>\frac{(1-\gamma)(v-\omega)}{\omega}-1 \tag{27}$$

are satisfied, then

$$\frac{1-|v|}{1-|\omega|}<\frac{1}{1-\delta}\left(-\Re\frac{z\left(\mathcal{L}_\mu^\varkappa(\gamma+1,ð)f(z)\right)'}{z\left(\mathcal{L}_\mu^\varkappa(\gamma+1,ð)f(z)\right)}-\delta\right)<\beta_2, \tag{28}$$

Where

$$\beta_2=\frac{1}{1-\delta}\left((\gamma+1-\delta)-\frac{1}{{}_2F_1\left(1,\frac{(1-\gamma)(v-\omega)}{\omega};a+1;\frac{\omega}{(\omega-1)}\right)}\right). \tag{29}$$

The bound for β_2 *is the best that can be found.*

Proof: The demonstration of this theorem mirrors that of **Theorem 4**. Hence, we will only highlight the essential aspects here. Consider

$$\Omega(z) = \frac{1}{1-\delta}\left(\frac{-z\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma+1,ð)f(z)\right)'}{\left(\mathcal{L}_{\mu}^{\varkappa}(\gamma+1,ð)f(z)\right)} - \delta\right). \tag{30}$$

In this case, using (7), we have

$$\Omega(z) + \frac{z\Omega'(z)}{(\Omega(z)-1)(\delta-1)+\gamma} \prec \frac{1+vz}{1+\omega z}, (z \in \Delta^*), \tag{31}$$

where $h_2(z)$ is the best dominant provided by (27). The proof of **Theorem 5**, part (i) is now complete. As for part (ii) of **Theorem 5**, we write

$$\Lambda(z) = (1+\omega z)^{\delta_1}\int_0^1 u^{\delta_2-1}(1-u)^{\varkappa_1-\delta_2-1}(1+\omega uz)^{-\delta_1}du \quad (z \in \Delta^*)$$

$$= \frac{\Gamma(\delta_2)}{\Gamma(\varkappa_1)}\,{}_2F_1\left(1,\delta_1;\varkappa_1;\frac{\omega z}{\omega z+1}\right)$$

in order to establish (28), with $\delta_1 = \frac{(1-\delta)(v-\omega)}{\omega}$, $\delta_2 = a$ and $\varkappa_1 = \delta_2 + 1$. The condition

$$\Re(\gamma) > \frac{(1-\delta)(v-\omega)}{\omega} - 1,$$

and $0 < \omega < 1$ implies $\Re(x_1) > \delta_1 > 0$. Hence,

$$\Lambda(z) = \int_0^1 \lambda(z,v)du(v),$$

where

$$\lambda(z,v) = \frac{1+\omega z}{1+(1-v)\omega z}, \quad (0 < u < 1)$$

and

$$du(v) = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)\Gamma(\varkappa_1-\delta_1)}u^{\delta_1-1}(1-u)^{\varkappa_1-\delta_1-1}du.$$

From this, by **Lemma 2**, we get

$$\inf_{z\in u}\left(\frac{1}{\Lambda(z)}\right) = \frac{\gamma}{{}_2F_1\left(1,\frac{(1-\gamma)(v-\omega)}{\omega};a+1;\frac{\omega}{\omega-1}\right)}. \tag{32}$$

From (32) the right-hand inequality of (28) follows. Due to the concept of subordination, the bound for β_2 is sharp. With this, the proof of **Theorem 5** is now complete. Following this, we can present the proof of an inclusion theorem with respect to the parameter variation ð.

Theorem 6. *(i) Suppose that* $0 \le \delta < 1, -1 \le \omega < v \le 1$ *and satisfy*

$$\frac{v-\omega}{1+\omega} \le \frac{\Re(ð)}{1-\delta}. \tag{33}$$

Let $f \in \Sigma_a^t(\gamma, ð+1; \delta; v, \omega)$ *and let the function* $\Lambda \in \Delta$ *be given by*

$$\Lambda(z) = \begin{cases} \int_0^1 u^{ð-1} \left(\frac{1+\omega u z}{1+\omega z}\right)^{-\frac{(1-\delta)(v-\omega)}{\omega}} du & \omega \neq 0, \\ \int_0^1 u^{ð-1} e^{-v(1-\delta)(u-1)z} du & \omega = 0. \end{cases} \tag{34}$$

Then,

$$\frac{1}{1-\delta}\left(\frac{-z\left(L\mu^{\varkappa}(\gamma,ð)f(z)\right)'}{z\left(\mathcal{L}_\mu^{\varkappa}(\gamma,ð)f(z)\right)} - \delta\right) \prec \frac{1}{1-\delta}\left((ð+1-\delta) - \frac{1}{\Lambda(z)}\right)$$

$$:= \lambda_3(z) \prec \frac{1+vz}{1+\omega z}, \quad (z \in \Delta^*, 0 < \omega < 1) \tag{35}$$

and the best dominant of (35) *is* $\lambda_3(z)$*. It follows that,*

$$\Sigma_a^t(\gamma, ð+1; \delta; v, \omega) \subset \Sigma_\mu^{\varkappa}(\gamma, ð; \delta; v, \omega).$$

(ii) Furthermore, if the additional restrictions $0 < \omega < 1$ *and*

$$\Re(ð) > \frac{(1-\gamma)(v-\omega)}{\omega} - 1 \tag{36}$$

are satisfied, then

$$\frac{1-|v|}{1-|\omega|} < \frac{1}{1-\delta}\left(-\Re \frac{z\left(\mathcal{L}_\mu^{\varkappa}(\gamma,ð+1)f(z)\right)'}{z\left(\mathcal{L}_\mu^{\varkappa}(\gamma,ð+1)f(z)\right)} - \delta\right) < \beta_3, \tag{37}$$

where

$$\beta_3 = \frac{1}{1-\delta}\left((ð+1-\delta) - \frac{1}{{}_2F_1\left(1, \frac{(1-\gamma)(v-\omega)}{\omega}; ð+1; \frac{\omega}{(\omega-1)}\right)}\right). \tag{38}$$

The bound for β_3 *is the best that can be found.*

Proof: Set

$$\Upsilon(z) = \frac{1}{1-\delta}\left(\frac{-z\left(\mathcal{L}_\mu^{\varkappa}(\gamma,ð+1)f(z)\right)'}{\left(\mathcal{L}_\mu^{\varkappa}(\gamma,ð+1)f(z)\right)} - \delta\right). \tag{39}$$

Using (8), we get

$$\frac{1}{1-\delta}\left(\frac{-z\left(\mathcal{L}_\mu^{\varkappa}(\gamma,ð+1)f(z)\right)'}{\left(\mathcal{L}_\mu^{\varkappa}(\gamma,ð+1)f(z)\right)} - \delta\right) = \Upsilon(z) + \frac{z\Upsilon'(z)}{(\Upsilon(z)-1)(\delta-1)+\gamma}$$

$$\prec \frac{1+vz}{1+\omega z}, \quad (z \in U^*). \tag{40}$$

As a result of **Lemma 1**, we acquire

$$\Upsilon(z) \prec \mu_3(z) \prec \frac{1+\nu z}{1+\omega z}, \quad (z \in \Delta^*) \tag{41}$$

where (35) gives the best dominant β_3. The proof of the second part of **Theorem 5** is identical to that of **Theorem 4**. Therefore **Theorem 6** is complete.

4 Potential applications

Recently, there have been numerous studies on the practical applications of subordination and inclusion theorems, with some of the results being highly intriguing and deserving of further investigation.

J. Morais and H.M. Zayed [19] have used the concepts of subordination and inclusion to solve complex fluid flow problems, including those involving a vortex and a source/sink. They provided an explicit formulation for the complex potential (complex velocity) and demonstrated how a univalent function could be used to create a fluid flow from a single source.

Another application of subordination and inclusion theorems can be found in the study of the Coulomb wave function. This solution of the Coulomb wave equation, named after Charles-Augustin de Coulomb, is used to explain the behavior of charged particles in a Coulomb potential. The Coulomb wave function can be described using confluent hypergeometric functions or Whittaker functions of imaginary argument [18].

The Gaussian hypergeometric series and differential subordination techniques are used to derive the subordination and inclusion theorems. One fascinating application of these theorems is in the field of electromagnetic cloaking. The cloaked object, a subset of the two-dimensional cloak, and the latter are interconnected areas in the complex plane, leading to the conclusion that any parallelepiped with the aforementioned basis could be a three-dimensional cloak [17].

The use of subordination and inclusion theorems is growing across a range of engineering and life science domains, including physics, due to their potential for practical applications.

5 Conclusion

In this research, we aimed to investigate the geometric properties of a subclass of meromorphic functions in terms of a complex linear operator, particularly those related to Hurwitz-Lerch Zeta and Kummer functions. Our approach was based on the convolution principle using the proposed operator, which allowed us to analyze the intriguing aspects of this subclass.

We discovered several new geometric subclasses and highlighted their various advantages and applications. These subclasses are linked to the subordination and inclusion theorems and univalent functions in the punctured unit disk Δ^*. The subordination and inclusion theorems play a crucial role in solving practical problems in various engineering and life science domains, including physics. Our findings have the potential to contribute to the further development of these theories and their applications.

To sum up, this study represents a significant step forward in understanding the geometric properties of a subclass of meromorphic functions and their connection to the subordination and inclusion theorems. We hope that our results will inspire further research and exploration in this field.

Conflicts of Interest

The authors declare no conflict of interest.

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Data Availability

No data were used to support the study.

Authors' contributions

All contributors have reviewed and approved the final version of the paper.

References

[1] Alb Lupa¸s, A., & Oros, G. I. (2021). Differential subordination and superordination results using fractional integral of confluent hypergeometric function. Symmetry, 13(2), 327. https://doi.org/10.3390/sym13020327

[2] Alb Lupa¸s, A., & Oros, G. I. (2021). New Applications of Salagean and Ruscheweyh Operators for Obtaining Fuzzy Differential Subordinations. Mathematics, 9(16), 2000. https://doi.org/10.3390/math9162000

[3] Alb Lupa¸s, A., Oros, G. I., & Oros, G. (2012). A note on special strong differential subordinations using multiplier transformation. Journal of Computational Analysis and Applications, 14, 261-265.

[4] Alb Lupa¸s, A., Oros, G. I., & Oros, G. (2012). On special strong differential subordinations using Salaagean and Ruscheweyh operators. Journal of Computational Analysis and Applications, 14, 266-270.

[5] Al-Janaby, H. F., Ghanim, F., & Darus, M. (2020). On The Third-Order Complex Differential Inequalities of ξ -Generalized-Hurwitz-Lerch Zeta Functions. Mathematics, 8(5), 845. https://doi.org/10.3390/math8050845

[6] Al-Janaby, H. F., Ghanim, F., & Darus, M. (2019). Some geometric properties of integral operators proposed by Hurwitz-Lerch Zeta function. IOP Conf. Series: Journal of Physics: Conf. Series, 1212, 012010. https://doi.org/10.1088/1742-6596/1212/1/0120106/1212/1/012010

[7] Aouf, M. K., El-Ashwah, R. M., & El-Deeb, S. M. (2010). Subordination results for certain subclasses of uniformly starlike and convex functions defined by convolution. European Journal of Pure and Applied Mathematics, 3(5), 903-917.

[8] Dziok, J., & Srivastava, H. M. (2003). Certain subclasses of analytic functions associated with the generalized hypergeometric function. Integral Transforms Spec. Funct., 14(1), 7-18.

[9] Ghanim, F., & Al-Janaby, H. F. (2019). A certain subclass of univalent meromorphic functions defined by a linear operator associated with the Hurwitz-Lerch zeta function. Rad Hrvatske akademije znanosti i umjetnosti. Matematicke znanosti, 23(538), 71-84.

[10] Ghanim, F., & Al-Janaby, H. F. (2020). Inclusion and Convolution Features of Univalent Meromorphic Functions Correlating with Mittag-Leffler Function. Filomat, 34, 2141-2150.

[11] Ghanim, F., & Al-Janaby, H. F. (2021). Some analytical merits of Kummer-type function associated with Mittag-Leffler parameters. Arab. J. Basic Appl. Sci., 28, 255-263.

[12] Ghanim, F., Al-Janaby, H. F., & Bazighifan, O. (2022). Geometric Properties of the Meromorphic Functions Class Through Special Functions Associated with a Linear Operator. Adv Cont Discr Mod., 17. https://doi.org/10.1186/s13662-022-03691-y

[13] Ghanim, F., Al-Shaqsi, K., Darus, M., & Al-Janaby, H. F. (2021). Subordination properties of meromorphic Kummer function correlated with Hurwitz-Lerch zeta-function. Mathematics, 9(2), 192. https://doi.org/10.3390/math9020192

[14] Liu, J. L., & Srivastava, H. M. (2004). Certain properties of the Dziok-Srivastava operator. Appl. Math. Comput., 159, 485-493.

[15] Liu, J. L., & Srivastava, H. M. (2004). Classes of meromorphically multivalent functions associated with the generalized hypergeometric function. Math. Comput. Modelling, 39(1), 21-34.

[16] Miller, S. S., & Mocanu, P. T. (2000). Differential subordinations: theory and applications. Series in Pure and Applied Mathematics, 225, Marcel Dekker, New York.

[17] Mishra, A. K., Panigrahi, T., & Mishra, R. K. (2013). Subordination and inclusion theorems for subclasses of meromorphic functions with applications to electromagnetic cloaking. Mathematical and Computer Modelling, 57(Issues 3-4), 945-962.

[18] Mondal, S. R. (2022). Subordination Involving Regular Coulomb Wave Functions. symmetry, 14(5), 1007. https://doi.org/10.3390/sym14051007

[19] Morais, J., & Zayed, H. M. (2021). Applications of differential subordination and superordination theorems to fluid mechanics involving a fractional higher-order integral operator. Alexandria Engineering Journal, 60(4), 3901-3914.

[20] El-Deeb, S., & Bulboaca, T. (2021). Differential subordination for subclasses of meromorphic functions connected with a convolution operator. Nonlinear Studies, 28, 1-14.

[21] Srivastava, H. M., & Attiya, A. A. (2007). An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination. Integral Transforms and Special Functions, 18(3), 207-216.

[22] Srivastava, H. M., & Choi, J. (2001). Series Associated with the Zeta and Related Functions. Kluwer Academic Publishers.

[23] Srivastava, H. M., Gaboury, S., & Ghanim, F. (2015). Partial sums of certain classes of meromorphic functions related to the Hurwitz-Lerch zeta function. Moroccan J. Pure Appl. Anal., 1, 1-13.

[24] Srivastava, H. M., Jankov, D., Pogany, T. K., & Saxena, R. K. (2011). Two-sided inequalities for the extended Hurwitz-Lerch Zeta function. Computers and Mathematics with Applications, 62(1), 516-522.

[25] Srivastava, H. M., Saxena, R. K., Pogany, T. K., & Saxena, R. (1980). Integral Transforms and Special Functions. J. London Math. Soc., 2(21), 287-290.

[26] Wilken, D. R., & Feng, J. (2011). A remark on convex and starlike functions. Appl. Math. Comput., 22(7), 487-506.

[27] Whittaker, E. T., & Watson, G. N. (1927). A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions, With an Account of the Principal Transcendental Functions (4th ed.). Cambridge University Press, Cambridge.