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# On functional equations in several variables arising in information theory

In this paper we deal with so-called sum form functional equations arising in information theory in connection with characterization of different types of information measures.

## 1. INTRODUCTION

*"The universality and importance of concept of information could be compared only with that of energy"* (Rényi [14]). *"The great inventions of civilisation serve either to transform store and transmit energy (fire, mechanisms like wheel, use of water and wind energy, electricity, nuclear energies, etc.), or they serve to transform store and transmit information (speech, writing, drum- and fire-signals, printing, telegraph, telephone, radio, photograph, film, television, computers, etc.)"* (Aczél-Daróczy [1]). It took a long time (until the middle of the nineteenth century) for the abstract concept of energy be developed, i.e. for it to be recognised that mechanical energy, electricity, atomic energy and so on are different forms of the same substance and they can be compared, measured with common measure.
In connection with the concept of information, this essentially happened a century later, with the works of Shannon ([15],[16]). One of the basic problems of the information theory is to find suitable information measures, and characterise these measures. The characterisation problem leads to different types of functional equations, for example to the so-called sum form functional equations investigating in this paper.

Let $k \geq 1$ be a fixed integer. We shall use the following notations:

$$J = [0,1]^k,\ J^0 = ]0,1[^k,\ \underline{c} = (c,\ldots,c) \in \mathbf{R}^k,\ (c \in \mathbf{R})$$

$$\Gamma_n = \Gamma_n^{[k]} = \left\{ (p_1,\ldots,p_n) \in J^n : \sum_{i=1}^{n} p_i = \underline{1} \right\},\ \ n=2,3\ldots$$

(the set of n-ary complete discrete probability distributions),

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$$\Gamma_n{}^0 = \Gamma_n{}^{0[k]} = \left\{ (p_1,\dots,p_n) \in (J^0)^n : \sum_{i=1}^{n} p_i = \underline{1} \right\}, \ \text{n=2,3…}$$

(the set of n-ary complete discrete probability distributions without zero probabilities). Referring to the elements of the set $\Gamma_n^{[k]}$ we shall use the following notations:

$$P = (p_1,\dots,p_n) = \begin{pmatrix} p_{11} & \cdots & p_{n1} \\ \vdots & & \vdots \\ p_{1k} & \cdots & p_{nk} \end{pmatrix} \in \Gamma_n^{[k]}$$

We define the operations $\otimes:\Gamma_n\times\Gamma_m\to\Gamma_{nm}$ and $\bullet$:J×J→J by

$$(p_1,\dots,p_n)\otimes(q_1,\dots,q_m)= (p_1q_1,\dots, p_1q_m, p_2q_1,\dots, p_2q_m,\dots, p_nq_1,\dots, p_nq_m)$$

and

$$(x_1,\dots,x_k) \bullet (y_1,\dots,y_k) = (x_1y_1,\dots, x_ky_k),$$

respectively.

If $x = (x_1,x_2,\dots,x_k)$ and $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_k)\in\mathbf{R}^k$ then

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_k^{\alpha_k}.$$

The following types of functions play central role in our investigations.
A function $\mathbf{A}:\mathbf{R}^k\to\mathbf{R}$ is **additive**, if

$$\mathbf{A(x+y)=A(x)+A(y)}$$

holds for all $x,y\in\mathbf{R}^k$.
A function $\mathbf{M}:J^0\to\mathbf{R}$ is **multiplicative on $J^0$**, if

$$\mathbf{M(x\bullet y)=M(x)\cdot M(y)}$$

holds for all $x,y\in J^0$.
A function $\mathbf{M}:J\to\mathbf{R}$ is **multiplicative on J**, if $\mathbf{M(\underline{0})=0}$, $\mathbf{M(\underline{1})=1}$ and

$$\mathbf{M(x\bullet y)=M(x)\cdot M(y)}$$

holds for all $x,y\in J$.
A function $\mathbf{L}$: $J^0\to\mathbf{R}$ is **logarithmic on $J^0$**, if

$$\mathbf{L(x\bullet y)=L(x)+L(y)}$$

holds for all $x,y\in J^0$.
A function $\mathbf{L}:J\to\mathbf{R}$ is **logarithmic on J**, if $\mathbf{L(\underline{0})=0}$ and

$$\mathbf{L(x\bullet y) = L(x)+L(y)}$$

holds for all $x,y \in J \setminus \{\underline{0}\}$.

## 2. Measures of information

The so-called entropy **E** of a single stochastic event **A** with the probability **p**$=P(A) \neq 0$ shows the measure of how unexpected the event was (or the measure of information yielded by the event). We suppose, that the entropy depends only upon the probability of the event considered: $\mathbf{E(A)=H(p)}$.
It is rather natural to suppose that the function **H** has the following properties:

**(1)** $H(p) \geq 0$ for all $p \in ]0,1]$

**(2)** $H(pq) = H(p)+H(q)$ for all $p,q \in ]0,1]$

**(3)** $H(1/2) = 1$.

It is well known that the only function that satisfies conditions **(1), (2)** and **(3) is**

$$H(p) = -\log_2 p\ , \quad p \in ]0,1]$$

(see [1]).
The concept of entropy of an experiment (or a system) introduced by Shannon in [15] is fundamental in information theory. The Shannon entropy is the sequence of functions $\mathbf{H_n:\Gamma_n^{[1]} \to R}$ or $\mathbf{H_n:\Gamma_n^{0[1]} \to R}$, **n=1,2,…** defined by

**(4)**
$$H_n(p_1,\ldots,p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i\,,$$

where $0 \cdot \log_2 0 = 0$.

The information measure is defined as a generalization of Shannon's concept (see [1]): a sequence of function $\mathbf{I_n:\Gamma_n \to R}$ or $\mathbf{I_n:\Gamma_n^0 \to R}$, n=2,3,… is called information measure. Beside the Shannon entropy the following information measures are also well-known (see [1] and [2]):

*Information measures depend upon one dimensional probability distributions:*

**ENTROPY OF DEGREE α**

$$H_n^{\alpha}(P) = \begin{cases} \dfrac{1}{2^{1-\alpha}-1}\left(\displaystyle\sum_{i=1}^{n} p_i^{\alpha} - 1\right), & \text{ha } \alpha \neq 1 \\ H_n(P), & \text{ha } \alpha = 1 \end{cases}\ , \ (0^{\alpha}=0),\ P \in \Gamma_n^{[1]}\ \textit{(or } \Gamma_n^{0[1]}\textit{)}$$

**ENTROPY OF DEGREE (α,β)**

$$H_n^{(\alpha,\beta)}(P)=\sum_{i=1}^{n} z_{(\alpha,\beta)}(p_i),\quad P\in\Gamma_n^{[1]}\ \textit{(or } \Gamma_n^{0[1]}\textit{)},\ \text{where}$$

$$z_{(\alpha,\beta)}(t)=\begin{cases} 0 & ,\text{ha} \quad t\in\{0,1\} \\ \dfrac{t^{\beta}-t^{\alpha}}{2^{1-\beta}-2^{1-\alpha}} & ,\text{ha} \quad t\in]0,1[,\ \alpha\neq\beta, \\ -2^{\alpha-1}t^{\alpha}\log_2 t & ,\text{ha} \quad t\in]0,1[,\ \alpha=\beta \end{cases}$$

*Information measures depend upon two dimensional probability distributions:*

**KERRIDGE'S INACCURACY**

$$K_n(P)=-\sum_{i=1}^{n} p_{1i}\log_2 p_{2i}\ ,\ P\in\Gamma_n^{[2]}\ \textit{(or } \Gamma_n^{0[2]}\textit{)}\ ,\ n=2,3,\ldots$$

**KULLBACH'S ERROR (OR DIRECTED DIVERGENCE)**

$$E_n(P)=\sum_{i=1}^{n} p_{1i}\log_2\frac{p_{1i}}{p_{2i}}\ ,\ P\in\Gamma_n^{[2]}\ \textit{(or } \Gamma_n^{0[2]}\textit{)}\ ,\ n=2,3,\ldots$$

**ERROR OF DEGREE α** ($\alpha\notin\{0,1\}$)

$$E_n^{\alpha}(P)=(2^{\alpha-1}-1)\left(\sum_{i=1}^{n} p_{1i}^{\alpha}p_{2i}^{1-\alpha}-1\right),\ P\in\Gamma_n^{[2]}\ \textit{(or } \Gamma_n^{0[2]}\textit{)},\ n=2,3,\ldots$$

*Information measures depend upon three dimensional probability distributions:*

**THEIL'S INFORMATION IMPROVEMENT (OR DIRECTED DIVERGENCE)**

$$G_n(P)=\sum_{i=1}^{n} p_{1i}\log_2\frac{p_{2i}}{p_{3i}}\ ,\quad P\in\Gamma_n^{[3]}\ \textit{(or } \Gamma_n^{0[3]}\textit{)},\ n=2,3,\ldots$$

**DIRECTED DIVERGENCE OF DEGREE α** ($\alpha\neq1$)

$$G_n^{\alpha}(P)=\frac{1}{2^{\alpha-1}-1}\sum_{i=1}^{n} p_{1i}p_{2i}^{\alpha-1}p_{3i}^{1-\alpha}\ ,\ P\in\Gamma_n^{[3]}\ \textit{(or } \Gamma_n^{0[3]}\textit{)},\ n=2,3,\ldots$$

## 3. ADDITIVITY PROPERTY AND SUM FORM INFORMATION MEASURES

The information measure $I_n:\Gamma_n$ *(or $\Gamma_n^0$)* $\to \mathbf{R}$, n=2,3,… is said to be

A1 **($\alpha$,n,m)-additive** ($\alpha\in\mathbf{R}$, n≥2, m≥2 fixed), if

$$I_{nm}(P\otimes Q) = I_n(P) + I_m(Q) + (2^{1-\alpha}-1)\cdot I_n(P)\cdot I_m(Q),$$

$P\in\Gamma_n$ *(or $\Gamma_n^0$)*, $Q\in\Gamma_m$ *(or $\Gamma_m^0$)*,

A2 **(n,m)-additive**, if (1,n,m)-additive (n≥2, m≥2 fixed),

A3 **($\alpha,\beta$,n,m)-additive** ($\alpha,\beta\in\mathbf{R}^k$, n≥2, m≥2 fixed), if

$$I_{nm}(P\otimes Q) = \sum_{i=1}^{n} p_i^{\alpha}\ I_m(Q) + \sum_{j=1}^{m} q_j^{\beta}\ I_n(P),$$

$P\in\Gamma_n$ *(or $\Gamma_n^0$)*, $Q\in\Gamma_m$ *(or $\Gamma_m^0$)*, $\underline{0}^{\alpha}=\underline{0}^{\beta}=0$,

A4 **weighted (n,m)-additive of type ($M_1$,$M_2$)**, if

$$I_{nm}(P\otimes Q) = \sum_{i=1}^{n} M_1(p_i)\, I_m(Q) + \sum_{j=1}^{m} M_2(q_j)\, I_n(P),$$

$P\in\Gamma_n$ *(or $\Gamma_n^0$)*, $Q\in\Gamma_m$ *(or $\Gamma_m^0$)*, where $M_1$ and $M_2$ are multiplicative functions.

The information measure $I_n:\Gamma_n$ *(or $\Gamma_n^0$)*$\to\mathbf{R}$, n=2,3,… has **sum property**, if there exists a function f:J *(or $J^0$)*$\to\mathbf{R}$, such that

$$I_n(p_1,\ldots,p_n) = \sum_{i=1}^{n} f(p_i), \qquad n=2,3,\ldots,$$

$(p_1,\ldots,p_n)\in\Gamma_n$ *(or $\Gamma_n^0$)*. **f** is a generating function of information measure $\{I_n\}$.

## 4. SUM FORM FUNCTIONAL EQUATIONS

If the sum form information measure $\{I_n\}$ – with generating function **f** – has the property A1, A2, A3, or A4, then the function **f** satisfies the following so-called sum form functional equations, respectively:

**Behara-Nath equation I.**

**(B-N I.)** $$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + \lambda \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) \ ,$$

$\lambda=2^{1-\alpha}-1$, $(p_1,\ldots,p_n)\in\Gamma_n$ *(or* $\Gamma_n^0$*)*, $(q_1,\ldots,q_m)\in\Gamma_m$ *(or* $\Gamma_m^0$*)* ,

**Chaundy-McLeod equation:**

**(C-M)** $$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \ ,$$

$(p_1,\ldots,p_n)\in\Gamma_n$ *(or* $\Gamma_n^0$*)*, $(q_1,\ldots,q_m)\in\Gamma_m$ *(or* $\Gamma_m^0$*)*,

**Behara-Nath equation III.**

**(B-N III.)** $$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^{\beta} \sum_{i=1}^{n} f(p_i),$$

$(p_1,\ldots,p_n)\in\Gamma_n$ *(or* $\Gamma_n^0$*)*, $(q_1,\ldots,q_m)\in\Gamma_m$ *(or* $\Gamma_m^0$*)*, ( $\underline{0}^{\alpha}=\underline{0}^{\beta}=0$ ),

**Sum form equation of multiplicative type**

**(M)** $$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) = \sum_{i=1}^{n} M_1(p_i) \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} M_2(q_j) \sum_{i=1}^{n} f(p_i),$$

$(p_1,\ldots,p_n)\in\Gamma_n$ *(or* $\Gamma_n^0$*)*, $(q_1,\ldots,q_m)\in\Gamma_m$ *(or* $\Gamma_m^0$*)*.

It is easy to see, that equation **(B-N III.)** is a special case of equation **(M)**, when the multiplicative functions are power functions.
With the definition $\mathbf{F(x)=x+\lambda\cdot f(x)}$, $x\in J$ *(or* $J^0$*, respectively)* equation **(B-N I.)** goes over into

**Behara-Nath equation II.**

**(B-N II.)** $$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i \bullet q_j) = \sum_{i=1}^{n} F(p_i) \sum_{j=1}^{m} F(q_j) \ ,$$

$(p_1,\ldots,p_n)\in\Gamma_n$ *(or* $\Gamma_n^0$*)*, $(q_1,\ldots,q_m)\in\Gamma_m$ *(or* $\Gamma_m^0$*)* .

## 5. General solution of equations (B-N II.) and (M)

The open domain case was investigated and the general solution of equations **(B-N II.)** and **(M)** were given by Ebanks, Sahoo and Sander in [2].

**THEOREM 1.** Let $n\geq 3$, $m\geq 3$ be fixed integers. The general solution of equation **(B-N II.)** where $f:J^0\to\mathbf{R}$ is unknown function and the equality holds for all $(p_1,\ldots,p_n)\in\Gamma_n^0$, $(q_1,\ldots,q_m)\in\Gamma_m^0$, is

$$f(p) = A_1(p) + b, \qquad p\in J^0,$$

or

$$f(p) = A_2(p) + M(p), \qquad p\in J^0,$$

where $A_1,A_2:\mathbf{R}^k\to\mathbf{R}$ are additive functions, $M:J^0\to\mathbf{R}$ is a multiplicative function, and $A_2(\underline{1})=0$, $A_1(\underline{1}) + nmb = [A_1(\underline{1})+nb]\cdot[A_1(\underline{1})+mb]$.

**THEOREM 2.** Let $n\geq 3$, $m\geq 3$ be fixed integers and let $M_1,M_2:J^0\to\mathbf{R}$ be multiplicative functions, $M_1$ or $M_2$ is different from projections of $J^0$. The general solution of equation **(M)**, where $f:J^0\to\mathbf{R}$ is an unknown function and the equality holds for all $(p_1,\ldots,p_n)\in\Gamma_n^0$, $(q_1,\ldots,q_m)\in\Gamma_m^0$, is

$$f(p) = A_1(p) + C(M_1(p)-M_2(p)), \qquad p\in J^0, \quad \text{if} \quad M_1\neq M_2,$$

$$f(p) = A_2(p) + M_1(p)L(p)-b, \qquad p\in J^0, \quad \text{if} \quad M_1=M_2,$$

where $A_1,A_2:\mathbf{R}^k\to\mathbf{R}$ are additive functions, $L:J^0\to\mathbf{R}$ is a logarithmic function, $C,b\in\mathbf{R}$ $A_1(\underline{1})=0$, furthermore

$$b=\frac{A_2(\underline{1})}{nm}, \text{ if } M\equiv 0, \qquad b=\frac{A_2(\underline{1})}{nm}(n+m-1), \text{ if } M\equiv 1 \text{ and}$$

$$b = A_2(\underline{1}) = 0, \text{ if } M\notin\{0,1\}.$$

On closed domain the general solution of equations **(B-N II.)** and **(M)** (when $n\geq 3$ and $m\geq 3$) is known only in the one dimensional case. The higher dimensional case for equation **(B-N II.)** has been recently investigated and partial result is obtained (in the two dimensional case) by the author.

In the one dimensional case for equation **(B-N II.)** and for equation **(M)** – when the multiplicative functions are power functions – the general solution was given by L. Losonczi and Gy. Maksa in [11] and in [12]. Applying the methods used in Losonczi-Maksa [12] with arbitrary multiplicative functions instead of power functions we have the general solution of equation **(M),** too.

**THEOREM 3.** Let $n\geq 3$, $m\geq 3$ be fixed integers and $k=1$. The general solution of equation

$$\textbf{(B-N II.)}\qquad \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i)\sum_{j=1}^{m} f(q_j),$$

$(p_1,\dots,p_n)\in\Gamma_n^{[1]}$, $(q_1,\dots,q_m)\in\Gamma_m^{[1]}$, where $f:[0,1]\to\mathbf{R}$ is unknown function, is

$$f(p) = A_1(p) + b, \qquad p\in[0,1],$$

or

$$f(p) = A_2(p) + M(p), \qquad p\in[0,1],$$

where $A_1,A_2:\mathbf{R}\to\mathbf{R}$ are additive functions, $M:[0,1]\to\mathbf{R}$ is a multiplicative function, $A_2(1) = 0$, $A_1(1) + nmb = (A_1(1)+nb)\cdot(A_1(1)+mb)$.

**THEOREM 4.** Let $n\geq 3$, $m\geq 3$ be fixed integers and let $M_1,M_2:[0,1]\to\mathbf{R}$ be multiplicative functions, $M_1$ or $M_2$ is different from the identity function of $[0,1]$. The general solution of equation **(M)**, where $f:[0,1]\to\mathbf{R}$ is unknown function, is

$$f(p) = A_1(p) + C(M_1(p)-M_2(p)), \qquad p\in[0,1], \qquad \text{if} \qquad M_1\neq M_2,$$

$$f(p) = A_2(p) + M_1(p)L(p)-b, \qquad p\in[0,1], \qquad \text{if} \qquad M_1=M_2,$$

where $A_1,A_2:\mathbf{R}\to\mathbf{R}$ are additive functions, $L:[0,1]\to\mathbf{R}$ is a logarithmic function, $C,b\in\mathbf{R}$ $A_1(1)=0$, furthermore

$$b=\frac{A_2(1)}{nm}, \text{ if } M\equiv 0, \qquad b=\frac{A_2(1)}{nm}(n+m-1), \text{ if } M\equiv 1,$$

$$b = A_2(1) = 0, \text{ if } M\notin\{0,1\}.$$

## 6. STABILITY OF EQUATIONS IN HYERS-ULAM SENSE

**ULAM'S PROBLEM** (1940):
Let **(G,o)** be a group, **(H,*)** be a metric group with metric **d**. Is the following statement true:
For arbitrary **ε**>0 there exists δ>0 such that for all function **f:G→H** satisfying the inequality

$$\mathbf{d( f(x\circ y) , f(x)*f(y) ) < \varepsilon, \quad x,y\in G}$$

there exists a homomorphism **A:G→H** satisfying the inequality

$$\mathbf{d( f(x) , A(x) ) < \delta, \ x\in G.}$$

Similar question can be asked in connection with other equations. See the following survey papers: Forti [3], Ger [4], Hyers-Isac-Rassias [6], and Székelyhidi [17].

We remark that Ulam did not deal with the connection between the constants $\varepsilon$ and $\delta$ in his question. Its clear that the stability result is more informative when $\delta$ can be expressed by $\varepsilon$. In many cases a constant $K\in\mathbf{R}$ can be found such that, $\delta=K\varepsilon$.

The most remarkable theorem in this area is the following result of Hyers:

**THEOREM** (Hyers [5]). Let **X** and **Y** be Banach spaces and $\varepsilon\geq 0$ be fixed. Suppose that the function $\mathbf{f:X\to Y}$ satisfies the inequality

$$\left\| \mathbf{f(x+y) - f(x) - f(y)} \right\| \leq \varepsilon$$

for all $x,y\in X$. Then there exists exactly one function $\mathbf{A:X\to Y}$, such that

$$\mathbf{A(x+y) = A(x) + A(y)} \qquad \text{for all } x,y\in X$$

and

$$\left\| \mathbf{f(x) - A(x)} \right\| \leq \varepsilon \qquad \text{for all } x\in X.$$

By the stability problem for equation **(B-N II.)** say we mean the following: Let $n\geq 3$, $m\geq 3$ be fixed integers and let $0\leq\varepsilon\in\mathbf{R}$ be fixed. Prove or disprove that the functions $f:J\to\mathbf{R}$ satisfying the inequality

$$\left| \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) - \sum_{i=1}^{n} f(p_i)\sum_{j=1}^{m} f(q_j) \right| \leq \varepsilon$$

for all $(p_1,\dots p_n)\in\Gamma_n$ , $(q_1,\dots q_m)\in\Gamma_m$ are the sum of a solution of equation **(B-N II.)** and a bounded function.

## 6. A SIMPLER SUM FORM EQUATION

In our investigations the stability of the sum form equation

**(S)** $$\sum_{i=1}^{n}\varphi(p_i) = d \qquad (p_1,\dots,p_n)\in \Gamma_n \ (or\ \Gamma_n^0)$$

plays central role, where $\varphi:J$ *(or $J^0$)*$\to\mathbf{R}$, $d\in\mathbf{R}$ since – for example in case of equation **(B-N II.)** – from the inequality

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) - \sum_{i=1}^{n} f(p_i)\sum_{j=1}^{m} f(q_j)\right| \le \varepsilon$$

with fixed $Q=(q_1,\ldots,q_m)\in\Gamma_m$ *(or $\Gamma_m^0$)*, for the function $\varphi(\cdot,Q):J$ *(or $J^0$)*$\to\mathbf{R}$ defined by

$$\varphi(p,Q) = \sum_{j=1}^{m}\left[f(p \bullet q_j) - M_1(p)f(q_j) - f(p)M_2(q_j)\right]$$

we get that

$$\left|\sum_{i=1}^{n}\varphi(p_i)\right| \le \varepsilon$$

for all $(p_1,\ldots,p_n)\in\Gamma_n$ *(or $\Gamma_n^0$)*.

The general solution of equation **(S)** – in the one dimensional case – was given in Losonczi-Maksa [11] on closed domain, and in Losonczi [10] on open domain.

**THEOREM 5.** Let $n\ge 3$ be fixed integer, $0\le\varepsilon\in\mathbf{R}$ be fixed and $k=1$. The function $\varphi:[0,1]$ *(or $]0,1[$)*$\to\mathbf{R}$ satisfies equation **(S)** if and only if there exists an additive function $A:\mathbf{R}\to\mathbf{R}$ such that

$$\varphi(p) = A(p) + \frac{d - A(1)}{n}$$

holds for all $p\in[0,1]$ *(or $]0,1[$)*.

It was proved by Ebanks, Sahoo and Sander in [2] that similar statement is valid in higher dimension. They dealt only with the open domain case:

**THEOREM 6.** Let $n\ge 3$ be fixed integer, $0\le\varepsilon\in\mathbf{R}$ be fixed. The function $\varphi:J^0\to\mathbf{R}$ satisfies equation **(S)** if and only if there exists an additive function $A:\mathbf{R}^k\to\mathbf{R}$ such that

$$\varphi(p) = A(p) + \frac{d - A(\underline{1})}{n} \qquad \textbf{(5)}$$

holds for all $p\in J^0$.

It is a simple consequence of Theorem 6. that the general solution of equation **(S)** can be given in the same form:

**THEOREM 7.** Let $n\geq 3$ be fixed integer, $0\leq\varepsilon\in\mathbf{R}$ be fixed. The function $\varphi:J\to\mathbf{R}$ satisfies equation **(S)** if and only if there exists an additive function $A:\mathbf{R}^k\to\mathbf{R}$ such that **(5)** holds for all $p\in J$.

In order to obtain this statement suppose that equality **(S)** holds for all $(p_1,\dots,p_n)\in\Gamma_n$. Applying Theorem 6. we get that **(5)** holds for all $p\in J^0$.
Let now $p\in J^*=[0,1[^k$, $p_i\in J^0$ $i=2,\dots,n$ such that $(p,p_2,\dots,p_n)\in\Gamma_n$. From equation **(S)** we have

$$0=\varphi(p)+\sum_{i=2}^{n}\varphi(p_i)=\varphi(p)-A(p)+\frac{A(1)}{n},$$

that is (5) holds for all $p\in J^*$. Finally let $p\in J\backslash J^*$, $p_i\in J^*$ $i=2,\dots,n$ and $(p,p_2,\dots,p_n)\in\Gamma_n$. in equation (3). With similar calculations we obtain that **(5)** holds for $p\in J$.
Conversely: if the function $\varphi:J\to\mathbf{R}$ have the form **(5)** then $\varphi$ satisfies equation **(S)** for all $(p_1,\dots,p_n)\in\Gamma_n$.

## 7. The stability of equation (S)

In the one dimensional case the stability of equation **(S)** has been proved. It was shown by Maksa in [13] on closed domain.

**THEOREM 8.** Let $0\leq\varepsilon\in\mathbf{R}$ be fixed and k=1. If the function $\varphi:[0,1]\to\mathbf{R}$ satisfies the inequality

$$\left|\sum_{i=1}^{n}\varphi(p_i)-d\right|\leq\varepsilon$$

for all $(p_1,\dots,p_n)\in\Gamma_n^{[1]}$, then there exist an additive function $A:\mathbf{R}\to\mathbf{R}$ such that

$$\left|\varphi(p)-a(p)+\frac{a(1)-d}{n}\right|\leq 18\cdot\varepsilon\ ,\qquad p\in[0,1].$$

The stability of equation **(S)** on open domain was proved by Kocsis in [7].

**THEOREM 9.** Let $0\leq\varepsilon\in\mathbf{R}$ be fixed and k=1. If the function $\varphi:]0,1[\to\mathbf{R}$ satisfies equality **(S)** for all $(p_1,\dots,p_n)\in\Gamma_n^{0[1]}$ then there exist an additive function $a:\mathbf{R}\to\mathbf{R}$ such that

$$\left|\varphi(p)-a(p)+\frac{a(1)-d}{n}\right|\leq 220\cdot\varepsilon\qquad p\in]0,1[.$$

The higher dimensional (closed domain case) was investigated and similar statement was presented by the author. This result is under publication.

## 8. STABILITY OF SUM FORM EQUATIONS (B-N II.) AND (M)

On closed domain **t**he stability of equation **(B-N II.)** was proved by Maksa in [13] in the one dimensional case.

**THEOREM 10.** Let $n\geq 3$, $m\geq 3$ be fixed integers, $0\leq\varepsilon\in\mathbf{R}$ be fixed and k=1. If the function $f:[0,1]\to\mathbf{R}$ satisfies the inequality

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} f(p_i)\sum_{j=1}^{m} f(q_j)\right| \leq \varepsilon$$

for all $(p_1,\ldots p_n)\in\Gamma_n[1]$ , $(q_1,\ldots q_m)\in\Gamma_m^{[1]}$ then

$$f(p) = a_1(p)+b(p), \qquad p\in[0,1],$$

or

$$f(p) = a_2(p)+M(p)+f(0), \qquad p\in[0,1],$$

where $a_1,a_2:\mathbf{R}\to\mathbf{R}$ are additive functions, $M:[0,1]\to\mathbf{R}$ is a multiplicative function, $b:[0,1]\to\mathbf{R}$ is a bounded function.

The stability of equation **(B-N II.)** in higher dimension (on closed domain) has been recently investigated and partial results were obtained by the author.

Finally consider the stability problem of the sum form equation of multiplicative type **(M)**. In connection with this equation only one dimensional results are known.

The stability of equation **(M)** – when the multiplicative functions are power functions – was shown in Kocsis-Maksa [9] and with arbitrary multiplicative functions (one of them is different from the identity function of [0,1]) was shown in Kocsis [8].

**THEOREM 11.** Let $n\geq 3$, $m\geq 3$ be fixed integers, $0\leq\varepsilon\in\mathbf{R}$ be fixed, k=1 and let $M_1,M_2:[0,1]\to\mathbf{R}$ be multiplicative ($M_1$ or $M_2$ is different from the identity function of [0,1]). If the function $f:[0,1]\to\mathbf{R}$ satisfies the inequality

$$\textbf{(6)}\qquad \left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} M_1(p_i)\sum_{j=1}^{m} f(q_j) - \sum_{j=1}^{m} M_2(q_j)\sum_{i=1}^{n} f(p_i)\right| \le \varepsilon,$$

for all $(p_1,\ldots,p_n)\in\Gamma_n^{[1]}$ and $(q_1,\ldots,q_m)\in\Gamma_m^{[1]}$ then there exist additive functions $a_1,a_2:\mathbf{R}\to\mathbf{R}$, a logarithmic function $L:[0,1]\to\mathbf{R}$, bounded functions $B_1,B_2:[0,1]\to\mathbf{R}$ and $C\in\mathbf{R}$ such that $a_1(1)=a_2(1)=0$, and

$$f(p) = a_1(p) + C(M_1(p) - M_2(p)) + B_1(p), \qquad p\in[0,1], \qquad \text{if} \qquad M_1\neq M_2,$$

$$f(p) = a_2(p) + M_1(p)L(p) + B_2(p), \qquad p\in[0,1], \qquad \text{if} \qquad M_1=M_2.$$

The stability of equation **(M)** in the open domain case was investigated in Kocsis [11]. The following theorem shows, that equation **(M)** is also stable on open domain in the case n=m and $M_1\neq M_2$.

**THEOREM 12.** Let $n=m\geq 3$ be fixed integer, $\varepsilon\geq 0$ be fixed, k=1 and let $M_1,M_2:]0,1[\to\mathbf{R}$ be multiplicative, $M_1\neq M_2$. If the function $f:]0,1[\to\mathbf{R}$ satisfies inequality **(6)** for all $(p_1,\ldots,p_m)\in\Gamma_m^{[1]}$ and $(q_1,\ldots,q_m)\in\Gamma_m^{0[1]}$ then there exists an additive function $a:\mathbf{R}\to\mathbf{R}$ such that

$$f(p)=a(p) + C(M_1(p)-M_2(p)) + B(p), \qquad p\in]0,1[.$$

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