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General Section

Common values of linear recurrences related to Shank's simplest cubics

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A B S T R A C T

Let $A, B, C \in \mathbb{Z}$ not all zeroes and let $F(u,n)=F(A,B,C,u,n)$ be the linear recursive sequence, which is defined by the initial terms $F(u,0) = A, F(u,1) = B, F(u,2) = C$ and whose characteristic polynomial is Daniel Shanks simplest cubic $S_u(X) = X^3 - (u-1)X^2 - (u+2)X - 1, u \in \mathbb{Z}$. We prove that there exists an effectively computable constant c depending only on $L = \max\{|A|, |B|, |C|\}$ such that if $|F(A,B,C,u,n)| = |F(A,B,C,u,m)|$ holds for some integers u, n, m with $n \neq m$ then $|n|, |m| < c$. For the choices $(A,B,C) \in \{(0,0,1), (1,-1,1)\}$ we solve the above equations completely. At the end we give an outlook to the equation $F(0,0,1,u,n) = F(0,0,1,v,m)$ for some fixed integers n, m.

1. Introduction

Let $S_u(X) = X^3 - (u-1)X^2 - (u+2)X - 1, u \in \mathbb{Z}$. Shanks [19] proved that the family of number fields generated by their roots are Galois and have nice arithmetical

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properties. He called them *simplest cubic fields*. The polynomial $S_u(X,Y) = Y^3 S_u(X/Y)$ is homogeneous and cubic. Since the breakdown work of Baker [1] it was known that Thue equations are effectively solvable, i.e. one can compute an upper bound for the size of their solutions. That bound is extremely large, but can be reduced with numerical diophantine approximation techniques, which enabled to solve concrete Thue equations. You find a good survey of the early development in the books of de Weger [30] and Smart [21].

Thomas [27] was the first, who successfully investigated an infinite family of Thue equations. He proved that

$$S_u(X,Y) = \pm 1$$

has only the solutions $(x,y) = (0,\pm 1), (\pm 1, 0), (\pm 1, \mp 1)$ if $u = 2, 4 \le u \le 1000$ or $u > 1.365 \cdot 10^7$. Moreover he established all solutions for $u = 0, 1, 3$. A bit later Mignotte [12] filled the gap $1000 < u \le 1.365 \cdot 10^7$ by proving that the above equation has only the trivial solutions in this range too.

Beside Thue equations one can associate to the simplest cubics an other family of interesting arithmetical objects, namely a family of linear recursive sequences. Let $A, B, C \in \mathbb{Z}$ not all zeroes and let $F(u,n) = F(A,B,C,u,n)$ be the linear recursive sequence (lrs in the sequel), which is defined by the initial terms $F(u,0) = A, F(u,1) = B, F(u,2) = C$ and by the recursion

$$F(u, n+3) = (u-1)F(u, n+2) + (u+2)F(u, n+1) + F(u,n), \quad n \ge 0.$$

The $F(u,n)$ are polynomials of degree at most $n-2$ in the indeterminate u with coefficients belonging to $\mathbb{Z}[A,B,C]$. The recursive formula allows to compute $F(u,n)$ for negative n-s, which are polynomials over $\mathbb{Z}[A,B,C]$ too because the coefficient of $F(u,n)$ is one. Setting

$$F'(u,n) = F(u,-n),$$

the sequence $(F'(u,n))$ is again linear recursive with initial terms $F'(u,0) = A$, $F'(u,-1) = B$, $F'(u,-2) = C$ and with the recursion

$$F'(u, n+3) = -(u+2)F'(u, n+2) - (u-1)F'(u, n+1) + F'(u,n), \quad n \ge -2.$$

The characteristic polynomial of $(F'(u,n))$ is $X^3 S_u(1/X)$.

In this paper we investigate the diophantine equation

$$|F(u,m)| = |F(u,n)| \tag{1}$$

in the unknown integers u, n, m. Throughout the article we assume that $n \neq m$.

For general starting values A, B, C we can prove the following effective theorem

Theorem 1. *Assume that A, B, C denote integers such that at most two of them are zero. There exists an effectively computable constant c depending only on $L = \max\{|A|, |B|, |C|\}$ such that if (1) holds for some integers u, n, m with $n \neq m$ then*

$$|n|, |m| < c.$$

The curiosity of Theorem 1 is that c does not depend on u, hence there are only finitely many pairs $(n, m) \in \mathbb{Z}^2$ for which there exist $u \in \mathbb{Z}$ such that (u, n, m) is a solution of (1). This does not mean that $|u|$ is bounded too. Depending on the value of $|F(u, n)|$, which is equal to $|F(u, m)|$ the integer u can be bounded or can be arbitrarily large. You see in Theorem 2 cases $(i), (iii), (vi)$ that for some values of (n, m) the u can be any integer.

To prove Theorem 1 we apply an A. Baker-type theorem for linear forms in logarithms of three algebraic numbers. However during the preparation of its proof, exactly in Lemma 4, we identify triples (A, B, C) for which linear forms in two logarithms of algebraic numbers are enough. That yields much better upper bounds and makes it possible to enumerate all solutions. These triples are given by

$$(0, 0, 1), (1, -1, 1), (1, -1, 2), (1, 0, 0), (1, 0, 1), (2, -1, 1), (2, -1, 2).$$

We have that $F(1, -1, 2, u, n) = -F(1, -1, 1, u, n - 1)$ and $F(2, -1, 1, u, n) = -F(1, -1, 1, u, n + 1)$, $-2 \leq n \leq 3$, hence both relations hold for all $n \in \mathbb{Z}$. By symmetry solving the problem for the triple $(0, 0, 1)$ yields also a solution for $(1, 0, 0)$. Hence, with respect to (1), only the triples

$$(0, 0, 1), (1, -1, 1), (1, -1, 2), (1, 0, 1)$$

are essentially different. In this article we show how to compute all solutions in such cases illustrating the method with the triples $(0, 0, 1)$ and $(1, -1, 1)$.

Theorem 2. *Let $(F(u, n))_{-\infty}^{\infty}, u \in \mathbb{Z}$ be the family of Shanks sequences with initial values $(A, B, C) \in \{(0, 0, 1), (1, -1, 1)\}$. Then the diophantine equation (1) has in $u, n, m \in \mathbb{Z}, u \geq 0, n > m$ the following solutions*

(i) $(A, B, C) = (0, 0, 1)$: $u \in \mathbb{Z}_{\geq 0}, (n, m) = (1, 0), (2, -1)$, $u = 0, (n, m) = (3, -1), (3, 2)$, $u = 1, (n, m) = (3, 0), (3, 1), (4, -2), (5, -1), (5, 2), (6, -3)$, $u = 2, (n, m) = (3, -1), (3, 2)$.
(ii) $(A, B, C) = (1, -1, 1)$: $u \in \mathbb{Z}_{\geq 0}, (n, m) = (1, 0), (2, 0), (2, 1), (3, -1)$, $u = 1, (n, m) = (4, -1), (4, 3), (5, -2)$, $u = 2, (n, m) = (4, 0), (4, 1), (4, 2), (5, -2)$, $u = 4, (n, m) = (4, 0), (4, 1), (4, 2)$ $u = 5, (n, m) = (4, -1), (4, 3)$.

We will see that the difficult part of the proof of Theorem 1 is the case $n \geq 0, m < 0$, i.e. to find common values of the recurrences $(F(u, n))$ and $(F'(u, n))$. Mignotte [10] was the first who proved that two lrs's have, under some condition, only finitely many effectively computable common values. His conditions hold for our recurrences for any fixed u, thus there is an effectively computable constant, $c > 0$, such that $|n|, |m| < c$ for all solutions $(n, m) \in \mathbb{Z}^2$ of (1). Unfortunately c depends on u, hence Mignotte's result does not exclude the infinitude of solutions (1) if u varies. In case of Berstel's sequence, defined by $B_0 = B_1 = 0, B_2 = 1$ and $B_n = 2B_{n-1} - 4B_{n-2} + 4B_{n-3}, n \geq 3$, Mignotte [11] completely determined all c for which $B_n = \pm c$ has at least two solutions, together with the corresponding n's.

Similar problem arises in the papers [3] and [8], where common values of k-generalized Fibonacci and k-generalized Pell sequences are established. For $k \geq 2$ let $(F_n^{(k)})$ denote the sequence of k-generalized Fibonacci numbers. It is defined by the initial terms $F_n^{(k)} = 0, n = -k+2, \ldots, 0,\ F_1^{(k)} = 1$ and by the recursion $F_n^{(k)} = F_{n-1}^{(k)} + \ldots + F_{n-k+1}^{(k)}$ for $n \geq 1$. The first k terms of the k-generalized Pell sequence $(P_n^{(k)})$ are the same, but its further members satisfy the recursion $P_n^{(k)} = 2P_{n-1}^{(k)} + \ldots + P_{n-k+1}^{(k)}$ for $n \geq 1$. The characteristic polynomials of both sequences are Pisot, i.e. have all but one root inside the unit circle, see e.g. [14], [15] as well as [6]. Both sequences are parametrized by the length or order of the recursion. In contrast the members of the Shanks family are all cubic, their coefficients vary. By referring to Mignotte [10] it is an easy exercise to prove that the equations $F_n^{(k)} = F_m^{(l)}$, $P_n^{(k)} = P_m^{(l)}$ and $F_n^{(k)} = P_m^{(l)}$ have for fixed $k, l \geq 2$ only finitely many effectively computable solutions in $(n, m) \in \mathbb{Z}^2, n, m \geq 0$. However, when k and l vary big problem arise, which were overcame by Marques [8] for the k-generalized Fibonacci numbers and by Bravo, Herrera and Luca [3] for the mixed case.

Equation (1) is the compact form of three equations, which are $|F(u, n)| = |F(u, m)|$, $|F(u, -n)| = |F(u, -m)|$ and $|F(u, n)| = |F(u, -m)|$ and in all cases $n, m \geq 0$. Because of $(|F(u, n)|)$ and $(|F'(u, n)|)$ are ultimately strictly increasing, the first two are easily solvable. In contrast, in the third case we have to use heavy tools of transcendental number theory, namely the theory of linear forms in logarithms of algebraic numbers. The linear form we derived appeared, up to a constant summand, which is also the logarithm of an algebraic number, to be the same as that of Thomas [27]. To prove the effective finiteness of the solutions is essential to show that $n \geq cu$ with a positive constant c. If the above mentioned constant summand is zero, i.e. in the proof of Theorem 2, we borrow the proof of $n \geq u \log u$ from Thomas' paper [27], and a part of the reduction of the upper bound from Mignotte [12].

The paper is organized as follows. In the next section we collect basic properties of the Shanks polynomials and the Shanks sequences. Special attention is devoted to the weights in the Binet formula for $F(u, n)$. Especially important are the Lemmata 4 and 5. After this we present the necessary background from transcendental number theory. Section 4 is devoted to the proof of Theorem 1. It is divided into four parts. Subsection 4.1 deals with the simple cases when n and m have the same sign. In the second part we derive upper bound for a linear form in logarithm of algebraic numbers. This together with the results of Lemma 4 enable us to prove $n \geq cu \log u$ with a positive constant c. The proof of Theorem 1 is finished in Subsection 4.4.

The logic of the proof of Theorem 2 is the same as of Theorem 1. Equation (1) is specialized to (13). To establish all solutions of it we have to overcome lot of technical difficulties. The proof of Theorem 2 starts in Section 5 with preparations. We establish here the solutions $(n,m) \in \mathbb{Z}^2$ of (13), such that $nm \geq 0$ or $nm < 0$, but either $|n|$ or $|m|$ is small. We prove strong lower and upper bounds for $|F_i(u,n)|$ too. The rest of the proof is divided into three parts. In Section 6 we prove that $u \leq 63103$ and $n \leq 579840$ provided $u \geq 1$. In the next section we prove $n \leq 46926$ provided $u = 0$. The proof of Theorem 2 is finalized in Section 8 describing a diophantine approximation technique for the reduction of the bound for n.

In the last Section 9 we give an outlook to the equation $F(u,n) = F(v,m)$ for fixed integers n, m.

2. Properties of the Shanks polynomials and the Shanks sequences

Denote $\lambda_i(u), i = 1,2,3$ the zeroes of the simplest cubics $S_u(X)$. Because the relation

$$-X^3 S_{-u}(1/X) = S_{u-1}(X),$$

which can be proved easily, it is enough to study the location of the roots for the non-negative parameter values. The following properties are well known, see e.g. [13]. The numbers $\lambda_i(u), i = 1,2,3$ can be enumerated such that

$$\lambda_1(u) \in (-1,0), \quad \lambda_2(u) \in (-2,-1), \quad \lambda_3(u) = \lambda \in \begin{cases} (u, u+2/u), & \text{if} \quad u > 0 \\ (1,2), & \text{if} \quad u = 0 \end{cases}$$

moreover

$$\begin{aligned} |\lambda_1(u)| < 1 < |\lambda_2(u)| < \lambda, &\quad \text{if} \quad u > 0 \\ |\lambda_1(u)| < 1 < \lambda < |\lambda_2(u)|, &\quad \text{if} \quad u = 0. \end{aligned}$$

We will assume this arrangement throughout this paper. If the parameter u is not specified then, to simplify the notation, we will write λ_i instead of $\lambda_i(u)$, $i = 1,2$ and λ instead of $\lambda_3(u)$.

The polynomials $S_u(X)$ are invariant under the transformation $\sigma : X \mapsto \frac{-1}{1+X}$, which satisfies $\sigma^2(X) = -\frac{1+X}{X}$ and $\sigma^3(X) = X$. Hence $\mathbb{Q}(\lambda)$ is a Galois field of degree three, and

$$\lambda_1 = -\frac{1}{\lambda+1}, \lambda_2 = -\frac{1}{\lambda_1+1} = -\frac{\lambda+1}{\lambda} \quad \text{and} \quad \lambda = -\left(1 + \frac{1}{\lambda_1}\right). \tag{2}$$

We will also use the following identities.

$$\frac{1}{\lambda} = \lambda^2 - (u-1)\lambda - (u-2), \ \frac{1}{\lambda+1} = -\lambda^2 + u\lambda + (u-2).$$

We now turn our attention to the Shanks' sequences. The terms $F(u,n)$ and $F'(u,n)$ are by their definition polynomials in u with coefficients belonging to $\mathbb{Z}[A,B,C]$, moreover their degrees are at most $n-2$ and $n-1$ respectively. We have also the remarkable relation

$$F'(A,B,C,u,n) = F(C,B,A,-u-1,n+2). \tag{3}$$

Indeed, as $F'(A,B,C,u,0) = A = F(C,B,A,-u-1,-2), F'(A,B,C,u,-1) = B = F(C,B,A,-u-1,-1), F'(A,B,C,u,-2) = C = F(C,B,A,-u-1,0)$ the assertion holds for $n = -2,-1,0$. Furthermore

$$S_{-u-1}(X) = X^3 + (u+2)X^2 + (u-1)X - 1 = X^3 S_u(1/X),$$

which is the characteristic polynomial of $(F'(u,n))$ thus the claim is true for all n.

Notice that by relation (3) studying the properties of $(F(u,n))$ for negative u's is equivalent the study of $(F(u,n))$ with negative n. Hence we may assume without loss of generality $u \geq 0$ in (1).

One can express the terms of linear recursive sequences as weighted sums of the powers of the zeroes of their characteristic polynomials. Actually we have

$$F(u,n) = a_1(u)\lambda_1(u)^n + a_2(u)\lambda_2(u)^n + a_3(u)\lambda^n,$$

which is called Binet formula. The next lemma summarizes the important properties of the weights. Similarly to the λ-s we write a_i instead of $a_i(u), i = 1,2,3$ provided this causes no confusion. Remark that while the λ-s depend only on u, the a-s depend on A,B,C as well.

Lemma 1. *We have*

$$a_1 = \frac{\lambda_2 - \lambda}{u^2+u+7} a_1',$$
$$a_2 = \frac{\lambda - \lambda_1}{u^2+u+7} a_2',$$
$$a_3 = \frac{\lambda_1 - \lambda_2}{u^2+u+7} a_3',$$

where

$$a_1' = -\lambda(A+B) - A + B + C + \frac{B}{\lambda},$$
$$a_2' = -B\lambda - A + C + \frac{A+B}{\lambda+1},$$
$$a_3' = B + C + \frac{A+B}{\lambda} + \frac{B}{\lambda+1}.$$

Proof. Setting $n = 0, 1, 2$ in the Binet formula we obtain the system of linear equations

$$a_1 + a_2 + a_3 = A$$
$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda = B$$
$$a_1\lambda_1^2 + a_2\lambda_2^2 + a_3\lambda^2 = C.$$

Observing that the determinant of the matrix of the coefficients equals $-(u^2 + u + 7)$, which is non-zero for any u,[1] and solving the system of equations by using Cramer's rule we get the expressions for the a_i-s. For example

$$\begin{aligned}
-a_1(u^2 + u + 7) &= A(\lambda_2\lambda^2 - \lambda_2^2\lambda) - B(\lambda^2 - \lambda_2^2) + C(\lambda - \lambda_2) \\
&= (\lambda - \lambda_2)\left(-A\frac{\lambda + 1}{\lambda}\lambda - B\left(\lambda - 1 - \frac{1}{\lambda}\right) + C\right) \\
&= (\lambda - \lambda_2)\left(-(A + B)\lambda - A + B + C + \frac{B}{\lambda}\right).
\end{aligned}$$

To prove the expressions for a_2 and a_3 one can repeat the argument for a_1 or, alternatively, one observes that $\sigma(a_j) = a_{j+1 \bmod 3}$. □

Let $A, B, C \in \mathbb{Z}$ be such that at least one is non-zero. Assume that $(u, n, m) \in \mathbb{Z}^3$ is a solution of (1). The sequence $(F(A, B, C, u, n))$ is homogeneous in the parameters A, B, C, i.e. $F(\alpha A, \alpha B, \alpha C, u, n) = \alpha F(A, B, C, u, n)$ holds for any $\alpha \in \mathbb{C}$. Hence (1) admits the same solutions than

$$\left|F\left(\frac{A}{D}, \frac{B}{D}, \frac{C}{D}, u, n\right)\right| = \left|F\left(\frac{A}{D}, \frac{B}{D}, \frac{C}{D}, u, m\right)\right|,$$

where $D = \gcd(A, B, C)$ or $D = -1$, thus we may assume in the rest of the paper that A, B, C are coprime and $C \geq 0$.

The next lemma shows that the $|a_i|$-s are uniformly bounded. We express the bounds here and in the forthcoming lemmata in the dependence of λ, but one can easily transform them into the dependence of u because $\lambda \in (u, u + 1)$.

Lemma 2. *If* $\lambda \geq 3L + 1$ *then*

$$|a_1|, |a_2|, |a_3| < 2(L + 1),$$
$$|a_1| > \frac{1}{(\lambda + 1)^2},$$
$$|a_3| > \frac{1}{(\lambda + 1)^4}.$$

[1] The square root of the discriminant of $S_u(X)$.

Proof. We mentioned in the proof of Lemma 1 that $-(u^2+u+7)$ is the square root of the discriminant of $S_u(X)$, i.e. $u^2+u+7 = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda_1-\lambda_2)$. Hence

$$\frac{\lambda_2-\lambda}{u^2+u+7} = \frac{-1}{(\lambda-\lambda_1)(\lambda_1-\lambda_2)}.$$

Using (2) one obtains

$$(\lambda-\lambda_1)(\lambda_1-\lambda_2) = \lambda\left(\frac{\lambda^2+\lambda+1}{\lambda^2+\lambda}\right)^2.$$

As the second factor is larger than one, we obtain the left hand side of the inequality

$$\lambda < |(\lambda-\lambda_1)(\lambda_1-\lambda_2)| < \lambda+1$$

immediately. One can verify the other inequality with a simple computation. Using this we obtain

$$\begin{aligned}|a_1| &= \frac{|a_1'|}{|(\lambda-\lambda_1)(\lambda_1-\lambda_2)|}\\ &< \frac{|-\lambda(A+B)-A+B+C+B/\lambda|}{\lambda}\\ &\le |A+B| + \frac{|-A+B+C|}{\lambda} + \frac{|B|}{\lambda^2}\\ &< 2(L+1).\end{aligned}$$

The last inequality is true because $|A+B| \le 2L$. Further we have $|-A+B+C| \le 3L < \lambda$ by the definition of L and by our assumption. Finally $|B| \le L < \lambda/3 \le \lambda^2$, because L is a positive integer, thus $\lambda > 3$. The proof of the upper estimates for $|a_2|$ and $|a_3|$ is similar and left to the reader.

In order to prove the lower bound for $|a_1|$ we notice first that $|a_1'| \ge \frac{1}{\lambda+1}$. Indeed, if $A+B \ne 0$, then $|-\lambda(A+B)| + |A-B-C| \ge \lambda - |A-B-C| \ge 3L+1-3L = 1$. Further, as $|B|/\lambda < 1/3$ we have $|a_1'| > 1-1/3 = 2/3 > 1/(\lambda+1)$. If $A+B=0$ and $A-B-C \ne 0$ then we argue similarly. Finally, if $A+B = A-B-C = 0$ then $B \ne 0$ and we get again the claimed conclusion.

The estimate $|a_1'| \ge \frac{1}{\lambda+1}$ together with the upper bound for $|(\lambda-\lambda_1)(\lambda_1-\lambda_2)|$ proves the lower bound for $|a_1|$ immediately. The proof of the lower bound for $|a_3|$ is similar after noticing that $|a_3'| \ge \frac{1}{\lambda(\lambda+1)}$. □

Lemma 3. *We have*

$$\frac{a_2}{a_3} = \lambda\frac{a_2'}{a_3'}.$$

If $\lambda > 4L$ *then*

$$\frac{1}{(\lambda+1)(2L+1)} < \left|\frac{a_2'}{a_3'}\right| < (\lambda+1)^3(L+2).$$

Proof. We have

$$\frac{a_2}{a_3} = \frac{\lambda_1-\lambda}{\lambda_2-\lambda_1}\frac{a_2'}{a_3'}$$

by Lemma 1. Expressing the numerator and the denominator of the first factor as functions of λ we get

$$\lambda_1-\lambda = -\frac{1}{\lambda+1}-\lambda = -\frac{\lambda^2+\lambda+1}{\lambda+1}$$
$$\lambda_2-\lambda_1 = -\frac{\lambda+1}{\lambda}+\frac{1}{\lambda+1} = -\frac{\lambda^2+\lambda+1}{\lambda(\lambda+1)},$$

and the first statement of the lemma is proved.

Plainly

$$\frac{1}{\lambda+1} \le |a_2'| < (L+2)\lambda$$

provided $|C-A| \le 2L < \lambda$ and $|A+B| \le 2L < \lambda(\lambda+1)$, but both inequalities follow from $\lambda > 3L$. As λ is a cubic algebraic number $a_3' \ne 0$. Further

$$\frac{1}{\lambda(\lambda+1)} = \frac{1}{\lambda}-\frac{1}{\lambda+1} \le |a_3'| \le 2L+1.$$

Combining the upper and the lower bounds for $|a_2'|$ and for $|a_3'|$ we obtain

$$\frac{1}{\lambda+1}\frac{1}{2L+1} \le \left|\frac{a_2'}{a_3'}\right| < \lambda^2(\lambda+1)(L+2) < (\lambda+1)^3(L+2). \quad \square$$

In the proof of our theorems the quotient $\frac{a_1}{a_3}$ plays distinguished role. In the next lemma we show that it is $\lambda+1$ times $\frac{a_1'}{a_3'}$, i.e. it is enough to control the latter quotient. For some values of A, B, C we are able to show, that $\frac{a_1'}{a_3'}$ is a unit, whence we are able to solve (1) completely. We checked other triplets too, but they do not lead to units.

Lemma 4. *We have*

$$\frac{a_1}{a_3} = -(\lambda+1)\frac{a_1'}{a_3'}.$$

Assume that $C \ge 0$ *and* $\gcd(A, B, C) = 1$.

(i) *If* $A+B \ne 0$ *and* $B+C \ne 0$ *and* $\lambda \ge 8L$ *then*

(a) $A \neq C$ *yields*

$$\left|\frac{a_1'}{\lambda a_3'} + \frac{A+B}{B+C}\right| \leq \frac{26L^2}{\lambda},$$

(b) $A = C, B \neq 0$ *and* $(A, B, C) \neq (3, -1, 3)$ *yields*

$$\left|\frac{a_1'}{\lambda a_3'} + 1 - \frac{A+3B}{A+B}\frac{1}{\lambda+1}\right| \leq \frac{9L^2}{\lambda(\lambda+1)},$$

(c) $A = C, B \neq 0$ *and* $(A, B, C) = (3, -1, 3)$ *yields*

$$\frac{a_1'}{\lambda a_3'} + 1 = \frac{-1}{\lambda(2\lambda^2+3\lambda+2)},$$

(d) *finally* $A = C$ *and* $B = 0$ *yields*

$$\frac{a_1'}{a_3'} = -\lambda.$$

(ii) *If* $A + B \neq 0$ *and* $A + 2B \neq 0$, *but* $B + C = 0$, *and* $\lambda > 3L$ *then*
(a) $B \neq 0$ *implies*

$$\left|\frac{a_1'}{\lambda(\lambda+1)a_3'} + \frac{A+B}{A+2B}\right| \leq \frac{4L^2}{\lambda}$$

(b) $B = 0$ *implies*

$$\frac{a_1'}{a_3'} = -\lambda(\lambda+1).$$

(iii) *If* $A + B \neq 0$, *but* $A + 2B = B + C = 0$, *then*

$$\frac{a_1'}{a_3'} = -(\lambda+1)^3.$$

(iv) *If* $A + B = 0$, *but* $B + C \neq 0$ *and* $\lambda \geq 6L$ *then*

$$\left|\frac{a_1'}{a_3'} - \frac{2B+C}{B+C}\right| \leq \frac{4L(L+1)}{\lambda}.$$

(v) *If* $A + B = B = 0$, *then*

$$\frac{a_1'}{a_3'} = 1.$$

(vi) *If* $A + B = B + C = 0$, *then*

$$\frac{a_1'}{a_3'} = \frac{\lambda + 1}{\lambda}.$$

(vii) *If* $A + B = 2B + C = 0$, *then*

$$\frac{a_1'}{a_3'} = \frac{\lambda + 1}{\lambda^2}.$$

Proof. Applying σ to the identity of Lemma 3 we obtain

$$\frac{a_3}{a_1} = \sigma\left(\frac{a_2}{a_3}\right) = \sigma(\lambda)\sigma\left(\frac{a_2'}{a_3'}\right) = -\frac{1}{\lambda+1}\frac{a_3'}{a_1'},$$

which proves our first identity.

For convenience of the reader we recall the definitions of a_1' and a_3'

$$a_1' = -\lambda(A + B) + \frac{B}{\lambda} - A + B + C$$
$$a_3' = \frac{A+B}{\lambda} + \frac{B}{\lambda+1} + B + C.$$

Considering a_1', a_3' functions of λ or u we can observe that their behavior is essentially different if $A + B = 0$ or $A + B \neq 0$. They need different treatise.

Case (i) (a), $A + B \neq 0, B + C \neq 0$**.**

Now we estimate the numerator, N, and denominator, D, of

$$\left|\frac{a_1'}{\lambda a_3'} + \frac{A+B}{B+C}\right|$$

separately. We have

$$|N| = \left|(B+C)(B+C-A) + (A+B)^2 + \frac{B(B+C)}{\lambda} + \frac{\lambda B(A+B)}{\lambda+1}\right|$$
$$\leq 6L^2 + 4L^2 + L^2 + 2L^2 = 13L^2,$$

where we used that $\lambda > 3$ and $\frac{\lambda}{\lambda+1} < 1$.

Further we have $\lambda \geq 8L \geq 4|A+B|$ and $\lambda + 1 \geq 8L + 1 > 4|B|$, thus

$$\left|\frac{A+B}{\lambda} + \frac{B}{\lambda+1}\right| \leq \left|\frac{A+B}{\lambda}\right| + \left|\frac{B}{\lambda+1}\right| < \frac{1}{2}.$$

As $B + C \neq 0$ we have $|B + C| \geq 1$, hence $|a_3'| > 1/2$, thus $|D| > \lambda/2$, which proves (i) (a).

Case (i) (b), (c), $A+B\neq 0, B+C\neq 0, A=C, B\neq 0$.

We have

$$\frac{a_1'}{\lambda a_3'}+1-\frac{A+3B}{(A+B)(\lambda+1)}=-\frac{B((A+3B)\lambda^2-(A+B)\lambda-(A+B))}{\lambda(\lambda+1)(A+B)((A+B)\lambda^2+(2A+3B)\lambda+A+B)}.$$

Inserting here $A=3, B=-1$ we get (c) immediately. To prove (b) divide the numerator and the denominator by λ^2 and use that $|A+B|\geq 1, |B|\leq L, |A+3B|\leq 4L$ and $|A+B|/\lambda^2, |A+B|/\lambda, |2A+3B|/\lambda \leq 1/4$.

The proof of (d) is straightforward.

Case (ii): $A+B\neq 0, A+2B\neq 0, B+C=0$. We denote by N and D the numerator as well as the denominator of

$$\left|\frac{a_1'}{\lambda(\lambda+1)a_3'}+\frac{A+B}{A+2B}\right|.$$

We have

$$\begin{aligned}N&=\left|-A(A+2B)+\frac{B(A+2B)}{\lambda}+(A+B)^2\right|\\&=\left|B^2+\frac{B(A+2B)}{\lambda}\right|\\&\leq L^2+L^2=2L^2,\end{aligned}$$

because $\lambda>3$.

To prove a lower bound for D we start with

$$\lambda(\lambda+1)|a_3'|=|(A+B)(\lambda+1)+B\lambda|=|(A+2B)(\lambda+1)-B|\geq\lambda+1-L,$$

because $A+2B\neq 0$. As $\lambda>3L$ we have $L<2(\lambda+1)$, hence

$$(\lambda+1)\left(1-\frac{L}{\lambda+1}\right)>\frac{\lambda+1}{2}>\frac{\lambda}{2}.$$

This together with $A+2B\neq 0$ implies

$$D=\lambda(\lambda+1)|a_3'||A+2B|>\lambda/2.$$

Case (iii): $A+B\neq 0, A+2B=B+C=0$.

The assumptions $\gcd(A,B,C)=1$ and $C\geq 0$ yield $(A,B,C)=(2,-1,1)$, i.e.

$$\frac{a_1'}{\lambda(\lambda+1)a_3'}=\frac{-\lambda-\frac{1}{\lambda}-2}{\lambda+1-\lambda}=-\frac{(\lambda+1)^2}{\lambda}.$$

Case (iv) $A+B=0, B+C\neq 0$**.** In this case our functions simplify to

$$a_1' = \frac{B}{\lambda} + 2B + C$$
$$a_3' = \frac{B}{\lambda+1} + B + C.$$

In the actual case

$$\begin{aligned}\left|\frac{a_1'}{a_3'} - \frac{2B+C}{B+C}\right| &= \left|\frac{B}{B+C}\frac{(B+C)\frac{\lambda+1}{\lambda} - (2B+C)}{B+(\lambda+1)(B+C)}\right| \\ &= \left|\frac{B}{B+C}\frac{-B+\frac{B+C}{\lambda}}{\lambda(B+C)+2B+C}\right| \\ &\leq \frac{2L(L+1)}{\lambda}.\end{aligned}$$

The last inequality holds, because $|2B+C| \leq 3L < \lambda/2$ and $1 \leq |B+C| \leq 2L < \lambda$.

Case (v) $A+B=B=0$**.** Then $A=B=0$, thus $C\neq 0$, which together with $\gcd(A,B,C)=1$ and $C\geq 0$ yields $(A,B,C)=(0,0,1)$, hence $a_1'=a_3'=1$.

Case (vi) $A+B=B+C=0$**.** Then $A=C=-B$, which together with $\gcd(A,B,C)=1$ and $C\geq 0$ yields $(A,B,C)=(1,-1,1)$, hence

$$\frac{a_1'}{a_3'} = \frac{1/\lambda+1}{1/(\lambda+1)} = \frac{\lambda+1}{\lambda}.$$

Case (vii) $A+B=2B+C=0$**.** Then $A=-B$ and $C=-2B$, which together with $\gcd(A,B,C)=1$ and $C\geq 0$ yields $(A,B,C)=(1,-1,2)$, hence

$$\frac{a_1'}{a_3'} = \frac{-1/\lambda}{-1/(\lambda+1)+1} = \frac{\lambda+1}{\lambda^2}. \quad \square$$

We finish this section with the estimation of the growth of $|F(u,n)| = |F(A,B,C,u,n)|$ and $|F(u,-n)| = |F(A,B,C,u,-n)|$.

Lemma 5. *Assume that* $u,n\in\mathbb{Z}, n\geq 0, u+1>\lambda>4L$*. If* $n\geq 4$ *then*

$$||F(u,-n)| - |a_1|(\lambda+1)^n| < 4(L+1). \tag{4}$$

If $n\geq 8$ *then*

$$||F(u,n)| - |a_3|\lambda^n| < |a_2||\lambda_2|^n + 1 < 2(L+1)|\lambda_2|^n + 1. \tag{5}$$

Proof. By the Binet formula and by (2)

$$F(u,m) = a_1\lambda_1^m + a_2\lambda_2^m + a_3\lambda^m = a_1\left(-\frac{1}{\lambda+1}\right)^m + a_2\left(-\frac{\lambda+1}{\lambda}\right)^m + a_3\lambda^m$$

holds for every integers m.

If $m = -n$ with a non-negative integer n then, as $\lambda > 4$ for $u \geq 4$, the first term tends exponentially to infinity, while the second and third to zero. More precisely by Lemma 2 we obtain

$$|F(u,-n) - a_1(-(\lambda+1))^n| < 4(L+1).$$

If the signs of $F(u,-n)$ and $a_1(-(\lambda+1))^n$ coincide then (4) holds. Otherwise

$$||F(u,-n)| + |a_1|(\lambda+1)^n| < 4(L+1),$$

especially $|a_1|(\lambda+1)^n < 4(L+1)$. Using again Lemma 2 and $\lambda > 4L$ we get $|a_1|(\lambda+1)^n > (\lambda+1)^{n-2}$, hence

$$(4L+1)^{n-2} < (\lambda+1)^{n-2} < 4(L+1),$$

which is absurd for $n \geq 4$.

Assume now that $m = n$ with a non-negative integer n, then as $u > 4L - 1 \geq 3$, the dominating term in the Binet formula is $a_3\lambda^n$, but the absolute value of the second term tends to infinity as well. Using Lemma 2 we obtain

$$\begin{aligned}|F(u,n) - a_3\lambda^n| &\leq |a_2||\lambda_2|^n + \frac{2(L+1)}{(\lambda+1)^n}\\ &< |a_2||\lambda_2|^n + 1\\ &< 2(L+1)|\lambda_2|^n + 1\end{aligned}$$

for $n \geq 1$. Indeed

$$\frac{2(L+1)}{(\lambda+1)^n} < \frac{2(L+1)}{(4L+1)^n},$$

which is obviously true for $n \geq 1$.

If $F(u,n)$ and $a_3\lambda^n$ have the same signs, then (5) holds. Otherwise

$$|a_3|\lambda^n < 2(L+1)\left(\frac{\lambda+1}{\lambda}\right)^n + 1.$$

Using the lower bound for $|a_3|$ from Lemma 2 and that the first term is larger than one we obtain

$$\frac{\lambda^n}{(\lambda+1)^4} < 4(L+1)\left(\frac{\lambda+1}{\lambda}\right)^n.$$

As $\lambda > 4L$ we have $4(L+1) < \lambda^2$, thus it is enough to solve the stronger inequality

$$\lambda^{n-6} < \left(\frac{\lambda+1}{\lambda}\right)^{n+4},$$

which is impossible if $n > 7$. $\square$

3. A Baker's type inequality

An important ingredient of the proof is A. Baker's theory on linear forms in logarithms of algebraic numbers. It has many variants, nowadays one of the sharpest is due to Matveev [9]. To prove Theorem 2 we need a sharp lower bound for a linear form in two logarithms. In this case the nearly thirty years old estimate of Laurent, Mignotte and Nesterenko [7] is better.

For an algebraic number η with minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(N)}) \in \mathbb{Z}[X]$$

with $a_0 > 0$ and with relative prime coefficients, write $h(\eta)$ for its *absolute logarithmic or Weil height* given by

$$h(\eta) = \frac{1}{N}\left(\log a_0 + \sum_{j=1}^{N} \max\{0, \log|\eta^{(j)}|\}\right).$$

If η is an algebraic integer then

$$h(\eta) \le \log\max\{|\eta^{(j)}|,\ j = 1, \ldots, N\}. \tag{6}$$

We provide some important and well known properties of the function h.

Lemma 6. *Let γ, η be algebraic numbers of degree at most d and $u \in \mathbb{Q}$. Then we have*

(1) $h(\gamma \pm \eta) \le h(\gamma) + h(\eta) + \log 2$,
(2) $h(\gamma\eta^{\pm 1}) \le h(\gamma) + h(\eta)$,
(3) $h(\gamma^u) = |u|h(\gamma)$.

For the proof see e.g. Waldschmidt [29], Ch. 3.2.

Let $\mathbb{K}$ be an algebraic number field of degree $d_{\mathbb{K}}$ and let $\eta_1, \eta_2, \ldots, \eta_t \in \mathbb{K} \setminus \{0\}$, and $e_1, \ldots, e_t$ be nonzero integers. Put

$$E = \max\{|e_1|, \dots, |e_t|, 3\} \qquad \text{and} \qquad \Gamma = \prod_{i=1}^{t} \eta_i^{e_i} - 1.$$

Let $F_1, \dots, F_t$ be such that

$$F_j \geq \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 1\}, \quad \text{for} \quad j = 1, \dots t.$$

Under the above mentioned circumstances, Matveev [9] proved the additive variant of the following theorem.

Lemma 7. *If* $\Gamma \neq 0$*, then*

$$\log |\Gamma| > -3 \cdot 30^{t+4}(t+1)^{5.5} d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log tE) F_1 F_2 \cdots F_t.$$

Now we recapitulate Corollaire 2. of Laurent, Mignotte and Nesterenko [7].

Lemma 8. *Assume that* η_1, η_2 *are multiplicatively independent, positive, real, algebraic numbers,* e_1, e_2 *positive integers and* $\Lambda = e_1 \log \eta_1 - e_2 \log \eta_2$*. Set* $\mathbb{K} = \mathbb{Q}(\eta_1, \eta_2)$ *and* $d_{\mathbb{K}} = [\mathbb{K} : \mathbb{Q}]$*. Set further*

$$\mathcal{E} = \frac{e_1}{d_{\mathbb{K}} \log A_2} + \frac{e_2}{d_{\mathbb{K}} \log A_1},$$

where $A_1, A_2 > 1$ *are reals and such that*

$$\log A_i \geq \max\left\{h(\eta_i), \frac{|\log \eta_i|}{d_{\mathbb{K}}}, \frac{1}{d_{\mathbb{K}}}\right\}, \quad i = 1, 2.$$

Then

$$\log |\Lambda| \geq -24.34 \cdot d_{\mathbb{K}}^4 \left(\max\left\{\log \mathcal{E} + 0.14, \frac{21}{d_{\mathbb{K}}}, \frac{1}{2}\right\}\right)^2 \log A_1 \log A_2.$$

We will complete the proof of our main result by using a reduction procedure, which is based on the following classical result of Lagrange (1770). For its proof see e.g. Hua [5].

Lemma 9. *Let* $\alpha \in \mathbb{R}$ *and denote* $\frac{p_n}{q_n}, n \geq 0$ *the sequence of its convergents. If* $n \geq 1, 0 < q \leq q_n$ *and* $p/q \neq p_n/q_n$ *then*

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p}{q}\right|.$$

The following technical lemma has many variants, its present form appeared in [18].

Lemma 10. *If $s \geq 1, T \geq (4s^2)^s$, and $x/(\log x)^s < T$, then*

$$x < 2^s T(\log T)^s.$$

We also need the following simple property of the logarithm function, for its proof see de Weger [30].

Lemma 11. *Let $a \in \mathbb{R}$. If $a < 1$ and $|x| < a$ then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a}|x|,$$

and

$$|x| < \frac{a}{1-e^{-a}}|e^x - 1|.$$

4. Proof of Theorem 1

We pointed out in Section 2 that to study (1) we may assume without loss of generality that A, B, C are coprime and $C \geq 0$. That parameters are fixed, while u, n, m varies.

We distinguish five cases according the signs of n, m, and the size of u. If n and m have the same signs then elementary considerations are enough to the proof. If u lies below a bound depending only on $L = \max\{|A|, |B|, |C|\}$, then an old result of Mignotte [10] implies our statement. That are called simple cases.

4.1. Simple cases

Case I. $n, m \geq 0$**.** We may assume without loss of generality that $0 \leq m < n$.

If $\lambda > 4L$, then we have by (1) and by (5)

$$\begin{aligned}|a_3|(\lambda^n - \lambda^m) &= ||F(u,n)| - |a_3|\lambda^n - (|F(u,m)| - |a_3|\lambda^m)| \\ &< 4(L+1)|\lambda_2|^n + 2.\end{aligned}$$

The left hand side is at least $\lambda^n|a_3|(1-1/\lambda) > \lambda^n/(2(\lambda+1)^4) > 1$ for all $n \geq 6$ by Lemma 4 and because $\lambda > 4L \geq 4$. Hence dividing by $|a_3|(\lambda^n - \lambda^m) \geq \lambda^n/(2(\lambda+1)^4)$ we obtain

$$1 < \frac{8(L+1)(\lambda+1)^4}{\lambda^n}\left(1+\frac{1}{\lambda}\right)^n + \frac{4(\lambda+1)^4}{\lambda^n}.$$

Because of $\lambda > 4L \geq 4$ we have $16 < \lambda^2, L+1 < \lambda$ and $(\lambda+1)^4 < \lambda^5$, hence the second summand is less than $1/2$ provided $n \geq 7$, while the first summand is less than $(\lambda+1)^{n+4}/(2\lambda^{2n-3})$, which is less than $1/2$ whenever

$$\frac{n+4}{4} \le \frac{2n-3}{5},$$

i.e. $n \ge 11$. Hence if $\lambda > 4L$ then (1) cannot hold for $n \ge 11$.

Fix $1 \le u < 4L$. Then $\lambda \le 4L$ and $\lambda > |\lambda_2| > 1 > |\lambda_1|$, hence there exists a function $c_0(u)$ depending only on u, such that

$$||F(u,n)| - |a_3|\lambda^n| < |a_2||\lambda_2|^n + 1,$$

for all $n \ge c_0(u)$. As the pair $(n,m) \in \mathbb{Z}^2$ is a solution of (1), the last inequality yields the upper estimate

$$|a_3|\lambda^n(1-1/\lambda) \le |a_3|(\lambda^n - \lambda^m) < 2|a_2||\lambda_2|^n + 2$$

while the lower estimate is obvious. There exists a function $c_1(u)$ depending only on u such that this inequality is impossible for all $n \ge c_1(u)$, hence (1) cannot hold for $n \ge \max\{c_0(u), c_1(u)\}$. If $u = 0$ then $|\lambda_2| > \lambda > |\lambda_1|$, but repeating the above argument we obtain constants $c_0(0), c_1(0)$ such that (1) cannot hold for $n \ge \max\{c_0(0), c_1(0)\}$ too. Summarizing, with the choice $c = \max\{11, c_0(u), c_1(u), 0 \le u \le 4L\}$ Theorem 1 holds in this case.

Case II. $n, m < 0$**.** The proof is the same as in the previous case, but we are working with (4) instead of (5). If $\lambda > 4L$ then we conclude that (1) cannot hold for $n \ge 6$. In the range $0 \le u \le 4L$ we get unspecified upper bounds for $|n|$ like in Case I.

Case III. $n \ge 0, m < 0$ **and** $0 \le u < 1664L^3$**.** The characteristic polynomial of $(F(u,n))_{n=0}^{\infty}$ is $S_u(X)$ with the dominating root λ_2, if $u = 0$ and λ for $u > 0$. Similarly the characteristic polynomial of $(F(u,-n))_{n=0}^{\infty}$ is $-X^3 S_u(1/X)$ with the dominating root λ_1. Thus for any fixed u in the range $0 \le u < 1664L^3$ by the Théorème of Mignotte [10] the equation (1) has only finitely many effectively computable solutions, i.e. there exist an effectively computable constant $c(u) > 0$ such that $\max\{n, |m|\} \le c(u)$ for any solutions $(n,m) \in \mathbb{Z}^2$ of (1). The choice $c = \max\{c(u), 0 \le u < 1664L^3\}$ verifies Theorem 1 in this case.

Case IV. $n \le 7, m \ge -3$ **and** $u \ge 1664L^3$**.**

Later we will apply Lemma 5, but it holds only for $n \ge 8$ or for $m \le -4$, hence to settle the uncovered range we need extra argument.

Assume first that $n \le 7$. A simple computation yields

$$\begin{aligned} F(u,7) = {} & (C+B)u^5 + (-C+B+A)u^4 + (9C+6B-A)u^3 \\ & + (-10C+6B+7A)u^2 + (21C-7A)u - 14C + 14B + 9A, \end{aligned}$$

thus $|F(u,7)| < 37L(u+1)^5$ for any $u \ge 0$. It is easy to see that $|F(u,j)| < 37L(u+1)^5 < u(u+1)^5$ is true for $0 \le j \le 7$ too for any $u \ge 1664L^3$. If $m \ge -9$ then we are done. Otherwise, by (4), by Lemma 2, and by $L \ge 1$ we have

$$|F(u,m)| > |a_1|(\lambda+1)^{-m} - 4(L+1) > (\lambda+1)^{-m-2} - 8L.$$

Hence, if $F(u,j) = F(u,m)$ for some triple $(u,j,m) \in \mathbb{Z}^2$ with $u \geq 1664L^3, 0 \leq j \leq 7$ and $m \geq -9$ then

$$u(u+1)^5 > (\lambda+1)^7 - 8L \geq (\lambda+1)^7 - u.$$

Dividing by $u(u+1)^5$ and taking into account that $u < \lambda$ and $u \geq 1664L^3$ we obtain

$$1 > \lambda + 1 - \frac{1}{(u+1)^5} > u \geq 1664,$$

which is absurd. Thus $m \geq -9$ and the theorem is proved whenever $n \leq 7$.

Assume now that $m \geq -3$. We have

$$F(u,-3) = -(A+B)u^3 - (4A+2B-C)u^2 - (10A+3B-3C)u - 11A + 3B + 5C,$$

which yields $|F(u,-3)| < 19L(u+1)^3$. The same upper bound holds for $|F(u,j)|, j = 0,-1,-2$ too. In the first step we showed that $n \geq 8$, but then

$$|F(u,n)| > |a_3|\lambda^n - |a_2|\lambda_2^n - 1$$

by (5). Similarly to the previous case we may assume $n \geq 11$, whence analogous argument yields

$$|a_3|\lambda^n < |a_2|\lambda_2^n + 19L(u+1)^3 + 1.$$

Dividing by $|a_3|\lambda^n$ and using Lemmata 3 and 2 as well as the inequality $(\lambda+1)^4 < \lambda^5$, which was proved in Case I, we get

$$\begin{aligned} 1 &< \lambda(\lambda+1)^3(L+2)\left(\frac{\lambda+1}{\lambda^2}\right)^n + 19L(\lambda+1)^4\frac{(u+1)^3}{\lambda^n} + \frac{(\lambda+1)^4}{\lambda^n} \\ &< \frac{1}{2}\frac{(\lambda+1)^{n+3}}{\lambda^{2n-2}} + \frac{u(\lambda+1)^7}{\lambda^n} + \frac{1}{3} \\ &< 1, \end{aligned}$$

which is a contradiction. Hence, if $m \geq -3$ then $n \leq 11$, i.e. the theorem is proved for this strip too.

4.2. The hard case

The investigation of the hard case starts here, but continues in the next subsections. From now on we assume $u \geq 1664L^3$ and $n \geq 8, m \leq -4$. Our aim here is to derive a sharp upper bound for a linear form in logarithms of algebraic numbers.

By our assumptions on n and m the inequalities (4) hold and (5) imply

$$|a_3|\lambda^n - |a_2||\lambda_2|^n - 1 < |F(u,n)| = |F(u,m)| < |a_1|(\lambda+1)^{-m} + 4(L+1)$$

and

$$|a_1|(\lambda+1)^{-m} - 4(L+1) < |F(u,m)| = F(u,n) < |a_3|\lambda^n + |a_2||\lambda_2|^n + 1,$$

hence

$$\left||a_3|\lambda^n - |a_1|(\lambda+1)^{-m}\right| < |a_2||\lambda_2|^n + 4L + 5.$$

Dividing by $|a_3|\lambda^n$ and using Lemma 1 we get

$$\left|\frac{|a_1'|}{|a_3'|}\frac{(\lambda+1)^{-m+1}}{\lambda^n} - 1\right| < \lambda\frac{|a_2'|}{|a_3'|}\left(\frac{\lambda+1}{\lambda^2}\right)^n + \frac{4L+5}{|a_3|\lambda^n}.$$

We estimate the first summand of the right hand side by using Lemma 3, the inequalities $n \geq 12, 2L+4 < 8L < \lambda$ and $2(\lambda+1)^4 < \lambda^5$ and obtain

$$\lambda\frac{|a_2'|}{|a_3'|}\left(\frac{\lambda+1}{\lambda^2}\right)^n < \lambda(\lambda+1)^3(L+2)\left(\frac{\lambda+1}{\lambda^2}\right)^n < \frac{(\lambda+1)^{n+3}}{2\lambda^{2n-2}} < \frac{1}{2\lambda^{n/2}}.$$

To estimate the second summand of the right hand side we invoke Lemma 2 and get

$$\frac{4L+5}{|a_3|\lambda^n} < \frac{(4L+5)(\lambda+1)^4}{\lambda^n} < \frac{(\lambda+1)^5}{\lambda^n}\frac{1}{2\lambda^{n/2}} < \frac{1}{2\lambda^{n/2}}.$$

Inserting these estimates we obtain,

$$\left|\frac{|a_1'|}{|a_3'|}\frac{(\lambda+1)^{-m+1}}{\lambda^n} - 1\right| < \frac{1}{\lambda^{n/2}},$$

which is weaker, but much more manageable than the earlier one.

In Lemma 4 we distinguished seven cases for the behavior of $|a_1'/a_3'|$, but we would like to investigate them uniformly. To be able to do this we write the last inequality in the form

$$\left|\Phi(u)\lambda^{-n-m+\tau}\left(\frac{\lambda+1}{\lambda}\right)^{-m+\mu} - 1\right| < \frac{1}{\lambda^{n/2}}, \tag{7}$$

Table 1
Values of parameters.

Region	$\Phi(u)$	τ	μ	ϕ	s
$A+B\neq 0,\ B+C\neq 0,\ A\neq C$	$\frac{\lvert a_1'\rvert}{\lambda\lvert a_3'\rvert}$	2	1	$\left\lvert\frac{A+B}{B+C}\right\rvert$	$26L^2$
$A+B\neq 0, B+C\neq 0, A=C, B\neq 0,$ $(A,B,C)\neq(3,-1,3)$	$\frac{\lvert a_1'\rvert}{\lambda\lvert a_3'\rvert}$	2	1	1	$26L^2$
$A+B\neq 0, B+C\neq 0, A=C, B\neq 0,$ $(A,B,C)=(3,-1,3)$	$\frac{\lvert a_1'\rvert}{\lambda\lvert a_3'\rvert}$	2	1	1	$\frac{1}{154}$
$A+B\neq 0, B+C\neq 0, A=C, B=0$	1	1	1	1	0
$A+B\neq 0, B+C=0, A+2B\neq 0, B\neq 0$	$\frac{\lvert a_1'\rvert}{\lambda(\lambda+1)\lvert a_3'\rvert}$	-1	0	$\left\lvert\frac{A+B}{A+2B}\right\rvert$	$4L^2$
$A+B\neq 0, B+C=0, A+2B\neq 0, B=0$	1	3	2	1	0
$A+B\neq 0, A+2B=B+C=0$	1	4	4	1	0
$A+B=0, B+C\neq 0$	$\frac{\lvert a_1'\rvert}{\lvert a_3'\rvert}$	1	1	$\left\lvert\frac{2B+C}{B+C}\right\rvert$	$4L(L+1)$
$A+B=B=0$	1	1	1	1	0
$A+B=B+C=0$	1	1	2	1	0
$A+B=2B+C=0$	1	0	2	1	0

where the values of $\Phi(u),\tau,\mu$ are given in Table 1. It includes two more columns ϕ and s, where $\phi=\lim_{u\to\infty}\Phi(u)$ and $s=0$, if $\Phi(u)$ is the constant one function, otherwise

$$|\Phi(u)-\phi|\leq\frac{s}{\lambda},\tag{8}$$

see Lemma 4.

4.3. *Strong lower bound for n*

Our aim in this subsection is to prove that $-m+\mu$ is large comparing to λ. By Table 1 ϕ is a rational number, which is either one or satisfies $\phi\in[1/(3L),1-1/(3L)]\cup[1+1/(3L),3L]$. The assumption $\lambda>1664L^3$ yields

$$|\Phi(u)-\phi|\leq\frac{s}{\lambda}\leq\frac{26L^2}{\lambda}<\frac{1}{6L}\leq\frac{\phi}{2}.$$

Hence

$$\Phi(u)=|\phi-(\phi-\Phi(u))|\geq\phi-|\phi-\Phi(u)|\geq\phi-\frac{\phi}{2}=\frac{\phi}{2}.$$

Our next claim is so important that we formulate it as a lemma.

Lemma 12. *If for some initial value $(A,B,C)\in\mathbb{Z}^3$ the $\Phi(u)$ is not the constant one function, then $\Phi(u)>1$ or $\Phi(u)<1$ for all $\lambda>u\geq 78L^3$.*

Proof. If $\lim_{u\to\infty}\Phi(u)=\phi\neq 1$, then we can prove this claim uniformly. If $\phi>1$, then $\phi\geq 1+1/(3L)$. The inequality (8) and $\lambda>78L^3$ imply

$$\Phi(u) \geq \phi - \frac{s}{\lambda} \geq 1 + \frac{1}{3L} - \frac{26L^2}{\lambda} > 1 + \frac{1}{3L} - \frac{26L^2}{78L^3} = 1.$$

The case $1/(3L) \leq \phi < 1$ can be handled similarly.

Assume in the sequel that $\phi = 1$, but the corresponding $\Phi(u)$ is not the constant one function, which happens only in (i) (c) and (i) (d) of Lemma 4. In the latter case we have

$$\frac{a_1'}{\lambda a_3'} + 1 = \frac{-1}{\lambda(2\lambda^2 + 3\lambda + 2)},$$

i.e. $\Phi(u) = |\frac{a_1'}{\lambda a_3'}| > 1$ for all $u \geq 0$, because $\lambda > 0$.

In the case (i) (c) we have $A+3B \neq 0$. Assuming $(A+3B)/(A+B) > 0$ the estimation of Lemma 4 yields

$$\begin{aligned}\Phi(u) = \left|\frac{a_1'}{\lambda a_3'}\right| &\leq 1 - \frac{A+3B}{A+B}\frac{1}{\lambda+1} + \frac{9L^2}{\lambda(\lambda+1)} \\ &\leq 1 - \frac{1}{\lambda+1}\left(\frac{1}{2L} - \frac{9L^2}{78L^3}\right) = 1 - \frac{10}{13}\frac{1}{\lambda+1}.\end{aligned}$$

Hence $\Phi(u) < 1$, whenever $\lambda \geq 78L^3$. If $(A+3B)/(A+B) < 0$, then one can prove $\Phi(u) > 1$ for all $\lambda \geq 78L^3$ in similar manner. □

With the notation

$$\Gamma = \Phi \cdot \lambda^{-n-m+\tau}\left(\frac{\lambda+1}{\lambda}\right)^{-m+\mu}$$

(7) becomes $|\Gamma - 1| < 1/\lambda^{n/2}$. As $\lambda > 8$ the right hand side is smaller than $1/2$ provided $n \geq 1$, hence we may apply Lemma 11 with $x = \Gamma - 1, a = 1/2$ and obtain

$$|\log\Gamma| = \left|(n+m-\tau)\log\lambda - (-m+\mu)\log\frac{\lambda+1}{\lambda} - \log\Phi\right| < \frac{2}{\lambda^{n/2}}. \tag{9}$$

The factor 2 is correct because $\frac{-\log(1-1/2)}{1/2} = 2\log 2 < 2$.

Lemma 13. *Let $(A,B,C) \in \mathbb{Z}^3$ be as in Theorem 1. Assume that $(n,m,u) \in \mathbb{Z}^3$ is a solution of (1) such that $n \geq 0, m < 0$. If $\lambda > 1664L^3$ then $n+m-\tau \geq 0$. Moreover, if $-m \geq 4L+3$ then inequality may hold only if $\phi < 1$ and $-m+\tau > \lambda/(4L)$.*

Proof. It is very easy to prove that $n+m-\tau \geq 0$. Indeed, otherwise

$$\Phi(u)\lambda^{-n-m+\tau}\left(\frac{\lambda+1}{\lambda}\right)^{-m+\mu} \geq \frac{\phi}{2}\lambda \geq \frac{1}{6L}52L^3 \geq 8L^2 \geq 8,$$

but this contradicts (7). The chain of inequalities holds because $\Phi(u) \geq \phi/2$ whenever $\lambda > 1664L^3 > 52L^3$ and $\phi > 1/(3L)$.

Until the end of the proof of this Lemma let us assume that $n+m-\tau = 0$. It is again very easy to exclude $\Phi(u) \geq 1$. Indeed, if $\Phi(u) \geq 1$ and $n+m-\tau = 0$ then

$$\Phi(u)\lambda^{-n-m+\tau}\left(\frac{\lambda+1}{\lambda}\right)^{-m+\mu} - 1 \geq \left(\frac{\lambda+1}{\lambda}\right)^{-m+\mu} - 1 \geq 1 + \frac{-m+\mu}{\lambda} - 1 \geq \frac{1}{\lambda},$$

which contradicts (7).

Hence, to fulfill (7) for some u the inequality $\Phi(u) < 1$ must hold, but because of $\lambda > 78L^3$ Lemma 12 implies $\Phi(u) < 1$ for all $\lambda > u \geq 78L^3$. Then $\phi = \lim_{u\to\infty} \Phi(u) \leq 1$. Assume that $\phi = 1$, which happens only in the cases (i) (c) of Lemma 4.[2] As $\Phi(u) < 1$ and $A + 3B \neq 0$ we have $(A+3B)/(A+B) > 0$. Thus Lemma 4, (i) (c) implies

$$\Phi(u) \geq 1 - \frac{A+3B}{A+B}\frac{1}{\lambda+1} - \frac{9L^2}{\lambda(\lambda+1)} \geq 1 - \frac{1}{\lambda+1}(2L+1+1) = 1 - \frac{2(L+1)}{\lambda+1} \tag{10}$$

because $\frac{A+3B}{A+B} = 1 + \frac{2B}{A+B} \leq 2L+1$ and because $\lambda > 78L^3$.

In the actual case we have $\tau = 2, \mu = 1$, thus, if $n+m-2 = 0$ then

$$\begin{aligned}\Phi(u)\lambda^{-n-m+2}\left(\frac{\lambda+1}{\lambda}\right)^{-m+1} - 1 &\geq \left(1 - \frac{2(L+1)}{\lambda+1}\right)\left(1 + \frac{-m+1}{\lambda}\right) - 1\\ &= \frac{-m+1}{\lambda} - \frac{2(L+1)}{\lambda+1} - \frac{2(L+1)(-m+1)}{\lambda(\lambda+1)}\\ &\geq \frac{-m+1}{2\lambda} - \frac{2(L+1)(-m+1)}{\lambda(\lambda+1)}\\ &= \frac{-m+1}{2\lambda}\left(1 - \frac{4(L+1)}{\lambda+1}\right) \geq \frac{-m+1}{4\lambda},\end{aligned}$$

provided $-m \geq 4L+3$. However, then the last inequality contradicts again (7), i.e. $-m \geq 4L+3$ forces $n+m-2 > 0$ in this case. On the other hand, if $-m < 4L+3$ then $n < 4L+5$.

It remains to deal with the case $\phi < 1$. Then $1/(3L) \leq \phi \leq 1 - 1/(3L)$ and

$$\frac{1}{3L} - \frac{26L^2}{\lambda} \leq \Phi(u) \leq 1 - \frac{1}{3L} + \frac{26L^2}{\lambda}$$

by (8). Assume that $-m + \tau \leq \frac{\lambda}{4L}$. Then

[2] Remark that in case (i) (d) a byproduct of the proof of Lemma 12 is $\Phi(u) > 1$ for all $u \geq 0$.

$$\left(\frac{\lambda+1}{\lambda}\right)^{-m+\tau} = \left(\left(1+\frac{1}{\lambda}\right)^{\lambda}\right)^{(-m+\tau)/\lambda}$$
$$< e^{(-m+\tau)/\lambda} \le e^{1/(4L)} = \sum_{j=0}^{\infty} \frac{1}{j!(4L)^j} = 1 + \frac{1}{4L} + \sum_{j=2}^{\infty} \frac{1}{j!(4L)^j}$$
$$< 1 + \frac{1}{4L} + \frac{1}{8L^2}\sum_{j=0}^{\infty} \frac{1}{(4L)^j} < 1 + \frac{1}{4L} + \frac{1}{2L(4L-1)} = 1 + \frac{8L-1}{8L(4L-1)}.$$

Hence

$$\Phi(u)\left(\frac{\lambda+1}{\lambda}\right)^{-m+\tau} < \left(1 - \frac{1}{3L} + \frac{26L^2}{\lambda}\right)\left(1 + \frac{8L-1}{8L(4L-1)}\right)$$
$$= 1 - \frac{8L-5}{24L(4L-1)} + \frac{26L^2}{\lambda}\left(1 + \frac{1}{4L-1}\right)$$
$$\le 1 - \frac{1}{24L} + \frac{104L^2}{3\lambda} < 1 - \frac{1}{48L}.$$

The last inequality holds whenever $\lambda \ge 1664L^3$.

Hence, if $m+n-\tau=0$ then

$$\left|\Phi(u)\left(\frac{\lambda+1}{\lambda}\right)^{-m+\tau} - 1\right| = 1 - \Phi(u)\left(\frac{\lambda+1}{\lambda}\right)^{-m+\tau} > \frac{1}{48L},$$

which contradicts again (7). All assertions of the lemma are proved. □

Now we are in the position to fulfill our goal of this section, i.e. to prove a strong lower bound for n. A partial result was proved in Lemma 13, namely if $n+m-\tau=0$ then $\phi<1$ and $n=-m+\tau>\lambda/(4L)$.

In the sequel we assume $n+m-\tau>0$, then (9) implies

$$(-m+\mu)\log\frac{\lambda+1}{\lambda} > (n+m-\tau)\log\lambda - \log\Phi(u) - \frac{2}{\lambda^{n/2}} \ge \log\lambda - \log\Phi(u) - \frac{2}{\lambda^{n/2}}.$$

The function $x-\log(1+x)$ is zero at $x=0$, and is strictly monotonically increasing in $(0,\infty)$, hence

$$\log\frac{\lambda+1}{\lambda} < \frac{1}{\lambda},$$

which implies

$$-m+\mu > \lambda\left(\log\lambda - \log\Phi(u) - \frac{2}{\lambda^{n/2}}\right). \tag{11}$$

If $\Phi(u) = 1$ for all u or if $\Phi(u) < 1$ for some u then $-m + \mu > \lambda \log(\lambda/e)$. As $\lambda > u \geq 1664L^3$ by our assumption, $\Phi(u) < 1$ for all u in the later case by Lemma 12.

Assume now $\Phi(u) > 1$ for some $u \geq 1664L^3$. Then, by Lemma 12, $\Phi(u) > 1$ for all $u \geq 1664L^3$. Hence $\phi \geq 1$, and because $|\Phi(u)/\phi - 1| < 26L^2/(\phi\lambda) < 1/2$ the application of Lemma 11 yields $|\log \Phi(u) - \log \phi| < 52L^2/(\phi\lambda)$.

If $\phi > 1$ then we have $1 + 1/(3L) \leq \phi \leq 3L$ by Table 1. Hence

$$\log \Phi(u) < \log \phi + \frac{52L^2}{\phi\lambda} < \log(3L) + \frac{52L^2}{\lambda},$$

which yields

$$\log \Phi(u) + \frac{2}{\lambda^{n/2}} < \log(3L) + \frac{52L^2}{\lambda} + 1 < \log(3L) + 2 < 2\log(3L).$$

The last inequality is true because $\log(3L) > 2$. Inserting this into (11) and taking into consideration $n + m - \tau > 0$ we conclude

$$\lambda \log(\lambda/(9L^2)) < -m + \tau \leq n. \tag{12}$$

Finally assume that $\phi = 1$. Then $\log \Phi < 52L^2/\lambda$, hence

$$\log \lambda - \log \Phi - \frac{2}{\lambda^{n/2}} > \log(\lambda/(9L^2)) + \log(9L^2) - \frac{52L^2}{\lambda} - 1 > \log(\lambda/(9L^2)),$$

i.e. (12) holds always provided $\lambda \geq 52L^2$. Notice that (12) is a bit sharper lower bound for n as $n > \lambda/(4L)$, which was proved in Lemma 13, hence in the sequel we have to work with that weaker estimate.

4.4. Finish the proof of Theorem 1

To finish the proof of Theorem 1 we will use Matveev's theorem (Lemma 7), but then we have to exclude the case $\Gamma = 1$, which we write now in the form $\frac{|a_1|}{|a_3|}\lambda^{-n}(\lambda+1)^{-m} = 1$. This is equivalent to

$$|a_1\lambda_1^m| = |a_1|(\lambda + 1)^{-m} = |a_3|\lambda^n = |a_3\lambda^n|.$$

As all numbers appearing in the equations are real we get $a_1\lambda_1^m = \pm a_3\lambda^n$. Applying to this equation the automorphism σ defined in Section 2 we obtain $a_2\lambda_2^m = \pm a_1\lambda_1^n$ and $a_3\lambda^m = \pm a_2\lambda_2^n$. Dividing the last equation by a_2 and using Lemma 3 we obtain

$$|\lambda_2^n| = \frac{|a_3|}{|a_2|}\lambda^m = \frac{1}{\lambda}\frac{|a_3'|}{|a_2'|}\lambda^m < \frac{1}{\lambda}(\lambda + 1)(2L + 1)\lambda^m,$$

which yields

Table 2
Parameters for Matveev's theorem.

	1	2	3
η	$\Phi(u)$	λ	$(\lambda+1)/\lambda$
e	1	$-n-m+\tau$	$-m+\mu$
$h(\eta)$	$\log(2^8\cdot L^{10}(\lambda+1)^2)$	$(\log(\lambda+1))/3$	$(\log(\lambda+1))/3$
F	$3\cdot h(\eta)$	$\log(\lambda+1)$	$\log(\lambda+1)$

$$|\lambda_2^{n-1}| < (2L+1)\lambda^m.$$

The right hand side is less than one for $m \le -1$, while the left hand side is always at least one, i.e. the inequality is only possible if $m = 0$, but then $a_1 = \pm a_3\lambda^n$ and $n \le 6$ because $|a_1/a_3| \le 2(L+1)(\lambda+1)^4$ by Lemma 2.

Hence in the rest we may assume $\Gamma \ne 1$, and Lemma 7 is applicable. First we identify the parameters:

We prove below that the values in rows three and four of Table 2 are correct. We have $t = 3$, $\mathbb{K} = \mathbb{Q}(\lambda)$, thus $d_{\mathbb{K}} = 3$. The properties of the Shank's polynomials yield

$$h(\lambda) = h\left(\frac{\lambda+1}{\lambda}\right) = (\log|\lambda_2\lambda|)/3 = |\log|\lambda_1||/3 = (\log(\lambda+1))/3,$$

hence we may choose $F_i = \log(\lambda+1), i = 2, 3$.

Now we turn to the estimation of $h(\Phi(u))$. If $\Phi(u) = 1$ then $h(\Phi(u)) = 0$, hence $F_1 = 1$. Otherwise $\Phi(u) = \kappa|a_1'/a_3'|$, where $\kappa = 1, 1/\lambda$ or $1/(\lambda(\lambda+1))$. Thus $h(\Phi(u)) \le h(|a_1'|) + h(|a_3'|) + h(\kappa) = h(a_1') + h(a_3') + h(\kappa)$ by Lemma 6, and because $h(x) = h(|x|)$ if $x \in \mathbb{R}$. The relation $\sigma^2(a_1') = a_3'$, which can be proved with a simple computation, leads to the further simplification $h(\Phi(u)) \le 2h(a_3') + h(\kappa)$. In the below estimation of $h(a_3')$ we use $h(A) = h(B) = h(C) \le \log L$, $h(\lambda) = h(\lambda+1) = (\log(\lambda+1))/3$, and several times Lemma 6:

$$\begin{aligned} h(a_3') &\le h(B) + h(C) + h(A+B) + h(\lambda) + h(B) + h(\lambda+1) + 3\log 2 \\ &\le 3\log L + h(A) + h(B) + 2(\log(\lambda+1))/3 + 4\log 2 \\ &\le 5\log L + 2(\log(\lambda+1))/3 + 4\log 2. \end{aligned}$$

Hence $h(\Phi(u)) \le 10\log L + 6(\log(\lambda+1))/3 + 8\log 2$, which is much larger than 0, hence we may take it in the case $\Phi(u) = 1$ too.

Remember that $n \ge 12$ and $m \le -4$, hence $-m+\mu > -n-m+\tau$. Further $-m+\mu \ge 3$, hence the choice $E = -m+\mu$ is allowed, and Lemma 7 yields

$$\begin{aligned} \log|\Gamma - 1| &> -3\cdot 30^7\cdot 3^3(1+\log 3)(1+\log(3(-m+\mu)))\log^2(\lambda+1) \\ &\quad \times 3\cdot\log(2^8\cdot L^{10}(\lambda+1)^2) \\ &> -4.47\cdot 10^{13}\cdot\log(-m+\mu)\log^2(\lambda+1)(4\log 2 + 5\log L + \log(\lambda+1)) \end{aligned}$$

$$> -1.8 \cdot 10^{14} \cdot \log(-m+\mu) \log^3(\lambda+1).$$

The last inequality holds because $\lambda > 52L^2$, hence $4\log 2 + 5\log L + \log(\lambda+1) < 4\log(\lambda+1)$.

On the other hand by (7) we have

$$\log|\Gamma - 1| < -\frac{n}{2}\log\lambda.$$

Comparing the lower and the upper bounds and using $n \geq -m+\tau$, which is true by Lemma 13, yields

$$\frac{n}{2}\log\lambda < 1.8 \cdot 10^{14} \cdot \log(-m+\mu)\log^3(\lambda+1) < 1.8 \cdot 10^{14} \cdot \log n \cdot \log^3(\lambda+1),$$

which can be written in the form

$$\frac{n}{\log n} < 4.5 \cdot 10^{14} \cdot \log^2(\lambda+1),$$

where we used $\log(\lambda+1) < 5 \cdot (\log\lambda)/4$.

The application of Lemma 10 with the parameters $x = n, s = 1, T = 4.5 \cdot 10^{14} \cdot \log^2(\lambda+1)$ is allowed because $T > 4$, hence

$$\begin{aligned} n &< 9 \cdot 10^{14} \cdot \log^2(\lambda+1) \cdot \log(4.5 \cdot 10^{14} \cdot \log^2(\lambda+1)) \\ &< 9 \cdot 10^{14} \cdot \log^2(\lambda+1) \cdot (34 + 2\log\log(\lambda+1)) \\ &< 3.06 \cdot 10^{16} \cdot \log^2(\lambda+1) \cdot \log\log(\lambda+1). \end{aligned}$$

In the last subsection we proved the lower bound $n \geq \lambda/(4L)$, hence

$$\lambda/(4L) < 3.06 \cdot 10^{16} \cdot \log^2(\lambda+1) \cdot \log\log(\lambda+1).$$

As $\lambda \geq 1664L^3$ we have $L < 12\lambda^{1/3}$, hence $\lambda/(4L)) > \lambda^{2/3}/48$. Further $\log(\lambda+1) < 5 \cdot (\log\lambda)/4$, and $\lambda \geq 1664$ yields $\log\log(\lambda+1) < 2(\log\lambda)^{1/3}$, hence

$$\lambda^{2/3} < 3.7 \cdot 10^{18} \cdot \log^{7/3}\lambda.$$

We rewrite it as

$$\lambda < 7.12 \cdot 10^{27} \cdot \log^{7/2}\lambda,$$

for which the application of Lemma 10 with the parameters $x = \lambda, s = 7/2, T = 7.12 \cdot 10^{27}$ yields

$$\lambda < 2^{7/2} \cdot 7.12 \cdot 10^{27} \log^{7/2}(7.12 \cdot 10^{27}) < 1.71 \cdot 10^{36}.$$

Some lines above we estimated n by λ, which actually yields

Table 3
Shanks sequences for $-3 \le n \le 4$.

(A,B,C)	-3	-2	-1	0	1	2	3	4
$(0,0,1)$	u^2+3u+5	$-u-2$	1	0	0	1	$u-1$	u^2-u+3
$(1,-1,1)$	$-u^2-4u-9$	$u+4$	-2	1	-1	1	-2	$-u+3$

$$n < 1.4 \cdot 10^{42}.$$

Finally we get

$$-m \le n - \tau \le 1.4 \cdot 10^{42} + 4.$$

Our Theorem 1 is completely proved. □

5. Preparation the proof of Theorem 2

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By Lemma 4 there are Shanks sequences, cases (iii), (v), (vi), (vii), for which we have hope to solve completely equation (1). Table 3 displays the values for $-3 \le n \le 4$ in cases of the triplets $(0,0,1), (1,-1,1)$.

To simplify the notation set $F_1(u,n) = F(0,0,1,u,n), F_2(u,n) = F(1,-1,1,u,n)$.

We are now in the position to settle the simple cases of Theorem 1.

Proposition 1. *Assume that* $0 \le n < m$ *or* $m < n \le -1$ *or*

$$(n,m) \in ([0,\infty) \times \{-1,-2\}) \cup ([0,n_{0i}] \times (-\infty,-1]),$$

where $n_{01} = \begin{cases} 4, & \text{if} \quad u = 0 \text{ or } u \ge 3 \\ 6, & \text{if} \quad u = 2 \\ 16, & \text{if} \quad u = 1 \end{cases}$ *and* $n_{02} = \begin{cases} 4, & \text{if} \quad u \ge 5 \\ 5, & \text{if} \quad u = 0,3,4 \\ 7, & \text{if} \quad u = 2 \\ 16, & \text{if} \quad u = 1. \end{cases}$

Then all solutions of the equations

$$|F_i(u,n)| = |F_i(u,m)|, i = 1,2, u > 0 \tag{13}$$

are

(i) $i = 1, u \in \mathbb{Z}_{\ge 0}, (n,m) = (0,1), F_1(u,n) = 0,$
(ii) $i = 1, u \in \mathbb{Z}_{\ge 0}, (n,m) = (2,-1), |F_1(u,n)| = 1,$
(iii) $i = 1, u = 0, (n,m) \in \{(3,-1),(3,2)\}, |F_1(u,n)| = 1,$
(iv) $i = 1, u = 1, (n,m) \in \{(5,-1),(5,2)\}, |F_1(u,n)| = 1,$
(v) $i = 1, u = 1, (n,m) = (4,-2), |F_1(u,n)| = 3,$
(vi) $i = 1, u = 1, (n,m) = (6,-3), |F_1(u,n)| = 9,$

(vii) $i = 1, u = 2, (n, m) \in \{(3, -1), (3, 2)\}, |F_1(u, n)| = 1,$
(viii) $i = 2, u \in \mathbb{Z}_{\geq 0}, (n, m) \in \{(0, 1), (0, 2), (1, 2)\}, |F_2(u, n)| = 1,$
(ix) $i = 2, u \in \mathbb{Z}_{\geq 0}, (n, m) = (3, -1), |F_2(u, n)| = 2,$
(x) $i = 2, u = 1, 4, (n, m) \in \{(4, -1), (4, 3)\}, |F_2(u, n)| = 2,$
(xi) $i = 2, u = 1, (n, m) = (5, -2), |F_2(u, n)| = 5,$
(xii) $i = 2, u = 2, (n, m) \in \{(4, 0), (4, 1), (4, 2)\}, |F_2(u, n)| = 1,$
(xiii) $i = 2, u = 2, (n, m) = (5, -2), |F_2(u, n)| = 6,$
(xiv) $i = 2, u = 5, (n, m) \in \{(4, -1), (4, 3)\}, |F_2(u, n)| = 2.$

Proof. We have $F_1(u, 2) = 1, F_1(u, 3) = u - 1, F_1(u, 4) = u^2 - u + 3$ by Table 3, hence $1 = F_1(u, 2) \leq F_1(u, 3) < F_1(u, 4)$ for all $u \geq 2$. Assume that $F_1(u, n-2) < F_1(u, n-1) < F_1(u, n)$ is true for some $n \geq 4$ and all $u \geq 2$. The recursive relation $F_1(u, n+1) = (u-1)F_1(u, n) + (u+2)F_1(u, n-1) + F_1(u, n-2)$ yields $F_1(u, n+1) > (u-1)F_1(u, n) + (u+2)F_1(u, n-1) > F_1(u, n)$, hence $(F_1(u, n))$ is strictly increasing whenever $u > 2$ and $n \geq 2$, thus $F_1(u, n) = F_1(u, m)$ cannot hold for $u > 2$ and $2 \leq n < m$ and for $u = 2$ and $3 \leq n < m$. If $u > 2$ and $0 \leq n \leq 2$, then

$$|F_1(u, n)| \leq 1 < 2 \leq u - 1 = F_1(u, 3) < F_1(u, m)$$

for any $m \geq 4$, and if $u = 2$ and $0 \leq n \leq 3$ then

$$|F_1(u, n)| \leq 1 < 5 = F_1(2, 4) < F_1(2, m)$$

for any $m \geq 5$, hence (13) has for i=1, $u \geq 2, n, m \geq 0$ only the solutions given in the Proposition.

The values of $F_1(u, n)$ and $F_2(u, n)$ for $u = 0, 1, 2, 3$ and $-4 \leq n \leq 11$, which we need in the sequel are given in the Tables 4 as well as 5.

By Table 4 we have $27 = F_1(1, 9) < F_1(1, 10) < F_1(1, 11)$, hence the same argument, which we used for $u \geq 2$, shows that $(F_1(1, n))_{n \geq 9}$ is strictly increasing too. Finally we claim that $F_1(0, 2n-1) < 0, F_1(0, 2n) > 0$ and $|F_1(0, 2n-1)| < F_1(0, 2n) < |F_1(0, 2n+1)|$ for all $n \geq 2$. The claim is true for $n = 2, 3$. Assume that $F_1(0, 2n-1) < 0, F_1(0, 2n) > 0, F_1(0, 2n+1) < 0$ and $-F_1(0, 2n+1) > F_1(0, 2n)$ hold for some $n \geq 2$. Then

$$\begin{aligned} F_1(0, 2n+2) &= -F_1(0, 2n+1) + 2F_1(0, 2n) + F_1(u, 2n-1) \\ &= -F_1(0, 2n+1) + F_1(0, 2n) + (F_1(0, 2n) + F_1(u, 2n-1)) \\ &> -F_1(0, 2n+1) \end{aligned}$$

because, by the induction hypothesis, all summands in the second row are positive. We should treat the case $F_1(0, 2n+3)$ too, but it can be done analogously, therefore we left it to the reader. With the same argument as was used for $u \geq 2$ one can prove that the Proposition holds for $i = 1, u = 0, 1$ and $n, m \geq 0$ too.

Table 4
Values of $F_1(u,n)$ for $u = 0,1,2,3$ and $-4 \le n \le 11$.

u/n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
0	-11	5	-2	1	0	0	1	-1	3	-4	9	-14	28	-47	89	-155
1	-26	9	-3	1	0	0	1	0	3	1	9	6	28	27	90	109
2	-55	15	-4	1	0	0	1	1	5	10	31	76	210	545	1461	3851
3	-104	23	-5	1	0	0	1	2	9	29	105	364	1282	4489	15752	55231

Table 5
Values of $F_2(u,n)$ for $u = 0,1,2,3$ and $4 \le n \le 11$.

u/n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
0	20	-9	4	-2	1	-1	1	-2	3	-6	10	-19	33	-61	108	-197
1	40	-14	5	-2	1	-1	1	-2	2	-5	4	-13	7	-35	8	-98
2	76	-21	6	-2	1	-1	1	-2	1	-6	-4	-27	-49	-161	-384	-1077
3	134	-30	7	-2	1	-1	1	-2	0	-9	-20	-85	-279	-1003	-3486	-12266

We have $F_1(u,-1)=1, F_1(u,-2)=-u-2, F_1(u,-3)=u^2+3u+5, F_1(u,-4)=-u^3-4u^2-10u-11$ by Table 3, hence $F_1(u,-(2n-1))>0, F_1(u,-2n)<0$ and $F_1(u,-(2n-1))<-F_1(u,-2n)$ hold for $n=1,2$. Using the recursive relation $F_1(u,-(n+3))=-(u+2)F_1(u,-(n+2))-(u-1)F_1(u,-(n+1))+F_1(u,-n)$ one can prove like some lines before that the members of the sequence $(F_1(u,-n))_{n\geq 1}$ have alternating signs moreover their absolute values are strictly monotone increasing.

One can prove similarly that the sequences $(|F_2(u,n)|)_{n\geq n_0}, (|F_2(u,-n)|)_{n\leq n_1}$ are strictly increasing for all $u\geq 0$, if $n_1\leq 0$, while $n_0=\begin{cases} 4, & \text{if} \quad u=0 \text{ or } u\geq 3\\ 6, & \text{if} \quad u=2 \\ 16, & \text{if} \quad u=1.\end{cases}$ Again with the same argument as was detailed for the case $i=1, u\geq 2, n,m\geq 0$ one can show that the statement is true whenever the signs of n and m are equal.

We finally deal with the case $n\geq 0, m<0$. We detail the proof only for $i=2$, because the other case can be handled similarly. Assume first that $-2\leq m\leq 0$. Because of $F_2(u,5)=-u^2+2u-6=-(u-1)^2-5$, we have

$$|F_2(u,5)|-F_2(u,-2)=u^2-2u+6-(u+4)=(u-3/2)^2-1/4,$$

hence the difference is zero for $u=1$, and positive if $u>1$. The same is true for $u=0$, because $F_2(0,5)=-6$ and $F_2(0,-2)=4$. As $(|F_2(u,n)|)_{n\geq 5}, u\neq 1$ is strictly increasing we obtain that for $i=2, -2\leq m<0\leq n, u\neq 1$ the equation (13) admits only the solutions given in the Proposition. We showed above that $(|F_2(1,n)|)_{n\geq 16}$ is strictly increasing, hence the former argument fits to the case $u=0, n\geq 16$ too. By Table 4 the inequality $|F_2(0,n)|>5=F_2(0,-2)$ holds already for $n\geq 7$, hence the assertion is true for $u=0$ too.

Secondly, assume that $u\geq 5$ and $0\leq n\leq 4$ and $m<0$. We have $F_2(u,4)=-u+3$, hence $|F_2(u,4)|=u-3\geq 2$ with equality only for $u=5$. Hence $|F_2(u,4)|\geq |F_2(u,n)|$ for all $n\leq 4$. On the other hand $F_2(u,-2)=u+4>|F_2(u,4)|$ for any $u\geq 0$ and $(|F_2(u,m)|)_{m\leq 0}$ is strict increasing, hence in the actual range the solutions of (13) satisfy $m=-1$.

For $u=0,3,4$ with $n\leq 5$, for $u=2$ with $n\leq 7$, and for $u=0$ with $n\leq 16$ repeating the former argumentation we get the same conclusion. $\square$

After Proposition 1 only the cases when n and m have opposite signs are left. To handle them successfully we have to prove for the growth of $|F_i(u,n)|$ more accurate estimations than (4) and (5).

We start with the Binet formulae for $F_i(u,n), i=1,2$. By Lemma 1, we have

$$F_i(u,n)=a_{i1}\lambda_1^n+a_{i2}\lambda_2^n+a_{i3}\lambda^n$$

with

$$a_{11}=-\frac{\Delta}{\lambda},\quad a_{12}=\frac{\Delta}{\lambda+1},\quad a_{13}=\frac{\Delta}{\lambda(\lambda+1)},$$

and

$$a_{21} = -\frac{\lambda+1}{\lambda}a_{11}, \quad a_{22} = \lambda a_{12}, \quad a_{23} = -\frac{1}{\lambda+1}a_{13},$$

where

$$\Delta = \frac{\lambda^2+\lambda+1}{u^2+u+7}.$$

If $u \geq 1$ then $u < \lambda < u + 2/u$ yields $0 < \Delta < 1$. Direct computation shows that the same inequality is true for $u = 0$ too.

The explicit values of a_{ij} make it possible to considerable improve the estimates (4) and (5). If $u > 1$ and $n \geq 2$ or $u = 1$ and $n \geq 6$ then we have

$$\left| |F_1(u,-n)| - \frac{\Delta}{\lambda}(\lambda+1)^{-n} \right| \leq \frac{\Delta}{\lambda+1}, \tag{14}$$

$$\left| F_1(u,n) - \frac{\Delta}{\lambda(\lambda+1)}\lambda^n \right| \leq \frac{\Delta}{\lambda}\left(\frac{\lambda+1}{\lambda}\right)^n, \tag{15}$$

$$\left| |F_2(u,-n)| - \frac{\Delta(\lambda+1)}{\lambda^2}(\lambda+1)^{-n} \right| \leq \Delta, \tag{16}$$

$$\left| |F_2(u,n)| - \frac{\Delta}{\lambda(\lambda+1)^2}\lambda^n \right| \leq \Delta\left(\frac{\lambda+1}{\lambda}\right)^n. \tag{17}$$

Notice that if $u = 0$ then $\lambda + 1 > \lambda > \lambda/(1+\lambda)$, i.e. (14) and (16) are true for $u = 0$ too. After this remark we prove only (14). Plainly

$$\begin{aligned} |F_1(u,-n) - a_{11}(-(\lambda+1))^{-n}| &\leq a_{12}\left(\frac{\lambda+1}{\lambda}\right)^n + a_{13}\lambda^{-n} \\ &= \frac{\Delta}{\lambda}\left(\left(\frac{\lambda+1}{\lambda}\right)^{n-1} + \frac{1}{\lambda^n(\lambda+1)^2}\right) \\ &< \frac{\Delta}{\lambda}\left(\frac{\lambda+1}{\lambda}\right)^n\left(\frac{\lambda}{\lambda+1} + \frac{1}{\lambda}\right). \end{aligned}$$

From here the proof of (14) is the same as of (5). The proof of the other inequalities are similar.

In the proof of Theorem 2 we need the following technical lemma.

Lemma 14. *Set*

$$g(n) = \frac{\lambda(\lambda+1)^2}{\lambda^n}\left(\left(1+\frac{1}{\lambda}\right)^n + 1\right).$$

Then

Table 6
Values of $n_1(u)$.

u	1	2	3	4	5	6	7	8	≥ 9
$n_1(u)$	30	13	11	10	9	9	9	9	8

$$g(n) < 2\lambda(\lambda+1)^2\left(\frac{\lambda+1}{\lambda^2}\right)^n, \tag{18}$$

and $g(n) \leq 1/(2\lambda^4)$ *for* $n \geq n_1(u)$ *given in Table 6. Finally, if* $n+1 > \lambda\log\lambda$ *then*

$$g(n) < \frac{(\lambda+1)^5}{\lambda^{n+1}}e^{(n+1)/\lambda}. \tag{19}$$

Proof. From the definition of $g(n)$ we have

$$\frac{g(n)}{g(n-1)} = \frac{1}{\lambda}\left(\left(\left(\frac{\lambda+1}{\lambda}\right)^n + 1\right) \Big/ \left(\left(\frac{\lambda+1}{\lambda}\right)^{n-1} + 1\right)\right).$$

The second factor is obviously larger than one and less than $(\lambda+1)/\lambda$, hence

$$\frac{1}{\lambda} < \frac{g(n)}{g(n-1)} < \frac{\lambda+1}{\lambda^2}.$$

Plainly $g(0) = 2\lambda(\lambda+1)^2$, which implies (18) immediately.

Set $f(n,x) = (x+1)^{n+2} + (x+1)^2x^n - x^{2n-5}/2$. We have $g(n) < 1/(2\lambda^4)$ if and only if $f(n,\lambda) < 0$. If $x > 1$ is fixed then $f(n,x)$ tends to $-\infty$, whenever $n \to \infty$. Hence for any $u \geq 1$ there is a $n_1 = n_1(u)$ such that $f(n,\lambda(u)) < 0$ for all $n \geq n_1$. As $f(8,x) = (x+1)^{10} + (x+1)^2x^8 - x^{11}/2$ is negative for $x \geq 8.55$ we have $f(n,x) < 0$ for $n \geq 9$ and $x \geq 8,55$. On the other hand $f(7,x) = (x+1)^9 + (x+1)^2x^7 - x^9/2 < 0$ holds whenever $x > 0$. Noticing that $\lambda(u) \geq \lambda(9) > 8.55$ for $u \geq 9$ we obtain $n_1(u) = 8$ for $u \geq 9$. A simple computation yields $n_1(u)$ for $1 \leq u \leq 8$. We present them in Table 6.

Finally we treat the case $n+1 > \lambda\log\lambda$. Our first observation is that

$$\begin{aligned} g(n) &< \lambda g(n+1) \\ &= \frac{(\lambda(\lambda+1))^2}{\lambda^n}\left(\left(\left(1+\frac{1}{\lambda}\right)^\lambda\right)^{(n+1)/\lambda} + 1\right) \\ &< \frac{(\lambda(\lambda+1))^2}{\lambda^n}\left(e^{(n+1)/\lambda} + 1\right) \end{aligned}$$

because $(1+1/x)^x < e$ for $x > 1$ and $\lambda > 1$.

We have plainly $e^{1/\lambda} > 1 + \frac{1}{\lambda}$ and the assumption $n+1 > \lambda\log\lambda$ yields $e^{(n+1)/\lambda} > \lambda$, which imply

$$e^{(n+1)/\lambda}(e^{1/\lambda}-1) > \lambda\frac{1}{\lambda} = 1.$$

We can rewrite this as $e^{(n+1)/\lambda}+1 < e^{(n+2)/\lambda}$, which together with the above inequality implies

$$g(n) < \frac{(\lambda(\lambda+1))^2}{\lambda^n}e^{(n+2)/\lambda} < \frac{(\lambda+1)^5}{\lambda^{n+1}}e^{(n+1)/\lambda}. \quad \square$$

6. Proof of Theorem 2, I, $u > 0$

Assume that $(u,n,m) \in \mathbb{Z}^3$ is a solution of (1) with F_1 or F_2 instead of F. We showed in the Introduction that it is enough to solve (1) for $u \geq 0$. Proposition 1 describes all solutions such that n and m have the same signs. Hence we only have to concentrate to the case when n and m have different signs. Because of symmetry reason we may assume $n \geq 0$ and $m < 0$. In Section 2 we pointed out that λ dominates among the zeroes of $S_u(X)$ for $u > 0$, but for $u = 0$ this role plays λ_2. Interestingly, the dominating zero of $X^3S_u(1/X)$ is always λ_1. The dominating zeroes of $S_u(X)$ and $X^3S_u(1/X)$ play central role in the further argumentation, hence we cannot avoid the distinction between the cases $u = 0$ and $u > 0$. We settle here $u > 0$ and postpone $u = 0$ to the next section.

In Proposition 1 we presented all solutions of (1) with $F = F_1$ and $F = F_2$ such that $n \leq n_{0,i}, i = 1,2$, or $m \geq -2$, thus we may assume in the sequel $n > n_{0,i}, i = 1,2$, and $m < -2$. Then the inequalities (14)-(17) hold, hence, similarly to (4), (5) in the proof of Theorem 1, enable us to derive strong upper bound for a linear form in logarithms of algebraic numbers. The essential difference is that there we had to do with a linear form in three variables, while now we will have only two. This seems to be is a small difference, but, together with a much more precise lower estimation for $\max\{n,|m|\}$, makes it possible to derive a bound for u and for $\max\{n,|m|\}$, which is small enough to numerical computation.

The inequalities (14), (15) imply

$$\frac{\Delta}{\lambda(\lambda+1)}\lambda^n - \frac{\Delta}{\lambda}\left(\frac{\lambda+1}{\lambda}\right)^n < F_1(u,n) = |F_1(u,m)| < \frac{\Delta}{\lambda}(\lambda+1)^{-m} + \frac{\Delta}{\lambda+1}$$

and

$$\frac{\Delta}{\lambda}(\lambda+1)^{-m} - \frac{\Delta}{\lambda+1} < |F_1(u,m)| = F_1(u,n) < \frac{\Delta}{\lambda(\lambda+1)}\lambda^n + \frac{\Delta}{\lambda}\left(\frac{\lambda+1}{\lambda}\right)^n.$$

Hence

$$\left|\frac{\Delta}{\lambda(\lambda+1)}\lambda^n - \frac{\Delta}{\lambda}(\lambda+1)^{-m}\right| < \frac{\Delta}{\lambda}\left(\frac{\lambda+1}{\lambda}\right)^n + \frac{\Delta}{\lambda+1}.$$

Dividing by the term including λ^n we get

$$\left|(\lambda+1)\frac{(\lambda+1)^{-m}}{\lambda^n}-1\right| = \left|\lambda^{-n-m+1}\left(1+\frac{1}{\lambda}\right)^{-m+1}-1\right| < \frac{(\lambda+1)^{n+1}}{\lambda^{2n}}+\frac{1}{\lambda^{n-1}}.$$

For F_2 the inequalities (16), (17) yield on the same way

$$\left|\lambda^{-n-m+2}\left(1+\frac{1}{\lambda}\right)^{-m+3}-1\right| < \frac{\lambda(\lambda+1)^2}{\lambda^n}\left(\left(1+\frac{1}{\lambda}\right)^n+1\right).$$

Observe that the right hand side is exactly $g(n)$.

The expressions in the absolute values are the same function considered at different arguments, moreover the second upper bound is obviously larger, hence from here until the last few steps of the proof the investigation of (1) can be joined, more exactly we continue with the inequality

$$\left|\lambda^{-E}\left(1+\frac{1}{\lambda}\right)^{C}-1\right| < g(n), \tag{20}$$

where

$$C = \begin{cases} -m+1 & \text{if} \quad i=1 \\ -m+3 & \text{if} \quad i=2 \end{cases}, \quad E = \begin{cases} n+m-1 & \text{if} \quad i=1 \\ n+m-2 & \text{if} \quad i=2 \end{cases}.$$

We assume $n \geq n_1$ and to the cases $n_{0,i}(u) < n < n_1(u), i = 1, 2$ we come back later. Set $\Gamma = \lambda^{-E}\left(1+\frac{1}{\lambda}\right)^C - 1$. Then $|\Gamma| < 1/(2\lambda^4)$ by the second assertion of Lemma 14. Plainly $C \geq 2$, hence

$$\left(\frac{\lambda+1}{\lambda}\right)^C = \left(1+\frac{1}{\lambda}\right)^C > 1+\frac{2}{\lambda}.$$

Thus, if $E \leq 0$ then the left hand side of (20) is at least $\frac{-m+1}{\lambda} \geq \frac{2}{\lambda}$, which contradicts that the right hand side of it is less than $1/(2\lambda^4)$. Hence $E > 0$.

The actual Γ is equal to $\Gamma - 1$ with the Γ from Section 4.3 and with the choice $\Phi = 1$. There we proved that $\Gamma \neq 1$, but only if u is large enough. In this special case we prove this for all u. We claim that $\Gamma \neq 0$. Otherwise we would have

$$(\lambda+1)^C = \lambda^{C+E}.$$

Applying to this equation the automorphism σ^2 and taking into account that $\sigma^2\lambda = \lambda_2 = -1 - 1/\lambda$ we get

$$(-1/\lambda)^C = (\lambda_2)^{C+E},$$

but this is impossible because $C, C+E > 0$, while $|\lambda_2| > 1$ and $|-1/\lambda| < 1$.

Recall that $|\Gamma| < g(n) < 1/(2\lambda^4) < 1/2$, hence we may apply Lemma 11 with $x = \Gamma, a = 1/2$ and obtain

$$|\Lambda| = \left| E \log \lambda - C \log \frac{\lambda + 1}{\lambda} \right| < 2g(n) \tag{21}$$

for all $u \geq 1$. The factor 2 is correct because $\frac{-\log(1-1/2)}{1/2} = 2\log 2 < 2$.

We now recapitulate tacitly the argumentation of Thomas [27]. Our notations coincide with his.

We wish to find the least value of C and E (both positive integers) that satisfy (21).- wrote Thomas. He continuous *Clearly, we minimize C by taking $E = 1$. In this case $C \geq C_0$, where C_0 is the largest integer such that*

$$C_0 \log(1 + 1/\lambda) < \log \lambda.$$

(We use here the fact that $1/\lambda > \log(1+1/\lambda) > 1/\lambda - 1/2\lambda^2 > 1/q^3\lambda^4$.) Hence $C_0 + 1 \geq C_0'$, where C_0' is the least integer such that $C_0/\lambda \geq \log \lambda$. Thus

$$C + 1 \geq C_0 + 1 \geq C_0 \geq \lambda \log \lambda.$$

Although our upper bound in (21) is $1/\lambda^4$ is worse than Thomas' bound $1/(q^3\lambda^4)$, where q denotes some positive integer, it is still enough for the same conclusion, i.e. we have $\lambda \log \lambda \leq \begin{cases} -m+1, & \text{if} \quad i = 1 \\ -m+3 & \text{if} \quad i = 2 \end{cases}$. As $1 \leq E = \begin{cases} n+m-1, & \text{if} \quad i = 1 \\ n+m-2 & \text{if} \quad i = 2 \end{cases}$ we obtain in both cases

$$\lambda \log \lambda \leq C + 1 \leq C + E \leq n + 1 \tag{22}$$

for all $u \geq 1, n \geq n_1(u)$.

We may assume in the sequel $\Lambda \neq 0$. Indeed, otherwise $\Gamma = 0$ would hold, which impossibility we proved above.

To prove an upper bound for u we wish to use Lemma 8. First we identify the parameters. The numbers $\eta_1 = \frac{\lambda+1}{\lambda}, \eta_2 = \lambda$ are algebraic integers, while the numbers $e_2 = E, e_1 = C$ are positive integers. With these choice $\mathbb{K} = \mathbb{Q}(\lambda)$, thus $d_{\mathbb{K}} = 3$. We have

$$h\left(\frac{\lambda+1}{\lambda}\right) = h(\lambda) = (\log|\lambda_2(u)\lambda|)/3 = |\log|\lambda_1(u)||/3 = (\log(\lambda+1))/3,$$

hence we may choose $A_i, i = 1, 2$ such that $\log A_i = (\log(\lambda+1))/3$. This yields

$$\mathcal{E} = \frac{C}{\log(\lambda+1)} + \frac{E}{\log(\lambda+1)} = \frac{C+E}{\log(\lambda+1)} \leq \frac{n+1}{\log(\lambda+1)}.$$

We claim that if $u \geq 10^3$, then

$$\max\left\{\log\mathcal{E}+0.14, \frac{21}{d_{\mathbb{K}}}, \frac{1}{2}\right\} = \log\mathcal{E}+0.14.$$

Indeed, we know that if $u \geq 1$ and $n \geq n_1(u)$ then $C+E \geq \lambda\log\lambda$. Thus, if $u \geq 10^3$, then $\mathcal{E} > 0.9998\cdot\lambda > 0.9998\cdot u > 999.8$ and $\log\mathcal{E} > 6.9$, hence

$$\max\left\{\log\mathcal{E}+0.14, \frac{21}{d_{\mathbb{K}}}, \frac{1}{2}\right\} \geq \max\{6.9+0.14, 7\} = 7.04,$$

and the claim is proved.

If $u \leq 10^3$ then the maximum can be 7 too, but in that case $\log\mathcal{E} \leq 7-0.14 = 6.86$, hence

$$n-2 \leq C+E = \mathcal{E}\cdot\log(\lambda+1) \leq 6587,$$

hence $n \leq 6589$.

Next we investigate the case when the maximum is $\log\mathcal{E}+0.14$ and $u \geq 1$. For the above parameters Lemma 8 yields

$$\begin{aligned}\log|\Lambda| &\geq -24.34\cdot 3^4\cdot(\log(n+1)-\log\log(\lambda+1)+0.14)^2\left(\frac{\log(\lambda+1)}{3}\right)^2\\ &= -219.1\cdot(\log(n+1)-\log\log(\lambda+1)+0.14)^2(\log(\lambda+1))^2.\end{aligned}$$

On the other hand by (21) and (19) we have

$$\log|\Lambda| < 5\log(\lambda+1)-(n+1)\log\lambda+\frac{n+1}{\lambda}.$$

The combination of the lower and upper bounds for $\log|\Lambda|$ yields

$$(c_1+c_4)(n+1) < c_2(\log(n+1)-c_3)^2+c_5 \tag{23}$$

with

$$\begin{aligned}c_1 &= \left(1-\frac{1}{\lambda}\right)\log\lambda,\\ c_2 &= 219.1\cdot(\log(\lambda+1))^2,\\ c_3 &= -\log\log(\lambda+1)+0.14,\\ c_4 &= \frac{1}{\log\lambda}\log\frac{\lambda}{e},\\ c_5 &= 5\log(\lambda+1).\end{aligned}$$

For fixed u inequality (23) yields an upper bound for n. To simplify the computation observe that if the left hand side is minored or the right hand side is majored then the solutions of the new inequality are solutions of (23). With this in mind we erase $c_4 > 0$ and divide by $c_1(\log(n+1) - c_3)^2$ and obtain

$$\frac{n+1}{(\log(n+1)-c_3)^2} < \frac{c_2}{c_1} + \frac{c_5}{c_1(\log(n+1)-c_3)^2}.$$

The division is allowed because $c_1 > 1$ for $u \geq 4$ and

$$\log(n+1) - c_3 \geq \log \mathcal{E} + \log\log(\lambda+1) - \log\log(\lambda+1) + 0.14 \geq 7.04.$$

Hence

$$\frac{c_5}{(\log(n+1)-c_3)^2} < \frac{5\log(\lambda+1)}{7.04^2} < 0.9 \cdot (\log(\lambda+1))^2.$$

This implies that if n is a solution of

$$\frac{n+1}{\log^2(n+1)} < \frac{c_2'}{c_1},$$

where $c_2' = c_2 + 0.9 \cdot (\log(\lambda+1))^2 = 220 \cdot (\log(\lambda+1))^2$ then it satisfies (23) too. A simple computation verifies that $\frac{c_2'}{c_1} > 256$ for $u \geq 4$, thus Lemma 10 is applicable with $s = 2$ and $T = \frac{c_2}{c_1'}$. Hence we get

$$n + 1 < N(u) = \frac{4c_2'}{c_1} \log^2 \frac{c_2'}{c_1}. \tag{24}$$

Comparing $N(u)$ with (22) we obtain

$$\lambda \log \lambda \leq n + 1 < \frac{4c_2'}{c_1} \log^2 \frac{c_2'}{c_1},$$

which implies $\lambda \leq 53278.19397$, i.e. $u \leq 53278$.[3] The function $N(u)$ is monotone increasing and $N(53278) = 579841.3877$, and, as n is an integer, we get $n \leq 579840$ for all solutions $n \geq 0, m < 0, u \geq 4$ of the equation $F(u,n) = F(u,m)$, provided $\log \mathcal{E} > 6.86$.

As 579840 is much larger than 6589, which we obtained as upper bound in the opposite case, $n \leq 579840$ holds for all solutions $n \geq 0, m < 0, u \geq 4$ of the equation $F(u,n) = F(u,m)$.

Finally, if $1 \leq u \leq 3$ direct computation leads to

[3] Notice that this bound is much better than $1.365 \cdot 10^7$ given by Thomas [27] and even than $3 \cdot 10^6$ given by Mignotte [12]. The reason lies in the application of the much sharper lower bound for linear forms in two algebraic numbers due to Laurent, Mignotte and Nesterenko [7], which was not available for Thomas and Mignotte.

u	1	2	3
$N(u)$	121510	30131	25107

Thus $n \leq 579840$ for all $1 \leq u \leq 63103$.

Using (21) and (19) it is easy to see that

$$\left| E \frac{\log \lambda}{\log(1+1/\lambda)} - C \right| < e^{-u},$$

which is the inequality in line 2 of page 176 of Mignotte [12]. He proved that his inequality has no solutions in positive integers C, E for $u > 10^3$. We derived this inequality under the assumptions $m < -2$ and $n > n_{0i}, i = 1, 2$. This implies $C > 0$, hence $E = 0$ by Mignotte's result, which is absurd. Thus our equations have in the actual range no solutions.

Mignotte [12] dealt only with $n > 10^3$ because Thomas [27] solved the Thue equation $y^3 S_u(x/y) = \pm 1$ in the range $[0, 1000]$ with a different argument. Unfortunately his method is not applicable in our situation, but a numerical reduction procedure, similar to that applied by Mignotte allow us to solve inequality (21) in the remaining range.

Before describing that we prove an analogous inequality to (21) for $u = 0$, which makes it possible to compute an upper bound for n and then reduce it with the same method as in the case $u \geq 1$.

7. Proof of Theorem 2, II, $u = 0$

In this section we are dealing exclusively with $u = 0$, therefore and to simplify the notation we omit the argument (0), e.g. $F_i(n)$ means $F_i(0, n)$. This case cannot be handled with the others because not λ_3, but λ_2 is the - in absolute value - dominating root. This does not affect the Binet formula and its weights, hence the method of the previous section works, with natural modifications, well. We pointed out earlier that (14) and (16) are true for $u = 0$ too. Unfortunately this is not true for (15) and (17), but a simple computation shows that we can replace them by

$$a_{2i}(|\lambda_2|^n - \lambda^n) < |F_i(n)| < a_{2i}(|\lambda_2|^n + \lambda^n),\ i = 1, 2, \tag{25}$$

which is true for $n \geq 2$.

We distinguish again three cases, $n, m \leq 0, n, m \geq 0$ and $n \geq 0, m \leq 0$. The first two were treated in Proposition 1 we have to deal only with the last one.

The inequalities (14) and (16) together with (25) imply

$$|a_{2i}|\lambda_2|^n - |a_{1i}||\lambda_1|^m| < a_{2i}(\lambda^n + 2),\ i = 1, 2.$$

Dividing by $a_{2i}|\lambda_2|^n$ and taking into account the expressions of λ_1, λ_2 and a_{1i}/a_{2i} with λ we obtain

$$\left|\left(\frac{\lambda+1}{\lambda}\right)^i(\lambda+1)^{-m}\left(\frac{\lambda+1}{\lambda}\right)^{-n}-1\right|<\left(\frac{\lambda}{|\lambda_2|}\right)^n+\frac{2}{|\lambda_2|^n}<2\left(\frac{\lambda}{|\lambda_2|}\right)^n,\ i=1,2.$$

The last inequality is true only if $n \geq 4$. Under the same assumption the right hand side is less than 1/2, thus

$$\left|(n+m-i)\log\left(\frac{\lambda+1}{\lambda}\right)+m\log\lambda\right|<4\left(\frac{\lambda}{|\lambda_2|}\right)^n,\ i=1,2. \tag{26}$$

The arguments of the logarithms of the linear form on the left hand side are the same as of Λ, thus we can apply Lemma 8 with the parameters $\eta_1, \eta_2, \mathbb{K}, d_{\mathbb{K}}$ like during its earlier application, while with $e_1 = n+m-i, e_2 = -m$, which are both positive. Thus $\log A_j = (\log(\lambda+1))/3,\ j=1,2$ and

$$E=\frac{n-i}{\log(\lambda+1)}\geq\frac{n-2}{\log(\lambda+1)}.$$

If $n-2>889$ then $\log E > 7$ and we obtain

$$-219.1\cdot(\log(\lambda+1))^2(\log(n-2)-\log\log(\lambda+1))^2<\log 4-n(\log|\lambda_2|-\log\lambda).$$

Inserting here the actual values $\lambda_2 = -1.801937736, \lambda = 1.246979604$ we obtain $n \leq 46926$. The reduction of this bound we postpone the next section.

8. Proof of Theorem 2, III, reduction and finalization

The inequalities (21), (26) have the common shape

$$\left|q\log\left(\frac{\lambda(u)+1}{\lambda(u)}\right)-p\log\lambda(u)\right|<g(n,u)$$

with positive integers p, q, and with an, for any fixed u, exponentially decreasing function $g(n,u)$.

Our goal is to find all solution $(p,q) \in \mathbb{Z}^2$ of it with $0<q<N(u)$ for all $0 \leq u \leq 1000$. The procedure below is essentially the same as of de Weger [30], Chapter 3.2. Setting $\alpha = \alpha(u) = \log(1+1/\lambda(u))/\log\lambda(u)$ and dividing by q we obtain the inequality

$$(\log\lambda(u))\left|\alpha(u)-\frac{p}{q}\right|<\frac{g(n,u)}{q}\leq g(n,u).$$

We showed in the last two sections that $M = 579840$ is an upper bound for q for all $0 \leq u \leq 1000$. Perform for all fixed $0 \leq u \leq 1000$ the following procedure

(1) Compute $\alpha(u)$ with at least six decimal digits precision.

(2) Compute the continued fraction expansion of $\alpha(u)$ and its convergents $\frac{p_i}{q_i}, i = 0, \dots, k$ such that $q_{k-1} < M \le q_k$.
(3) From the inequality

$$(\log \lambda(u)) \left| \alpha(u) - \frac{p_k}{q_k} \right| < g(n, u)$$

compute a new upper bound for n. The explanation of this step see below.
(4) Iterate the Steps (2)-(3) until the bound for n stabilizes.

The denominators of the convergents of $\alpha(u)$ grow exponentially, thus we will find k with $q_{k-1} < M \le q_k$ quickly except when $\alpha(u)$ is rational, but this did not happen in our computation. If $p/q \ne p_k/q_k$ then we get

$$(\log \lambda(u)) \left| \alpha(u) - \frac{p_k}{q_k} \right| < (\log \lambda(u)) \left| \alpha(u) - \frac{p}{q} \right| < g(n, u)$$

by Lemma 9. Otherwise $p/q = p_k/q_k$ and we have equality instead of the first inequality, thus the inequality of Step (3) always holds. Notice that a concrete real number stays on the left hand side, thus the inequality yields indeed an upper bound for n.

Let us provide some details of the computation. We start with the special case $u = 0$, here we obtained a sharper bound, hence we take $M = 46925$. The first appropriate convergent has $p_k = 744489$ and $q_k = 279058$. The new bound for n is 79. In the second round we get the convergent $p_k/q_k = 739/277$ and an improved bound for n this time is 41. It turns out that a new reduction step does not improve the bound, therefore we stop and enumerate the small solutions both for $i = 1, 2$. The computation time was less than 30 seconds.

If $1 \le u \le 1000$, then the bound is given by $M = 579840$. We implemented the reduction procedure in the computer algebra package SageMath [20], the parallel computation was completed in 6 minutes using 8 cores of a Core i7 notebook. As an example consider the case with $u = 47$. The first step reduced the bound to 10 using the convergent

$$\frac{p_k}{q_k} = \frac{35608}{6519105}.$$

The second round yields the new bound 5, the corresponding convergent is

$$\frac{p_k}{q_k} = \frac{1}{183}.$$

The last step did not improve the bound.

Remark 1. We note that the ternary recurrence sequence $F_1(0, n)$ is in The On-Line Encyclopedia of Integer Sequences, its entry is A006053 [16]. If we consider the sequence of the positive integer elements of $F_1(0, n)$, then we get the entry A094790, and the negative elements (in absolute value) give the entry A094789. All these sequences have some combinatorial interpretation in terms of counting pathes in certain graphs.

9. The equation $F(u,m) = F(v,n)$

Previously we considered equations of the form $F(u,n) = F(u,m)$, that is the coefficients of the recurrence relation is fixed. Now we fix m and n and we determine the possible pairs (u,v) such that $F(u,m) = F(v,n)$.

Theorem 3. *Let* $(m,n) \in \{(4,5),(4,6),(4,7),(4,8),(5,8),(6,8)\}$*. The equation*

$$F(u,m) = F(v,n)$$

has exactly the following solutions

(m,n)	$(u,v) \in$
$(4,5)$	$\emptyset$
$(4,6)$	$\{(-153,-12),(-8,-2),(-2,0),(-2,1),(3,0),(3,1),(9,-2),(154,-12)\}$
$(4,7)$	$\emptyset$
$(4,8)$	$\emptyset$
$(5,8)$	$\emptyset$
$(6,8)$	$\emptyset$

Proof. Let $(m,n) = (4,5)$. We have the equation

$$u^2 - u + 3 = v^3 - v^2 + 5v - 4.$$

It can be transformed as follows

$$(8u-4)^2 = (4v)^3 - 4(4v)^2 + 80(4v) - 432.$$

That is we get an elliptic curve. To determine all integral points on a given elliptic curve one can follow a method developed by Gebel, Pethő and Zimmer [4] and independently by Stroeker and Tzanakis [25]. Implementations can be found in the computer algebra packages Magma [2] and SageMath [20]. It turns out that there is no integral solution in case of the above curve.

Let $(m,n) = (4,6)$. The equation is as follows

$$u^2 - u + 3 = v^4 - v^3 + 7v^2 - 7v + 9.$$

This time we obtain an elliptic curve given by the quartic model

$$(4u-2)^2 = (2v)^4 - 2(2v)^3 + 28(2v)^2 - 56(2v) + 100.$$

Tzanakis [28] provided a method to determine all integral points on quartic models, the algorithm is implemented in Magma as `IntegralQuarticPoints`. We get the integral solutions

$$(u,v) \in \{(-153,-12),(-8,-2),(-2,0),(-2,1),(3,0),(3,1),(9,-2),(154,-12)\}.$$

Let $(m,n)=(4,7)$. The corresponding equation is a genus 2 curve given by

$$C: \quad (2u-1)^2 = 4v^5 - 4v^4 + 36v^3 - 40v^2 + 84v - 67.$$

The rank of the Mordell-Weil group is 0 and we get that $C(\mathbb{Q}) = \{\infty\}$. The Magma procedures used to compute these data are based on Stoll's papers [22], [23], [24].

Let $(m,n)=(4,8)$. The equation can be written as

$$(2u-1)^2 = 4v^6 - 4v^5 + 44v^4 - 52v^3 + 148v^2 - 140v + 101 := g(v).$$

Here we apply Runge's method [17] to determine all the integral solutions. Consider the polynomials

$$\begin{aligned} h_1(v) &= 16v^3 - 8v^2 + 86v - 60, \\ h_2(v) &= 16v^3 - 8v^2 + 86v - 62. \end{aligned}$$

It follows that

$$\begin{aligned} 64g(v) - h_1(v)^2 &= -32v^3 + 1116v^2 + 1360v + 2864, \\ 64g(v) - h_2(v)^2 &= 32v^3 + 1084v^2 + 1704v + 2620. \end{aligned}$$

Therefore we get that

$$\begin{aligned} h_1(v)^2 < 64g(v) = (16u-8)^2 < h_2(v)^2 &\text{ if } v < -33, \\ h_2(v)^2 < 64g(v) = (16u-8)^2 < h_1(v)^2 &\text{ if } v > 37. \end{aligned}$$

We do not obtain any integral solutions with $v \in [-33\ldots37]$. If $v \notin [-33\ldots37]$, then it follows that $16u-8 = \pm(16v^3 - 8v^2 + 86v - 61)$, a contradiction modulo 2.

Now we deal with the case $(m,n)=(5,8)$. The equation is as follows

$$u^3 - u^2 + 5u - 4 = v^6 - v^5 + 11v^4 - 13v^3 + 37v^2 - 35v + 28 := F(v,8).$$

This is also a Runge type equation and one may apply the approach described in [26]. We approximate the polynomial $u^3 - u^2 + 5u - 4$ by cubes as

$$\begin{aligned} (u^3 - u^2 + 5u - 4) - u^3 &= -u^2 + 5u - 4, \\ (u^3 - u^2 + 5u - 4) - \left(u - \frac{2}{3}\right)^3 &= u^2 + \frac{11}{3}u - \frac{100}{27}. \end{aligned}$$

If $u \notin [-5 \ldots 4]$, then

$$\left(u-\frac{2}{3}\right)^3 < u^3-u^2+5u-4 = F(u,5) < u^3.$$

The same idea can be used in case of the degree 6 polynomial in v, here we have

$$F(v,8)-\left(v^2-\frac{1}{3}v+\frac{31}{9}\right)^3 = \frac{1}{3}v^4-\frac{164}{27}v^3+\frac{7}{27}v^2-\frac{1874}{81}v-\frac{9379}{729},$$
$$F(v,8)-\left(v^2-\frac{1}{3}v+\frac{33}{9}\right)^3 = -\frac{1}{3}v^4-\frac{152}{27}v^3-\frac{41}{9}v^2-\frac{194}{9}v-\frac{575}{27}.$$

If $v \notin [-17 \ldots 19]$, then

$$\left(v^2-\frac{1}{3}v+\frac{31}{9}\right)^3 < F(v,8) < \left(v^2-\frac{1}{3}v+\frac{33}{9}\right)^3.$$

We obtain the system of inequalities

$$\left(u-\frac{2}{3}\right)^3-\left(v^2-\frac{1}{3}v+\frac{33}{9}\right)^3 < F(u,5)-F(v,8)=0 < u^3-\left(v^2-\frac{1}{3}v+\frac{31}{9}\right)^3.$$

Thus

$$\frac{31}{3} < 3u-3v^2+v < 13.$$

To determine all integral solution of the equation $F(u,5)=F(v,8)$ it remains to compute the solutions with $u \in [-5,4], v \in [-17,19]$ and $3u-3v^2+v \in \{11,12\}$. We do not obtain any integral solution.

The equation in case of $(m,n)=(6,8)$ is as follows

$$u^4-u^3+7u^2-7u+9 = v^6-v^5+11v^4-13v^3+37v^2-35v+28.$$

If $u \notin [-13 \ldots 16]$, then

$$\left(u^2-\frac{1}{2}u+\frac{13}{4}\right)^2 < F(u,6) < \left(u^2-\frac{1}{2}u+\frac{7}{2}\right)^2.$$

If $v < -33$, then we have

$$\left(v^3-\frac{1}{2}v^2+\frac{43}{8}v-\frac{15}{4}\right)^2 < F(v,8) < \left(v^3-\frac{1}{2}v^2+\frac{43}{8}v-\frac{31}{8}\right)^2.$$

If $v > 37$, then we get

$$\left(v^3 - \frac{1}{2}v^2 + \frac{43}{8}v - \frac{31}{8}\right)^2 < F(v,8) < \left(v^3 - \frac{1}{2}v^2 + \frac{43}{8}v - \frac{15}{4}\right)^2.$$

Using the fact that $F(u,6) - F(v,8) = 0$ and the well-known identity $a^2 - b^2 = (a - b)(a + b)$ we need to handle the inequalities

$$\begin{aligned} \frac{1}{4} &< \ \ v^3 + u^2 - \tfrac{1}{2}v^2 - \tfrac{1}{2}u + \tfrac{43}{8}v \ < \frac{5}{8}, \\ \frac{3}{8} &< \ \ v^3 + u^2 - \tfrac{1}{2}v^2 - \tfrac{1}{2}u + \tfrac{43}{8}v \ < \frac{1}{2}, \\ -\frac{29}{4} &< -v^3 + u^2 + \tfrac{1}{2}v^2 - \tfrac{1}{2}u - \tfrac{43}{8}v < -\frac{57}{8}, \\ -\frac{59}{8} &< -v^3 + u^2 + \tfrac{1}{2}v^2 - \tfrac{1}{2}u - \tfrac{43}{8}v < -7. \end{aligned}$$

The first one implies that $8v^3 + 8u^2 - 4v^2 - 4u + 43v \in \{3, 4\}$. To determine the possible integral solutions we compute the resultant of the polynomial $F(u,6) - F(v,8)$ and $8v^3 + 8u^2 - 4v^2 - 4u + 43v - 3$ with respect to u. There is no integral solution. Similarly in case of the polynomial $8v^3 + 8u^2 - 4v^2 - 4u + 39$. The remaining inequalities can be solved in a similar way. Finally we check the cases with $-13 \le u \le 16$ and those with $-33 \le v \le 37$. We do not get any integral solutions. □

Data availability

No data was used for the research described in the article.

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