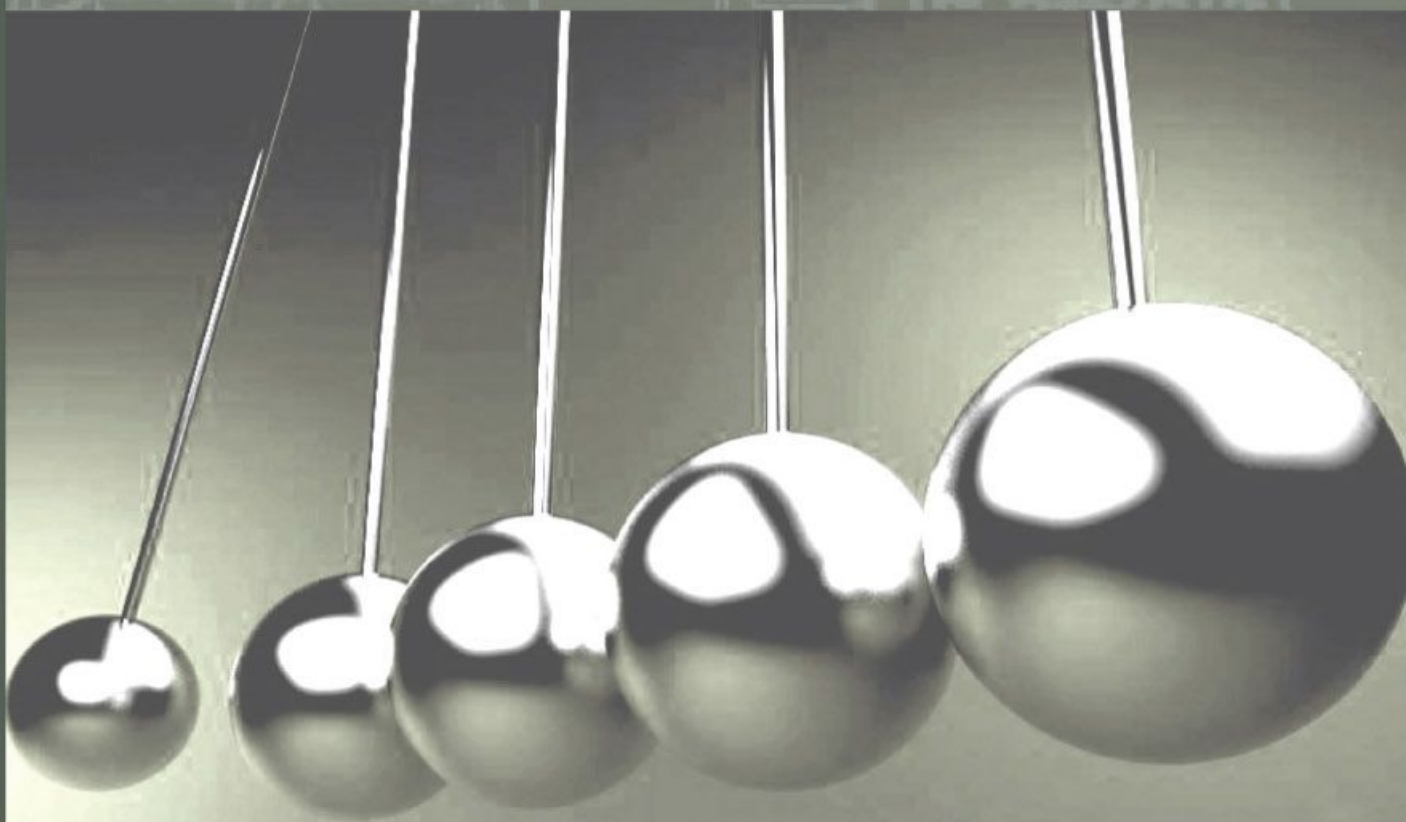


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Technical Mechanics I.  
**Statics**



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TECHNICAL MECHANICS I. –  
STATICS



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# 1. Mathematical introduction – Vector algebra

## 1.1 Concept of vector

A **vector** is a mathematical object that has **magnitude**, **direction** and **sense**. Geometrically, a vector can be represented as a directed line segment, whose length is the magnitude, its direction is a line which is parallel to it and an arrow indicates its sense.

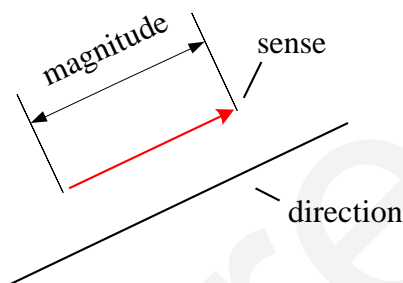


Figure 1 The characteristics of a vector

## 1.2 Description of a vector with coordinates

For the description of a vector we introduce the  $\bar{i}$ ,  $\bar{j}$ ,  $\bar{k}$  **basis vectors** as follows:

- $\bar{i}$ ,  $\bar{j}$  and  $\bar{k}$  are unit vectors:  $|\bar{i}| = |\bar{j}| = |\bar{k}| = 1$
- $\bar{i}$ ,  $\bar{j}$  and  $\bar{k}$  are perpendicular to each other
- $\bar{i}$ ,  $\bar{j}$  and  $\bar{k}$  form a right handed system

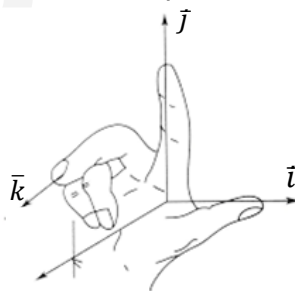


Figure 2 Right handed system

Vector  $\bar{v}$  can be written with the  $\bar{i}$ ,  $\bar{j}$ ,  $\bar{k}$  unit vectors as follows (Figure 2):

$$\bar{v} = v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k} \quad (1.1)$$

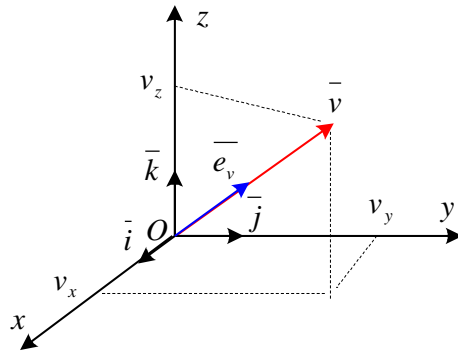


Figure 3 Coordinates of a vector

In equation (1.1)  $v_x$ ,  $v_y$  and  $v_z$  are the **coordinates** of vector  $\bar{v}$ . In another notation:

$$\bar{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad (1.2)$$

The magnitude of vector  $\bar{v}$  can be calculated from its coordinates by equation (1.3):

$$|\bar{v}| = v = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad (1.3)$$

**Unit vector in the direction of vector  $\bar{v}$**  is defined as:

$$\bar{e}_v = \frac{\bar{v}}{v}$$

Let's denote the angle between vector  $\bar{v}$  and the  $x$ ,  $y$  and  $z$  axes with  $\alpha_x$ ,  $\alpha_y$  and  $\alpha_z$  respectively (Figure 4).

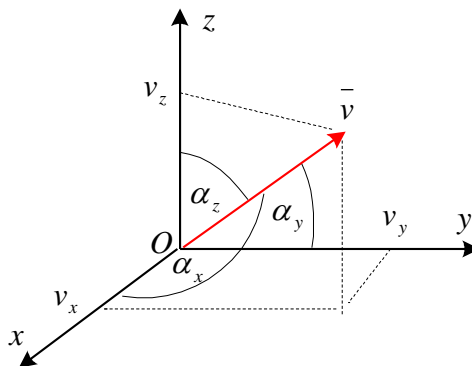


Figure 4 Angles between vector  $\bar{v}$  and the coordinate axes

$\cos \alpha_x$ ,  $\cos \alpha_y$  and  $\cos \alpha_z$  can be calculated as:

$$\cos \alpha_x = \frac{v_x}{v}, \cos \alpha_y = \frac{v_y}{v}, \cos \alpha_z = \frac{v_z}{v} \quad (1.4)$$

Unit vector  $\bar{e}_v$  can be expressed by the  $\bar{i}$ ,  $\bar{j}$ ,  $\bar{k}$  basis vectors:

$$\bar{e}_v = \frac{\bar{v}}{v} = \frac{v_x \bar{i} + v_y \bar{j} + v_z \bar{k}}{v} = \cos \alpha_x \cdot \bar{i} + \cos \alpha_y \cdot \bar{j} + \cos \alpha_z \cdot \bar{k} \quad (1.5)$$

From equation (1.3) and (1.5):

$$|\bar{e}_v| = \sqrt{\cos^2 \alpha_x + \cos^2 \alpha_y + \cos^2 \alpha_z} = 1 \quad (1.6)$$

From equation (1.6) we get the following identity:

$$\cos^2 \alpha_x + \cos^2 \alpha_y + \cos^2 \alpha_z = 1 \quad (1.7)$$

### 1.3 Vector operations

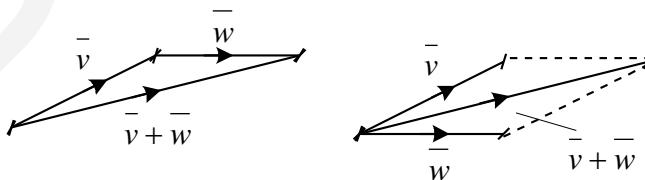
#### **Adding vectors:**

*With calculation:*

$$\begin{aligned} \bar{v} + \bar{w} &= (v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k}) + (w_x \cdot \bar{i} + w_y \cdot \bar{j} + w_z \cdot \bar{k}) \\ &= (v_x + w_x) \cdot \bar{i} + (v_y + w_y) \cdot \bar{j} + (v_z + w_z) \cdot \bar{k} \end{aligned}$$

$$\bar{v} + \bar{w} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} + \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = \begin{pmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{pmatrix} \quad (1.8)$$

*With construction:*



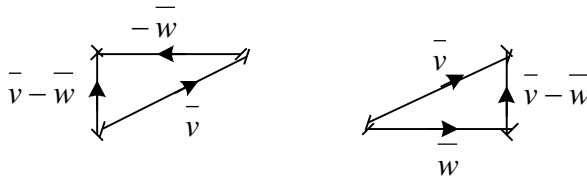
#### **Subtracting vectors:**

*With calculation:*

$$\begin{aligned} \bar{v} - \bar{w} &= (v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k}) - (w_x \cdot \bar{i} + w_y \cdot \bar{j} + w_z \cdot \bar{k}) \\ &= (v_x - w_x) \cdot \bar{i} + (v_y - w_y) \cdot \bar{j} + (v_z - w_z) \cdot \bar{k} \end{aligned}$$

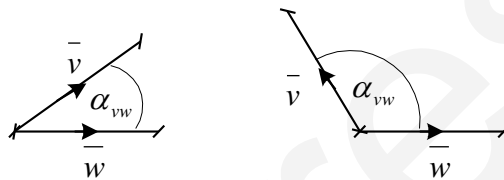
$$\vec{v} + \vec{w} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} + \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = \begin{pmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{pmatrix} \quad (1.9)$$

With construction:



### Angle between vectors:

The angle between two vectors is varied between  $0^\circ$  and  $180^\circ$ .



### Scalar product of vectors:

The **scalar product** of vector  $\vec{v}$  and  $\vec{w}$  is defined as:

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos \alpha_{vw} \quad (1.10)$$

The above definition has the following consequences:

- If  $0^\circ \leq \alpha_{vw} < 90^\circ$  then  $0 < \vec{v} \cdot \vec{w}$
- If  $\alpha_{vw} = 90^\circ$  then  $\vec{v} \cdot \vec{w} = 0$
- If  $90^\circ < \alpha_{vw} \leq 180^\circ$  then  $\vec{v} \cdot \vec{w} < 0$

### Calculation of scalar product from vector coordinates:

*Theorem 1.1:*

The scalar product of vector  $\vec{v}$  and  $\vec{w}$  is calculated by the following formula from the coordinates of the vectors:

$$\vec{v} \cdot \vec{w} = v_x \cdot w_x + v_y \cdot w_y + v_z \cdot w_z \quad (1.11)$$

*Proof:*

$$\bar{i} \cdot \bar{j} = |\bar{i}| \cdot |\bar{j}| \cdot \overbrace{\cos 90^\circ}^0 = 0 \rightarrow \bar{i} \cdot \bar{k} = \bar{j} \cdot \bar{k} = 0$$

$$\bar{i} \cdot \bar{i} = \overbrace{|\bar{i}|}^1 \cdot \overbrace{|\bar{i}|}^1 \cdot \overbrace{\cos 0^\circ}^1 = 1 \rightarrow \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$$

$$\begin{aligned} \bar{v} \cdot \bar{w} &= (v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k}) \cdot (w_x \cdot \bar{i} + w_y \cdot \bar{j} + w_z \cdot \bar{k}) = \\ &= v_x \cdot w_x \cdot \overbrace{\bar{i} \cdot \bar{i}}^1 + v_x \cdot w_y \cdot \overbrace{\bar{i} \cdot \bar{j}}^0 + v_x \cdot w_z \cdot \overbrace{\bar{i} \cdot \bar{k}}^0 + v_y \cdot w_x \cdot \overbrace{\bar{j} \cdot \bar{i}}^0 + v_y \cdot w_y \cdot \overbrace{\bar{j} \cdot \bar{j}}^1 \\ &+ v_y \cdot w_z \cdot \overbrace{\bar{j} \cdot \bar{k}}^0 + v_z \cdot w_x \cdot \overbrace{\bar{k} \cdot \bar{i}}^0 + v_z \cdot w_y \cdot \overbrace{\bar{k} \cdot \bar{j}}^0 + v_z \cdot w_z \cdot \overbrace{\bar{k} \cdot \bar{k}}^1 \\ &= v_x \cdot w_x + v_y \cdot w_y + v_z \cdot w_z \end{aligned}$$

### **Calculation of the angle between vectors from their coordinates:**

*Theorem 1.2:*

The angle between vector  $\bar{v}$  and  $\bar{w}$  is calculated by the following formula from the coordinates of the vectors:

$$\cos \alpha_{vw} = \frac{v_x \cdot w_x + v_y \cdot w_y + v_z \cdot w_z}{\sqrt{v_x^2 + v_y^2 + v_z^2} \cdot \sqrt{w_x^2 + w_y^2 + w_z^2}}, \quad \bar{v} \neq \bar{0}, \bar{w} \neq \bar{0} \quad (1.12)$$

*Proof:*

$$\bar{v} \cdot \bar{w} = |\bar{v}| \cdot |\bar{w}| \cdot \cos \alpha_{vw} \rightarrow \cos \alpha_{vw} = \frac{\bar{v} \cdot \bar{w}}{|\bar{v}| \cdot |\bar{w}|} = \frac{v_x \cdot w_x + v_y \cdot w_y + v_z \cdot w_z}{\sqrt{v_x^2 + v_y^2 + v_z^2} \cdot \sqrt{w_x^2 + w_y^2 + w_z^2}}$$

### **Cross product of vectors:**

The **cross product** of vector  $\bar{v}$  and  $\bar{w}$  is vector  $\bar{v} \times \bar{w}$  defined as:

- $|\bar{v} \times \bar{w}| = |\bar{v}| \cdot |\bar{w}| \cdot \sin \alpha_{vw}$
- $\bar{v} \times \bar{w}$  is normal to the plane of  $\bar{v}$  and  $\bar{w}$
- $\bar{v}$ ,  $\bar{w}$  and  $\bar{v} \times \bar{w}$  form a right-handed system

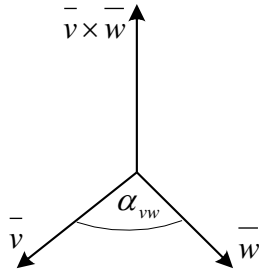


Figure 5 Direction and sense of vector  $\bar{v} \times \bar{w}$

**Theorem 1.3:**

$$\bar{v} \times \bar{w} = -\bar{w} \times \bar{v}$$

**Proof:**

It comes from the definition of cross product and Figure 5.

**Theorem 1.4:**

If  $\bar{v}$  is parallel to  $\bar{w}$  then  $\bar{v} \times \bar{w} = \bar{0}$ .

**Proof:**

If  $\bar{v}$  is parallel to  $\bar{w}$  then  $\alpha_{vw} = 0^\circ$  or  $180^\circ$ .  $\rightarrow |\bar{v} \times \bar{w}| = |\bar{v}| \cdot |\bar{w}| \cdot \overbrace{\sin \alpha_{vw}}^0 = 0$   
 $\rightarrow \bar{v} \times \bar{w} = \bar{0}$

**Calculation of cross product from vector coordinates:**

**Theorem 1.5:**

The cross product of vector  $\bar{v}$  and  $\bar{w}$  is calculated by the following formula from the coordinates of the vectors:

$$\bar{v} \times \bar{w} = \begin{pmatrix} v_y \cdot w_z - v_z \cdot w_y \\ v_z \cdot w_x - v_x \cdot w_z \\ v_x \cdot w_y - v_y \cdot w_x \end{pmatrix} \quad (1.13)$$

**Proof:**

$$\bar{i} \times \bar{j} = \bar{k}, \quad \bar{i} \times \bar{k} = -\bar{j}, \quad \bar{j} \times \bar{k} = \bar{i}$$

$$\bar{j} \times \bar{i} = -\bar{k}, \quad \bar{k} \times \bar{i} = \bar{j}, \quad \bar{k} \times \bar{j} = -\bar{i} \quad (\text{These equations are consequences of Theorem 1.3})$$

$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = \bar{0}$  (These equations are consequences of *Theorem 1.4*)

$$\begin{aligned}\bar{v} \times \bar{w} &= (v_x \cdot \bar{i} + v_y \cdot \bar{j} + v_z \cdot \bar{k}) \times (w_x \cdot \bar{i} + w_y \cdot \bar{j} + w_z \cdot \bar{k}) = \\ &= v_x \cdot w_x \cdot \overbrace{\bar{i} \times \bar{i}}^{\bar{0}} + v_x \cdot w_y \cdot \overbrace{\bar{i} \times \bar{j}}^{\bar{k}} + v_x \cdot w_z \cdot \overbrace{\bar{i} \times \bar{k}}^{-\bar{j}} + v_y \cdot w_x \cdot \overbrace{\bar{j} \times \bar{i}}^{-\bar{k}} + v_y \cdot w_y \\ &\cdot \overbrace{\bar{j} \times \bar{j}}^{\bar{0}} + v_y \cdot w_z \cdot \overbrace{\bar{j} \times \bar{k}}^{\bar{i}} + v_z \cdot w_x \cdot \overbrace{\bar{k} \times \bar{i}}^{\bar{j}} + v_z \cdot w_y \cdot \overbrace{\bar{k} \times \bar{j}}^{-\bar{i}} + v_z \cdot w_z \cdot \overbrace{\bar{k} \times \bar{k}}^{\bar{0}} \\ &= (v_y \cdot w_z - v_z \cdot w_y) \cdot \bar{i} + (v_z \cdot w_x - v_x \cdot w_z) \cdot \bar{j} + (v_x \cdot w_y - v_y \cdot w_x) \cdot \bar{k}\end{aligned}$$

Consequently:

$$\bar{v} \times \bar{w} = \begin{pmatrix} v_y \cdot w_z - v_z \cdot w_y \\ v_z \cdot w_x - v_x \cdot w_z \\ v_x \cdot w_y - v_y \cdot w_x \end{pmatrix} \quad (1.14)$$

### **Method for the calculation of cross product:**

The cross product of two vectors can be calculated with *Formula 1.12*. The problem with this formula is that it is hard to remember it. Because of that we show an easy method for the calculation of cross product applying a procedure which is similar to the calculation procedure of determinants. The method is:

$$\begin{aligned}\bar{v} \times \bar{w} &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \bar{i} \cdot \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} - \bar{j} \cdot \begin{vmatrix} v_x & v_z \\ w_x & w_z \end{vmatrix} + \bar{k} \cdot \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} \\ &= \bar{i} \cdot (v_y \cdot w_z - v_z \cdot w_y) - \bar{j} \cdot (v_x \cdot w_z - v_z \cdot w_x) + \bar{k} \cdot (v_x \cdot w_y - v_y \cdot w_x) \\ &= \begin{pmatrix} v_y \cdot w_z - v_z \cdot w_y \\ v_z \cdot w_x - v_x \cdot w_z \\ v_x \cdot w_y - v_y \cdot w_x \end{pmatrix}\end{aligned}$$

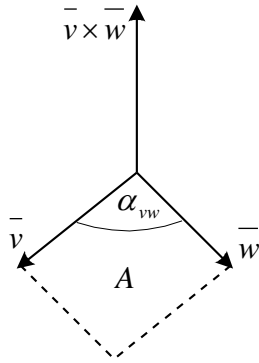
In the above method notation refers to the determinant of a matrix.

### **Geometric representation of the magnitude of cross product:**

*Theorem 1.6:*

The area of the parallelogram, which is determined by vector  $\bar{v}$  and  $\bar{w}$ , is equal to the magnitude of their cross product.

$$A = |\bar{v} \times \bar{w}|$$



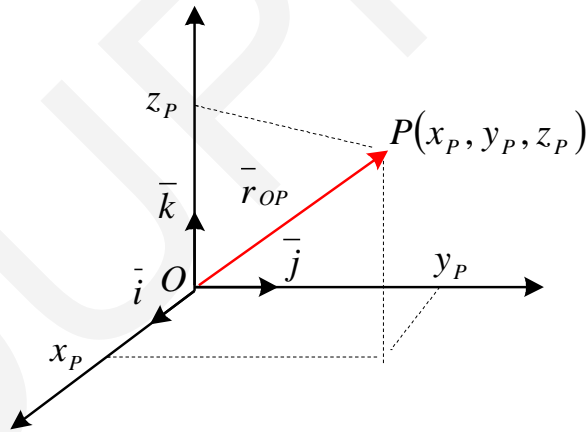
*Proof:*

It comes from basic geometry that:

$$A = |\vec{v}| \cdot |\vec{w}| \cdot \sin \alpha_{vw} = |\vec{v} \times \vec{w}| \quad (1.15)$$

**Position vector:**

The  $\vec{r}_{OP}$  **position vector** is defined as the vector which point from the origin of the coordinate system to point  $P$ .



*Figure 6 Position vector*

$$\vec{r}_{OP} = x_P \cdot \vec{i} + y_P \cdot \vec{j} + z_P \cdot \vec{k},$$

$$\vec{r}_{OP} = \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix} \quad (1.16)$$

## 1.4 Numerical problems

*Problem 1:*

Three vectors are given with their coordinates:

$$\bar{a} = 2.5 \cdot \bar{i} + 4.2 \cdot \bar{j} - 3 \cdot \bar{k}$$

$$\bar{b} = -3.5 \cdot \bar{i} + 5 \cdot \bar{j} + 6 \cdot \bar{k}$$

$$\bar{c} = 3.2 \cdot \bar{i} + 2 \cdot \bar{j} - 4.8 \cdot \bar{k}$$

- Calculate vector  $(\bar{a} - \bar{b}) \cdot \bar{c}$ .
- Calculate vectors  $\bar{a} \times \bar{b}$  and  $\bar{b} \times \bar{a}$ .
- Calculate unit vectors  $\bar{e}_1$  and  $\bar{e}_2$  which are normal to the plane of  $\bar{a}$  and  $\bar{b}$ .
- Calculate the angle between the  $\bar{e}_1$  and  $\bar{c}$  vectors.

*Solution:*

a)

$$\bar{a} = \begin{pmatrix} 2.5 \\ 4.2 \\ -3 \end{pmatrix}, \bar{b} = \begin{pmatrix} -3.5 \\ 5 \\ 6 \end{pmatrix}, \bar{c} = \begin{pmatrix} 3.2 \\ 2 \\ -4.8 \end{pmatrix}$$

$$\begin{aligned} (\bar{a} - \bar{b}) \cdot \bar{c} &= \left( \begin{pmatrix} 2.5 \\ 4.2 \\ -3 \end{pmatrix} - \begin{pmatrix} -3.5 \\ 5 \\ 6 \end{pmatrix} \right) \cdot \begin{pmatrix} 3.2 \\ 2 \\ -4.8 \end{pmatrix} = \begin{pmatrix} 6 \\ -0.8 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 3.2 \\ 2 \\ -4.8 \end{pmatrix} \\ &= 6 \cdot 3.2 + (-0.8) \cdot 2 + (-9) \cdot (-4.8) = 60.8 \end{aligned}$$

b)

$$\begin{aligned} \bar{a} \times \bar{b} &= \begin{pmatrix} 2.5 \\ 4.2 \\ -3 \end{pmatrix} \times \begin{pmatrix} -3.5 \\ 5 \\ 6 \end{pmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2.5 & 4.2 & -3 \\ -3.5 & 5 & 6 \end{vmatrix} \\ &= \bar{i} \cdot \begin{vmatrix} 4.2 & -3 \\ 5 & 6 \end{vmatrix} - \bar{j} \cdot \begin{vmatrix} 2.5 & -3 \\ -3.5 & 6 \end{vmatrix} + \bar{k} \cdot \begin{vmatrix} 2.5 & 4.2 \\ -3.5 & 5 \end{vmatrix} \\ &= \bar{i} \cdot (4.2 \cdot 6 - (-3 \cdot 5)) - \bar{j} \cdot (2.5 \cdot 6 - (-3 \cdot -3.5)) + \bar{k} \\ &\quad \cdot (2.5 \cdot 5 - (4.2 \cdot -3.5)) = 40.2 \cdot \bar{i} - 4.5 \cdot \bar{j} + 27.2 \cdot \bar{k} \end{aligned}$$

$$\bar{a} \times \bar{b} = \begin{pmatrix} 40.2 \\ -4.5 \\ 27.2 \end{pmatrix}$$

$$\bar{b} \times \bar{a} = -(\bar{a} \times \bar{b}) = \begin{pmatrix} -40.2 \\ 4.5 \\ -27.2 \end{pmatrix}$$

c)

$$\bar{e}_1 = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|} = \frac{\begin{pmatrix} 40.2 \\ -4.5 \\ 27.2 \end{pmatrix}}{\sqrt{(40.2)^2 + (-4.5)^2 + (27.2)^2}} = \frac{\begin{pmatrix} 40.2 \\ -4.5 \\ 27.2 \end{pmatrix}}{48.73} = \begin{pmatrix} 0.825 \\ -0.092 \\ 0.558 \end{pmatrix}$$

$$\bar{e}_2 = -\bar{e}_1 = \begin{pmatrix} -0.825 \\ 0.092 \\ -0.558 \end{pmatrix}$$

d)

$$\cos \alpha_{e_1c} = \frac{\bar{e}_1 \cdot \bar{c}}{|\bar{e}_1| \cdot |\bar{c}|} = \frac{0.825 \cdot 3.2 + (-0.092 \cdot 2) + (0.558 \cdot -4.8)}{1 \cdot \sqrt{(3.2)^2 + (2)^2 + (-4.8)^2}} = -0.0367$$

$$\rightarrow \alpha_{e_1c} = 92.1^\circ$$

**Problem 2:**

Two points are given with their coordinates:

$$A = (-2; 4; 8), B = (4; -8; 4).$$

- Calculate the distance between points A and B.
- Calculate the angle between vectors  $\bar{r}_{OA}$  and  $\bar{r}_{OB}$ .
- Calculate the unit vectors which are normal to plane OAB.

**Solution:**

It is advised to make a figure before solving the problem.

a)

$$\bar{r}_{OA} = \begin{pmatrix} -2 \\ 4 \\ 8 \end{pmatrix}, \bar{r}_{OB} = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}$$

$$d_{AB} = |\bar{r}_{AB}| = |\bar{r}_{OB} - \bar{r}_{OA}| = \left| \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 8 \end{pmatrix} \right| = \left| \begin{pmatrix} 6 \\ -12 \\ -4 \end{pmatrix} \right| = \sqrt{(6)^2 + (-12)^2 + (-4)^2} \\ = 14$$

b)

$$\cos \alpha_{AOB} = \frac{\bar{r}_{OA} \cdot \bar{r}_{OB}}{|\bar{r}_{OA}| \cdot |\bar{r}_{OB}|} = -0.089 \rightarrow \alpha_{AOB} = 95.1^\circ$$

c)

$$\bar{e}_1 = \frac{\bar{r}_{OA} \times \bar{r}_{OB}}{|\bar{r}_{OA} \times \bar{r}_{OB}|} = \begin{pmatrix} 0.894 \\ 0.447 \\ 0 \end{pmatrix}$$

$$\bar{e}_2 = -\bar{e}_1 = \begin{pmatrix} -0.894 \\ -0.447 \\ 0 \end{pmatrix}$$

**Problem 3:**

Three points are given with their coordinates:

$$A = (3; 5; 0), B = (0; -2; 3), C = (-4; 0; 2)$$

Calculate the area of triangle ABC.

**Solution:**

It is advised to make a figure before solving the problem.

$$\bar{r}_{OA} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}, \bar{r}_{OB} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}, \bar{r}_{OC} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$$

$$\bar{r}_{AB} = \bar{r}_{OB} - \bar{r}_{OA} = \begin{pmatrix} -3 \\ -7 \\ 3 \end{pmatrix}, \bar{r}_{AC} = \bar{r}_{OC} - \bar{r}_{OA} = \begin{pmatrix} -7 \\ -5 \\ 2 \end{pmatrix}$$

$$T_{\Delta} = \frac{|\vec{r}_{AB}| \cdot |\vec{r}_{AC}| \cdot \sin \alpha_{BAC}}{2} = \frac{|\vec{r}_{AB} \times \vec{r}_{AC}|}{2} = 18.59$$

## 2. Newton's laws of motion. Force formulas. Equilibrium state of a particle

### 2.1 Newton's laws

**Newton's laws** are the basic laws (axioms) of technical mechanics. Newton's basic hypothesis is that there exists always a **frame of reference** in which his laws are valid. The above frame of reference is called as an **inertial frame of reference**. In technical mechanics a frame of reference which is fixed to Earth, or moving with a constant velocity relative to it, can be considered as an inertial frame of reference. All the frames of references which are accelerating relative to Earth are not inertial ones.

#### **Newton's first law:**

The momentum of a body remains constant until it gets into mechanical interaction with another body. Thus without interaction:

$$\vec{p}(t) = \vec{p} = \text{constant} \quad (2.1)$$

Thus, if the mass of the body is constant ( $m(t) = m = \text{constant}$ ), then its velocity is also constant:

$$\vec{v}(t) = \vec{v} = \text{constant} \quad (2.2)$$

Thus the body is in rest or its **centre of mass** is moving uniformly in a straight line.

#### **Newton's second law:**

If a body is in mechanical interaction with other bodies the **rate of change of its momentum** is equal to the **resultant force** that the other bodies exert on it.

$$\vec{F} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{p}}{\Delta t}, \quad \vec{F} = [\text{N}] \quad (2.3)$$

If the mass of the body is constant, then:

$$\bar{F} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \bar{p}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta(m \cdot \bar{v})}{\Delta t} = m \cdot \lim_{\Delta t \rightarrow 0} \frac{\Delta \bar{v}}{\Delta t} = m \cdot \bar{a} \quad (2.4)$$

### **Newton's third law:**

During the mechanical interaction of two bodies the equation below is valid:

$$\bar{F}_{12} = -\bar{F}_{21} \quad (2.5)$$

In the above equation  $\bar{F}_{12}$  is the force that body 1 exerts on body 2. Thus the two forces have the same magnitude and direction but opposite sense.

### **Newton's fourth law:**

If a body is in mechanical interaction with several different bodies, then the resultant force acting on it is calculated as:

$$\bar{F} = \sum_i \bar{F}_i = \bar{F}_1 + \bar{F}_2 + \dots + \bar{F}_i + \dots \quad (2.6)$$

In the above equation  $\bar{F}_i$  is the force that is exerted by body  $i$  in the absence of the other bodies.

## **2.2 Force formulas**

**Force formulas** give the force as a function of the parameters of the given mechanical interaction. Force formulas are usually determined experimentally. In the following we present the formulas of the most important forces in technical mechanics.

### **Gravitational force:**

The magnitude of **gravitational force** acting between two bodies with masses  $M$  and  $m$  is calculated as:

$$|\bar{F}_g| = F_g = \gamma \cdot \frac{M \cdot m}{r^2}, \quad \gamma = 6.674 \cdot 10^{-11} \left[ \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \right] \quad (2.7)$$

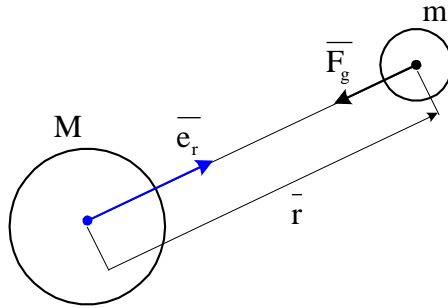


Figure 7 Gravitational force

In the above equation  $r$  is the distance between the centres of masses of the two bodies and  $\gamma$  is the gravitational constant. Introducing the radial unit vector as  $\vec{e}_r = \frac{\vec{r}}{r}$  the formula of gravitational force is as follows:

$$\vec{F}_g = -\gamma \cdot \frac{M \cdot m}{r^2} \cdot \vec{e}_r = -\gamma \cdot \frac{M \cdot m}{r^3} \cdot \vec{r} \quad (2.8)$$

where  $\vec{r}$  is the position vector which points from the centre of mass of body M to the centre of mass of body m. If one of the bodies is Earth and the other one is a small object on its surface then formula (2.8) can be written as:

$$\vec{F}_g = \left( -\gamma \frac{M_E}{R_E^2} \cdot \vec{e}_r \right) \cdot m = m \cdot \vec{g} \quad (2.9)$$

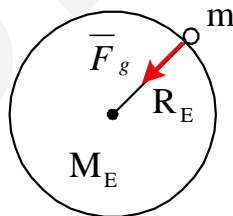


Figure 8 Gravitational force on the surface of Earth

In formula (2.9)  $M_E$  and  $R_E$  are the mass and radius of Earth and  $\vec{g}$  is the gravitational acceleration. Since Earth is not a perfect globe the magnitude of  $\vec{g}$  depends on the global position. In the territory of Hungary the magnitude of  $\vec{g}$  is:

$$g = |\vec{g}| = 9,81 \left[ \frac{\text{m}}{\text{s}^2} \right]$$

### Spring force:

The magnitude of the force exerted by an **ideal spring** is calculated as (Figure 9):

$$|\bar{F}_D| = F_D = D \cdot |r - r_0| = D \cdot \Delta r \quad (2.10)$$

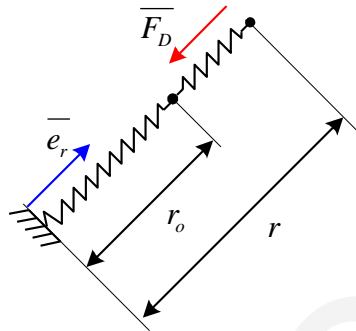


Figure 9 Spring force

where  $D$  is the **spring constant**,  $r_0$  is the length of the spring without **deformation**  $r$  is the **deformed length** and  $\Delta r$  is the **deformation** of the spring. The direction of spring force is parallel to the direction of the spring and its sense is opposite to the  $(r - r_0) \cdot \bar{e}_r$  vector. Thus the formula of spring force is as follows:

$$\bar{F}_D = -D \cdot (r - r_0) \cdot \bar{e}_r \quad (2.11)$$

### Reaction forces:

To define the concept of **reaction force** first we should introduce the concept of **rigid body** and **constraint**. A body is rigid if the distance between any of its two points is constant at any kinds of mechanical loading on it.

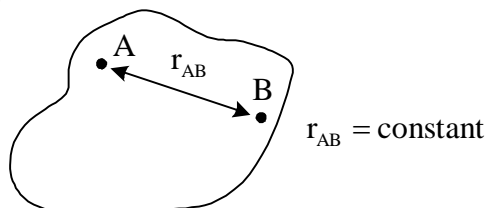


Figure 10 Rigid body

A constraint is a rigid body which serves to restrict the motion or position of another body along to a given curve or surface. The force which the

constraint exerts on the body is called as a **reaction force**. Reaction forces are usually unknown but they can be determined if the other forces acting on the body are known.

*Types of constraints:*

1) *Inextensible cable*

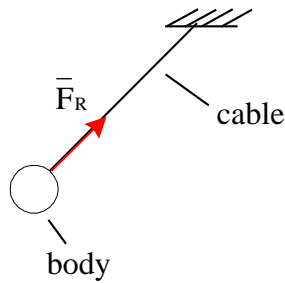


Figure 11 Inextensible cable

*The direction of the reaction force exerted by a cable is parallel to the cable. A cable can only pull but never push.*

2) *Rigid rod*

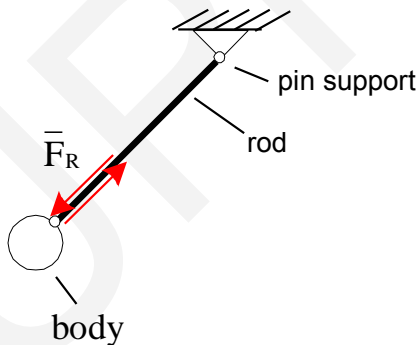


Figure 12 Rigid rod

*If the rod is connected to an ideal pin support then the direction of the reaction force exerted by a rod is parallel to the rod. A rod can either pull or push.*

3) *Smooth or rough rigid surface:*

*Smooth:*

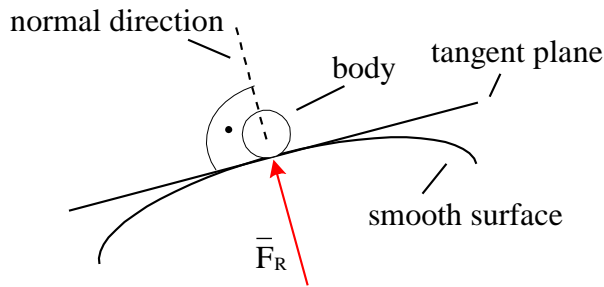


Figure 13 Smooth rigid surface

The reaction force exerted by a **smooth rigid surface** has normal direction. (That is normal to the tangent plane of the surface.) A surface can only push but never pull.

*Rough:*

The reaction force exerted by a **rough rigid surface** has both normal and tangential component. If the body is **in rest** (Figure 14) the tangential component has a maximum value above which the body slips.

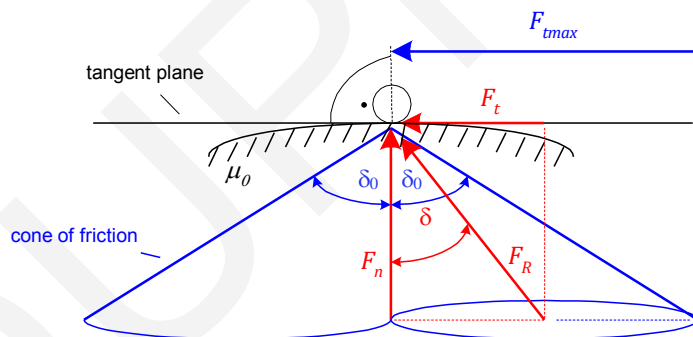


Figure 14 Rough rigid surface – the body is in rest

This maximum value is linearly proportional with the normal component of the reaction force (Equation 2.12).

$$F_{tmax} = \mu_0 \cdot F_n \quad (2.12)$$

In the above equation  $\mu_0$  is the **coefficient of static friction**. The body is in static equilibrium, if and only if, the inequality below is valid:

$$|F_t| \leq F_{tmax} = \mu_0 \cdot F_n \quad (2.13)$$

We introduce the **cone of friction** (Figure 14) the half-angle of which is defined by Equation 2.14.

$$\delta_0 = \mu_0 = \tan^{-1} \left( \frac{F_{t\max}}{F_n} \right) \quad (2.14)$$

If the body is in rest the reaction force ( $\bar{F}_R$ ) is inside or on the superficies of the cone of friction.

If the body **slips** on the rough surface with velocity  $\bar{v}$  then the tangential component of the reaction force is linearly proportional with the normal component and the tangential component has the same direction but opposite sense as the velocity vector (Figure 15, Equation 2.15).

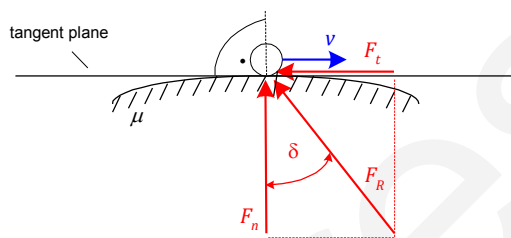


Figure 15 Rough rigid surface – the body is slipping

$$F_t = \mu \cdot F_n \quad (2.15)$$

In the above equation  $\mu$  is the **coefficient of kinetic friction**. In the table below static and kinetic friction coefficients are given for different material pairs under dry and lubricated conditions.

**Approximate coefficients of friction [11]:**

Materials		Static friction		Kinetic friction	
		Dry and clean	Lubricated	Dry and clean	Lubricated
Aluminium	Steel	0.61	-	0.47	-
Brass	Steel	0.35-0.51	0.19	0.44	-
Concrete	Rubber	1	0.3	0.6-0.85	0.45-0.75
Steel	Steel	0.74-0.8	0.16	0.42-0.62	-
Wood	Metal	0.2-0.6	0.2	-	-
Wood	Wood	0.25-0.5	0.2	-	-

On the bases of the table above it can be stated that:

- The static friction coefficient is always bigger, then the kinetic one
- A friction coefficient is always bigger under dry conditions then under lubricated ones

### 2.3 Equilibrium state of a particle

If the dimensions of a body is negligible compared to the other dimensions in the mechanical problem, then the body can be modelled as a **particle**. A particle is a geometric point with mass. A particle is in **equilibrium** if the resultant force acting on it is equal to zero (*Equation 2.16*).

$$\bar{F} = \sum_i \bar{F}_i = \bar{0} \quad (2.16)$$

From Newton's second law it follows that in equilibrium state the acceleration of the particle is equal to zero. Thus it is in rest or moving uniformly in a straight line. If the particle is in rest the equilibrium state is called **static**.

## 2.4 Numerical problems

*Problem 1 (Calculation of the resultant force of a 3D force system)*

The system of forces below acts on a particle which is in the origin of the coordinate system.

$$\bar{F}_1 = 5\bar{i} + 3\bar{j} - \bar{k}[\text{N}]$$

$$\bar{F}_2 = 30 \cdot (0.8\bar{i} - 0.6\bar{k})[\text{N}]$$

$$|\bar{F}_3| = F_3 = 50[\text{N}] \text{ and } \bar{F}_3 \text{ goes through point } P_3(0,6,2)$$

$$|\bar{F}_4| = F_4 = 20[\text{N}] \text{ and } \alpha_{4x} = 90^\circ, \alpha_{4y} = 45^\circ, \alpha_{4z} = 135^\circ.$$

Calculate the resultant of the force system.

*Solution*

$$\bar{F}_1 = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} [\text{N}]$$

$$\bar{F}_2 = \begin{pmatrix} 24 \\ 0 \\ -18 \end{pmatrix} [\text{N}]$$

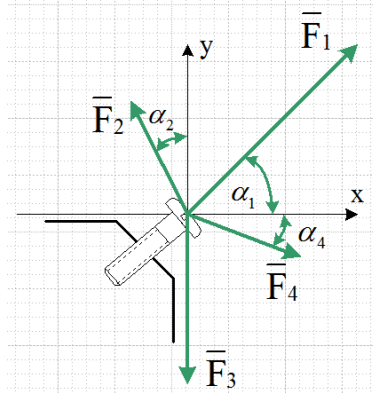
$$\bar{F}_3 = F_3 \cdot \bar{e}_3 = 50 \cdot \frac{\bar{r}_{O3}}{|\bar{r}_{O3}|} = 50 \cdot \frac{\begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}}{\sqrt{0^2 + 6^2 + 2^2}} = \begin{pmatrix} 0 \\ 47.43 \\ 15.81 \end{pmatrix} [\text{N}]$$

$$\begin{aligned} \bar{F}_4 &= F_4 \cdot \bar{e}_4 = F_4 \cdot (\cos \alpha_{4x} \cdot \bar{i} + \cos \alpha_{4y} \cdot \bar{j} + \cos \alpha_{4z} \cdot \bar{k}) \\ &= 20 \cdot (\cos 90^\circ \cdot \bar{i} + \cos 45^\circ \cdot \bar{j} + \cos 135^\circ \cdot \bar{k}) \\ &= 0 \cdot \bar{i} + 14.14 \cdot \bar{j} - 14.14 \cdot \bar{k} = \begin{pmatrix} 0 \\ 14.14 \\ -14.14 \end{pmatrix} [\text{N}] \end{aligned}$$

$$\begin{aligned} \bar{F} &= \sum_i \bar{F}_i = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4 = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 24 \\ 0 \\ -18 \end{pmatrix} + \begin{pmatrix} 0 \\ 47.43 \\ 15.81 \end{pmatrix} + \begin{pmatrix} 0 \\ 14.14 \\ -14.14 \end{pmatrix} \\ &= \begin{pmatrix} 29 \\ 64.57 \\ -17.33 \end{pmatrix} [\text{N}] \end{aligned}$$

*Problem 2 (Calculation of the resultant force of a 2D force system)*

Four forces act on a screw head as it is shown in the figure below:



*Data:*  $F_1 = 150[\text{N}]$ ,  $F_2 = 80[\text{N}]$ ,  $F_3 = 110[\text{N}]$ ,  $F_4 = 100[\text{N}]$ ,  $\alpha_1 = 30^\circ$ ,  $\alpha_2 = 20^\circ$ ,  $\alpha_4 = 15^\circ$ .

- Calculate the resultant force of the force system.
- Calculate the magnitude of the resultant force.
- Construct the resultant.

### Solution

Remark: We can solve this problem similarly as *Problem 1* but in case of a 2D force system there is also an easier way. We present it bellow.

a)

$$\vec{F}_1 = F_1 \cdot \cos \alpha_1 \cdot \vec{i} + F_1 \cdot \sin \alpha_1 \cdot \vec{j} = \begin{pmatrix} F_1 \cdot \cos \alpha_1 \\ F_1 \cdot \sin \alpha_1 \end{pmatrix} [\text{N}]$$

$$\vec{F}_2 = -F_2 \cdot \sin \alpha_2 \cdot \vec{i} + F_2 \cdot \cos \alpha_2 \cdot \vec{j} = \begin{pmatrix} -F_2 \cdot \sin \alpha_2 \\ F_2 \cdot \cos \alpha_2 \end{pmatrix} [\text{N}]$$

$$\vec{F}_3 = 0 \cdot \vec{i} - F_3 \cdot \vec{j} = \begin{pmatrix} 0 \\ -F_3 \end{pmatrix} [\text{N}]$$

$$\vec{F}_4 = F_4 \cdot \cos \alpha_4 \cdot \vec{i} - F_4 \cdot \sin \alpha_4 \cdot \vec{j} = \begin{pmatrix} F_4 \cdot \cos \alpha_4 \\ -F_4 \cdot \sin \alpha_4 \end{pmatrix} [\text{N}]$$

$$\vec{F} = \sum_i \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$$

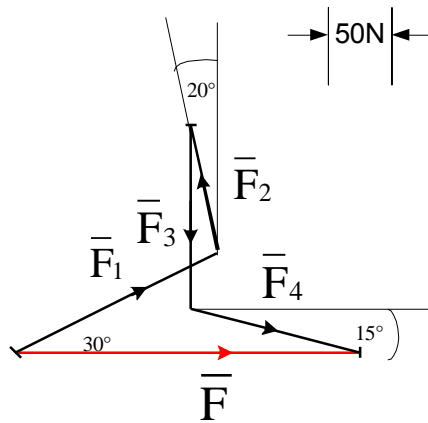
$$= \begin{pmatrix} F_1 \cdot \cos \alpha_1 \\ F_1 \cdot \sin \alpha_1 \end{pmatrix} + \begin{pmatrix} -F_2 \cdot \sin \alpha_2 \\ F_2 \cdot \cos \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -F_3 \end{pmatrix} + \begin{pmatrix} F_4 \cdot \cos \alpha_4 \\ -F_4 \cdot \sin \alpha_4 \end{pmatrix}$$

$$= \begin{pmatrix} 200.26 \\ 14.29 \end{pmatrix} [\text{N}]$$

b)

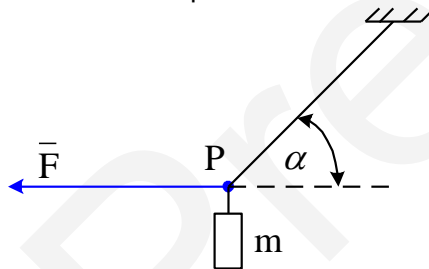
$$F = |\vec{F}| = \sqrt{(200.26)^2 + (14.29)^2} = 200.78[\text{N}]$$

c) We give the lengths of the vectors compared to a force scale.



**Problem 3**

Particle P in the figure is in static equilibrium.

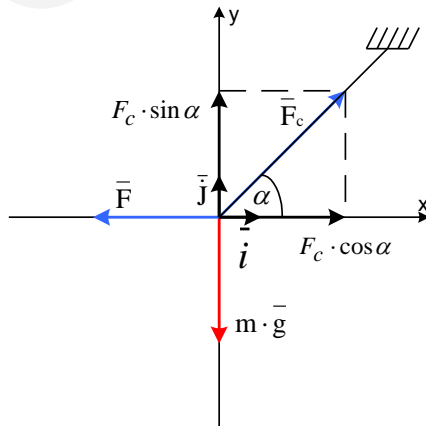


Data:  $F = 150[\text{N}]$ ,  $\alpha = 30^\circ$ ,  $g = 9.81 \left[ \frac{\text{m}}{\text{s}^2} \right]$ .

- Calculate the tensile force in the cable and the mass of the particle.
- Construct the tensile and gravitational force.

**Solution**

a)



$$\vec{F} = -F \cdot \vec{i} + 0 \cdot \vec{j} = \begin{pmatrix} -F \\ 0 \end{pmatrix} = \begin{pmatrix} -150 \\ 0 \end{pmatrix} [\text{N}]$$

$$\vec{F}_c = F_c \cdot \cos \alpha \cdot \vec{i} + F_c \cdot \sin \alpha \cdot \vec{j} = \begin{pmatrix} F_c \cdot \cos 30^\circ \\ F_c \cdot \sin 30^\circ \end{pmatrix} [\text{N}]$$

$$m \cdot \vec{g} = 0 \cdot \vec{i} - m \cdot g \cdot \vec{j} = \begin{pmatrix} 0 \\ -m \cdot 9.81 \end{pmatrix} [\text{N}]$$

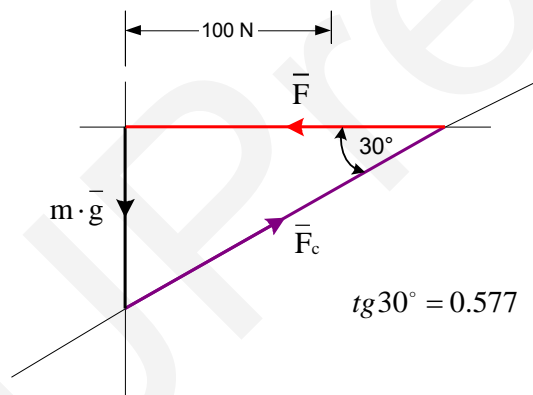
$$\vec{F} = \sum_i \vec{F}_i = \begin{pmatrix} -150 \\ 0 \end{pmatrix} + \begin{pmatrix} F_c \cdot \cos 30^\circ \\ F_c \cdot \sin 30^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ -m \cdot 9.81 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

I.  $-150 + F_c \cdot \cos 30^\circ = 0 \rightarrow F_c = 173.2 [\text{N}]$

II.  $F_c \cdot \sin 30^\circ - m \cdot 9.81 = 0 \rightarrow m = 8.83 [\text{kg}]$

b)

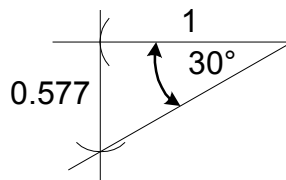
In equilibrium the sum of the forces is zero, thus the three forces form a vector triangle.



Steps of the construction:

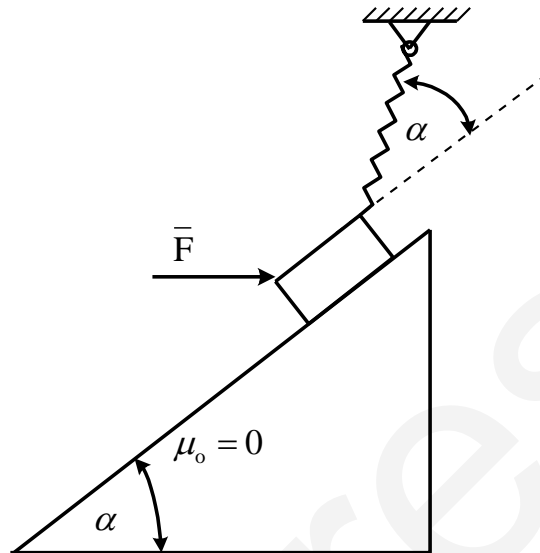
- 1) Drawing force  $\vec{F}$
- 2) Constructing angle  $\alpha$
- 3) Drawing the line of  $\vec{F}_c$  and  $m \cdot \vec{g}$  and constructing their intersection
- 4) Reading the magnitude of  $\vec{F}_c$  and  $m \cdot \vec{g}$ , then comparing their length with the unit

The construction of angle  $\alpha$ :



*Problem 4*

In the figure below a particle with mass  $m$  is in static equilibrium on a smooth surfaced incline. The  $\Delta r$  compression and  $D$  constant of the spring are known.

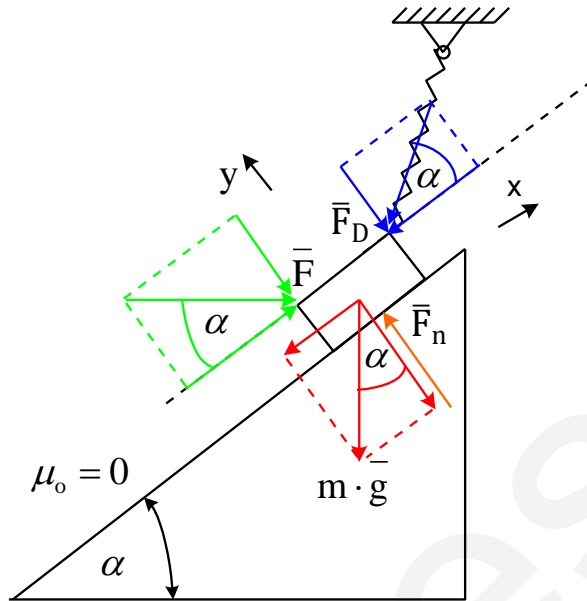


*Data:*  $m = 40[\text{kg}]$ ,  $\alpha = 30^\circ$ ,  $D = 1000 \left[ \frac{\text{N}}{\text{mm}} \right]$ ,  $\Delta r = 2[\text{mm}]$ ,  $g = 9.81 \frac{\text{m}}{\text{s}^2}$ .

- Calculate the magnitude of force  $\vec{F}$  ( $\vec{F}$  is horizontal).
- Calculate the magnitude of the reaction force ( $\vec{F}_n$ ) that the incline exerts on the particle.
- Construct forces  $\vec{F}$  and  $\vec{F}_n$ .

*Solution*

- and b)



$$\vec{F} = F \cdot \cos \alpha \cdot \vec{i} - F \cdot \sin \alpha \cdot \vec{j} = \begin{pmatrix} F \cdot \cos 30^\circ \\ -F \cdot \sin 30^\circ \end{pmatrix} [\text{N}]$$

$$m \cdot \vec{g} = -m \cdot g \cdot \sin \alpha \cdot \vec{i} - m \cdot g \cdot \cos \alpha \cdot \vec{j} = \begin{pmatrix} -40 \cdot 9.81 \cdot \sin 30^\circ \\ -40 \cdot 9.81 \cdot \cos 30^\circ \end{pmatrix} [\text{N}]$$

$$\vec{F}_D = -F_D \cdot \cos \alpha \cdot \vec{i} - F_D \cdot \sin \alpha \cdot \vec{j} = \begin{pmatrix} -D \cdot \Delta r \cdot \cos \alpha \\ -D \cdot \Delta r \cdot \sin \alpha \end{pmatrix} = \begin{pmatrix} -2000 \cdot \cos 30^\circ \\ -2000 \cdot \sin 30^\circ \end{pmatrix} [\text{N}]$$

$$\vec{F}_n = 0 \cdot \vec{i} + F_n \cdot \vec{j} = \begin{pmatrix} 0 \\ F_n \end{pmatrix} [\text{N}]$$

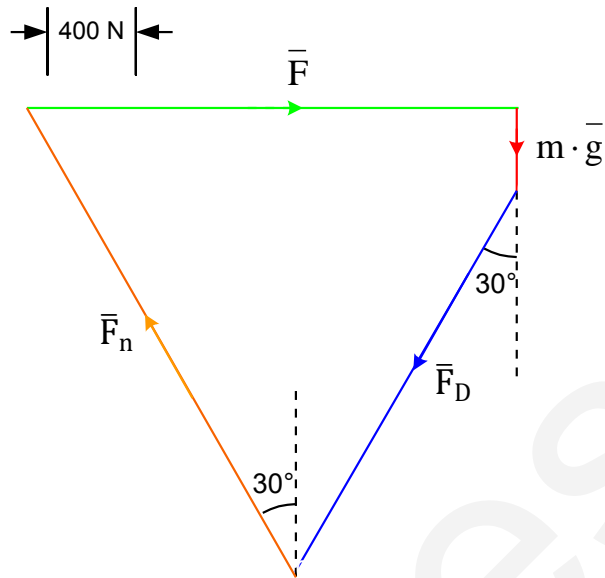
$$\vec{F} = \sum_i \vec{F}_i = \begin{pmatrix} F \cdot \cos 30^\circ \\ -F \cdot \sin 30^\circ \end{pmatrix} + \begin{pmatrix} -40 \cdot 9.81 \cdot \sin 30^\circ \\ -40 \cdot 9.81 \cdot \cos 30^\circ \end{pmatrix} + \begin{pmatrix} -2000 \cdot \cos 30^\circ \\ -2000 \cdot \sin 30^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ F_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I.} \quad F \cdot \cos 30^\circ - 40 \cdot 9.81 \cdot \sin 30^\circ - 2000 \cdot \cos 30^\circ = 0 \rightarrow F = 2227 [\text{N}]$$

$$\text{II.} \quad -F \cdot \sin 30^\circ - 40 \cdot 9.81 \cdot \cos 30^\circ - 2000 \cdot \sin 30^\circ + F_n = 0 \rightarrow F_n =$$

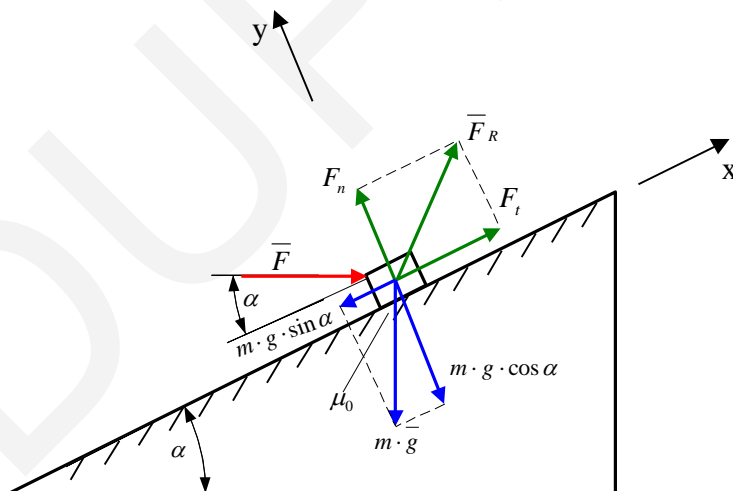
$$2453.3 [\text{N}]$$

c)



**Problem 5**

In the figure below a particle with mass  $m$  is in static equilibrium on a rough inclined surface.



**Data:**  $\alpha = 30^\circ$ ,  $m \cdot g = 100$  [N],  $\mu_0 = 0.3$

a) Calculate the minimum and maximum value between the magnitude of force  $\vec{F}$  can be varied ( $F_{\min}$  and  $F_{\max}$ ) if the particle is in static equilibrium. ( $\vec{F}$  is horizontal)

- b) Calculate the magnitude of the reaction force – that the incline exerts on the particle – at the above minimum and maximum values.  
 c) Construct  $F_{\min}$  and  $F_{\max}$ .

*Solution:*

a)

$$\sum \vec{F} = \vec{F}_R + \vec{F} + m \cdot \vec{g} = \vec{0} = \begin{pmatrix} F_t \\ F_n \end{pmatrix} + \begin{pmatrix} F \cdot \cos \alpha \\ -F \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} -m \cdot g \cdot \sin \alpha \\ -m \cdot g \cdot \cos \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_t + F \cdot \cos \alpha - m \cdot g \cdot \sin \alpha = 0 \rightarrow F_t = m \cdot g \cdot \sin \alpha - F \cdot \cos \alpha = 50 - 0.866 \cdot F$$

$$\text{II. } F_n - F \cdot \sin \alpha - m \cdot g \cdot \cos \alpha = 0 \rightarrow F_n = m \cdot g \cdot \cos \alpha + F \cdot \sin \alpha = 86.6 + 0.5 \cdot F$$

$$|F_t| \leq \mu_0 \cdot F_n \rightarrow -\mu_0 \cdot F_n \leq F_t \leq \mu_0 \cdot F_n$$

*Case 1:*

$$-\mu_0 \cdot F_n \leq F_t$$

$$-0.3 \cdot (86.6 + 0.5 \cdot F) \leq 50 - 0.866 \cdot F$$

$$0.716 \cdot F \leq 76$$

$$F \leq F_{\max} = 106.14 \text{ [N]}$$

*Case 2:*

$$F_t \leq \mu_0 \cdot F_n$$

$$50 - 0.866 \cdot F \leq 0.3 \cdot (86.6 + 0.5 \cdot F)$$

$$24 \leq 1.016 \cdot F$$

$$23.62 \text{ [N]} = F_{\min} \leq F$$

$$23.62 \text{ [N]} \leq F \leq 106.14 \text{ [N]}$$

b)

$$F_{t\min} = 50 - 0.866 \cdot F_{\min} = 50 - 0.866 \cdot 23.62 = 29.55 \text{ [N]}$$

$$F_{t\max} = 50 - 0.866 \cdot F_{\max} = 50 - 0.866 \cdot 106.14 = -41.92 \text{ [N]}$$

$$F_{n\min} = 86.6 + 0.5 \cdot F_{\min} = 86.6 + 0.5 \cdot 23.62 = 98.41 \text{ [N]}$$

$$F_{n\max} = 86.6 + 0.5 \cdot F_{\max} = 86.6 + 0.5 \cdot 106.14 = 139.67 \text{ [N]}$$

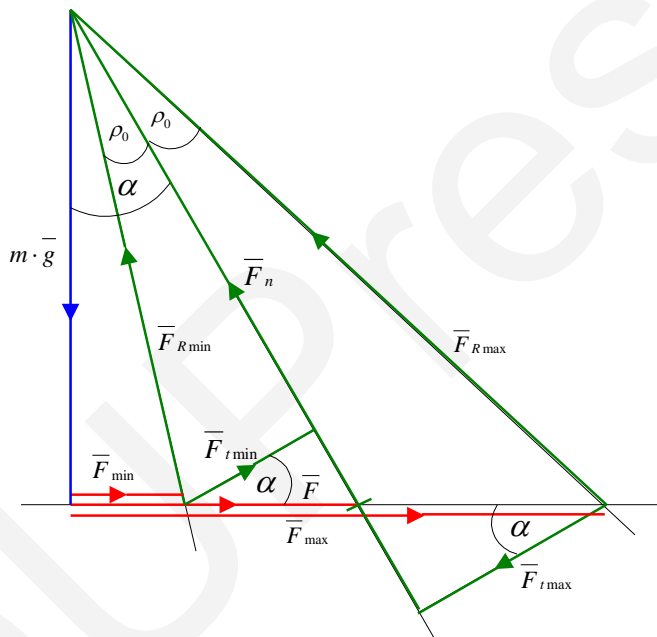
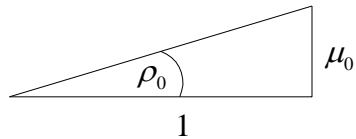
$$F_{R\min} = \sqrt{F_{t\min}^2 + F_{n\min}^2} = \sqrt{29.55^2 + 98.41^2} = 102.75 \text{ [N]}$$

$$F_{R\max} = \sqrt{F_{t\max}^2 + F_{n\max}^2} = \sqrt{(-41.92)^2 + 139.67^2} = 145.83 \text{ [N]}$$

c)

The construction method of angle  $\rho_0$  and values  $F_{\min}$  and  $F_{\max}$  is presented below:

$$\tan \rho_0 = \frac{F_{t \max}}{F_n} = \mu_0$$

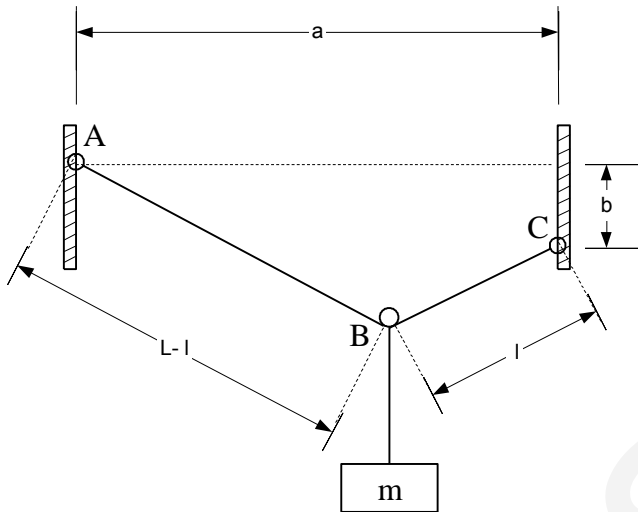


Clicking on the link below the construction above can be followed as a Power Point animation step by step:

Animation 2.1

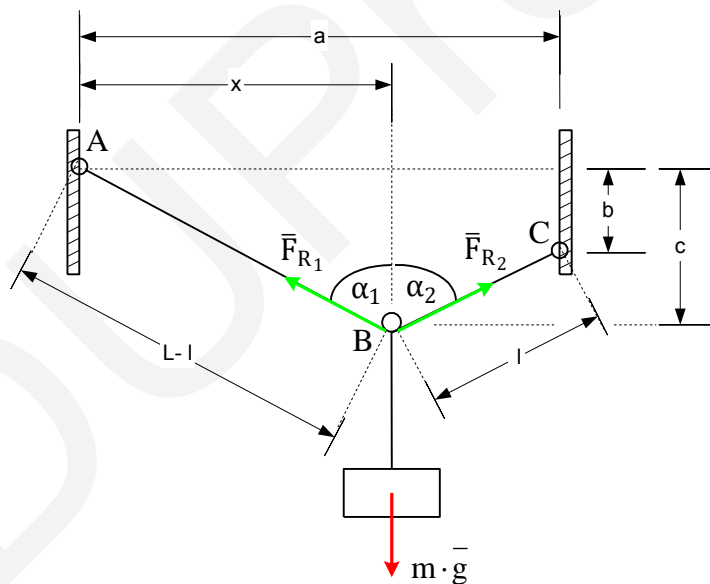
### Problem 6

The small roller (B) on rope AC is in static equilibrium. The mass ( $m$ ) of the load which is hanging on the roller, the length of the rope ( $L$ ), and all the other necessary geometric data are given below.



*Data:*  $m = 100[\text{kg}]$ ,  $L = 15[\text{m}]$ ,  $a = 10[\text{m}]$ ,  $b = 3.354[\text{m}]$ .  
 Calculate the magnitude of the tensile force in the rope.

*Solution*



Since AB and BC are parts of rope AC:

$$|\vec{F}_{R1}| = |\vec{F}_{R2}| = F_R$$

$$\vec{F}_{R1} = -F_R \cdot \sin \alpha_1 \cdot \vec{i} + F_R \cdot \cos \alpha_1 \cdot \vec{j} = \begin{pmatrix} -F_R \cdot \sin \alpha_1 \\ F_R \cdot \cos \alpha_1 \end{pmatrix} [\text{N}]$$

$$\bar{F}_{R2} = F_R \cdot \sin \alpha_2 \cdot \bar{i} + F_R \cdot \cos \alpha_2 \cdot \bar{j} = \begin{pmatrix} F_R \cdot \sin \alpha_2 \\ F_R \cdot \cos \alpha_2 \end{pmatrix} [\text{N}]$$

$$m \cdot \bar{g} = 0 \cdot \bar{i} - m \cdot g \cdot \bar{j} = \begin{pmatrix} 0 \\ -m \cdot 9.81 \end{pmatrix} [\text{N}]$$

$$\bar{F} = \sum_i \bar{F}_i = \begin{pmatrix} -F_R \cdot \sin \alpha_1 \\ F_R \cdot \cos \alpha_1 \end{pmatrix} + \begin{pmatrix} F_R \cdot \sin \alpha_2 \\ F_R \cdot \cos \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -m \cdot 9.81 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I.} \quad -F_R \cdot \sin \alpha_1 + F_R \cdot \sin \alpha_2 = 0 \rightarrow \alpha_1 = \alpha_2 = \alpha$$

$$\text{II.} \quad F_R \cdot \cos \alpha_1 + F_R \cdot \cos \alpha_2 - m \cdot 9.81 = 0 \rightarrow F_R = \frac{m \cdot 9.81}{2 \cdot \cos \alpha}$$

The geometric equations below are valid:

$$\text{I.} \quad a^2 + (2c - b)^2 = L^2$$

$$\text{II.} \quad \tan \alpha = \frac{x}{c} = \frac{a-x}{c-b}$$

From the first equation:

$$c = \frac{\sqrt{L^2 - a^2} + b}{2} = 7.267 [\text{m}]$$

From the second equation:

$$\frac{x}{7.267} = \frac{10 - x}{7.267 - 3.354} \rightarrow x = 6.5 [\text{m}]$$

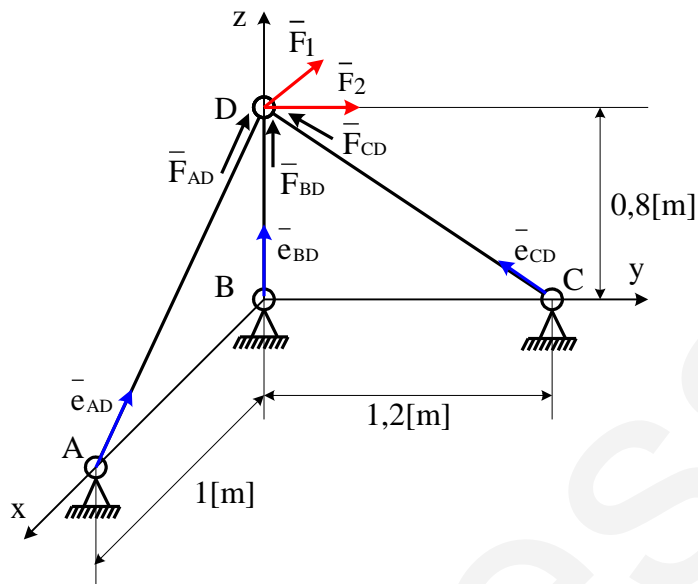
$$\tan \alpha = \frac{x}{c} = 0.894 \rightarrow \alpha = 41.81^\circ$$

From equation II:

$$F_R = \frac{100 \cdot 9.81}{2 \cdot \cos 41.81^\circ} = 658.1 [\text{N}]$$

### Problem 7

The figure below shows a structure which is built up of three bars. One end of each bar is connected to a pin support (points A, B, C), the other ends are connected to particle D. Forces  $\bar{F}_1$  and  $\bar{F}_2$  act on particle D which is in static equilibrium.



Data:  $\vec{F}_1 = -1000 \cdot \vec{i}[\text{N}]$ ,  $\vec{F}_2 = 1000 \cdot \vec{j}[\text{N}]$

Calculate the magnitude of the reaction forces ( $\vec{F}_{AD}$ ,  $\vec{F}_{BD}$ ,  $\vec{F}_{CD}$ ) acting in the bars.

**Solution**

The unit vectors in the direction of the bars are:

$$\vec{e}_{AD} = \frac{\vec{r}_{AD}}{|\vec{r}_{AD}|} = \frac{\begin{pmatrix} -1 \\ 0 \\ 0.8 \end{pmatrix}}{\sqrt{(-1)^2 + (0)^2 + (0.8)^2}} = \begin{pmatrix} -0.781 \\ 0 \\ 0.625 \end{pmatrix}$$

$$\vec{e}_{BD} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{e}_{CD} = \frac{\vec{r}_{CD}}{|\vec{r}_{CD}|} = \frac{\begin{pmatrix} 0 \\ -1.2 \\ 0.8 \end{pmatrix}}{\sqrt{(0)^2 + (-1.2)^2 + (0.8)^2}} = \begin{pmatrix} 0 \\ -0.832 \\ 0.555 \end{pmatrix}$$

Reaction forces  $\vec{F}_{AD}$ ,  $\vec{F}_{BD}$  and  $\vec{F}_{CD}$  can be written as:

$$\vec{F}_{AD} = F_{AD} \cdot \vec{e}_{AD} = F_{AD} \cdot \begin{pmatrix} -0.781 \\ 0 \\ 0.625 \end{pmatrix}$$

$$\vec{F}_{BD} = F_{BD} \cdot \vec{e}_{BD} = F_{BD} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\bar{F}_{CD} = F_{CD} \cdot \bar{e}_{CD} = F_{CD} \cdot \begin{pmatrix} 0 \\ -0.832 \\ 0.555 \end{pmatrix}$$

Particle D is in static equilibrium:

$$\bar{F} = \sum_i \bar{F}_i = \bar{F}_{AD} + \bar{F}_{BD} + \bar{F}_{CD} + \bar{F}_1 + \bar{F}_2 = F_{AD} \cdot \begin{pmatrix} -0.781 \\ 0 \\ 0.625 \end{pmatrix} + F_{BD} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + F_{CD} \cdot \begin{pmatrix} 0 \\ -0.832 \\ 0.555 \end{pmatrix} + \begin{pmatrix} -1000 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1000 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{I.} \quad -0.781 \cdot F_{AD} - 1000 = 0 \rightarrow F_{AD} = -1281[\text{N}]$$

$$\text{II.} \quad -0.832 \cdot F_{CD} + 1000 = 0 \rightarrow F_{CD} = -1202[\text{N}]$$

$$\text{III.} \quad 0.625 \cdot F_{AD} + F_{BD} + 0.555 \cdot F_{CD} = 0 \rightarrow F_{BD} = 133.5[\text{N}]$$

### 3. Moment of a force, equivalence and resultant of a force system

#### 3.1 Moment of a force

In the following we deal with the statics of rigid bodies. While a particle can have only translational (linear) motion a rigid body can also rotate. The rotating effect of a force is described by its **moment** (Figure 16, Equation 3.1).

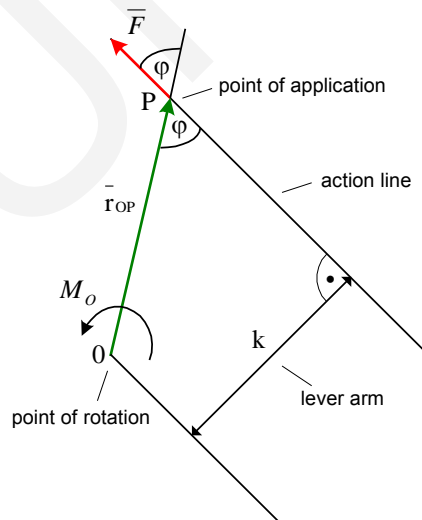


Figure 16 Moment of a force

$$\bar{M}_O = \bar{r}_{OP} \times \bar{F} \quad (3.1)$$

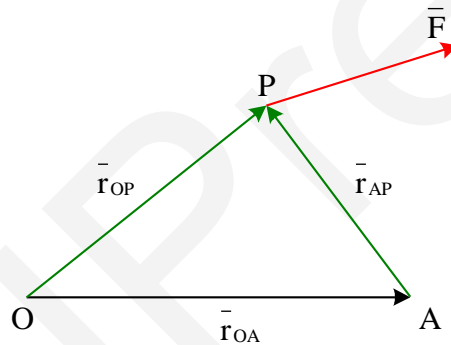
Thus the moment of force  $\vec{F}$  relative to point 0 is the cross product of the  $\vec{r}_{OP}$  position vector and the force. Thus moment is a vector quantity. The magnitude of moment  $\bar{M}_O$  is calculated with the following formula:

$$|\bar{M}_O| = \overbrace{|\vec{F}|}^F \cdot \overbrace{|\vec{r}_{OP}| \cdot \sin \varphi}^k = F \cdot k \quad (3.2)$$

where  $F$  is the magnitude of force  $\vec{F}$  and  $k$  is its **lever arm**. The lever arm is defined as the distance between the point of rotation and the **action line** of force  $\vec{F}$ .

**Connection between moments of a force relative to different points:**

On the basis of *Figure 17* we can find connection between the moments of force  $\vec{F}$  relative to points 0 and A (*Equation 3.3*).



*Figure 17 Moments of force  $\vec{F}$  relative to different points*

$$\bar{M}_A = \vec{r}_{AP} \times \vec{F} = (\vec{r}_{OP} - \vec{r}_{OA}) \times \vec{F} = \vec{r}_{OP} \times \vec{F} - \vec{r}_{OA} \times \vec{F} = \bar{M}_O + \vec{r}_{AO} \times \vec{F} \quad (3.3)$$

**3.2 Resultant force and moment of a force system**

The **resultant force** was defined earlier as:

$$\vec{F} = \sum_i \vec{F}_i = \vec{0}$$

The definition of the **resultant moment** of a force system relative to point 0 is:

$$\bar{M}_O = \sum_i \bar{M}_{O_i} = \sum_i (\bar{r}_{O_i} \times \bar{F}_i) \quad (3.4)$$

where  $\bar{r}_{O_i}$  is the position vector which points from the point of rotation (O) to the point of application ( $P_i$ ) of force  $\bar{F}_i$ .

**Connection between the resultant moments of a force system relative to different points:**

Let's write the resultant moment of a force system relative to point A applying Equation 3.3 for each force  $\bar{F}_i$  of the force system.

$$\bar{M}_A = \sum_i \bar{M}_{A_i} = \sum_i (\bar{M}_{O_i} + \bar{r}_{A O} \times \bar{F}_i) = \sum_i \bar{M}_{O_i} + \sum_i (\bar{r}_{A O} \times \bar{F}_i) = \overbrace{\sum_i \bar{M}_{O_i}}^{\bar{M}_O} + \bar{r}_{A O} \times \overbrace{\sum_i \bar{F}_i}^{\bar{F}} \quad (3.5)$$

Finally we get Equation 3.6:

$$\bar{M}_A = \bar{M}_O + \bar{r}_{A O} \times \bar{F} \quad (3.6)$$

Thus in case of a force system we got the same formula as in case of a single force.

### 3.3 Equivalence and resultant of force systems

Two force systems are **equivalent** if their resultant moments are equal to each other relative to any point A of space.

$$\bar{M}_A' = \bar{M}_A'' \text{ relative to any point A} \quad (3.7)$$

In Equation 3.7 the ' and '' signs refer to force system 1 and 2.

*Theorem 3.1:*

Two force systems are equivalent if their resultant forces and moments are equal to each other relative to a given point O of space.

$$\bar{F}' = \bar{F}'' \text{ and } \bar{M}_O' = \bar{M}_O'' \rightarrow \text{Force system ' and '' are equivalent} \quad (3.7)$$

*Proof:*

$$\bar{M}_A' = \bar{M}_O' + \bar{r}_{A O} \times \bar{F}', \quad \bar{M}_A'' = \bar{M}_O'' + \bar{r}_{A O} \times \bar{F}''$$

If  $\bar{M}_O' = \bar{M}_O''$  and  $\bar{F}' = \bar{F}''$  then from the above equations it comes that  $\bar{M}_A' = \bar{M}_A''$  relative to any point A. It means that force system ' and '' are equivalent.

*Theorem 3.2:*

Two force systems are equivalent if their resultant moments are equal to each other relative to three different non-collinear points of space.

*Remark:* Non-collinear means that the points are not on the same line

$$\bar{M}_A' = \bar{M}_A'', \bar{M}_B' = \bar{M}_B'', \bar{M}_C' = \bar{M}_C'',$$

$A \neq B \neq C$ , A, B and C are non-collinear points  $\rightarrow$

$\rightarrow$  Force system ' and '' are equivalent (3.8)

*Proof:*

$$\bar{M}_B' = \bar{M}_A' + \bar{r}_{BA} \times \bar{F}', \bar{M}_B'' = \bar{M}_A'' + \bar{r}_{BA} \times \bar{F}''$$

$$\bar{M}_C' = \bar{M}_A' + \bar{r}_{CA} \times \bar{F}', \bar{M}_C'' = \bar{M}_A'' + \bar{r}_{CA} \times \bar{F}''$$

$$\bar{M}_A' = \bar{M}_A'', \bar{M}_B' = \bar{M}_B'', \bar{M}_C' = \bar{M}_C'' \rightarrow \bar{r}_{BA} \times \bar{F}' = \bar{r}_{BA} \times \bar{F}'' \text{ and } \bar{r}_{CA} \times \bar{F}' = \bar{r}_{CA} \times \bar{F}''$$

$A \neq B \neq C$  and A, B and C are non-collinear  $\rightarrow \bar{r}_{BA} \neq \bar{0}$ ,  $\bar{r}_{CA} \neq \bar{0}$  and  $\bar{r}_{BA}$  is non-parallel to  $\bar{r}_{CA}$ .

We prove that  $\bar{F}' = \bar{F}''$ :

*Case I:*

If  $\bar{r}_{BA}$  is non-parallel to  $\bar{F}'$  then from equation  $\bar{r}_{BA} \times \bar{F}' = \bar{r}_{BA} \times \bar{F}''$  -regarding that  $\bar{r}_{BA} \neq \bar{0}$  - it comes that  $\bar{F}' = \bar{F}''$ .

*Case II:*

If  $\bar{r}_{BA}$  is parallel to  $\bar{F}'$  then  $\bar{r}_{CA}$  is non-parallel to it. (Since  $\bar{r}_{BA}$  is non-parallel to  $\bar{r}_{CA}$ ). Consequently from equation  $\bar{r}_{CA} \times \bar{F}' = \bar{r}_{CA} \times \bar{F}''$  - regarding that  $\bar{r}_{CA} \neq \bar{0}$  - it comes that  $\bar{F}' = \bar{F}''$ .

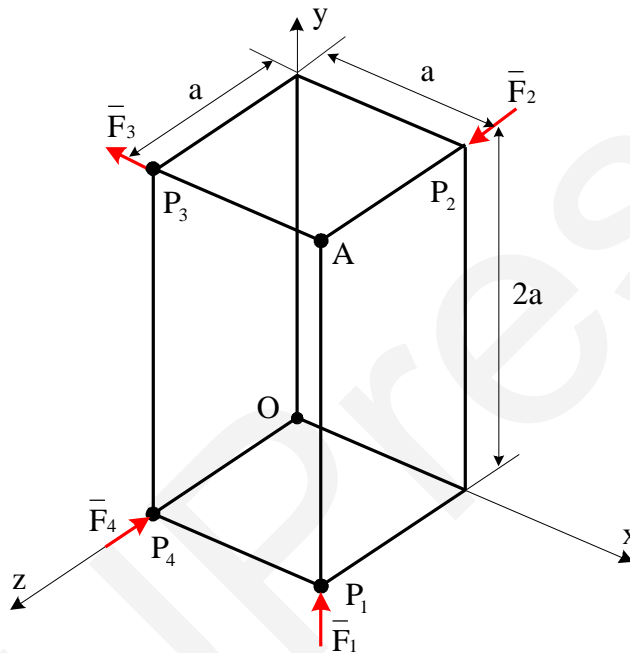
Thus it is proved that  $\bar{F}' = \bar{F}''$ .

Since  $\bar{F}' = \bar{F}''$  and  $\bar{M}_A' = \bar{M}_A''$  from *Theorem 3.1* it comes that the two force systems are equivalent.

On the basis of *Theorem 3.1* and *3.2* we can check whether two force systems are equivalent or not.

The **resultant of a force system** is defined as the simplest force system which is equivalent to it.

### 3.4 Numerical problems



*Data:*  $F_1 = 5[\text{N}]$ ,  $F_2 = 5[\text{N}]$ ,  $F_3 = 10[\text{N}]$ ,  $F_4 = 10[\text{N}]$ ,  $a = 2[\text{m}]$ .

a) Calculate the resultant force and resultant moment of the force system relative to point O.

b) Calculate the resultant moment of the force system relative to point A using the formula:

$$\vec{M}_A = \vec{M}_O + \vec{r}_{AO} \times \vec{F}$$

c) Calculate the resultant moment of the force system relative to point A on the basis of the definition.

*Solution:*

a)

$$\bar{F}_1 = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} [N]; \quad \bar{r}_{OP_1} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} [m]; \quad \rightarrow \quad \bar{M}_{1O} = \bar{r}_{OP_1} \times \bar{F}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \\ 10 \end{pmatrix} [Nm]$$

$$\bar{F}_2 = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} [N]; \quad \bar{r}_{OP_2} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} [m]; \quad \rightarrow \quad \bar{M}_{2O} = \bar{r}_{OP_2} \times \bar{F}_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 20 \\ -10 \\ 0 \end{pmatrix} [Nm]$$

$$\bar{F}_3 = \begin{pmatrix} -10 \\ 0 \\ 0 \end{pmatrix} [N]; \quad \bar{r}_{OP_3} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} [m]; \quad \rightarrow \quad \bar{M}_{3O} = \bar{r}_{OP_3} \times \bar{F}_3 = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} -10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -20 \\ 40 \end{pmatrix} [Nm]$$

$$\bar{F}_4 = \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} [N]; \quad \bar{r}_{OP_4} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} [m]; \quad \rightarrow \quad \bar{M}_{4O} = \bar{r}_{OP_4} \times \bar{F}_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} [Nm]$$

$$\bar{F} = \sum_{i=1}^4 \bar{F}_i = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} -10 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} [N]$$

$$\bar{M}_O = \sum_{i=1}^4 \bar{M}_{Oi} = \begin{pmatrix} -10 \\ 0 \\ 10 \end{pmatrix} + \begin{pmatrix} 20 \\ -10 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -20 \\ 40 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -30 \\ 50 \end{pmatrix} [Nm]$$

b)

$$\bar{M}_A = \bar{M}_O + \bar{r}_{AO} \times \bar{F} = \begin{pmatrix} 10 \\ -30 \\ 50 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 40 \\ -20 \\ 0 \end{pmatrix} [Nm]$$

c)

$$\bar{M}_A = \overbrace{\bar{r}_{AP_1} \times \bar{F}_1}^{\bar{0}} + \overbrace{\bar{r}_{AP_2} \times \bar{F}_2}^{\bar{0}} + \overbrace{\bar{r}_{AP_3} \times \bar{F}_3}^{\bar{0}} + \bar{r}_{AP_4} \times \bar{F}_4 = \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 40 \\ -20 \\ 0 \end{pmatrix} [Nm]$$

## 4. Classification of force systems

### 4.1 Couple and wrench

The force systems are classified on the basis of their resultant force and moment. Before performing the classification, we have to define the concept of a **couple** and **wrench**. A **couple** is a pair of forces with the same magnitude and direction but with opposite sense.



Figure 18 Couple

*Theorem 4.1:*

The resultant moment of a couple has the same value relative to any point A of space. (In other words: The resultant moment of a couple is independent of the point of rotation.)

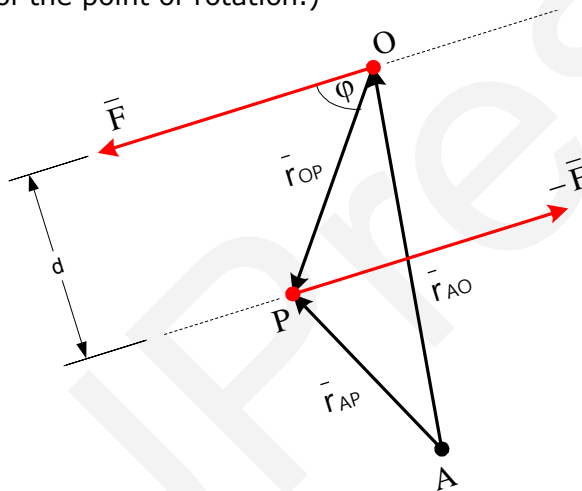


Figure 19 Couple

*Proof:*

We prove that the resultant moment of the couple in *Figure 19* relative to any point A has the same value as relative to point O. The resultant moment of the couple relative to point O is as follows:

$$\bar{M}_O = \overbrace{\bar{r}_{OO}}^{\bar{0}} \times \bar{F} + \bar{r}_{OP} \times (-\bar{F}) = \bar{r}_{OP} \times (-\bar{F}) \quad (4.1)$$

The resultant moment of the couple relative to point A:

$$\bar{M}_A = \bar{r}_{AO} \times \bar{F} + \bar{r}_{AP} \times (-\bar{F}) = (\bar{r}_{AP} - \bar{r}_{AO}) \times (-\bar{F}) = \bar{r}_{OP} \times (-\bar{F}) \quad (4.2)$$

Thus:

$$\bar{M}_A = \bar{M}_O, \quad \text{relative to any point A.} \quad (4.3)$$

The magnitude of the resultant moment of the couple can be calculated as:

$$|\bar{M}_O| = |\bar{r}_{OP} \times (-\bar{F})| = \overbrace{|\bar{F}|}^F \cdot \overbrace{|\bar{r}_{OP}| \cdot \sin \varphi}^d = F \cdot d \quad (4.4)$$

where  $d$  is the distance between the action lines of  $\bar{F}$  and  $-\bar{F}$ . The resultant moment of a couple is called as a **torque**. The torque of a couple is always normal to the plane of the couple (*Figure 20*).

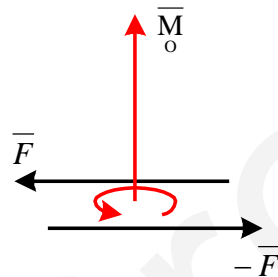


Figure 20 Torque

A force and a couple whose torque is parallel to the action line of the force constitute a **wrench** (*Figure 21*).

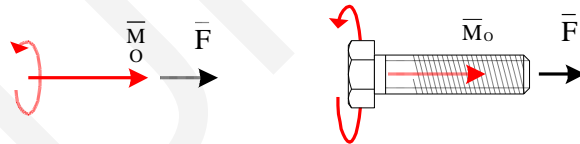


Figure 21 Wrench

Now we are going to state and prove a general theorem which gives the classification of force systems. Let's denote the resultant force and moment of the force system relative to point  $O$  with  $\bar{F}$  and  $\bar{M}_O$ .

## 4.2 Classes of force systems

*Theorem 4.2:*

All the force systems can be classified into the following three classes:

1. If  $\bar{F} = \bar{0}$  then the resultant of the force system is a couple with  $\bar{M}_O$  torque.

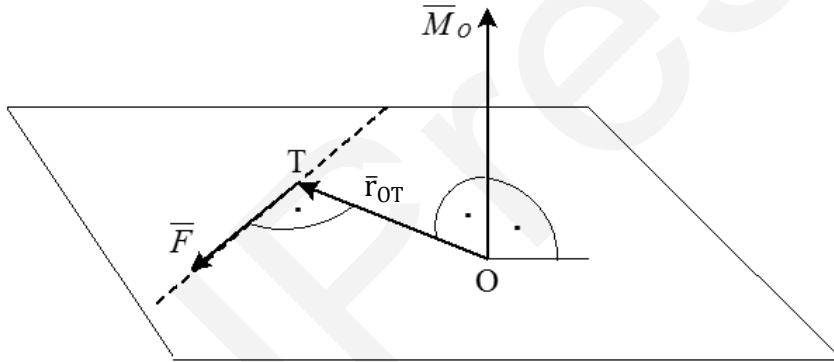
2. If  $\bar{F} \neq \bar{0}$  and  $\bar{M}_O$  is perpendicular to  $\bar{F}$  then the resultant of the force system is  $\bar{F}$  on a fixed action line in space
3. If  $\bar{F} \neq \bar{0}$  and  $\bar{M}_O$  is non-perpendicular to  $\bar{F}$  then the resultant of the force system is a wrench

*Proof:*

1. Let's apply formula (3.6):

$\bar{M}_A = \bar{M}_O + \bar{r}_{AO} \times \bar{F} = \bar{M}_O \rightarrow \bar{M}_A = \bar{M}_O$  relative to any point A  $\rightarrow$  The resultant of the force system is a couple with  $\bar{M}_O$  torque.

2. Let's fix a plane which passing through point O and normal to  $\bar{M}_O$ . Since  $\bar{F}$  is perpendicular to  $\bar{M}_O$  it can be represented in the above plane. In *Figure 22* point T is denotes the orthogonal projection of point O on the action line of force  $\bar{F}$ .



*Figure 22*

Since  $\bar{M}_O$  is normal to the plane of  $\bar{r}_{OT}$  and  $\bar{F}$  it can be calculated as the moment of resultant force  $\bar{F}$  relative to point O (*Equation 4.5*).

$$\bar{M}_O = \bar{r}_{OT} \times \bar{F} \quad (4.5)$$

In the above equation vector  $\bar{r}_{OT}$  assigns the position of the action line of force  $\bar{F}$ .

Let's express  $\bar{r}_{OT}$  from the above equation:

$$\bar{F} \times \bar{M}_O = \bar{F} \times (\bar{r}_{OT} \times \bar{F}) = \bar{r}_{OT} \cdot \overbrace{(\bar{F} \cdot \bar{F})}^{F^2} - \bar{F} \cdot \overbrace{(\bar{F} \cdot \bar{r}_{OT})}^0 = \bar{r}_{OT} \cdot F^2$$

From the above equation the  $\bar{r}_{OT}$  position vector:

$$\bar{r}_{OT} = \frac{\bar{F} \times \bar{M}_O}{F^2} \quad (4.6)$$

From *Theorem 3.1* it comes, that applying on the action line which is fixed by the  $\vec{r}_{OT}$  position vector in *Equation 4.6*, resultant force  $\vec{F}$  is the resultant of the force system.

- Let's decompose resultant moment  $\vec{M}_O$  into two components. One of the components ( $\vec{M}_{O//}$ ) is parallel to resultant force  $\vec{F}$  the other one ( $\vec{M}_{O\perp}$ ) is perpendicular to it (*Figure 23*).

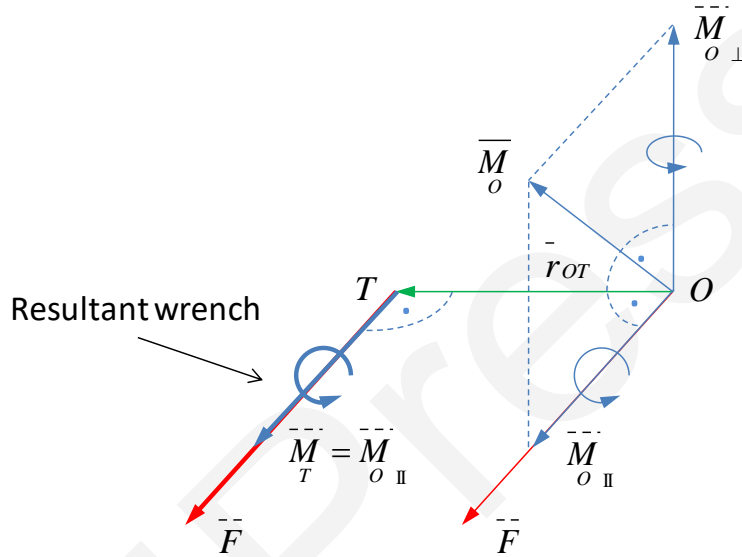


Figure 23

If we define point T in the same way as in case 2 then the formulas below are valid:

$$\vec{M}_{O\perp} = \vec{r}_{OT} \times \vec{F} \rightarrow \vec{r}_{OT} = \frac{\vec{F} \times \vec{M}_{O\perp}}{F^2} = \frac{\vec{F} \times \vec{M}_{O\perp} + \vec{F} \times \vec{M}_{O//}}{F^2} = \frac{\vec{F} \times (\vec{M}_{O\perp} + \vec{M}_{O//})}{F^2} = \frac{\vec{F} \times \vec{M}_O}{F^2}$$

The resultant moment of the force system at point T:

$$\vec{M}_T = \vec{M}_O + \vec{r}_{TO} \times \vec{F} = \vec{M}_{O//} + \vec{M}_{O\perp} + \vec{r}_{TO} \times \vec{F} = \vec{M}_{O//} + \overbrace{\vec{M}_{O\perp} - \vec{r}_{OT} \times \vec{F}}^{\vec{0}} = \vec{M}_{O//}$$

Thus, the resultant wrench at point T:

$$(\vec{F}, \vec{M}_T) = (\vec{F}, \vec{M}_{O//})$$

The position of point T is given by the following formula:

$$\vec{r}_{OT} = \frac{\vec{F} \times \vec{M}_O}{F^2}$$

### 4.3 Numerical problems

Let's determine the resultant wrench of the force system in section 3.4. The resultant force and moment of the above force system was calculated previously relative to point  $O$ :

$$\bar{F} = \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} [\text{N}], \quad \bar{M}_O = \begin{pmatrix} 10 \\ -30 \\ 50 \end{pmatrix} [\text{Nm}].$$

The unit vector in the direction of resultant force  $\bar{F}$ :

$$\bar{e}_F = \frac{\bar{F}}{F} = \frac{\begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix}}{\sqrt{(-10)^2 + (5)^2 + (-5)^2}} = \begin{pmatrix} -0.816 \\ 0.408 \\ -0.408 \end{pmatrix}$$

The components of resultant moment  $\bar{M}_O$ :

$$\begin{aligned} \bar{M}_{O//} &= (\bar{M}_O \cdot \bar{e}_F) \cdot \bar{e}_F = \overbrace{(10 \cdot (-0.816) + (-30) \cdot 0.408 + 50 \cdot (-0.408))}^{-40.8} \cdot \begin{pmatrix} -0.816 \\ 0.408 \\ -0.408 \end{pmatrix} \\ &= \begin{pmatrix} 33.29 \\ -16.64 \\ 16.64 \end{pmatrix} [\text{Nm}] \end{aligned}$$

$$\bar{M}_{O\perp} = \bar{M}_O - \bar{M}_{O//} = \begin{pmatrix} 10 \\ -30 \\ 50 \end{pmatrix} - \begin{pmatrix} 33.29 \\ -16.64 \\ 16.64 \end{pmatrix} = \begin{pmatrix} -23.29 \\ -13.36 \\ 33.36 \end{pmatrix} [\text{Nm}]$$

The position of the resultant wrench:

$$\bar{r}_{OT} = \frac{\bar{F} \times \bar{M}_O}{F^2} = \frac{\begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} \times \begin{pmatrix} 10 \\ -30 \\ 50 \end{pmatrix}}{150} = \begin{pmatrix} 0.667 \\ 3 \\ 1.667 \end{pmatrix} [\text{m}]$$

The resultant wrench:

$$(\bar{F}, \bar{M}_T) = (\bar{F}, \bar{M}_{O//}) = \left( \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} [\text{N}], \begin{pmatrix} 33.28 \\ -16.64 \\ 16.64 \end{pmatrix} [\text{Nm}] \right)$$

The resultant moment as a scalar moment about the action line of resultant force  $\bar{F}$ :

$$M_T = M_{O//} = \bar{e}_F \cdot \bar{M}_{O//} = \begin{pmatrix} -0.816 \\ 0.408 \\ -0.408 \end{pmatrix} \cdot \begin{pmatrix} 33.28 \\ -16.64 \\ 16.64 \end{pmatrix} = -40.73 [\text{Nm}]$$

Thus, the resultant wrench:

$$(\bar{F}, M_T) = \left( \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} [\text{N}], -40.73 [\text{Nm}] \right)$$

## 5. Coplanar force systems

### 5.1 Classification of coplanar force systems

A force system is called as coplanar if all the forces of the system are in the same plane (S).

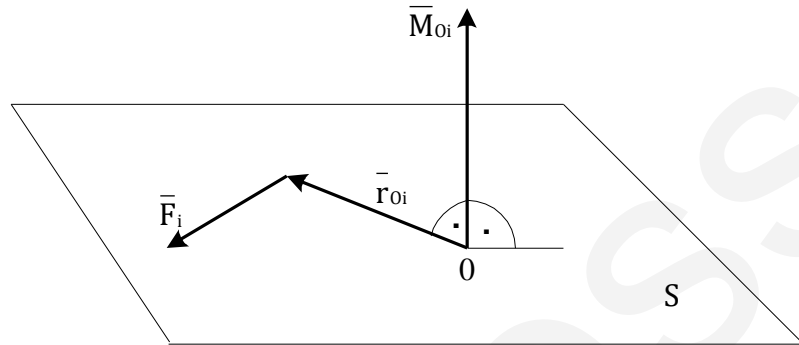


Figure 24

*Theorem 5.1:*

The resultant force and moment of a coplanar force system are perpendicular to each other.

*Proof:*

Let's denote the resultant force and moment of the coplanar force system relative to point O with  $\bar{F}$  and  $\bar{M}_O$  respectively ( $O \in S$ ).

$$\bar{r}_{O_i}, \bar{F}_i \in S \text{ and } \bar{M}_{O_i} = \bar{r}_{O_i} \times \bar{F}_i \rightarrow \bar{M}_{O_i} \text{ is normal to } S \rightarrow \bar{M}_O = \sum_i \bar{M}_{O_i} \text{ is also normal to } S.$$

Consequently:

$$\bar{F} = \sum_i \bar{F}_i \in S \text{ and } \bar{M}_O \text{ is normal to } S \rightarrow \bar{M}_O \text{ is perpendicular to } \bar{F}.$$

*Theorem 5.2:*

All the coplanar force systems can be classified into the following two classes:

1. If  $\bar{F} = \bar{0}$  then the resultant of the force system is a couple with  $\bar{M}_O$  torque.
2. If  $\bar{F} \neq \bar{0}$  then the resultant of the force system is  $\bar{F}$  applied on an action line which going through point T defined by formula (5.1).

$$\bar{r}_{OT} = \frac{\bar{F} \times \bar{M}_O}{F^2} \quad (5.1)$$

*Proof:*

1. It comes from *Theorem 4.2/1* directly.
2. It comes from *Theorems 5.1 and 4.2/2* directly.

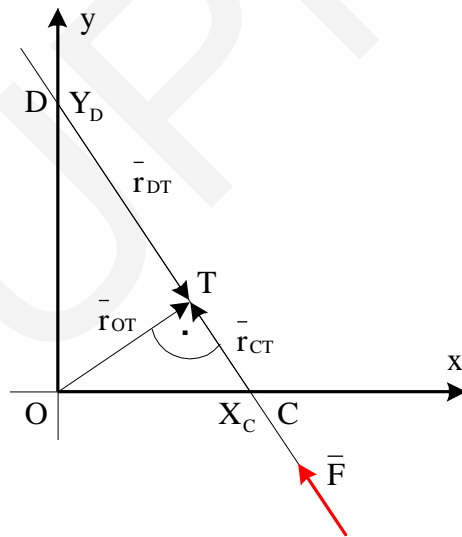
*Theorem 5.3:*

If the resultant force and moment of a coplanar force system are given as:

$$\bar{F} = \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix}, \quad \bar{M}_O = \begin{pmatrix} 0 \\ 0 \\ M_O \end{pmatrix}$$

then the intersections of the action line of resultant force  $\bar{F}$  with the x and y axes are calculated as (*Figure 25*):

$$x_C = \frac{M_O}{F_y} \quad y_D = -\frac{M_O}{F_x} \quad (5.2 \text{ and } 5.3)$$



*Figure 25*

*Proof:*

$$\bar{r}_{OT} = \frac{\bar{F} \times \bar{M}_O}{F^2} = \frac{\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ F_x & F_y & 0 \\ 0 & 0 & M_O \end{vmatrix}}{F^2} = \frac{\bar{i} \cdot F_y \cdot M_O - \bar{j} \cdot F_x \cdot M_O}{F^2} = \begin{pmatrix} \frac{F_y}{F^2} \cdot M_O \\ -\frac{F_x}{F^2} \cdot M_O \\ 0 \end{pmatrix}$$

Thus the  $\bar{r}_{OT}$ ,  $\bar{r}_{CT}$  and  $\bar{r}_{DT}$  vectors can be written as:

$$\bar{r}_{OT} = \begin{pmatrix} x_{OT} \\ y_{OT} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{F_y}{F^2} \cdot M_O \\ -\frac{F_x}{F^2} \cdot M_O \\ 0 \end{pmatrix}, \quad \bar{r}_{CT} = \begin{pmatrix} x_{OT} - x_C \\ y_{OT} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{F_y}{F^2} \cdot M_O - x_C \\ -\frac{F_x}{F^2} \cdot M_O \\ 0 \end{pmatrix},$$

$$\bar{r}_{DT} = \begin{pmatrix} x_{OT} \\ y_{OT} - y_D \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{F_y}{F^2} \cdot M_O \\ -\frac{F_x}{F^2} \cdot M_O - y_D \\ 0 \end{pmatrix}$$

Since  $\bar{r}_{OT}$  is perpendicular to  $\bar{r}_{CT}$ :

$$\bar{r}_{OT} \cdot \bar{r}_{CT} = \begin{pmatrix} \frac{F_y}{F^2} \cdot M_O \\ -\frac{F_x}{F^2} \cdot M_O \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{F_y}{F^2} \cdot M_O - x_C \\ -\frac{F_x}{F^2} \cdot M_O \\ 0 \end{pmatrix} = 0$$

$$\frac{F_y}{F^2} \cdot M_O \cdot \left( \frac{F_y}{F^2} \cdot M_O - x_C \right) + \left( -\frac{F_x}{F^2} \cdot M_O \right) \cdot \left( -\frac{F_x}{F^2} \cdot M_O \right) = 0$$

$$\frac{F_y^2}{F^4} \cdot M_O^2 + \frac{F_x^2}{F^4} \cdot M_O^2 = \frac{F_y}{F^2} \cdot M_O \cdot x_C$$

$$M_O^2 \cdot \frac{\overbrace{F_x^2 + F_y^2}^{F^2}}{F^4} = \frac{F_y}{F^2} \cdot M_O \cdot x_C$$

$$M_O = F_y \cdot x_C \rightarrow x_C = \frac{M_O}{F_y}$$

Since  $\bar{r}_{OT}$  and  $\bar{r}_{DT}$  are also perpendicular to each other in a same way we get:

$$y_D = -\frac{M_O}{F_x}$$

## 5.2 Scalar moment of a force

The **scalar moment** describes the rotating effect of a force about an axis (*Figure 26*). If the axis is given by unit vector  $\bar{e}$  then the scalar moment is calculated as:

$$M_{Oe} = \bar{M}_O \cdot \bar{e} = |\bar{M}_O| \cdot \overset{1}{|\bar{e}|} \cdot \cos \theta = |\bar{M}_O| \cdot \cos \theta \quad (5.4)$$

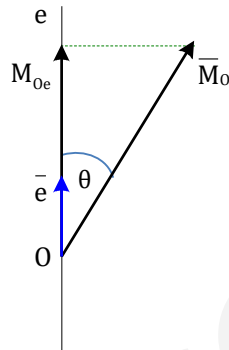


Figure 26

where  $\bar{M}_O$  is the vector moment of the force relative to point  $O$ . Thus, the scalar moment is the orthogonal projection of the vector moment onto axis  $e$ .

In case of a coplanar force system in the  $xy$  plane the scalar moment of force  $\bar{F}_i$  is defined as its scalar moment about the  $z$  axis. Thus the scalar moment of  $\bar{F}_i$  is calculated as (*Figure 27*):

$$\begin{aligned} M_{Oz} &= \bar{M}_{Oz} \cdot \bar{k} = (\bar{r}_{Oz} \times \bar{F}_i) \cdot \bar{k} = \left( \begin{pmatrix} x_i \\ y_i \\ 0 \end{pmatrix} \times \begin{pmatrix} F_{xi} \\ F_{yi} \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F_{yi} \cdot x_i - F_{xi} \cdot y_i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= F_{iy} \cdot x_i - F_{ix} \cdot y_i \end{aligned}$$

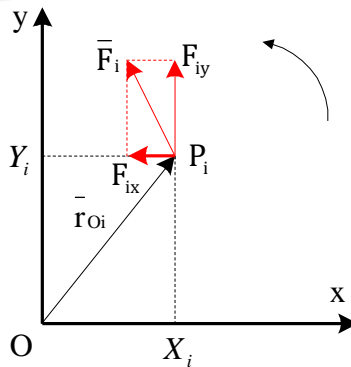


Figure 27

If we define the positive sense of rotation as counter-clockwise (*Figure 28*) then  $F_{iy} \cdot x_i$  is the scalar moment of the  $F_{iy}$ , while  $-F_{ix} \cdot y_i$  is the scalar moment of the  $F_{ix}$  component.

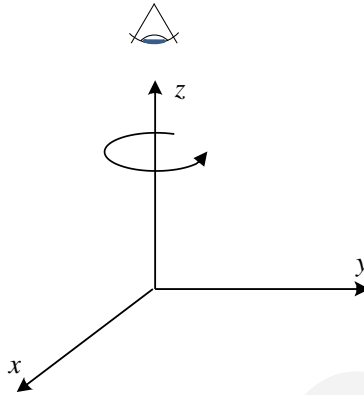


Figure 28

Thus the scalar moment of  $\bar{F}_i$  can be calculated as the sum of the moments of its  $F_{ix}$  and  $F_{iy}$  components.

### 5.3 Construction of the resultant of a coplanar force system

In the following we give method for the construction of the resultant of a coplanar force system. As we proved it in chapter 5.1, the resultant of a coplanar force system can be:

- 1) a couple with  $\bar{M}_0$  torque, if  $\bar{F} = \bar{0}$
- 2)  $\bar{F}$  applied on a given action line – defined by formula 5.1 – if  $\bar{F} \neq \bar{0}$

( $\bar{F}$  and  $\bar{M}_0$  denote the resultant force and moment of the force system.)

Thus, the result of the construction can be a couple or resultant force  $\bar{F}$  on a given action line. The basis of the construction method is *Theorem 5.3*.

*Theorem 5.3:*

Let  $\bar{F}_1$  and  $\bar{F}_2$  be non-zero, coplanar forces with intersecting action lines (*Figure 28*). The resultant of  $\bar{F}_1$  and  $\bar{F}_2$  is force  $\bar{F}$  if:

1.  $\bar{F} = \bar{F}_1 + \bar{F}_2$
2.  $\bar{F}$  goes through the intersection point of the action lines of  $\bar{F}_1$  and  $\bar{F}_2$

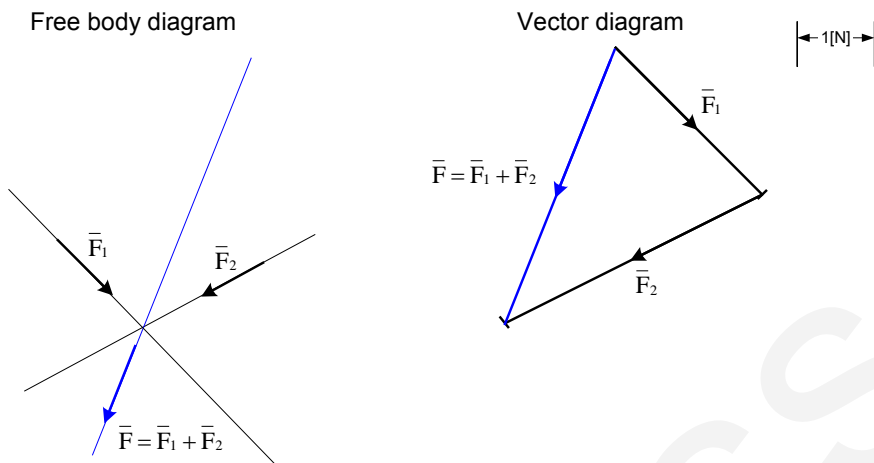


Figure 29

*Proof:*

$\bar{F} = \bar{F}_1 + \bar{F}_2 \neq \bar{0}$ , thus the resultant is  $\bar{F}$  applied on a given action line (*Theorem 5.2*). Since the resultant moment of  $\bar{F}_1$  and  $\bar{F}_2$  is zero relative to the intersection point of their action lines, resultant force  $\bar{F}$  has to go through this point too.

As it is shown in *Figure 29* resultant force  $\bar{F}$  and its action line are constructed separately. The diagram for the construction of  $\bar{F}$  is called as **vector diagram**, while the one for the construction of its action line is as **free body diagram**. For the accurate drawing of vector diagram we need to fix a force scale (unit). In *Figure 29* 1 [cm] corresponds to 1 [N], thus the magnitude of  $\bar{F}_1$  and  $\bar{F}_2$  are approximately 2.7 and 3.7 cm respectively.

In case of three or more forces (*Figure 30*) the resultant of  $\bar{F}_1$  and  $\bar{F}_2$  has to be constructed first, and after that the resultant of  $\bar{F}_1 + \bar{F}_2$  and  $\bar{F}_3$ , then the resultant of  $\bar{F}_1 + \bar{F}_2 + \bar{F}_3$  and  $\bar{F}_4$  etc.

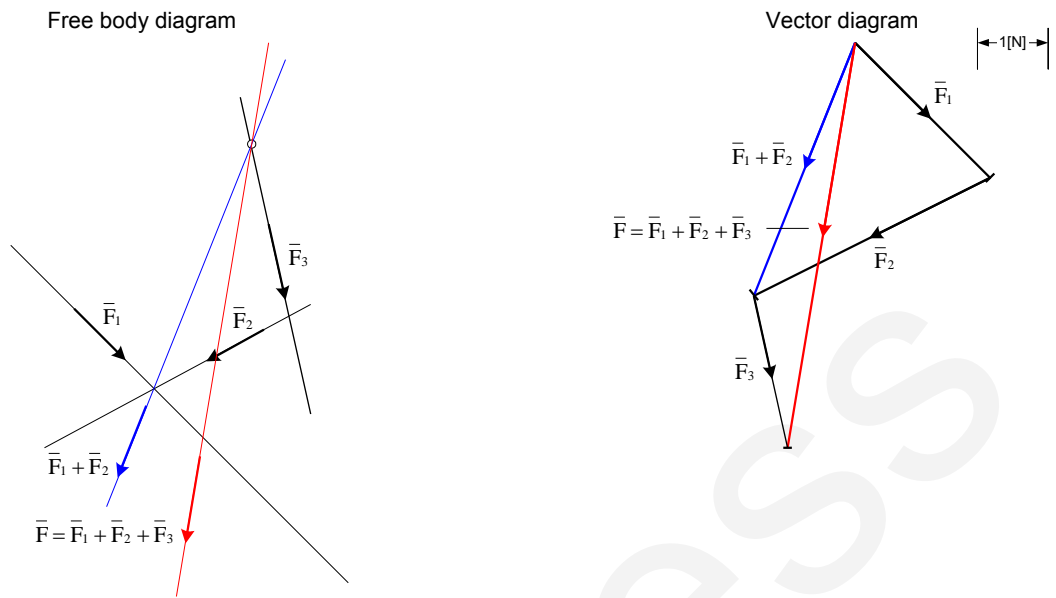


Figure 30

If the action lines of all the forces are parallel to each other we cannot apply the above presented construction method directly. In that case we have to add two fictive forces ( $\vec{s}_1$  and  $\vec{s}_2$ ) – with the same action line and magnitude but with opposite sense – to the original force system (Figure 31). It can be easily seen that the addition of these fictive forces doesn't vary the resultant force and moment, thus the resultant of the original force system.

Free body diagram

Vector diagram

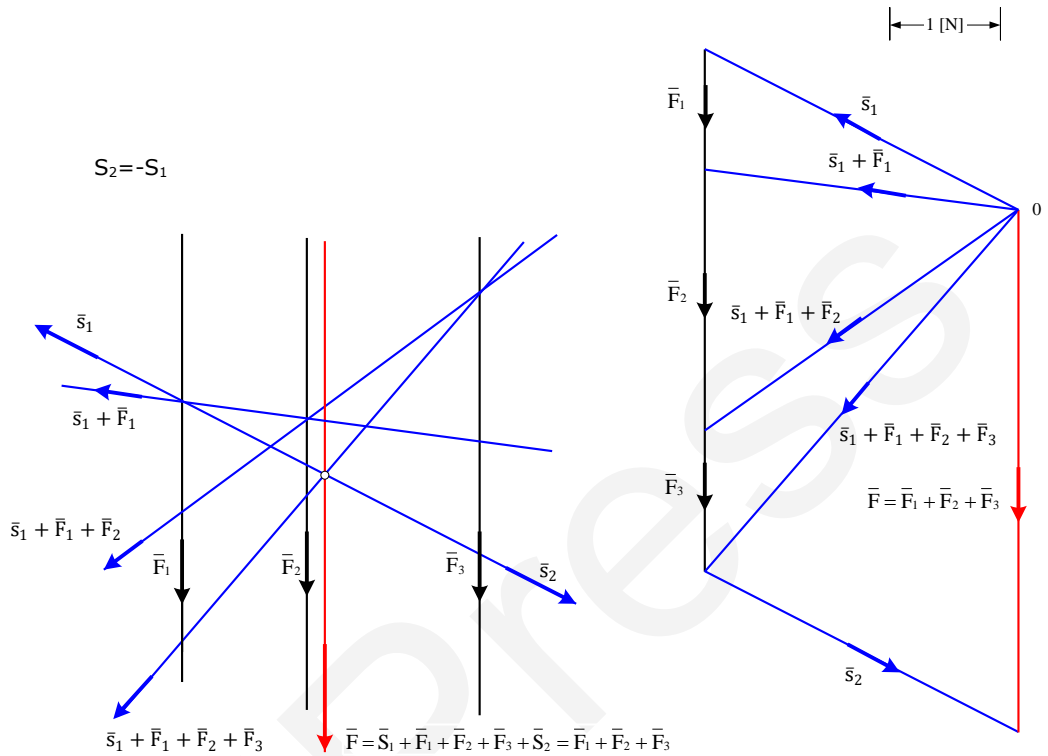


Figure 31

In engineering practice simplified notations are applied; instead of  $\vec{s}_1$ ,  $\vec{s}_1 + \vec{F}_1$ ,  $\vec{s}_1 + \vec{F}_1 + \vec{F}_2$ , and  $\vec{s}_1 + \vec{F}_1 + \vec{F}_2 + \vec{F}_3$  arabic numbers 1,2,3 and 4 are written. Furthermore,  $\vec{s}_2$  and  $\vec{F}$  are not presented in the vector diagram. Figure 32 shows the construction in Figure 31 applying simplified notations.

Free body diagram

Vector diagram

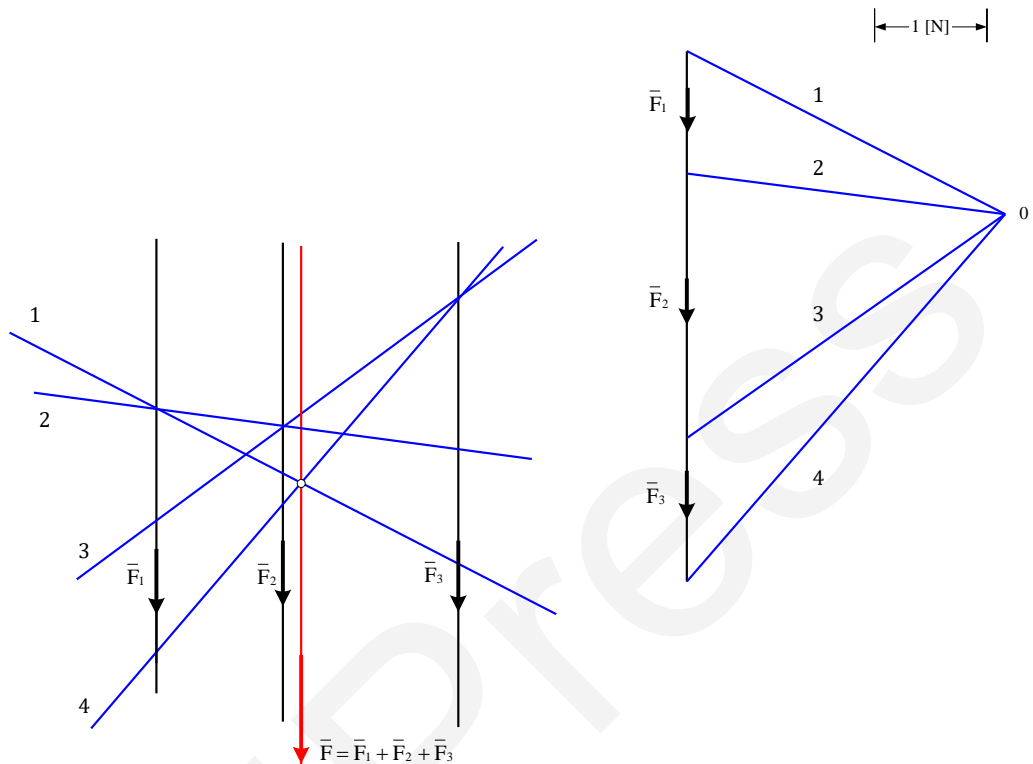


Figure 32

In case of a parallel force system the steps of the construction are as follows:

1. Fixing a force scale (unit) and drawing forces  $\bar{F}_1, \bar{F}_2, \bar{F}_3 \dots$
2. Fixing point O and drawing lines 1,2,3,4,... in the vector diagram (The position of point O is optional, the sequence of lines have to follow the sequence of forces)
3. Drawing the action lines of forces  $\bar{F}_1, \bar{F}_2, \bar{F}_3 \dots$  in the free body diagram
4. Drawing line 1 in the free body diagram. The position of line 1 is optional.
5. Drawing line 2 through the intersection of line 1 and the action line of  $\bar{F}_1$
6. Drawing line 3 through the intersection of line 2 and the action line of  $\bar{F}_2$
7. Drawing line 4 through the intersection of line 3 and the action line of  $\bar{F}_3$
8. Drawing the action line of resultant force  $\bar{F}$  through the intersection of line 4 and line 1

As we mentioned it before if  $\bar{F} = \bar{0}$  the resultant is a couple. *Figure 33 and 34* shows the results of the construction if  $\bar{F} = \bar{0}$ ,  $\bar{M}_0 \neq \bar{0}$  and  $\bar{F} = \bar{0}$ ,  $\bar{M}_0 = \bar{0}$  respectively. In both cases the resultant is a couple with non-zero and zero torque respectively.

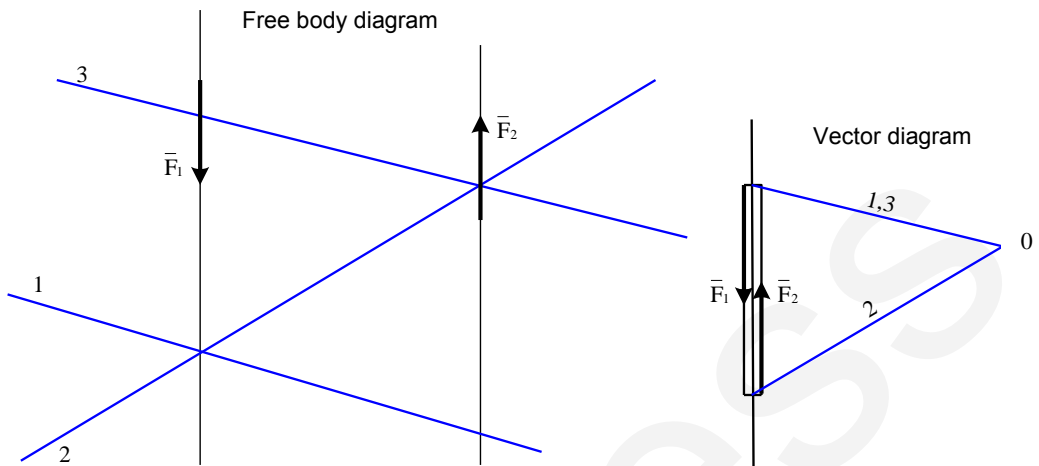


Figure 33

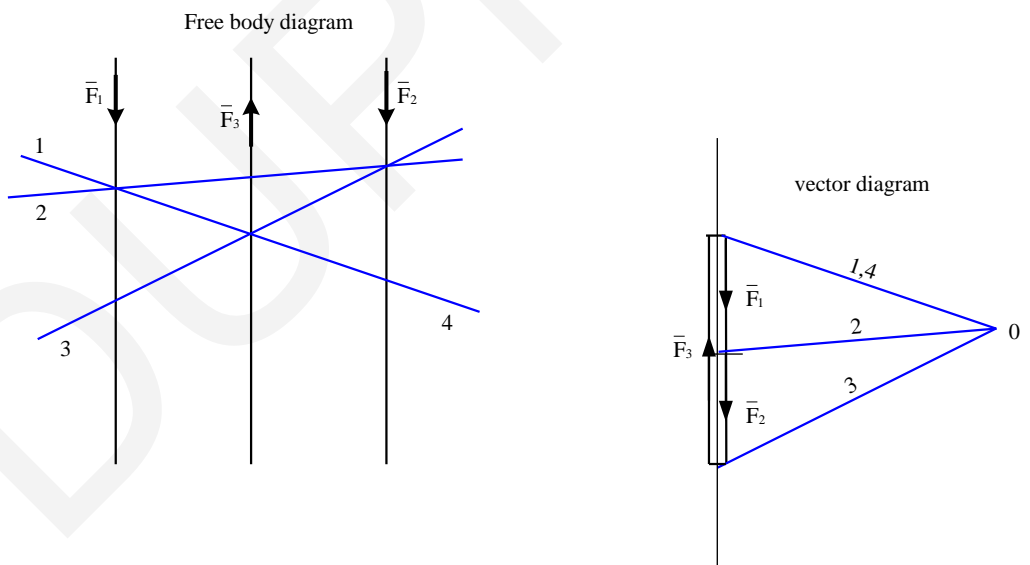


Figure 34

Observing *Figure 33* ( $\bar{F} = \bar{0}$ ,  $\bar{M}_0 \neq \bar{0}$ ) we can realise that line 1 and 4 are the same in the vector diagram and they are parallel to each other in the free body diagram. In case of *Figure 34* ( $\bar{F} = \bar{0}$ ,  $\bar{M}_0 = \bar{0}$ ) line 1 and 4 are the same in both diagrams. The above observation in *Figure 34* is very important

when constructing unknown forces in static equilibrium, so, we state it as a theorem:

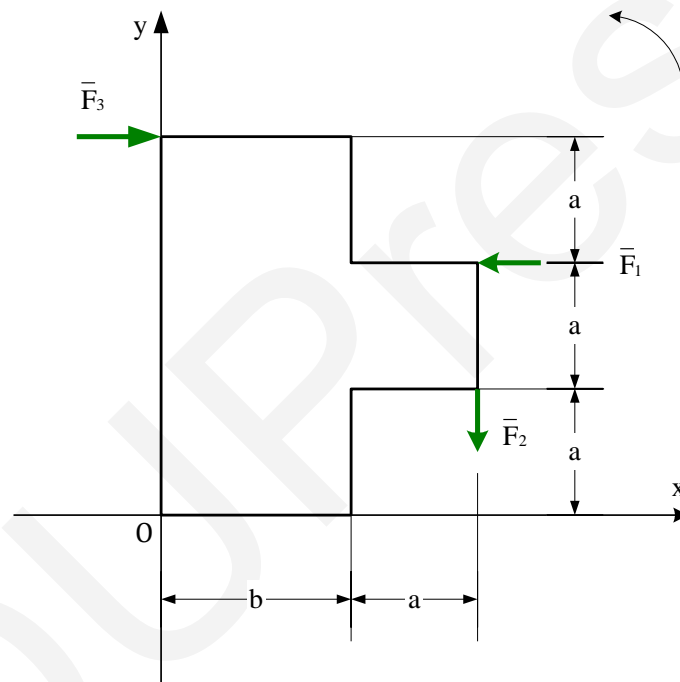
*Theorem 5.4:*

If both the resultant force and moment of a coplanar force system are equals zero then the first and last construction lines are the same in both the vector and free body diagrams.

## 5.4 Numerical problems

*Problem 1:*

Three forces act on the rigid plate in the figure. The dimensions of the plate and the magnitude of the forces are given below.



*Data:*  $F_1 = 4[N]$ ,  $F_2 = 5[N]$ ,  $F_3 = 10[N]$ ,  $a = 2[m]$ ,  $b = 3[m]$ .

- Calculate the resultant force and moment (scalar and vector) of the force system relative to point O.
- Calculate the intersections of the action line of the resultant force with the x and y axes.
- Construct the resultant of the force system.

*Solution:*

a)

$$\vec{F}_1 = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} [N], \vec{F}_2 = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} [N], \vec{F}_3 = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} [N]$$

Resultant force:

$$\vec{F} = \sum_{i=1}^3 \vec{F}_i = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 0 \end{pmatrix} [N]$$

Resultant scalar moment relative to point O:

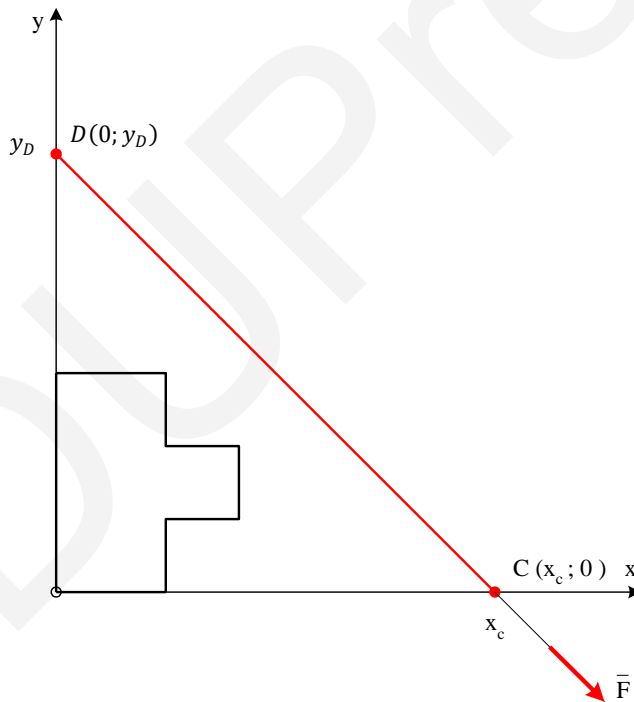
$$M_O = F_1 \cdot 2a - F_2 \cdot (a + b) - F_3 \cdot 3a = 4 \cdot 2 \cdot 2 - 5(2 + 3) - 10 \cdot 3 \cdot 2 = -69 [\text{Nm}]$$

Resultant vector moment relative to point O:

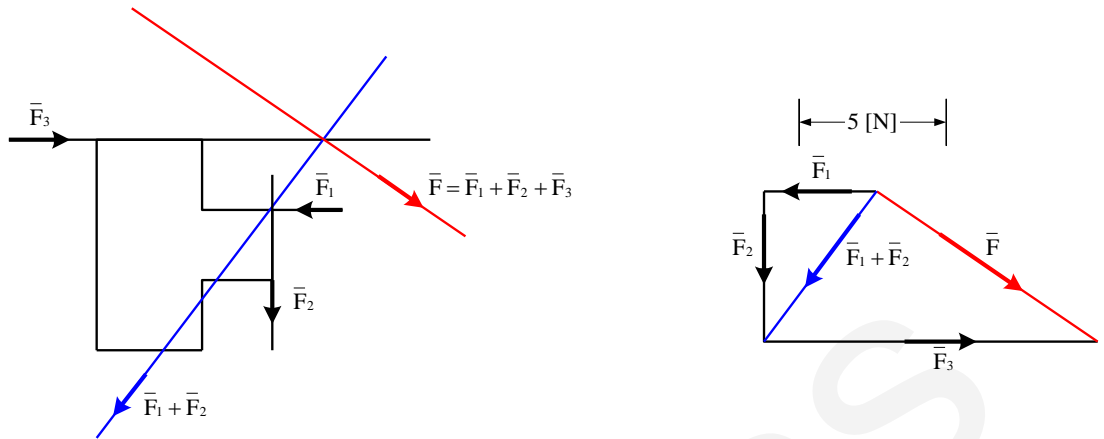
$$\vec{M}_O = M_O \cdot \vec{k} = \begin{pmatrix} 0 \\ 0 \\ -69 \end{pmatrix} [\text{Nm}]$$

b)

$$x_c = \frac{M_O}{F_y} = \frac{-69}{-5} = 13.8 [\text{m}], \quad y_D = -\frac{M_O}{F_x} = -\frac{-69}{6} = 11.5 [\text{m}]$$

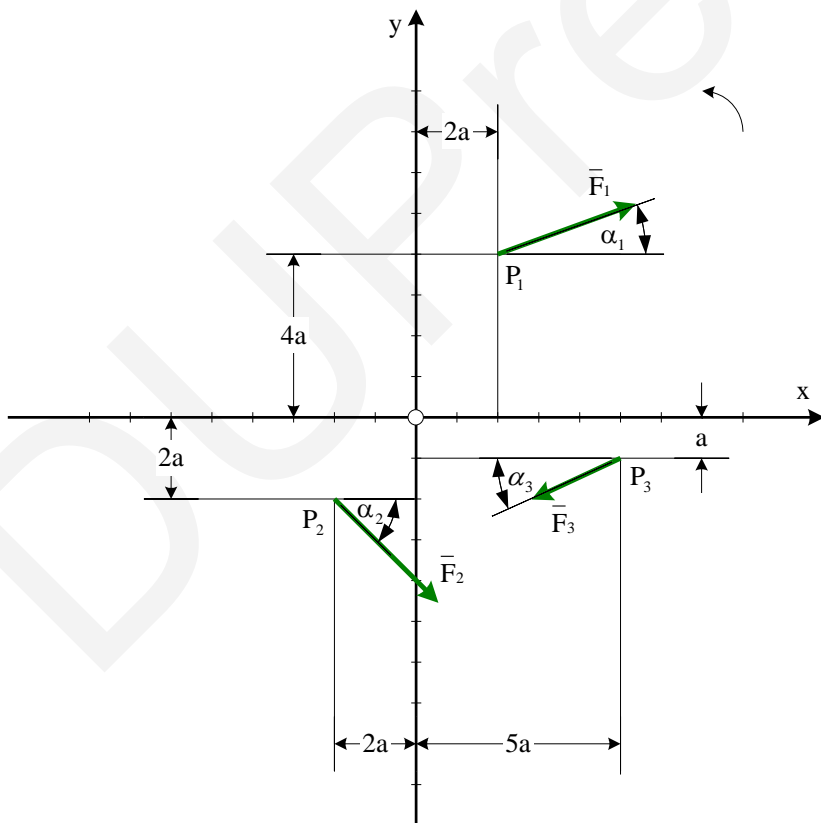


c)



**Problem 2:**

Three forces and their points of application are given in the figure below.



**Data:**  $F_1 = 3[N]$ ,  $\alpha_1 = 20^\circ$ ,  $F_2 = 3[N]$ ,  $\alpha_2 = 45^\circ$ ,  $F_3 = 2[N]$ ,  $\alpha_3 = 25^\circ$ ,  $a = 0.1[m]$ .

- Calculate the resultant force and moment (scalar and vector) of the force system relative to point O.
- Calculate the intersections of the action line of the resultant force with the x and y axes.
- Construct the resultant of the force system.

*Solution:*

a)

$$\vec{F}_1 = \begin{pmatrix} F_1 \cdot \cos \alpha_1 \\ F_1 \cdot \sin \alpha_1 \end{pmatrix} = \begin{pmatrix} 2.819 \\ 1.026 \end{pmatrix} [N]$$

$$\vec{F}_2 = \begin{pmatrix} F_2 \cdot \cos \alpha_2 \\ -F_2 \cdot \sin \alpha_2 \end{pmatrix} = \begin{pmatrix} 2.121 \\ -2.121 \end{pmatrix} [N]$$

$$\vec{F}_3 = \begin{pmatrix} -F_3 \cdot \cos \alpha_3 \\ -F_3 \cdot \sin \alpha_3 \end{pmatrix} = \begin{pmatrix} -1.813 \\ -0.845 \end{pmatrix} [N]$$

Resultant force:

$$\vec{F} = \sum_{i=1}^3 \vec{F}_i = \begin{pmatrix} 2.819 \\ 1.026 \end{pmatrix} + \begin{pmatrix} 2.121 \\ -2.121 \end{pmatrix} + \begin{pmatrix} -1.813 \\ -0.845 \end{pmatrix} = \begin{pmatrix} 3.127 \\ -1.940 \end{pmatrix} [N]$$

Resultant scalar moment relative to point O:

$$M_O = -F_1 \cdot \cos \alpha_1 \cdot 4a + F_1 \cdot \sin \alpha_1 \cdot 2a + F_2 \cdot \cos \alpha_2 \cdot 2a + F_2 \cdot \sin \alpha_2 \cdot 2a - F_3 \cdot \cos \alpha_3 \cdot a - F_3 \cdot \sin \alpha_3 \cdot 5a = -0.678 [\text{Nm}]$$

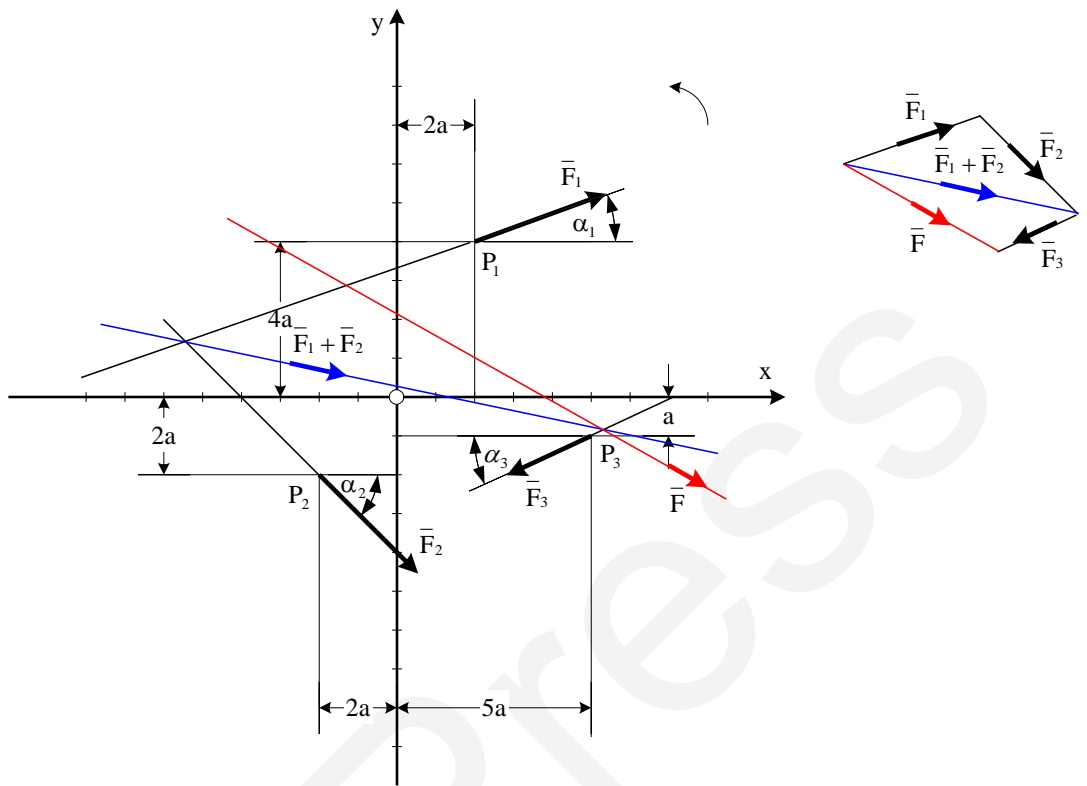
Resultant vector moment relative to point O:

$$\vec{M}_O = M_O \cdot \vec{k} = \begin{pmatrix} 0 \\ 0 \\ -0.678 \end{pmatrix} [\text{Nm}]$$

b)

$$x_c = \frac{M_O}{F_y} = \frac{-0.678}{-1.940} = 0.349 [\text{m}], \quad y_D = -\frac{M_O}{F_x} = -\frac{-0.678}{3.127} = 0.217 [\text{m}]$$

c)



## 6. Resultant of a homogeneous gravitational force system. Centre of gravity

### 6.1 Resultant of a homogeneous gravitational force system

In this section formulas are derived for the resultant force and moment of a homogeneous gravitation force system and then we give its resultant. Let's divide the body in *Figure 35* into parts which dimensions are negligible compared to the dimensions of the body. The mass of part  $i$  is denoted by  $m_i$ . In case of a gravitational force system the resultant force is traditionally denoted by  $\bar{G}$ .

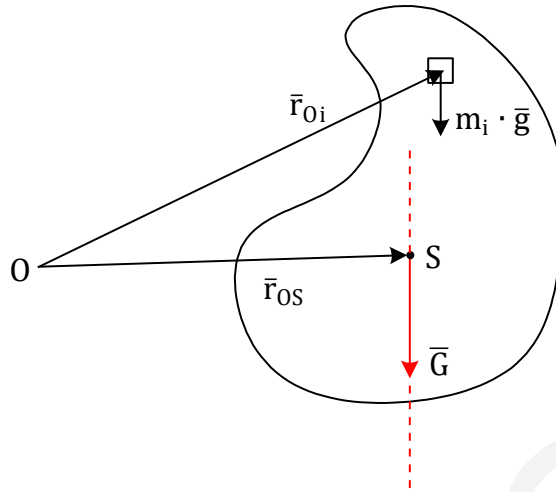


Figure 35

On the basis of *Figure 35* the resultant force and moment of a gravitational force system are as follows:

$$\bar{G} = \sum_i (m_i \cdot \bar{g}) = \overbrace{(\sum_i m_i)}^m \cdot \bar{g} = m \cdot \bar{g} \quad (6.1)$$

$$\bar{M}_O = \sum_i \bar{M}_{O_i} = \sum_i (\bar{r}_{O_i} \times m_i \cdot \bar{g}) = \sum_i (m_i \cdot \bar{r}_{O_i} \times \bar{g}) = (\sum_i m_i \cdot \bar{r}_{O_i}) \times \bar{g} \quad (6.2)$$

From the above formula it comes that  $\bar{M}_O$  is perpendicular to  $\bar{g}$  thus it is also perpendicular to  $\bar{G}$ . Since  $\bar{G} \neq \bar{0}$  and  $\bar{M}_O$  is perpendicular to  $\bar{G}$  from *Theorem 4.2* it comes that the resultant is  $\bar{G}$  applied on a fixed action line in space.

## 6.2 Centre of gravity

*Figure 36* represents the body in three different positions. To the different positions belong different action lines which intersect each other at the same point (S). Point S is called as the **centre of gravity**.

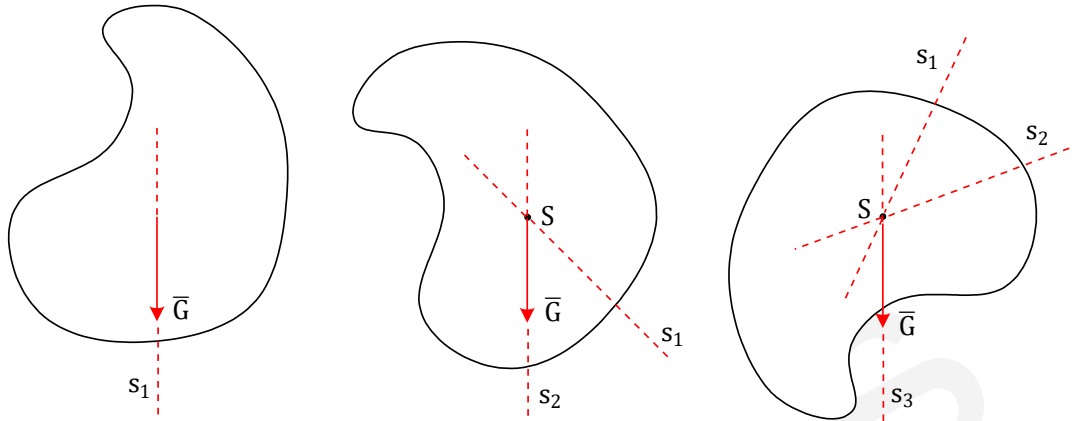


Figure 36

The resultant moment of a homogeneous gravitational force system can be written in two different forms as follows (see Figure 34 and Equation 6.2):

$$\begin{aligned} \text{I.} \quad & \bar{M}_O = (\sum_i m_i \cdot \bar{r}_{Oi}) \times \bar{g} \\ \text{II.} \quad & \bar{M}_O = \bar{r}_{OS} \times \bar{G} = \bar{r}_{OS} \times m \cdot \bar{g} = \bar{r}_{OS} \times (\sum_i m_i) \cdot \bar{g} = (\sum_i m_i) \cdot \bar{r}_{OS} \times \bar{g} \quad (6.3) \end{aligned}$$

Comparing the two equations:

$$(\sum_i m_i) \cdot \bar{r}_{OS} \times \bar{g} = (\sum_i m_i \cdot \bar{r}_{Oi}) \times \bar{g} \quad (6.4)$$

Expressing  $\bar{r}_{OS}$  from the above equation we get the formula for the position of the centre of gravity:

$$\bar{r}_{OS} = \frac{\sum_i m_i \cdot \bar{r}_{Oi}}{\sum_i m_i} \quad (6.5)$$

Expressing the mass of each part with its volume and density:

$$\bar{r}_{OS} = \frac{\sum_i \rho_i \cdot V_i \cdot \bar{r}_{Oi}}{\sum_i \rho_i \cdot V_i} \quad (6.6)$$

If the mass distribution of the body is homogeneous ( $\rho_i = \rho = \text{constant}$ ):

$$\bar{r}_{OS} = \frac{\sum_i \rho \cdot V_i \cdot \bar{r}_{Oi}}{\sum_i \rho \cdot V_i} = \frac{\rho \cdot \sum_i V_i \cdot \bar{r}_{Oi}}{\rho \cdot \sum_i V_i} = \frac{\sum_i V_i \cdot \bar{r}_{Oi}}{\sum_i V_i} \quad (6.7)$$

If the body is a homogeneous plate with constant  $d$  thickness (mathematically it is a plane figure):

$$\bar{r}_{OS} = \frac{\sum_i d \cdot A_i \bar{r}_{Oi}}{\sum_i d \cdot A_i} = \frac{d \cdot \sum_i A_i \bar{r}_{Oi}}{d \cdot \sum_i A_i} = \frac{\sum_i A_i \bar{r}_{Oi}}{\sum_i A_i} \quad (6.8)$$

In the above equation  $\sum_i A_i = A$  is the total area of the plate. From Equation (6.8) the  $x_S$  and  $y_S$  coordinates of the  $\bar{r}_{OS}$  position vector are as follows:

$$x_S = \frac{\sum_i A_i x_i}{\sum_i A_i} \quad (6.9), \quad y_S = \frac{\sum_i A_i y_i}{\sum_i A_i} \quad (6.10)$$

If the plate is divided into pieces with infinitesimal areas, then formula (6.9) and (6.10) can be written as follows:

$$x_S = \frac{\int_A x dA}{\int_A dA} \quad (6.11), \quad y_S = \frac{\int_A y dA}{\int_A dA} \quad (6.12)$$

### 6.3 Centre of gravity of plane figures

In *Figure 37* we marked the position of the centre of gravity of basic plane figures. In case of the circle, rectangle and general triangle the position of the centre of gravity comes from basic geometry, while in case of the right angled triangle, sector and semi circle it is determined by integration calculus.

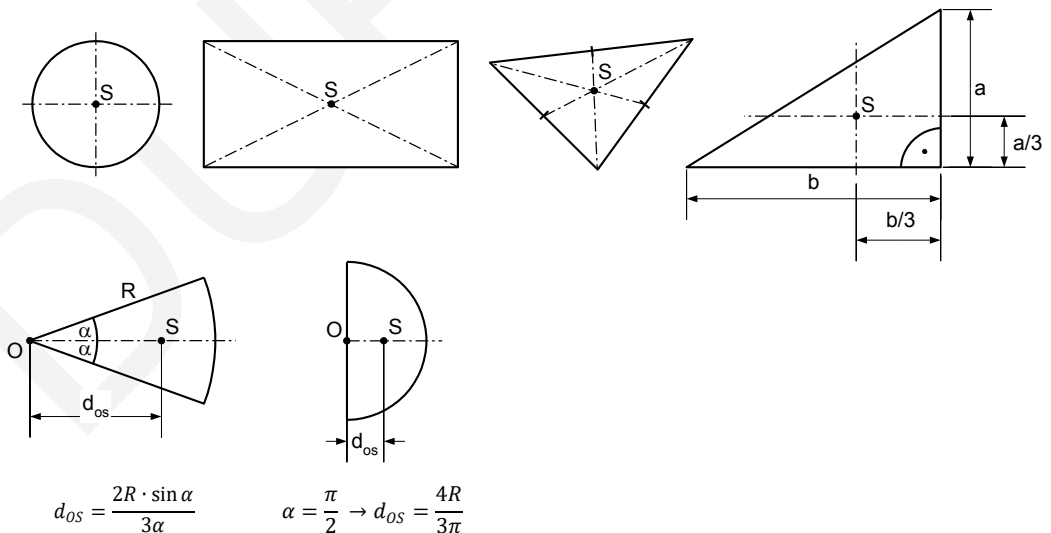
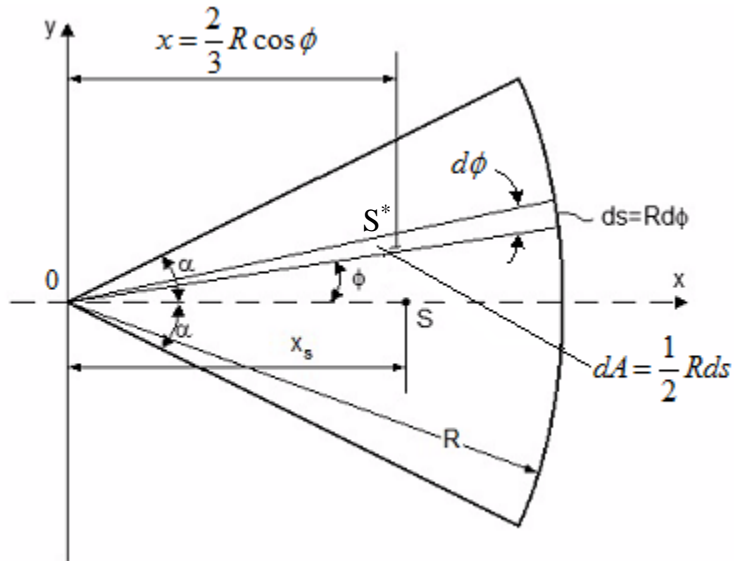


Figure 37

As an example a formula is derived for the position of the centre of gravity of the sector. Let the x axis be the symmetry axis of the sector as it is shown in *Figure 38*.



*Figure 38*

The area of the sector with central angle  $d\phi$  and radius  $R$  is calculated as:

$$dA = \frac{1}{2} R ds = \frac{1}{2} R^2 d\phi \quad (6.13)$$

The distance between the origin of the coordinate system and the centre of gravity ( $S^*$ ) of the above sector is  $\frac{2}{3}R$ . (It is true because the above small sector can be regarded as an isosceles triangle.)

Thus, the distance between the y axis and the centre of gravity ( $S^*$ ) of the above sector is:

$$x = \frac{2}{3} R \cdot \cos \phi \quad (6.14)$$

Thus, the centre of gravity of the sector with central angle  $2\alpha$  and radius  $R$  is calculated as (see Equation 6.11):

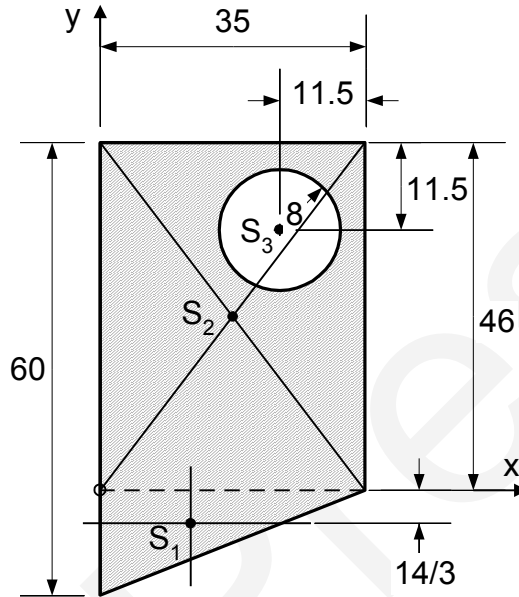
$$x_S = \frac{\int_A x dA}{\int_A dA} = \frac{\int_{-\alpha}^{\alpha} \frac{1}{3} R^3 \cdot \cos \phi d\phi}{\int_{-\alpha}^{\alpha} \frac{1}{2} R^2 d\phi} = \frac{2}{3} R \cdot \frac{\int_{-\alpha}^{\alpha} \cos \phi d\phi}{\int_{-\alpha}^{\alpha} d\phi} = \frac{2}{3} R \cdot \frac{2 \cdot \sin \alpha}{2\alpha} = \frac{2R \cdot \sin \alpha}{3\alpha}$$

Specially, the position of the centre of gravity of the semicircle ( $\alpha = \frac{\pi}{2}$ ) is:

$$x_S = \frac{2R \cdot \sin \frac{\pi}{2}}{3 \cdot \frac{\pi}{2}} = \frac{4R}{3\pi} \quad (6.15)$$

## 6.4 Numerical problems

Problem 1:



- Calculate the coordinates of the centre of gravity of the homogeneous plate in the figure.
- Check your results with construction.

*Solution:*

- Let's divide the plate into three parts. The centres of gravity of the right angled triangle, rectangle and circle is denoted as  $S_1$ ,  $S_2$  and  $S_3$  respectively. The circle is cut out of the plate, thus its area is negative. The following table contains the areas and coordinates of the above three parts.

i	$A_i$ [mm <sup>2</sup> ]	$x_{si}$ [mm]	$y_{si}$ [mm]
1	245	35/3	-14/3
2	1610	35/2	23
3	-201	23.5	34.5

From the above data coordinates  $x_S$  and  $y_S$  of the centre of gravity of the plate are calculated as:

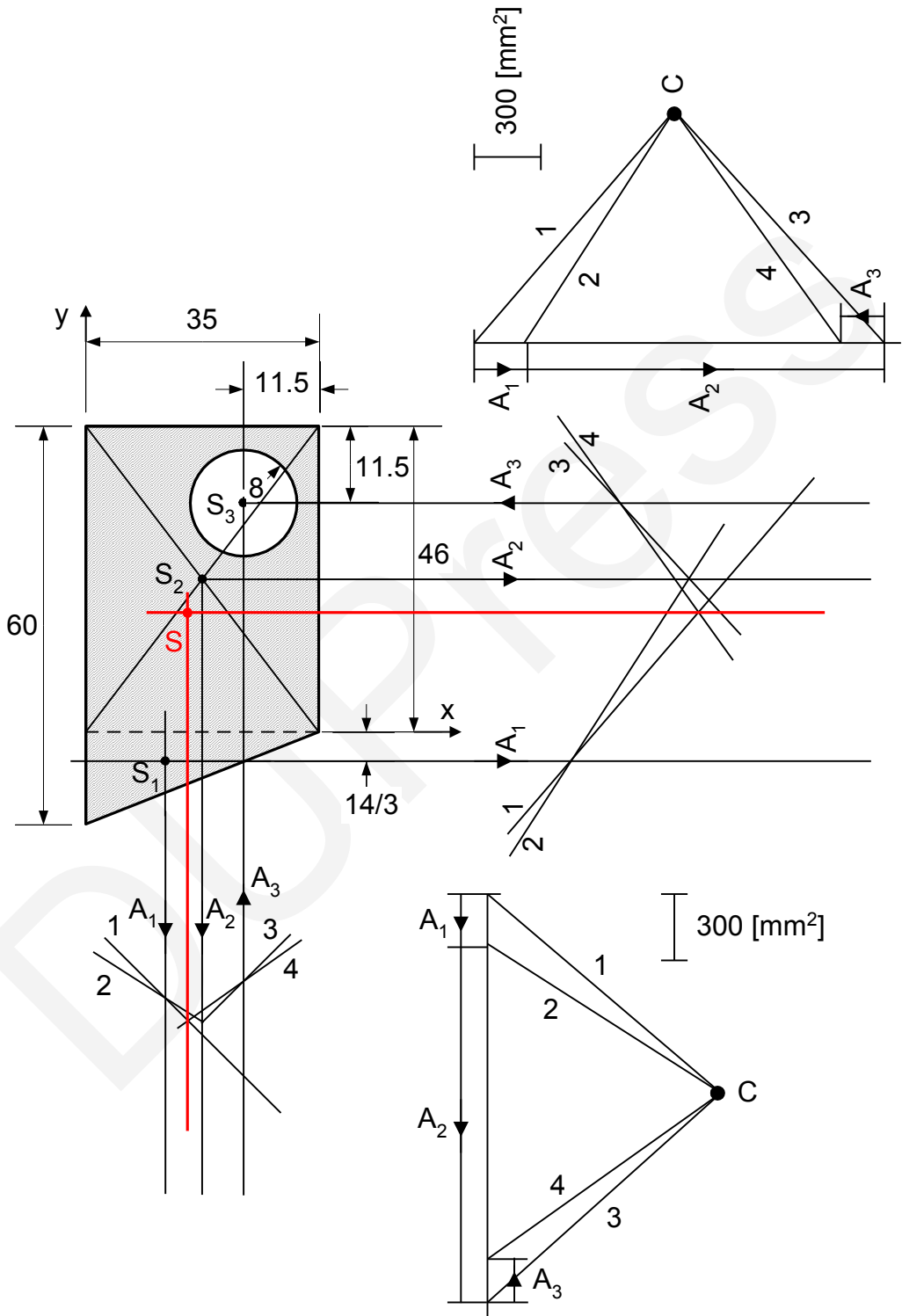
$$x_s = \frac{\sum_{i=1}^3 x_{si} \cdot A_i}{\sum_{i=1}^3 A_i} = \frac{x_{s1} \cdot A_1 + x_{s2} \cdot A_2 + x_{s3} \cdot A_3}{A_1 + A_2 + A_3} = \frac{\frac{35}{3} \cdot 245 + \frac{35}{2} \cdot 1610 - 23.5 \cdot 201}{245 + 1610 - 201}$$

$$= 16.77 \text{ [mm]}$$

$$y_s = \frac{\sum_{i=1}^3 y_{si} \cdot A_i}{\sum_{i=1}^3 A_i} = \frac{\frac{-14}{3} \cdot 245 + 23 \cdot 1610 - 34.5 \cdot 201}{245 + 1610 - 201} = 17.5 \text{ [mm]}$$

b)

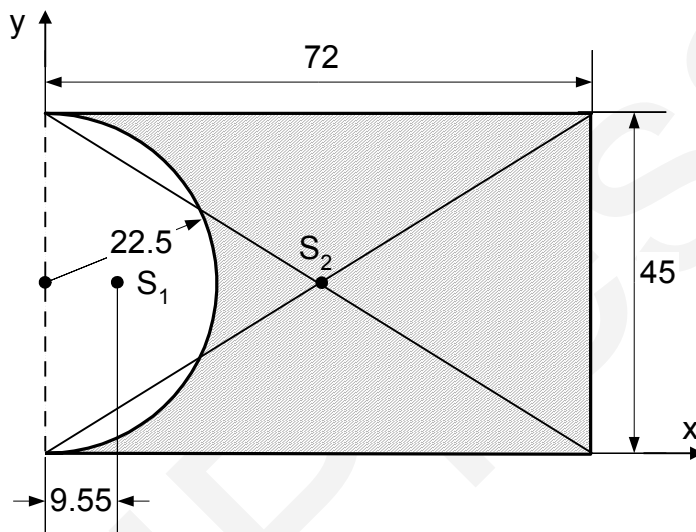
Since a gravitational force system is a parallel force system, we apply the method presented in *Figure 31* in Section 5.3 for the construction. Obviously, the construction have to be performed both in  $x$  and  $y$  direction, and instead of forces, the "area vectors" ( $\bar{A}_1, \bar{A}_2, \bar{A}_3$ ) of the different parts of the plate have to be used. The figure bellow shows the construction in details.



Clicking on the link below the construction above can be followed as a Power Point animation step by step:

Animation 6.1

Problem 2:



- Calculate the coordinates of the centre of gravity of the plate in the figure.
- Check your results with construction.

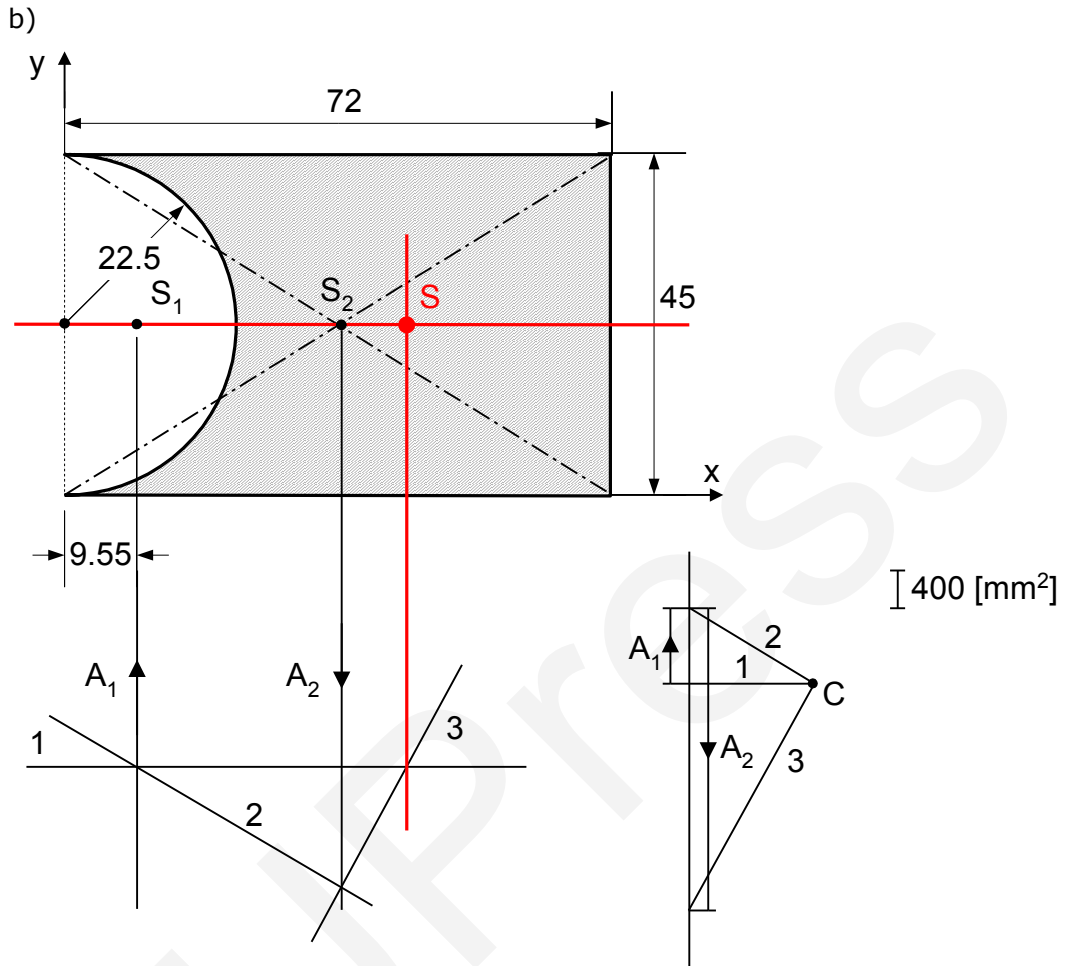
Solution:

a)

i	$A_i$ [mm <sup>2</sup> ]	$x_{si}$ [mm]	$y_{si}$ [mm]
1	-795.2	9.55	22.5
2	3240	36	22.5

$$x_s = \frac{\sum_{i=1}^2 x_{si} \cdot A_i}{\sum_{i=1}^2 A_i} = \frac{-9.55 \cdot 795.2 + 36 \cdot 3240}{-795.2 + 3240} = 44.6 \text{ [mm]}$$

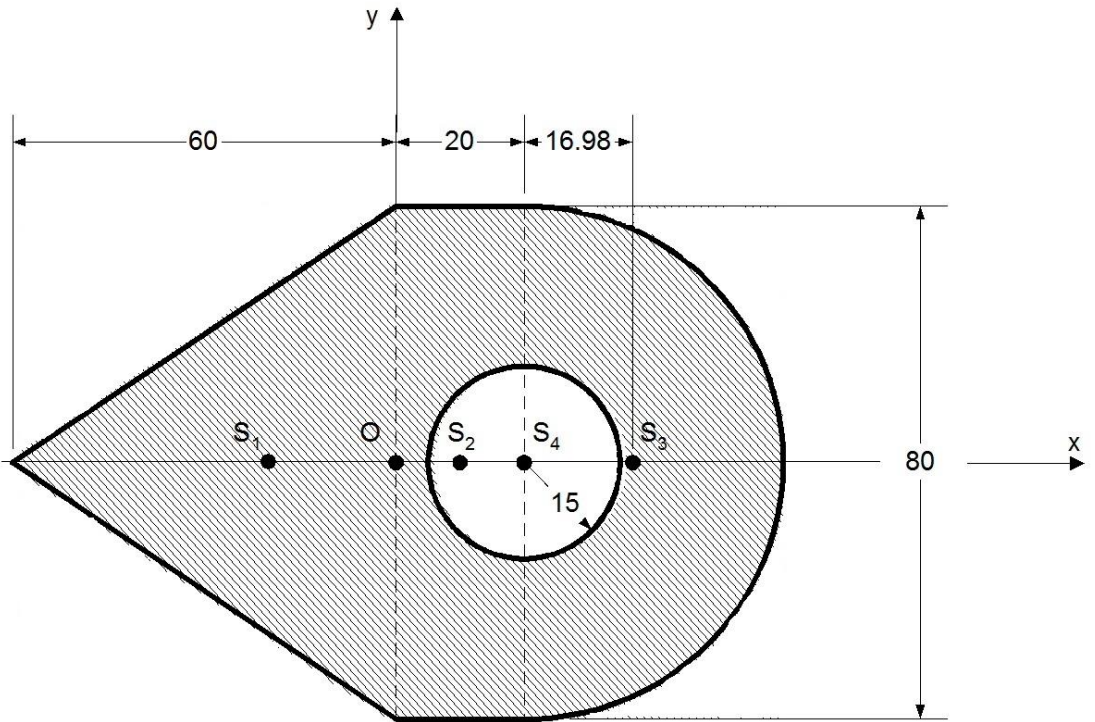
$$y_s = \frac{\sum_{i=1}^2 y_{si} \cdot A_i}{\sum_{i=1}^2 A_i} = \frac{-22.5 \cdot 795.2 + 22.5 \cdot 3240}{-795.2 + 3240} = 22.5 \text{ [mm]}$$



The Power Point animation of the construction is presented below:

Animation 6.2

Problem 3:



- Calculate the coordinates of the centre of gravity of the plate in the figure.
- Check your results with construction.

Solution:

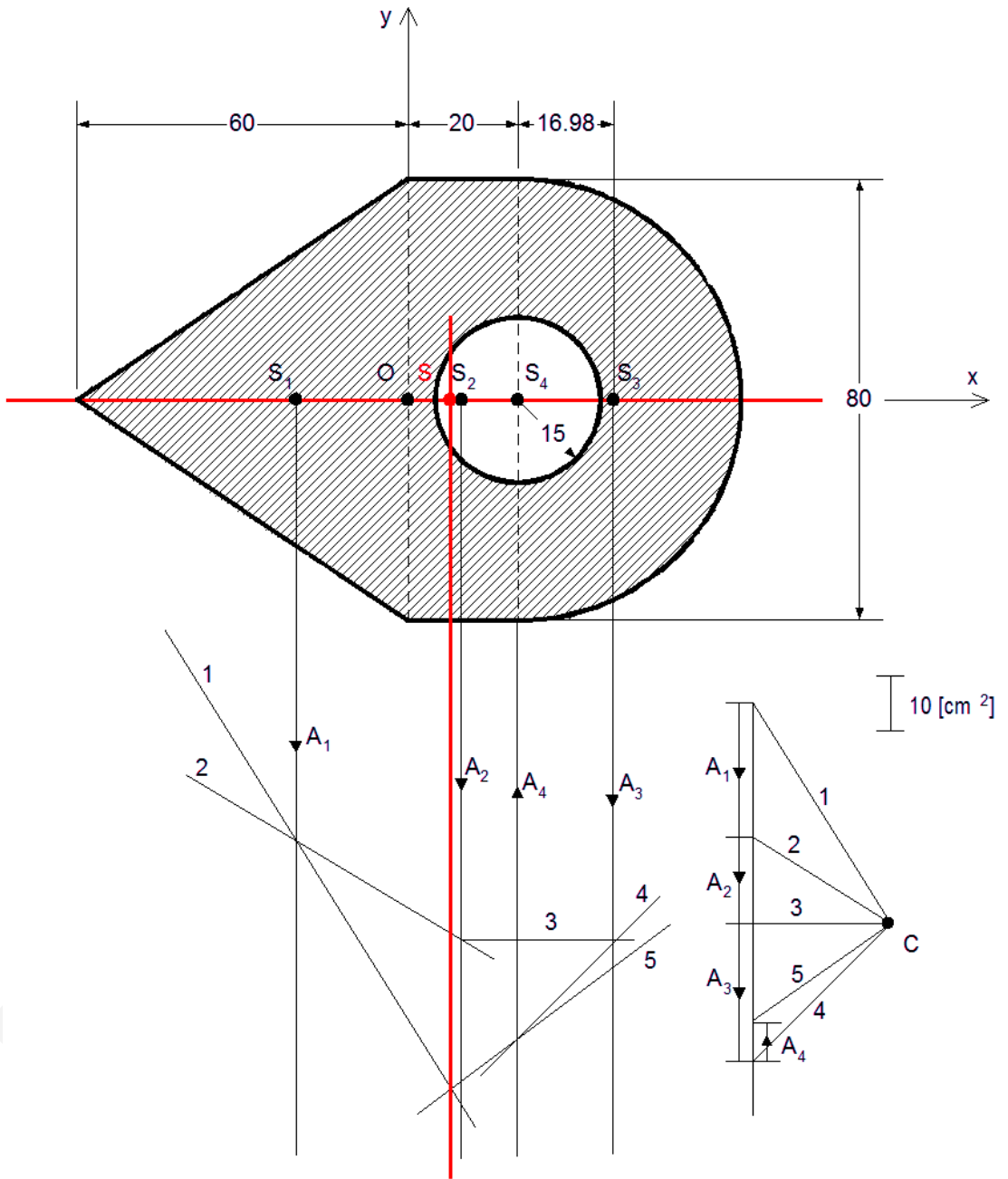
a)

i	$A_i[\text{cm}^2]$	$x_{si}[\text{cm}]$	$y_{si}[\text{cm}]$
1	24	-2	0
2	16	1	0
3	25.13	3.698	0
4	-7.069	2	0

$$x_s = \frac{\sum_{i=1}^4 x_{si} \cdot A_i}{\sum_{i=1}^4 A_i} = \frac{24 \cdot (-2) + 16 \cdot 1 + 25.13 \cdot 3.698 - 7.069 \cdot 2}{24 + 16 + 25.13 - 7.069} = 0.8059 [\text{cm}]$$

$$y_s = 0 [\text{cm}]$$

b)



## 7. Equilibrium state and equations of a rigid body

### 7.1 Equilibrium state of a rigid body

Equilibrium (definition): A rigid body is in equilibrium if the resultant moment of the force system that acts on the body is equal to zero for any point A of space.

$$\vec{M}_A = \vec{0} \text{ relative to any point A} \rightarrow \text{Equilibrium}$$

*Theorem 7.1:*

A rigid body is in equilibrium if the resultant force and resultant moment of the force system that acts on the body is equal to zero for a given point O of space

$$\vec{F} = \vec{0} \text{ and } \vec{M}_O = \vec{0} \rightarrow \text{Equilibrium}$$

*Proof:*

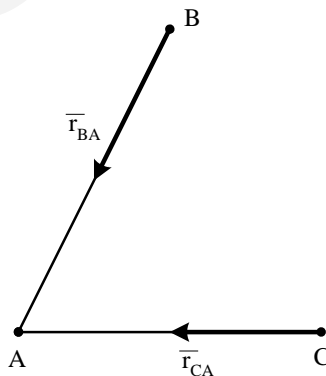
$$\vec{M}_A = \vec{M}_O + \vec{r}_{AO} \times \vec{F} = \vec{0} \text{ relative to any point A of space} \rightarrow \text{Equilibrium}$$

*Theorem 7.2:*

If the resultant moment of the force system that acts on the rigid body is equal to zero relative to three different non-collinear points (A, B, C) of space then the body is in equilibrium.

*Proof:*

Let's consider *Figure 39*:



*Figure 39*

1) A, B and C are different points  $\rightarrow \vec{r}_{BA} \neq \vec{0}, \vec{r}_{CA} \neq \vec{0}$

2) A, B and C are non-collinear points  $\rightarrow \vec{r}_{BA} \nparallel \vec{r}_{CA}$

3)  $\vec{M}_A = \vec{0}, \vec{M}_B = \vec{0}, \vec{M}_C = \vec{0}$

$$\vec{M}_B = \vec{M}_A + \vec{r}_{BA} \times \vec{F} \rightarrow \vec{r}_{BA} \times \vec{F} = \vec{0}$$

Case 1:

If  $\vec{r}_{BA} \nparallel \vec{F}$  then  $\vec{r}_{BA} \times \vec{F} = \vec{0}$  and  $\vec{r}_{BA} \neq \vec{0} \rightarrow \underline{\vec{F} = \vec{0}}$

Case 2:

If  $\vec{r}_{BA} \parallel \vec{F}$  then  $\vec{r}_{CA} \nparallel \vec{F}$  (It is true because  $\vec{r}_{CA} \nparallel \vec{r}_{BA}$ .)

$$\vec{M}_C = \vec{M}_A + \vec{r}_{CA} \times \vec{F} \rightarrow \vec{r}_{CA} \times \vec{F} = \vec{0}$$

$\vec{r}_{CA} \times \vec{F} = \vec{0}$  and  $\vec{r}_{CA} \nparallel \vec{F} \rightarrow \underline{\vec{F} = \vec{0}}$

Consequently from the theorem it comes that resultant force  $\vec{F}$  is equal to zero.

Applying *Theorem 7.1* from  $\vec{F} = \vec{0}$  and  $\vec{M}_A = \vec{0}$  it comes that the rigid body is in equilibrium.

## 7.2 Equilibrium equations of a rigid body

On the basis of *Theorem 7.1* the rigid body is in equilibrium if:

$$\vec{F} = \vec{0}, \quad \vec{M}_O = \vec{0} \text{ relative a given point O off space.}$$

In case of a 3 dimensional force system from  $\vec{F} = \vec{0}$  it comes:

$$\sum_i F_{ix} = 0, \quad \sum_i F_{iy} = 0, \quad \sum_i F_{iz} = 0$$

From  $\vec{M}_O = \vec{0}$  it comes that:

$$\sum_i M_{Oix} = 0, \quad \sum_i M_{Oiy} = 0, \quad \sum_i M_{Oiz} = 0$$

Consequently, in case of a 3 dimensional force system there are six independent equilibrium equations:

$$\sum_i F_{ix} = 0, \quad \sum_i F_{iy} = 0, \quad \sum_i F_{iz} = 0, \quad \sum_i M_{Oix} = 0, \quad \sum_i M_{Oiy} = 0, \quad \sum_i M_{Oiz} = 0$$

In case of a 2D force system from the above six equations we get the following three ones:

$$\sum_i F_{ix} = 0, \quad \sum_i F_{iy} = 0, \quad \sum_i M_{O_i} = \sum_i M_{O_{iz}} = 0$$

On the basis of *Theorem 7.2* the rigid body is in equilibrium if:

$$\bar{M}_A = \bar{0}, \quad \bar{M}_B = \bar{0}, \quad \bar{M}_C = \bar{0}$$

In case of a 3D force system from the above vector equations it comes:

$$\begin{aligned} \sum_i M_{A_{ix}} &= 0, & \sum_i M_{A_{iy}} &= 0, & \sum_i M_{A_{iz}} &= 0 \\ \sum_i M_{B_{ix}} &= 0, & \sum_i M_{B_{iy}} &= 0, & \sum_i M_{B_{iz}} &= 0 \\ \sum_i M_{C_{ix}} &= 0, & \sum_i M_{C_{iy}} &= 0, & \sum_i M_{C_{iz}} &= 0 \end{aligned}$$

Consequently, in case of a 3D force system there are nine equilibrium equations but there are only six independent ones out of them.

In case of a 2D force system from the above nine equations we get the following 3 ones:

$$\sum_i M_{A_i} = \sum_i M_{A_{iz}} = 0, \quad \sum_i M_{B_i} = \sum_i M_{B_{iz}} = 0, \quad \sum_i M_{C_i} = \sum_i M_{C_{iz}} = 0$$

Thus, we can conclude that in case of a 2D and 3D force systems there are three and six independent equilibrium equations respectively.

A mechanical structure is **statically determinate** if all the unknowns can be determined from the equilibrium equations.

Consequently, in case of 2D and 3D structures three and six unknowns can be determined respectively. If there are more unknowns, then the structure is **statically indeterminate**.

### 7.3 Supports and connections

In section 2.2 the concept of a constraint was defined and inextensible cable, rigid rod and surface were mentioned as examples. In the following we introduce further constraints, which are generally called as **supports, - or connections -** and define their properties.

*Roller support:*

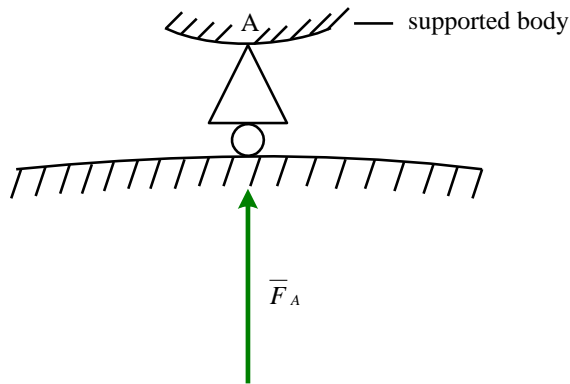


Figure 40

A roller support is equipped with an ideal wheel which lets the supported body moving freely in tangential direction, and also rotate freely about the point of connection (A), but resists vertical forces. Thus, a roller means one unknown parameter.

*Pinned support:*

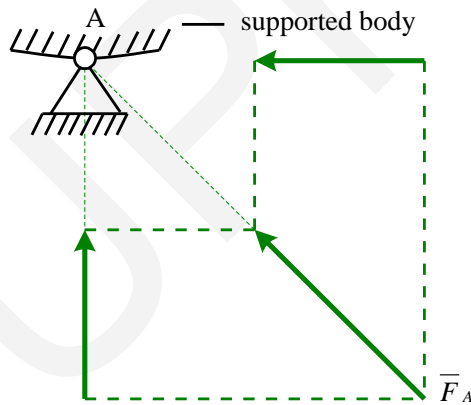


Figure 41

A pinned support lets the supported body rotate freely about the point of connection (A), but resist both horizontal and vertical forces. Thus, a pinned support means 2 unknown parameters.

Fixed support:

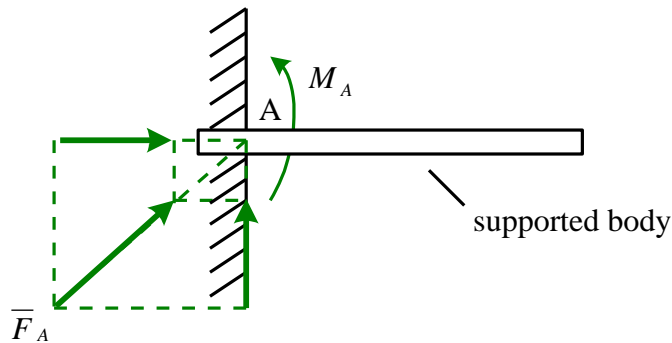


Figure 42

A fixed support resist both horizontal and vertical forces as well as a moment. The moment that the support exerts on the body is usually referred to as a reaction moment. Thus, a fixed support means 3 unknown parameters.

#### 7.4 Calculation of reaction forces

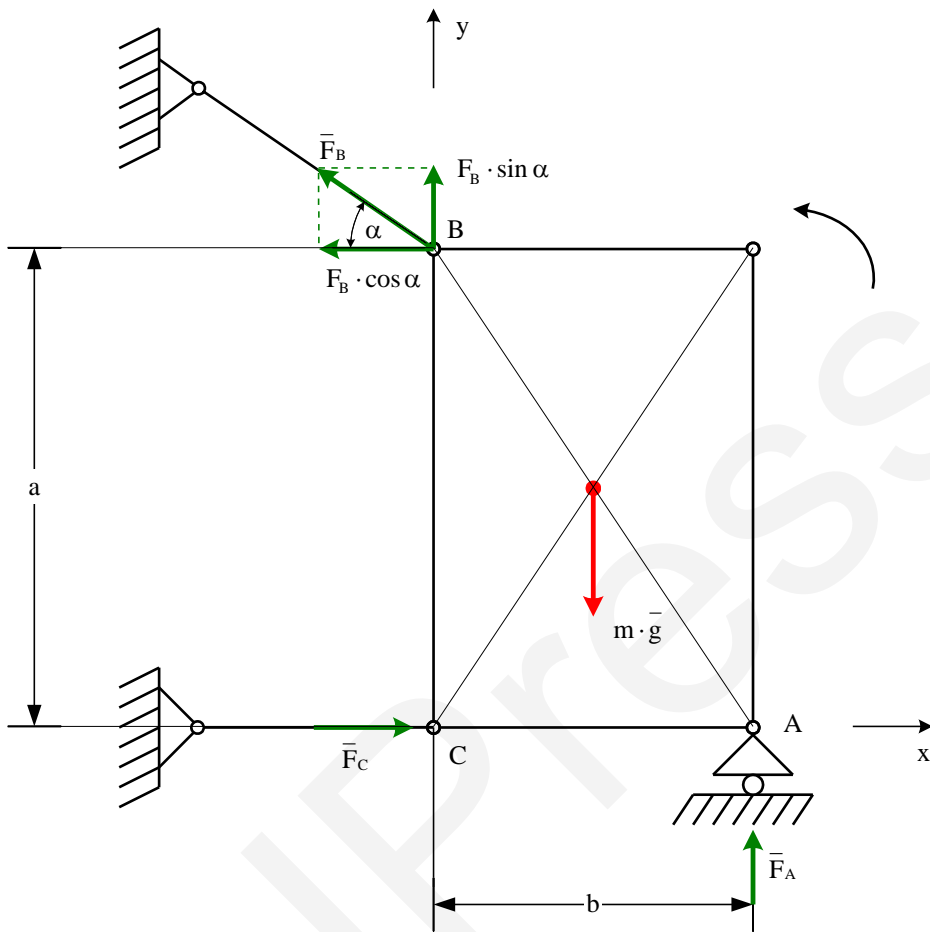
When calculating the unknown reaction forces, acting on the mechanical structure, we have to follow the steps below:

- 1) Analysis of constraints (supports)
- 2) Deciding whether the structure is statically determinate
- 3) Writing the equilibrium equations (Remark: It is practical to write the  $\sum_i M_A$  equation relative to the intersection point (A) of the action lines of unknown forces)
- 4) Solving the equations
- 5) Checking the results

#### 7.5 Numerical problems

*Problem 1:*

Calculate the magnitude of the  $\bar{F}_A$ ,  $\bar{F}_B$  and  $\bar{F}_C$  reaction forces that act on the rectangular shaped rigid plate in the figure below. The plate is fixed with two ideal rods connected to points B and C with ideal pin support (hinges) as well as a roller support at point A.



Data:  $m \cdot g = 500 [N]$ ,  $a = 3 [m]$ ,  $b = 2 [m]$ ,  $\alpha = 30^\circ$ .

Solution:

- 1) The two rods and the roller support altogether mean three unknowns.
- 2) The structure is statically determinate. (We have got a 2-dimensional force system, thus 3 scalar equilibrium equations.)
- 3)

$$\sum_i \vec{F}_i = \vec{F}_A + \vec{F}_B + \vec{F}_C + m \cdot \vec{g} = \vec{0}$$

$$\begin{pmatrix} 0 \\ F_A \end{pmatrix} + \begin{pmatrix} -F_B \cdot \cos \alpha \\ F_B \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} F_C \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -m \cdot g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sum_i M_i = F_B \cdot \cos \alpha \cdot a - F_B \cdot \sin \alpha \cdot b + m \cdot g \cdot \frac{b}{2} = 0$$

$$\text{I. } -F_B \cdot \cos \alpha + F_C = 0$$

$$\text{II. } F_A + F_B \cdot \sin \alpha - m \cdot g = 0$$

$$\text{III. } F_B \cdot \cos \alpha \cdot a - F_B \cdot \sin \alpha \cdot b + m \cdot g \cdot \frac{b}{2} = 0$$

4)

$$\text{III. } F_B \cdot (\cos \alpha \cdot a - \sin \alpha \cdot b) = -\frac{m \cdot g \cdot b}{2}$$

$$F_B = -\frac{m \cdot g \cdot b}{2 \cdot (\cos \alpha \cdot a - \sin \alpha \cdot b)} = -312.9 \text{ [N]}$$

$$\text{II. } F_A = -F_B \cdot \sin \alpha + m \cdot g = 656.5 \text{ [N]}$$

$$\text{I. } F_C = F_B \cdot \cos \alpha = -271 \text{ [N]}$$

Thus the  $\bar{F}_A$ ,  $\bar{F}_B$  and  $\bar{F}_C$  reaction forces are:

$$\bar{F}_A = \begin{pmatrix} 0 \\ 656.5 \end{pmatrix} \text{ [N]}, \quad \bar{F}_B = \begin{pmatrix} -271 \\ 156.4 \end{pmatrix} \text{ [N]}, \quad \bar{F}_C = \begin{pmatrix} -271 \\ 0 \end{pmatrix} \text{ [N]}$$

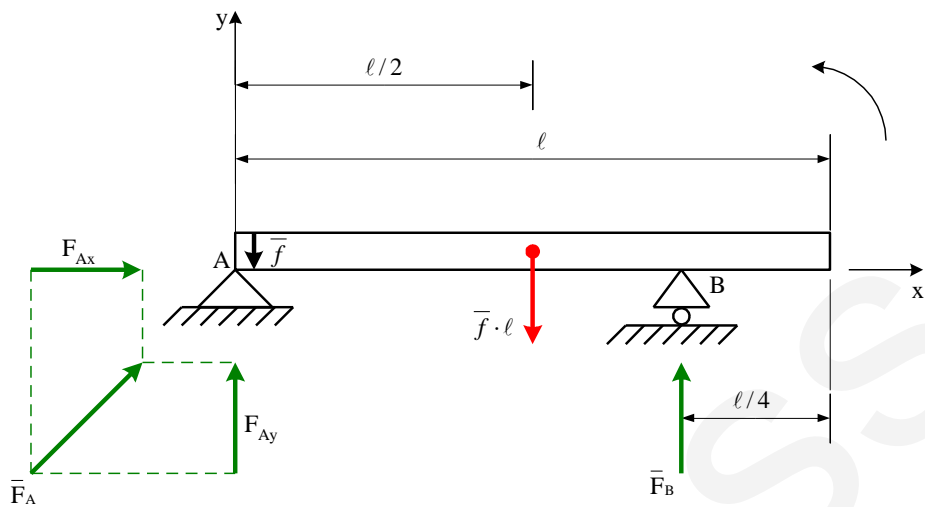
The negative sign of  $F_B$  and  $F_C$  refer to the opposite sense of  $\bar{F}_B$  and  $\bar{F}_C$  compared to the assumed ones.

5) Checking:

$$\sum_i M_i = F_A \cdot b + F_C \cdot a - m \cdot g \cdot \frac{b}{2} = 656.5 \cdot 2 + (-271) \cdot 3 - 500 \cdot 1 = 0$$

*Problem 2:*

Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces that act on the supported beam in the figure below.



Data:  $f = 300 \frac{[N]}{[m]}$ ,  $l = 4 [m]$ .

Solution:

- 1) The pinned and the roller support altogether mean three unknown parameters.
- 2) The structure is statically determinate.
- 3)

$$\sum_i \bar{F}_i = \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} 0 \\ F_B \end{pmatrix} + \begin{pmatrix} 0 \\ -f \cdot l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sum_i M_i = -f \cdot l \cdot \frac{l}{2} + F_B \cdot \frac{3l}{4} = 0$$

$$\text{I. } F_{Ax} = 0$$

$$\text{II. } F_{Ay} + F_B - f \cdot l = 0$$

$$\text{III. } -f \cdot \frac{l}{2} + \frac{3}{4} F_B = 0$$

4)

$$\text{III. } -f \cdot \frac{l}{2} + \frac{3}{4} F_B = 0 \rightarrow F_B = \frac{2f \cdot l}{3} = 800 [N](\uparrow)$$

$$\text{II. } F_{Ay} = -F_B + f \cdot l = 400 [N](\uparrow)$$

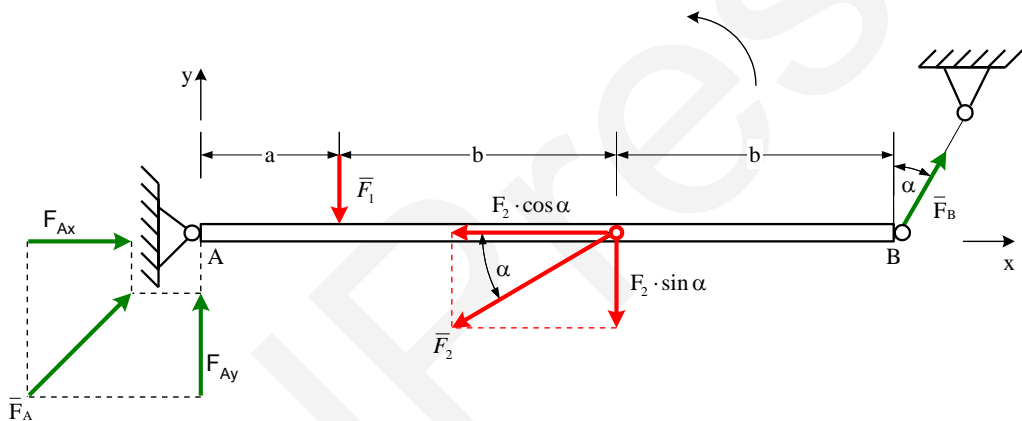
$$\bar{F}_A = \begin{pmatrix} 0 \\ 400 \end{pmatrix} [N], \quad \bar{F}_B = \begin{pmatrix} 0 \\ 800 \end{pmatrix} [N]$$

5) Checking:

$$\sum_i M_i = -F_{Ay} \cdot \frac{3l}{4} + f \cdot \frac{l}{4} = -400 \cdot \frac{3 \cdot 4}{4} + 300 \cdot \frac{4}{4} = 0$$

*Problem 3:*

Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces act on the horizontal weightless rigid rod in the figure. The rod is fixed with a pinned support at point A and with an ideal cable at point B.



*Data:*  $a=1[m]$ ,  $b=2[m]$ ,  $F_1 = 5[kN]$ ,  $F_2 = 10[kN]$ ,  $\alpha = 30^\circ$ .

*Solution:*

- 1) The pinned support and the cable altogether mean three unknown parameters.
- 2) The structure is statically determinate.
- 3)

$$\sum_i \bar{F}_i = \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} 0 \\ -F_1 \end{pmatrix} + \begin{pmatrix} -F_2 \cdot \cos \alpha \\ -F_2 \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} F_B \cdot \sin \alpha \\ F_B \cdot \cos \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sum_i M_i = -F_1 \cdot a - F_2 \cdot \sin \alpha \cdot (a + b) + F_B \cdot \cos \alpha \cdot (a + 2b) = 0$$

$$\text{I. } F_{Ax} - F_2 \cdot \cos \alpha + F_B \cdot \sin \alpha = 0$$

$$\text{II. } F_{Ay} - F_1 - F_2 \cdot \sin \alpha + F_B \cdot \cos \alpha = 0$$

$$\text{III. } -F_1 \cdot a - F_2 \cdot \sin \alpha (a + b) + F_B \cdot \cos \alpha (a + 2b) = 0$$

4)

$$\text{III. } F_B \cdot \cos \alpha \cdot (a + 2b) = F_1 \cdot a + F_2 \cdot \sin \alpha \cdot (a + b)$$

$$F_B = \frac{F_1 \cdot a + F_2 \cdot \sin \alpha (a + b)}{\cos \alpha \cdot (a + 2b)} = 4.619 \text{ [kN]}$$

$$\text{II. } F_{Ay} = F_1 + F_2 \cdot \sin \alpha - F_B \cdot \cos \alpha = 6 \text{ [kN]}$$

$$\text{I. } F_{Ax} = F_2 \cdot \cos \alpha - F_B \cdot \sin \alpha = 6.351 \text{ [kN]}$$

$$\bar{F}_A = \begin{pmatrix} 6.351 \\ 6 \end{pmatrix} \text{ [kN]}, \quad \bar{F}_B = \begin{pmatrix} 2.310 \\ 4 \end{pmatrix} \text{ [kN]}$$

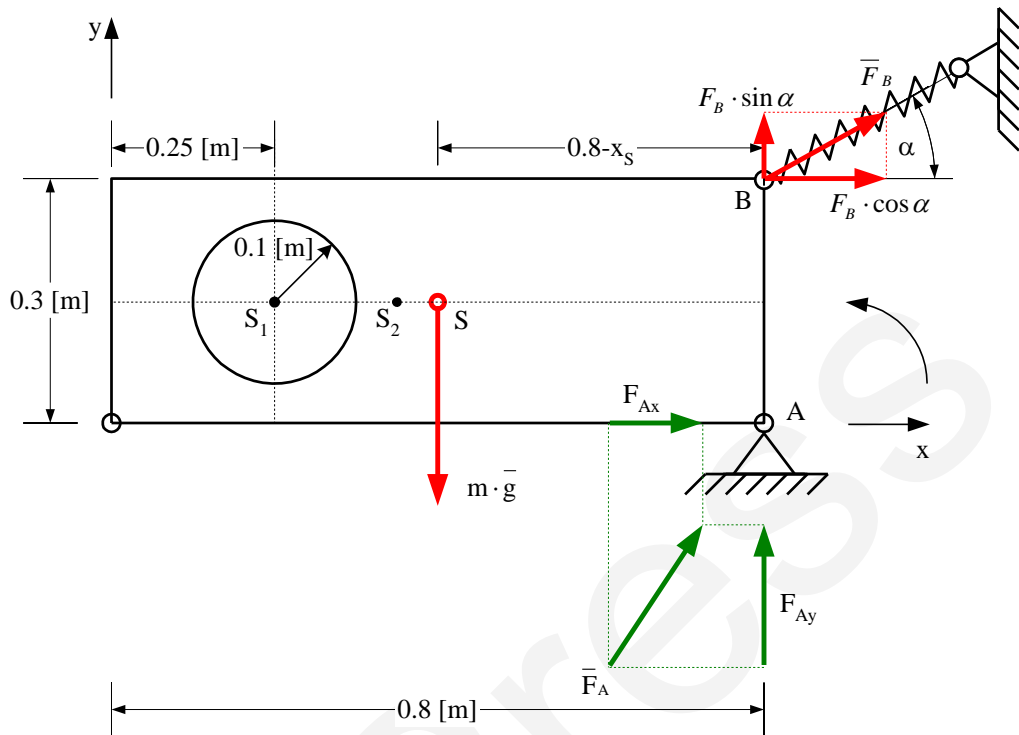
5) Checking:

$$\begin{aligned} \sum_i M_i &= -F_{Ay} \cdot (a + 2b) + F_1 \cdot 2b + F_2 \cdot \sin \alpha \cdot b \\ &= -6 \cdot (1 + 4) + 5 \cdot 4 + 10 \cdot 2 \cdot \sin 30^\circ = 0 \end{aligned}$$

*Problem 4:*

The rigid plate in the figure below is fixed with a pinned support at a point A and with a linear spring at point B.

- Calculate the  $x_s$  and  $y_s$  coordinates of the centre of gravity of the plate.
- Calculate the  $\bar{F}_A$  reaction and  $\bar{F}_B$  spring forces.
- Calculate the  $\Delta r$  extension of the spring.



Data:  $\alpha = 30^\circ$ ,  $m \cdot g = 50 \text{ [N]}$ ,  $D = 100 \left[ \frac{\text{N}}{\text{m}} \right]$ .

Solution:

a)

i	$A_i [\text{m}^2]$	$x_{si} [\text{m}]$	$y_{si} [\text{m}]$
1	-0.0314	0.25	0.15
2	0.24	0.4	0.15

$$x_s = \frac{\sum_i A_i x_{si}}{\sum_i A_i} = \frac{-0.0314 \cdot 0.25 + 0.24 \cdot 0.4}{-0.0314 + 0.24} = 0.4226 \text{ [m]}$$

$$y_s = \frac{\sum_i A_i y_{si}}{\sum_i A_i} = \frac{-0.0314 \cdot 0.15 + 0.24 \cdot 0.15}{-0.0314 + 0.24} = 0.15 \text{ [m]}$$

b)

1) The pinned support and the spring altogether mean three unknown parameters.

2) The structure is statically determinate.

3)

$$\sum_i \bar{F}_i = m \cdot \bar{g} + \bar{F}_A + \bar{F}_B = \begin{pmatrix} 0 \\ -m \cdot g \end{pmatrix} + \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} F_B \cdot \cos \alpha \\ F_B \cdot \sin \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sum_i M_{Ai} = (0.8 - x_S) \cdot m \cdot g - F_B \cdot \cos \alpha \cdot 0.3 = 0$$

4)

$$\text{I. } F_{Ax} + F_B \cdot \cos 30^\circ = 0 \rightarrow F_{Ax} = -F_B \cdot \cos 30^\circ = -62.9 \text{ [N]}$$

$$\text{II. } -m \cdot g + F_{Ay} + F_B \cdot \sin 30^\circ = 0 \rightarrow F_{Ay} = m \cdot g - F_B \cdot \sin 30^\circ = 13.68 \text{ [N]}$$

$$\text{III. } 0.3774 \cdot 50 - F_B \cdot \cos 30^\circ \cdot 0.3 = 0 \rightarrow F_B = 72.63 \text{ [N]}$$

$$\bar{F}_A = \begin{pmatrix} -62.90 \\ 13.68 \end{pmatrix} \text{ [N]}, \quad \bar{F}_B = \begin{pmatrix} 62.90 \\ 36.31 \end{pmatrix} \text{ [N]}$$

5) Checking:

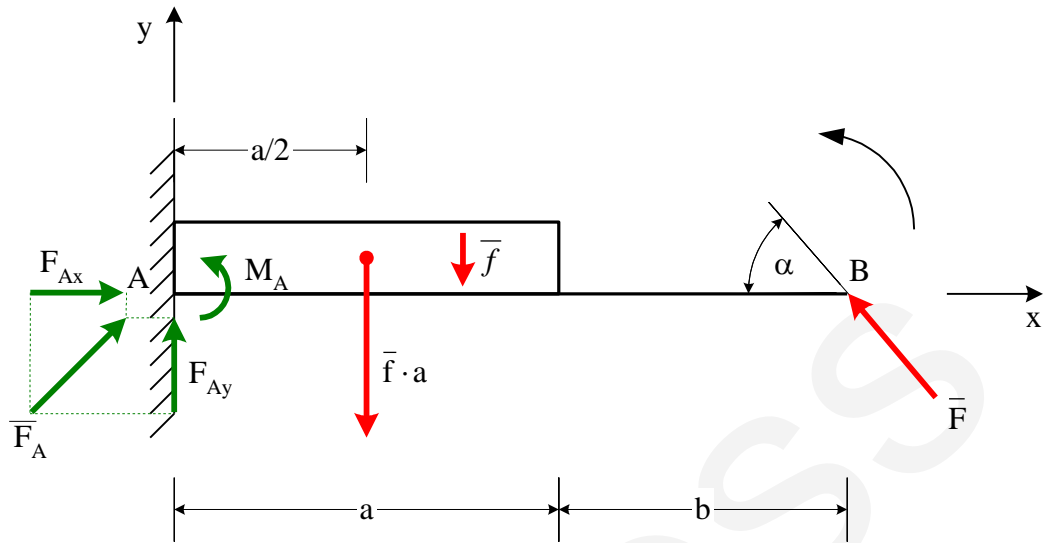
$$\sum_i M_{Bi} = F_{Ax} \cdot 0.3 + m \cdot g \cdot (0.8 - x_S) = 0$$

c)

$$F_B = D \cdot \Delta r \rightarrow \Delta r = \frac{F_B}{D} = 0.726 \text{ [m]}$$

*Problem 5:*

Calculate the  $\bar{F}_A$  reaction force and  $M_A$  reaction moment at fixed end A of the cantilever in the figure below.



Data:  $F=600$  [N],  $f=200$  [N/m],  $\alpha = 50^\circ$ ,  $a=4$  [m],  $b=3$  [m].

Solution:

- 1) The fixed end means three unknown parameters.
- 2) The structure is statically determinate.
- 3)

$$\sum_i \bar{F}_i = \bar{F} + \bar{f} \cdot a + \bar{F}_A = \begin{pmatrix} -F \cdot \cos \alpha \\ F \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} 0 \\ -f \cdot a \end{pmatrix} + \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } -F \cdot \cos \alpha + F_{Ax} = 0$$

$$\text{II. } F \cdot \sin \alpha - f \cdot a + F_{Ay} = 0$$

$$\text{III. } \sum_i M_i = M_A - f \cdot a \cdot \frac{a}{2} + F \cdot \sin \alpha \cdot (a + b) = 0$$

4)

$$\text{I. } F_{Ax} = F \cdot \cos \alpha = 385.67 \text{ [N]}$$

$$\text{II. } F_{Ay} = f \cdot a - F \cdot \sin \alpha = 340.37 \text{ [N]}$$

$$\text{III. } M_A = f \cdot \frac{a^2}{2} - F \cdot \sin \alpha \cdot (a + b) = -1617.38 \text{ [Nm]}$$

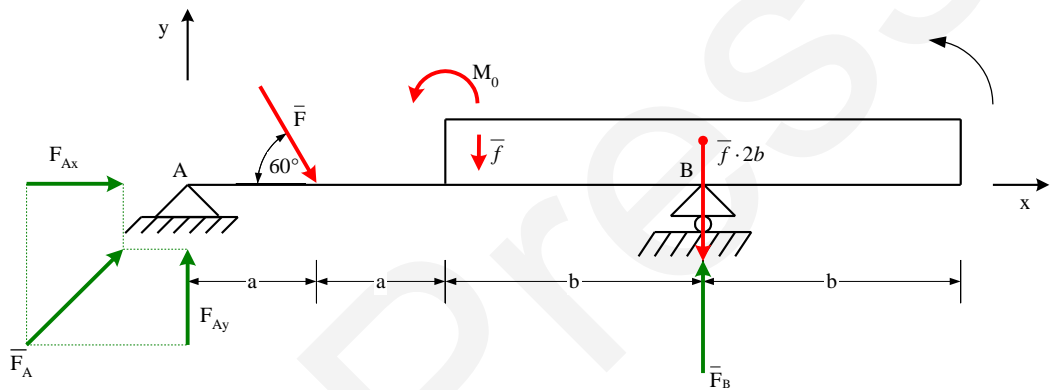
$$\bar{F}_A = \begin{pmatrix} 385.67 \\ 340.37 \end{pmatrix} \text{ [N]}$$

5) Checking:

$$\begin{aligned}\sum_i M_B &= -F_{Ay} \cdot (a + b) + f \cdot a \cdot \left(b + \frac{a}{2}\right) + M_A = \\ &= -340.37 \cdot (4 + 3) + 200 \cdot 4 \cdot \left(3 + \frac{4}{2}\right) + (-1617.38) \\ &= 0\end{aligned}$$

*Problem 6:*

Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces that act on the supported beam in the figure.



*Data:*  $a=1$  [m],  $b=2$  [m],  $F=40$  [kN],  $f=8$  [kN/m],  $M_0=5$  [kNm]

*Solution:*

- 1) The pinned and the roller support altogether mean three unknown parameters.
- 2) The structure is statically determinate.
- 3)

$$\begin{aligned}\sum_i \bar{F}_i &= \bar{F} + F_A + \bar{f} \cdot 2b + F_B = \begin{pmatrix} F \cdot \cos 60 \\ -F \cdot \sin 60 \end{pmatrix} + \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} 0 \\ -f \cdot 2b \end{pmatrix} + \begin{pmatrix} 0 \\ F_B \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned}$$

$$\text{I. } F \cdot \cos 60^\circ + F_{Ax} = 0$$

$$\text{II. } -F \cdot \sin 60^\circ + F_{Ay} - f \cdot 2b + F_B = 0$$

$$\text{III. } \sum_i M_i = -F \cdot \sin 60^\circ \cdot a + F_B \cdot (2a + b) - f \cdot 2b \cdot (2a + b) + M_O = 0$$

4)

$$\text{III. } F_B = \frac{F \cdot \sin 60^\circ \cdot a + f \cdot 2b \cdot (2a + b) - M_O}{2a + b} = 39.41 \text{ [kN]}$$

$$\text{II. } F_{Ay} = F \cdot \sin 60^\circ + f \cdot 2b - F_B = 27.23 \text{ [kN]}$$

$$\text{I. } F_{Ax} = -F \cdot \cos 60^\circ = -20 \text{ [kN]}$$

$$\bar{F}_A = \begin{pmatrix} -20 \\ 27.23 \end{pmatrix} \text{ [kN]}, \quad \bar{F}_B = \begin{pmatrix} 0 \\ 39.41 \end{pmatrix} \text{ [kN]}$$

5) Checking

$$\begin{aligned} \sum_i M_B &= -F_{Ay} \cdot (2a + b) + F \cdot \sin \alpha \cdot (a + b) + M_O + f \cdot b \cdot \frac{b}{2} - f \cdot b \cdot \frac{b}{2} = \\ &= -27.23 \cdot (2 + 2) + 40 \cdot \sin 60^\circ \cdot (1 + 2) + 5 = 0 \end{aligned}$$

## 8. Construction of reaction forces acting on a structure in equilibrium states

In the following we give method for the construction of reaction forces. The method will be presented in the following two cases:

- Construction of the reaction forces in an intersecting force system. (The action lines of at least two of the forces intersect each other.)
- Construction of the reaction forces in a parallel force system. (The action lines of all the forces parallel to each other.)

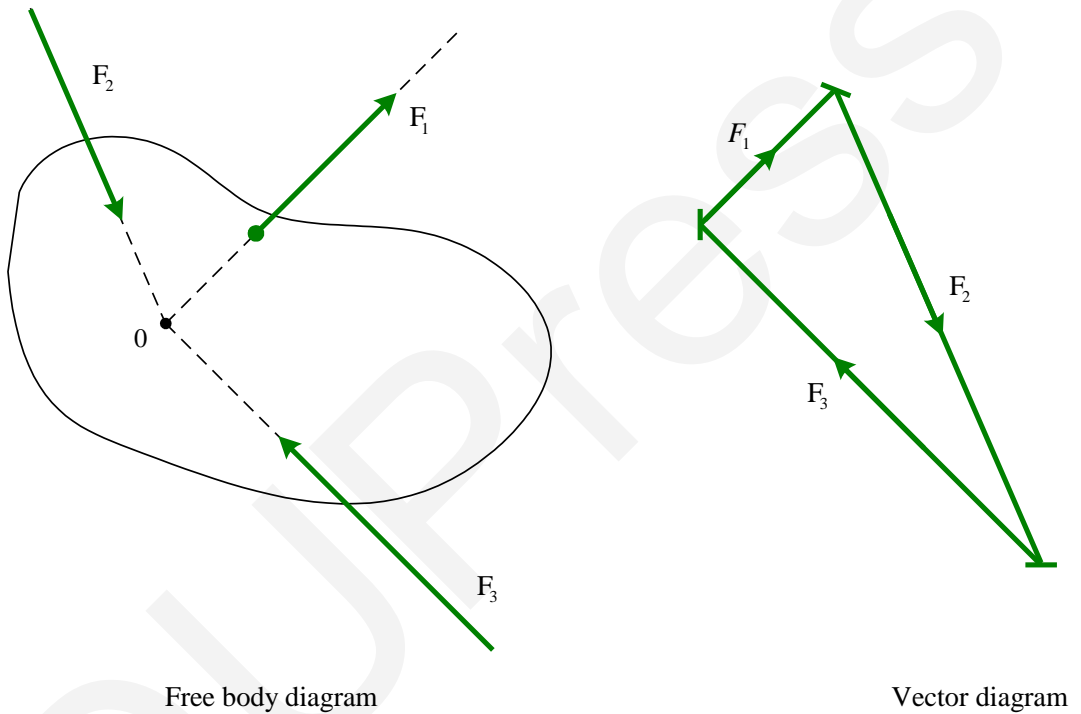
### 8.1 Intersecting force system

First, we deal with the case when only three forces acting on the analyzed mechanical structure (see *Figure 43 and Theorem 8.1*). If there are more than three forces, then the construction method can be derived back the one that is applied in the case of three forces.

*Theorem 8.1:*

If the force system consists of three forces, then the structure is in equilibrium if and only if:

- 1) The three forces form a closed vector triangle.
- 2) The action lines of the forces intersect each other at the same point (O).



*Figure 43*

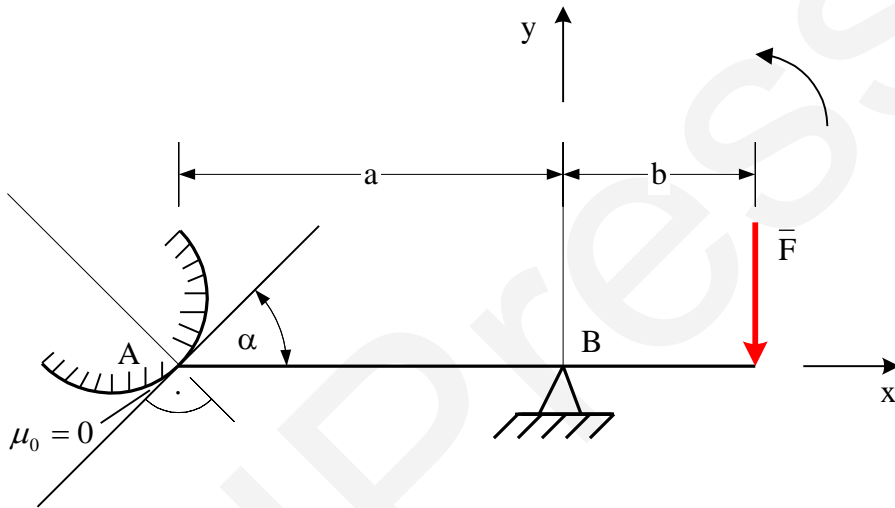
Condition 1) and 2) comes from equations  $\sum_i \bar{F}_i = \bar{0}$  and  $\sum_i \bar{M}_i = \bar{0}$  respectively.

## Numerical problems

### Problem 1:

The figure below shows a beam which is supported by a simple support – in a shape of a circle – at point A and by a pinned support at point B. The beam is in equilibrium.

Calculate and construct the unknown  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces.



Data:  $F=100$  [N],  $a=2$  [m],  $b=1$  [m],  $\alpha=45^\circ$

Solution:

Calculation:

$$\sum_i \bar{F}_i = \bar{F}_A + \bar{F}_B + \bar{F} = \begin{pmatrix} F_A \cdot \cos \alpha \\ -F_A \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} -F_{Bx} \\ F_{By} \end{pmatrix} + \begin{pmatrix} 0 \\ -F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_A \cdot \cos \alpha - F_{Bx} = 0 \rightarrow F_{Bx} = F_A \cdot \cos \alpha = 50 \text{ [N]}$$

$$\text{II. } -F_A \cdot \sin \alpha + F_{By} - F = 0 \rightarrow F_{By} = F + F_A \cdot \sin \alpha = 150 \text{ [N]}$$

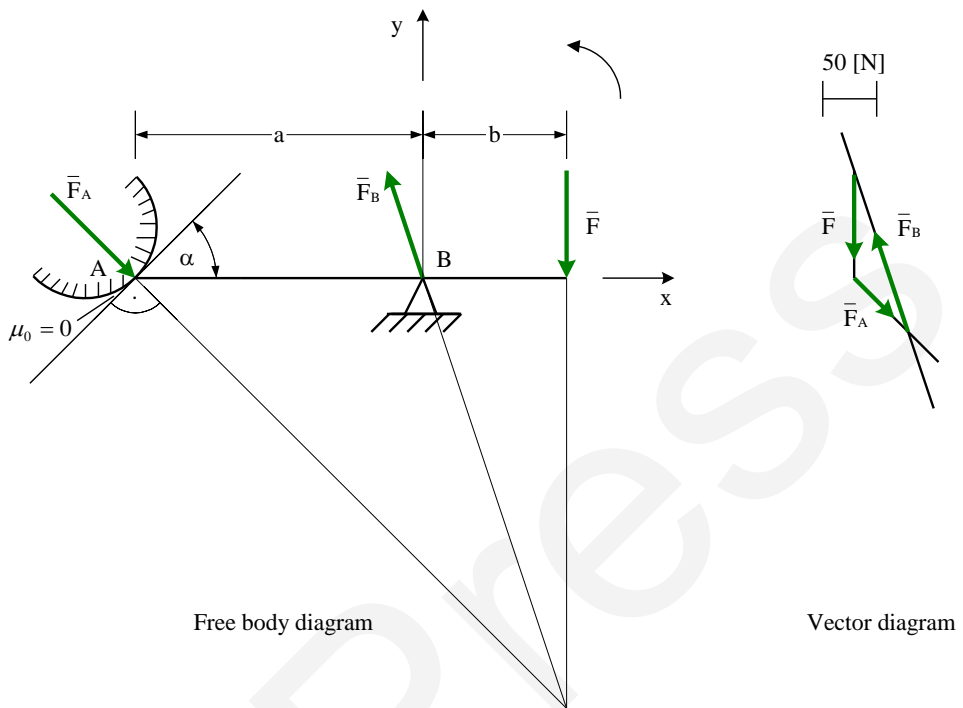
$$\text{III. } \sum_i M_i = F_A \cdot \sin \alpha \cdot a - F \cdot b = 0 \rightarrow F_A = \frac{F \cdot b}{\sin \alpha \cdot a} = 70.70 \text{ [N]}$$

Checking:

$$\sum_i M_A = F_{By} \cdot a - F \cdot (a + b) = 150 \cdot 2 - 100 \cdot (2 + 1) = 0$$

Construction:

The construction is shown in the figure below both for the free body and vector diagram (see *Theorem 8.1*).



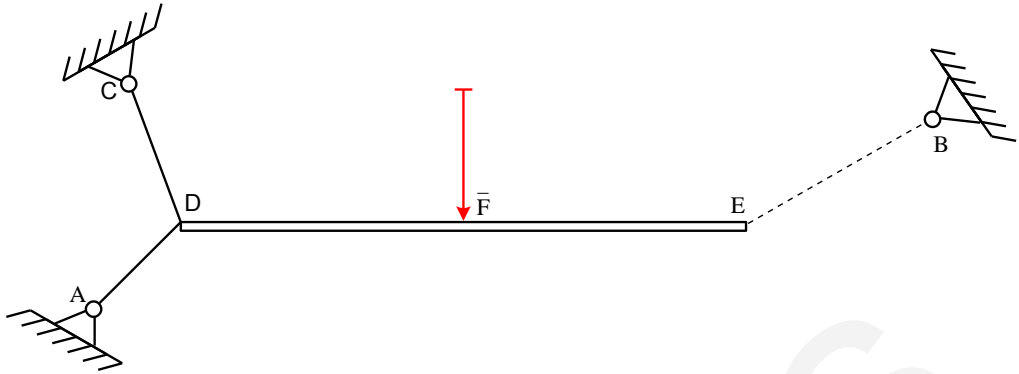
Clicking on the link below the construction above can be followed as a Power Point animation step by step:

[Animation 8.1](#)

*Problem 2:*

The horizontal beam in the figure is supported by an ideal cable at point E and by two ideal rods at point D. The other ends of the rods are connected two points A and C through ideal pin supports (hinges). The beam is loaded with vertical force  $\vec{F}$  at its midpoint.

Construct the unknown  $\vec{F}_A$ ,  $\vec{F}_C$  and  $\vec{F}_E$  reaction forces.



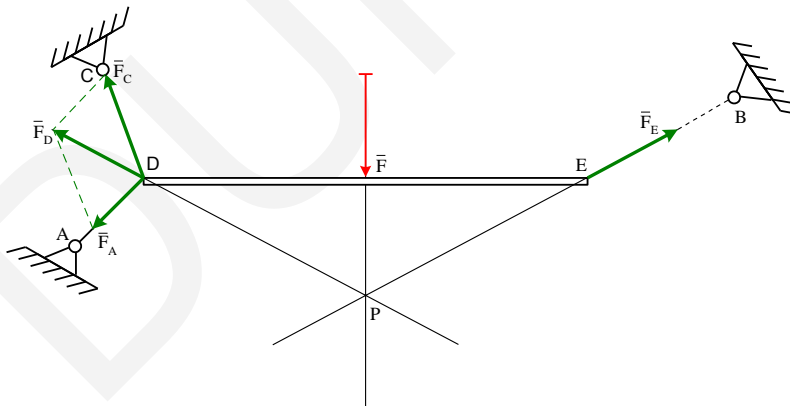
*Solution:*

The construction method can be traced back to the one in *Problem 1*.

The steps of construction are as follows:

(The resultant of  $\vec{F}_A$  and  $\vec{F}_C$  is denoted by  $\vec{F}_D$ )

- 1) Constructing the  $\vec{F}_D$  and  $\vec{F}_E$  forces applying the procedure in *Problem 1*.
- 2) Constructing the  $\vec{F}_A$  and  $\vec{F}_C$  reaction forces.
- 3) Reading the magnitude of  $\vec{F}_A$ ,  $\vec{F}_C$  and  $\vec{F}_E$  from the vector diagram.



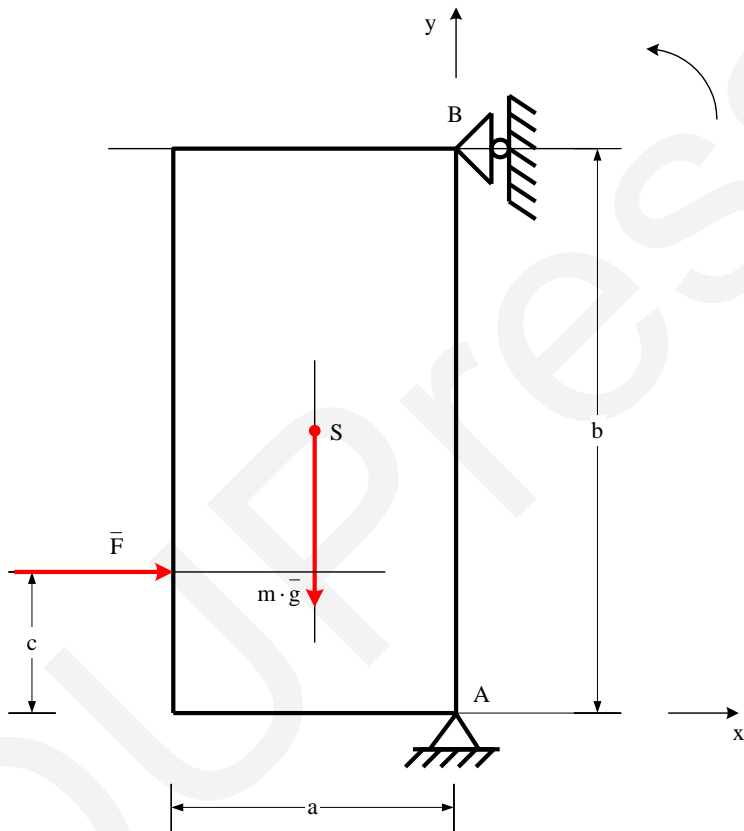
Free body diagram

Vector diagram

*Problem 3:*

The figure below shows a rigid plate in the shape of a rectangle which is supported by a roller and a pinned support. The plate is in static equilibrium.

Calculate and construct the unknown  $\vec{F}_A$  and  $\vec{F}_B$  reaction forces.



*Data:*  $F=100$  [N],  $m \cdot g=200$  [N],  $a=2$  [m],  $b=4$  [m],  $c=1$  [m]

*Solution:*

Calculation:

$$\sum_i \vec{F}_i = \vec{F} + m \cdot \vec{g} + \vec{F}_A + \vec{F}_B = \begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -m \cdot g \end{pmatrix} + \begin{pmatrix} -F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} F_B \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F - F_{Ax} + F_B = 0 \rightarrow F_{Ax} = F + F_B = 125 \text{ [N]}$$

$$\text{II. } -m \cdot g + F_{Ay} = 0 \rightarrow F_{Ay} = m \cdot g = 200 \text{ [N]}$$

$$\text{III. } \sum_i M_i = m \cdot g \cdot \frac{a}{2} - F \cdot c - F_B \cdot b = 0$$

---

$$\text{III. } F_B = \frac{m \cdot g \cdot \frac{a}{2} - F \cdot c}{b} = 25 \text{ [N]}$$

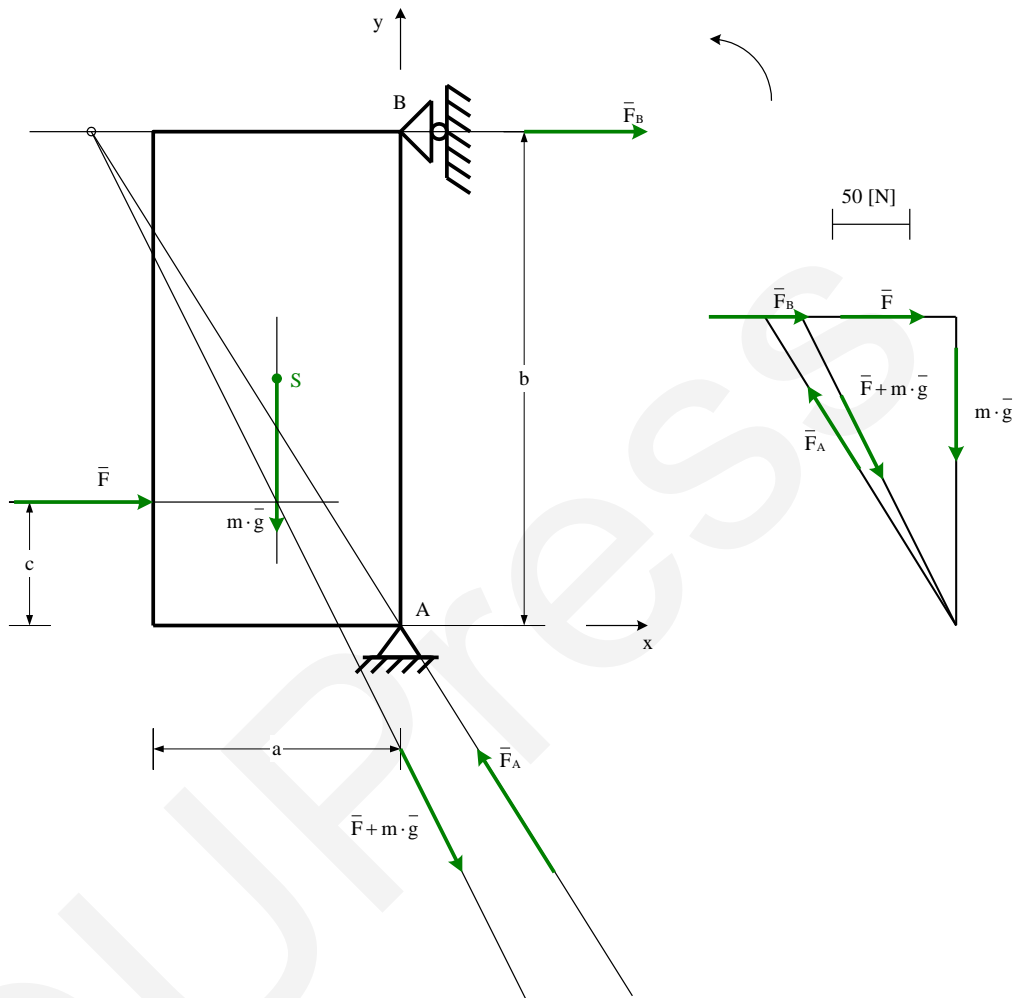
Checking:

$$\begin{aligned} \sum_i M_i &= -F_{Ax} \cdot b + F \cdot (b - c) + m \cdot g \cdot \frac{a}{2} \\ &= -125 \cdot 4 + 100 \cdot (4 - 1) + 200 \cdot 1 = 0 \end{aligned}$$

Construction:

The steps of the construction are as follows:

- 1) Constructing the resultant of the  $\vec{F}$  and  $m \cdot \vec{g}$  forces.
- 2) Constructing the action line of reaction force  $\vec{F}_A$  applying that the action lines of  $\vec{F} + m \cdot \vec{g}$ ,  $\vec{F}_A$  and  $\vec{F}_B$  intersect each other at the same point.
- 3) Drawing the vector diagram.
- 4) Reading the magnitude of reaction forces  $\vec{F}_A$  and  $\vec{F}_B$  from the vector diagram, comparing their length with the unit.



Clicking on the link below the construction above can be followed as a Power Point animation step by step:

Animation 8.2

## 8.2 Parallel force system

In the case of a parallel force system the construction method – that was presented in Section 8.1. – cannot be applied. Instead of that, we apply the method for the construction of the resultant of a parallel force system (Section 5.3) using the statement that in equilibrium the first and last construction line is the same both in the free-body and vector diagram (Theorem 5.4).

Let's denote the construction lines as: 1, 2, ... n-1, n.

The steps of the construction are as follows:

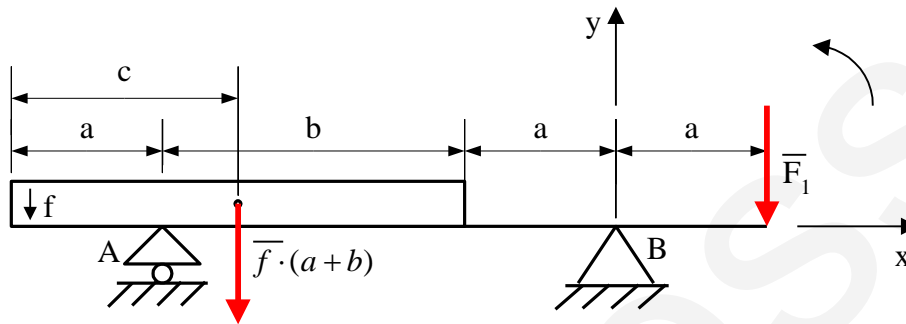
- 1) We start constructing the resultant of the known forces applying the procedure in Section 5.3.
- 2) We stop the procedure after drawing the construction line that points to the terminal point of the last known force in the vector diagram and also, the one which is parallel to that in the free body diagram.
- 3) We find the intersection point of the above (last) construction line in the free body diagram with the action line of one of the unknown forces.
- 4) We find the intersection point of the first construction line (line 1) in the free body diagram with the action line of the other (remaining) unknown force.
- 5) Connecting the intersection points in the free body diagram – that were constructed in steps 3) and 4) – we get construction line n-1.
- 6) We draw construction line n-1 in the vector diagram and read the magnitude of the unknown forces.

*Remark:* The reason for step 5) is the fact that construction lines 1 and n are the same both in the vector and in the free body diagram (see *Theorem 5.4.*).

## Numerical problems

*Problem 4:*

Calculate and construct the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces that act on the supported beam in the figure.



*Data:*  $F_1 = 9$  [kN],  $f = 6$  [kN/m],  $a = 1$  [m],  $b = 2$  [m],

*Solution:*

*Calculation:*

$$f \cdot (a + b) = 18 \text{ [kN]}$$

$$c = \frac{a + b}{2} = 1.5 \text{ [m]}$$

$$\sum_i \bar{F}_i = \bar{F}_A + \bar{f} \cdot (a + b) + \bar{F}_B + \bar{F}_1 = \begin{pmatrix} 0 \\ F_A \end{pmatrix} + \begin{pmatrix} 0 \\ -f \cdot (a + b) \end{pmatrix} + \begin{pmatrix} F_{Bx} \\ F_{By} \end{pmatrix} + \begin{pmatrix} 0 \\ -F_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_{Bx} = 0$$

$$\text{II. } F_A - f \cdot (a + b) + F_{By} - F_1 = 0$$

$$\text{III. } \sum_i M_i = -F_A \cdot (a + b) + f \cdot (a + b) \cdot (2a + b - c) - F_1 \cdot a = 0$$

$$\text{III. } -F_A \cdot 3 + 18 \cdot 2.5 - 9.1 = 0 \rightarrow F_A = 12 \text{ [kN]}$$

$$\text{II. } F_{By} = F_1 + f \cdot (a + b) - F_A = 9 + 18 - 12 = 15 \text{ [kN]}$$

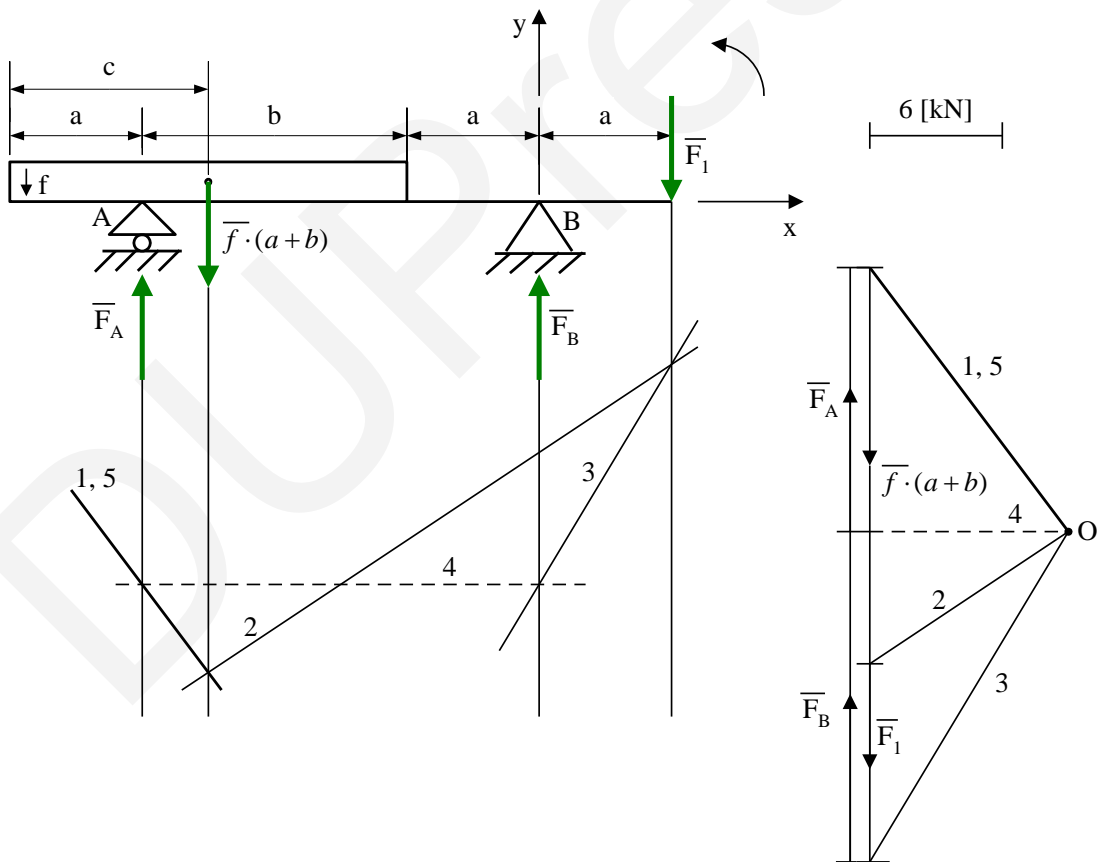
Checking:

$$\sum_i M_A = -f \cdot (a + b) \cdot (c - a) + F_{By} \cdot (a + b) - F_1 \cdot (2a + b) + f$$

$$\cdot \left(a + \frac{a}{2}\right) =$$

$$= -6 \cdot (1 + 2) \cdot (1.5 - 1) + 15 \cdot (1 + 2) - 9 \cdot (2 + 2) + 6 \cdot 1.5 = 0$$

Construction:

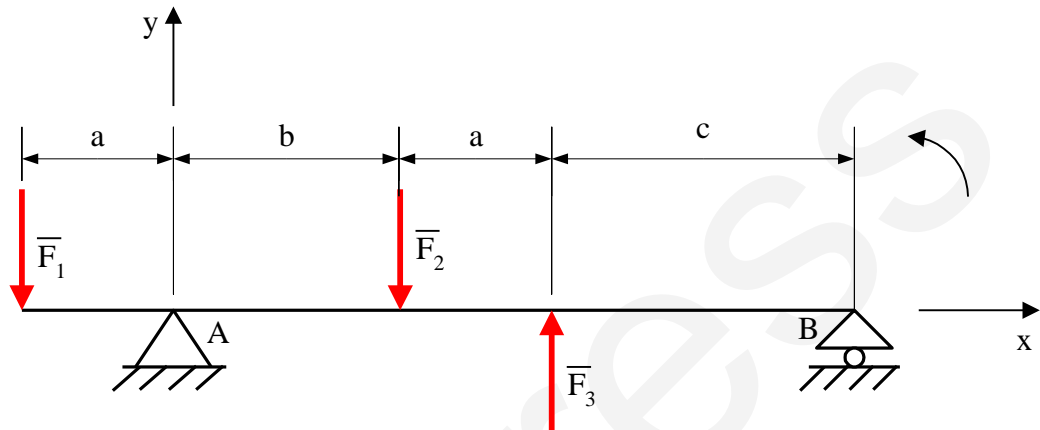


Clicking on the link below the construction above can be followed as a Power Point animation step by step:

Animation 8.3

*Problem 5:*

Calculate and construct the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces that act on the supported beam in the figure.



*Data:*  $a=1[m]$ ,  $b=1,5[m]$ ,  $c=2[m]$ ,  $F_1 = 20 [kN]$ ,  $F_2 = 50 [kN]$ ,  $F_3 = 30 [kN]$

*Solution:*

*Calculation:*

$$\sum_i \bar{F}_i = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_A + \bar{F}_B = \begin{pmatrix} 0 \\ -F_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -F_2 \end{pmatrix} + \begin{pmatrix} 0 \\ F_3 \end{pmatrix} + \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} 0 \\ F_B \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

I.  $F_{Ax} = 0$

II.  $-F_1 - F_2 + F_3 + F_{Ay} + F_B = 0$

III.  $\sum_i M_A = F_1 \cdot a - F_2 \cdot b + F_3 \cdot (a + b) + F_B \cdot (a + b + c) = 0$

$$\text{III. } 20 \cdot 1 - 50 \cdot 1.5 + 30 \cdot (1 + 1.5) + F_B \cdot (1 + 1.5 + 2) = 0$$

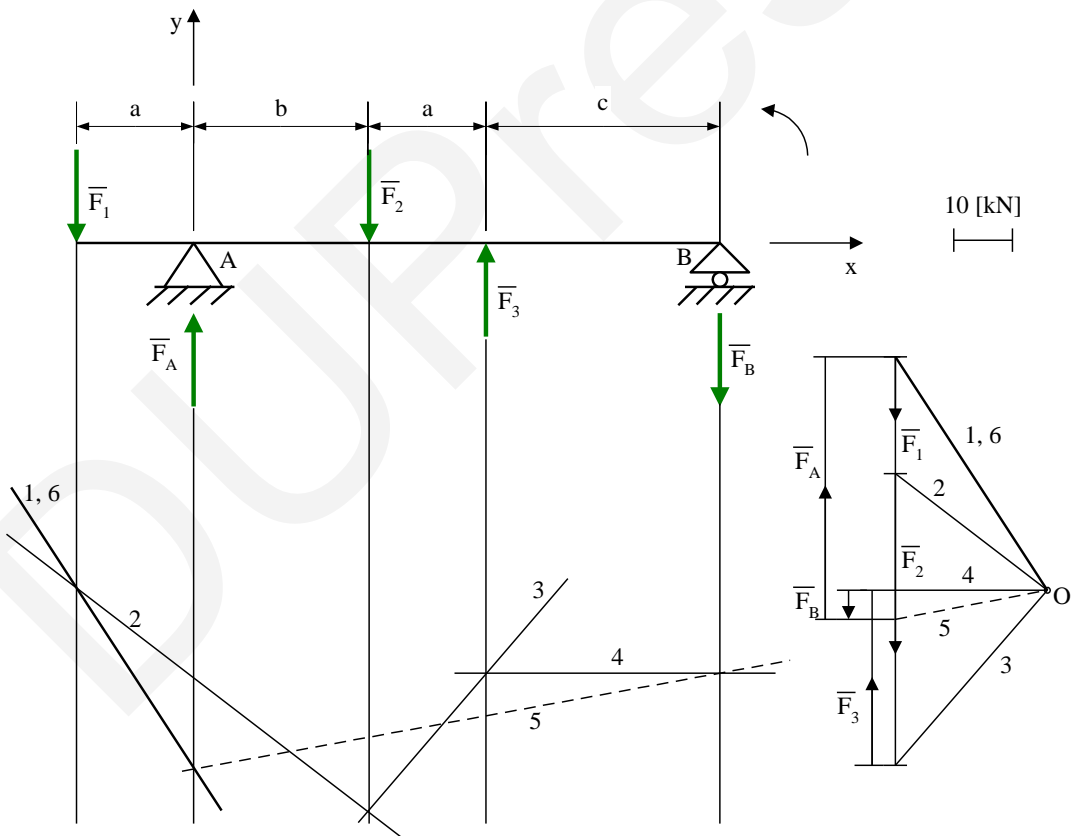
$$F_B = \frac{-20 + 75 - 75}{4.5} = -4.44 \text{ [kN]}$$

$$\text{II. } F_{Ay} = -F_B + F_1 + F_2 - F_3 = 4.44 + 20 + 50 - 30 = 44.44 \text{ [kN]}$$

Checking:

$$\begin{aligned} \sum_i M_B &= F_1 \cdot (2a + b + c) - F_{Ay} \cdot (a + b + c) + F_2 \cdot (a + c) - F_3 \cdot c = \\ &= 20 \cdot 5.5 - 44.44 \cdot 4.5 + 50 \cdot 3 - 30 \cdot 2 = 0 \end{aligned}$$

Construction:



Clicking on the link below the construction above can be followed as a Power Point animation step by step:  
Animation 8.4

## 9. Constraints with friction

In this section, we analyse practical constraints like a rough surface, a pin or rope with friction, or a wheel (tyre) with rolling resistance.

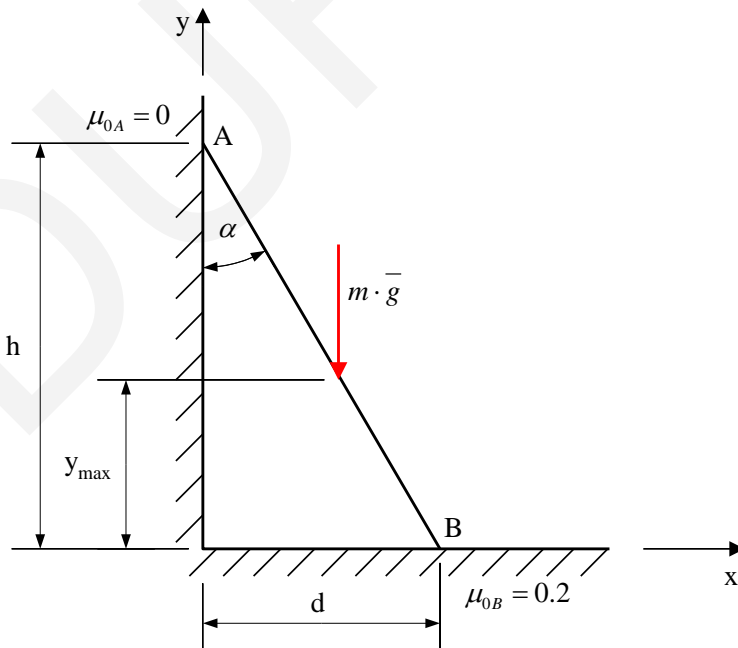
### 9.1 Simple support

The theoretical part of section 9.1 was presented in section 2.2. and a numerical problem in this topic for the equilibrium of a particle was also solved in section 2.4. (*Problem 5*). In this section, we present problems for the static equilibrium of rigid disks on a rough surface.

#### Numerical problems

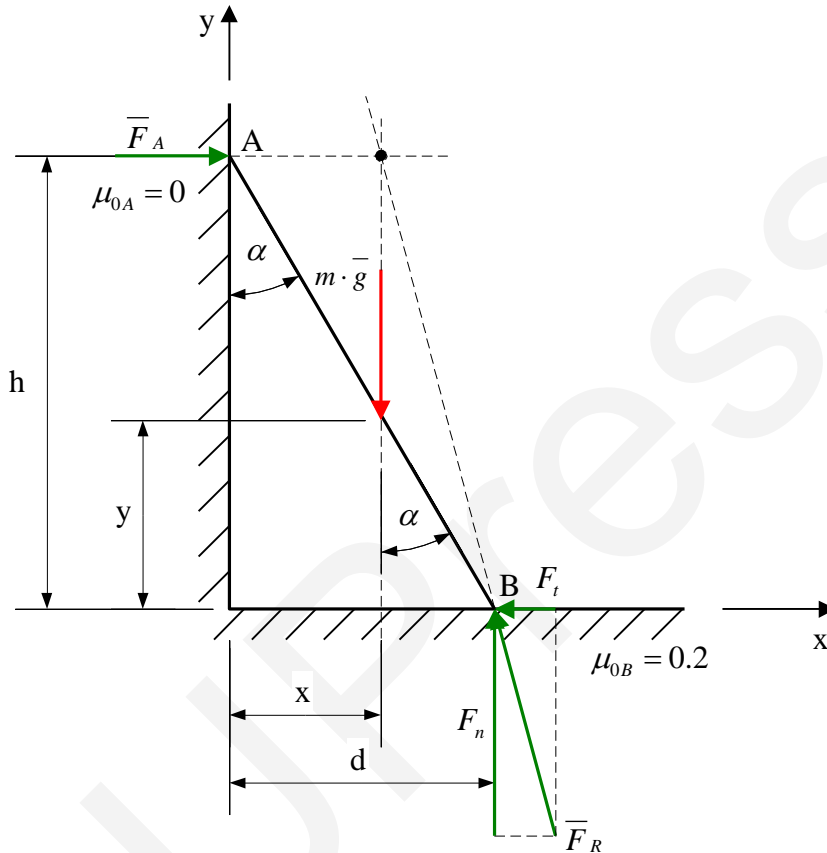
##### *Problem 1:*

A man of mass  $m$  climbs up on ladder  $BA$ . Calculate and construct the maximum height of climbing ( $y_{max}$ ) above which the ladder slips. The mass of the ladder is negligible. The vertical wall is smooth while the horizontal floor is rough. The coefficient of static friction between the floor and the ladder is denoted by  $\mu_{0B}$ .



Data:  $m \cdot g = 1000$  [N];  $d = \sqrt{3}$  [m];  $h = 3$  [m];  $\mu_{0B} = 0,2$ .

Solution:



Calculation:

$$\sum \vec{F} = \vec{F}_A + \vec{F}_R + m \cdot \vec{g} = \vec{0} = \begin{pmatrix} F_A \\ 0 \end{pmatrix} + \begin{pmatrix} -F_t \\ F_n \end{pmatrix} + \begin{pmatrix} 0 \\ -m \cdot g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_A - F_t = 0 \rightarrow F_t = F_A$$

$$\text{II. } F_n - m \cdot g = 0 \rightarrow F_n = m \cdot g$$

$$\sum_i M_B = m \cdot g \cdot (d - x) - F_A \cdot h = 0 \rightarrow F_A = \frac{m \cdot g \cdot (d - x)}{h}$$

$$\tan \alpha = \frac{d}{h} = \frac{d - x}{y} \rightarrow d - x = y \cdot \tan \alpha$$

$$F_A = \frac{m \cdot g \cdot y \cdot \tan \alpha}{h}$$

$$|F_t| \leq \mu_{0B} \cdot F_n \rightarrow -\mu_{0B} \cdot F_n \leq F_t \leq \mu_{0B} \cdot F_n$$

Case 1:

$$F_t \leq \mu_{0B} \cdot F_n$$

$$F_A \leq \mu_{0B} \cdot m \cdot g$$

$$\frac{m \cdot g \cdot y \cdot \tan \alpha}{h} \leq \mu_{0B} \cdot m \cdot g$$

$$\frac{y \cdot \tan \alpha}{h} \leq \mu_0 \rightarrow y \leq \frac{h \cdot \mu_{0B}}{\tan \alpha} = \frac{h \cdot \mu_{0B}}{\frac{d}{h}} = \frac{h^2 \cdot \mu_{0B}}{d} = 1.039 \text{ [m]} = y_{max}$$

Case 2:

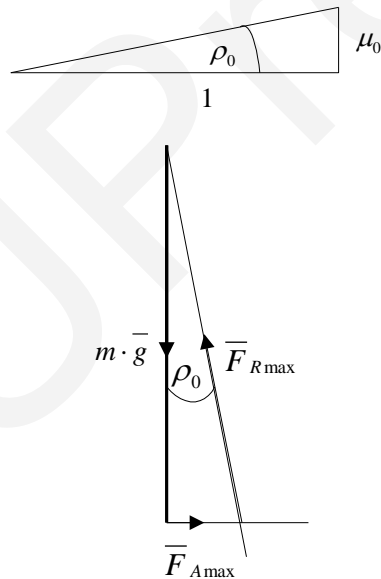
$$-\mu_{0B} \cdot F_n \leq F_t$$

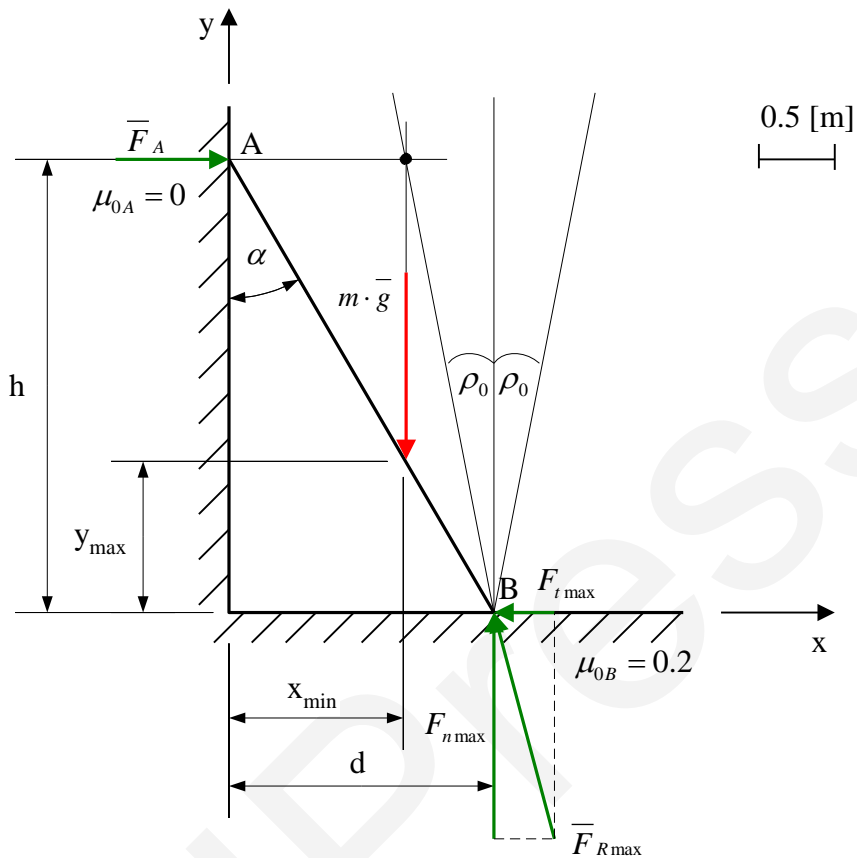
$$-\mu_{0B} \cdot m \cdot g \leq F_A$$

$$-\mu_{0B} \cdot m \cdot g \leq \frac{m \cdot g \cdot y \cdot \tan \alpha}{h}$$

$$-\mu_{0B} \leq \frac{y \cdot \tan \alpha}{h} \rightarrow y \geq \frac{-\mu_{0B} \cdot h}{\tan \alpha} = \frac{-h^2 \cdot \mu_{0B}}{d} = -1.039 \text{ [m]} \rightarrow \text{It is not a real solution.}$$

Construction:

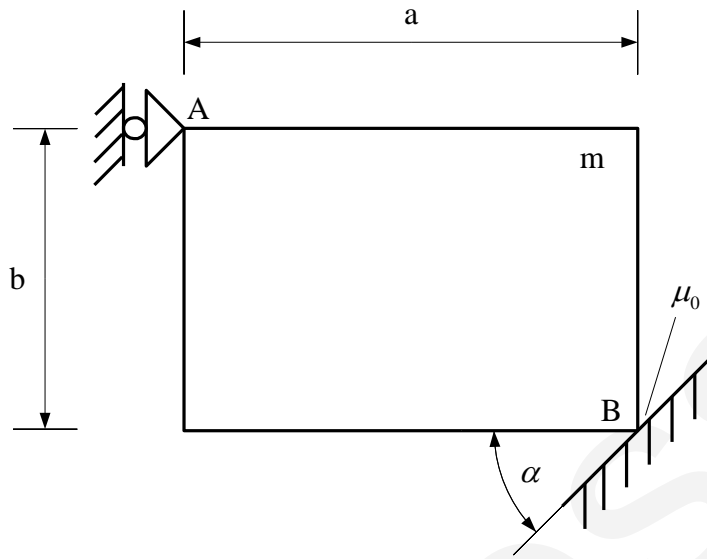




Comparing the length of  $y_{\max}$  with the unit we get that:  $y_{\max} \approx 1 \text{ [m]}$

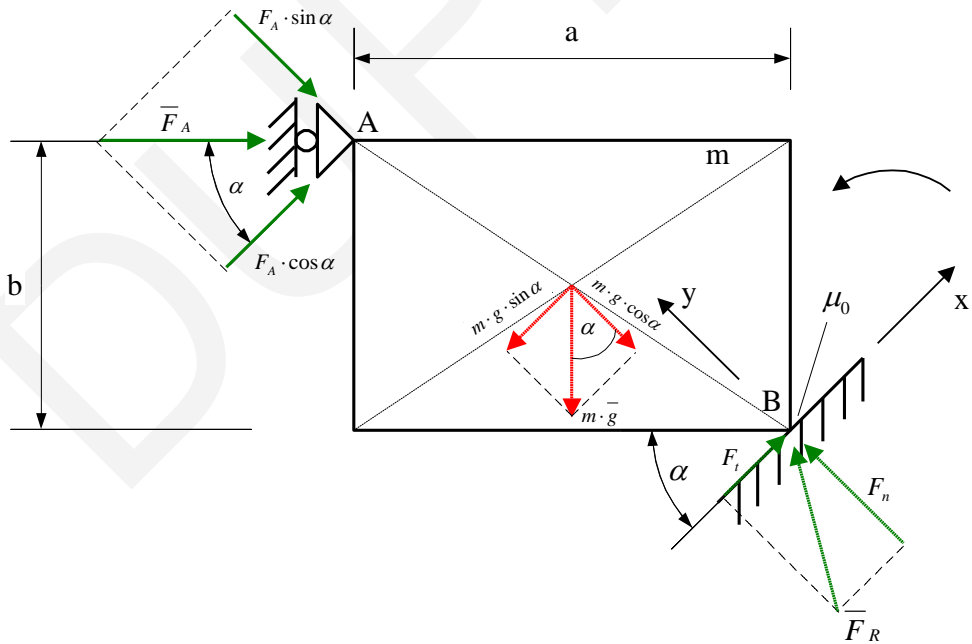
*Problem 2:*

Point A of the rigid plate in the figure is supported by a roller while point B of it leans against a rough surfaced incline. Calculate and construct the minimum value of the coefficient of static friction at which the plate is in static equilibrium.



Data:  $a=60$  [mm],  $b=40$  [mm],  $m=5$  [kg],  $\alpha=45^\circ$

Solution:



Calculation:

$$\sum \vec{F} = \vec{F}_A + \vec{F}_R + m \cdot \vec{g} = \vec{0} = \begin{pmatrix} F_A \cdot \cos \alpha \\ -F_A \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} F_t \\ F_n \end{pmatrix} + \begin{pmatrix} -m \cdot g \cdot \sin \alpha \\ -m \cdot g \cdot \cos \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_A \cdot \cos \alpha + F_t - m \cdot g \cdot \sin \alpha = 0$$

$$\text{II. } -F_A \cdot \sin \alpha + F_n - m \cdot g \cdot \cos \alpha = 0$$

$$\sum_i M_B = -F_A \cdot b + m \cdot g \cdot \frac{a}{2} = 0 \rightarrow F_A = m \cdot g \cdot \frac{a}{2b} = 5 \cdot 9.81 \cdot \frac{60}{80} = 36.79 \text{ [N]}$$

$$\text{I. } F_A \cdot \cos \alpha + F_t - m \cdot g \cdot \sin \alpha = 0 \rightarrow F_t = 8.669 \text{ [N]}$$

$$\text{II. } -F_A \cdot \sin \alpha + F_n - m \cdot g \cdot \cos \alpha = 0 \rightarrow F_n = 60.698 \text{ [N]}$$

$$|F_t| \leq \mu_0 \cdot F_n \rightarrow -\mu_0 \cdot F_n \leq F_t \leq \mu_0 \cdot F_n$$

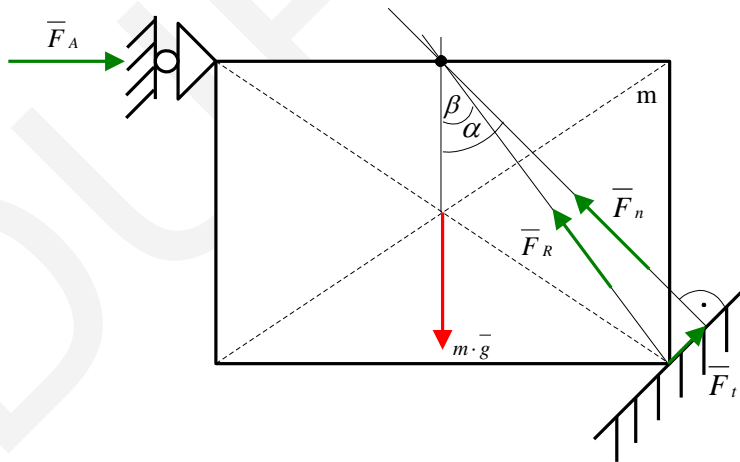
Case 1:

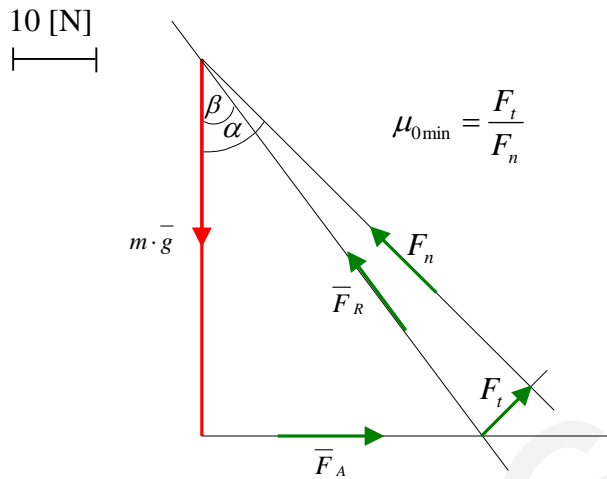
$$-\mu_0 \cdot F_n \leq F_t \rightarrow -\mu_0 \cdot 60.698 \leq 8.669 \rightarrow \mu_0 \geq -0.1428 \text{ (It doesn't give a solution.)}$$

Case 2:

$$F_t \leq \mu_0 \cdot F_n \rightarrow 8.669 \leq \mu_0 \cdot 60.698 \rightarrow 0.1428 \leq \mu_0 \rightarrow \mu_{0min} = 0.1428$$

Construction:





Comparing the length of  $F_t$  and  $F_n$  with the unit we get that:

$$\frac{F_t}{F_n} = \mu_{0min} \approx 0.14$$

## 9.2 Rolling friction (resistance)

To be able to understand the phenomenon of rolling friction we compare the ideal situation, - when the disc and ground is ideally rigid - with the real one, when one of them or both is soft.

*Ideal case:*

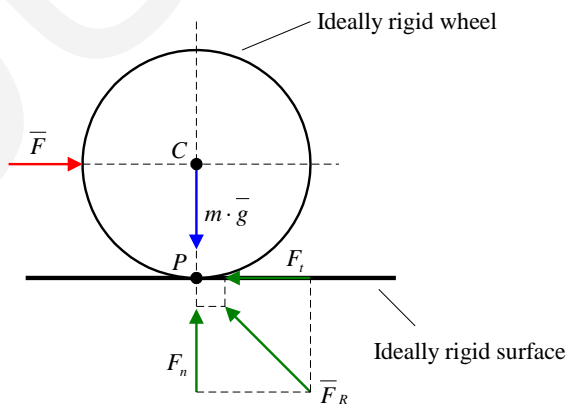


Figure 44

- The wheel and the ground is ideally rigid.
- The wheel touches the ground in an ideal line which goes through point P.
- The action line of component  $F_n$  goes through point P and C. Consequently the moment of  $F_n$  is zero relative to points P and C.
- The wheel is in static equilibrium, if and only if, horizontal force  $\bar{F}$  is equal to zero.

*Real case:*

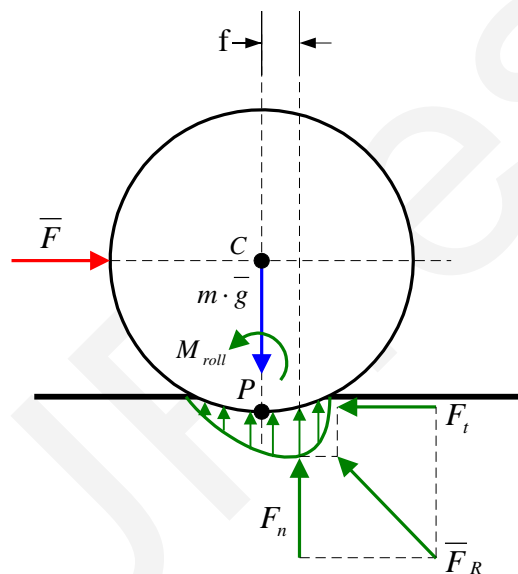


Figure 45

- The wheel or the surface is soft.
- The wheel touches the ground along a surface. Let's denote the resultant of the distributed force system along the surface by  $F_n$ .
- The action line of  $F_n$  is at distance  $f$  of point P and C.
- The wheel is in static equilibrium if and only if  $f \leq f_0$ , where  $f_0$  is the possible maximum value of  $f$  is static equilibrium. ( $f$  is called as the arm of rolling friction)
- Consequently, the wheel won't roll, if and only if:

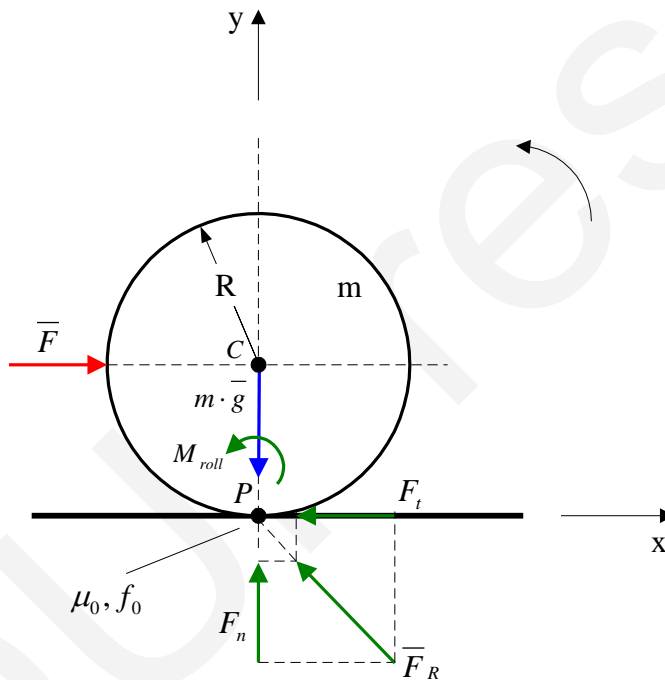
$$|M_{roll}| = f \cdot F_n \leq f_0 \cdot F_n$$

where  $M_{\text{roll}}$  is the moment of rolling friction.

## Numerical problems

### Problem 1:

Calculate the maximum magnitude of force  $\bar{F}$  at which the disc with radius  $R$  is in static equilibrium on the rough horizontal surface. The coefficient of static friction ( $\mu_0$ ) and arm of rolling resistance ( $f_0$ ) are known.



Data:  $m=20$  [kg],  $R=0.5$  [m],  $g=9.81$  [m/s<sup>2</sup>],  $\mu_0=0.6$ ,  $f_0=0.02$  [m].

Solution:

In static equilibrium:  $F_{\min} \leq F \leq F_{\max}$

Equilibrium equations:

$$\sum \bar{F} = \bar{F} + \bar{F}_R + m \cdot \bar{g} = \bar{0} = \begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} -F_t \\ F_n \end{pmatrix} + \begin{pmatrix} 0 \\ -m \cdot g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

I.  $F - F_t = 0 \rightarrow F_t = F$

II.  $F_n - m \cdot g = 0 \rightarrow F_n = m \cdot g$

$$\sum_i M_i = M_{roll} - F \cdot R = 0 \rightarrow M_{roll} = F \cdot R$$

The wheel won't start with rolling, if and only if:

$$|M_{roll}| \leq F_n \cdot f_0$$

Case 1:

$$M_{roll} \leq F_n \cdot f_0$$

$$F \cdot R \leq m \cdot g \cdot f_0$$

$$F \leq \frac{m \cdot g \cdot f_0}{R}$$

Case 2:

$$-F_n \cdot f_0 \leq M_{roll}$$

$$-m \cdot g \cdot f_0 \leq F \cdot R$$

$$F \geq \frac{-m \cdot g \cdot f_0}{R}$$

$$\frac{-m \cdot g \cdot f_0}{R} \leq F \leq \frac{m \cdot g \cdot f_0}{R}$$

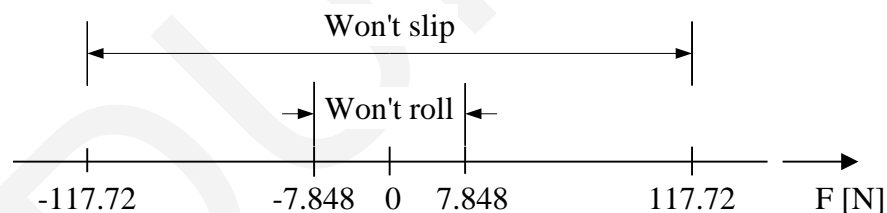
$$-7.848 [N] \leq F \leq 7.848 [N]$$

The wheel won't start with slipping, if and only if:

$$|F_t| \leq \mu_0 \cdot F_n$$

$$-\mu_0 \cdot m \cdot g \leq F \leq \mu_0 \cdot m \cdot g$$

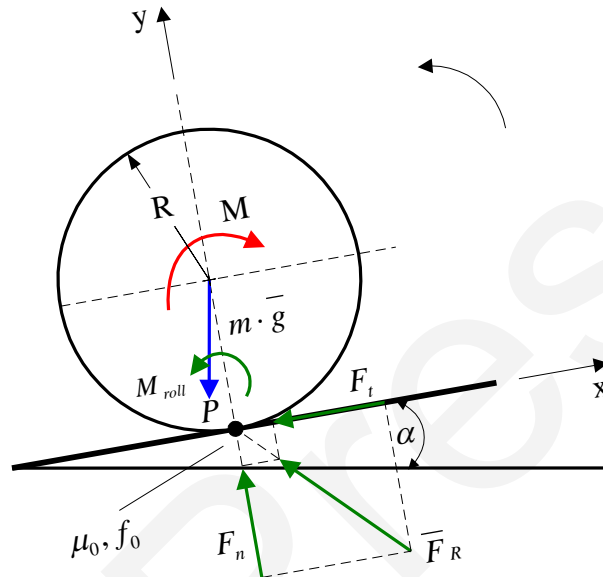
$$-117.72 [N] \leq F \leq 117.72 [N]$$



The wheel is in static equilibrium if it won't roll and slip. Consequently, in static equilibrium  $-7.848 [N] \leq F \leq 7.848 [N] \rightarrow F_{max} = 7.848 [N]$ .

Problem 2:

Calculate the minimum and maximum value of torque  $M$  at which the wheel with radius  $R$  is in static equilibrium on the rough surfaced incline. The coefficient of static friction ( $\mu_0$ ) and arm of rolling friction ( $f_0$ ) are known.



Data:  $R=200$  [mm],  $m \cdot g=5$  [kN],  $\alpha=10^\circ$ ,  $\mu_0=0.3$ ,  $f_0=1.5$  [cm].

Solution:

$$\sum \bar{F} = \bar{F}_R + m \cdot \bar{g} = \bar{0} = \begin{pmatrix} -F_t \\ F_n \end{pmatrix} + \begin{pmatrix} -m \cdot g \cdot \sin \alpha \\ -m \cdot g \cdot \cos \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_t = -m \cdot g \cdot \sin \alpha = -0.868 \text{ [kN]}$$

$$\text{II. } F_n = m \cdot g \cdot \cos \alpha = 4.924 \text{ [kN]}$$

$$\sum_i M_P^i = -M + m \cdot g \cdot \sin \alpha \cdot R + M_{roll} = 0$$

$$M_{roll} = M - m \cdot g \cdot \sin \alpha \cdot R = M - 0.1736 \text{ [kNm]}$$

The wheel won't start with rolling:

$$|M_{roll}| \leq F_n \cdot f_0 \rightarrow -F_n \cdot f_0 \leq M_{roll} \leq F_n \cdot f_0$$

$$-0.07386 \leq M - 0.1736 \leq 0.07386$$

$$0.09974 \text{ [kNm]} \leq M \leq 0.2475 \text{ [kNm]}$$

The wheel won't start with slipping:

$$|F_t| \leq \mu_0 \cdot F_n$$

$$-\mu_0 \cdot m \cdot g \cdot \cos \alpha \leq -m \cdot g \cdot \sin \alpha \leq \mu_0 \cdot m \cdot g \cdot \cos \alpha$$

$$-\mu_0 \leq -\tan \alpha \leq \mu_0 \rightarrow \mu_0 \geq \tan \alpha \geq -\mu_0$$

$$-16.7^\circ \leq \alpha \leq 16.7^\circ$$

$10^\circ \leq 16.7^\circ \rightarrow$  The wheel won't slip.

The wheel is in static equilibrium, if:

$$0.09974 [Nm] \leq M \leq 0.2475 [Nm]$$

### 9.3 Rope friction

The figure below shows a rope on a rough surface cylinder. The position of the cylinder is fixed, the segment of the rope, which is in contact with the cylinder, is described by angle  $\alpha$ . The rope is strained by forces  $\vec{F}_1$  and  $\vec{F}_2$ .

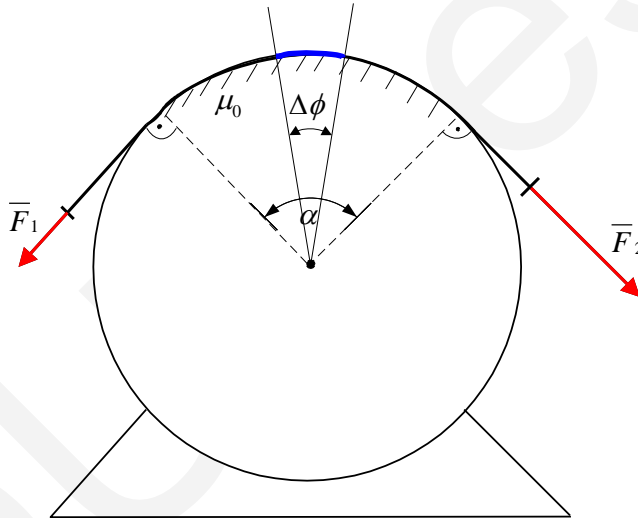


Figure 46

Let's analyse the equilibrium state of an infinitesimally small segment of the rope regarding it as a particle (blue coloured segment). Forces, acting on the rope segment, are presented in Figure 47.

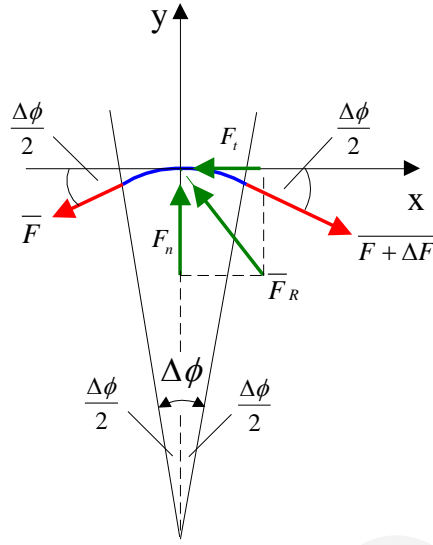


Figure 47

The equilibrium equation of the rope segment is as follows:

$$\sum \bar{F} = \bar{F} + \overline{F + \Delta F} + \bar{F}_R = \bar{0} = \begin{pmatrix} -F \cdot \cos \frac{\Delta\varphi}{2} \\ -F \cdot \sin \frac{\Delta\varphi}{2} \end{pmatrix} + \begin{pmatrix} (F + \Delta F) \cdot \cos \frac{\Delta\varphi}{2} \\ -(F + \Delta F) \cdot \sin \frac{\Delta\varphi}{2} \end{pmatrix} + \begin{pmatrix} -F_t \\ F_n \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } -F \cdot \cos \frac{\Delta\varphi}{2} + (F + \Delta F) \cdot \cos \frac{\Delta\varphi}{2} - F_t = 0$$

$$\text{II. } -F \cdot \sin \frac{\Delta\varphi}{2} - (F + \Delta F) \cdot \sin \frac{\Delta\varphi}{2} + F_n = 0$$

$$\text{I. } F_t = -F \cdot \cos \frac{\Delta\varphi}{2} + (F + \Delta F) \cdot \cos \frac{\Delta\varphi}{2} = \Delta F \cdot \cos \frac{\Delta\varphi}{2}$$

$$\text{II. } F_n = F \cdot \sin \frac{\Delta\varphi}{2} + (F + \Delta F) \cdot \sin \frac{\Delta\varphi}{2} = 2 \cdot F \cdot \sin \frac{\Delta\varphi}{2} + \Delta F \cdot \sin \frac{\Delta\varphi}{2}$$

$$\text{I. } dF_t = \lim_{\Delta\varphi \rightarrow 0} \left( \Delta F \cdot \cos \frac{\Delta\varphi}{2} \right) = dF \cdot 1 = dF$$

$$\text{II. } dF_n = \lim_{\Delta\varphi \rightarrow 0} \left( 2 \cdot F \cdot \sin \frac{\Delta\varphi}{2} + \Delta F \cdot \sin \frac{\Delta\varphi}{2} \right) = 2 \cdot F \cdot \sin \frac{d\varphi}{2} = 2 \cdot F \cdot \frac{d\varphi}{2} = F \cdot d\varphi$$

Remark: If  $\Delta\varphi \rightarrow 0$  then  $\sin \frac{\Delta\varphi}{2} = \frac{\Delta\varphi}{2}$

Hence  $\lim_{\Delta\varphi \rightarrow 0} (\Delta F \cdot \sin \frac{\Delta\varphi}{2})$  is a small quantity of second orders it is negligible.

The rope segment is in static equilibrium if and only if:

$$|dF_t| \leq \mu_0 \cdot dF_n$$

$$|dF| \leq \mu_0 \cdot F \cdot d\varphi$$

$$-\mu_0 \cdot F \cdot d\varphi \leq dF \leq \mu_0 \cdot F \cdot d\varphi$$

$$-\mu_0 \cdot d\varphi \leq \frac{dF}{F} \leq \mu_0 \cdot d\varphi$$

$$-\mu_0 \cdot \int_0^\alpha d\varphi \leq \int_{F_1}^{F_2} \frac{dF}{F} \leq \mu_0 \cdot \int_0^\alpha d\varphi$$

$$-\mu_0 \cdot \alpha \leq \ln \frac{F_2}{F_1} \leq \mu_0 \cdot \alpha$$

$$e^{-\mu_0 \cdot \alpha} \leq \frac{F_2}{F_1} \leq e^{\mu_0 \cdot \alpha}$$

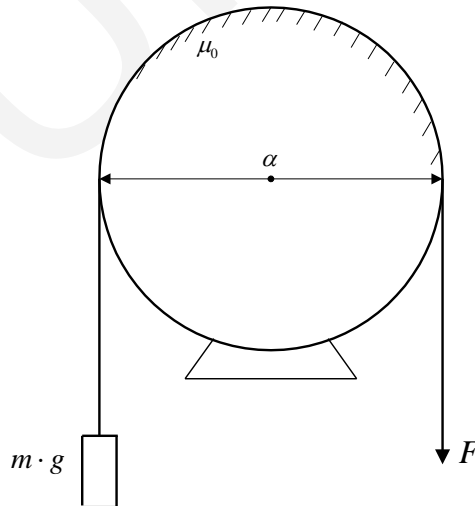
Thus, the rope is in static equilibrium if, and only if:

$$F_1 \cdot e^{-\mu_0 \cdot \alpha} \leq F_2 \leq F_1 \cdot e^{\mu_0 \cdot \alpha}$$

## Numerical problems

### Problem 1:

Calculate the minimum and maximum values ( $F_{min}, F_{max}$ ), between the magnitude of force  $F$  can be varied, if the mechanical system below is in static equilibrium. (The rough surface cylinder is fixed.)



Data:  $m \cdot g = 100$  [N],  $\alpha = \pi$ ,  $\mu_0 = 0.3$

*Solution:*

The system is in static equilibrium, if and only if:

$$m \cdot g \cdot e^{-\mu_0 \cdot \alpha} \leq F \leq m \cdot g \cdot e^{\mu_0 \cdot \alpha}$$

$$100 \cdot e^{-0.9425} \leq F \leq 100 \cdot e^{0.9425}$$

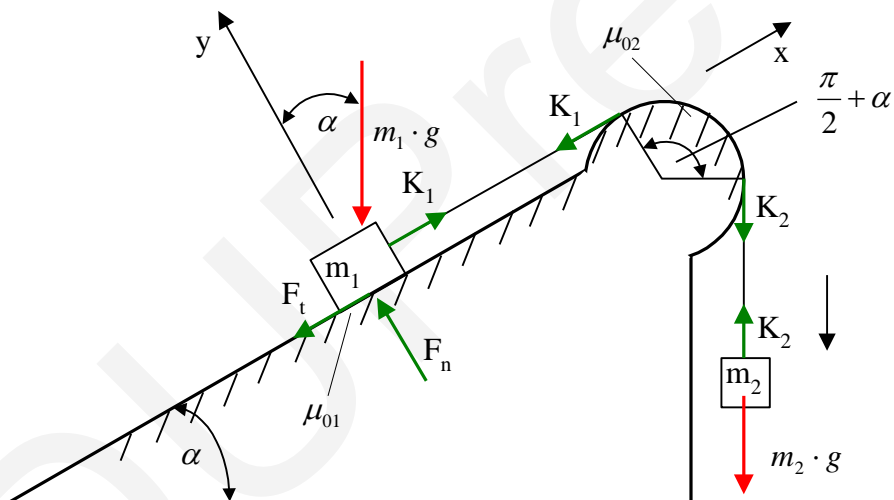
$$38.96 \text{ [N]} \leq F \leq 256.64 \text{ [N]}$$

$$F_{\min} = 38.96 \text{ [N]}$$

$$F_{\max} = 256.64 \text{ [N]}$$

*Problem 2:*

Calculate the minimum and maximum values ( $m_{2\min}, m_{2\max}$ ), between mass  $m_2$  can be varied, if the mechanical system below is in static equilibrium.



*Data:*  $m_1 = 10 \text{ [kg]}$ ,  $\alpha = 30^\circ$ ,  $\mu_{01} = 0.2$ ,  $\mu_{02} = 0.3$

*Solution:*

In static equilibrium:

$$m_{2\min} \leq m_2 \leq m_{2\max}$$

Equilibrium equations for the particle with mass  $m_1$ :

$$\text{I. } \sum_i F_{ix} = K_1 - F_t - m_1 \cdot g \cdot \sin \alpha = 0 \rightarrow F_t = K_1 - m_1 \cdot g \cdot \sin \alpha$$

$$\text{II. } \sum_i F_{iy} = F_n - m_1 \cdot g \cdot \cos \alpha = 0 \rightarrow F_n = m_1 \cdot g \cdot \cos \alpha$$

$$\text{III. } |F_t| \leq \mu_{01} \cdot F_n$$

Equilibrium equations for the particle with mass  $m_2$ :

$$\text{IV. } \sum_i F_i = m_2 \cdot g - K_2 = 0 \rightarrow K_2 = m_2 \cdot g$$

The rope friction is taken into consideration with the inequality below:

$$\text{V. } K_1 \cdot e^{-\mu_{02} \cdot (\frac{\pi}{2} + \alpha)} \leq K_2 \leq K_1 \cdot e^{\mu_{02} \cdot (\frac{\pi}{2} + \alpha)}$$

$$\text{If } F_t \leq \mu_{01} \cdot F_n$$

$$K_1 - m_1 \cdot g \cdot \sin \alpha \leq \mu_{01} \cdot m_1 \cdot g \cdot \cos \alpha$$

$$K_1 \leq m_1 \cdot g \cdot (\sin \alpha + \mu_{01} \cdot \cos \alpha)$$

$$K_1 \leq 66 [N] = K_{1max}$$

$$\text{If } -\mu_{01} \cdot F_n \leq F_t$$

$$-\mu_{01} \cdot m_1 \cdot g \cdot \cos \alpha \leq K_1 - m_1 \cdot g \cdot \sin \alpha$$

$$K_1 \geq m_1 \cdot g \cdot (\sin \alpha - \mu_{01} \cdot \cos \alpha)$$

$$K_1 \geq 32 [N] = K_{1min}$$

$$32 [N] \leq K_1 \leq 66 [N]$$

$$K_{1min} \cdot e^{-\mu_{02} \cdot (\frac{\pi}{2} + \alpha)} \leq K_2 \leq K_{1max} \cdot e^{\mu_{02} \cdot (\frac{\pi}{2} + \alpha)}$$

$$17 [N] \leq K_2 \leq 123.7 [N]$$

$$K_{2min} = 17 [N] \rightarrow m_{2min} = 1.73 [kg]$$

$$K_{2max} = 123.7 [N] \rightarrow m_{2max} = 12.6 [kg]$$

$$1.73 [kg] \leq m_2 \leq 12.6 [kg]$$

## 9.4 Pin friction

The figure below shows the interior of a sleeve bearing.

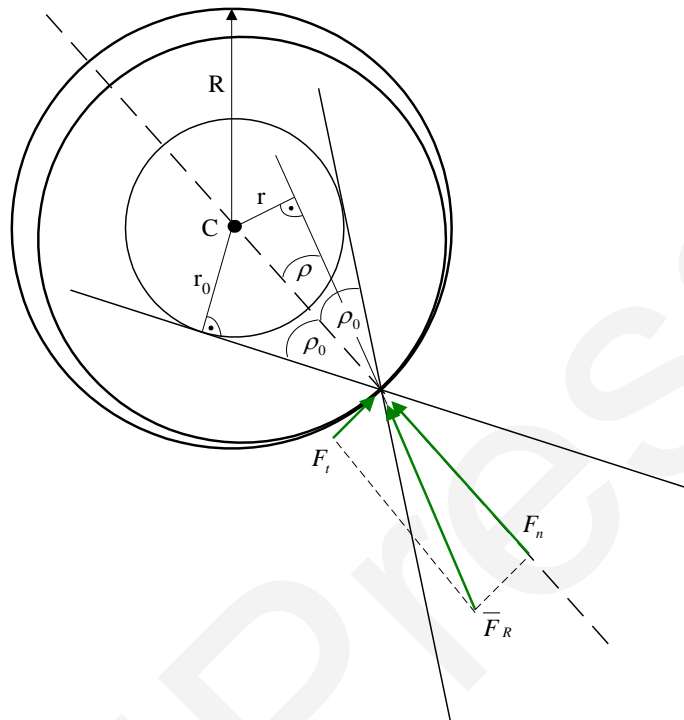


Figure 48

Because of the friction between the housing and the shaft the action line of the reaction force – exerted by the housing on the shaft – is at a distance  $r$  from the centre of the bearing ( $C$ ). Consequently, the reaction force has a moment ( $M_c$ ) relative to centre  $C$ . The shaft is in equilibrium if, and only if:

$$|M_c| = F_R \cdot r \leq F_R \cdot r_0 = F_R \cdot \sin \rho_0 \cdot R$$

In the above equation:

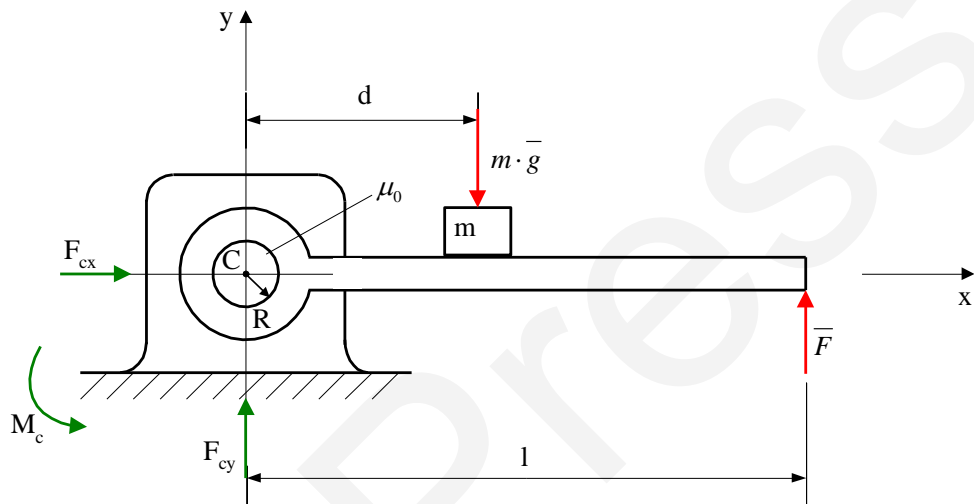
$$\sin \rho_0 = \frac{r_0}{R}, \quad \rho_0 = \arctan \mu_0 \quad F_R = \sqrt{F_n^2 + F_t^2},$$

where  $r_0$  is the maximum value of  $r$  in static equilibrium.

## Numerical problems

### Problem 1:

Calculate the minimum and maximum values ( $F_{min}, F_{max}$ ), between the magnitude of force  $F$  can be varied, if the mechanical system below is in static equilibrium.



Data:  $m=50$  [kg],  $l=2$  [m],  $d=0.8$  [m],  $R=0.01$  [m],  
 $\mu_0=0.3=\tan\rho_0 \rightarrow \rho_0=16.7^\circ$

### Solution:

In static equilibrium:  $F_{min} \leq F \leq F_{max}$

Equilibrium equations:

$$\text{I. } \sum_i F_{ix} = F_{cx} = 0 \rightarrow F_{cx} = 0$$

$$\text{II. } \sum_i F_{iy} = F_{cy} - m \cdot g + F = 0$$

$$F_c = \sqrt{F_{cx}^2 + F_{cy}^2} = m \cdot g - F$$

$$\text{III. } \sum_i M_i = M_c - m \cdot g \cdot d + F \cdot l = 0 \rightarrow M_c = m \cdot g \cdot d - F \cdot l$$

$$-F_c \cdot R \cdot \sin \rho_0 \leq M_c \leq F_c \cdot R \cdot \sin \rho_0$$

*Case 1:*

$$\text{If } -F_C \cdot R \cdot \sin \rho_0 \leq M_C$$

$$-(m \cdot g - F) \cdot R \cdot \sin \rho_0 \leq m \cdot g \cdot d - F \cdot l$$

$$-m \cdot g \cdot R \cdot \sin \rho_0 + F \cdot R \cdot \sin \rho_0 \leq m \cdot g \cdot d - F \cdot l$$

$$-1.41 + 0.0029 \cdot F \leq 392.4 - 2 \cdot F$$

$$2.0029 \cdot F \leq 393.81$$

$$F \leq 196.62 \text{ [N]} = F_{max}$$

*Case 2:*

$$\text{If } M_C \leq F_C \cdot R \cdot \sin \rho_0$$

$$m \cdot g \cdot d - F \cdot l \leq (m \cdot g - F) \cdot R \cdot \sin \rho_0$$

$$m \cdot g \cdot d - F \cdot l \leq m \cdot g \cdot R \cdot \sin \rho_0 - F \cdot R \cdot \sin \rho_0$$

$$392.4 - 2 \cdot F \leq 1.41 - 0.0029 \cdot F$$

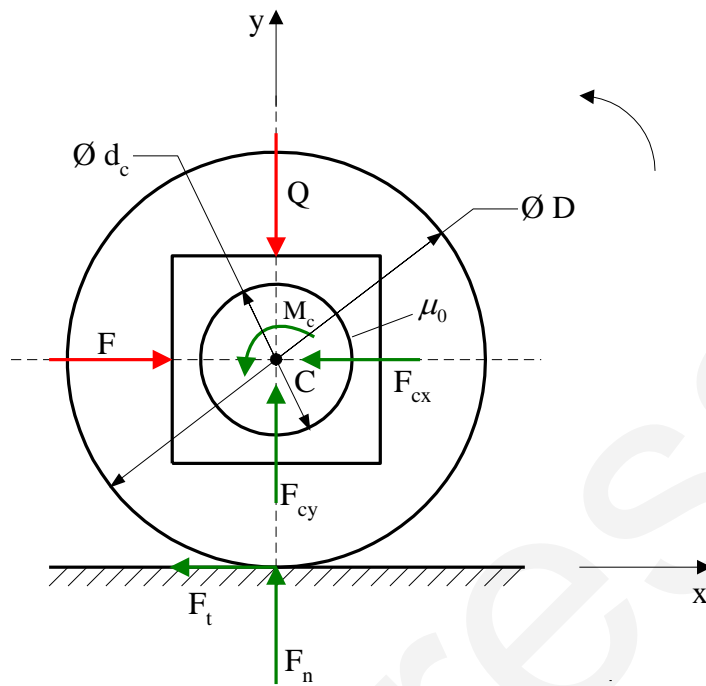
$$-1.9971 \cdot F \leq -390.99$$

$$F \geq 195.78 \text{ [N]} = F_{min}$$

$$195.78 \text{ [N]} \leq F \leq 196.62 \text{ [N]}$$

*Problem 2:*

Calculate the minimum and maximum values ( $F_{min}, F_{max}$ ), between the magnitude of force F can be varied, if the wheel in the figure doesn't roll. Rolling resistance and the mass of the wheel is negligible, and the wheel cannot slip.



Data:  $D=300$  [mm],  $d_c=50$  [mm],  $Q=10$  [kN],  $\mu_0=0.1$

Solution:

Equilibrium equations for the wheel:

$$\sum_i F_{ix} = F - F_t = 0 \rightarrow F_t = F$$

$$\sum_i F_{iy} = F_n - Q = 0 \rightarrow F_n = Q$$

$$\sum_i M_c = M_c - F_t \cdot \frac{D}{2} = 0 \rightarrow M_c = F_t \cdot \frac{D}{2}$$

$$\mu_0 = 0.1 = \tan \rho_0 \rightarrow \rho_0 = 5.71^\circ$$

Since the mass of the wheel is negligible:

$$F_n = F_{cy}, F_t = F_{cx}$$

$$F_c = \sqrt{F_{cx}^2 + F_{cy}^2} = \sqrt{F^2 + Q^2}$$

The wheel don't roll, if and only if:

$$|M_C| \leq F_C \cdot \frac{d_c}{2} \cdot \sin \rho_0$$

Case 1:

$$M_C \leq F_C \cdot \frac{d_c}{2} \cdot \sin \rho_0$$

$$F_t \cdot \frac{D}{2} \leq \sqrt{F^2 + Q^2} \cdot \frac{d_c}{2} \cdot \sin \rho_0$$

$$0.15 \cdot F \leq \sqrt{F^2 + 100} \cdot 0.0025$$

$$60 \cdot F \leq \sqrt{F^2 + 100}$$

If  $F > 0$

$$60 \leq \frac{\sqrt{F^2 + 100}}{F}$$

$$60 \leq \sqrt{\frac{F^2 + 100}{F^2}}$$

$$3600 \leq \frac{F^2 + 100}{F^2}$$

$$F \leq \sqrt{\frac{100}{3599}}$$

$$0 \leq F \leq 0.1667 \text{ [kN]}$$

If  $F < 0$ : This case is unnecessary to analyze.

Case 2:

$$-F_C \cdot \frac{d_c}{2} \cdot \sin \rho_0 \leq M_C$$

$$-\sqrt{F^2 + Q^2} \cdot \frac{d_c}{2} \cdot \sin \rho_0 \leq F_t \cdot \frac{D}{2}$$

$$-\sqrt{F^2 + 100} \cdot 0.0025 \leq 0.15 \cdot F$$

If  $F < 0$

$$60 \leq \frac{-\sqrt{F^2 + 100}}{F}$$

$$60 \leq -\sqrt{\frac{F^2 + 100}{F^2}}$$

$$3600 \geq \frac{F^2 + 100}{F^2}$$

$$3600 \cdot F^2 \geq F^2 + 100$$

$$3599 \cdot F^2 \geq 100$$

$$F^2 \geq 0.0277$$

$$-0.1667 \text{ [kN]} \leq F \leq 0$$

$$-0.1667 \text{ [kN]} \leq F \leq 0.1667 \text{ [kN]}$$

## 9.5 Wedge

Figure 49 shows a wedge with negligible mass. Calculate and construct the minimum and maximum values between vertical force  $F$  can be varied if the wedge is in static equilibrium.

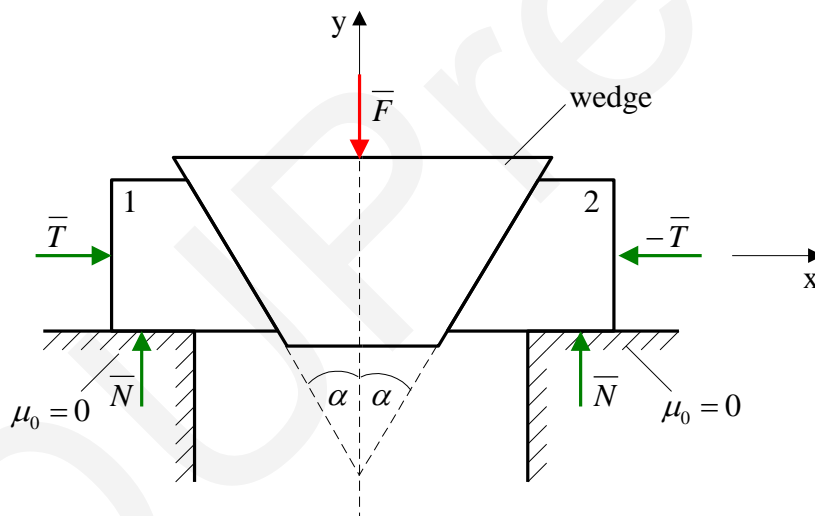


Figure 49

The mechanical system which consists of the wedge and also block 1 and 2 is in static equilibrium:

$$\sum_i \vec{F}_i = \vec{T} - \vec{T} + \vec{F} + 2 \cdot \vec{N} = \vec{0}$$

$$\begin{pmatrix} 0 \\ -F \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \cdot N \end{pmatrix} = \vec{0} \rightarrow N = \frac{F}{2}$$

Block 1 is also in static equilibrium:

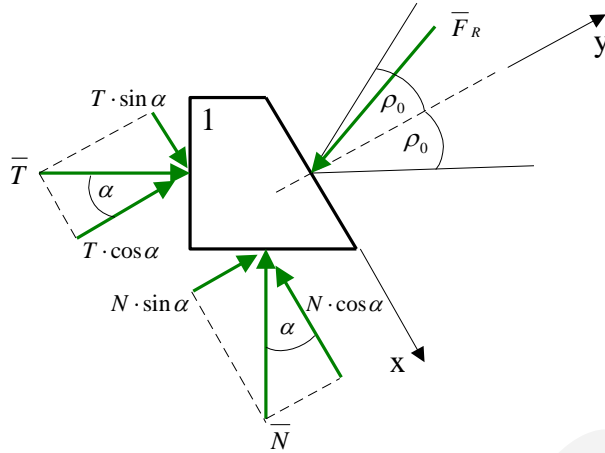


Figure 50

$$\sum_i \bar{F}_i = \bar{T} + \bar{N} + \bar{F}_R = \bar{0}$$

$$\begin{pmatrix} T \cdot \sin \alpha \\ T \cdot \cos \alpha \end{pmatrix} + \begin{pmatrix} -N \cdot \cos \alpha \\ N \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} F_t \\ -F_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

I.  $F_t = N \cdot \cos \alpha - T \cdot \sin \alpha$

II.  $F_n = T \cdot \cos \alpha + N \cdot \sin \alpha$

The wedge is in equilibrium if, and only if:

$$|F_t| \leq \mu_0 \cdot F_n$$

Case 1:

$$N \cdot \cos \alpha - T \cdot \sin \alpha \leq \mu_0 \cdot (T \cdot \cos \alpha + N \cdot \sin \alpha)$$

$$N \cdot \cos \alpha - T \cdot \sin \alpha \leq \mu_0 \cdot T \cdot \cos \alpha + \mu_0 \cdot N \cdot \sin \alpha$$

$$\frac{F}{2} - T \cdot \tan \alpha \leq T \cdot \tan \rho_0 + \tan \rho_0 \cdot \frac{F}{2} \cdot \tan \alpha$$

$$\frac{F}{2} - \frac{F}{2} \cdot \tan \alpha \cdot \tan \rho_0 \leq T \cdot \tan \rho_0 + T \cdot \tan \alpha$$

$$\frac{F}{2} \cdot (1 - \tan \alpha \cdot \tan \rho_0) \leq T \cdot (\tan \rho_0 + \tan \alpha)$$

$$F \leq 2 \cdot T \cdot \frac{\tan \rho_0 + \tan \alpha}{1 - \tan \alpha \cdot \tan \rho_0}$$

$$F \leq 2 \cdot T \cdot \tan(\alpha + \rho_0)$$

Case 2:

$$-\mu_0 \cdot F_n \leq F_t$$

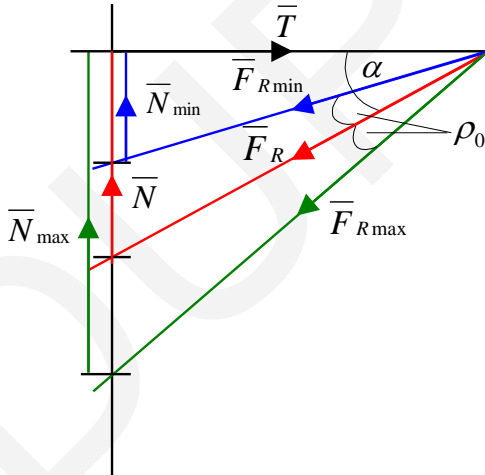
$$\begin{aligned}
-\mu_0 \cdot (T \cdot \cos \alpha + N \cdot \sin \alpha) &\leq N \cdot \cos \alpha - T \cdot \sin \alpha \\
-\mu_0 \cdot T \cdot \cos \alpha - \mu_0 \cdot N \cdot \sin \alpha &\leq N \cdot \cos \alpha - T \cdot \sin \alpha \\
-T \cdot \tan \rho_0 - \tan \rho_0 \cdot \frac{F}{2} \cdot \tan \alpha &\leq \frac{F}{2} - T \cdot \tan \alpha \\
T \cdot \tan \alpha - T \cdot \tan \rho_0 &\leq \frac{F}{2} + \frac{F}{2} \cdot \tan \alpha \cdot \tan \rho_0 \\
T \cdot (\tan \alpha - \tan \rho_0) &\leq \frac{F}{2} \cdot (1 + \tan \alpha \cdot \tan \rho_0) \\
2 \cdot T \cdot \frac{\tan \alpha - \tan \rho_0}{1 + \tan \alpha \cdot \tan \rho_0} &\leq F \\
2 \cdot T \cdot \tan(\alpha - \rho_0) &\leq F
\end{aligned}$$

Thus, the wedge is in equilibrium if and only if:

$$2 \cdot T \cdot \tan(\alpha - \rho_0) \leq F \leq 2 \cdot T \cdot \tan(\alpha + \rho_0)$$

If  $F_{\min} = 2 \cdot T \cdot \tan(\alpha - \rho_0) \leq 0 \rightarrow \alpha \leq \rho_0$  then the wedge is considered to be self-locking. It means that decreasing the magnitude of force  $\bar{F}$  down to zero the wedge remains in static equilibrium.

*Construction:*



$$F_{\min} = 2 \cdot N_{\min}$$

$$F_{\max} = 2 \cdot N_{\max}$$

Figure 51

## 9.6 Groove

Figure 52 shows a groove in a steel block. In the groove, there is a wedge with mass  $m$ . Calculate the maximum value of horizontal force  $F$  at which the wedge is in static equilibrium in the groove.

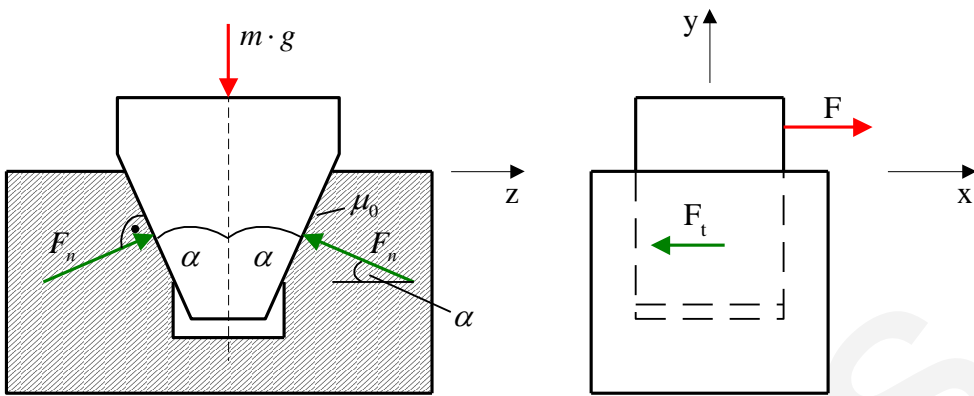


Figure 52

Equilibrium equations for the wedge:

$$\sum_i F_{ix} = F - 2 \cdot F_t = 0$$

$$\sum_i F_{iy} = -m \cdot g + 2 \cdot F_n \cdot \sin \alpha = 0$$

$$\sum_i F_{iz} = F_n \cdot \cos \alpha - F_n \cdot \cos \alpha = 0$$

The wedge is in static equilibrium if, and only if:

$$|F_t| \leq \mu_0 \cdot F_n$$

$$\text{If } F_t \leq \mu_0 \cdot F_n$$

$$F \leq \frac{\mu_0}{\sin \alpha} \cdot m \cdot g$$

$$\text{If } -\mu_0 \cdot F_n \leq F_t$$

$$-\frac{\mu_0}{\sin \alpha} \cdot m \cdot g \leq F$$

$$-\frac{\mu_0}{\sin \alpha} \cdot m \cdot g \leq F \leq \frac{\mu_0}{\sin \alpha} \cdot m \cdot g$$

Let's introduce the virtual coefficient of static friction as follows:

$$\mu'_0 = \frac{\mu_0}{\sin \alpha} > \mu_0$$

If  $\alpha = 90^\circ$  then  $\mu'_0 = \mu_0$ . The figure below represents this special case.

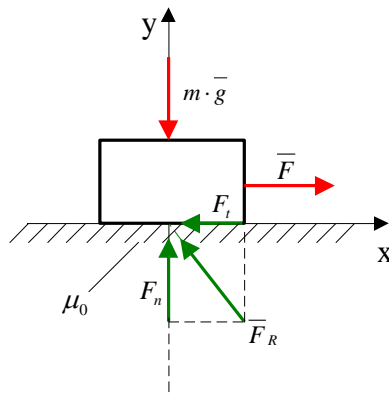


Figure 53

## 9.7 Screw

The figure below represents a screw with flat thread:

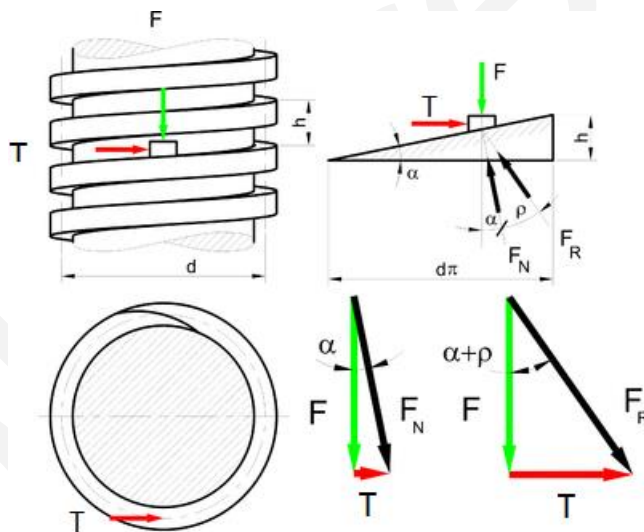


Figure 54

On the basis of figure 53 the following equations are valid.

$$\tan \alpha = \frac{h}{d\pi}, \quad \mu = \tan \rho, \quad \tan(\alpha + \rho) = \frac{T}{F}, \quad T = F \cdot \tan(\alpha + \rho)$$

The torque required to tighten the screw with force  $F$  is as follows:

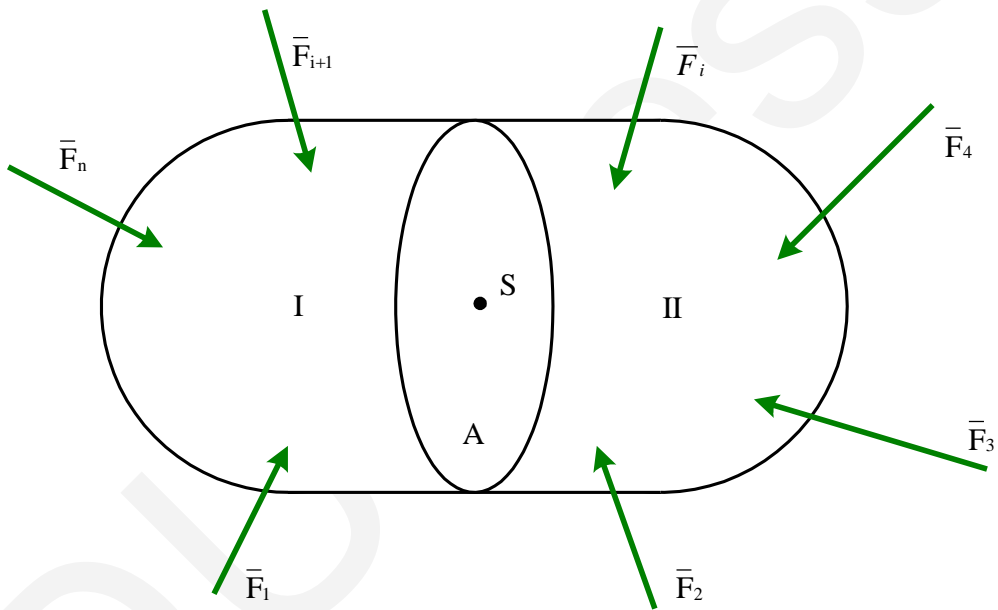
$$M_1 = \frac{d}{2} \cdot F \cdot \tan(\alpha + \rho)$$

The torque required to loosen the screw is:

$$M_2 = \frac{d}{2} \cdot F \cdot \tan(\alpha - \rho)$$

## 10. Loading diagrams of beams

In the previous sections we have learned how to calculate the unknown external forces and moments acting on a mechanical structure in equilibrium. In this section, as a next step, we deal with the calculation of the internal forces and moments. *Figure 55* illustrates a rigid body which is loaded by external forces  $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_i, \vec{F}_{i+1}, \dots, \vec{F}_n$ .



*Figure 55*

Let's divide the body into two parts with plane A. The centre of gravity of the intersecting plane A is denoted by S.

Since the body is in equilibrium:

$$\sum_{i=1}^n \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_i + \vec{F}_{i+1} + \dots + \vec{F}_n = \vec{0} \quad (10.1)$$

$$\sum_{i=1}^n \vec{M}_i = \sum_{i=1}^n (\vec{r}_{Si} \times \vec{F}_i) = \vec{0} \quad (10.2)$$

( $\vec{r}_{Si}$  denotes the vector which points from point  $S$  to the terminal point of force  $\vec{F}_i$ )

The body and all of its parts are in equilibrium. The equilibrium of part I is ensured by the  $\vec{F}_1, \vec{F}_{i+1} \dots \vec{F}_n$  external forces and also by the internal force system which is distributed along plane A. The resultant force and moment of the above internal force system, relative to point  $S$ , is denoted with  $\vec{F}_{int}$  and  $\vec{M}_{int}$ .

$$\sum_i \vec{F}_i = \vec{F}_1 + \vec{F}_{i+1} + \dots + \vec{F}_n + \vec{F}_{int} = \vec{0} \quad (10.3)$$

$$\sum_i \vec{M}_i = \vec{M}_1 + \vec{M}_{i+1} + \dots + \vec{M}_n + \vec{M}_{int} = \vec{0} \quad (10.4)$$

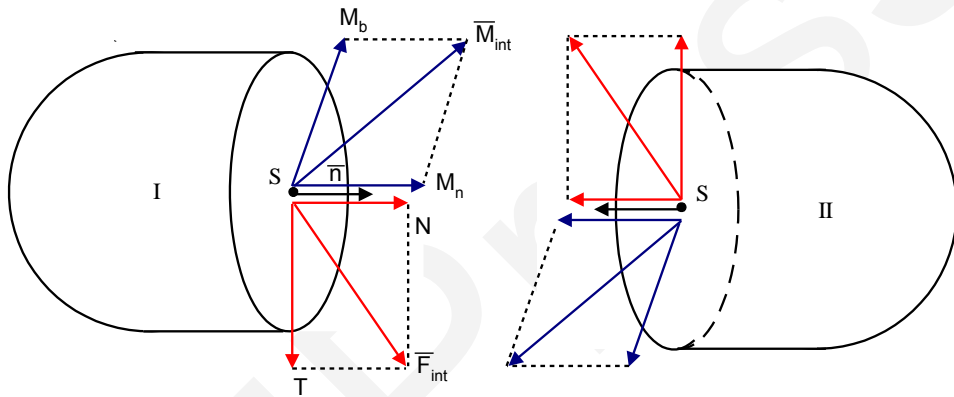


Figure 56

Let's decompose the  $\vec{F}_{int}$  and  $\vec{M}_{int}$  vectors into components which are parallel and normal to plane A. Components  $T$  and  $N$  of  $\vec{F}_{int}$  are called as shear and normal force, while components  $M_b$  and  $M_n$  of  $\vec{M}_{int}$  are as bending and torsional moment.

Specially, if the body is a beam:

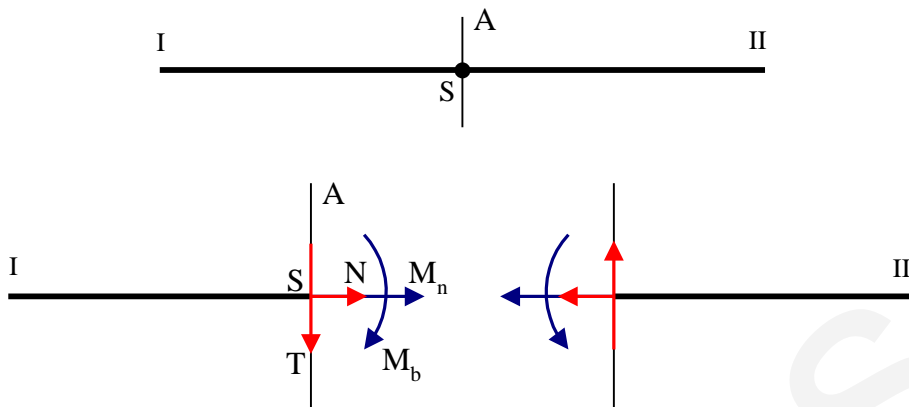


Figure 57

Vector  $\bar{M}_b$  is both perpendicular to N and T and in the case of part I points inwards while in the case of part II outwards the plane of the figure.

Figure 58 represents the connection between the scalar and vector bending moments.

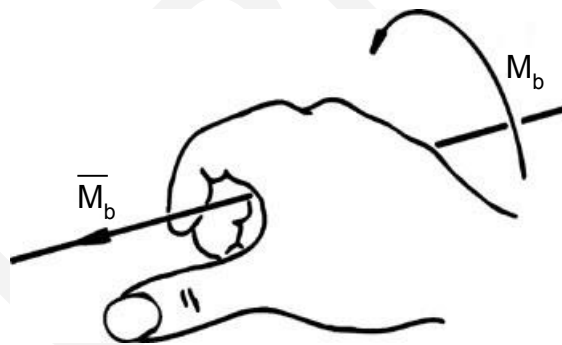


Figure 58

In the following table the sign conventions are summarized. A component is regarded as positive in the cases below:

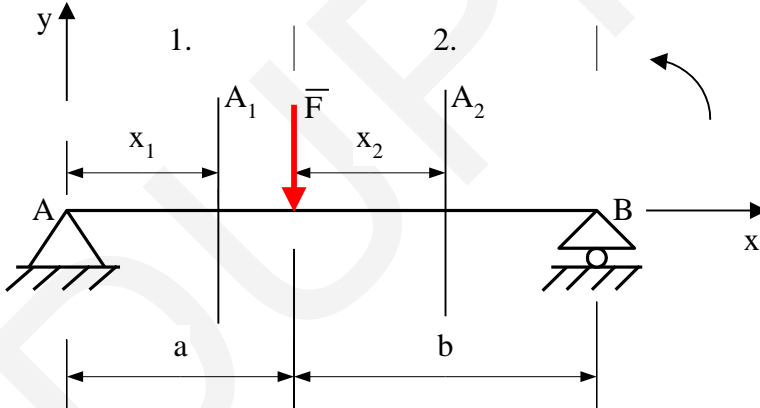
$N > 0$			or
$T > 0$			or

$M_n > 0$			or
$M_b > 0$			or

### Numerical problems

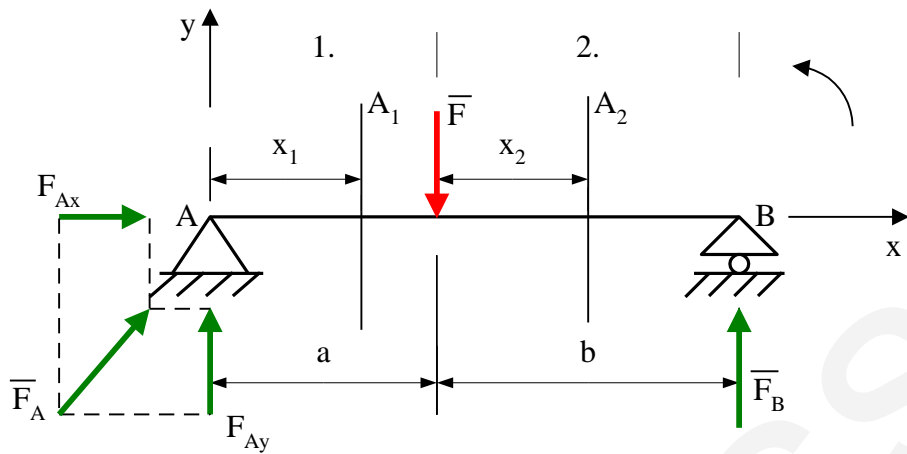
#### Problem 1:

- Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces acting on the supported beam in the figure.
- Calculate the normal force, shear force and bending moment as a function of coordinate  $x$ .
- Draw the normal force, shear force and bending moment diagrams of the beam.



Data:  $F=100$  [N],  $a=3$ [m],  $b = 4$  [m]

Solution:



a)

$$\sum_i \bar{F}_i = \bar{F} + \bar{F}_A + \bar{F}_B = \begin{pmatrix} 0 \\ -F \end{pmatrix} + \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} 0 \\ F_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_{Ax} = 0$$

$$\text{II. } -F + F_{Ay} + F_B = 0 \rightarrow F_{Ay} = F - F_B = 57.14 \text{ [N]}$$

$$\text{III. } \sum_i M_i = -F \cdot a + F_B \cdot (a + b) = 0 \rightarrow F_B = F \cdot \frac{a}{a+b} = 42.86 \text{ [N]}$$

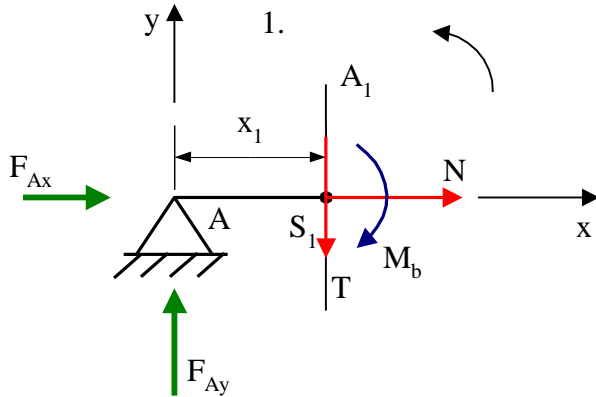
Checking:

$$\sum_i M_B = -F_{Ay} \cdot (a + b) + F \cdot b = -57.14 \cdot (3 + 4) + 100 \cdot 4 = 0$$

b)

Concentrated and distributed forces and torques divide a beam into sections.

In this problem force  $\bar{F}$  divides the beam into two sections (section 1 and 2). Let's intersect section 1 and 2 with planes  $A_1$  and  $A_2$  at arbitrary distances  $x_1$  and  $x_2$  from their initial points. Plane  $A_1$  divides beam A-B into two parts. Both parts are in static equilibrium. Let's write equilibrium equations for the A-S<sub>1</sub> section of the beam taking into consideration the external and internal forces and moments.

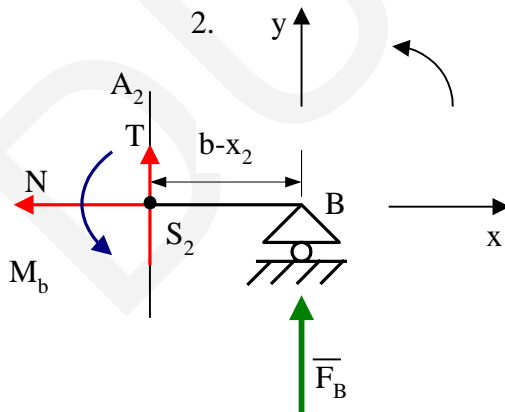


$$\sum_i F_{ix} = F_{Ax} + N = 0 \rightarrow N = -F_{Ax} = 0 \text{ [N]}$$

$$\sum_i F_{iy} = F_{Ay} - T = 0 \rightarrow T = F_{Ay} = 57.14 \text{ [N]}$$

$$\sum_i M_{S_1} = -F_{Ay} \cdot x_1 - M_b = 0 \rightarrow M_b = -F_{Ay} \cdot x_1 = -57.14 \cdot x_1 \text{ [Nm]}$$

Plane A<sub>2</sub> divides beam A-B into two parts. Both parts are in static equilibrium. Let's write equilibrium equations for the S<sub>2</sub>-B section of the beam.



$$\sum_i F_{ix} = -N = 0 \rightarrow N = 0 \text{ [N]}$$

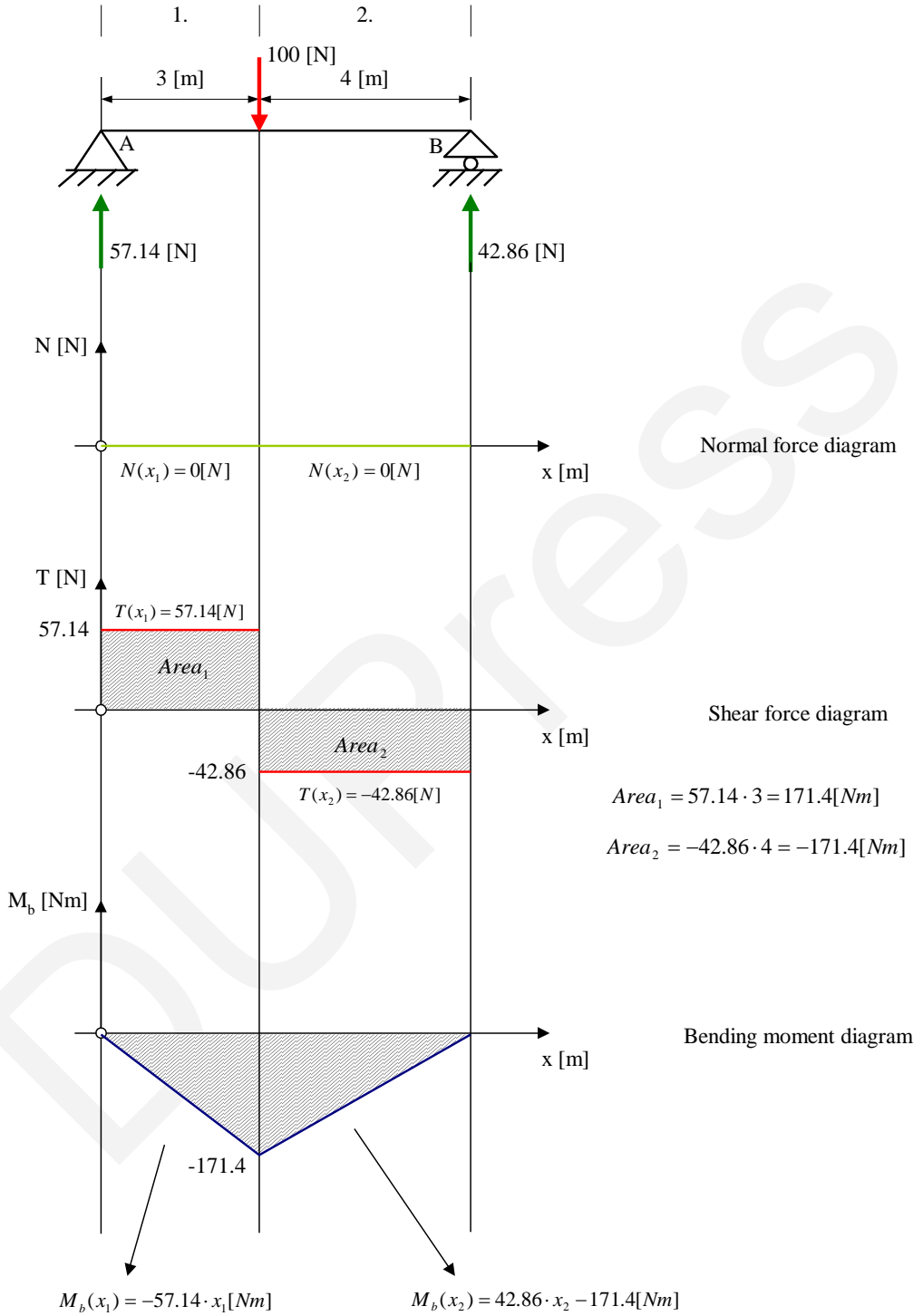
$$\sum_i F_{iy} = T + F_B = 0 \rightarrow T = -F_B = -42.86 \text{ [N]}$$

$$\sum_i M_{S_2} = M_b + F_B \cdot (b - x_2) = 0 \rightarrow M_b = -F_B \cdot (b - x_2)$$

$$= F_B \cdot x_2 - F_B \cdot b = 42.86 \cdot x_2 - 171.4 \text{ [Nm]}$$

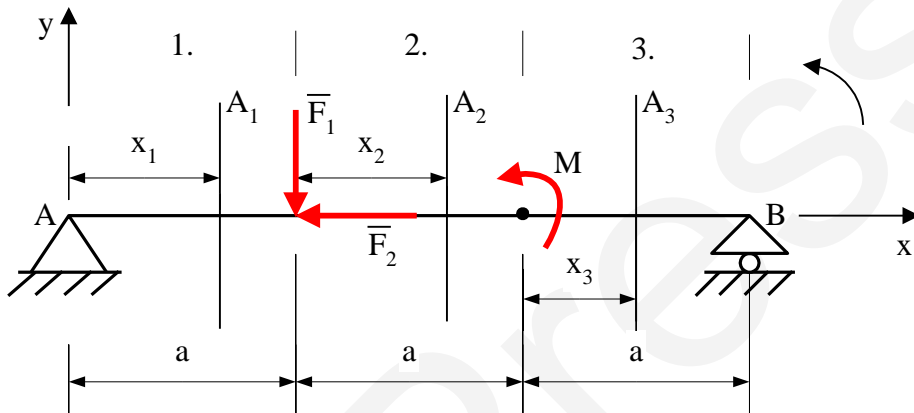
c)

DUPRESS



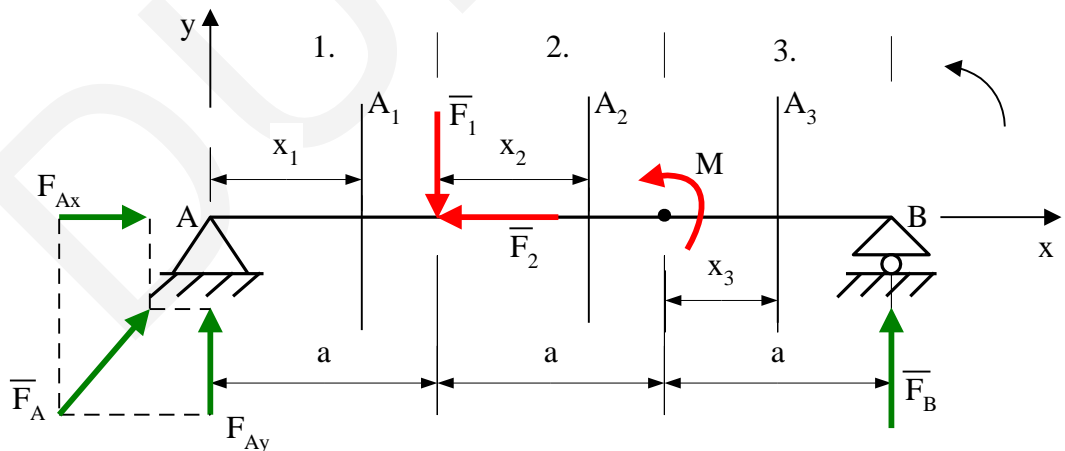
Problem 2:

- Calculate the  $\vec{F}_A$  and  $\vec{F}_B$  reaction forces acting on the supported beam in the figure.
- Calculate the normal force, shear force and bending moment as a function of coordinate  $x$ .
- Draw the normal force, shear force and bending moment diagrams of the beam.



Data:  $F_1=100$  [N],  $F_2=100$  [N],  $M=30$  [Nm],  $a=3$ [m].

Solution:



a)

$$\sum_i \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_A + \vec{F}_B = \begin{pmatrix} 0 \\ -F_1 \end{pmatrix} + \begin{pmatrix} -F_2 \\ 0 \end{pmatrix} + \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} 0 \\ F_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } -F_2 + F_{Ax} = 0 \rightarrow F_{Ax} = F_2 = 100 \text{ [N]}$$

$$\text{II. } -F_1 + F_{Ay} + F_B = 0 \rightarrow F_{Ay} = F_1 - F_B = 70 \text{ [N]}$$

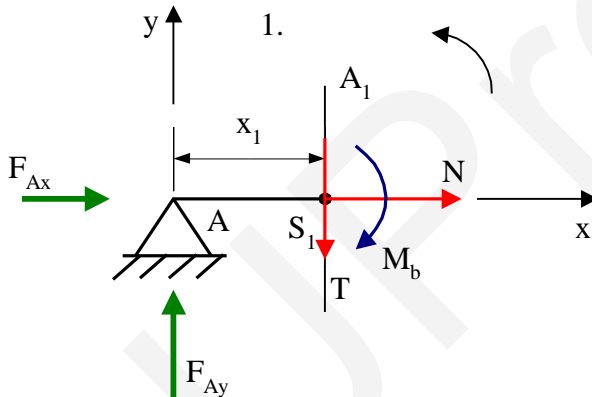
$$\text{III. } \sum_i M_i = -F_1 \cdot a + M + F_B \cdot 3a = 0 \rightarrow F_B = \frac{F_1 \cdot a - M}{3a} = 30 \text{ [N]}$$

Checking:

$$\sum_i M_B = M + F_1 \cdot 2a - F_{Ay} \cdot 3a = 30 + 100 \cdot 6 - 70 \cdot 9 = 0$$

b)

Forces  $\bar{F}_1$  and  $\bar{F}_2$  and torque  $M$  divide the beam into three sections. Let's intersect section 1, 2 and 3 with planes  $A_1$ ,  $A_2$  and  $A_3$  at arbitrary distances  $x_1$ ,  $x_2$  and  $x_3$  from their initial points. Plane  $A_1$  divides beam A-B into two parts. Let's write equilibrium equations for the A- $S_1$  section of the beam taking into consideration the external and the internal forces and moments.

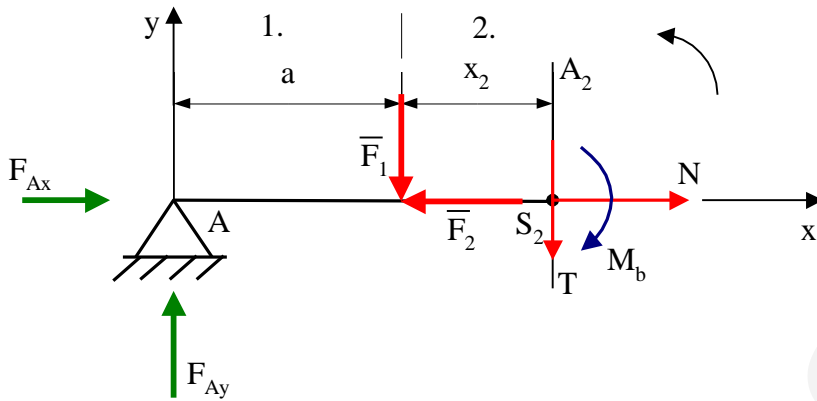


$$\sum_i F_{ix} = F_{Ax} + N = 0 \rightarrow N = -F_{Ax} = -100 \text{ [N]}$$

$$\sum_i F_{iy} = F_{Ay} - T = 0 \rightarrow T = F_{Ay} = 70 \text{ [N]}$$

$$\sum_i M_{S_1} = -F_{Ay} \cdot x_1 - M_b = 0 \rightarrow M_b = -F_{Ay} \cdot x_1 = -70 \cdot x_1 \text{ [Nm]}$$

Let's write equilibrium equations for the A- $S_2$  section of the beam.



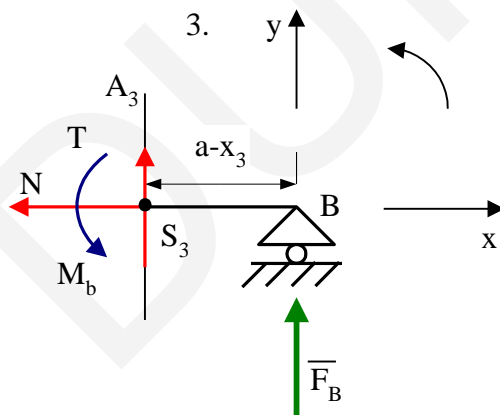
$$\sum_i F_{ix} = F_{Ax} - F_2 + N = 0 \rightarrow N = F_2 - F_{Ax} = 0 \text{ [N]}$$

$$\sum_i F_{iy} = F_{Ay} - F_1 - T = 0 \rightarrow T = F_{Ay} - F_1 = -30 \text{ [N]}$$

$$\sum_i M_i = -F_{Ay} \cdot (a + x_2) + F_1 \cdot x_2 - M_b = 0 \rightarrow$$

$$M_b = 30 \cdot x_2 - 210 \text{ [Nm]}$$

Let's write equilibrium equations for the  $S_3$ -B section of the beam.



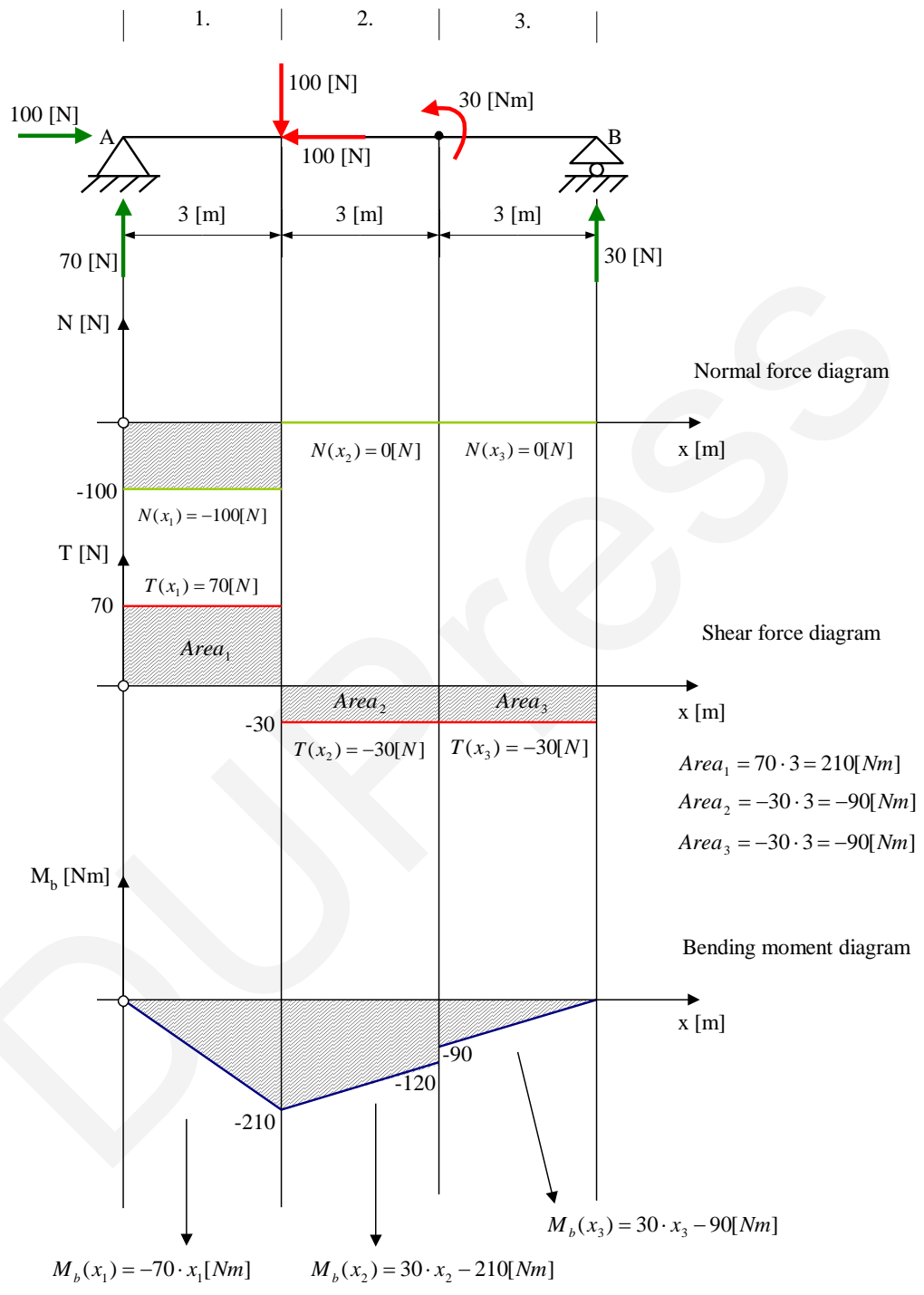
$$\sum_i F_{ix} = -N = 0 \rightarrow N = 0 \text{ [N]}$$

$$\sum_i F_{iy} = T + F_B = 0 \rightarrow T = -F_B = -30 [N]$$

$$\sum_i M_{S_3} = M_b + F_B \cdot (a - x_3) = 0 \rightarrow M_b = 30 \cdot x_3 - 90 [Nm]$$

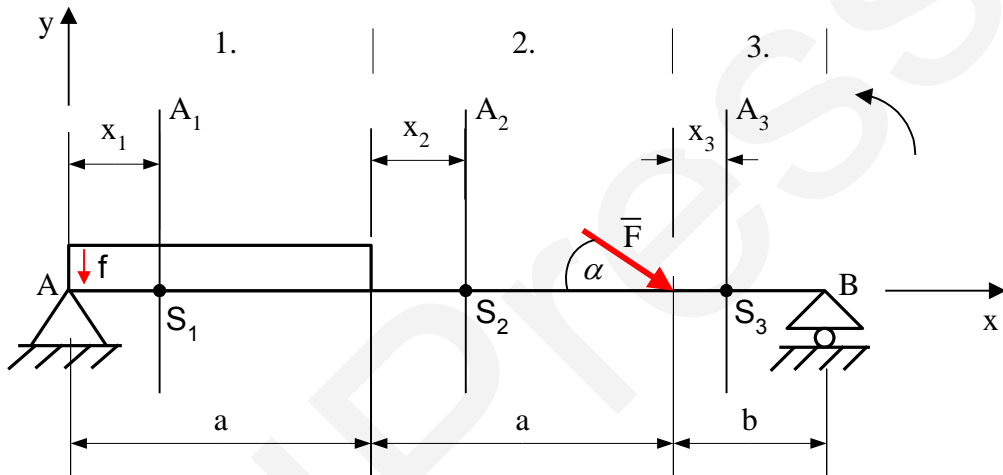
c)

DUPRESS



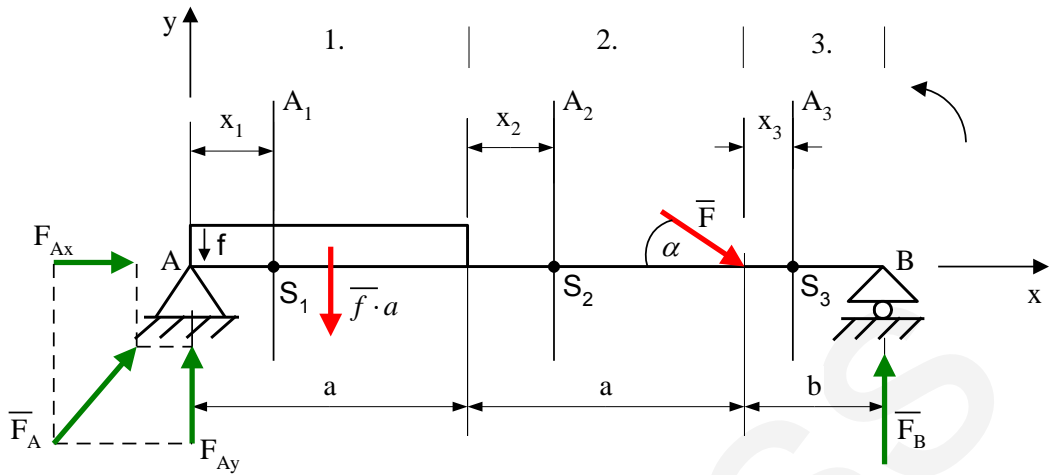
*Problem 3:*

- Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces acting on the supported beam in the figure.
- Calculate the normal force, shear force and bending moment as a function of coordinate  $x$ .
- Draw the normal force, shear force and bending moment diagrams of the beam.



*Data:*  $a=2$  [m],  $b=1$  [m],  $\alpha=30^\circ$ ,  $f=5$  [kN/m],  $F=10$  [kN].

*Solution:*



a)

$$\sum_i \bar{F}_i = \bar{F}_A + \bar{f} \cdot a + \bar{F} + \bar{F}_B = \begin{pmatrix} F_{Ax} \\ F_{Ay} \end{pmatrix} + \begin{pmatrix} 0 \\ -f \cdot a \end{pmatrix} + \begin{pmatrix} F \cdot \cos \alpha \\ -F \cdot \sin \alpha \end{pmatrix} + \begin{pmatrix} 0 \\ F_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{I. } F_{Ax} + F \cdot \cos \alpha = 0 \rightarrow F_{Ax} = -F \cdot \cos \alpha = -8.66 \text{ [kN]}$$

$$\text{II. } F_{Ay} - f \cdot a - F \cdot \sin \alpha + F_B = 0 \rightarrow$$

$$\rightarrow F_{Ay} = f \cdot a + F \cdot \sin \alpha - F_B = 9 \text{ [kN]}$$

$$\text{III. } \sum_i M_A = -f \cdot a \cdot \frac{a}{2} - F \cdot \sin \alpha \cdot 2a + F_B \cdot (2a + b) = 0 \rightarrow$$

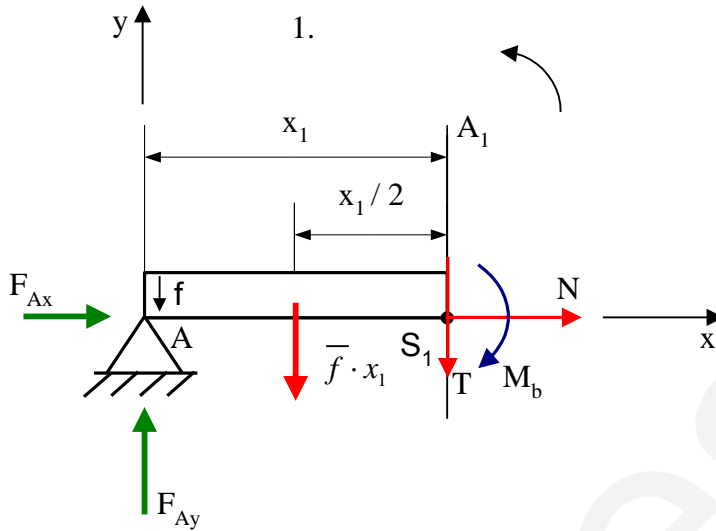
$$\rightarrow F_B = \frac{\frac{f \cdot a^2}{2} + F \cdot \sin \alpha \cdot 2a}{2a + b} = 6 \text{ [kN]}$$

Checking:

$$\sum_i M_B = -F_{Ay} \cdot (2a + b) + f \cdot a \cdot \left( \frac{a}{2} + a + b \right) + F \cdot \sin \alpha \cdot b =$$

$$= -9 \cdot 5 + 5 \cdot 2 \cdot 4 + 10 \cdot \sin 30^\circ \cdot 1 = 0$$

b)

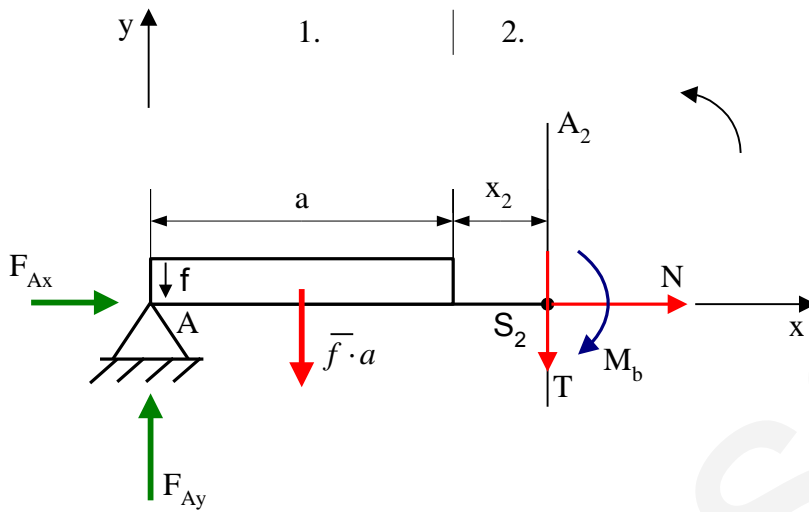


Equilibrium equations for the A-S<sub>1</sub> section of the beam:

$$\sum_i F_{ix} = F_{Ax} + N = 0 \rightarrow N = -F_{Ax} = 8.66 \text{ [kN]}$$

$$\sum_i F_{iy} = F_{Ay} - T - f \cdot x_1 = 0 \rightarrow T = F_{Ay} - f \cdot x_1 = 9 - 5 \cdot x_1 \text{ [kN]}$$

$$\begin{aligned} \sum_i M_i = -F_{Ay} \cdot x_1 + f \cdot x_1 \cdot \frac{x_1}{2} - M_b = 0 &\rightarrow M_b = f \cdot \frac{x_1^2}{2} - F_{Ay} \cdot x_1 \\ &= \frac{5}{2} \cdot x_1^2 - 9 \cdot x_1 \text{ [kNm]} \end{aligned}$$



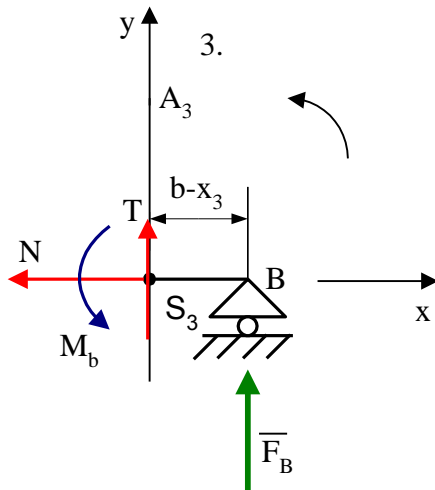
Equilibrium equations for the A-S<sub>2</sub> section of the beam:

$$\sum_i F_{ix} = F_{Ax} + N = 0 \rightarrow N = -F_{Ax} = 8.66 \text{ [kN]}$$

$$\sum_i F_{iy} = F_{Ay} - f \cdot a - T = 0 \rightarrow T = F_{Ay} - f \cdot a = -1 \text{ [kN]}$$

$$\sum_i M_{S_2} = -F_{Ay} \cdot (a + x_2) + f \cdot a \cdot \left(\frac{a}{2} + x_2\right) - M_b = 0$$

$$M_b = (f \cdot a - F_{Ay}) \cdot x_2 + f \cdot \frac{a^2}{2} - F_{Ay} \cdot a \rightarrow M_b = x_2 - 8 \text{ [kNm]}$$



Equilibrium equations for the  $S_3$ -B section of the beam:

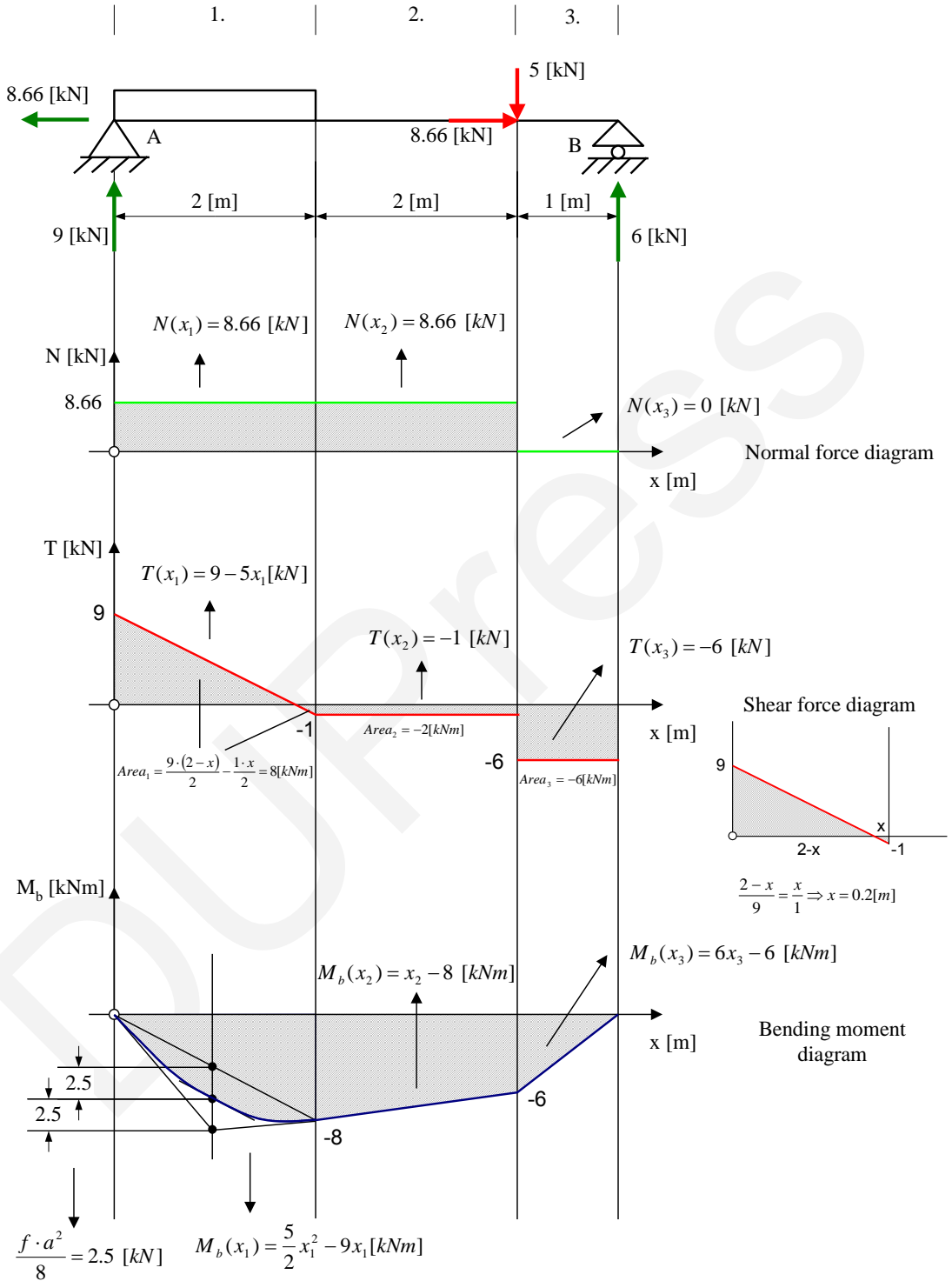
$$\sum_i F_{ix} = -N = 0 \rightarrow N = 0 \text{ [kN]}$$

$$\sum_i F_{iy} = T + F_B = 0 \rightarrow T = -F_B = -6 \text{ [kN]}$$

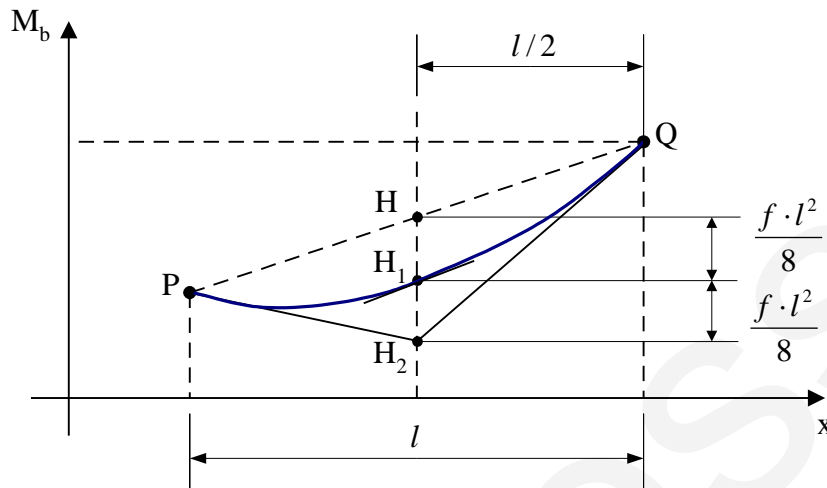
$$\sum_i M_i = M_b + F_B \cdot (b - x_3) = 0 \rightarrow M_b = F_B \cdot x_3 - F_B \cdot b \rightarrow$$

$$M_b = 6 \cdot x_3 - 6 \text{ [kNm]}$$

c)



The construction method of the parabolic arc is presented below. In the figure  $f$  denotes the intensity of the distributed force on the given section.



Steps of the construction:

- 1) Connecting the initial and end points (P and Q) with a dashed line.
- 2) Drawing a vertical line through the midpoint (H) of the above line.
- 3) Measuring down the value of expression  $\frac{f \cdot l^2}{8}$  from point H twice (points H<sub>1</sub> and H<sub>2</sub>).
- 4) Connecting points P and Q to point H<sub>2</sub>. Line segments P-H<sub>2</sub> and H<sub>2</sub>-Q are tangent lines of the parabola at points P and Q respectively.
- 5) Drawing a line parallel to segment P-Q through point H<sub>1</sub> we get the tangent line of the parabola at point H<sub>1</sub>.
- 6) Drawing the parabola arc touching the tangent lines at points P, H<sub>1</sub> and Q.

## 11. Simple rules for the drawing of the normal force, shear force and bending moment diagrams

Observing the loading diagrams that was obtained in Section 10 we can realize simple drawing rules. Applying the above rules, we can draw the diagrams easily without calculating the normal force, shear force and bending moment functions.

*Rule 1:*

When drawing a diagram, we always start from zero.

*Rule 2:*

We take into consideration the effects of the different kinds of forces and torques as it is shown in the figure below:

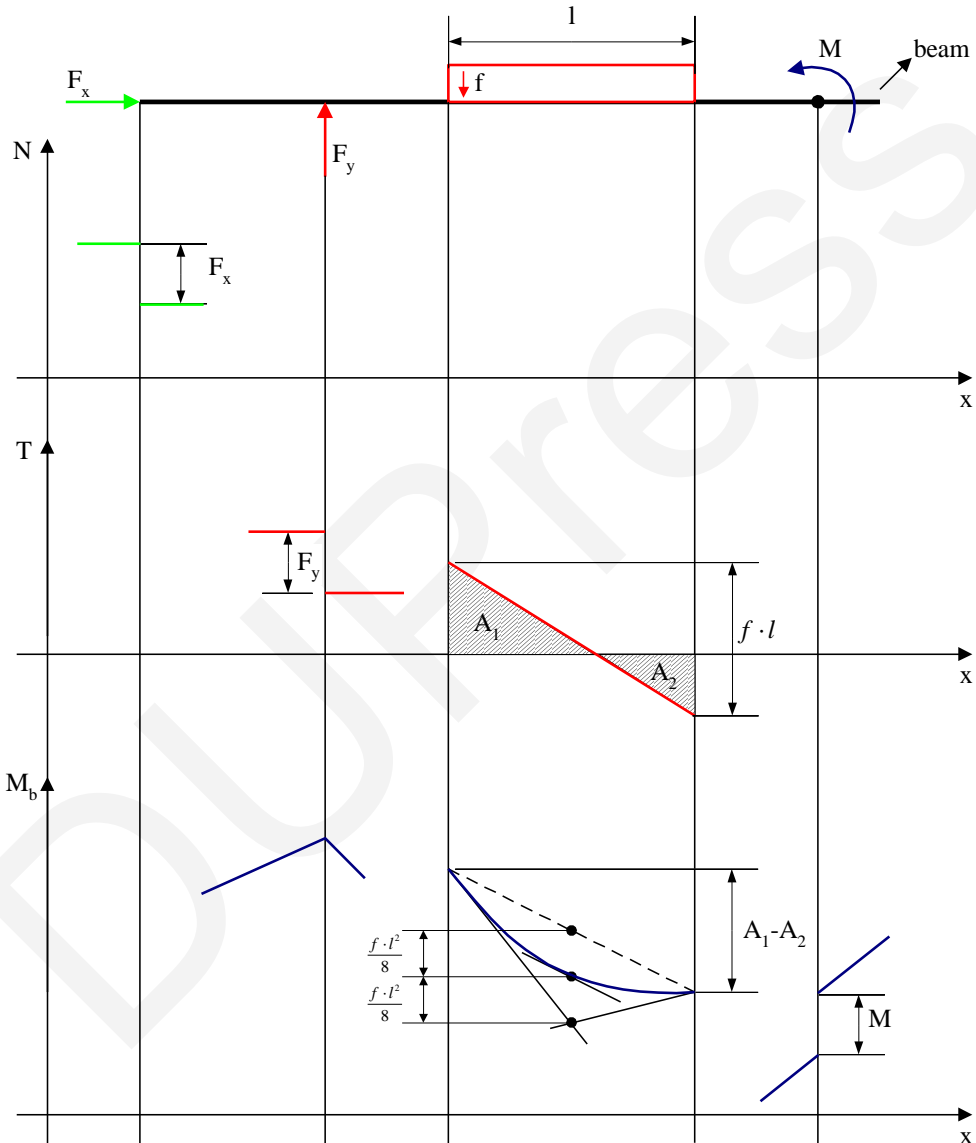


Figure 59

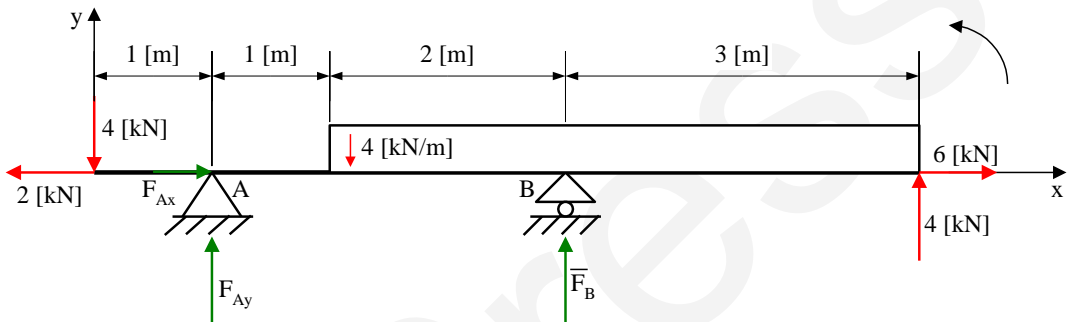
*Rule 3:*

When finish drawing a diagram, we always arrive into zero.

## Numerical problems

### Problem 1:

Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces acting on the supported beam in the figure. Draw the normal force, shear force and bending moment diagrams of the beam.



*Solution:*

*Calculation:*

$$\sum_i F_{ix} = -2 + F_{Ax} + 6 = 0 \rightarrow F_{Ax} = -4[kN]$$

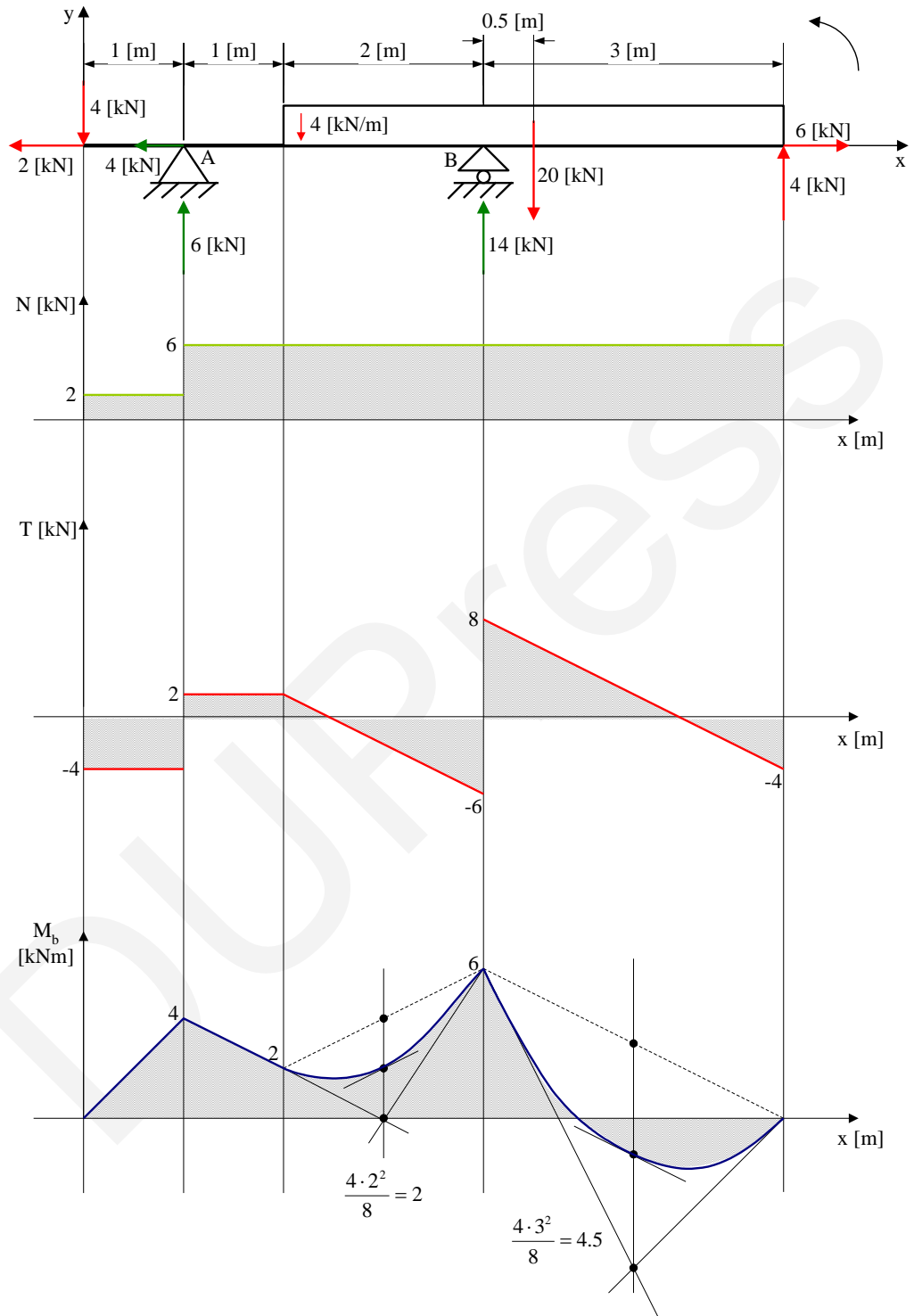
$$\sum_i F_{iy} = -4 + F_{Ay} - 4 \cdot 5 + F_B + 4 = 0 \rightarrow F_{Ay} = 6[kN]$$

$$\begin{aligned} \sum_i M_A^i &= 4 \cdot 1 - 4 \cdot (2 + 3) \cdot (1 + 2 + 0.5) + F_B \cdot 3 + 4 \cdot (1 + 2 + 3) = \\ &= 0 \rightarrow F_B = 14[kN] \end{aligned}$$

*Checking:*

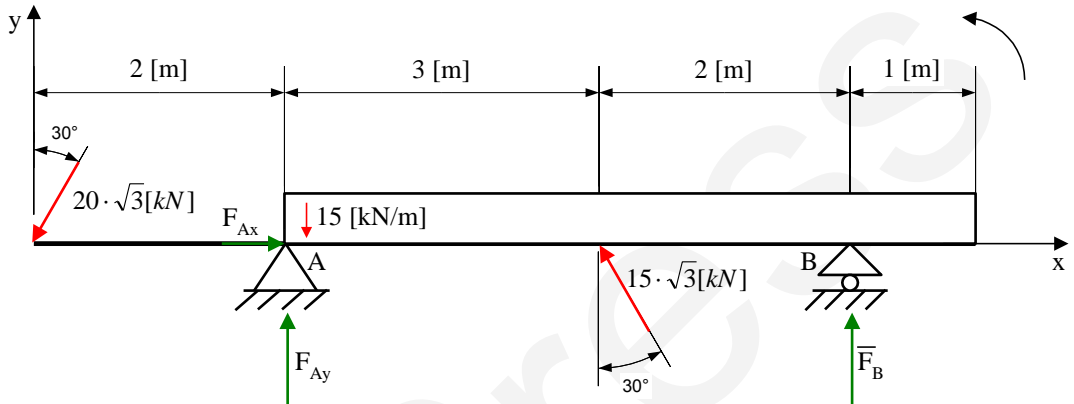
$$\sum_i M_B^i = 4 \cdot (1 + 1 + 2) - F_{Ay} \cdot (1 + 2) - 4 \cdot (2 + 3) \cdot 0.5 + 4 \cdot 3 = 0$$

*Loading diagrams:*



*Problem 2:*

Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces acting on the supported beam in the figure. Draw the normal force, shear force and bending moment diagrams of the beam.



*Solution:*

*Calculation:*

$$\sum_i F_{ix} = -20 \cdot \sqrt{3} \cdot \sin 30^\circ + F_{Ax} - 15 \cdot \sqrt{3} \cdot \sin 30^\circ = 0 \rightarrow F_{Ax} = 30.31 \text{ [kN]}$$

$$\sum_i F_{iy} = -20 \cdot \sqrt{3} \cdot \cos 30^\circ + F_{Ay} - 15 \cdot (3 + 2 + 1) + 15 \cdot \sqrt{3} \cdot \cos 30^\circ + F_{By} = 0 \rightarrow$$

$$\rightarrow F_{Ay} = 69 \text{ [kN]}$$

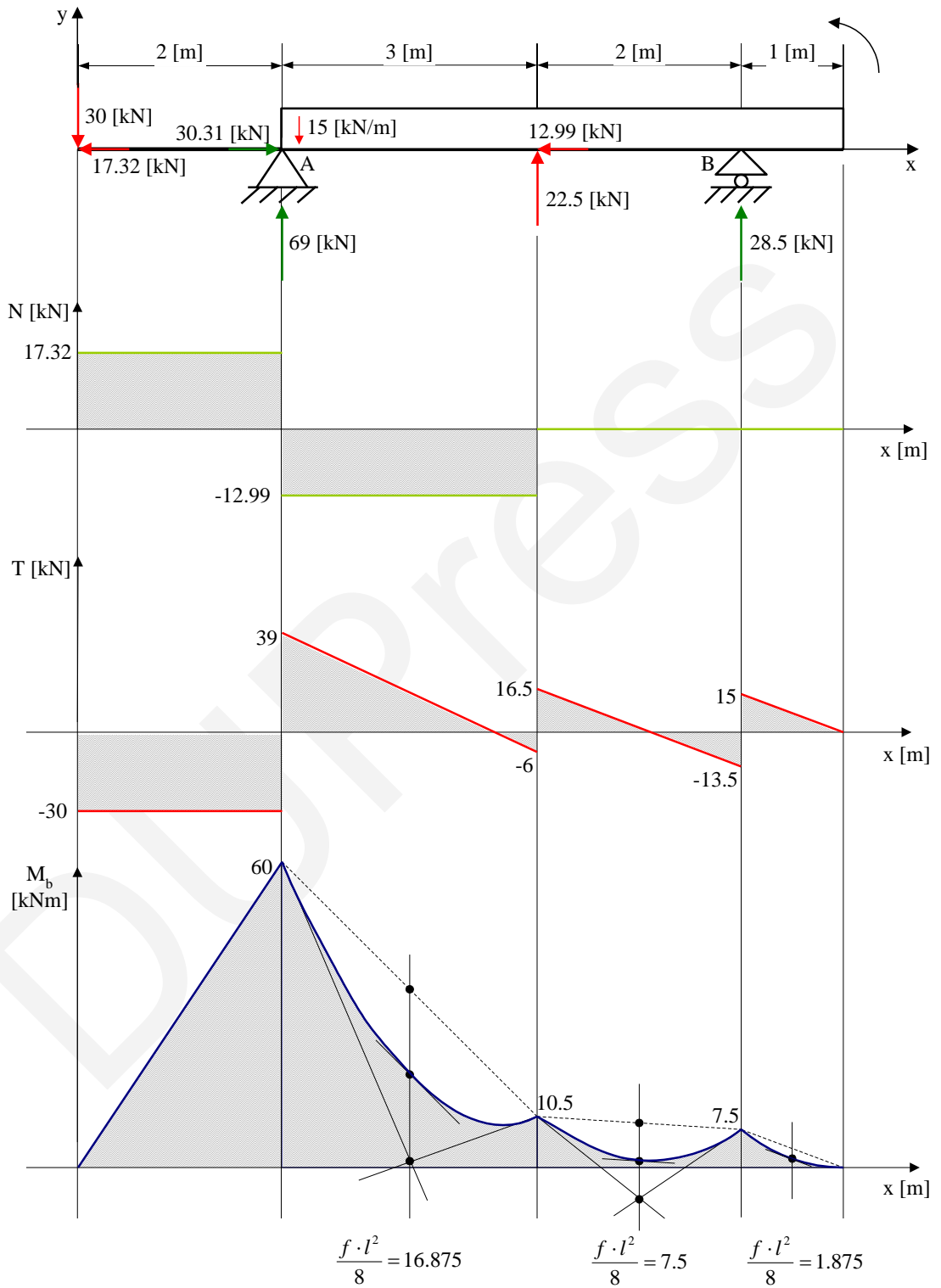
$$\sum_i M_A = 20 \cdot \sqrt{3} \cdot \cos 30^\circ \cdot 2 + 15 \cdot \sqrt{3} \cdot \cos 30^\circ \cdot 3 + F_{By} \cdot 5 - 15 \cdot (3 + 2 + 1) \cdot 3 = 0 \rightarrow F_{By} = 28.5 \text{ [kN]}$$

*Checking:*

$$\sum_i M_B = -15 \cdot 1 \cdot 0.5 + 15 \cdot 5 \cdot 2.5 - 15 \cdot \sqrt{3} \cdot \cos 30^\circ \cdot 2 - 69 \cdot 5 + 20 \cdot \sqrt{3} \cdot \cos 30^\circ \cdot 7 = 0$$

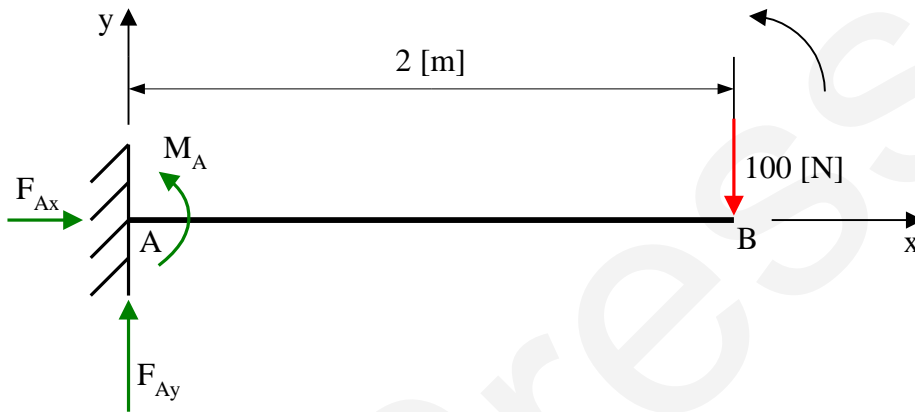
*Loading diagrams:*

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*Problem 3:*

Calculate the  $\bar{F}_A$  reaction force and  $M_A$  reaction moment acting on the cantilever at point A. Draw the normal force, shear force and bending moment diagrams of the cantilever.



*Solution:*

*Calculation:*

$$\sum_i M_{Ai} = M_A - 100 \cdot 2 = 0 \rightarrow M_A = 200 \text{ [Nm]}$$

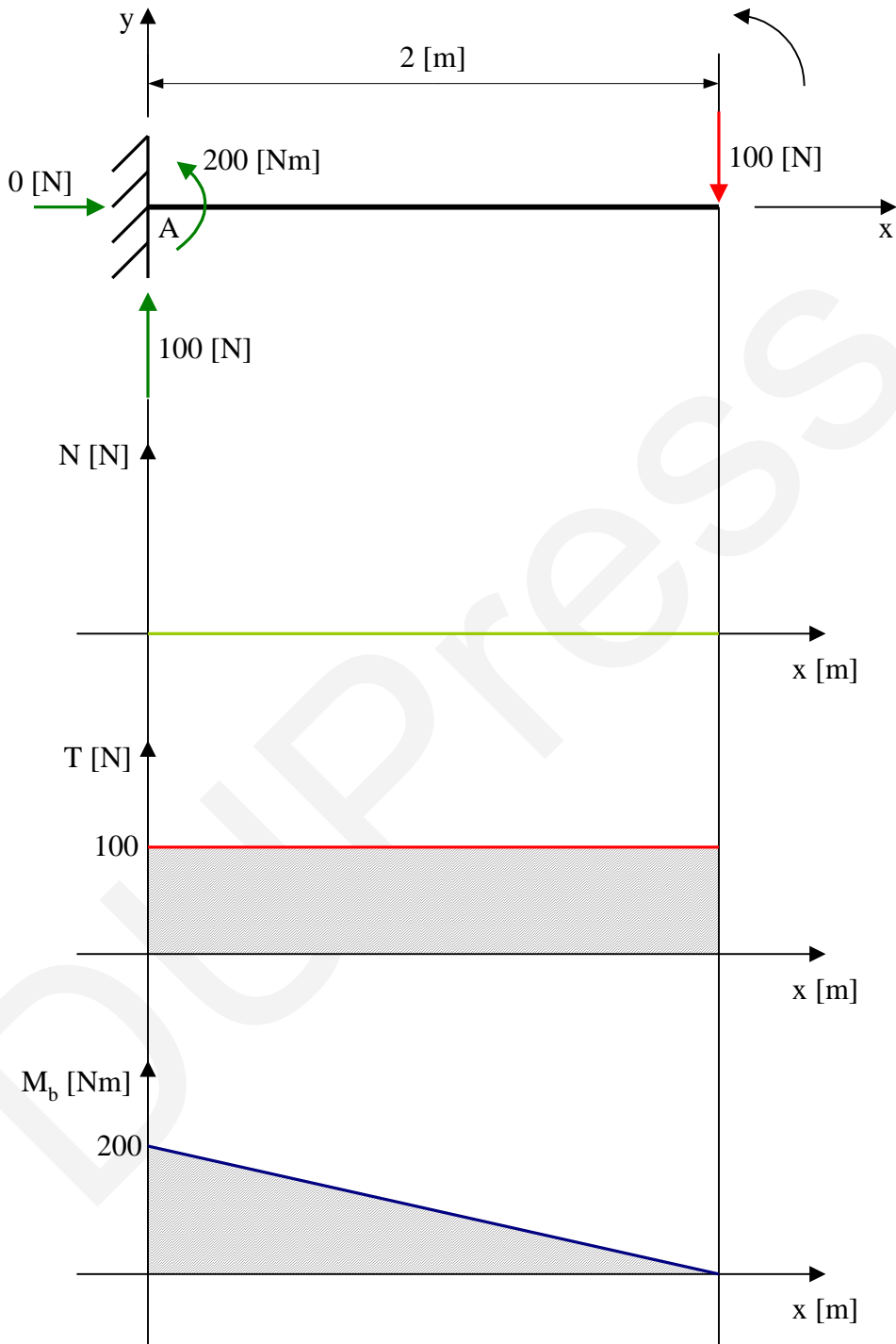
$$\sum_i F_{ix} = F_{Ax} = 0 \text{ [N]}$$

$$\sum_i F_{iy} = F_{Ay} - 100 = 0 \rightarrow F_{Ay} = 100 \text{ [N]}$$

*Checking:*

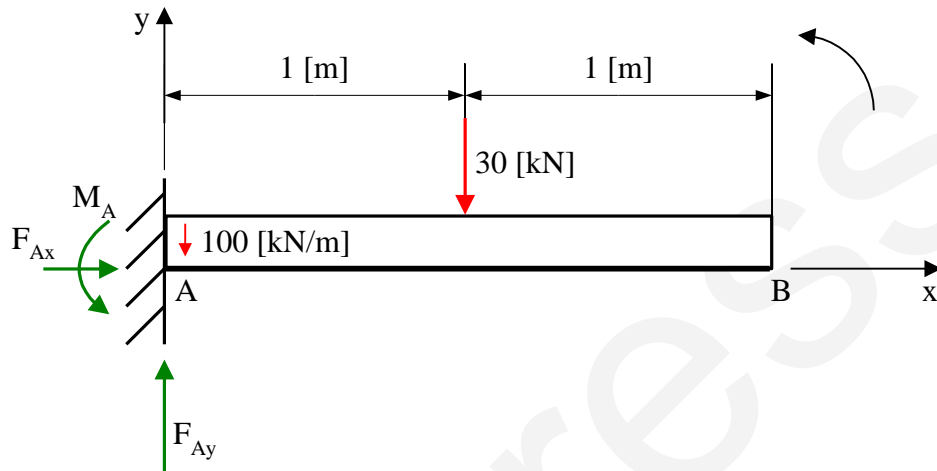
$$\sum_i M_{Bi} = -100 \cdot 2 + 200 = 0$$

*Loading diagrams:*



Problem 4:

Calculate the  $\bar{F}_A$  reaction force and  $M_A$  reaction moment acting on the cantilever at point A. Draw the normal force, shear force and bending moment diagrams of the cantilever.



*Solution:*

*Calculation:*

$$\sum_i M_A = M_A - 30 \cdot 1 - 100 \cdot (1 + 1) \cdot 1 = 0 \rightarrow M_A = 230 \text{ [kNm]}$$

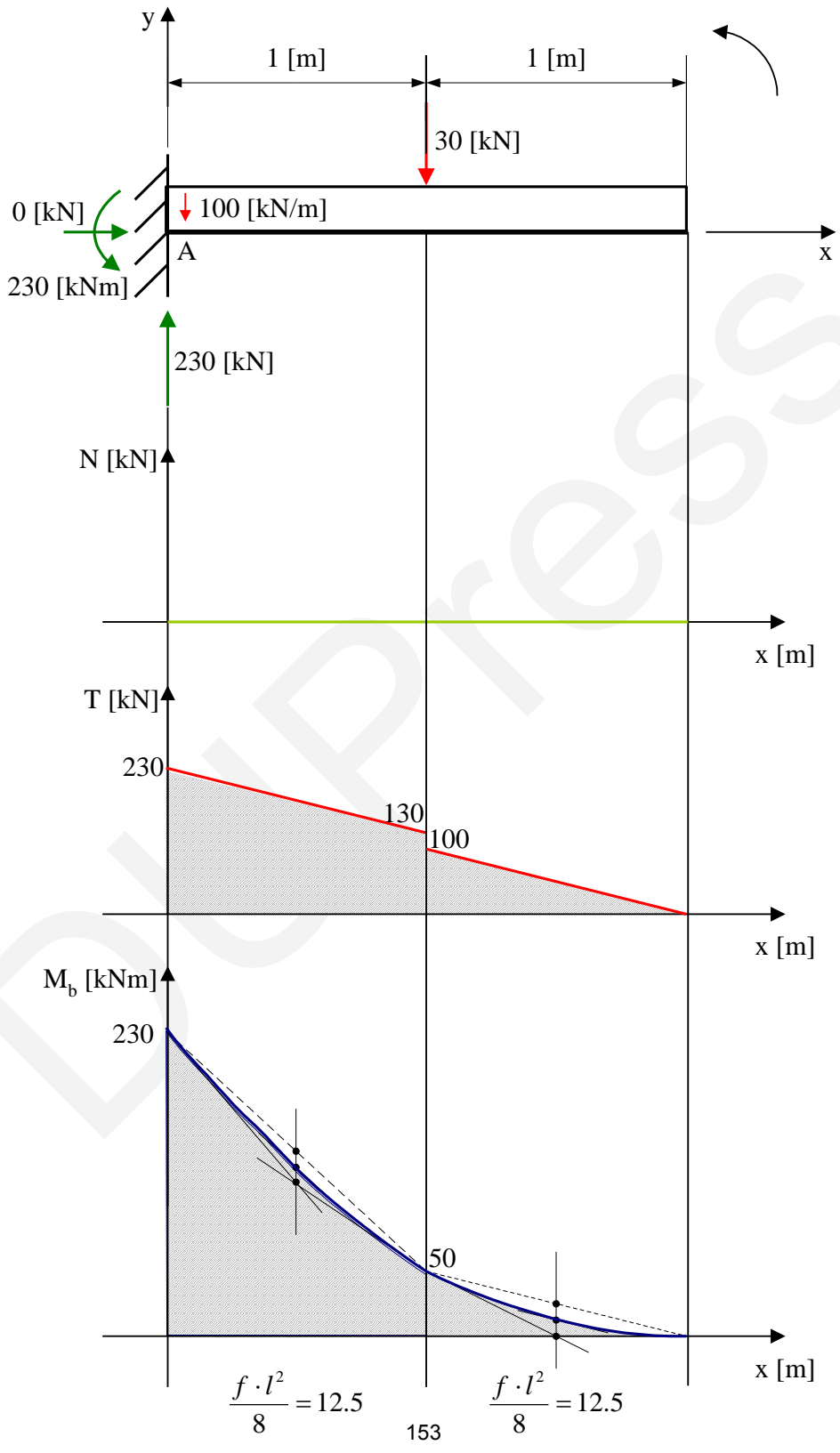
$$\sum_i F_{ix} = F_{Ax} = 0 \text{ [kN]}$$

$$\sum_i F_{iy} = F_{Ay} - 30 - 100 \cdot (1 + 1) = 0 \rightarrow F_{Ay} = 230 \text{ [kN]}$$

*Checking:*

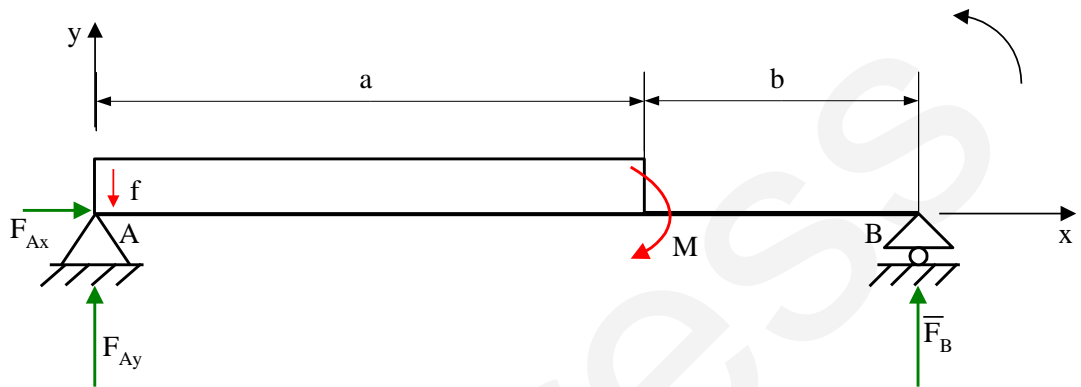
$$\sum_i M_B = 30 \cdot 1 + 100 \cdot 2 \cdot 1 - 230 \cdot 2 + 230 = 0$$

*Loading diagrams:*



**Problem 5:**

Calculate the  $\bar{F}_A$  and  $\bar{F}_B$  reaction forces acting on the supported beam in the figure. Draw the normal force, shear force and bending moment diagrams of the beam.



**Data:**  $f=10$  [kN/m],  $a=0.4$  [m],  $b=0.2$  [m],  $M=2$  [kNm]

**Solution:**

**Calculation:**

$$\sum_i M_i = -f \cdot a \cdot \frac{a}{2} - M + F_B \cdot (a + b) = 0 \rightarrow F_B = 4.666 \text{ [kN]}$$

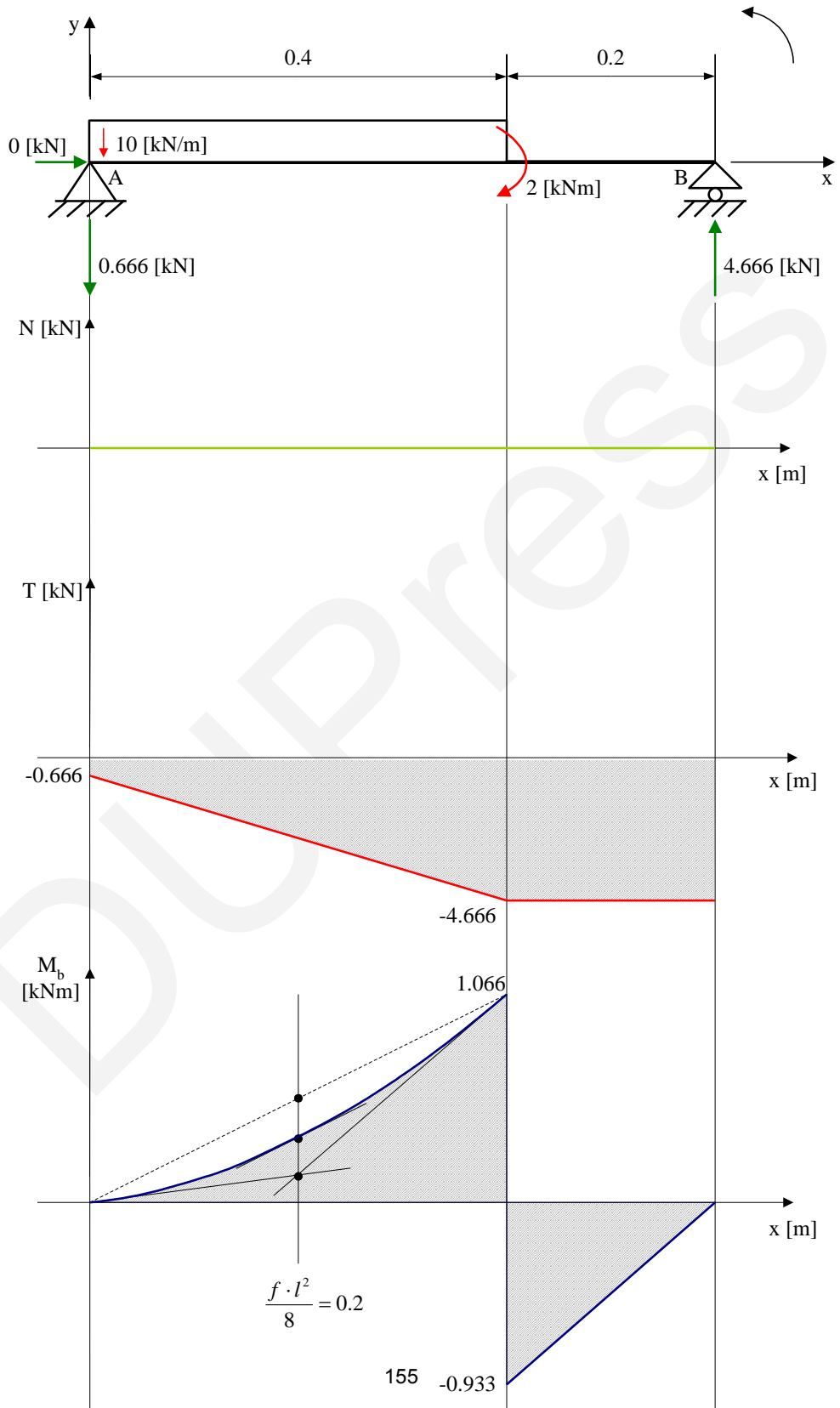
$$\sum_i F_{ix} = F_{Ax} = 0 \text{ [kN]}$$

$$\sum_i F_{iy} = F_{Ay} - f \cdot a + F_B = 0 \rightarrow F_{Ay} = -0.666 \text{ [kN]}$$

**Checking:**

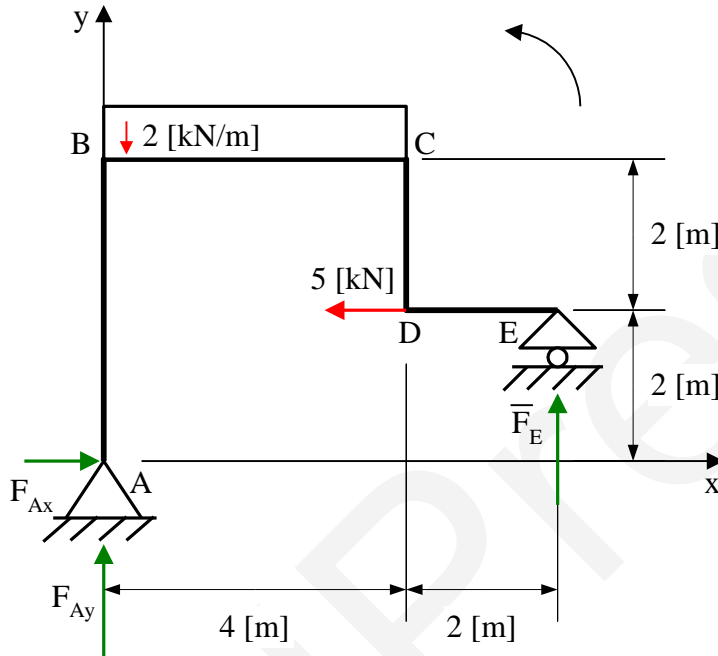
$$\sum_i M_i = 10 \cdot 0.4 \cdot 0.4 + 0.666 \cdot 0.6 - 2 = 0$$

**Loading diagrams:**



**Problem 6:**

Calculate the  $\bar{F}_A$  and  $\bar{F}_E$  reaction forces acting on the fractioned line beam in the figure. Draw the normal force, shear force and bending moment diagrams of the beam.



**Solution:**

**Calculation:**

$$\sum_i M_A^i = -2 \cdot 4 \cdot 2 + 5 \cdot 2 + F_E \cdot 6 = 0 \rightarrow F_E = 1 \text{ [kN]}$$

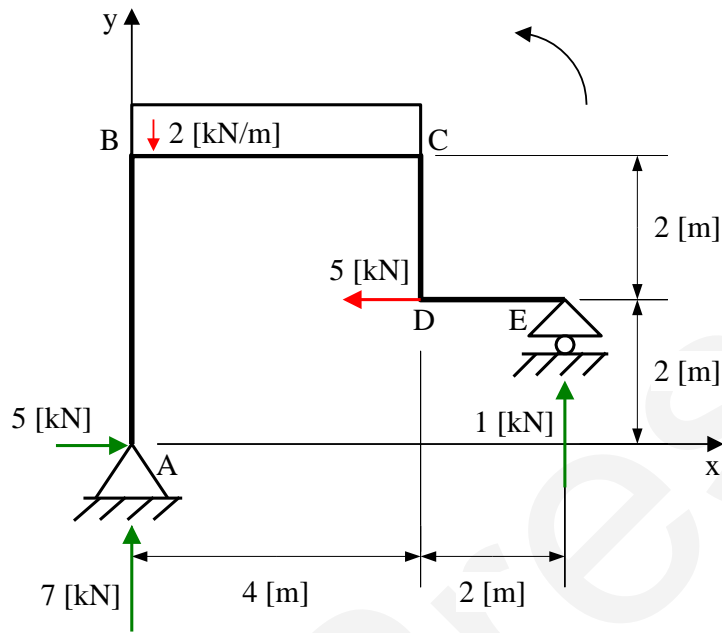
$$\sum_i F_{ix} = F_{Ax} - 5 = 0 \rightarrow F_{Ax} = 5 \text{ [kN]}$$

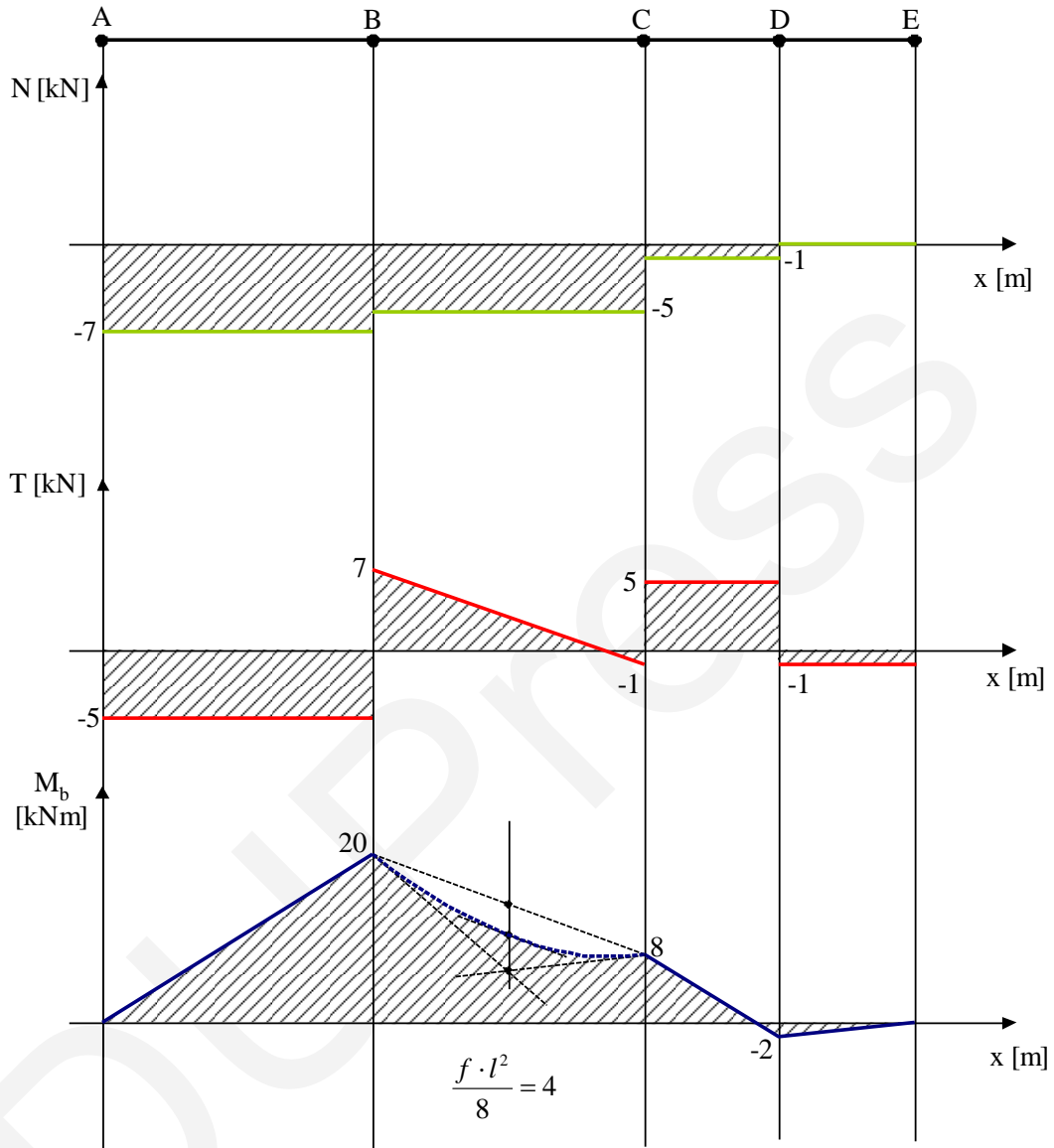
$$\sum_i F_{iy} = F_{Ay} - 2 \cdot 4 + F_E = 0 \rightarrow F_{Ay} = 7 \text{ [kN]}$$

**Checking:**

$$\sum_i M_E^i = 2 \cdot 4 \cdot 4 - 7 \cdot 6 + 5 \cdot 2 = 0$$

Loading diagrams:





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