BIOMICS

**Biological and Mathematical Basis of Interaction Computing**

**WP1: Algebraic Structure of Continuous and Discrete Models of Cellular Pathways**

**Deliverable D1.3.2**
**Examples Based on the Chevalley Correspondence between Lie Groups and SNAGs**

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# Chapter 1

# Finite Chevalley Groups

## 1.1 Chevalley bases of simple Lie algebras

Let $L$ be a finite simple Lie algebra over $\mathbb{C}$ and a fixed Cartan subalgebra $H$. Related to $H$ we have the unique (up to isomorphism) Cartan decomposition of $L$ that is

$$L = H \oplus \bigoplus_{r \in \Phi} L_r.$$

Now, $H$ is a Cartan subalgebra that is a maximal Abelian subalgebra. where $\Phi$ is the system of roots. Let $\Pi$ be a fundamental root systen in $\Phi$.

We have some useful facts:

1. Every element of $\Phi$ is a linear combination of $\Pi$ such that all multipliers are integer and all of them are positive or all of them are negative.

2. $\dim H = |\Pi| = l$ is called the rank of the root system $\Phi$. This equals to the rank of the Lie algebra $L$.

Now, we can define $\Phi^+$ such that the set of linear combinations of elements $\Pi$ using only positive coefficients. Of course, we can define $\Phi^-$ as well. From the preceding facts, we obtain that $\Phi = \Phi^+ \cup \Phi^-$.

**Example.** Let $L = \mathfrak{sl}(3, \mathbb{C})$ thus $L$ comsists of all matrices of $3 \times 3$ with zero trace.

$$L = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| a_{11} + a_{22} + a_{33} = 0, a_{ij} \in \mathbb{C}; 1 \leq i, j \leq 3 \right\}$$

Now, the Cartan decomposition

$$L = H \oplus \bigoplus_{\substack{1 \leq i,j \leq 3 \\ i \neq j}} \mathbb{C} e_{ij}$$

where $H$ consists of all diagonal matrices of $3 \times 3$ with zero trace. That is,

$$H = \mathbb{C}(e_{11} - e_{22}) \oplus \mathbb{C}(e_{22} - e_{33})$$

where $e_{ij}$ is the elementary matrix with 1 in the $(i, j)$ position and 0 everywhere else.

The roots $r_{ij}$ are linear maps for all $1 \leq i \neq j \leq 3$ from thr Cartan subalgebra $H$ onto $\mathbb{C}$.

$$r_{ij}\colon H \to \mathbb{C}, \qquad r_{ij}(h) = a_{ii} - a_{jj}, \qquad \text{if } h = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The root spaces are

$$L_{r_{ij}} = \mathbb{C}e_{ij} 1 \leq i \neq j \leq 3.$$

The system of roots

$$\Phi = \{ r_{12}, r_{13}, r_{21}, r_{23}, r_{31}, r_{32} \}$$

Since $r_{ji} = -r_{ij}$ and $r_{ik} = r_{ij} - r_{jk}$ where $ii, j, k$ are distinct numbers, a fundamental root system (base) can be chosen for example

$$\Pi = \{ r_{12}, r_{23} \}$$

This means that the rank of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ equals to 2.

It easy to see that

$$\Phi^+ = \{ r_{12}, r_{13}, r_{23} \}, \qquad \Phi^- = \{ r_{21}, r_{23}, r_{31} \}.$$

We need the following useful technical lemma:

**Lemma 1.** *Any positive root $r \in \Phi^+$ can be expressed as a sum of fundamental roots*

$$r = p_{i_1} + p_{i_2} + \cdots + p_{i_k}$$

*in such a way that $p_{i_1} + p_{i_2} + \cdots + p_{i_a}$ is a (positive) root for all $a \leq k$.*

For each root $r$ let $e_r$ be a non-zero element of $L_r$. If $e_r$ is already chosen for $r \in \Phi^+$ there is a unique element $e_{-r} \in L_{-r}$ such that $[e_r, e_{-r}] = h_r$, and we shall suppose that $e_{-r}$ chosen in this way. The set

$$\{ h_r : r \in \Pi \} \cup \{ e_r : r \in \Phi \}$$

is a basis for $L$. For the elements of basis, we have the following rules:

3. $[h_r, h_s] = 0$, $r, s \in \Pi$
4. $[h_r, e_s] = A_{rs} e_s$, where $A_{rs} = \frac{2(r,s)}{(r,r)}$ is an integer for $r \in \Pi$, $s \in \Phi$
5. $[e_r, e_{-r}] = h_r$, $r \in \Phi$
6. Since $[L_r, L_s] = L_{r+s}$ we have that $[e_r, e_s] = N_{r,s} e_{r+s}$, $r, s \in \Phi$.

   Note that, if $r + s \notin \Phi$ then $L_{r+s} = 0$ thus let $N_{r,s}$ be zero for this case.

Since we have a significant freedom on the choice of the set $\{ e_r : r \in \Phi^+ \}$, our goals is to find a set where the **structure constants** $N_{r,s}$ are nice integers such as $0, \pm 1, \pm 2, \pm 3$. A base with this property is called **Chevalley base** $\mathcal{B}_C$.

Let

$$-pr + s, \ldots, s, \ldots qr + s$$

be the **r-chain of roots through s**.

**Example.** If $L = \mathfrak{sl}(3, \mathbb{C})$ then

$$h_{r_{ij}} = e_{ii} - e_{jj} \qquad e_{r_{ij}} = e_{ij} \qquad 1 \leq i \neq j \leq 3$$

Let denote $\delta_{ij}$ the **Kronecker-delta** whicg equals to 1 if $i = j$ and 0 otherwise. Now, the values $A_{ij}$and $Nij$ can be calculated using the following easy calculation rules

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$$

The following tables show the values of $A_{ij}$ and $N_{ij}$for $1 \leq i \neq j \leq 3$:

| | $e_{r_{12}}$ | $e_{r_{13}}$ | $e_{r_{21}}$ | $e_{r_{23}}$ | $e_{r_{31}}$ | $e_{r_{32}}$ |
|---|---|---|---|---|---|---|
| $h_{r_{12}}$ | 2 | 1 | -2 | -1 | -1 | 1 |
| $h_{r_{13}}$ | 1 | 2 | -1 | 1 | -2 | -1 |
| $h_{r_{21}}$ | -2 | -1 | 2 | 1 | -2 | -1 |
| $h_{r_{23}}$ | -1 | 1 | 1 | 2 | -1 | -2 |
| $h_{r_{31}}$ | -1 | -2 | 1 | -1 | 2 | 1 |
| $h_{r_{32}}$ | 1 | -1 | -1 | -2 | 1 | 2 |

| | $r_{12}$ | $r_{13}$ | $r_{21}$ | $r_{23}$ | $r_{31}$ | $r_{32}$ |
|---|---|---|---|---|---|---|
| $r_{12}$ | - | 1,0 | - | 0,1 | 0,1 | 1,0 |
| $r_{13}$ | 1,0 | - | 0,1 | 1,0 | - | 0,1 |
| $r_{21}$ | - | 0,1 | - | 1,0 | 1,0 | 0,1 |
| $r_{23}$ | 0,1 | 1,0 | 1,0 | - | 0,1 | - |
| $r_{31}$ | 0,1 | - | 1,0 | 0,1 | - | 1,0 |
| $r_{32}$ | 1,0 | 0,1 | 0,1 | - | 1,0 | - |

The last table in this example shows the values of $p$ and $q$ fort the r-chain of roots through s.

Now, $A_{rs} = p - q$ and the length of a chain of a roots is at most 4, moreover

**Lemma 2.**

$$\frac{(r+s, r+s)}{(s,s)} = \frac{p+1}{q}.$$

Using the preceding facts and the Jacobi identity many times we obtain the following theorem:

**Theorem 3.** *The structure constants of a basis of a simple Lie algebra $L$ over $\mathbb{C}$ satisfy the following relations:*

7. $N_{s,r} = -N_{r,s}$, $r, s \in \Phi$

8. $$\frac{N_{r_1,r_2}}{(r_3,r_3)} = \frac{N_{r_2,r_3}}{(r_1,r_1)} = \frac{N_{r_3,r_1}}{(r_1,r_1)}$$

   *if* $r_1, r_2, r_3 \in \Phi$ *satisfy* $r_1 + r_2 + r_3 = 0$.

9. $N_{r,s} N_{-r,-s} = -(p+1)^2$, $r, s, r+s \in \Phi$

*10.*
$$\frac{N_{r_1,r_2}N_{r_3,r_4}}{(r_1+r_2,r_1+r_2)}+\frac{N_{r_2,r_3}N_{r_1,r_4}}{(r_2+r_3,r_2+r_3)}+\frac{N_{r_3,r_1}N_{r_2,r_4}}{(r_3+r_1,r_3+r_1)}=0$$
*if* $r_1, r_2, r_3, r_4 \in \Phi$ *satisfy* $r_1 + r_2 + r_3 + r_4 = 0$ *and if no pair are opposite.*

Using the first lemma and the isomorphism theorem for simple Lie algebras and the previous theorem one can prove that

**Theorem 4** (Chevalley)**.** *The elements* $\{ e_r : r \in \Phi^+ \}$ *can be chosen such that* $N_{-r,-s} = -N_{r,s}$ *for all* $r, s \in \Phi$*. This means that* $N_{r,s} = \pm(p + 1)$*.*

An interesting question, what we can say about the signs of the structure constants $N_{r,s}$ above. Suppose a total ordering giving on the space containing roots ( e.g the lexical graphic extension of an arbitrary total ordering on the fundamental roots). An ordered pair $(r, s)$ is called **special pair** if $r + s \in \Phi$ and $0 \leq r \leq s$. An ordered pair $(r, s)$ is called **extra special pair** if $(r, s)$ is a special pair and if for all special pairs $(r_1, s_1)$ with $r + s = r_1 + s_1$ we have $r \leq r_1$.

**Theorem 5.** *The signs of the structure constants* $N_{r,s}$ *may be chosen arbitrarily for extraspecial pairs* $(r, s)$*, and then the structure constants for all pairs are uniquely determined.*

To define the Chevalley groups associated to finite dimensional simple Lie-algebras we shall use the exponentional map. Recall that an automorhism of a Lie algebra $L$ is a bijective linear map $\theta$ on $L$ satisfying
$[\theta x, \theta y] = \theta[x, y]$.
The set of all automorphism of $L$ clearly forms a group.

**Lemma 6.** *Let* $L$ *be a Lie algebra over a field of characteristic* 0 *and* $\delta$ *be a derivation of* $L$ *which is nilpotent (*$\delta^n = 0$ *for some* $n$*). Then*

$$\exp\delta = id + \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^{(n-1)}}{(n-1)!}$$

*is an automorpism of* $L$*.*

**Example.** Let $L = \mathfrak{sl}(3, \mathbb{C})$. Let denote $x = e_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Take the **adjoint map**

$$\delta = \mathrm{ad}_x \colon L \to L$$
$$\mathrm{ad}_x(h) = [h, x],$$
$$\delta\colon \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} -a_{21} & a_{11}-a_{22} & -a_{23} \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{pmatrix},$$
$$\delta^2\colon \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} 0 & -2a_{21} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\delta^3 = 0$ we have that $\delta$ is nilpotent. Thus the exponential map $\exp\delta$ is an automorphism of the Lie algebra $L$.

$$\exp\delta = id + \delta + \frac{\delta^2}{2!},$$
$$(\exp\delta)^{-1} = id - \delta + \frac{\delta^2}{2!},$$
$$\exp\delta\colon \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}-a_{21} & a_{11}+a_{12}-a_{21}-a_{22} & a_{13}-a_{23} \\ a_{21} & a_{21}+a_{22} & a_{23} \\ a_{31} & a_{31}+a_{32} & a_{33} \end{pmatrix}.$$

As a result, taking only the set of integer linear combinations of $\mathcal{B}_C$ in $L$, this set will be closed under Lie multiplication, so one gets a Lie algebra $L(\mathbb{Z})$ over $\mathbb{Z}$. Now, one can get a Lie algebra $L(K)$ of type $L$ over an arbitrary field $K$ by simply taking the tensor product $L(\mathbb{Z}) \otimes_{\mathbb{Z}} K$. In particular, for each prime power $q = p^f \in \mathbb{N}$, one gets the finite Lie algebra $L(q) = L(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_q$ of type $L$ over the finite field $\mathbb{F}_q$. Note that $L(q)$ will not be a simple Lie algebra in general! A further remark is that one can choose a Chevalley basis in many ways in the simple Lie algebra $L$, but the resulting $L(\mathbb{Z})$ will always be the same up to isomorphism. This results in the fact that the isomorphism type of $L(q)$ is also uniquely defined.

## 1.2 The Chevalley groups

Now, starting from the simple Lie algebra $L = L(\mathbb{C})$ and a Chevalley basis $\mathcal{B}_C = \{\, h_r, e_s \mid r \in \Pi, s \in \Phi \,\}$ of $L$, Since if $r, s \in \Phi$ then $(q+1)r + s$ is not a root, it is clear that $\operatorname{ad} e_s$ thus $\zeta \cdot \operatorname{ad} e_s$ is a nilpotent derivation of $L$ for every $s \in \Phi$ and $\zeta \in \mathbb{C}$. Therefore, $x_s(\zeta) := \exp(\zeta \operatorname{ad} e_s)$ is an automorphism of $L$. Now, the Chevalley group $G_L$ of $L$ is defined as

$$G_L = \langle x_s(\zeta) \mid s \in \Phi,\ \zeta \in \mathbb{C} \rangle \leq \operatorname{Aut} L.$$

Now, we turn to the case when $K$ is a field of characteristic $p$. At first sight the above construction cannot be used in general, since we cannot substitute $\operatorname{ad} e_s$ into the power series of the exponential function if $p$ is smaller than the nilpotency class of $\operatorname{ad} e_s$. (This can only occur for $p \in \{\, 2, 3 \,\}$ by Lemma **??** and Theorem **??**.) However, by calculating the action of $x_s(\zeta) \in G_L$ on the elements of the Chevalley basis $\mathcal{B}_C \subset L = L(\mathbb{C})$ we get the following equalities. (Here $r$ and $s$ are linearly independent roots.)

$$x_s(\zeta)(e_s) = e_s;$$
$$x_s(\zeta)(e_{-s}) = e_{-s} + \zeta h_s - \zeta^2 e_s;$$
$$x_s(\zeta)(h_s) = h_s - 2\zeta e_s;$$
$$x_s(\zeta)(h_r) = h_r - A_{rs}\zeta e_s;$$
$$x_s(\zeta)(e_r) = \sum_{i=0}^{q} \pm \binom{p+i}{i} \zeta^i e_{is+r}.$$

Now, fixing an arbitrary field $K$, we can define the map $x_s(t)\colon L(K) \to L(K)$ by simply exchanging $t$ for $\zeta$ in the above coefficients. We have to show that in this way $x_s(t)$ really defines an automorphism of $L(K)$. Since $x_s(\zeta)x_s(-\zeta) = 1$, we have that $x_s(t)x_s(-t) = 1$ as well.This means that $x_s(t)$ is non-singular, thus $x_s(i)$ is a linear bijection of $L(K)$. A technical calculation shows (see [2] pp. 63–64) that $x_s(t)$ preserves the Lie bracket, therefore $x_s(t)$ is a Lie algebra automrphism of $L(K)$. Now, the Chevalley group (of adjoint type) of $L(K)$ is simply defined as

$$G_L(K) = \langle x_s(t) \mid s \in \Phi,\, t \in K \rangle$$

like it was defined over $\mathbb{C}$.

It turns out that for the case in which $K$ is a finite field the Chevalley groups $G_L(K)$ are generally finite simple groups (apart from a few exceptions). In this way one gets 9 classes of finite simple groups of Lie type: $A_l(q)$, $B_l(q)$, $C_l(q)$, $D_l(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$. Note that these are not all the finite simple group of Lie type; other groups can be obtained from graph isomorphisms of the Dynkin diagrams. In this way one can get other classes of simple groups. The Steinberg groups ${}^2A_l(q^2)$, ${}^2D_l(q^2)$, ${}^3D_4(q^3)$, ${}^2E_6(q^2)$ and the Suzuki-Ree groups ${}^2B_2(2^{2n+1})$, ${}^2F_4(2^{2n+1})$, ${}^2G_2(3^{3n+1})$.

### 1.2.1 The groups of $A_1(K)$

These are the simplest Chevalley groups. Recall that the simple Lie algebra $A_1$ over $\mathbb{C}$ can be represented as the algebra of $2 \times 2$ matrices of trace 0 under Lie multiplication $[x, y] = xy - yx$. Here a Chevalley basis $\mathcal{B}_C = \{ h_r, e_r, e_{-r} \}$ where

$$h_r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_{-r} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we have

$$[h_r, e_r] = 2e_r, \qquad [h_r, e_{-r}] = -2e_{-r}, \qquad [e_r, e_{-r}] = h_r.$$

**Theorem 7.** *$A_1(K) \simeq PSL_2(K)$.*

To prove this theorem we require the following lemma:

**Lemma 8.** *Let $L$ be a simple Lie algebra over $\mathbb{C}$ and suppose we have a representation of $L$ by matrices under Lie multiplication. Suppose that $y \in L$ is represented by a nilpotent matrix. Then $\operatorname{ad} y$ is a nilpotent derivation of $L$ and*

$$\exp(\operatorname{ad} y)(x) = \exp y \cdot x \cdot (\exp y)^{-1}$$

*for all $x \in L$. This means that the image of $x$ under the automorphism $\exp(\operatorname{ad} y)$ is given by transforming by $\exp y$.*

## 1.3 Unipotent Subgroups

Let $G = G_L(K)$ be the Chevalley group of type $L$ over $K$. Then

$$G = \langle x_r(t) : r \in \Phi, t \in K \rangle$$

An easy calculation shows that

$$x_r(t_1) \cdot x_r(t_2) = x_r(t_1 + t_2)$$

Let $X_r = \langle x_r(t) : t \in K \rangle$ and $X_r$ isomorphic to the additive subgroup of $K$. The subgroups of $X_r$ are called the **root subgroups** of $G$.

Recall that a linear transformation $\theta$ of a vectorspace is called **unipotent** if $\theta - 1$ is nilpotent.

Recall that every root $r$ in $\Phi$ is the unique integer linear combination of elements $p_1, P_2, \ldots p_l$ of the base $\Pi$ so $r = \sum \lambda_i p_i$ we can define a **height function h** such that $h(r) = \sum \lambda_i$. Take a Cartan decomposition

$$L = H \oplus \bigoplus_{r \in \Phi} L_r.$$

We define $L_0 = H$ and

$$L_i = \sum_{h(r)=i} L_r$$

for each $i \neq 0$. Then

$$L = \bigoplus_{i \in \mathbb{Z}} L_i.$$

Now if $r \in \Phi^+$ and $x \in L_i$ it is clear that

$$x_r(t)(x) - x \in \sum_{j>i} L_j.$$

Since each element of $U$ is a product of elements $x_r(t)$ for positive roots $r$, we have for each $u \in U$, $x \in L_i$:

$$u(x) - x \in \sum_{j>i} L_j.$$

Thus $u - 1$ is a nilpotent linear transformation of $L$, whence $u$ is umipotent. Similarly each element of $V$ is a unipotent transformation of $L$.

**Lemma 9.** *Let $U =< X_r; r \in \Phi^+ >$ and $V =< X_r; r \in \Phi^+ >$. Then every element of $U$ and $V$ is unipotent transformation on $L(K)$.*

To describe the structure of $U$ and $V$ we will need the following lemma, the special case of the Campbell-Hausdorff formula and the Chevalley's commutator formula.

**Lemma 10.** *Let $L$ be a simple Lie algebra over $\mathbb{C}$. Let $y$ be an element of $L$ such that* $\operatorname{ad} y$ *is nilpotent and let $\theta$be an automorphism of $L$. Then*

$$\theta \exp(ady)\theta^{-1} = \exp(\operatorname{ad} \theta y).$$

**Theorem 11** (Campbell–Hausdorff)**.** *Let $V$ be a vector space over a field of characteristic 0 and $\alpha, \beta$ linear transformations of $V$ such that $\alpha, \beta$ and $[\alpha, \beta] = \alpha\beta - \beta\alpha$ are nilpotent and $[\alpha, \beta]$ commutes with $\alpha$ and $\beta$. Then $\alpha + \beta$ is also nilpotent, and*

$$\exp(\alpha + \beta) = \exp \alpha \cdot \exp \beta \cdot \exp(-\frac{1}{2}[\alpha, \beta]).$$

Using the previous lemma and theorem many times we get the Chevalley's commutator formula:

**Theorem 12** (Chevalley)**.** *If $r, s \in \Phi^+$ are linearly independent then*

$$[x_s(u), x_r(t)] = \prod_{i,j>0} x_{ir+js}(C_{ijrs}(-t)^i u^j).$$

*where the product taken over all positive roots of the form $ir+js, i > 0, j > 0$, in increasing order. Each $C_{ijrs}$ is one of $\pm 1, \pm 2, \pm 3$.*

Since every element of $\Phi^+$ is a positive integer linear combination of the fundamental roots, the heigth function $h\colon \Phi^+ \to Z^+$ is the number of the fundamental roots of the positive integer linear combination.

As a consequence of the Chevalley commutator formula we obtain

**Theorem 13.** *Let $G = G_L(K)$ be a Chevalley group, $U$ be the subgroup of $G$ generated by the root subgroups $X_r$ with $r \in \Phi^+$, and $U_m$ be the subgroup generated by $X_r$ with $r \in \Phi^+$ and $h(r) \geq m$. Then*

1. *$U$ is nilpotent and*
$$U = U_1 \supset U_2 \supset \dots U_h \supset 1$$
*is a central seeries for for $U$, where $h$ is the greatest height of a root of $L$.*

2. *$U = \prod_{r\in\Phi^+} X_r$ moreover each element of $U$ is uniquely expressible in the form*
$$\prod_{r_i\in\Phi^+} x_{r_i}(t_i),$$
*where the product is taken over all positive roots in increasing order.*

**Example.** Let $L = \mathfrak{sl}(3, \mathbb{C})$ and the fundamental base

$$\Pi = \{\, r_{12}, r_{23} \,\}$$

and

$$\Phi = \{\, r_{12}, r_{13}, r_{21}, r_{23}, r_{31}, r_{32} \,\}$$

Using that

$$r_{ij}\colon H \to \mathbb{C}, \qquad\qquad r_{ij}\colon \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \mapsto a_{ii} - a_{jj}$$

for all $1 \leq i \neq j \leq 3$, we have

$$r_{13} = r_{12} + r_{23}, \quad r_{21} = -r_{12}, \quad r_{31} = -r_{12} - r_{13}, \quad r_{32} = -r_{23}$$

and

$$h(r_{12}) = 1, \quad h(r_{13}) = 2, \quad h(r_{21}) = -1, \quad h(r_{23}) = 1, \quad h(r_{31}) = -2, \quad h(r_{32}) = -1.$$

Therefore

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$$

where

$$L_{-2} = \mathbb{C}e_{31}, \quad L_{-1} = \mathbb{C}e_{21} \oplus \mathbb{C}e_{32}, \quad L_0 = H, \quad L_1 = \mathbb{C}e_{12} \oplus \mathbb{C}e_{23}, \quad L_2 = \mathbb{C}e_{13}$$

Since

$$\Phi^+ = \{\, r_{12}, r_{13}, r_{23} \,\}$$

and for $r, s \in \Phi^+$ and positive integers $i, j$

$$ir + js \in \Phi^+ \quad \text{if and only if} \quad i = j = 1 \quad r \neq s \quad \text{and} \quad r, s \in \{ r_{12}, r_{23} \}$$

using the Chevalley commutatore formula, we have

$$[x_{r_{12}}(t), x_{r_{13}}(u)] = 0, \quad [x_{r_{12}}(t), x_{r_{23}}(u)] = x_{r_{13}}(tu), \quad [x_{r_{13}}(t), x_{r_{23}}(u)] = 0$$

Now

$$U =< X_r \mid r \in \Phi^+ >=< X_{r_{12}}, X_{r_{13}}, X_{r_{23}} >=< x_{r_{12}}(t), x_{r_{13}}(u), x_{r_{23}}(v) \mid t, u, v \in \mathbb{C} >$$

It easy to define an injective homomorphism from $U$ into the **special linear group of degree 3** $\mathrm{SL}(3, \mathbb{C})$, which consists of all complex matrices of $3 \times 3$ with determinant 1.

$$\phi\colon U \to \mathrm{SL}(3, \mathbb{C})$$

$$\phi(x_{r_{12}}(t)) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \phi(x_{r_{13}}(u)) = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \phi(x_{r_{12}}(v)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}.$$

This means that $\phi(U)$ is the set of the **upper unitriangular matrices** of $3 \times 3$

$$\phi(U) = \left\{ \begin{pmatrix} 1 & t & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbb{C} \right\}$$

Consequently $U$ is nilpotent and a central series for $U$:

$$U_0 = U \supset U_1 =< X_{r_{13}} >\supset U_2 = 1$$

Since

$$\begin{pmatrix} 1 & t & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & u - tv \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have if $g \in U$ and $\phi(g) = \begin{pmatrix} 1 & t & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$ then the unique decomposation:

$$g = x_{r_{12}}(t) x_{r_{23}}(v) x_{r_{13}}(u - tv)$$

Similarly, we obtain that the subgroup $V$ of $G$ is isomorphic to the lower unitriangular matrices of $3 \times 3$.

Consequently, using that

$$G =< U, V >$$

$G$ is isomorphic to the $\mathrm{SL}(3, \mathbb{C})$.

## 1.4 The structure of the Chevalley groups

Using the classification of simple Lie algebras over the complex field $\mathbb{C}$ one can see that a simple Lie-algebra actually a direct sum of some subspaces which are isomorphic to $\mathfrak{sl}(2, K)$. By the definition of Chevalley groups, it is not surprising that a Chevalley group contains lots of homomorph images of $Sl(2, K)$. This fact will help us to understand the strucure of Chevalley groups since the calculation rules are straightforward in the group of matrices of $2 \times 2$.

### 1.4.1 The homomorph images of $Sl(2, K)$

In this section the theorems can be proven when the field $K$ is the complex field $\mathbb{C}$ and then a longer proof shows that the complex field can be changed to an arbirary field .

**Theorem 14.** *The subgroup* $< X_r, X_{-r} >$ *of the Chevalley group* $G_L(K)$ *is a homomorph image of* $\mathrm{Sl}_2(\mathrm{K})$ *such that*

$$\phi_r \colon \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_r(t)$$

$$\phi_r \colon \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-r}(t)$$

We introduce the elements $h_r(t), n_r(t)$ of the Chevalley group $G_L(K)$ which are the images under the homomorphism $\phi_r$ of the special (diagonal and anti-diagonal) matrices such that Let

$$h_r(t) = \phi_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

$$n_r(t) = \phi_r \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}.$$

### 1.4.2 Special subgroups

Using the homomorph images of the $Sl(2, K)$ we will define special subgroups of the Chevalley group $G_L(K)$ such as $H, B$ and $N$. The $(B, N)$-pairs have very special and surprising properties. Based on this properties we have an argument to calculate the orders of the Chevalley groups and using a theorem of Chevalley one can see the simplicity of the Chevalley groups apart from a few exceptions.

Let the subgroup $H$ be generated by the elements $h_r(t)$, the subgroup $N$ be generated by the elements $n_r(t)$ and $B$ is the subgroup generated by $U$ and $H$. An interesting fact that $B = UH$. The subgroups $(B, N)$ have the following nice and surprising properties:

7. $G$ is generated by $B$ and $N$

8. $B \cap N$ is normal subgroup in $N$

9. The group $W = N/B \cap N$ is generated by a set of element $w_i$ $i \in I$ such that $w_i^2 = 1$

10. If $n_i \in N$ maps to $w_i$ under the natural homomorphism of $N$ into $W$, and if $n$ is any element of $N$, then

$$Bn_iB \cdot BnB \subseteq Bn_iNB \cup BnB,$$

11. $n_iBn_i \neq B$.

### 1.4.3 Groups with a $(B, N)$-pair

Now, similarly to the abstract method of the root system it is useful to introduce the general concept of $(B, N)$-pair in an arbitrary group. A pair of subgroups $(B, N)$-pair if the five properties in the last sections as axioms are satiesfied.

From the previous section we have

**Lemma 15.** *The Chevalley group $G_L(K)$ has a $(B, N)$-pair*

For a subset $J \subseteq I$ let $W_J$ be generated by the elements $w_j$ for $j \in J$. By the third property, we have $W = W_I$. Let denote $\psi\colon N \to B \cap N$ the natural homomorphism.

**Theorem 16.** *Let $G$ be a group with a $(B, N)$-pair. Then*

12. *$G = BNB$*

13. *Let $W_J = \psi(N_J)$. Then $P_J = BN_JB$ is a subgroup of $G$.*

14. *Let $n, n'$ be elements of $N$. Then $BnB = Bn'B$ if and only if $\psi(n) = \psi(n')$. This means there is a natural bijection between double cosets od $b$ and elements of $W$.*

15. *The subgroups $P_J$ are the only subgroups of $G$ containing $B$*

16. *Each subgroup $P_J$ of $G$ equals to its normalizer. Furthermore distinct subgroups $P_J$, $P_K$ cannot be conjugate in $G$.*

17. *The subgroups $P_J$ for distinct subset $J$ of $I$ are distinct. Furthermore we have $P_J \cap P_K = P_{J \cap K}$. This means that the subgroups $P_J$ form a lattice isomorphic to the lattice of subsets $I$.*

**Example.** Let $L = \mathfrak{sl}(3, \mathbb{C})$. Then $G = G_L(\mathbb{C}) = \mathrm{SL}(3, \mathbb{C})$. The special subgroups of $G$

$$U = \left\{ \begin{pmatrix} 1 & t & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbb{C} \right\},$$
$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{C}, abc = 1 \right\}.$$

The subgroup $B = UH$ is the group of the upper triangle matrices with determinant 1.

Let $P$ be the **permutation matrix group** of $3\times3$. Then $N$ is the group of **monomial matrices** of $3 \times 3$ with determinant 1.

$$N = HP \cap \mathrm{SL}(3, \mathbb{C}) = (H \rtimes P) \cap \mathrm{SL}(3, \mathbb{C})$$

The Weyl group $W = N/N \cap B = N/H$ is isomorphic to the symmetric group of degree 3.

Thus $W$ is generated by two transpotions

$$W =< (12), (23) >$$

Let $n_1 \in \psi^{-1}((12))$ and $n_2 \in \psi^{-1}((23))$ such that

$$n_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Now the subgroups $P_1$ and $P_2$ of $G$ containing the subgroup $B$

$$P_1 = B\langle n_1 \rangle B = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} : a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} = 1 \right\}$$

and

$$P_2 = B\langle n_2 \rangle B = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{pmatrix} : a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} = 1 \right\}.$$

### 1.4.4 A canonical form in the Chevalley groups

In this section we return to the situation in wfich $G$ is the Chevalley group $G_L(K)$. Then $G$ has a $(B, N)$ pair. Thus every element $g$ of $G$ can be written in the form $g = b_1 n b_2$ where $b_1, b_2 \in B$ and $n \in N$. However this is not a unique decomposition we can express an element of $G$ in such form in a number of different ways. We seek a canonical form ie. a decomposition of elements od $G$ such that each element has a unique decomposition in a given form. This decomposition play an important role in the understanding of the structure of the Chevalley groups, f or example using this decomposition we will calculate the order of the finite Chevalley groups.

**Definition 1.** Let $\Phi$ be a root system. A subset $\Psi \subseteq \Phi$ is called a closed set of roots if $r, s \in \Psi$ and $ir + jr \in \Phi$ ($i, j$ are positive integers) then $ir + jr \in \Psi$ as well.

Let $w \in W$ and define

$$\Psi_1 = \left\{ r \in \Phi^+ \mid w(r) \in \Phi^+ \right\},$$
$$\Psi_2 = \left\{ r \in \Phi^+ \mid w(r) \in \Phi^+ \right\}.$$

Now, $\Psi_1, \Psi_2$ are closed subsets and $\Phi = \Psi_1 \cup \Psi_2$. We define

$$U_w^+ = \prod_{r \in \Psi_1} X_r,$$
$$U_w^- = \prod_{r \in \Psi_2} X_r.$$

Then $U = U_w^+ U_w^-$ and $U_w^+ \cap U_w^- = 1$.

**Theorem 17.** *For each $w \in W$ choose a coset representative $n_w \in N$ such that $\psi(n_w) = w$. Then each element $g \in G$ can be expressible in just one way in the form*

$$g = b n_w u$$

*, where $b \in B$ and $u \in U_w^-$*

**Corollary 18.** *Each element $g \in G$ has a unique form $g = u_1 h n_w u$, where $u_1 \in U, h \in H, w \in W, u \in U_w^-$.*

**Corollary 19.** *Two elements of $H$ which are conjugate in $G$ are conjugate in $N$.*

Let $L_J$ be generated by $H$ and the root subgroups $X_r$ for all $r \in \Phi_J$ and $U_J$ be generated by $X_r$ for all $r \in \Phi^+ \setminus \Phi_J$. Using the Chevalley commutator formula, we have

$$U_J = \prod_{r \in \Phi^+ \setminus \Phi_J} .$$

Now, we have the following properties:

12. $U_J$ is a normal subgroup of $P_J$
13. $P_J = U_J L_J$ and $U_J \cap L_J = 1$
14. $P_J$ is the normalizer of $U_J$ in $G$.

These properties above mean that $P_J$ is a semi-direct product of $U_J$ and $L_J$ This is called Levi decomposition:

$$P_J = U_J \rtimes L_J.$$

### 1.4.5 The order of a finite Chevalley group

We now consider the case in which the base field $K$ is a finite field with $q$ elements, where $q = p^f$ for a prime $p$. Using that a group is a disjoint union of the double cosets of two fixed subgroups and the unique canonical form from the previous section, we have

$$|G| = \sum_{w \in W} |Bn_w B| = \sum_{w \in W} |UHn_w U_w^-| = |U| \cdot |H| \sum_{w \in W} |U^- w|$$

Now by the Theorem ... , each element of $U$ is uniquely expressible in the form

$$\prod_{r_i \in \Phi^+} x_{r_i}(t_i)$$

for some element $t_i \in K$. Thus

$$U = q^N \quad \text{where} \quad N = |\Phi^+|.$$

Also each element of $U_w^-$ is uniquely expressible in the form

$$\prod_{\substack{r_i \in \Phi^+ \\ w(r_i) \in \Phi^+}} x_{r_i}(t_i) \quad \text{thus} \quad U = q^{l(w)}.$$

Using linear $K$-characters one can prove that

$$|H| = \frac{1}{d}(q-1)^l$$

where $d$ is a divisor of $q - 1$ as the following table shows:

| | $d$ |
|---|---|
| $A_l$ | $(l+1, q+1)$ |
| $B_l$ | $(2, q-1)$ |
| $C_l$ | $(2, q-1)$ |
| $D_{2k+1}$ | $(4, q-1)$ |
| $D_{2k}$ | $(2, q-1)^2$ |
| $G_2$ | $1$ |
| $F_4$ | $1$ |
| $E_6$ | $(3, q-1)$ |
| $E_7$ | $(2, q-1)$ |
| $E_8$ | $1$ |

Putting the previous things together we obtain that

**Theorem 20.** *Let $G = G_L(K)$. Then*

$$|G| = \frac{1}{d} q^N (q-1)^t \sum_{w \in W} q^{l(w)}.$$

Now the polynomial

$$\sum_{w \in W} t^{l(w)}$$

is considered as an element of $\mathbb{Q}[t]$. The following two theorems show more useful forms of this polynom.

**Theorem 21** (Solomon)**.**

$$\sum_{w \in W} t^{l(w)} = \prod_{r \in \Phi^+} \frac{t^{h(r)+1} - 1}{t^{h(r)} - 1},$$

*where $h(r)$ is the height of the root $r$.*

**Theorem 22** (Solomon)**.**

$$\sum_{w \in W} t^{l(w)} = \prod_{i=1}^{l} \left( \frac{t^{m_i+1} - 1}{t - 1} \right),$$

*where the integers $m_1, \dots, m_l$ are as follows:*

| | $m_1, m_2, \dots, m_l$ |
|---|---|
| $A_l$ | $1, 2, \dots, l$ |
| $B_l$ | $1, 3, 5, \dots, 2l-1$ |
| $C_l$ | $1, 3, 5 \dots, 2l-1$ |
| $D_l$ | $1, 2, \dots, 2l-3, l-1$ |
| $G_2$ | $1, 5$ |
| $F_4$ | $1, 5, 7, 11$ |
| $E_6$ | $1, 4, 5, 7, 8, 11$ |
| $E_7$ | $1, 5, 7, 9, 11, 13, 17$ |
| $E_8$ | $1, 7, 11, 13, 17, 19, 23, 29$ |

### 1.4.6 The simplicity of the Chevalley groups

First, we show a general criterion for simplicity valid for arbitrary groups:

**Theorem 23.** *Let $G$ be a group with a $(B, N)$-pair satisfying the following conditions:*

15. $G = G'$,
16. *$B$ is soluble,*
17. $\bigcap_{g \in G} gBg^{-1} = 1$,
18. *the set $I$ cannot be decomposed into two non-empty complementary subsets $J, K$ such that $w_j$ commutes with $w_k$ for all $j \in J$, $k \in K$. (As in the previous section, the quotient group $W = B/B \cap N$ is generated by a set of element $w_i$ $i \in I$ such that $w_i^2 = 1$).*

*Then $G$ is simple.*

**Theorem 24.** *Let $L$ be a simple Lie algebra over $\mathbb{C}$ and $K$ be an arbitrary field.*

15. *Then the Chevalley group $G_L(K)$ is simple, except for $A_1(2), A_1(3), B_2(2), G_2(2)$*
16. *Each Chevalley group (even a non-simple one) has trivial centre.*

### 1.4.7 Abstract definition of the Chevalley groups by generators and relations

The Chevalley group $G = G_L(K)$ is generated by the elements $x_r(t) = \exp(\operatorname{ad} te_r)$ for all $r \in \Phi$, $t \in K$. In this section our goal is to finding generators and relations to define $G$ as an abstract group. Let $h_r(t)$, $n_r(t)$ be defined as in the section .... by

$$h_r(t) = \phi_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

$$n_r(t) = \phi_r \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$$

Since

$$x_r(t) = \phi_r \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad x_{-r}(t) = \phi_r \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

we have

$$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t),$$
$$h_r(t) = n_r(t)n_r(-1)$$

The following relations hold is $G$:

$$x_r(t_1)x_r(t_2) = x_r(t_1 + t_2),$$

$$[x_s(u), x_r(t)] = \prod_{i,j>0} x_{ir+js}(C_{ijrs}(-t)^i u^j)$$

$$h_r(t_1)h_r(t_2) = h_r(t_1 t_2)$$

**Theorem 25** (Steinberg)**.** *Let $L$ be a simple Lie algebra with $L \neq A_1$ and let $K$ be a field. For each root $r$ of $L$ and each element $t$ of $K$ introduce a symbol $\overline{x_r}(t)$. Let $\overline{G}$ be the abstract group generated by the elements $\overline{x}_r(t)$ subject to relations*

$$\overline{x_r}(t_1)\overline{x}_r(t_2) = \overline{x}_r(t_1 + t_2),$$

$$[\overline{x}_s(u), \overline{x}_r(t)] = \prod_{i,j>0} \overline{x}_{ir+js}(C_{ijrs}(-t)^i u^j) \quad \text{(Chevalley commutator formula)}$$

*where*

$$\overline{h}_r(t) = \overline{n}_r(t)\overline{n}_r(-1)$$

*and*

$$\overline{n}_r(t) = \overline{x}_r(t)\overline{x}_{-r}(-t^{-1})\overline{x}_r(t).$$

*Let $\overline{Z}$ be the center of $\overline{G}$. Then $\overline{G}/\overline{Z}$ is isomorphic to the Chevalley group $G = G_L(K)$.*

The group $\overline{G}$ defined in the previous theorem is called **universal Chevalley group** of type $L$ over $K$

**Theorem 26.** *Let $\overline{G}$ be the universal Chevalley group. Then*

15. *each element of $\overline{H}$ has a unique expression of the form*

$$\overline{h}_{p_1}(t_1)\overline{h}_{p_2}(t_2)\ldots\overline{h}_{p_l}(t_l)$$

*and therefore*

$$\overline{H} \simeq \underbrace{K^* \oplus K^* \oplus \cdots \oplus K^*}_{l}$$

*where $K^*$ is the multiplicative group of $K$*

16. *an element $\overline{h}_{p_1}(t_1)\overline{h}_{p_2}(t_2)\ldots\overline{h}_{p_l}(t_l)$ of $\overline{H}$ is in the centre $\overline{Z}$ of $\overline{G}$ if and only if*

$$\prod_{i=1}^{l} t_i^{\frac{2(p_i,p_j)}{(p_i,p_i)}} = 1$$

# Chapter 2

# Classification of real (semi)simple Lie algebras and Lie groups

## 2.1 The Classical Lie groups and Lie algebras

In this section we give a description of the so called classical Lie algebras and Lie groups, which provide us most of the classes of simple Lie algebras and Lie groups. In fact, for a fixed field $\mathbb{K}$ there are only finitely many non-isomorphic Lie algebras (resp. groups of Lie type) which are exceptional, i.e. not belong to these series. Throughout this section we restrict our attention to the case when the base field is $\mathbb{R}$ or $\mathbb{C}$, but we mention that similar constructions can be given over other fields, (e.g. over finite fields), as well.

Basically, a classical Lie group over a field $\mathbb{K}$ is the automorphism group of a geometric space over the field $\mathbb{K}$. For each such Lie group there is a corresponding Lie algebra over $\mathbb{K}$. (We note that the classical Lie groups can also be defined as algebraic groups, which allows one to define their tangent space in an algebraic way over any field, (e.g. over fields of positive characteristic).

In what follows, we give a description of the classical Lie groups and their Lie algebras over the field of complex and real numbers. It turns out, that apart from finitely many exceptional cases, these provide us all the simple Lie algebras and Lie groups over these fields.

### 2.1.1 The general and special linear Lie groups and Lie algebras

Let $V$ be an $n$ dimensional vector space over the field $\mathbb{K}$. Then the general linear group $GL(V)$ is the group of all automorphisms of $V$, that is the group of all invertible linear maps $V \mapsto V$. By fixing a basis of $V$, this group can be identified with the group of all $n \times n$ invertible matrices over the field $\mathbb{K}$, so

$$GL(V) \simeq GL(n, \mathbb{K}) := \{ A \in M_n(\mathbb{K}) \mid \det(A) \neq 0 \} .$$

Now, the special linear group $SL(n, \mathbb{K}) \leq GL(n, \mathbb{K})$ is the subgroup of all matrices having determinant one. If one writes the matrix form of a linear map $\varphi \colon V \to V$ with respect to a basis in $V$, then the determinant of the matrix does only depend on $\varphi$, not on the chosen basis. This means that there is a corresponding subgroup $SL(V) \simeq SL(n, \mathbb{K})$ of $GL(V)$, which we also call a special linear group. It is easy to prove that $SL(V)$ (resp. $SL(n, \mathbb{K})$) is the commutator subgroup of $GL(V)$ (resp. $GL(n, \mathbb{K})$.)

By calculating the tangent spaces of $GL(n, \mathbb{K})$ and $SL(n, \mathbb{K})$ one gets the general and special Lie algebras, denoted by $\mathfrak{gl}(n, \mathbb{K})$ and $\mathfrak{sl}(n, \mathbb{K})$. The Lie bracket in these Lie algebras can be calculated with the help of the usual matrix multiplication by the rule $[X, Y] := XY - YX$. Thus, we get the classical Lie groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$ with corresponding Lie algebras $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{R})$. Here, we have

$$\begin{aligned} \mathfrak{sl}(n, \mathbb{R}) &= \{ X \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{Tr} X = 0 \}, \\ \mathfrak{sl}(n, \mathbb{C}) &= \{ X \in \mathfrak{gl}(n, \mathbb{C}) \mid \operatorname{Tr} X = 0 \}. \end{aligned}$$

We mention here another construction for a real Lie algebra, which uses the division ring of quaternions. (One can learn more on quaternions in Section 3.3.1, where we present the Cayley–Dickson process, which gives a unified way to define the complex, quaternion and octonion numbers. The octonions are connected to the simple Lie algebra $\mathfrak{g}_2$. A number of interesting facts about these numbers are shown in Section 3.3.1.) The construction of the quaternions is the following. Their base set is

$$\mathbb{H} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \},$$

so $\mathbb{H}$ is a four dimensional vector space over $\mathbb{R}$ with basis $1, i, j, k$. Now, the multiplication between the basis elements is given by the rules

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j,$$

and this multiplication is extended to the whole $\mathbb{H}$ by using the associative and distributive laws. For a quaternion element $z = a + bi + cj + dk$ we call $\Re z = a$ and $\Im z = bi + cj + dk$ the real and imaginary part of $z$, respectively. Finally, like in the case of complex numbers one can define a conjugation operation on $\mathbb{H}$ such by the rule $\bar{z} = \Re z - \Im z$.

Now, one can define the (right) vector space $\mathbb{H}^n$ as the set of all column vectors of length $n$ containing quaternion elements in their coordinates, with pointwise addition and scalar multiplication by elements of $\mathbb{H}$. The set of all $\mathbb{H}$-linear maps $\mathbb{H}^n \to \mathbb{H}^n$ can be identified with the set of all $n \times n$ matrices acting on $\mathbb{H}^n$ by the usual matrix multiplication from the left. The set of these matrices form a real Lie algebra $\mathfrak{gl}(n, \mathbb{H})$ with Lie bracket $[X, Y] := XY - YX$. Note that this Lie bracket does not respect the multiplication by elements of $\mathbb{C}$, so this is not a complex Lie algebra. Finally, one can define the one-codimensional subalgebra $\mathfrak{sl}(n, \mathbb{H}) \leq \mathfrak{gl}(n, \mathbb{H})$ as

$$\mathfrak{sl}(n, \mathbb{H}) = \{ X \in \mathfrak{gl}(n, \mathbb{H}) \mid \Re(\operatorname{Tr} X) = 0 \}.$$

### 2.1.2 Bilinear and hermitian forms and related Lie groups and Lie algebras

In this section we give a construction for all the classical Lie algebras different from the general and special linear ones. Here, the general idea is to choose a bilinear function on $V$ and to take the set of all linear transformations of $V$ which leave this bilinear form invariant. Precise definitions and some of the most important facts follows in the following list.

**Summary 27.** *Let $\mathbb{K}$ denote either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let $V$ be an $n$ dimensional vector space over $\mathbb{K}$.*

1. *A map* $(.,.) \colon V \times V \to \mathbb{K}$ *is called bilinear, if for all* $a, b, c \in V$ *and* $k \in \mathbb{K}$ *the following holds:*

$$(a+b,c) = (a,c) + (b,c), \quad (a,b+c) = (a,b) + (a,c), \quad (ka,b) = k(a,b) = (a,kb).$$

2. *A bilinear function is called symmetric, if* $(a,b) = (b,a)$ *for every* $a, b \in V$*; and is called symplectic (or antisymmetric) if* $(a,b) = -(b,a)$ *for every* $a, b \in V$*.*

3. *A map* $(.\,,.) \colon V \times V \to \mathbb{K}$ *is called hermitian, if for all* $a, b, c \in V$ *and* $k \in \mathbb{K}$ *the following holds:*

$$(a+b,c) = (a,c) + (b,c), \qquad (a,b+c) = (a,b) + (a,c),$$
$$(ka,b) = k(a,b), \quad \text{and} \quad (b,a) = \overline{(a,b)}.$$

*(Now,* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{K} = \mathbb{H}$*.)*

4. *Two vectors* $(a,b) \in V$ *are orthogonal with respect to the bilinear or hermitian form* $(.\,,.)$*, if* $(a,b) = 0$*.*

5. *The radical of a bilinear or hermitian form is* $\{\, a \in V \mid (a,b) = 0 \,\forall b \in V \,\}$*, which is a subspace of* $V$*. The bilinear form is called non-degenerate if its radical is the null space.*

6. *A vector space with a non-degenerate symmetric (resp. symplectic) bilinear form is called a symmetric (resp. a symplectic) space. Similarly, a hermitian space is a vector space endowed with a non-degenerate hermitian form.*

7. *By choosing a basis* $B = \{\, b_1, b_2, \ldots, b_n \,\}$ *in* $V$*, one can define the matrix of the bilinear or hermitian form* $(.\,,.)$ *as* $A = (a_{ij})$ *with* $a_{ij} := (b_i, b_j)$*. Now, identifying* $V$ *and the space of column vectors* $\mathbb{K}^n$ *with respect to this basis, the bilinear (resp. hermitian) form can be calculated as* $(u,v) = u^T A v$ *(resp. as* $(u,v) = u^T A \overline{v}$*).*

Now, let $V$ be an $n$ dimensional space over $\mathbb{K}$ with bilinear or hermitian form $(.\,,.)$. Then one can define the corresponding Lie group $G$ as the elements of $GL(V)$ leaving $(.\,,.)$ invariant, i.e.

$$G = \{\, \varphi \in GL(V) \mid (\varphi(x), \varphi(y)) = (x,y) \;\forall\; x, y \in V \,\}\,.$$

Then the Lie algebra $\mathfrak{g}$ of this Lie group can be defined as

$$\mathfrak{g} = \{\, \varphi \in \mathrm{End}(V) \mid (\varphi(x), y) + (x, \varphi(y)) = 0 \;\forall\; x, y \in V \,\}\,.$$

One can get the “special versions” of these Lie groups and Lie algebras by taking the intersections $G \cap SL(V)$ and $\mathfrak{g} \cap \mathfrak{sl}(V)$, respectively.

If the bilinear or hermitian form is given by a matrix $A$ with respect to a fixed basis, then the corresponding Lie group and Lie algebra can be defined as a set of matrices in the following way.
For a bilinear form we have

$$G = \left\{\, X \in GL(n, \mathbb{K}) \mid X^T A X = A \,\right\}, \quad \mathfrak{g} = \left\{\, X \in M_n(\mathbb{K}) \mid X^T A + A X = 0 \,\right\}.$$

For a hermitian form we have

$$G = \left\{ X \in GL(n,\mathbb{K}) \mid X^T A \overline{X} = A \right\}, \quad \mathfrak{g} = \left\{ X \in M_n(\mathbb{K}) \mid X^T A + A\overline{X} = 0 \right\}.$$

Now, using this construction one can define a number of examples to classical Lie algebras. Let $I_n$ be the identity matrix and $p, q \in \mathbb{N},\ p+q = n$. Let us define the following matrices.

$$J_{n,n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad K_{n,n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{2.1}$$

Furthermore, for any matrix $A \in GL(n,\mathbb{K}), (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$ we define its adjoint $A^*$ as its conjugate transpose i.e. $A^* = \bar{A}^T$. (So $A^* = A^T$ for $\mathbb{K} = \mathbb{R}$, otherwise we use the conjugate operation on $\mathbb{C}$ or on $\mathbb{H}$ to every element of the matrix.)

Now, one can define the following list of classical semisimple Lie algebras over the complex (Table 2.1) and real numbers (Table 2.2).

Table 2.1: Classical Lie algebras over $\mathbb{C}$

$$\begin{aligned}
\mathfrak{sl}(n,\mathbb{C}) &= \{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr} X = 0 \}, \\
\mathfrak{so}(n,\mathbb{C}) &= \left\{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid X + X^T = 0 \right\}, \\
\mathfrak{sp}(n,\mathbb{C}) &= \left\{ X \in \mathfrak{gl}(2n,\mathbb{C}) \mid X^T J_{n,n} + J_{n,n} X = 0 \right\}.
\end{aligned}$$

Table 2.2: Classical Lie algebras over $\mathbb{R}$

$$\begin{aligned}
\mathfrak{sl}(n,\mathbb{R}) &= \{ X \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr} X = 0 \}, \\
\mathfrak{sl}(n,\mathbb{H}) &= \{ X \in \mathfrak{gl}(n,\mathbb{H}) \mid \operatorname{Re} \operatorname{Tr} X = 0 \}, \\
\mathfrak{su}(p,q) &= \{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid X^* I_{p,q} + I_{p,q} X = 0 \}, \\
\mathfrak{so}(p,q) &= \{ X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^* I_{p,q} + I_{p,q} X = 0 \}, \\
\mathfrak{so}^*(2n) &= \left\{ X \in \mathfrak{su}(n,n) \mid X^T K_{n,n} + K_{n,n} X = 0 \right\}, \\
\mathfrak{sp}(p,q) &= \{ X \in \mathfrak{gl}(n,\mathbb{H}) \mid X^* I_{p,q} + I_{p,q} X = 0 \}, \\
\mathfrak{sp}(n,\mathbb{R}) &= \left\{ X \in \mathfrak{gl}(2n,\mathbb{R}) \mid X^T J_{n,n} + J_{n,n} X = 0 \right\}.
\end{aligned}$$

In the following first we give a hint how it can be proved that these Lie algebras are semisimple, then for each such algebras we give the bilinear or symplectic space which defines it. To prove that these algebras are really semisimple, we need the following definition.

**Definition 28** (reductive Lie algebra)**.** A Lie algebra is called reductive if for every ideal $\mathfrak{a}$ in $\mathfrak{g}$ there is an ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$.

Reductive and semisimple Lie algebras are closely connected to each other as the following proposition shows.

**Proposition 29.** *A Lie algebra* $\mathfrak{g}$ *is reductive if and only if* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + Z(\mathfrak{g})$ *for semisimple* $[\mathfrak{g}, \mathfrak{g}]$*. (Here,* $Z(\mathfrak{g})$ *denotes the centre of the Lie algebra.)*

Therefore, the semisimplity of each of the above Lie algebras follows from that they have trivial centres (this can be easily checked) and that they are reductive. To see this later property, one needs to check that each of these Lie algebras are closed under taking conjugate transpose and then use the following proposition.

**Proposition 30.** *If* $\mathfrak{g} \leq \mathfrak{gl}(n, \mathbb{K})$ *(where* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$*) is a real matrix Lie algebra closed under conjugate transpose, then* $\mathfrak{g}$ *is reductive.*

## 2.2 Connection with the complex case

In this section we give a description of the connection of semisimple Lie algebras over the real and those over the complex numbers. We will see that every (semi)simple real Lie algebra is connected to a unique (semi)simple complex Lie algebra, thus we can use the description of all simple complex Lie algebras via Dynkin diagrams to classify all the simple real ones [6].

### 2.2.1 Complexification and realification, real forms

Let $\mathfrak{g}_0$ be a real Lie algebra. Then one can define its **complexification** $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C}) := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ by extending scalar multiplication to $\mathbb{C}$. Note that $\mathfrak{g}_0(\mathbb{C}) \simeq \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as vector spaces. The Lie bracket of $\mathfrak{g}_0$ extends to $\mathfrak{g}_0(\mathbb{C})$ by the rule

$$[x_1 + iy_1, x_2 + iy_2] = [x_1, y_1] - [x_2, y_2] + i([x_1, y_2] + [y_1, x_2]), \qquad \forall\ x_1, y_1, x_2, y_2 \in \mathfrak{g}_0.$$

Conversely, if $\mathfrak{g}$ is a complex Lie algebra, then one can take its **realification** $\mathfrak{g}_{\mathbb{R}}$ by "forgetting" the multiplication with non-real scalars, that is, viewing it as a real Lie algebra. Choosing a basis of the real space $\mathfrak{g}_0$ provides a basis of the complex space $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$, thus $\dim_{\mathbb{R}}(\mathfrak{g}_0) = \dim_{\mathbb{C}}(\mathfrak{g})$. On the other hand, if $b_1, \ldots, b_n$ is a basis of the complex Lie algebra $\mathfrak{g}$, then $b_1, ib_1, b_2, ib_2, \ldots, b_n, ib_n$ is a basis of its realification $\mathfrak{g}_{\mathbb{R}}$, i.e. $\dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} = 2 \dim_{\mathbb{C}} \mathfrak{g}$.

Starting from a complex Lie algebra $\mathfrak{g}$, one might look for a real Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$. This problem leads us to the concept of real forms, which turns out to be central in the classification of (semi)simple real Lie algebras. We say that a real subalgebra $\mathfrak{g}_0 \leq \mathfrak{g}_{\mathbb{R}}$ is a **real form** of $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. A trivial observation is that if $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, then the $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$. Conversely, if $\mathfrak{g}$ is isomorphic to the complexification of the real Lie algebra $\mathfrak{g}_0$, then $\mathfrak{g}_0$ is isomorphic to a real form of $\mathfrak{g}$. Table 2.3 summarizes these notations for future reference.

Now one can use the Killing form to show that complexification and realification connects semisimple real Lie algebras with semisimple complex Lie algebras. (For the definition and properties of the Killing form see [6, Section 3.2])

The connection between Killing form of a real (resp. complex) Lie algebra and the Killing form of its complexification (resp. realification) can easily be computed and is described as follows.

**Proposition 31.** *Let* $\mathfrak{g}$ *be a complex Lie algebra with Killing form* $\kappa_{\mathfrak{g}}$*.*

Table 2.3: Notations for complexification and realification and real form

| | |
|---|---|
| $\mathfrak{g}_0$ | real Lie algebra, |
| $\mathfrak{g}_0(\mathbb{C}) = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ | **complexification of** $\mathfrak{g}_0$. |
| $\mathfrak{g}$ | complex Lie algebra, |
| $\mathfrak{g}_{\mathbb{R}}$ | **realification of** $\mathfrak{g}$, |
| $\mathfrak{g}_0 \leq \mathfrak{g}_{\mathbb{R}}$ for which $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ | **real form of** $\mathfrak{g}$. |

1. *If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ then its Killing form $\kappa_{\mathfrak{g}_0}$ satisfies*

$$\kappa_{\mathfrak{g}_0} = \kappa_{\mathfrak{g}} \mid_{\mathfrak{g}_0 \times \mathfrak{g}_0} .$$

2. *If $\kappa_{\mathfrak{g}_{\mathbb{R}}}$ is the Killing form of the realification $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$, then it can be computed as*

$$\kappa_{\mathfrak{g}_{\mathbb{R}}} = 2\Re\kappa_{\mathfrak{g}}.$$

*As a consequence,*

$$\kappa_{\mathfrak{g}_0} \textit{ is non-degenerate} \iff \kappa_{\mathfrak{g}} \textit{ is non-degenerate} \iff \kappa_{\mathfrak{g}_{\mathbb{R}}} \textit{ is non-degenerate}$$

Using Cartan's criterion (see [6, Theorem 19]) we obtain

**Corollary 32.** *A real (resp. a complex) Lie algebra is semisimple if and only if its complexification (resp. its realification) is semisimple.*

*Remark* 33. In fact, Cartan's criterion was formulated in [6] for Lie alegbras over $\mathbb{C}$, but one can show that it is still true for a Lie algebra over any field of characteristic 0. (For positive characteristic, it is still true that if the Killing form is non-degenerate, then the Lie algebra is semisimple, but the reverse implication does not necessarily hold.)

Moreover, if the Killing form of a Lie algebra (over any field) is non-degenerate, then one can decompose the Lie algebra to a direct sum of simple Lie algebras. The trick is that one can take the orthogonal complement of an ideal with respect to the Killing form. This complement turns out to be an ideal, as well, and thus the whole Lie algebra can be decomposed as the direct sum of these two ideals. Finally, one can use induction on the dimension by remarking that the restriction of the Killing form to each component in the direct sum is non-degenerate. As a special case one obtains that every semisimple real Lie algebra is a direct sum of some simple real Lie algebras.

Now, we formulate a theorem for simple Lie algebras similar to Corollary 32. We do not present a full proof here, but we mention that most of it can be proved by using the fact that both realification and complexification maps ideals into ideals. In fact, the only non-trivial claim is that if the complexification of a simple real Lie algebra $\mathfrak{g}_0$ is not simple, then there is a complex structure on $\mathfrak{g}_0$, i.e. it is a realification of a (simple) complex Lie algebra. The proof of this fact uses some concepts, which we will introduce in the next section.

**Theorem 34.** *A real Lie algebra $\mathfrak{g}_0$ is simple if and only if it is isomorphic to either a real form or a realification of a complex simple Lie algebra.*

*Remark* 35. In fact, the two possibilities in the theorem exclude each other. That is, the complexification of a realification of a simple Lie algebra is never simple. To be more precise, if $\mathfrak{g}$ is any complex Lie algebra, then $\mathfrak{g}_{\mathbb{R}}(\mathbb{C}) \simeq \mathfrak{g} \times \overline{\mathfrak{g}}$, where $\overline{\mathfrak{g}}$ is the complex conjugate Lie algebra of $\mathfrak{g}$, which means that $\mathfrak{g}_{\mathbb{R}} \simeq \overline{\mathfrak{g}}_{\mathbb{R}}$, but the multiplication by complex numbers in $\overline{\mathfrak{g}}$ is defined as $c * x := \bar{c} \cdot x$, where $*$ and $\cdot$ denotes the multiplication by scalars in the Lie algebras $\overline{\mathfrak{g}}$ and $\mathfrak{g}$, respectively.

Thus, we already have four (or three) infinite series and five exceptional examples for simple real Lie algebras, namely, the realifications of the simple complex Lie algebras classified by Dynkin diagrams (see [6]). In what follows, we give a method to give a description of all the real forms of complex simple Lie algebras.

### 2.2.2 Real forms and and antiinvolutions

In this section we show a connection between real forms of a complex Lie algebra and its second order automorphisms (and antiautomorphisms). Throughout the rest of the chapter $\mathfrak{g}$ and $\mathfrak{g}_0$ will always denote a Lie algebra over the complex and over the real numbers, respectively. Usually, we will assume that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. We begin this section with a couple of definitions.

**Definition 36.**

1. We say that $\alpha \in \mathrm{Aut}(\mathfrak{g})$ (resp. $\alpha \in \mathrm{Aut}(\mathfrak{g}_0)$) is an **involution** of $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) if it is a Lie algebra automorphism of $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) with the property $\alpha^2 = \mathrm{id}$.

2. $\sigma$ is an **antiinvolution** of the complex Lie algebra $\mathfrak{g}$ if it is an involution of $\mathfrak{g}_{\mathbb{R}}$, but it is antilinear, i.e. $\sigma(cx) = \bar{c}\sigma(x)$ for every $c \in \mathbb{C},\ x \in \mathfrak{g}$. Sometimes we will also call such a $\sigma$ a **real structure** on $\mathfrak{g}$ since, as it turns out, it is strongly connected to a real form of $\mathfrak{g}$.

3. For a complex Lie algebra $\mathfrak{g}$ with real form $\mathfrak{g}_0$ one can define the **conjugation** on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ as

$$\sigma(u + iv) = u - iv \qquad u, v \in \mathfrak{g}_0.$$

**Example.** Let $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$ (or $\mathfrak{g}_0 = \mathfrak{gl}(3, \mathbb{R})$). Then $\alpha\colon \mathfrak{g} \to \mathfrak{g},\ \alpha(X) = -X^T$ is an involution. It is clearly linear, and $\alpha([X,Y]) = -(XY - YX)^T = -(Y^T X^T - X^T Y^T) = [-X^T, -Y^T] = [\alpha(X), \alpha(Y)]$, so $\alpha \in \mathrm{Aut}(\mathfrak{g})$. Finally, $\alpha^2(X) = \alpha(-X^T) = -(-X^T)^T = X$, so $\alpha$ is an involution.

To see an antiinvolution of $\mathfrak{g}$, one can choose $\sigma\colon \mathfrak{g} \to \mathfrak{g},\ \sigma(X) = \overline{X}$. Then $\sigma(aX+Y) = \overline{a}\overline{X} + \overline{Y}$ for $a \in \mathbb{C},\ X, Y \in \mathfrak{g}$, so $\sigma$ is antilinear. Similarly, $\sigma([X,Y]) = \overline{XY - YX} = \overline{X}\ \overline{Y} - \overline{Y}\ \overline{X} = [\sigma(X), \sigma(Y)]$. Finally, $\sigma^2(X) = \sigma(\overline{X}) = X$. It is worth noting that $\sigma$ is just the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

Finally, one can easily find another antiinvolution $\tau := \alpha\sigma$. The composition of an automorphism and an antiautomorphism is always an antiautomorphism. Moreover, $\tau^2(X) = \tau(-\overline{X}^T) = X$ also holds. (This last property depends on the fact that $\alpha\sigma = \sigma\alpha$.)

If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ then the conjugation $\sigma$ with respect to $\mathfrak{g}_0$ is an antiinvolution of $\mathfrak{g}$ with the property

$$\mathfrak{g}_0 = \mathfrak{g}^\sigma := \{\, x \in \mathfrak{g} \mid \sigma(x) = x \,\}.$$

Conversely, if $\sigma$ is any antiinvolution of the Lie algebra $\mathfrak{g}$, then one can define a real form of $\mathfrak{g}$ as $\mathfrak{g}_0 := \mathfrak{g}^\sigma$. This means that the real forms and antiinvolutions of a complex Lie algebra are in a one to one correspondence with each other: whenever we find an antiinvolution $\sigma$ of a complex Lie algebra $\mathfrak{g}$, we can define a real form of $\mathfrak{g}$ by taking the set of fixed points of $\sigma$.

*Remark* 37. By using the concept of the complex conjugate $\mathfrak{g}$ defined in Remark 35 of Section 2.2.1, another connection can be found. As $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ has the same base set, any map $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ can be viewed also as a map $\alpha\colon \mathfrak{g} \to \overline{\mathfrak{g}}$. Now, $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ is an antiisomorphism if and only if $\alpha\colon \mathfrak{g} \to \overline{\mathfrak{g}}$ is an isomorphism, and vice versa. In particular, if $\mathfrak{g}$ possesses a real form, then $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ are isomorphic. (Conversely, if there is an isomorphism $\alpha\colon \mathfrak{g} \to \overline{\mathfrak{g}}$ with $\alpha^2 = \mathrm{id}$, then $\mathfrak{g}$ possesses a real form.)

The question arises, that if we find two different antiinvolutions of the same Lie algebra, when the corresponding real forms are isomorphic. This question is answered by the following theorem.

**Theorem 38.** *Let $\mathfrak{g}_0, \mathfrak{g}_1$ be two real forms of the same complex Lie algebra $\mathfrak{g}$ with corresponding conjugations $\sigma_0, \sigma_1$. Then $\mathfrak{g}_0 \simeq \mathfrak{g}_1$ if and only if there is an $\alpha \in \mathrm{Aut}(\mathfrak{g})$ such that $\sigma_1 = \alpha\sigma_0\alpha^{-1}$.*

*Thus, in order to classify all the simple real Lie algebras, we need to find all the antiinvolutions of the simple complex Lie algebras up to conjugacy by the automorphism group of the Lie algebra.*

### 2.2.3 Compact and split real forms

Until now, we did not see any real forms of simple Lie algebras. Now, we show two constructions by using the structure and isomorphism theorems developed for the complex case (for details, see [6] or e.g. [7]). First we collect the most important facts, which we will use throughout the remainder of chapter. We will also fix the notations and will use them without any further reference.

**Summary 39.** *Let $\mathfrak{g}$ be any complex simple Lie algebra.*

1. *There is a* ***root space decomposition***

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

*where $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$ (called a* ***Cartan subalgebra****), and $\mathfrak{g}_\alpha$ are one dimensional $\mathrm{ad}\,\mathfrak{h}$-invariant subspaces (called* ***root spaces****) indexed by the system of* ***roots*** *$\Phi \subset \mathfrak{h}^*$ such that $[h, x] = \alpha(h) \cdot x$ for every $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$. Here, $\mathfrak{h}^*$ denotes the set of $\mathfrak{h} \to \mathbb{C}$ linear functions.*

2. *There is a subset $\Pi \subset \Phi$ such that $\Pi$ forms a basis of $\mathfrak{h}^*$ and $\Phi$ is decomposed as $\Phi = \Phi^+ \cup \Phi^-$ where every element of $\Phi^+$ (resp. $\Phi^-$) can be written as an integer linear combination of the elements of $\Pi$ using only non-negative (resp. non-positive) coefficients only. The elements of $\Pi$, $\Phi^+$ and $\Phi^-$ are called* ***fundamental, positive and negative roots*** *(with respect to $\Pi$). We also note that $\Phi^- = \{ -\alpha \mid \alpha \in \Phi^+ \}$.*

3. *One can choose a basis*

$$B := \{ h_\alpha \in \mathfrak{h} \mid \alpha \in \Pi \} \cup \{ e_\beta \in \mathfrak{g}_\beta \mid \beta \in \Phi \}$$

*of* $\mathfrak{g}$ *such that the Lie bracket of any two of them is a real linear combination of the elements from* $B$. *That is, the structure constants of* $\mathfrak{g}$ *with respect to* $B$ *are all reals. (In fact, one can even choose a* ***Chevalley basis****, where all the structure constants are integers, which makes it possible to define Chevalley algebras over finite fields, a key step to the construction of finite simple groups of Lie type. But we do not need this result to deal with the real case.)*

4. *The subset*

$$\{ h_\alpha \mid \alpha \in \Pi \} \cup \{ e_\alpha \mid \alpha \in \Pi \} \cup \{ e_{-\alpha} \mid \alpha \in \Pi \}$$

*generates* $\mathfrak{g}$ *as a Lie algebra. We call this set the* ***canonical generator set for*** $\mathfrak{g}$ ***associated with*** $(\mathfrak{g}, \mathfrak{h}, \Pi)$.

5. *The restriction of the Killing form is non-degenerate on* $\mathfrak{h}$. *This results that one can define with its help a non-degenerate bilinear form* $(.\,,.)$ *on* $\mathfrak{h}^*$, *which turns out to be positive definite on the real subspace* $E = \langle \Phi \rangle_{\mathbb{R}} \leq \mathfrak{h}^*_{\mathbb{R}}$. *Thus it makes* $E$ *to an Euclidean space with scalar product* $(.\,,.)$.

6. *Let* $\Pi = \{ \alpha_1, \ldots, \alpha_l \} \subset E$ *be the system of fundamental roots. Then one can define the* ***Cartan integers*** *and the* ***Cartan matrix***

$$a_{ij} = \langle \alpha_i, \alpha_j \rangle := \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \qquad A := (a_{ij}).$$

7. *Then the* ***Dynkin diagram*** $\Gamma$ ***associated to*** $(\mathfrak{g}, \mathfrak{h}, \Pi)$ *is defined as the following directed graph.*
   - $\Gamma$ *has vertex set* $\Pi$.
   - *For every* $\alpha \neq \beta \in \Pi$, *the number of edges between* $\alpha$ *and* $\beta$ *is* $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle$. *The edge is undirected, if* $|\alpha| = |\beta|$, *otherwise it is directed towards the shorter root.*

*(You can find a picture in [6, Theorem 33] from all the Dynkin diagrams.)*

On the one hand, the Existence Theorem [6, Theorem 26] for every Dynkin diagram $A_l, \ldots, G_2$ there is a corresponding complex simple Lie algebra. (Occasionally, we will denote them by $A_l(\mathbb{C}), \ldots, G_2(\mathbb{C})$.) On the other hand, by the Uniqueness Theorem [6, Theorem 27] the Dynkin diagram uniquely defines the corresponding complex simple Lie algebra up to isomorphism. That is, non-isomorphic complex simple Lie algebras have different Dynkin diagrams. We state this latter theorem in a more technical form, which helps to better understand the remaining material of this chapter.

**Theorem 40.** *Suppose that two triples* $(\mathfrak{g}, \mathfrak{h}, \Pi)$ *and* $(\mathfrak{g}', \mathfrak{h}', \Pi')$ *of complex simple Lie algebras with Cartan subalgebras and fundamental systems are given. Let* $\{ h_\alpha,\ e_\alpha,\ e_{-\alpha} \mid \alpha \in \Pi \}$ *and* $\left\{ h'_\alpha,\ e'_\alpha,\ e'_{-\alpha} \mid \alpha \in \Pi' \right\}$ *be the associated canonical generator sets and let* $A$, $A'$ *be the associated Cartan matrices. If* $A = A'$, *then there is a unique isomorphism* $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ *satisfying*

$$\varphi(h_\alpha) = h'_\alpha, \quad \varphi(e_\alpha) = e'_\alpha, \quad \varphi(e_{-\alpha}) = e'_{-\alpha}, \quad \forall\ \alpha \in \Pi.$$

Now, we define the split and the compact real forms of any simple complex Lie algebra as follows.

**Definition 41.** A real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is called a **split real form** if there exists a Cartan subalgebra $\mathfrak{h}$ and associated root system $\Phi$ such that for every $\alpha \in \Phi$, the restriction of $\alpha$ to $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$ is real-valued.

Now, one can easily define a split real form $\mathfrak{g}_0$ of a complex simple Lie algebra $\mathfrak{g}$ as follows. We start from a triple $(\mathfrak{g}, \mathfrak{h}, \Phi)$, and a basis $B$ of $\mathfrak{g}$ satisfying the property as in item 3 of Summary 39 and we take $\mathfrak{g}_0$ as the real subspace generated by $B$. By the property mentioned in item 3 of Summary 39, $\mathfrak{g}_0$ is closed to the Lie bracket, so it is a subalgebra of $\mathfrak{g}_{\mathbb{R}}$. As $\mathfrak{g}_0$ contains the basis $B$ of $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, i.e. $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$.

*Remark* 42. We note that if $\mathfrak{g}_0$ is a split real form of $\mathfrak{g}$, then the restriction of the Killing form to $\mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathfrak{h}$ is clearly positive definite. On the other hand, its restriction to the whole split real form $\mathfrak{g}_0$ is not necessarily positive definite.

**Example.** Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ be the (complex) Lie algebra of all $3 \times 3$ matrices over $\mathbb{C}$ with zero trace. Then it is easy to check that $\mathfrak{g}_0 := \mathfrak{sl}(3, \mathbb{R})$ is a real subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0 = \mathfrak{g}_0(\mathbb{C})$, thus, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$.

We claim that it is a split real form of $\mathfrak{g}$. To check the criterion in Definition 41, let

$$\mathfrak{h} = \{ \operatorname{diag}(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{C},\ x_1 + x_2 + x_3 = 0 \}$$

be the Cartan subalgebra of diagonal matrices in $\mathfrak{g}$. The associated system of roots can be given as

$$\Phi = \{ r_{ij} \colon \operatorname{diag}(x_1, x_2, x_3) \mapsto x_i - x_j \mid 1 \le i \ne j \le 3 \} \subset \mathfrak{h}^*.$$

Now, $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ is the subalgebra of all real diagonal matrices in $\mathfrak{g}$. It is clear that $r_{ij}(x) \in \mathbb{R}$ for every $r_{ij} \in \Phi$ and for every $x \in \mathfrak{h}_0$.

To see that the Killing form $\kappa$ is positive definite on $\mathfrak{h}_0$, we use that the Killing form on $\mathfrak{sl}(3, \mathbb{C})$ is $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \cdot \operatorname{ad} y) = 6 \operatorname{Tr}(xy)$ (this was calculated in [6, page 45]). Thus, for any non-zero element $x = \operatorname{diag}(x_1, x_2, x_3) \in \mathfrak{h}_0$ we have

$$\kappa(x, x) = 6(x_1^2 + x_2^2 + x_3^2) > 0.$$

Finally, the Killing form is not positive definite on the whole $\mathfrak{g}_0$. For example, if $x = E_{12} - E_{21} \in \mathfrak{h}_0$, then

$$\kappa(x, x) = 6 \operatorname{Tr}(x^2) = 6 \operatorname{Tr}(-E_{11} - E_{22}) = -12.$$

Now, we define the concept of compact real forms, which will have a very important role in the following theory.

**Definition 43.** We say that a **real form $\mathfrak{u}_0$ of $\mathfrak{g}$ is compact** if the restriction of the Killing form to $\mathfrak{u}_0$ is negative definite.

*Remark* 44. The explanation for the name is that a real Lie algebra is compact if and only if it is the Lie algebra of a compact Lie group.

In order to show that every simple complex Lie algebra has a compact real form we look for a suitable antiinvolution. Let $\mathfrak{g}_0$ be a split real form of $\mathfrak{g}$ and let $\sigma$ be the conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. According to the Uniqueness theorem (Theorem 40), there is a unique automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ satisfying

$$\omega(h_\alpha) = -h_\alpha,\ \omega(e_\alpha) = -e_{-\alpha},\ \omega(e_{-\alpha}) = -e_\alpha \qquad \forall\ \alpha \in \Pi. \tag{2.2}$$

Now, it is easy to check that $\omega$ is an involution commuting with $\sigma$. Thus, $\tau = \sigma\omega = \omega\sigma$ is also an antiinvolution on $\mathfrak{g}$, which we will call a **Cartan involution of $\mathfrak{g}_\mathbb{R}$**. As we saw in Section 2.2.2, $\mathfrak{u}_0 := \mathfrak{g}^\tau = \{\, x \in \mathfrak{g} \mid \tau(x) = x \,\}$ is a real form of $\mathfrak{g}$. By using the basis $\{\, h_\alpha \in \mathfrak{h} \mid \alpha \in \Pi \,\} \cup \{\, e_\beta \in \mathfrak{g}_\beta \mid \beta \in \Phi \,\}$, the elements of $\mathfrak{u}_0$ can be given as

$$\mathfrak{u}_0 = \sum_{\alpha\in\Pi} \mathbb{R} i h_\alpha + \sum_{\alpha\in\Phi^+} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha\in\Phi^+} \mathbb{R} i(e_\alpha + e_{-\alpha}).$$

**Example.** Again, let $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$. We use the notation and method seen in the last few paragraphs to find a compact real form of $\mathfrak{g}$. First, as we have already shown, $\mathfrak{g}_0 = \mathfrak{sl}(3,\mathbb{R})$ is a split real form of $\mathfrak{g}$. Then the conjugation $\sigma$ with respect to $\mathfrak{g}_0$ is just the usual complex conjugation of matrices element-wise:

$$\sigma\colon \mathfrak{g} \to \mathfrak{g}, \quad X = (x_{ij}) \mapsto \overline{X} = (\overline{x_{ij}}).$$

Now, $\omega\colon \mathfrak{g} \to \mathfrak{g}$ is the unique Lie algebra automorphism of $\mathfrak{g}$ satisfying

$$\begin{gathered}\omega(E_{11} - E_{22}) = -E_{11} + E_{22},\ \omega(E_{22} - E_{33}) = -E_{22} + E_{33}\\ \omega(E_{12}) = -E_{21},\ \omega(E_{23}) = -E_{32},\ \omega(E_{21}) = -E_{12},\ \omega(E_{32}) = -E_{23}.\end{gathered}$$

As the map $X \mapsto -X^T$ (as a Lie algebra automorphism on $\mathfrak{g}$) satisfies all of these requirements, it must be equal to $\omega$. Thus, $\tau = \sigma\omega$ is defined as $\tau(X) := -\overline{X}^T$. Hence we obtain the compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$ as

$$\begin{aligned}\mathfrak{u}_0 &= \left\{\, X \in \mathfrak{sl}(3,\mathbb{C}) : X = -\overline{X}^T \,\right\}\\ &= \left\{ X = \begin{pmatrix} ia_{11} & a_{12} + ib_{12} & a_{13} + ib_{13}\\ -a_{12} + ib_{12} & ia_{22} & a_{23} + ib_{23}\\ -a_{13} + ib_{13} & -a_{23} + ib_{23} & ia_{33}\end{pmatrix} : a_{ij}, b_{ij} \in \mathbb{R},\ \sum_i a_{ii} = 0 \right\}\end{aligned}$$

By calculating the Killing form $\kappa(X,X)$ for a non-zero $X \in \mathfrak{u}_0$ we obtain

$$\kappa(X,X) = 6\,\mathrm{Tr}(X^2) = -6\left(\sum_i a_{ii}^2 - \sum_{i<j}(a_{ij}^2 + b_{ij}^2)\right) < 0.$$

We close this section by Table 2.4, containing the compact and split real forms for all the classical Dynkin diagrams. (For completeness, we also indicate the related complex simple Lie algebra.)

### 2.2.4 Describing real forms with conjugacy classes of involutions

We already learned that the isomorphism types of real forms of a complex simple Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with the conjugacy classes of antiinvolutions of $\mathfrak{g}$ (where conjugacy is understood by elements of $\mathrm{Aut}(\mathfrak{g})$.

Table 2.4: Compact and split real forms for classical Dynkin diagrams

| Diagram | Complex simple | Compact real form | Split real form |
|---|---|---|---|
| $A_l$ | $\mathfrak{sl}(l+1,\mathbb{C})$ | $\mathfrak{su}(l+1)$ | $\mathfrak{sl}(l+1,\mathbb{R})$ |
| $B_l$ | $\mathfrak{so}(2l+1,\mathbb{C})$ | $\mathfrak{so}(2l+1)$ | $\mathfrak{so}(n+1,n)$ |
| $C_l$ | $\mathfrak{sp}(l,\mathbb{C})$ | $\mathfrak{sp}(l)$ | $\mathfrak{sp}(l,\mathbb{R})$ |
| $D_l$ | $\mathfrak{so}(2l,\mathbb{C})$ | $\mathfrak{so}(2l)$ | $\mathfrak{so}(n,n)$ |

Before we turn to a concrete example, we give another possible description of the different real forms, namely by using conjugacy classes of involutions (i.e. elements of $\mathrm{Aut}(\mathfrak{g})$) instead of antiinvolutions, which makes calculations easier. In Section 2.2.3 we defined two real structures $\sigma$ and $\tau$ corresponding to a split and a compact real forms of $\mathfrak{g}$, respectively. We saw that $\sigma = \omega\tau$ where $\omega$ is an involution commuting with the compact real structure $\tau$. (We say that a real structure is compact, if the related real structure is compact.) On the other hand, if we find a $\theta \in \mathrm{Aut}(\mathfrak{g})$ satisfying $\theta^2 = \mathrm{id}$ and $\theta\tau = \tau\theta$, then $\theta\tau$ is always a real structure on $\mathfrak{g}$, so we can find a real form $\mathfrak{g}^{\theta\tau}$ of $\mathfrak{g}$. In fact, in this way all of the real forms can be found up to isomorpism.

**Theorem 45.** *Let $\mathfrak{g}$ be a complex simple Lie algebra with fixed compact real structure $\tau$.*

1. *If $\tau'$ is any other compact real structure on $\mathfrak{g}$ with $\tau\tau' = \tau'\tau$, then $\tau = \tau'$.*
2. *If $\sigma$ is any real structure on $\mathfrak{g}$, then there exists an $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that $\sigma' := \alpha\sigma\alpha^{-1}$ satisfies $\sigma'\tau = \tau\sigma'$.*
3. *Similarly, if $\theta \in \mathrm{Aut}(\mathfrak{g})$ is an involution, then there exists an $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that $\theta' := \alpha\theta\alpha^{-1}$ satisfies $\theta'\tau = \tau\theta'$.*
4. *For any real structure $\sigma$ on $\mathfrak{g}$ let us choose a $\sigma' = \alpha\sigma\alpha^{-1}$ satisfying $\sigma'\tau = \tau\sigma'$ and let $\theta = \sigma'\tau$ be the corresponding involution. Then the map $\mathcal{C}(\sigma) \mapsto \mathcal{C}(\theta)$ is a well-defined bijection between the conjugacy classes of antiinvolutions and the conjugacy classes of involutions by automorphisms of $\mathfrak{g}$.*

*Remark* 46. Observe that items 1 and 2 of Theorem 45 imply that any two compact real forms of a complex simple Lie algebra are isomorphic.

In view of item 4 of Theorem 45, one can find all the real forms of a complex simple Lie algebra $\mathfrak{g}$ by determining the conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{g})$ and by choosing an involution in each class which is commuting with $\tau$. Thus, we need to know the automorphism groups of the complex simple Lie algebras. The last theorem of this section is helpful in this problem.

**Theorem 47.** *Let $\mathfrak{g}$ be a complex simple Lie algebra with Dynkin diagram $\Gamma$. Then*

$$\mathrm{Aut}(\mathfrak{g}) \simeq \mathrm{Int}(\mathfrak{g}) \rtimes \mathrm{Aut}(\Gamma).$$

*In particular,* $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Int}(\mathfrak{g})$ *unless* $\mathfrak{g} \simeq A_l\,(l > 1),\ D_l\,(l > 3)$ *or* $E_6$*, when* $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \simeq C_2$ *acts non-trivially on the Dynkin diagram.*

## 2.3 Example: The real forms of $\mathfrak{sl}(n,\mathbb{C})$

To better understand the theory we developed so far, now we describe all the real forms of the simple Lie algebra $A_l(\mathbb{C}) \simeq \mathfrak{sl}(n,\mathbb{C})$ for $n := l+1$. We obtain each of these real forms by finding the related real form of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ and taking its intersection with $\mathfrak{sl}(n,\mathbb{C})$. We assume that the reader is familiar with the natural root space decomposition of this Lie algebra. For details, see e.g. [6, Appendix D].

1. An obvious real form of $\mathfrak{gl}(n,\mathbb{C})$ is the subalgebra $\mathfrak{gl}(n,\mathbb{R})$. Its intersection with $\mathfrak{sl}(n,\mathbb{C})$ is $\mathfrak{sl}(n,\mathbb{R})$ which is just the split real form defined in Section 2.2.3. The corresponding real structure is given as
$$\sigma_0\colon X = (x_{ij}) \mapsto \overline{X} = (\overline{x_{ij}}).$$

2. One can obtain a series of real forms of $\mathfrak{gl}(n,\mathbb{C})$ by choosing a $p, q \in \mathbb{N}$ with $p+q = n,\ p \leq q$ and defining
$$\mathfrak{u}(p,q) = \left\{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid XI_{p,q} + I_{p,q}\overline{X}^T = 0 \right\} \quad \text{for } I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$
One can define the corresponding real forms of $\mathfrak{sl}(n,\mathbb{C})$ as $\mathfrak{su}(p,q) := \mathfrak{u}(p,q) \cap \mathfrak{sl}(n,\mathbb{C})$. As a special case, we can define the Lie algebras of skew-hermitian matrices
$$\mathfrak{u}(n) = \mathfrak{u}(0,n) = \left\{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid X = -\overline{X}^T \right\},$$
$$\mathfrak{su}(n) = \mathfrak{su}(0,n) = \left\{ X \in \mathfrak{sl}(n,\mathbb{C}) \mid X = -\overline{X}^T \right\}.$$
We claim that $\mathfrak{su}(n)$ is the compact real form of $\mathfrak{sl}(n,\mathbb{C})$. To see this, let $\omega(X) = -X^T$. It is easy to check that $\omega$ is an involution on $\mathfrak{sl}(n,\mathbb{C})$. Moreover, $\omega$ satisfies Equations (2.2). As we saw in Section 2.2.3, this results in $\omega\sigma_0 = \sigma_0\omega$ (here $\sigma_0\colon X \mapsto \overline{X}$ is the real structure associated to the split real form $\mathfrak{sl}(n,\mathbb{R})$). Thus, $\tau := \omega\sigma_0$ is a compact real structure on $\mathfrak{sl}(n,\mathbb{C})$. Clearly, $\tau(X) = -\overline{X}^T$, hence $\mathfrak{su}(n) = \mathfrak{sl}(n,\mathbb{C})^\tau$ and the result follows.

   Now, $\mathfrak{su}(p,q) = \mathfrak{gl}(n,\mathbb{C})^\sigma$ holds for the real structure $\sigma(X) = -I_{p,q}\overline{X}^T I_{p,q}$. Clearly $\sigma = \theta\tau$, where $\theta\colon X \mapsto I_{p,q}XI_{p,q}$ is the conjugation by $I_{p,q}$ (hence is an involution of $\mathfrak{sl}(n,\mathbb{C})$). As $\theta$ trivially commutes with $\tau$, we obtain that $\sigma$ is a real structure on $\mathfrak{sl}(n,\mathbb{C})$, so $\mathfrak{su}(p,q)$ is really a real form of $\mathfrak{sl}(n,\mathbb{C})$.

   The elements of $\mathfrak{u}(p,q)$ can be given as
$$\mathfrak{u}(p,q) = \left\{ \begin{pmatrix} A & B \\ \overline{B}^T & C \end{pmatrix} : A \in \mathfrak{u}(p),\ C \in \mathfrak{u}(q),\ B \in M_{p\times q}(\mathbb{C}) \right\}.$$

3. Let $n = 2m$ and let $\mathbb{H}^m$ be the $m$ dimensional right vector space of column vectors over the quaternions. Then one can define an isomorphism
$$\mathbb{H}^m \to \mathbb{C}^{2m} \text{ as } (q_1,\dots,q_m) \mapsto (x_1,\dots,x_m,y_1,\dots,y_m),$$
where $q_i \in \mathbb{H}, x_i, y_i \in \mathbb{C}$ and $q_i = x_i + jy_i$ for every $1 \leq i \leq m$. Now, the multiplication by $j$ on $\mathbb{H}^m$ defines an antiisomorphism $J$ on $\mathbb{C}^{2m}$ satisfying $J^2 = -\mathrm{id}$. The action of $J$ is given as (see equation 2.1)
$$J\left((x_1,\dots,x_m,y_1,\dots,y_m)^T\right) := (-\overline{y}_1,\dots,-\overline{y}_m,\overline{x}_1,\dots,\overline{x}_m)^T, \text{ i.e. } J(v) = J_{m,m}\overline{v}.$$

Now, $\mathfrak{gl}(m, \mathbb{H})$ is naturally isomorphic with the real Lie algebra

$$\{ X \in \mathfrak{gl}(2m, \mathbb{C}) \mid XJ = JX \} \leq \mathfrak{gl}(2m, \mathbb{C})_{\mathbb{R}}.$$

By choosing $\sigma \colon X \mapsto JXJ^{-1} = -J_{m,m}\overline{X}J_{m,m}$, it is easy to see that $\sigma$ is an antiinvolution on $\mathfrak{gl}(2m, \mathbb{C})$, and $\mathfrak{gl}(2m, \mathbb{C})^{\sigma} \simeq \mathfrak{gl}(m, \mathbb{H})$, which proves that $\mathfrak{gl}(m, \mathbb{H})$ is a real form of $\mathfrak{gl}(2m, \mathbb{C})$. As in the previous case, one can easily find a decomposition $\sigma = \theta\tau$ with $\theta \colon X \mapsto -J_{m,m}X^TJ_{m,m}^{-1}$ and the Cartan involution $\tau \colon X \mapsto -\overline{X}^T$.

The elements of $\mathfrak{gl}(2m, \mathbb{C})^{\sigma}$ can be given as

$$\mathfrak{gl}(2m, \mathbb{C})^{\sigma} = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} : A, B \in \mathfrak{gl}(m, \mathbb{C}) \right\}.$$

Now, the real form $\mathfrak{sl}(m, \mathbb{H})$ of $\mathfrak{sl}(2m, \mathbb{C})$ is formed by the matrices in $\mathfrak{gl}(2m, \mathbb{C})^{\sigma}$ with the additional condition $\Re(\operatorname{Tr} A) = 0$.

By using the connection described in Section 2.2.4, we prove that we found all the real forms of $\mathfrak{sl}(n, \mathbb{C})$ up to isomorphism. By item 4 of Theorem 45 (see also the paragraph before Theorem 47) in order to see this it is enough to prove that every involution is conjugate with exactly one of the involutions $\theta$ correspondig to the real forms described in this section. As a reminder, in Table 2.5 we list these real forms along with the correspondig real structures $\sigma = \theta\tau$ and involutions $\theta$. As before, $\tau(X) = -\overline{X}^T$ is the usual compact real form on $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.

Table 2.5: Real forms of $\mathfrak{sl}(n, \mathbb{C})$

| real form | real structure | involution |
|---|---|---|
| $\mathfrak{g}^{\sigma} = \mathfrak{sl}(n, \mathbb{R})$ | $\sigma(X) = \overline{X}$ | $\theta_1(X) = \omega(X) = -X^T$ |
| $\mathfrak{g}^{\sigma} = \mathfrak{su}(p, q)$ | $\sigma(X) = -I_{p,q}\overline{X}^TI_{p,q}$ | $\theta_{p,q}(X) = I_{p,q}XI_{p,q}$ |
| $\mathfrak{g}^{\sigma} = \mathfrak{sl}(m, \mathbb{H})$, $(n = 2m)$ | $\sigma(X) = -J_{m,m}\overline{X}J_{m,m}$ | $\theta_2(X) = J_{m,m}(-X^T)J_{m,m}^{-1}$ |

First we classify the conjugacy classes of outer involutions of $\mathfrak{g}$. As the Dynkin diagram of $\mathfrak{sl}(2, \mathbb{C})$ consists only of one node, it has a trivial automorphism group. By Theorem 47, we have that every automorphism of $\mathfrak{sl}(2, \mathbb{C})$ is inner, so we assume $n \geq 3$. Then $|\operatorname{Aut}(\mathfrak{g}) : \operatorname{Int}(\mathfrak{g})| = 2$ by Theorem 47. In this case $\omega(X) = -X^T$ is an outer automorphism of $\mathfrak{g}$. (Since, for example, $X = \operatorname{diag}(2, -1, -1, 0, \dots, 0)$ and $\omega(X)$ have different eigenvalues.)

Now, if $\alpha$ is an outer involution of $\mathfrak{g}$, then $\alpha = \varphi\omega$ for some $\varphi \in \operatorname{Int}(\mathfrak{g})$. This means that $\varphi(X) = AXA^{-1}$ for some $A \in SL(n, \mathbb{C})$. Using $\alpha^2 = \mathrm{id}$ we obtain

$$X = \alpha^2(X) = \alpha(-AX^TA^{-1}) = A(A^{-1})^TXA^TA^{-1} = \operatorname{Ad} A(A^T)^{-1}(X)$$

for every $X \in SL(n, \mathbb{C})$. This implies that $A(A^T)^{-1} = \lambda I_n$ for some $\lambda \in \mathbb{C}^{\times}$. Thus, $A = \lambda A^T = \lambda^2 A$ and we obtain $\lambda = \pm 1$.

First, let $\lambda = 1$, i.e. let $A = A^T$ be a symmetric matrix. Then $A = UU^T$ for some $U \in SL(n, \mathbb{C})$. (This is equivalent with the fact that there is only one nondegenerate symmetric bilinear function on $\mathbb{C}^n$.) Therefore,

$$\alpha(X) = -AX^TA^{-1} = -UU^TX^T(U^T)^{-1}U^{-1} = \operatorname{Ad} U\omega \operatorname{Ad} U^{-1}(X)$$

This shows that $\alpha$ is conjugate to $\theta_1 = \omega$ (by $\operatorname{Ad} U \in \operatorname{Int}(\mathfrak{g})$), so the real structure correspondig to $\alpha$ is isomorphic to $\mathfrak{g}^{\omega\tau} = \mathfrak{sl}(n, \mathbb{R})$.

For $\lambda = -1$ we get $A = -A^T$ is a skew-symmetric matrix. Then $n = 2m$ is even and $A = UJ_{m,m}U^T$ for some $U \in SL(n, \mathbb{C})$. A similar calculation shows that $\alpha = \operatorname{Ad} U \operatorname{Ad} J_{m,m}\omega \operatorname{Ad} U^{-1}$, so $\alpha$ is conjugate to $\operatorname{Ad} J_{m,m}\omega$ by $\operatorname{Ad} U \in \operatorname{Int}(\mathfrak{g})$. But $\operatorname{Ad} J_{m,m}\omega(X) = -J_{m,m}X^T J_{m,m}^{-1} = \theta_2(X)$. Hence the real structure correspondig to $\alpha$ is isomorphic to $\mathfrak{g}^{\theta_2\tau} \simeq \mathfrak{sl}(m, \mathbb{H})$.

Now, let $n \geq 2$ be arbitrary and let $\alpha$ be an inner involution of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ given as $\alpha(X) = AXA^{-1}$ for some $A \in SL(n, \mathbb{C})$. As $\alpha^2 = \mathrm{id}$, we have $A^2 = \lambda I_n$ for some $\lambda \in \mathbb{C}^\times$ Choosing $\frac{1}{\sqrt{\lambda}}A$ instead of $A$ we can ensure that $A^2 = I_n$. Then $A$ is diagonalizable with eigenvalues $\pm 1$. So $A = UI_{p,q}U^{-1}$ for some $U \in SL(n, \mathbb{C})$ and $p + q = n$. Then $\alpha$ is conjugate to $\theta_{p,q} = \operatorname{Ad} I_{p,q}$ and we obtain the real form $\mathfrak{su}(p, q)$.

Finally, we note that $\operatorname{Ad} I_{p,q}$ and $\operatorname{Ad} I_{q,p}$ are conjugate via $\omega$, so they correspond to isomorphic real forms $\mathfrak{su}(p, q)$ and $\mathfrak{su}(q, p)$. Except of this, the above real forms are pairwise non-isomorphic for $n \geq 3$, while for $n = 2$ we have $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{su}(1, 1)$ and $\mathfrak{sl}(1, \mathbb{H}) \simeq \mathfrak{su}(2)$.

## 2.4 A classification via Vogan diagrams

Now, we give a possible way to classify the simple real Lie algebras with the help of Vogan diagrams. Vogan diagrams are simply Dynkin diagrams, but occasionally some of their nodes are painted black while some of the white nodes are connected with arrows in pairs. On the one hand, any such abstract Vogan diagram defines a simple real Lie algebra, which is unique up to isomorphism, and which is a real form of the complex simple Lie algebra associated to the underlying Dynkin diagram. On the other hand, unlike to the complex case, non-isomorphic Vogan diagrams might still define isomorphic Lie algebras. This redundancy can be handled with the Borel–de Siebenthal Theorem (Theorem 66), which helps one to give a reduced list of Vogan diagrams having one-to-one correspondence with the isomorphism classes of real forms of simple complex Lie algebras.

One might suspect from this description that only finitely many distinct real forms may exist for the same complex Lie algebra. (In fact, we already saw this for $A_l(\mathbb{C}) \simeq \mathfrak{sl}(l+1, \mathbb{C})$.) As we will see, when a real form $\mathfrak{g}_0$ has a Cartan subalgebra $\mathfrak{h}_0$, then the additional information coded in the Vogan diagrams is the values of the roots on this subalgebra. As an example, remember the split ($\mathfrak{g}_0$) and compact ($\mathfrak{u}_0$) real forms introduced in Section 2.2.3. It is easy to check that the restriction of the roots on the intersections $\mathfrak{g}_0 \cap \mathfrak{h}$ and $\mathfrak{u}_0 \cap \mathfrak{h}$ have only real and purely imaginary values, respectively.

After this long introduction we go into details. During this section, $\mathfrak{g}$ will always denote a simple complex Lie algebra with compact real form $\mathfrak{u}_0$ and arbitrary real form $\mathfrak{g}_0$.

### 2.4.1 Cartan involutions and Cartan decompositions

**Definition 48.** Let $\theta$ be an involution of $\mathfrak{g}_0$ (i.e. $\theta \in \operatorname{Aut}(\mathfrak{g}_0)$ with $\theta^2 = \mathrm{id}$) and let $\kappa$ be the Killing form on $\mathfrak{g}_0$. Then we define a bilinear function $\kappa_\theta$ on $\mathfrak{g}_0$ associated to $\theta$ as

$$\kappa_\theta(u, v) := -\kappa(u, \theta(v)).$$

Then we call $\theta$ a **Cartan involution** of $\mathfrak{g}_0$ if $\kappa_\theta$ is positive definite on $\mathfrak{g}_0$.

Now, if $\mathfrak{u}_0$ is a compact real form of the complex Lie algebra $\mathfrak{g}$, and we choose $\theta := \tau$ as the conjugation with respect to $\mathfrak{u}_0$, then it can easily be verified that $\kappa_\tau$ is positive definite on $\mathfrak{g}_\mathbb{R}$, thus, $\tau$ is a Cartan involution of $\mathfrak{g}_\mathbb{R}$. Conversely, taken any Cartan involution $\theta$ of $\mathfrak{g}_\mathbb{R}$, it is clear that $\mathfrak{g}^\tau$ is a compact real form of $\mathfrak{g}$. In other words, the Cartan involutions of $\mathfrak{g}_\mathbb{R}$ correspond to the compact real forms of $\mathfrak{g}$ and the existence of a Cartan involution of $\mathfrak{g}_\mathbb{R}$ is equivalent with the existence of a compact real form of $\mathfrak{g}$. (Which always exists, as we saw earlier in Section 2.2.3.)

Now, we would like to prove that every simple real Lie algebra possesses a Cartan involution. The key observation is the following theorem, which can be viewed as a generalization of items 2 and3 of Theorem 45.

**Theorem 49.** *If $\theta$ is a Cartan involution and $\sigma$ is any involution of a real semisimple algebra $\mathfrak{g}_0$, then there exists an $\alpha \in \mathrm{Int}(\mathfrak{g}_0)$ such that $\alpha\theta\alpha^{-1}$ commutes with $\sigma$.*

By choosing $\mathfrak{g}_\mathbb{R}$ and $\sigma$ to be the conjugation with respect to any real form $\mathfrak{g}_0$, one can use this theorem to show the existence of a special compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$ such that the conjugation $\tau$ with respect to $\mathfrak{u}_0$ commutes with $\sigma$. Now, an easy calculation shows that the restriction $\tau|_{\mathfrak{g}_0}$ is a Cartan involution of $\mathfrak{g}_0$. In particular, every real (semi)simple Lie algebra has a Cartan involution. Another direct consequence of this theorem is that the Cartan involutions of a real (semi)simple Lie algebra $\mathfrak{g}_0$ form a single class with respect to the conjugation with elements of $\mathrm{Int}(\mathfrak{g}_0)$.

Now, let $\theta$ be a Cartan involution of $\mathfrak{g}_0$. Viewing it as an element of $\mathrm{End}(\mathfrak{g}_0)$, its minimal polynomial is $x^2-1$. Consequently, its matrix form is diagonalizable with eigenvalues $\pm 1$. This results in a decomposition

$$\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0 \quad \text{where } \mathfrak{n}_0 = \{\, x \in \mathfrak{g}_0 \mid \theta(x) = x \,\}, \quad \mathfrak{p}_0 = \{\, x \in \mathfrak{g}_0 \mid \theta(x) = -x \,\}.$$

This idea leads us to the concept of Cartan decomposition.

**Definition 50.** Let $\mathfrak{g}_0$ be a real (semi)simple Lie algebra. We say that the vector space decomposition $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ is a **Cartan decomposition** of $\mathfrak{g}_0$ if the following two properties are satisfied.

1. The following inclusions hold

$$[\mathfrak{n}_0, \mathfrak{n}_0] \subseteq \mathfrak{n}_0, \qquad [\mathfrak{n}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \qquad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{n}_0,$$

2. For the Killing form $\kappa$ of $\mathfrak{g}$ we have that

$$\kappa \text{ is } \begin{cases} \text{negative definite on } \mathfrak{n}_0; \\ \text{positive definite on } \mathfrak{p}_0. \end{cases}$$

Now, Cartan involutions and Cartan decompositions of $\mathfrak{g}_0$ are intimately related. First, if $\theta$ is a Cartan involution of $\mathfrak{g}_0$, then the motivational decomposition shown before Definition 50 defines a Cartan decomposition of $\mathfrak{g}_0$. Conversely, if $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition, then one can define a corresponding Cartan involution as

$$\theta(x) := \begin{cases} x & \text{for } x \in \mathfrak{n}_0, \\ -x & \text{for } x \in \mathfrak{p}_0, \end{cases}$$

and extend it linearly to the whole $\mathfrak{g}_0$.

The Cartan decomposition has the following nice property, which can easily be proved.

**Theorem 51.** *If* $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ *is a Cartan decomposition of* $\mathfrak{g}_0$ *with Cartan involution* $\theta$*, then* $\operatorname{ad} X$ *is Hermitian for* $X \in \mathfrak{p}_0$*, and* $\operatorname{ad} X$ *is skew-Hermitian for* $X \in \mathfrak{n}_0$ *with respect to the bilinear form* $\kappa_\theta$*. In particular, if* $X \in \mathfrak{p}_0$ *(resp.* $X \in \mathfrak{n}_0$*), then every eigenvalue of* $\operatorname{ad} X$ *is real (resp. is purely imaginary).*

We close this section by mentioning that if $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, then $\mathfrak{n}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g}_0(\mathbb{C})$. Conversely, if $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ is not a Cartan decomposition, then $\mathfrak{n}_0 \oplus i\mathfrak{p}_0$ is not compact.

### 2.4.2 Cartan subalgebras

Let us consider a real (semi)simple algebra $\mathfrak{g}_0$ with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ and corresponding Cartan involution $\theta$.

As in the complex case, it will be helpful to choose a big Abelian subalgebra in $\mathfrak{g}_0$, which acts nicely on $\mathfrak{g}_0$ via the ad map. Since diagonalizability cannot be guaranteed, we define the concept of Cartan subalgebra in the following way.

**Definition 52.** We say that a real subalgebra $\mathfrak{h}_0$ is a **Cartan subalgebra** of $\mathfrak{g}_0$ if its complexification $\mathfrak{h}_0(\mathbb{C})$ is a Cartan subalgebra of $\mathfrak{g}_0(\mathbb{C})$.

Unlike to the complex case, the Cartan subalgebras are not all conjugate by $\operatorname{Aut}(\mathfrak{g}_0)$. Still, the following is true.

**Theorem 53.** *Any Cartan subalgebra of* $\mathfrak{g}_0$ *is conjugate via* $\operatorname{Int}(\mathfrak{g}_0)$ *to a* $\theta$*-stable Cartan subalgebra of* $\mathfrak{g}_0$*. (A Cartan subalgebra* $\mathfrak{h}_0$ *is called* $\theta$***-stable*** *if* $\theta(\mathfrak{h}_0) = \mathfrak{h}_0$*.)*

Thus, without loss of generality in the following we can assume that $\mathfrak{h}_0 \leq \mathfrak{g}_0$ is a $\theta$-invariant ($\theta$-stable) Cartan subalgebra. Then it is clear that

$$\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \text{ with } \mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{n}_0 \text{ and } \mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0. \tag{2.3}$$

By using Theorem 51 we obtain that every root $\alpha \in \Phi$ is real valued on $\mathfrak{a}_0 \oplus i\mathfrak{t}_0$, where $\Phi$ denotes the root system corresponding to the Cartan subalgebra $\mathfrak{h}_0(\mathbb{C})$ of $\mathfrak{g}_0(\mathbb{C})$. In view of this, the following definitions seem reasonable.

**Definition 54.** Using the notations introduced in (2.3), we call $\dim_{\mathbb{R}} \mathfrak{t}_0$ the **compact dimension** and $\dim_{\mathbb{R}} \mathfrak{a}_0$ the **non-compact dimension** of $\mathfrak{h}_0$. Furthermore, a $\theta$-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is called **maximally noncompact** if its noncompact dimension is maximal; it is called **maximally compact** if its compact dimension is maximal.

In the following, we will choose a *maximally compact* $\theta$-stable Cartan subalgebra. This can be found from the Cartan decomposition by using the following theorem.

**Theorem 55.** *Let* $\mathfrak{t}_0$ *be a maximal Abelian subspace of* $\mathfrak{n}_0$*. Then*

$$\mathfrak{h}_0 := Z_{\mathfrak{g}_0}(\mathfrak{t}_0) = \{\, A \in \mathfrak{g}_0 \mid AB = BA \;\forall B \in \mathfrak{t}_0 \,\}$$

*(the* ***centralizer*** *of* $\mathfrak{t}_0$ *in* $\mathfrak{g}_0$*) is a maximally compact* $\theta$*-stable Cartan subalgebra of* $\mathfrak{g}_0$*. Moreover, every maximally compact* $\theta$*-stable Cartan subalgebra can be found in this form.*

In the following let us fix a maximally compact $\theta$-stable Cartan subalgebra $\mathfrak{h}_0$ and let $\mathfrak{h}_0 := \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the decomposition to its compact and non-compact parts. By taking complexifications, we obtain a Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{g}$. Let $\Phi$ denote the root system associated to $\mathfrak{h}$.

**Definition 56.** We say that a root $\alpha \in \Phi$ is

- **real** if it takes only real values on $\mathfrak{h}_0 \iff \alpha$ vanishes on $\mathfrak{t}_0$,
- **imaginary** if it takes only purely imaginary values on $\mathfrak{h}_0 \iff \alpha$ vanishes on $\mathfrak{a}_0$,
- **complex** otherwise.

The maximally compact $\theta$-stable Cartan subalgebras can be characterized by roots as follows.

**Proposition 57.** *A $\theta$-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is maximally compact if and only if there are no real roots (with respect to the Cartan subalgebra $\mathfrak{h}_0(\mathbb{C})$ of $\mathfrak{g}_0(\mathbb{C})$).*

### 2.4.3 Definition of Vogan diagrams

Now, we define Vogan diagrams, which can be used to classify all the noncomplex real simple Lie algebras.

As before, let $\mathfrak{g}_0$ be a real simple Lie algebra with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ and corresponding Cartan involution $\theta$. Furthermore, let us choose a maximally compact $\theta$-stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, and let $\Phi$ denote the system of roots associated to $\mathfrak{h}_0(\mathbb{C})$. Then $\theta$ defines an action on $\Phi$ as follows.

**Proposition 58.** 1. *Let $\alpha \in \Phi$. The map $\theta(\alpha)\colon \mathfrak{h} \to \mathbb{C}$ is defined by the rule*

$$\theta(\alpha)(x) = \alpha\left(\theta^{-1}(x)\right) \qquad \forall x \in \mathfrak{h}.$$

*Then $\theta(\alpha) \in \Phi$, as well. In that way $\theta$ acts on $\Phi$ (and is of order at most two).*

2. *If $\alpha$ is purely imaginary, then $\theta(\alpha) = \alpha$ and either $\mathfrak{g}_\alpha \leq \mathfrak{n}_0(\mathbb{C})$ or $\mathfrak{g}_\alpha \leq \mathfrak{p}_0(\mathbb{C})$. (Here, $\mathfrak{g}_\alpha$ is the subspace defined in item 1 of Summary 39.)*

3. *If $\alpha$ is complex, then $\theta(\alpha) \neq \alpha$. Moreover, $\theta(\alpha)|_\mathfrak{a} = -\alpha|_\mathfrak{a}$ and $\theta(\alpha)|_\mathfrak{t} = \alpha|_\mathfrak{t}$.*

**Definition 59.** In view of item 2 of Proposition 58, we say that an imaginary root $\alpha$ is **compact** if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}_0(\mathbb{C})$ and it is **noncompact** if $\mathfrak{g}_\alpha \subseteq \mathfrak{p}_0(\mathbb{C})$.

In the next step, we define an ordering on $\Phi$, which can be used to define a system of fundamental roots $\Pi$. Recall that every $\alpha \in \Phi$ is real valued on $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$.

**Definition 60.** Let $\{ h_1, h_2, \dots, h_t \}$ and $\{ h_{t+1}, \dots, h_n \}$ be bases of $i\mathfrak{t}_0$ and $\mathfrak{a}_0$, respectively. Thus, $\{ h_1, \dots, h_n \}$ is a basis of $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$. (And it is also a basis of the complex space $\mathfrak{h}$.) Now, this basis defines a total ordering on $\Phi$ (in fact, on the whole space $\mathfrak{h}_0^*$) by using the lexicographical ordering. More precisely, for $\alpha, \beta \in \Phi$ let $\alpha > \beta$ if $\exists l$ such that $\alpha(h_l) > \beta(h_l)$ but $\alpha(h_j) = \beta(h_j)$ for every $j < l$.

By using this ordering we can define the system of fundamental, positive and negative roots (denoted by $\Pi, \Phi^+, \Phi^-$). In that way we obtain a triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi)$.

Now, thanks to the ordering chosen in Definition 60, one can prove that $\theta(\Phi^+) = \Phi^+$, which results in $\theta(\Pi) = \Pi$. In other words, $\theta$ acts on the vertices of the Dynkin diagram associated to the triple $(\mathfrak{g}, \mathfrak{h}, \Pi)$. The Vogan diagram associated to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi)$ collects the properties of this action.

**Definition 61.** With the above notation, the **Vogan diagram** associated to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi)$ is defined as follows.

1. The underlying graph is the Dynkin diagram associated to the triple $(\mathfrak{g}, \mathfrak{h}, \Pi)$.
2. Every complex root $\alpha \in \Pi$ is connected with $\theta(\alpha) \neq \alpha$ with an arrow $\leftrightarrow$.
3. Any noncompact imaginary root is painted to black.

We finish this section with the following two easy observations.

**Proposition 62.**

1. *The compact real forms correspond to usual Dynkin diagrams.*
2. *$\theta$ defines a graph automorphism on the Dynkin diagram of order two. Thus $\theta$ must be the identity unless the Dynkin diagram is of type $A_l$, $(l > 1)$, $D_l\,(l > 3)$ or $E_6$.*

### 2.4.4 Existence and uniqueness theorems

In order to complete the description of simple real noncomplex Lie algebras with Vogan diagrams, we formulate here some existence and uniqueness theorems. For existence, we need a precise definition of abstract Vogan diagrams.

**Definition 63.** An **abstract Vogan diagram** is a Dynkin diagram possibly with

- a graph automorphism of order two, which is indicated with arrows connecting the two element orbits;
- some of the one-element orbits painted to black.

**Theorem 64** (Existence theorem)**.** *If an abstract Vogan diagram is given, then there exists a real semisimple algebra $\mathfrak{g}_0$ with Cartan involution $\theta$, a maximally compact $\theta$-stable Cartan subalgebra $\mathfrak{h}_0$ with related root system $\Phi$, and a suitable ordering as in Definition 60 defining $\Pi$ such that the Vogan diagram associated to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi)$ is the same as the original one.*

We also have some uniqueness.

**Theorem 65** (Uniqueness theorem)**.** *Let us assume that two triples $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', \Pi')$ are given.*

- *If there is an isomorphism between them (which means isomorphism in each coordinate), then the corresponding Vogan diagrams are isomorphic.*
- *Conversely, if the Vogan diagrams defined by the two triples are the same, then $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are isomorphic.*

Unfortunately, there is no other uniqueness as in the complex case: there are non-isomorphic Vogan diagrams which define isomorphic real simple Lie algebras. The main reason for this is that the ordering in Definition 60 can be chosen in many different ways. This redundancy can be handled by the following theorem which can help to eliminate some unnecessary Vogan diagrams.

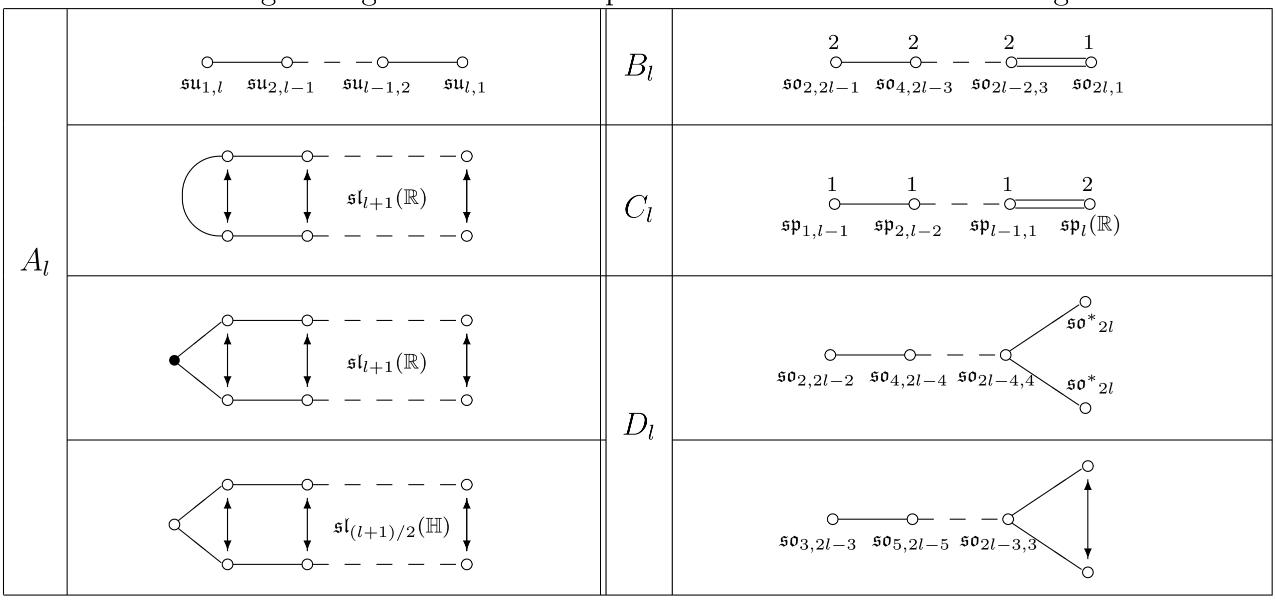
**Theorem 66** (Borel–de Siebenthal theorem)**.** *Let the triple* $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi)$ *be given.*

1. *Let* $\Phi$ *be the root system corresponding to* $\mathfrak{h}_0(\mathbb{C})$*. Then the ordering given in Definition 60 can be chosen so that it defines a system of fundamental roots* $\Pi' \subset \Phi$ *such that there is* at most one painted node *in the Vogan diagram related to the triple* $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi')$*.*

2. *Let us assume that the action of* $\theta$ *on the Dynkin diagram is the identity, and exactly one node is painted. Let* $\Pi = \{ \alpha_1, \ldots, \alpha_l \}$ *and let* $\{ \omega_1, \ldots, \omega_l \}$ *be the dual basis with respect to* $\langle ., . \rangle$ *given by* $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$*. Then the single painted node* $\alpha_i$ *can be chosen such that there is no* $j$ *with* $\langle \omega_i - \omega_j, \omega_j \rangle > 0$*.*

Now, we give a picture of all the noncompact Vogan diagrams surviving the Borel de Siebenthal conditions. For Vogan diagrams of classical type, we indicated the corresponding classical Lie algebras (Table 2.6). For these diagrams, the group written to the node $\alpha_i$ has a Vogan diagram with single painted node $\alpha_i$. For the exceptional Vogan diagrams, we used their usual name in the literature (Table 2.7).

Table 2.6: Vogan diagrams of non-compact real forms of classical Lie algebras



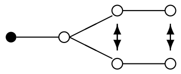
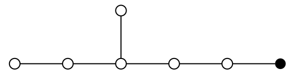
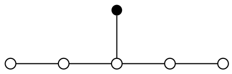
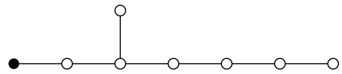
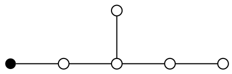
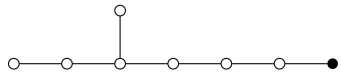
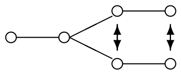
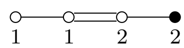
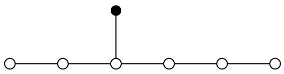
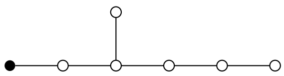

As a conclusion, we finish this chapter with a Classification theorem for the simple real Lie algebras.

**Theorem 67** (Classification Theorem)**.** *The simple real Lie algebras up to isomorphism are the following.*

1. *The realification* $\mathfrak{g}_{\mathbb{R}}$ *of a complex simple Lie algebra of type* $A_n$ *for* $n \geq 1$*,* $B_n$ *for* $n \geq 2$*,* $C_n$ *for* $n \geq 3$*,* $D_n$ *for* $n \geq 4$*,* $E_6$*,* $E_7$*,* $E_8$*,* $F_4$*,* $G_2$*.*

2. *The compact real forms of the simple complex Lie algebras as in item 1.*

Table 2.7: Vogan diagrams of non-compact real forms of exceptional Lie algebras

| | | | |
|---|---|---|---|
| E I |  | EVII |  |
| E II |  | EVIII |  |
| E III |  | EIX |  |
| E IV |  | FI |   |
| E V |  | FII |   |
| E VI |  | G |   |

3. *The classical matrix algebras:*

$$\begin{array}{lll} \mathfrak{su}(p,q) & \textit{with} & p \geq q > 0,\ p+q \geq 2 \\ \mathfrak{so}(p,q) & \textit{with} & p \geq q > 0,\ p+q \geq 5 \ \textit{odd} \\ & \textit{or with} & p \geq q > 0,\ p+q \geq 8 \ \textit{even} \\ \mathfrak{sp}(p,q) & \textit{with} & p \geq q > 0,\ p+q \geq 3 \\ \mathfrak{sp}(n,\mathbb{R}) & \textit{with} & n \geq 3 \\ \mathfrak{so}^*(2n) & \textit{with} & n \geq 4 \\ \mathfrak{sl}(n,\mathbb{R}) & \textit{with} & n \geq 3 \\ \mathfrak{sl}(n,\mathbb{H}) & \textit{with} & n \geq 2 \end{array}$$

*with Vogan diagrams given in Table 2.6.*

4. *One of the 12 exceptional noncompact noncomplex Lie algebras with Vogan diagrams given in Table 2.7*

*The only isomporhism among these is* $\mathfrak{so}^*(8) \simeq \mathfrak{so}(6,2)$.

*Remark* 68. Vogan diagrams is not the only possible way to characterize the non-complex simple real Lie algebras. By choosing a maximally noncompact $\theta$-stable Cartan subalgebra in the real form $\mathfrak{g}_0$ one can define the **Satake diagram of $\mathfrak{g}_0$**. Satake diagrams are similar to Vogan diagrams, but they are not the same. They are also based on Dynkin diagrams with some vertices painted and some connected by an arrow.

# Chapter 3

# More examples

## 3.1 Realisation of Vogan diagrams based on $A_{n-1}$ as classical Lie algebras

As an example we check that the Vogan diagrams of type $A_{n-1}$ really correspond to the matrix algebras given in the first column of Table 2.6. In particular, (up to the Existence and Uniqueness theorems of Section 2.4.4), we will see that previously we found all the real forms of $A_{n-1}(\mathbb{C}) \simeq \mathfrak{sl}(n, \mathbb{C})$ in Section 2.3.

### 3.1.1 Vogan diagrams of type $A_{n-1}$ with trivial automorphism

First, let $n = p + q \geq 2$ and consider the real form

$$\mathfrak{g}_0 = \mathfrak{su}(p, q) = \left\{ X \in \mathfrak{sl}(n, \mathbb{C}) : X = \begin{pmatrix} A & B \\ \overline{B}^T & C \end{pmatrix} \,\middle|\, A \in \mathfrak{u}(p),\ C \in \mathfrak{u}(q),\ B \in M_{p \times q}(\mathbb{C}) \right\}$$

of $A_{n-1}$. One can check that the map $\theta(X) := -\overline{X}^T$ is a Cartan involution on $\mathfrak{g}_0$. The corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{p}_0$ is given as

$$\mathfrak{n}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} : A \in \mathfrak{u}(p),\ C \in \mathfrak{u}(q) \right\} \quad \text{and} \quad \mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & B \\ \overline{B}^T & 0 \end{pmatrix} : B \in M_{p \times q}(\mathbb{C}) \right\}.$$

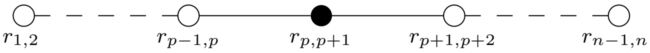
Now, let $\mathfrak{h}_0$ be the (real) subalgebra of diagonal matrices in $\mathfrak{g}_0$. Note that every element of $\mathfrak{h}_0$ contains only purely imaginary values in its main diagonal. Then $\mathfrak{h} = \mathfrak{h}_0(\mathbb{C})$ is just the usual Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ (the set of all diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$), thus, $\mathfrak{h}_0$ is a Cartan subalgebra in $\mathfrak{g}_0$. Since $\mathfrak{h}_0 \leq \mathfrak{n}_0$, it is clearly maximally compact.

Using the same notation as in [6, Appendix D], we see that the roots $r_{ij} \in \mathfrak{h}^*$, $r_{ij}(\text{diag}(\lambda_1, \ldots, \lambda_n)) = \lambda_i - \lambda_j$ take only purely imaginary values on $\mathfrak{h}_0$, i.e. every root is imaginary. As a consequence, $\theta$ induces the trivial automorphism on $A_l$ and there are no arrows in the correspondig Vogan diagram. Now, we must choose an ordering on the set of roots $\Phi$ which accepts the condition given in Definition 60 as well as a system of fundamental roots $\Pi$. It is easy to check that the natural one ($r_{ij} \in \Phi^+ \iff i < j$) is a suitable ordering, which defines the system of fundamental roots

$$\Pi = \{ r_{12} > r_{23} > \cdots > r_{n-1,n} \}.$$

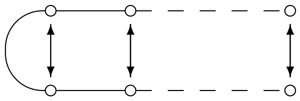
The correspondig root spaces are $\mathfrak{g}_{r_{i,i+1}} = \mathbb{C}e_{i,i+1}$ for $1 \leq i < n$. It is trivial to check that $\mathfrak{g}_{r_{i,i+1}} \in \mathfrak{n}_0(\mathbb{C})$ if and only if $r \neq p$. In other words, every element but the $p$-th of

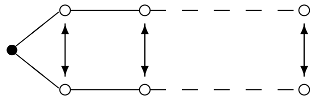
$\Pi$ is compact. By the Definition 61 the Vogan diagram associated to triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Pi)$ is

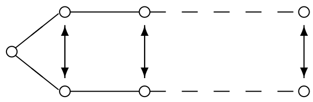



### 3.1.2 Vogan diagrams of type $A_{n-1}$ with nontrivial automorphism

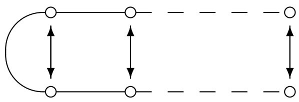
Now, we would like to find the real forms of $A_{n-1}$ correspondig to the remaining Vogan diagrams, i.e. to the following ones:







First, if $n = 2m+1$ is odd, then the first one is the only Vogan diagram which we did not realize as a matrix algebra yet. As we did not find the Vogan diagram of only the split real form $\mathfrak{sl}(2m+1, \mathbb{R})$, the Vogan diagram



must correspond to $\mathfrak{sl}(2m+1, \mathbb{R})$.

Now, let $n = 2m$ be even and consider the split real form $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R}) \leq \mathfrak{sl}(n, \mathbb{C})$. With respect to the Cartan involution $\theta(X) = -\overline{X}^T$, the Cartan decomposition can be written as

$$\mathfrak{n}_0 = \left\{ A \in \mathfrak{sl}(n, \mathbb{R}) \mid A = -A^T \right\} \quad \text{and} \quad \mathfrak{p}_0 = \left\{ A \in \mathfrak{sl}(n, \mathbb{R}) \mid A = A^T \right\}.$$

Thus, $\mathfrak{n}_0$ and $\mathfrak{p}_0$ are simply the sets of antisymmetric and symmetric matrices of trace zero, respectively.

In order to find a maximally compact Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{g}_0$, we introduce a compact structure on the vector space $\mathbb{R}^n$ and view it as a complex space $V = \mathbb{C}^m$. In this way $\mathfrak{gl}(m, \mathbb{C}) \simeq GL(V)$ is included into $\mathfrak{gl}(n, \mathbb{R})$.

By choosing a basis $\{ e_1, \ldots, e_m \}$ of the complex space $\mathbb{C}^m$, we can obtain the basis $\{ e_1, ie_1, \ldots, e_m, ie_m \}$ of the real space $\mathbb{R}^n$. If $X = (x_{ij}) \in \mathfrak{gl}(m, \mathbb{C})$ is a complex matrix, then we obtain the real matrix correspondig to this basis by replacing each element $x_{ij} = a + ib$ $(a, b \in \mathbb{R})$ with the $2 \times 2$ matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. The image of the diagonal matrices of

$\mathfrak{gl}(m,\mathbb{C})$ gives us the Cartan subalgebra

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} a_1 & b_1 & & & \\ -b_1 & a_1 & & & \\ & & \ddots & & \\ & & & a_m & b_m \\ & & & -b_m & a_m \end{pmatrix} : a_1, b_1, \ldots, a_m, b_m \in \mathbb{R},\ \sum_i a_i = 0 \right\}.$$

Clearly this is a $\theta$-stable Cartan subalgebra of $\mathfrak{g}_0$. Thus, we have the decomposition $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ where $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{n}_0$ and $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$ consist of the matrices corresponding to purely imaginary and real diagonal matrices in $\mathfrak{gl}(m,\mathbb{C})$, i.e.

$$\mathfrak{t}_0 = \left\{ \begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & b_m \\ & & & -b_m & 0 \end{pmatrix} \right\} \text{ and } \mathfrak{a}_0 = \left\{ \begin{pmatrix} a_1 & 0 & & & \\ 0 & a_1 & & & \\ & & \ddots & & \\ & & & a_m & 0 \\ & & & 0 & a_m \end{pmatrix} \right\}.$$

One can easily prove that $\mathfrak{h}_0$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$ by using Theorem 55. In order to desribe the system of roots correspondig to the complex Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0(\mathbb{C})$ of $\mathfrak{sl}(n,\mathbb{C})$, we define the linear functionals $e_1, \ldots, e_m, f_1, \ldots, f_m \colon \mathfrak{h} \to \mathbb{C}$ as follows. For every element of $X \in \mathfrak{h}$ and for every $1 \leq j \leq m$, both $e_j$ and $f_j$ acts trivially on all but the $j$-th block of $X$. Their actions on the $j$-th block are given as

$$e_j \begin{pmatrix} x_j & y_j \\ -y_j & x_j \end{pmatrix} = iy_j, \qquad f_j \begin{pmatrix} x_j & y_j \\ -y_j & x_j \end{pmatrix} = x_j$$

Then the root system with respect to $\mathfrak{h}$ can be given as

$$\Phi = \{ \pm e_j \pm e_k \pm (f_j - f_k) \mid 1 \leq j \neq k \leq m \} \cup \{ \pm 2e_l \mid 1 \leq l \leq m \}.$$

(Note that the first part of $\Phi$ is closed to the permutation $(j,k)$ of the indices; in other words, every root, except of the $\pm 2e_l$, appears above twice.)

In order to describe the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ we introduce the following notation. For every $X \in M^{2\times 2}(\mathbb{C})$ we denote by $E_{jk}(X) \in M^{2m\times 2m}(\mathbb{C})$ the matrix built up from $2 \times 2$ blocks such that it contains $X$ in the $(j,k)$-th position; every other blocks are zero matrices. An easy calculation shows that the root spaces $\mathfrak{g}_\alpha \leq \mathfrak{sl}(2k,\mathbb{C})$ are the following:

$$\begin{aligned} \mathfrak{g}_{+2e_l} &= \mathbb{C} \cdot E_{ll} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} & \mathfrak{g}_{-2e_l} &= \mathbb{C} \cdot E_{ll} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\ \mathfrak{g}_{e_j-e_k+(f_j-f_k)} &= \mathbb{C} \cdot E_{jk} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} & \mathfrak{g}_{e_j+e_k+(f_j-f_k)} &= \mathbb{C} \cdot E_{jk} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \\ \mathfrak{g}_{-e_j-e_k+(f_j-f_k)} &= \mathbb{C} \cdot E_{jk} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} & \mathfrak{g}_{-e_j+e_k+(f_j-f_k)} &= \mathbb{C} \cdot E_{jk} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \end{aligned}$$

Now, each root is real on $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$, hence there are no real roots. (This gives us a second proof to the fact that $\mathfrak{h}_0$ is maximally compact.) In accordance with Definition 60, we fix the ordered basis of $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$ as

$$E_{11} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} > E_{22} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} > \ldots > E_{mm} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} >$$

$$> E_{11} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - E_{22} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > \ldots > E_{m-1,m-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - E_{mm} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The correspondig ordering on $\Phi$ (or even on $(i\mathfrak{t}_0 \oplus \mathfrak{a}_0)^*$ can be defined by extending lexicographically the ordering

$$e_1 > e_2 > \ldots > e_m > f_1 > \ldots > f_m.$$

This results the system of positive roots

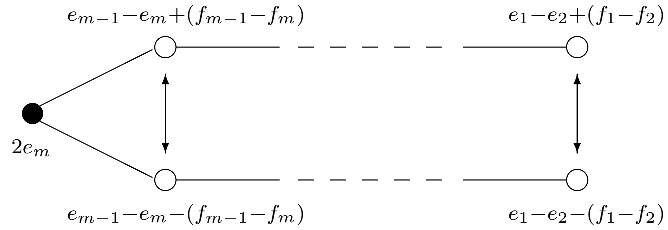
$$\Phi^+ = \begin{cases} e_j + e_k \pm (f_j - f_k), & \text{for all } j \neq k, \\ e_j - e_k \pm (f_j - f_k), & \text{for all } j < k, \\ 2e_l, & \text{for all } 1 \leq l \leq m, \end{cases}$$

and the system of fundamental roots

$$\Pi = \{\, e_1 - e_2 + (f_1 - f_2) > e_1 - e_2 - (f_1 - f_2) > \ldots > e_{m-1} - e_m + (f_{m-1} - f_m) > e_{m-1} - e_m - (f_{m-1} - f_m) > 2e_m \,\}.$$

Now, $\theta$ maps each $e_j$ into $e_j$ and each $f_j$ into $-f_j$. Therefore, the $\theta$ orbits on $\Pi$ are $\{\, e_j - e_{j+1} + (f_j - f_{j+1}), e_j - e_{j+1} - (f_j - f_{j+1}) \,\}$ for $1 \leq j < m$ and $\{\, 2e_m \,\}$. Moreover, $\mathfrak{g}_{2e_m} \leq \mathfrak{p}_0$ and $2e_m$ is a noncompact imaginary root, hence the node correspondig to it is painted to black in the Vogan diagram.

Summarizing all these, we obtain the Vogan diagram




Finally, the real form $\mathfrak{sl}(m, \mathbb{H})$ of $\mathfrak{sl}(2m, \mathbb{C})$ clearly must correspond to the (remaining) Vogan diagram

## 3.2 Vogan diagrams eliminated by the Borel de Siebenthal theorem

## 3.3 Describing real forms of $G_2$ via Octonions

### 3.3.1 The classical division algebras

Here we apply the Cayley-Dickson process to the field $\mathbb{R}$ of real numbers in order to construct the field $\mathbb{C}$ of complex numbers, the skew field $\mathbb{H}$ of quaternions and the non-associative field $\mathbb{O}$ of octonions. They are called the classical division algebras. We characterize them.

**The Cayley-Dickson construction** produces a sequence $\mathbb{F}_m$ of algebras over $\mathbb{R}$ such that each of them has an involutory antiautomorphism $a \mapsto \bar{a}$, called *conjugation*. One starts with $\mathbb{F}_0 = \mathbb{R}$, $\overline{a} = a$ for $a \in \mathbb{R}$. We assume that $\mathbb{F}_{m-1}$, $m \geq 1$, with its conjugation has been constructed. One puts $\mathbb{F}_m := \mathbb{F}_{m-1} \times \mathbb{F}_{m-1}$ and defines the addition, the multiplication and the conjugation in $\mathbb{F}_m$ as follows:

$$
\begin{aligned}
(a,b)+(c,d) &:= (a+c, b+d) \\
(a,b)(c,d) &:= (ac - \overline{d}b, da + b\overline{c}) \\
\overline{(a,b)} &:= (\overline{a}, -b),
\end{aligned}
\tag{3.1}
$$

where $a, b, c, d \in \mathbb{F}_{m-1}$. The dimension of $\mathbb{F}_m$ over $\mathbb{R}$ is $2^m$. The map $\mathbb{F}_{m-1} \to \mathbb{F}_m : a \mapsto (a, 0)$ shows that $\mathbb{F}_{m-1}$ may be identified with a subalgebra of $\mathbb{F}_m$. $\mathbb{R} = \mathbb{F}_0$ is a central subfield of all the algebras $\mathbb{F}_m$. The conjugation is an antiautomorphism of $\mathbb{F}_m$ and the set of its fixed elements is $\{x \in \mathbb{F}_m | \ \bar{x} = x\} = \mathbb{R}$. The algebra $\mathbb{F}_1 := \mathbb{C}$ of complex numbers is commutative and associative. The algebra $\mathbb{F}_2 := \mathbb{H}$ of Hamilton's *quaternions* is associative. For $m = 3$ we get the algebra of *octonions* (or Cayley numbers) $\mathbb{F}_3 := \mathbb{O}$. It is not associative, but alternative and consequently biassociative.

**Proposition 69.** *For $m \leq 3$ the algebra $\mathbb{F}_m$ has no zero divisors and therefore it is a division algebra in the sense that, for $a \neq 0$, the $\mathbb{R}$-linear maps $x \mapsto ax$ and $x \mapsto xa$ are bijective.*

This assertion follows from the fact that for $m \leq 3$ the algebra $\mathbb{F}_m$ is alternative. To the proof of this fact we introduce the notion of the norm of the algebras $\mathbb{F}_m$. For $x = (a, b) \in \mathbb{F}_m$, $a, b \in \mathbb{F}_{m-1}$ one has

$$x\overline{x} = (a,b)(\overline{a}, -b) = (a\overline{a} + b\bar{b}, 0), \ \overline{x}x = (\overline{a}, -b)(a,b) = (\overline{a}a + b\bar{b}, 0).$$

By induction on $m$, one gets $x\overline{x} = \overline{x}x$ and for $x \neq 0$ this is a positive element $||x||^2$ of the subfield $\mathbb{R}$. The map $x \mapsto ||x||^2$ is a positive definite quadratic form (the so-called *norm form*) on the $\mathbb{R}$-vector space $\mathbb{F}_m$. The associated bilinear form is $\langle x|y\rangle := ||x+y||^2 - ||x||^2 - ||y||^2 = \overline{x}y + \overline{y}x$. It is a positive definite inner product such that $\langle x|x\rangle = 2||x||^2$. Since conjugation is involutory we obtain $||\bar{x}||^2 = ||x||^2$, $\langle \bar{x}|\bar{y}\rangle = \langle x|y\rangle$. Moreover, for $a, b \in \mathbb{F}_{m-1}$ we have $||(a,b)||^2 = ||a||^2 + ||b||^2$. The subspaces $\mathbb{F}_{m-1} \times \{0\}$ and $\{0\} \times \mathbb{F}_{m-1}$ of $\mathbb{F}_m$ are orthogonal with respect to the inner product. For $0 \neq x \in \mathbb{F}_m$ we obtain that $x^{-1} := ||x||^{-2}\bar{x}$ is a two-sided multiplicative inverse and that $||x^{-1}||^2 = (||x||^2)^{-1}$.

$\mathbb{F}_m$ is a **quadratic algebra**, i.e. every element of $\mathbb{F}_m$ satisfies a quadratic equation with real coefficients. For every $x \in \mathbb{F}_m$

$$x + \bar{x} \in \mathbb{R} \textit{ so that } \bar{x} \in \mathbb{R} + \mathbb{R}x.$$

Consequently, $x^2 = (x + \bar{x})x - ||x||^2 \in \mathbb{R} + \mathbb{R}x$.

The orthogonal complement of the subspace $\mathbb{R}$ of $\mathbb{F}_m$ with respect to the inner product is the subspace

$$\mathrm{Pu}\mathbb{F}_m := \{x \in \mathbb{F}_m | \bar{x} = -x\}$$

of *pure* elements. From the above quadratic equation it follows that this subspace can be written as

$$\mathrm{Pu}\mathbb{F}_m := \{x \in \mathbb{F}_m |\ x^2 \in \mathbb{R},\ x^2 = -||x||^2 \le 0\}. \tag{3.2}$$

For $x \in \mathbb{F}_m$ the equality

$$x^2 = -||x||^2 \text{ holds if and only if } x \in \mathrm{Pu}\mathbb{F}_m. \tag{3.3}$$

For $u, v \in \mathrm{Pu}\mathbb{F}_m$ we have

$$\langle u|v\rangle = 0 \text{ precisely if } uv = -vu \text{ precisely if } uv \in \mathrm{Pu}\mathbb{F}_m. \tag{3.4}$$

Now we deal with the associative properties of the algebras $\mathbb{F}_m$. Every algebra $\mathbb{F}_m$, $m \ge 0$, is *mono-associative.*

**Proposition 70.** *For $x \in \mathbb{F}_m \setminus \mathbb{R}$, the span $\mathbb{R} + \mathbb{R}x$ of 1 and $x$ is an associative and commutative subalgebra of $\mathbb{F}_m$ and this is isomorphic to $\mathbb{C}$.*

*Proof.* The intersection of the 2-dimensional subspace $A = \mathbb{R} + \mathbb{R}x$ with the hyperplane $\mathrm{Pu}\mathbb{F}_m$ is a 1-dimensional subspace $\mathbb{R}u$ with $u \in \mathrm{Pu}\mathbb{F}_m$ and $||u||^2 = 1$. Hence $\{1, u\}$ is a basis of $A$ and $u^2 = -||u||^2 = -1 \in \mathbb{R}$ (cf. (3.2)). Hence $A$ is a subalgebra which is associative and commutative. It is isomorphic to $\mathbb{C}$. □

To get stronger associative properties of $\mathbb{F}_m$ for $m \le 3$ we use alternativity.

**Proposition 71.** *Let $\mathbb{F}_{m-1}$ be an associative algebra. Then for all $x, y \in \mathbb{F}_m$ one has*

$$\bar{x}(xy) = (\bar{x}x)y = ||x||^2 y \tag{3.5}$$

*and by conjugation*

$$(yx)\bar{x} = y(x\bar{x}) = ||x||^2 y. \tag{3.6}$$

*Equivalently, $\mathbb{F}_m$ is alternative, i.e. the following identities hold:*

$$x(xy) = x^2 y \quad \text{and} \quad (yx)x = yx^2.$$

*As a consequence of alternativity of $\mathbb{F}_m$ it is flexible, i.e. the following identities hold: $x(yx) = (xy)x$.*

*Proof.* Let $x = (a, b)$, $y = (c, d)$ for $a, b, c, d \in \mathbb{F}_{m-1}$. Direct computation and the associativity of $\mathbb{F}_{m-1}$ yield that

$$\bar{x}(xy) = (\bar{a}ac - \bar{a}\bar{d}b + \bar{a}\bar{d}b + c\bar{b}b, da\bar{a} + b\bar{c}\bar{a} - b\bar{c}\bar{a} + b\bar{b}d).$$

As $a\bar{a} = \bar{a}a = ||a||^2$ and $b\bar{b} = \bar{b}b = ||b||^2$ are scalars and $||a||^2 + ||b||^2 = ||x||^2$ we get $\bar{x}(xy) = (||a||^2 c + ||b||^2 c, ||a||^2 d + ||b||^2 d) = ||x||^2(c, d) = ||x||^2 y$ which proves (3.5). Applying conjugation (3.6) follows. The alternative laws are identical to (3.5) and (3.6) if $x$ is a pure element, i.e. if $x = -\bar{x}$. Decomposing $x \in \mathbb{F}_m$ into a scalar in $\mathbb{R}$ and a pure element the general case can be proved.

The flexibility of $\mathbb{F}_m$ follows from alternativity by computing the two sides of the equation $(x + y)((x + y)x) = (x + y)^2 x$. □

**Proposition 72** (Biassociativity.)**.** *For all $x \in \mathbb{O} \setminus \mathbb{R}$ and $y \in \mathbb{O} \setminus (\mathbb{R} + \mathbb{R}x)$, the span $\mathbb{R} + \mathbb{R}x + \mathbb{R}y + \mathbb{R}xy$ is an associative subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.*

*Proof.* Intersecting $\mathbb{R} + \mathbb{R}x$ and $\mathbb{R} + \mathbb{R}x + \mathbb{R}y$ with the hyperplane Pu$\mathbb{O}$, we obtain elements $u, v \in \mathrm{Pu}\mathbb{O}$ for which $||u||^2 = 1 = ||v||^2$ and $\langle u|v\rangle = 0$ such that $x = r_0 + r_1 u$, $y = s_0 + s_1 u + s_2 v$ for suitable scalars $r_\nu, s_\nu \in \mathbb{R}$, $\nu = 0, 1, 2$ with $r_1 \neq 0$, $s_2 \neq 0$. By (3.2) we have $u^2 = -||u||^2 = -1$ and therefore $\mathbb{R} + \mathbb{R}x + \mathbb{R}y + \mathbb{R}xy = \mathbb{R} + \mathbb{R}u + \mathbb{R}v + \mathbb{R}uv$. Hence it suffices to prove the following assertion:

**Lemma 73.** *Assume that $2 \leq m \leq 3$ and let $u, v \in Pu\mathbb{F}_m$ be such that*

$$||u||^2 = 1 = ||v||^2 \quad \textit{and} \quad \langle u|v\rangle = 0. \tag{3.7}$$

*Then the product $w := uv$ satisfies*

$$w \in Pu\mathbb{F}_m, \quad \langle u|w\rangle = 0 = \langle v|w\rangle, \;\; \textit{and} \; ||w||^2 = 1. \tag{3.8}$$

*Moreover, we have the following multiplication table:*

| | $u$ | $v$ | $w$ |
|---|---|---|---|
| $u$ | $-1$ | $w$ | $-v$ |
| $v$ | $-w$ | $-1$ | $u$ |
| $w$ | $v$ | $-u$ | $-1$ |

*The span $\mathbb{R} + \mathbb{R}u + \mathbb{R}v + \mathbb{R}w$ is an associative subalgebra of $\mathbb{F}_m$ isomorphic to $\mathbb{H}$.*

**Definition 74.** A triple $u, v, w$ with these properties is called a *Hamilton triple*.

*Proof.* At a first stage we shall assume in addition that $\mathbb{F}_m$, $m = 2, 3$, is already known to be alternative. This assumption will later prove. From (3.7), (3.2) and (3.4) we get $u^2 = -1 = v^2$ and $w \in \mathrm{Pu}\mathbb{F}_m$. Applying alternativity one obtains $uw = u(uv) = u^2 v = -v$ and $wv = (uv)v = uv^2 = -u$. By (3.4) it follows that $u$, $v$ and $w$ are mutually orthogonal with respect to $\langle\ |\ \rangle$ and anticommuting. Using alternativity we get that $w^2 u = w(wu) = wv = -u$. Since $w^2$ is a scalar by (3.3), this means that $w^2 = -1$ and $||w||^2 = 1$. Thus the multiplication table is proved and it follows that

$$A := \mathbb{R} + \mathbb{R}u + \mathbb{R}v + \mathbb{R}w$$

is a flexible subalgebra.

To prove the associativity of $A$ it suffices to show that triple products composed of $u$, $v$ and $w$ are associative. If two of the factors of such a triple product coincide, then this is true by alternativity and flexibility. For triple products with different factors we have, for instance,

$$(uv)w = w^2 = -1 = u^2 = u(vw) \;\text{ and }\; (vu)w = -w^2 = -1 = -v^2 = v(uw).$$

All the other triple products with different factors are obtained from these by cyclic permutation of $u$, $v$ and $w$ under which the corresponding equalities remain valid because of the symmetry of the multiplication table. Thus $A$ is associative.

Now we show our supplementary hypothesis of alternativity. Since $\mathbb{F}_1 = \mathbb{C}$ is associative, $\mathbb{F}_2$ is alternative by Proposition 71 and so the above arguments are valid in $\mathbb{F}_2$. In this case $1, u, v, w$ span the 4-dimensional algebra $\mathbb{F}_2 = \mathbb{H}$ since they are linearly independent by (3.7) and (3.8), thus $\mathbb{F}_2 = A$. In particular, from the above, $\mathbb{F}_2 = \mathbb{H}$ is associative.

Proposition 71 yields that also $\mathbb{F}_3 = \mathbb{O}$ is alternative. Thus our previous arguments apply here we get that the subalgebra $A$ spanned by 1 and by element2s $u, v, w$ with the properties (3.7) and (3.8) is associative. Any two such algebras are isomorphic since multiplication is entirely determined by the given multiplication table. As $\mathbb{H}$ is also spanned by such a basis every such subalgebra of $\mathbb{F}_3 = \mathbb{O}$ is isomorphic to $\mathbb{H}$. □

Since there are multiplicative inverses of the algebras $\mathbb{F}_m$ we obtain that $\mathbb{C}$ is a commutative field and $\mathbb{H}$ is a skew field. As $\mathbb{F}_3 = \mathbb{O}$ is alternative the assertion of Proposition 69 follows because if $xy = 0$, then $||x||^2 y = \bar{x}(xy) = 0$ so that $x = 0$ or $y = 0$.

Summarizing this discussion we obtain:

**Proposition 75.** *$\mathbb{O}$ is an alternative field, i.e. an alternative division algebra and it is biassociative.*

**Proposition 76. Multiplicativity of the norm.** *If $m \leq 3$, then for $x, y \in \mathbb{F}_m$ one has*

$$||xy||^2 = ||x||^2||y||^2. \tag{3.9}$$

*Moreover, for each $a \in \mathbb{F}_m$,*

$$\langle x|\bar{a}y\rangle = \langle ax|y\rangle \quad \text{and} \quad \langle x|y\bar{a}\rangle = \langle xa|y\rangle. \tag{3.10}$$

*Proof.* All the algebras $\mathbb{F}_m$, $m \leq 3$ are at least biassociative (cf. Proposition 72). Hence one has $||xy||^2 = \overline{(xy)}(xy) = \bar{y}(\bar{x}x)y = ||x||^2\bar{y}y = ||x||^2||y||^2$ which proves (3.9). For (3.10) we may assume $a \neq 0$. Put $z = a^{-1}y$, so that $az = a(a^{-1}y) = y$. Using (3.9), we obtain $\langle ax|y\rangle = \langle ax|az\rangle = ||ax + az||^2 - ||ax||^2 - ||az||^2 = ||a||^2(||x + z||^2 - ||x||^2 - ||z||^2) = ||a||^2\langle x|z\rangle = \langle x|||a||^2 z\rangle = \langle x|(\bar{a}a)z\rangle = \langle x|\bar{a}(az)\rangle = \langle x|\bar{a}y\rangle$. The first equality of (3.10) is thus proved. The second is obtained by applying conjugation which preserves the inner product (see the properties of the *norm form*). □

The following result is a motivation of the Cayley-Dickson process.

**Proposition 77** (Constructing $\mathbb{O}$ from the quaternion subfields.)**.** *Let $H$ be a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$ and let $z \in Pu\mathbb{O}$ be of unit length $||z||^2 = 1$ and orthogonal to $H$. Then the 4-dimensional $\mathbb{R}$-linear subspace $Hz$ is orthogonal to $H$ so that the $\mathbb{R}$-vector space $\mathbb{O}$ decomposes into the direct sum*

$$\mathbb{O} = H \oplus Hz.$$

*For $a, b, c, d \in H$ we have $(a + bz)(c + dz) = (ac - \bar{d}b) + (da + b\bar{c})z$.*

*Proof.* $H$ is invariant under conjugation (see the part $\mathbb{F}_m$ is a **quadratic algebra**). By (3.10) we have $\langle a|bz\rangle = \langle \bar{b}a|z\rangle = 0$. Hence $H$ and $Hz$ are orthogonal. Using alternativity we obtain for $x, y, w \in \mathbb{O}$,

$$||x + y||^2 w = (x + y)((\bar{x} + \bar{y})w) = (||x||^2 + ||y||^2)w + x(\bar{y}w) + y(\bar{x}w).$$

Now $||x + y||^2 = ||x||^2 + ||y||^2$ whenever $\langle x|y\rangle = 0$. Hence if $\langle x|y\rangle = 0$, then $x(\bar{y}w) = -y(\bar{x}w)$. For $w = 1$ if $y \in \mathrm{Pu}\mathbb{O}$ and $\langle x|y\rangle = 0$, then $xy = y\bar{x}$. Using these and alternativity we may compute the terms on the left hand side of the asserted product formula as follows.

$$(bz)c = \bar{c}(bz) = \bar{c}(z\bar{b}) = -\bar{z}(c\bar{b}) = -(b\bar{c})\bar{z} = (b\bar{c})z,$$

$$a(dz) = a(z\bar{d}) = -\bar{z}(\bar{a}\bar{d}) = z(\bar{d}\bar{a}) = (da)z,$$

$$(bz)(dz) = -\bar{d}((\bar{z}\bar{b})z) = -\bar{d}((b\bar{z})z) = -\bar{d}((b||z||^2) = -\bar{d}b.$$

□

**Definition 78.** *Cayley triples* are triples $u, v, z \in \mathrm{Pu}\mathbb{O}$, $||u||^2 = ||v||^2 = ||z||^2 = 1$ such that $u$ and $v$ are mutually orthogonal and such that $z$ is orthogonal to $u$, $v$ and $uv$.

This notion is symmetric in the sense that a permutation of a Cayley triple is another Cayley triple. For example by Proposition 77 and Lemma 73 we obtain the following identities: $u(vz) = (vu)z = -(uv)z$ (which shows that $\mathbb{O}$ is non-associative), $(vz)u = -(vu)z$. Thus one gets $u(vz) = -(vz)u$. From (3.4) we get that the pure element $vz$ is orthogonal to $u$. By Lemma 73 and Proposition 72 for a given Cayley triple $u, v, z$ the pure octonions $u$, $v$ and $w := uv$ form a Hamilton triple and together with 1 they span a subalgebra $H$ isomorphic to $\mathbb{H}$. Furthermore, by Proposition 77 the elements 1, $u$, $v$, $w$, $z$, $uz$, $vz$, $wz$ constitute a basis of $\mathbb{O}$ as a vector space over $\mathbb{R}$. From Lemma 73 and Proposition 77 one can compute a multiplication table for this basis. The following assertion is true. *For all Cayley triples $u$, $v$, $z$ one obtains the same multiplication table with respect to the basis* 1, $u$, $v$, $w$, $z$, $uz$, $vz$, $(uv)z$. This shows that for any two Cayley triples the $\mathbb{R}$-linear transformation of $\mathbb{O}$ mapping the basis obtained in this way from the first Cayley triple onto the basis corresponding to the second Cayley triple is an automorphism of $\mathbb{O}$. Hence we get the following:

**Corollary 79.** *For any two Cayley triples of $\mathbb{O}$, there is a unique automorphism of $\mathbb{O}$ mapping the first Cayley triple onto the second.*

The *standard Cayley triple* and the associated basis of $\mathbb{O}$ are obtained as follows. Let $i \in \mathbb{C}$ be an imaginary unit, that is $i \in \mathrm{Pu}\mathbb{C}$ with $i^2 = -1$. In $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ one considers the basis

$$1 = (1, 0), \quad i = (i, 0), \quad j := (0, 1), \quad k := ij = (i, 0)(0, 1) = (0, i).$$

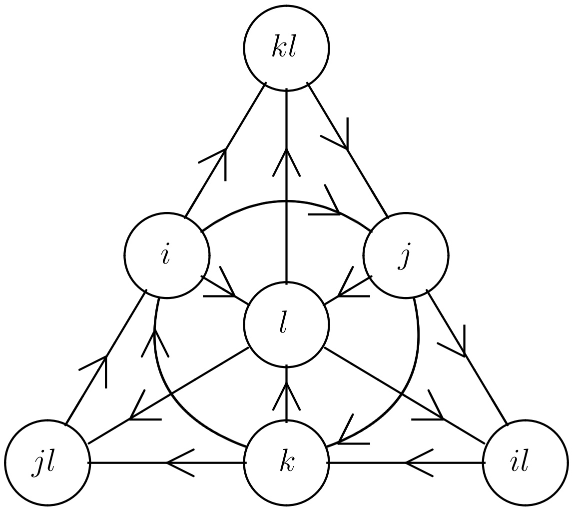
The triple $i$, $j$, $k$ is clearly a Hamilton triple as in Lemma 73. In $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ by identify $\mathbb{H}$ with the subalgebra $\mathbb{H} \times \{0\}$ we find these elements again, namely

$$i \widehat{=} (i, 0), \quad j \widehat{=} (j, 0), \quad k \widehat{=} (k, 0).$$

They are elements of $\mathrm{Pu}\mathbb{O}$. Putting $l = (0, 1) \in \mathrm{Pu}\mathbb{O}$, we note that $i$, $j$, $l$ is a Cayley triple and we compute the following products.

$$il = (i, 0)(0, 1) = (0, i), \; jl = (j, 0)(0, l) = (0, j), \; kl = (k, 0)(0, 1) = (0, k).$$

Following an idea of Freudenthal [5] 1.5.13, p. 19, one may represent this multiplication table graphically using the projective plane with 7 points.



Here is a simple explanation how this figure should one read. The "circle" in this figure shows the multiplication of the elements $i, j, k$ in the generated quaternion algebra $\mathbb{H}$. For each pair of elements $a, b \in \{i, j, k\}$ their product is the third one $c$ if $a, b, c$ is ordered in the way the arrow suggests it; otherwise the product is the opposite of the third one. As an example, $ij = k$ but $kj = -i$. By choosing one among the six "lines", we should think of it rather as an oriented circle, as well, by closing it with an arrow from the third one to the first one. (In fact, both "circles" and "lines" refers to projective lines of the above projective plane.)

Then the multiplication in such a "circle" is defined in the same way as for the "circle" containing $i, j, k$. So, $(il)k = jl$, $(jl)(il) = k$, while $(il)(jl) = -k$. (Note that the use of brackets is necessary now, because $\mathbb{O}$ is not associative.) Note that three nodes of this figure form a Hamilton triple if and only if they are on the same projective line.

### 3.3.2 The group of automorphisms of octonions

Now we study the group $\mathrm{Aut}(\mathbb{F}_m)$, $m \leq 3$ of automorphisms of the ring $\mathbb{F}_m$, that is we consider automorphisms with respect to addition and multiplication in $\mathbb{F}_m$, linearity over $\mathbb{R}$ is not presupposed. The only automorphism of $\mathbb{F}_0 = \mathbb{R}$ is the identity. The only automorphism of the field $\mathbb{F}_1 = \mathbb{C}$ which fixes the elements of $\mathbb{R}$ are the identity and conjugation. They map $i$ onto $i$ or $-i$.

**Proposition 80.** *Let $m \geq 2$. Then an automorphism $\alpha \in Aut(\mathbb{F}_m)$ fixes every element of the centre $\mathbb{R}$ of $\mathbb{F}_m$ and so is $\mathbb{R}$-linear. Moreover, $\alpha$ leaves $Pu\mathbb{F}_m$ invariant, commutes with conjugation and is orthogonal with respect to the inner product of $\mathbb{F}_m$.*

*Proof.* The center $\mathbb{R}$ of $\mathbb{F}_m$ is invariant under $\alpha$ and $\alpha|_{\mathbb{R}} = \mathrm{id}$ because $\mathbb{R}$ has no other automorphism. Hence $\alpha$ is $\mathbb{R}$-linear. By (3.2) the subspace $\mathrm{Pu}\mathbb{F}_m$ is invariant under $\alpha$. Therefore conjugation which on $\mathrm{Pu}\mathbb{F}_m$ induces $-\mathrm{id}$, commutes with $\alpha$. $\alpha$ is orthogonal since $||x^\alpha||^2 = x^\alpha \overline{x^\alpha} = x^\alpha \overline{x}^\alpha = (x\overline{x})^\alpha = (||x||^2)^\alpha = ||x||^2$. □

Let $\mathbb{F}$ be one of the algebras $\mathbb{F}_m$ for $0 \leq m \leq 3$. Using the multiplication in $\mathbb{F}$ we can describe of certain groups of orthogonal transformations acting on the $\mathbb{R}$-vector space $\mathbb{F} = \mathbb{R}^n$, where $n = 2^m$, $m \in \{0, 1, 2, 3\}$. The group of $\mathbb{R}$-linear transformations of $\mathbb{F}$ which are orthogonal with respect to the inner product $\langle \mid \rangle$ on $\mathbb{F}$ is denoted by $O_n(\mathbb{R}) = \{C : \mathbb{F} \to \mathbb{F} | C \text{ is } \mathbb{R} - \text{linear}, \ \forall x \in \mathbb{F} : \ ||Cx||^2 = ||x||^2\}$. The normal subgroup $SO_n(\mathbb{R}) = \{C \in O_n(\mathbb{R}) | \det C = 1\}$ of $O_n(\mathbb{R})$ has index 2.

**Lemma 81.** *(a) For each $a \in \mathbb{F}$ with $||a||^2 = 1$ the transformations $x \mapsto ax$ and $x \mapsto xa$ of $\mathbb{F} = \mathbb{R}^n$ belong to $O_n(\mathbb{R})$ and even to $SO_n(\mathbb{R})$ for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

*(b) The group $SO_n(\mathbb{R})$ is generated by the transformations $\mathbb{F} \to \mathbb{F} : x \mapsto axa$, where $a \in \mathbb{F}$, $||a||^2 = 1$.*

*Proof.* As the norm is multiplicative (cf. (3.9)) the $\mathbb{R}$-linear transformations in (a) with $||a|| = 1$ are elements of $O_n(\mathbb{R})$.

For $\mathbb{R}$ one has $SO_1(\mathbb{R}) = \{\mathrm{id}\}$. In general the transformations considered in (b) belong to $SO_n(\mathbb{R})$ by (a). In order to show that they generate $SO_n(\mathbb{R})$ we use reflections in hyperplanes. These reflections do not belong to $SO_n(\mathbb{R})$. For any $a \in \mathbb{F}$ with $||a||^2 = 1$, the reflection in the hyperplane orthogonal to $a$ is the mapping

$$\rho_a : x \mapsto x - 2a \cdot \frac{\langle x|a \rangle}{\langle a|a \rangle} = x - 2a \cdot \frac{(\bar{x}a + \bar{a}x)}{2||a||^2} = x - a(\bar{x}a + \bar{a}x) = -a\bar{x}a.$$

Every element of $SO_n(\mathbb{R})$ is the product of an even number of such hyperplane reflections, i.e. the product of transformations having the form $\rho_a\rho_b$ for $a, b \in \mathbb{F}$ with $||a||^2 = 1 = ||b||^2$. Now $x^{\rho_a\rho_b} = -b(\overline{-a\overline{x}a})b = b(\overline{a}x\overline{a})b$ so that $\rho_a\rho_b$ can also be written as the composition of the transformations $x \mapsto \bar{a}x\bar{a}$ and $x \mapsto bxb$. □

**Corollary 82.** *For $\mathbb{F} = \{\mathbb{C}, \mathbb{H}\}$ the group $SO_n(\mathbb{R})$ consists precisely of the transformations $\mathbb{F} \to \mathbb{F} : x \mapsto b^{-1}xa$, where $a, b \in \mathbb{F}$, $||a||^2 = 1 = ||b||^2$.*

*Proof.* These transformations belong to $SO_n(\mathbb{R})$ (cf. Lemma 81) and they form a subgroup of $SO_n(\mathbb{R})$. (Here the associativity of $\mathbb{F}$ is required). This subgroup contains the transformations of Lemma 81 (b) (put $b = a^{-1}$) which generate $SO_n(\mathbb{R})$. □

Using $\mathbb{F} = \mathbb{H}$ we obtain a description of $SO_3(\mathbb{R})$ from Corollary 82. The subspace $\mathrm{Pu}\mathbb{H} \cong \mathbb{R}^3$ of the $\mathbb{R}$-vector space $\mathbb{H}$ is the orthogonal complement of $\mathbb{R}$ in $\mathbb{H}$ (cf. definition of *pure* elements). The group of $\mathbb{R}$-linear transformations of $\mathbb{H}$ which are orthogonal with respect to the inner product and which fix 1 may be identified with the group of orthogonal transformations of $\mathrm{Pu}\mathbb{H} \cong \mathbb{R}^3$. We denote both groups by $O_3(\mathbb{R})$. The normal subgroup $SO_3(\mathbb{R})$ of elements of determinant 1 has index 2. Now the transformation $\mathbb{H} \to \mathbb{H} : x \mapsto b^{-1}xa$ in Corollary 82 fixes 1 precisely if $b = a$. Therefore we have:

**Corollary 83.** *$SO_3(\mathbb{R})$ consists precisely of the transformations $int(a) : \mathbb{H} \to \mathbb{H} : x \mapsto a^{-1}xa$, where $a \in \mathbb{H}$ satisfies $||a||^2 = 1$.*

Hence $SO_3(\mathbb{R})$ is an epimorphic image of the unit sphere $\mathbb{S}^3 = \{a \in \mathbb{H} |\ ||a||^2 = 1\}$ of $\mathbb{H} = \mathbb{R}^4$. By multiplicativity of the norm it is a subgroup of the multiplicative group $\mathbb{H}^*$.

**Corollary 84.** *The kernel of the epimorphism $int : \mathbb{S}^3 \to SO_3(\mathbb{R}) : a \mapsto int(a)$ is $\{1, -1\}$.*

*Proof.* An element $a \in \mathbb{S}^3$ belongs to the kernel if and only if $a^{-1}xa = x$ for all $x \in \mathbb{H}$. This is the case if $a$ belongs to the centre of $\mathbb{H}$, which is $\mathbb{R}$. Now one has $\mathbb{R} \cap \mathbb{S}^3 = \{1, -1\}$. □

Now we determine $\mathrm{Aut}(\mathbb{H})$. Because of associativity of $\mathbb{H}$ conjugation by an element $a \in \mathbb{H} \setminus \{0\}$ is an automorphism of the skew field $\mathbb{H}$, the *inner automorphism* $\mathrm{int}(a)$. Let $\mathrm{Int}(\mathbb{H})$ be the group of all these inner automorphisms. Since $||a||$ belongs to the centre $\mathbb{R}$ of $\mathbb{H}$ we have $\mathrm{int}\left(\frac{1}{||a||}a\right) = \mathrm{int}(a)$. Now $\frac{1}{||a||}a$ has norm 1, so that

$$\mathrm{Int}(\mathbb{H}) = \left\{\mathrm{int}(a) | a \in \mathbb{H}, ||a||^2 = 1\right\}.$$

By Proposition 80 and Corollary 83 we know that $\mathrm{Aut}(\mathbb{H})$ is contained in the group $O_3(\mathbb{R})$ and that $\mathrm{Int}(\mathbb{H}) = SO_3(\mathbb{R})$.

**Proposition 85.** *$Aut(\mathbb{H}) = Int(\mathbb{H}) = SO_3(\mathbb{R})$.*

*Proof.* $SO_3(\mathbb{H})$ has index 2 in $O_3(\mathbb{H})$ whence if $\mathrm{Aut}(\mathbb{H})$ were bigger then $SO_3(\mathbb{R})$ we would have $\mathrm{Aut}(\mathbb{H}) = O_3(\mathbb{R})$. In particular, conjugation (which on Pu $\mathbb{H}$ induces $-\mathrm{id} \in O_3(\mathbb{R})$) would have to be an element of $\mathrm{Aut}(\mathbb{H})$, but conjugation is an antiautomorphism and not an automorphism, as $\mathbb{H}$ is not commutative. □

Now we study $\mathrm{Aut}(\mathbb{O})$. We start our discussion with transitivity properties which follows from the fact that $\mathrm{Aut}(\mathbb{O})$ acts sharply transitively on the set of Cayley triples (see Corollary 79).

**Lemma 86.** *(a) $Aut(\mathbb{O})$ acts transitively on the set*

$$\left\{(u,v)|\ u,v \in Pu\ \mathbb{O},\ u \perp v,\ ||u||^2 = 1 = ||v||^2\right\},$$

*i.e. on the 6-sphere $\{u \in Pu\ \mathbb{O}, ||u||^2 = 1\}$ and the stabilizer $Aut(\mathbb{O})_i$ is transitive on the 5-sphere $\{u \in Pu\ \mathbb{O}, ||u||^2 = 1, u \perp i\}$.*

*(b) The stabilizer $Aut(\mathbb{O})_{i,j}$ is sharply transitive on the 3-sphere $\{(0,b) \in \mathbb{H} \subset \mathbb{H} || ||b||^2 = 1\}$.*

*Proof.* To verify (a) we use the fact that for any two pure orthogonal elements $u$, $v$ with $||u||^2 = 1 = ||v||^2$ there is a Cayley triple having $u$, $v$ as the first two elements. The elements of the 3-sphere given in (b) are precisely those elements $z$ of Pu$\mathbb{O}$ with unit length which satisfy the property $i, j, z$ is a Cayley triple because $i$, $j$, $ij$ span Pu$\mathbb{H} \times \{0\}$ and $\{0\} \times \mathbb{H}$ is the orthogonal space of Pu $\mathbb{H} \times \{0\}$ in Pu$\mathbb{O}$. □

**Lemma 87.** *(a) The automorphism group $\Lambda = Aut(\mathbb{O})$ acts transitively on the set $\mathcal{H}$ of subalgebras $H$ of $\mathbb{O}$ with $H \cong \mathbb{H}$ so that the stabilizers $\Lambda_H$ of such subalgebras are conjugate. These stabilizers cover $\Lambda$.*

*(b) The stabilizer of $\mathbb{H} = \mathbb{H} \times \{0\} \subset \mathbb{H} \times \mathbb{H} = \mathbb{O}$ is*

$$\Lambda_{\mathbb{H}} = \{(x,y) \mapsto (a^{-1}xa, b^{-1}ya)|\ a,b \in \mathbb{H}, ||a||^2 = 1 = ||b||^2\},$$

*where $(x,y) \in \mathbb{O} = \mathbb{H} \times \mathbb{H}$. It is isomorphic to $SO_4\mathbb{R}$.*

*(c) The stabilizer of $i$ and $j$ is $\Lambda_{i,j} = \{(x,y) \mapsto (x, b^{-1}y)|\ b \in \mathbb{H}, ||b||^2 = 1\}$.*

*(d) All involutions of $\Lambda$ are conjugate and $\Lambda$ is generated by them.*

*(e) $Aut(\mathbb{O})$ is a subgroup of $SO_8\mathbb{R}$.*

*Proof.* A subalgebra $H \cong \mathbb{H}$ intersects Pu$\mathbb{O}$ in a 3-dimensional subspace and hence contains orthogonal pure elements $u, v$ of $\mathbb{O}$ such that $||u||^2 = 1 = ||v||^2$. $H$ is a span of $1, u, v$ and $uv$ (see Proposition 73) and $\Lambda = \mathrm{Aut}(\mathbb{O})$ is transitive on the set $\mathcal{H}$ of such subalgebras (see Lemma 87 (a)). In particular the stabilizers $\Lambda_H$ of $H \in \mathcal{H}$ are conjugate. It remains to show that every element $\lambda \in \Lambda$ leaves some subalgebra $H \in \mathcal{H}$ invariant. Since $\lambda$ is orthogonal with respect to the inner product of $\mathbb{O}$ and leaves invariant the 7-dimensional subspace Pu$\mathbb{O}$ of $\mathbb{O}$ (cf. Proposition 80), Pu$\mathbb{O}$ decomposes into 1- and 2-dimensional invariant subspaces. Moreover, there is a 2-dimensional subspace spanned by orthogonal pure vectors $u, v$ of norm 1. The subalgebra $H$ spanned by $1, u, v, uv$ is isomorphic to $\mathbb{H}$ (see Proposition 73) and invariant under $\lambda$. This proves assertion (a).

The definition of the multiplication in $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ yields that the right hand side of $\Lambda_{\mathbb{H}}$ in (b) is a subgroup $M$ of $\mathrm{Aut}(\mathbb{O})$. The stabilizer $M_{i,j}$ fixes every element on the subalgebra $\mathbb{H} \times \{0\}$ which is spanned by $1, i, j, ij$. Thus, $M_{i,j}$ consists of the transformations described in assertion (b) with $a = \pm 1$ (cf. Corollary 84). This proves assertion (c). Moreover, $M_{i,j}$ acts sharply transitively on the unit sphere of $\{0\} \times \mathbb{H}$. By Lemma 86 the larger group $\Lambda_{i,j} \supseteq M_{i,j}$ is sharply transitively on this unit sphere. Hence one has $\Lambda_{i,j} = M_{i,j}$. $M$ leaves $\mathbb{H} \times \{0\}$ invariant and induces the full automorphism group $\mathrm{Int}(\mathbb{H}) = SO_3(\mathbb{R})$ (see Proposition 85). Moreover, $M$ is transitive on the set of pairs of pure orthogonal elements of norm 1 contained in $\mathbb{H} = \mathbb{H} \times \{0\} \subset \mathbb{H} \times \mathbb{H} = \mathbb{O}$ and this set is invariant under $\Lambda_{\mathbb{H}}$ (cf. Proposition 80). Hence one gets $\Lambda_{\mathbb{H}} = M$. Restriction to $\{0\} \times \mathbb{H}$ gives an epimorphism of $\Lambda_{\mathbb{H}}$ onto $SO_4(\mathbb{R})$ (cf. Corollary 82). If $b^{-1}ya = y$ for all $y \in \mathbb{H}$, then $a = b$. Hence we get an isomorphism and assertion (b) is proved.

Let $\iota \in \Lambda$ be an involution. As $\iota$ is orthogonal (see Proposition 80), the $\mathbb{R}$-vector space $\mathbb{O}$ is the orthogonal sum of the eigenspaces $F_+$ and $F_-$ of $\iota$ corresponding to the eigenvalues 1 and $-1$ and $\iota$ is uniquely determined by the fixed space $F_+$. This is a subalgebra of $\mathbb{O}$. Multiplication by an element $a \in F_- \setminus \{0\}$ is a vector space isomorphism between $F_+$ and $F_-$. Hence the subalgebra $F_+$ has dimension 4 over $\mathbb{R}$ and therefore $F_+$ is isomorphic to $\mathbb{H}$ (cf. Propositions 72 and 73). As $\Lambda$ acts transitively on the set $\mathcal{H}$ of such subalgebras (see Lemma 87 (a)) it follows that all involutions are conjugate. The fact that the stabilizer $\Lambda_{\mathbb{H}} \cong SO_4(\mathbb{R})$ is generated by its involutions is proved in [3] (Chap. II, 6, no. 1, p. 51). Then the same holds for the conjugates $\Lambda_H$, $H \in \mathcal{H}$ and for their union $\Lambda$ (see Lemma 87 (a)). Hence the assertion (d) is proved.

According to Proposition 80 $\mathrm{Aut}(\mathbb{O})$ is a subgroup of $O_8(\mathbb{R})$. Now assertions (a) and (b) imply that every automorphism has determinant 1, which proves assertion (e). □

**Theorem 88.** *Aut($\mathbb{O}$) is a compact simple Lie group of dimension* 14. *By the classification of simple Lie groups it is isomorphic to the exceptional compact Lie group* $G_2$.

*Proof.* We know that $\mathrm{Aut}(\mathbb{O}) \subseteq O_8(\mathbb{R})$. As multiplication in $\mathbb{O}$ is continuous, $\mathrm{Aut}(\mathbb{O})$ is closed in the usual topology of $O_8(\mathbb{R}) \subseteq GL_8(\mathbb{R})$. $\mathrm{Aut}(\mathbb{O})$ is compact because $O_8(\mathbb{R})$ is compact (cf. [10], Proposition 17.8, p. 337). Hence $\mathrm{Aut}(\mathbb{O})$ is a Lie group since it is a closed linear subgroup.

Now we prove that $\mathrm{Aut}(\mathbb{O})$ is a simple group. Assume that $N \neq \{\mathrm{id}\}$ be a normal subgroup of $\Lambda = \mathrm{Aut}(\mathbb{O})$. We have to prove that $N = \mathrm{Aut}(\mathbb{O})$. By Lemma 87 (d) it suffices to show that $N$ contains an involution. From Lemma 87 (a) we get that $N$ intersects some and therefore every of the conjugate subgroup $\Lambda_H$, $H \in \mathcal{H}$ non-trivially. Now we prove that every non-trivial normal subgroup of $SO_4(\mathbb{R}) \cong \Lambda_{\mathbb{H}}$ contains an involution. One has $SO_4(\mathbb{R}) = \{\mathbb{H} \to \mathbb{H} : y \mapsto b^{-1}ya|\ a, b \in \mathbb{H}, ||a||^2 = 1 = ||b||^2\}$ (cf. Corollary 82). The subgroups $A := \{\mathbb{H} \to \mathbb{H} : y \mapsto ya|\ a \in \mathbb{H}, ||a||^2 = 1\}$ and $B := \{\mathbb{H} \to \mathbb{H} : y \mapsto b^{-1}y|\ b \in \mathbb{H}, ||b||^2 = 1\}$ are normal in $SO_4(\mathbb{R})$ and $AB = SO_4(\mathbb{R})$. The centralizer $C_{SO_4(\mathbb{R})}(B) = A$ and $C_{SO_4(\mathbb{R})}(A) = B$. If $N$ is normal in $SO_4(\mathbb{R})$ such that $N \cap B = \{\mathrm{id}\}$, then one has $N \subseteq C_{SO_4(\mathbb{R})}(B) = A$. As $A$ and $B$ are isomorphic to $\mathbb{S}^3 = \{a \in \mathbb{H}, ||a||^2 = 1\}$ it suffices to prove that every non-trivial normal subgroup $H$ of $\mathbb{S}^3$ contains an involution. We consider the epimorphism $\mathrm{int} : \mathbb{S}^3 \to SO_3(\mathbb{R})$ with kernel $\{1, -1\}$ (cf. Corollary 84). If $H$ contains the involution $-1$ the proof is finished. If not, then $H$ is mapped isomorphically onto a non-trivial normal subgroup of $SO_3(\mathbb{R})$ by int. As $SO_3(\mathbb{R})$ is simple (see [1], p. 57) we have $\mathrm{int}(H) = SO_3(\mathbb{R})$. Hence $H$ must contain involutions because $SO_3(\mathbb{R})$ does. (This latter case does not occur, but this does not affect the argument).

The dimension of $\mathrm{Aut}(\mathbb{O})$ can be computed by applying the dimension formula for stabilizers to the transitive actions given in Lemma 86. We have $\dim(\mathrm{Aut}(\mathbb{O})_{i,j}) = \dim(\mathbb{S}^3) = 3$ (cf. Lemma 87 (c) or Lemma 86 (b)). $\mathrm{Aut}(\mathbb{O})_{i,j}$ is a stabilizer of the transitive action of $\mathrm{Aut}(\mathbb{O})_i$ on the 5-sphere of pure unit quaternions orthogonal to $i$. Hence we have $\dim(\mathrm{Aut}(\mathbb{O})_i) = \dim(\mathrm{Aut}(\mathbb{O})_{i,j}) + 5 = 8$. As $\mathrm{Aut}(\mathbb{O})$ acts transitively on the 6-dimensional unit sphere of Pu $\mathbb{O}$ (cf. Lemma 86 (a)) we obtain that $\dim(\mathrm{Aut}(\mathbb{O})) = \dim(\mathrm{Aut}(\mathbb{O})_i) + 6 = 14$. There is precisely one almost simple Lie group of dimension 14, the exceptional Lie group $G_2$ (cf. the classification of the almost simple Lie groups in [11], 94.32 and 94.33). □

# Bibliography


[1] E. Artin. *Geometric algebra.* Interscience Publishers, Inc., New York-London, 1957.

[2] R. W. Carter. *Simple groups of Lie type.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Reprint of the 1972 original, A Wiley-Interscience Publication.

[3] J. A. Dieudonné. *La géométrie des groupes classiques.* Springer-Verlag, Berlin-New York, 1971. Troisième édition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5.

[4] Á. Figula, E. Gselmann, G. Horváth, and M. Zoltán. *D1.2: Lie Group Analysis of Biological Systems.* BIOMICS Deliverable, European Commission, 2014.

[5] H. Freudenthal. Oktaven, Ausnahmegruppen und Oktavengeometrie. *Geom. Dedicata*, 19(1):7–63, 1985.

[6] Z. Halasi, G. Horváth, and K. Podoski. *D1.3.1: Introduction to the Chevalley Correspondence between Lie Groups and SNAGs.* BIOMICS Deliverable, European Commission, 2013.

[7] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1978. Second printing, revised.

[8] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1996.

[9] A. L. Onishchik. *Lectures on real semisimple Lie algebras and their representations.* ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2004.

[10] I. R. Porteous. *Topological geometry.* Cambridge University Press, Cambridge-New York, second edition, 1981.

[11] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, and M. Stroppel. *Compact projective planes*, volume 21 of *de Gruyter Expositions in Mathematics.* Walter de Gruyter & Co., Berlin, 1995. With an introduction to octonion geometry.