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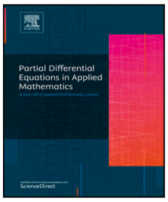

A hybrid analytical method for fractional order Klein–Gordon and Burgers equations

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ABSTRACT

In this study, fractional-order partial differential equations (FPDEs), specifically the Klein–Gordon equation (KGE) and the Burgers equation, are analytically solved using a modified and combined version of the Elzaki Decomposition Technique (ETADM). To assess the efficacy and robustness of the proposed approach, several examples are provided to obtain analytical and numerical results related to the KGE and the Burgers equation. Furthermore, the proposed techniques yield convergent series solutions with well-defined components, without the need for perturbation or linearization. Additionally, we compare several methods for solving differential equations arising in physics and engineering, including ETADM, the Variational Iteration Method (VIM), and the Adomian Decomposition Method (ADM). For comparison and validation, three examples are presented, along with the results obtained using both ETADM and VIM.

1. Introduction

Nonlinear differential equations (NLFDEs), which describe how variables change over time and are often challenging and unpredictable to solve, are most frequently approximated using linear equations. A common fundamental approach to solving NLFDEs is either to transform the problem into a form that can be treated as a linear equation or to modify the variables to simplify the solution process, making it resemble that of a linear equation.[1,2]

Occasionally, the problem is transformed into one or more ordinary differential equations, which may or may not be further solvable. A notable example of a nonlinear system is weather forecasting, where minor changes in one component of the system can lead to complex and unpredictable effects throughout the entire system. Some parameters within this system exhibit completely random behavior, making it impossible to produce precise long-term weather forecasts, even with the use of modern, sophisticated technologies.[3,4] Therefore, studying the exact solutions of nonlinear differential equations (NLDEs) is crucial when analyzing nonlinear systems such as the Van der Pol oscillator, general relativity, nonlinear optics, and others.[5,6]

The Elzaki Transform (ZT), first introduced in Ref. 7, is utilized both independently and in combination with other techniques. The Elzaki method is applied to partial differential equations (PDEs) in Ref. 8, while nonlinear PDEs of fractional order are addressed in Ref. 9. To solve systems of PDEs, the Homotopy Perturbation Method combined with ZT is employed in Ref. 10. A linear fractionally damped oscillator is analytically solved using the ZT approach in Ref. 11. The Elzaki Variational Iteration Method has also been demonstrated in Refs. 5, 12, 13. Moreover, a numerical technique for solving nonlinear PDEs arising in spatial flow, based on the combination of ZT and He's Homotopy Perturbation Method (HPM), is established in Ref. 14. The Elzaki Decomposition Method (EDM), originally developed in Ref. 2, integrates the ZT with the Adomian Decomposition Method. EDM provides a simple and effective approach for solving both linear and nonlinear differential equations. Compared to other methods, it is highly recommended for obtaining solutions to linear and nonlinear differential equations with reduced computational effort.[6,15]

Because PDEs may more accurately simulate a wide range of physical and biological phenomena than conventional differential equations, they have garnered significant attention within the mathematical community. Fractional calculus, in particular, is widely employed in fields such as biology, engineering, and economics.[16–19] Recently, hybrid analytical and numerical methods have gained significant attention for

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solving complex physical, biological, and epidemiological models.[20,21] Various approaches, including Elzaki transform-based techniques and fuzzy fractional numerical schemes, have been developed to solve nonlinearities and memory effects in such systems.[20–22] Many problems in applied science involve linear and nonlinear fractional partial differential equations, and over the past few decades, there have been substantial advancements in the development of analytical, approximate, and numerical methods for solving these types of problems.[23] The ZT Decomposition Method (ZTDM) is an analytical-numerical technique proposed by Targi Elzaki to study solutions analytically. This method is highly efficient and powerful for obtaining exact and approximate solutions to a variety of problems in physics and natural sciences. Techniques such as the Adomian Decomposition Method (ADM), the Variational Iteration Method (VIM), and the Elzaki Transform Adomian Decomposition Method (ETADM) are also notable in this context.[24]

The rest of the paper is structured as follows. In Section 2, we present some definitions and auxiliary results related to the ZT from the literature. In Section 3, we discuss and analyze the convergence of the proposed method. In Section 4, a step-by-step derivation of the proposed ETADM is provided. Section 5 presents the results and discussion of three examples. Finally, the conclusion is given in Section 6.

2. Some properties of ZT

In this section, we recall some definitions of ZT from Ref. 7 that we will use in the derivation of the new modified method.

Definition 2.1. The ZT is a function of s and defined by $E[f]$;

$$E\left[f(t)\right] = T\left(s\right) = s\int_0^{\infty} f\left(\vartheta\right) e^{-\frac{\vartheta}{s}}\, d\vartheta$$

Definition 2.2. ZT represents a function as an infinite sum of terms $f\left(t\right) = \sum_{n=0}^{\infty} a_n t^n$, On the now integral transform ZT given by:

$$E\left[f(t)\right] = T\left(\vartheta\right) = \sum_{n=0}^{\infty} n!\ a_n\ \vartheta^{n+2}$$

Definition 2.3. Let $T_n\left(\vartheta\right)$ denote ZT of nth derivative $f^{(n)}(t)$, then for $n \geq 1$.

$$T_n\left(\vartheta\right) = \frac{T(\vartheta)}{\vartheta^n} - \sum_{k=0}^{n-1} \vartheta^{2-n+k} f^{(k)}\left(0\right)$$

Then ZT of partial derivative define as

$$E\left(\frac{\partial f(x,t)}{\partial t}\right) = \frac{1}{\upsilon}T\left(x,\upsilon\right) - \upsilon f\left(x,0\right)$$
$$E\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = \frac{1}{\upsilon^2}T\left(x,\upsilon\right) - f\left(x,0\right) - \upsilon\frac{\partial f\left(x,0\right)}{\partial t}$$

3. Convergence analysis

The ZT Decomposition Method (ETDTM) is an analytical-numerical approach proposed by Targi Elzaki for studying solutions analytically. This technique is exceptionally efficient and powerful for obtaining exact and approximate solutions to various problems in physics and the natural sciences, providing rapidly convergent successive approximations. The necessary conditions for the convergence of ETDTM are presented here.

Theorem 3.1. *Let us assume Γ is a Banach space (BS). Then, the expansion result of $\varphi\left(s,\vartheta\right)$ converges uncertainty; there become $\delta, 0 < \delta < 1$, so that $\|\varphi_i\left(s,\vartheta\right)\| \leq \delta\|\varphi_{i-1}\left(s,\vartheta\right)\|$, for all $i \in \mathbb{N}$.*

Proof. Consider

$$\Phi_i\left(s,\vartheta\right) = \varphi_0\left(s,\vartheta\right) + \varphi_1\left(s,\vartheta\right) + \varphi_2\left(s,\vartheta\right) + \cdots + \varphi_i\left(s,\vartheta\right) \tag{1}$$

and Cauchy series in BS is

$$\begin{aligned}\|\Phi_{i+1}\left(s,\vartheta\right) - \Phi_i\left(s,\vartheta\right)\| &\leq \|\Phi_{i+1}\left(s,\vartheta\right)\| \leq \delta\|\Phi_i\left(s,\vartheta\right)\| \\ &\leq \delta^2\|\Phi_{i-1}\left(s,\vartheta\right)\| \leq \cdots \\ &\leq \|\Phi_0\left(s,\vartheta\right)\|\end{aligned} \tag{2}$$

we obtained $\forall i,j \in \mathbb{N},\ i \leq j$

$$\begin{aligned}\|\Phi_i\left(s,\vartheta\right) - \Phi_j\left(s,\vartheta\right)\| &\leq \|\Phi_{j+1}\left(s,\vartheta\right) - \Phi_j\left(s,\vartheta\right)\| \\ &\quad + \|\Phi_{j+2}\left(s,\vartheta\right) - \Phi_{j+1}\left(s,\vartheta\right)\| \\ &\quad + \cdots + \|\Phi_i\left(s,\vartheta\right) - \Phi_{i+1}\left(s,\vartheta\right)\|\end{aligned} \tag{3}$$

Using the triangle inequality, then the inequality (3) transforms into the inequality (2) and can be represented as follows

$$\|\Phi_i\left(s,\vartheta\right) - \Phi_j\left(s,\vartheta\right)\| \leq \delta^{j+1}(1 + \delta + \delta^2 + \cdots + \delta^{i-j-1})\|\Phi_0\left(s,\vartheta\right)\|$$

Where

$$\left(\frac{1-\delta^{i-j}}{1-\delta}\right) = 1 + \delta + \delta^2 + \cdots + \delta^{i-j-1}$$

Thus, $\|\Phi_i\left(s,\vartheta\right) - \Phi_j\left(s,\vartheta\right)\| \leq \delta^{j+1}\left(\frac{1-\delta^{i-j}}{1-\delta}\right)\|\Phi_0\left(s,\vartheta\right)\|$

Hence it is acquired as $0 < \delta < 1$, and $1 - \delta^{i-j} \leq 1$. Then

$$\|\Phi_i\left(s,\vartheta\right) - \Phi_j\left(s,\vartheta\right)\| \leq \frac{\delta^{j+1}}{1-\delta}\|\varphi_0\left(s,\vartheta\right)\|$$

Since $\varphi_0\left(s,\vartheta\right)$ is bounded, it is obtained as

$$\lim_{i,j\to\infty}\|\Phi_i\left(s,\vartheta\right) - \Phi_j\left(s,\vartheta\right)\| = 0$$

Thus, $\left\{\Phi_i\right\}$ is a Cauchy series in BS. ☐

4. Solutions of fractional differential equations by ETADM

Consider the fractional non-linear differential equation to illustrate ETADM,

$$D_\tau^\omega \yen(\tau) + \mathcal{N}\yen(\tau) + \mathcal{W}\yen(\tau) = \mathcal{H}\yen(\tau), \tag{4}$$

where $D_\tau^\omega(.)$ the fractional derivative of Caputo type, $\mathcal{N}$ is the linear operator and $\mathcal{W}$ means a non-linear operator with the derivatives of fractional order less than ω.

Taking the ZT of Eq. (4)

$$E(D_\tau^\omega \yen(\tau) + \mathcal{N}\yen(\tau) + \mathcal{W}\yen(\tau)) = E(\mathcal{H}(\tau)),$$

and by ZT

$$E[\yen(\tau)] = \delta^\omega \sum_{k=0}^{m-1} \delta^{2-\omega-k}\yen^{(k)}(0) - \delta^\omega E[\mathcal{N}\yen(\tau) + \mathcal{W}\yen(\tau) - \mathcal{H}(\tau)]. \tag{5}$$

Taking inverse of ZT

$$\yen(\tau) = \mathcal{H}(\tau) - E^{-1}\left[\delta^\omega E[\mathcal{N}\yen(\tau) + \mathcal{W}\yen(\tau)]\right] \tag{6}$$

In this context, $\mathcal{H}(\tau)$ represents the component derived from the source term and the specified initial condition. The solution to Eq. (6) is expressed as an infinite series, detailed below:

$$\yen(\tau) = \sum_{j=0}^{\infty} \yen(\tau) = \yen_0(\tau) + \yen_1(\tau) + \cdots + \yen_j(\tau) + \cdots$$

Now, use decompose $\mathcal{W}\yen(\tau)$ in Adomian polynomials as:

$$\mathcal{W}\yen(\tau) = \sum_{j=0}^{\infty} A_j,$$

where

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\Psi^j}\left(\mathcal{W}\sum_{j=0}^{\infty}(\Psi^j \yen_j)\right)\right]_{\Psi=0}, \quad j = 0, 1, 2, \ldots$$

Substituting in Eq. (6)

$$\sum_{j=0}^{\infty} \yen_{j+1}(\tau) = h(\tau) - E^{-1}\left[\delta^\omega E\left(\mathcal{N}\sum_{j=0}^{\infty}\yen(\tau) + \sum_{j=0}^{\infty} A_j\right)\right],$$

where

$$\yen_0(\tau) = h(\tau) = E^{-1}\left[\delta^{\omega}\sum_{k=0}^{m-1}\delta^{2-\omega-k}\yen^{(k)}(0) + \delta^{\omega}E(\mathcal{H}(\tau))\right],$$

and

$$\yen_{j+1}(\tau) = -E^{-1}\left(\delta^{\omega}E\left[\mathcal{N}\yen(\tau) + A_j\right]\right), \quad j \geq 1.$$

5. Results and discussion

As there are numerous phenomena in physics, biology, engineering and others applied that are represented by some types of ODEs, several other problems are impossible to be dealt with unless using PDEs. In this section, numerical examples will be considered to ensure the accuracy of the proposed ETADM method. We will present multiple examples to demonstrate the accuracy and simplicity of our proposed method. Some examples are given to illustrate the iteration of ETADM for solving PDE's.

Example 5.1. Consider the following equation

$$\yen_J(\vartheta, J) + \yen(\vartheta, J)\yen_{\vartheta} - \yen_{\vartheta\vartheta} = 0, \tag{7}$$

with the initial condition $\yen(\vartheta, 0) = \vartheta$.

By applying Adomian Decomposition method as follows:

$$L_J^{-1} = \int_0^J (.)\,dJ.$$

Taking $L_J^{-1}$to both side of above equation

$$L_J^{-1}\left(\yen_J(\vartheta, J) + \yen(\vartheta, J)\yen_{\vartheta} - \yen_{\vartheta\vartheta}\right) = 0$$
$$\yen(\vartheta, J) - \yen(\vartheta, 0) = -L_J^{-1}\left(\yen(\vartheta, J)\yen_{\vartheta} + \yen_{\vartheta\vartheta}\right).$$

The symbol $\yen\yen_{\vartheta}$ represents a nonlinear term that can be expressed using a series of Adomian polynomials.

$$\yen\yen_{\vartheta} = \sum_{m=0}^{\infty} A_m(\yen)$$
$$\sum_{m=0}^{\infty}\yen_m = \vartheta - L_J^{-1}\left(\sum_{m=0}^{\infty} A_m(\yen) + \sum_{m=0}^{\infty}(\yen)_{\vartheta\vartheta}\right)$$
$$\yen_0(\vartheta, J) = \vartheta$$
$$\yen_1(\vartheta, J) = -\vartheta J$$
$$\yen_2(\vartheta, J) = \vartheta J^2$$
$$\yen_3(\vartheta, J) = -\vartheta J^3$$
$$\yen_4(\vartheta, J) = \vartheta J^4$$
$$\yen_5(\vartheta, J) = -\vartheta J^5.$$

The series solution is given by

$$\yen(\vartheta, J) = \sum_{m=0}^{\infty}\yen_m = \vartheta(1 - J + J^2 - J^3 + \cdots), \tag{8}$$

which its closed form solution is

$$\yen(\vartheta, J) = \frac{\vartheta}{1 - J}. \tag{9}$$

Now, by applying the ZT

$$E_J\left(\yen_t(\vartheta, J) + \yen(\vartheta, J)\yen_{\vartheta} - \yen_{\vartheta\vartheta}\right) = 0$$
$$\frac{1}{s}E_J\left(\yen(\vartheta, J)\right) - s\yen(\vartheta, 0) = -E_t\left(\yen(\vartheta, J)\yen_{\vartheta} + \yen_{\vartheta\vartheta}\right).$$

Taking the inverse Elzaki

$$\yen(\vartheta, J) = \vartheta - E^{-1}\{sE_J\left(\yen(\vartheta, J)\yen_{\vartheta} + \yen_{\vartheta\vartheta}\right)\}$$
$$\sum_{m=0}^{\infty}\yen_m = \vartheta - E^{-1}\left[sE_J\left(\sum_{m=0}^{\infty} A_m(\yen) + \sum_{m=0}^{\infty}(\yen)_{\vartheta\vartheta}\right)\right].$$

The first few components of $A_m(\yen)$ polynomials is given by (8) which its closed form solution is (9)

Now, by using the variational iteration method

$$\yen_{n+1}(\vartheta, J) = \yen_n(\vartheta, J) - \int_0^J \left[\left(\yen_n(\vartheta, s)\right)_s + \yen_n(\vartheta, s)\left(\yen_n(\vartheta, s)\right)_{\vartheta} - \left(\yen_n(\vartheta, s)\right)_{\vartheta\vartheta}\right] ds.$$

$$\yen_0(\vartheta, J) = \vartheta$$
$$\yen_1(\vartheta, J) = \yen_0(\vartheta, J) - \int_0^J \left[\left(\yen_0(\vartheta, s)\right)_s + \yen_0(\vartheta, s)\left(\yen_0(\vartheta, s)\right)_{\vartheta} - \left(\yen_0(\vartheta, s)\right)_{\vartheta\vartheta}\right] ds$$
$$= \vartheta - \vartheta J$$
$$\yen_2(\vartheta, J) = \vartheta - \vartheta J + \vartheta J^2 - \frac{\vartheta J^3}{3}$$
$$\yen_3(\vartheta, J) = \vartheta - \vartheta J + \vartheta J^2 - \vartheta J^3 + \frac{11\vartheta J^4}{12} - \frac{13\vartheta J^5}{30} + \frac{\vartheta J^6}{6} - \frac{\vartheta J^7}{36}$$
$$\yen_4(\vartheta, J) = \vartheta - \vartheta J + \vartheta J^2 - \vartheta J^3 + \vartheta J^4 - \frac{29\vartheta J^5}{30} + \frac{47\vartheta J^6}{60} - \frac{121\vartheta J^7}{210} + \cdots$$
$$\yen_5(\vartheta, J) = \vartheta - \vartheta J + \vartheta J^2 - \vartheta J^3 + \vartheta J^4 - \vartheta J^5 + \cdots.$$

The series solution $\yen(\vartheta, J) = \yen_m$ is given by (8) which its closed form solution is (9). Figs. 1 and 2 show the approximate solutions y_4 and y_5 and Fig. 3 shows the absolute error.

The comparison between Adomian Decomposition and Elzaki with variational iteration method for Burgers equation in Example 5.1 shows that the Adomian Decomposition method is better than variational iteration method as we can notice that in Figs. 1 and 2 for the approximate solutions y_4 and y_5 and Fig. 3 for the absolute error.

Example 5.2. Consider the following equation

$$y_{JJ}(\vartheta, J) - y_{\vartheta\vartheta}(\vartheta, J) + y(\vartheta, J) = 2\sin\sin\vartheta, \tag{10}$$

with the initial conditions $y(\vartheta, 0) = \sin\vartheta,\ y_J(\vartheta, 0) = 1$.

By applying Adomian Decomposition method as follows

$$L_J^{-1} = \int_0^J \int_0^J (.)\,dJ\,dJ.$$

Applying L_J^{-1} to both side of above equation

$$L_J^{-1}\left(y_{JJ}(\vartheta, J) - y_{\vartheta\vartheta}(\vartheta, t) + y(\vartheta, J)\right) = L^{-1}(2\sin\sin\vartheta),$$
$$y(\vartheta, J) = y(\vartheta, 0) + Jy_J(\vartheta, 0) + L_J^{-1}(2\sin\sin\vartheta) + L_J^{-1}\left(y_{\vartheta\vartheta}(\vartheta, J) - y(\vartheta, J)\right),$$
$$\sum_{m=0}^{\infty} y_m = \sin\vartheta + J + J^2\sin\sin\vartheta + L_J^{-1}\left(\sum_{m=0}^{\infty}(y_m)_{\vartheta\vartheta} - \sum_{m=0}^{\infty} y_m\right),$$

$$y_0(\vartheta, J) = \sin\vartheta + J + J^2\sin\sin\vartheta$$
$$y_1(\vartheta, J) = -J^2\sin\sin\vartheta - \frac{J^4\sin\sin\vartheta}{6} - \frac{J^3}{3!}$$
$$y_2(\vartheta, J) = \frac{J^4\sin\sin\vartheta}{6} + \frac{J^6\sin\sin\vartheta}{90} + \frac{J^5}{5!}$$
$$y_3(\vartheta, J) = -\frac{J^6\sin\sin\vartheta}{90} - \frac{J^8\sin\sin\vartheta}{2520} - \frac{J^7}{7!}$$
$$y_4(\vartheta, J) = -\frac{J^8\sin\sin\vartheta}{2520} + \frac{J^{10}\sin\sin\vartheta}{113400} + \frac{J^9}{9!}$$
$$y_5(\vartheta, J) = -\frac{J^{10}\sin\sin\vartheta}{113400} - \frac{J^{12}\sin\sin\vartheta}{7484400} - \frac{J^{11}}{11!}.$$

The series solution is given by

$$y(\vartheta, J) = \sum_{m=0}^{\infty} y_m = \sin\vartheta + \left(J - \frac{J^3}{3!} + \frac{J^5}{5!} - \frac{J^7}{7!} + \frac{J^9}{9!} - \frac{J^{11}}{11!} + \cdots\right), \tag{11}$$

which its closed form solution is

$$y(\vartheta, J) = \sin\vartheta + \sin J. \tag{12}$$

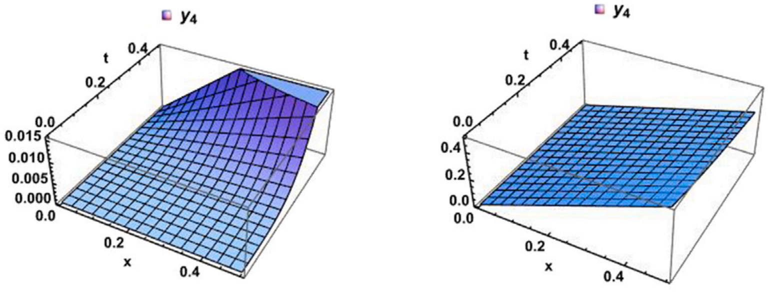

Fig. 1. Comparison of approximate solutions between the ADM (Left) and ZT with the VIM (Right) for y_4.

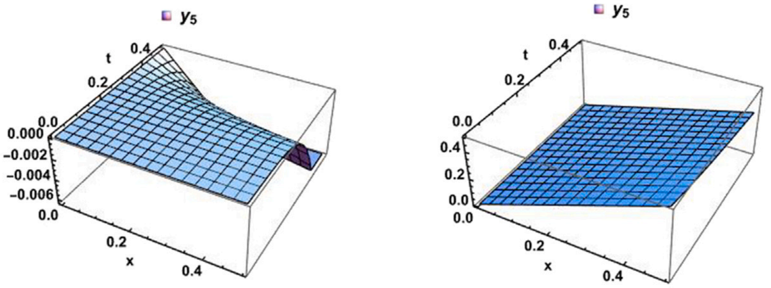

Fig. 2. Comparison of approximate solutions between the ADM (Left) and ZT with the VIM (Right) for y_5.

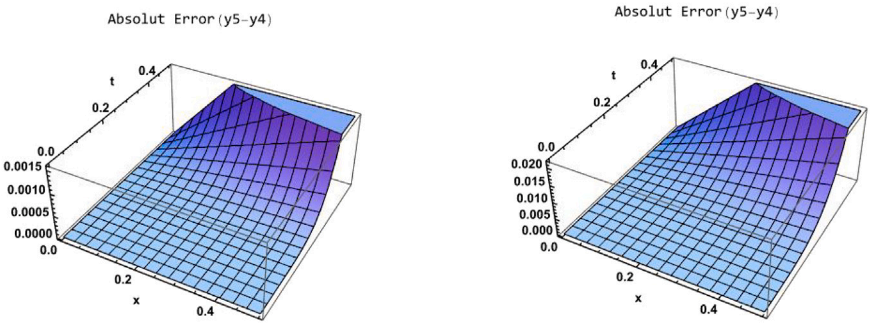

Fig. 3. Left: ADM and Elzaki, Right: VIM. Comparison of sequential error between the ADM and ZT with VIM for y_4 and y_5 and the results of the left side are the best.

Now, by applying the ZT

$$E_J\left(y_{JJ}(\vartheta,J)-y_{\vartheta\vartheta}(\vartheta,J)+y(\vartheta,J)\right)=E_J(2\sin\sin\vartheta),$$

$$\frac{1}{s^2}E_J\left(y(\vartheta,J)\right)-y(\vartheta,0)-sy_J(\vartheta,0)=E_J\left(y_{\vartheta\vartheta}(\vartheta,J)-y(\vartheta,J)\right)+E_J(2\sin\sin\vartheta\ ),$$

Taking the inverse Elzaki

$$y(\vartheta,t)=sin\vartheta+J+J^2\sin\sin\vartheta+E^{-1}\left\{s^2E_J\left(y_{\vartheta\vartheta}(\vartheta,J)-y(\vartheta,J)\right)\right\}$$

$$\sum_{m=0}^{\infty}y_m=sin\vartheta+J+J^2\sin\sin\vartheta+\left\{s^2E_J\left(\sum_{m=0}^{\infty}\left(y_m\right)_{\vartheta\vartheta}-\sum_{m=0}^{\infty}y_m\right)\right\}.$$

Then

$$y_0(\vartheta,J)=sin\vartheta+J+J^2\sin\sin\vartheta$$

$$y_1(\vartheta,J)=-J^2\sin\sin\vartheta-\frac{J^4\sin\sin\vartheta}{6}-\frac{J^3}{3!}$$

$$y_2(\vartheta,J)=\frac{J^4\sin\sin\vartheta}{6}+\frac{J^6\sin\sin\vartheta}{90}+\frac{J^5}{5!}$$

$$y_3(\vartheta,J)=-\frac{J^6\sin\sin\vartheta}{90}-\frac{J^8\sin\sin\vartheta}{2520}-\frac{J^7}{7!}$$

$$y_4(\vartheta,J)=\frac{J^8\sin\sin\vartheta}{2520}+\frac{J^{10}\sin\sin\vartheta}{113400}+\frac{J^9}{9!}$$

$$y_5(\vartheta,J)=-\frac{J^{10}\sin\sin\vartheta}{113400}-\frac{J^{12}\sin\sin\vartheta}{7484400}-\frac{J^{11}}{11!}.$$

The series solution is given by (11) which its closed form solution is (12). Now, we continue by using the variational iteration method

$$y_{n+1}(\vartheta,J)=y_n(\vartheta,J)+\int_0^J\left((J-s)\left(y_n(\vartheta,s)\right)_{ss}-\left(y_n(\vartheta,s)\right)_{\vartheta\vartheta}+y_n(\vartheta,s)-2\sin\sin\vartheta\right)ds,$$

$$y_0(\vartheta,J)=sin\vartheta+J,$$

$$y_1(\vartheta,J)=y_0(\vartheta,J)+\int_0^J\left((s-J)\left(y_0(\vartheta,s)\right)_{ss}-\left(y_0(\vartheta,s)\right)_{\vartheta\vartheta}+y_0(\vartheta,s)-2\sin\sin\vartheta\right)ds$$

$$y_1(\vartheta,J)=sin\vartheta+J-\frac{J^3}{3!}$$

$$y_2(\vartheta,J)=sin\vartheta+J-\frac{J^3}{3!}+\frac{J^5}{5!}$$

$$y_3(\vartheta,J)=sin\vartheta+J-\frac{J^3}{3!}+\frac{J^5}{5!}-\frac{J^7}{7!}$$

$$y_4(\vartheta,J)=sin\vartheta+J-\frac{J^3}{3!}+\frac{J^5}{5!}-\frac{J^7}{7!}+\frac{J^9}{9!}$$

$$y_5(\vartheta,J)=sin\vartheta+J-\frac{J^3}{3!}+\frac{J^5}{5!}-\frac{J^7}{7!}+\frac{J^9}{9!}-\frac{J^{11}}{11!}.$$

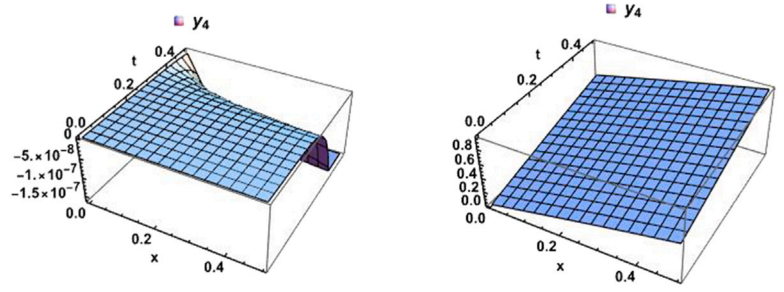

Fig. 4. Comparison of approximate solutions between the ADM (Left) and ZT with the VIM (Right) for y_4.

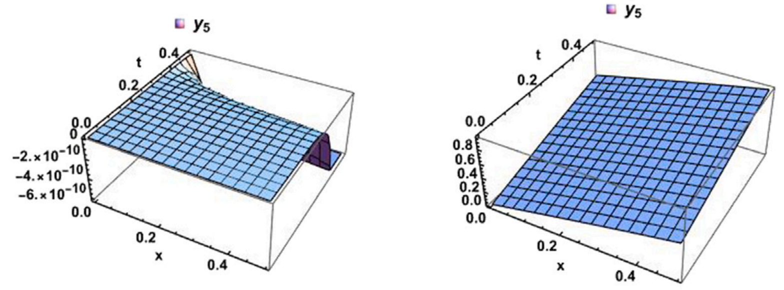

Fig. 5. Comparison of approximate solutions between the ADM (Left) and ZT with the VIM (Right) for y_5.

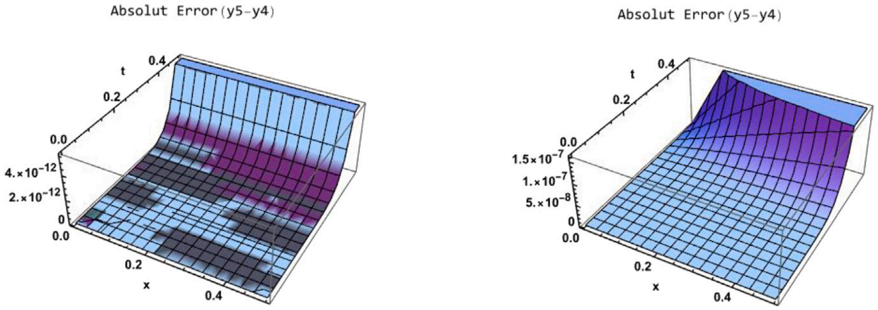

Fig. 6. Comparison of sequential error between the ADM and ZT (Left) with VIM (Right) for y_4 and y_5 and this shows the results to the left side are the best.

The series solution $y(\vartheta, J) = y_m$ is given by (11) which its closed form solution is (12). Figs. 4 and 5 show the approximate solutions y_4 and y_5 and Fig. 6 shows the absolute error.

The comparison between Adomian Decomposition and Elzaki with variational iteration method for linear non-homogeneous KGE in Example 5.2 show that the Adomian Decomposition method is better than variational iteration method as we can notice in Figs. 4 and 5 for the approximate solutions y_4 and y_5 and Fig. 6 the absolute error.

Example 5.3. Consider the following equation

$$\Psi_{JJJ}(s,J) - 3\Psi(s,J)\Psi_{ss}(s,J) = 0, \tag{13}$$

with the initial conditions $\Psi(s,0) = \frac{1}{s}$, $\Psi_J(s,0) = \frac{1}{s^2}, \Psi_{JJ}(s,0) = \frac{2}{s^3}$.

Again, we start by applying Adomian Decomposition method as follows

$$L_J^{-1} = \int_0^J \int_0^J \int_0^J (.)\, dJ\, dJ\, dJ.$$

Applying L_J^{-1} to both side of above equation

$$L_J^{-1}(\Psi_{JJJ}(s,J) - 3\Psi(s,J)\Psi_{ss}(s,J)) = 0$$

$$\Psi(s,J) - \ \Psi(s,0) - J\Psi_J(s,0) - \frac{J^2}{2}\Psi_{JJ}(s,0) = L_J^{-1}(3\Psi(s,J)\Psi_{ss}(s,J)).$$

The symbol $\Psi\Psi_{ss}$ represents a nonlinear term that can be expressed using a series of Adomian polynomials.

$$\Psi\Psi_{ss} = \sum_{m=0}^{\infty} A_m(\Psi), \quad \sum_{m=0}^{\infty} \Psi_m = \frac{1}{s} + \frac{J}{s^2} + \frac{J^2}{s^3} + L_J^{-1}\left(3\sum_{m=0}^{\infty} A_m(\Psi)\right)$$

$$\begin{aligned}
\Psi_0 &= \frac{1}{s} + \frac{J}{s^2} + \frac{J^2}{s^3}, \quad \Psi_1 = L_J^{-1}(3A_0) = \frac{J^3}{s^4} + \frac{J^4}{s^5} + \frac{J^5}{s^6} + \frac{9J^6}{20s^7} + \frac{6J^7}{35s^8}\\
\Psi_2 &= \frac{11J^6}{20s^7} + \frac{29J^7}{35s^8} + \frac{J^8}{s^9} + \frac{387J^9}{560s^{10}} + \frac{2207J^{10}}{5600s^{11}} + \frac{513J^{11}}{3850s^{12}} + \frac{9J^{12}}{275s^{13}}\\
\Psi_3 &= \frac{173J^9}{560s^{10}} + \frac{3393J^{10}}{5600s^{11}} + \frac{3337J^{11}}{3850s^{12}} + \frac{873J^{12}}{1100s^{13}} + \frac{928049J^{13}}{1601600s^{14}}\\
&\quad + \frac{1363431J^{14}}{4484480s^{15}} + \cdots\\
\Psi_4 &= \frac{191J^{12}}{1100s^{13}} + \frac{673551J^{13}}{1601600s^{14}} + \frac{3121049J^{14}}{4484480s^{15}} + \frac{2173581J^{15}}{2802800s^{16}}\\
&\quad + \frac{8764303J^{16}}{12812800s^{17}} + \cdots\\
\Psi_5 &= \frac{1371367J^{15}}{14014000s^{16}} + \frac{126317043J^{16}}{448448000s^{17}} + \frac{1615171837J^{17}}{3049446400s^{18}}\\
&\quad + \frac{4202009943J^{18}}{6098892800s^{19}} + \cdots.
\end{aligned}$$

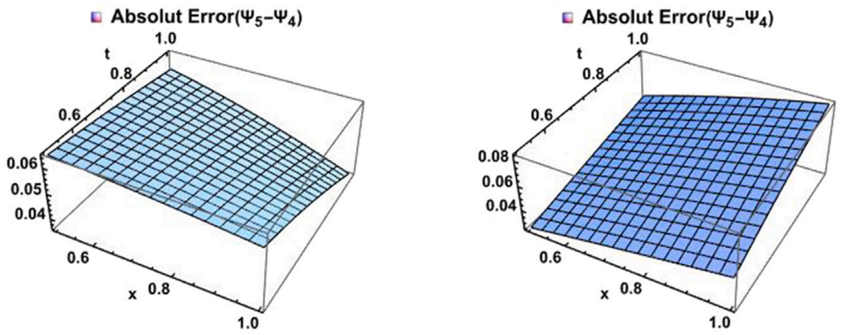

Fig. 7. Comparison of sequential error between ADM and ZT (Left) with the VIM (Right) for Ψ_4 and Ψ_5 and this shows that the results to the left side are the best.

The series solution is given by

$$\Psi(s,J)=\sum_{m=0}^{\infty}\Psi_m=\frac{1}{s}1+\frac{J}{s}+\left(\frac{J}{s}\right)^2+\left(\frac{J}{s}\right)^3+\left(\frac{J}{s}\right)^4+\left(\frac{J}{s}\right)^5+\cdots, \tag{14}$$

which its closed form solution is

$$\Psi(s,J)=\frac{1}{s+J}. \tag{15}$$

Now, by applying the ZT and by taking ZT of the previous equation and using its properties, we get

$$\sum_{m=0}^{\infty}\Psi_m(s,J)=\frac{1}{s}+\frac{1}{s^2}+\frac{2}{s^3}+3E^{-1}\left(v^3E\left(\sum_{m=0}^{\infty}A_m(\Psi)\right)\right),$$

$$\Psi_0=\frac{1}{s}+\frac{J}{s^2}+\frac{J^2}{s^3},$$

$$\Psi_1=\frac{1}{s}\left[\frac{J^3}{s^3}+\frac{J^4}{s^4}+\frac{J^5}{s^5}+\frac{9J^6}{20s^6}+\frac{6J^7}{35s^7}\right],$$

$$\Psi_2=\frac{1}{s}\left[\frac{11J^6}{20s^6}+\frac{29J^7}{35s^7}+\frac{J^8}{s^8}+\frac{387J^9}{560s^9}+\frac{2207J^{10}}{5600s^{10}}+\frac{513J^{11}}{3850s^{11}}+\frac{63J^{12}}{275s^{12}}\right]$$

$$\Psi_3=\frac{173J^9}{560s^{10}}+\frac{3393J^{10}}{5600s^{11}}+\frac{3337J^{11}}{3850s^{12}}+\frac{873J^{12}}{1100s^{13}}+\frac{928049J^{13}}{1601600s^{14}}+\frac{1363431J^{14}}{4484480s^{15}}+\cdots$$

$$\Psi_4=\frac{191J^{12}}{1100s^{13}}+\frac{673551J^{13}}{1601600s^{14}}+\frac{3121049J^{14}}{4484480s^{15}}+\frac{2173581J^{15}}{2802800s^{16}}+\frac{8764303J^{16}}{12812800s^{17}}+\cdots$$

$$\Psi_5=\frac{1371367J^{15}}{14014000s^{16}}+\frac{126317043J^{16}}{448448000s^{17}}+\frac{1615171837J^{17}}{3049446400s^{18}}+\frac{4202009943J^{18}}{6098892800s^{19}}+\cdots.$$

The series solution is given by (14) which its closed form solution is (15).

Now, by using the variational iteration method

$$\Psi_{n+1}(s,J)=\Psi_n(s,J)+\int_0^J\left[\left(\frac{(J-\vartheta)^2}{2}\right)\left(\Psi_n(s,J)\right)_{\vartheta\vartheta\vartheta}-3\Psi_n(s,J)\left(\Psi_n(s,J)\right)_{ss}\right]d\vartheta$$

$$\Psi_0=\sum_{k=0}^{2}\frac{J^k}{s^{k+1}},$$

$$\Psi_1=\sum_{k=0}^{5}\frac{J^k}{s^{k+1}}+\frac{9J^6}{20s^7}+\frac{6J^7}{35s^8},$$

$$\Psi_2=\sum_{k=0}^{8}\frac{J^k}{s^{k+1}}+\frac{1361J^9}{1680s^{10}}+\frac{10121J^{10}}{16800s^{11}}+\frac{4759J^{11}}{11550s^{12}}+\cdots,$$

$$\Psi_3=\sum_{k=0}^{11}\frac{J^k}{s^{k+1}}+\frac{571J^{12}}{600s^{13}}+\cdots,$$

$$\Psi_4=\sum_{k=0}^{14}\frac{J^k}{s^{k+1}}+\frac{15J^{15}}{28s^{16}}+\cdots,$$

$$\Psi_5=\sum_{k=0}^{15}\frac{J^k}{s^{k+1}}+\frac{3000J^{16}}{27534s^{17}}+\cdots.$$

The series solution $\Psi(s,J)=\Psi_m$ is given by (14) which its closed form solution is (15). Foe the comparison, Fig. 7 shows the absolute error.

The comparison between ADM and Elzaki with VIM in Example 5.2 shows that the ADM is better than VIM as we can notice that in Fig. 7 for the absolute error.

6. Conclusion

In this study, we have presented approximate-analytical solutions for fractional order Klein–Gordon and Burgers equations using a hybrid analytical approach, the Elzaki Decomposition Technique with a modified structure (ETADM). The accuracy, efficiency, and reliability of ETADM were evaluated using a comparative analysis with Variational Iteration Method (VIM) and Adomian Decomposition Method (ADM). The obtained results demonstrate that ETADM gives highly accurate solutions with minimal computational effort while avoiding discretization, perturbation, or linearization. Moreover, the method is straightforward, computationally efficient, and capable of handling both linear and nonlinear fractional PDEs. The obtained solutions in series form further give the robustness of the approach, making ETADM an effective method for solving complex differential equations. Future research may study extending this method to a broader class of nonlinear fractional partial differential equations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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