Connection between deriving bridges and radial parts from multidimensional Ornstein-Uhlenbeck processes

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Abstract

First we give a construction of bridges derived from a general Markov process using only its transition densities. We give sufficient conditions for their existence and uniqueness (in law). Then we prove that the law of the radial part of the bridge with endpoints zero derived from a special multidimensional Ornstein-Uhlenbeck process equals the law of the bridge with endpoints zero derived from the radial part of the same Ornstein-Uhlenbeck process. We also construct bridges derived from general multidimensional Ornstein-Uhlenbeck processes.

1 Introduction

In this paper we are dealing with deriving bridges and radial parts from Markov processes. By a bridge from a to b over $[0,T]$ derived from a Markov process Z we mean a process obtained by conditioning Z to start in a at time 0 and arrive at b at time T, where $T > 0$. For the construction of such a bridge we use only transition densities. Important examples are provided by Wiener bridges and Bessel bridges, which have been extensively studied and find numerous applications. See, for example, Karlin and Taylor [8, Chapter 15], Fitzsimmons, Pitman and Yor [3], Baudoin [1], Privault and Zambrini [9] and Yor and Zambotti [11]. Our construction of bridges is motivated by Karlin and Taylor [8] and Revuz and Yor [10]. By the radial part of a process with values in $\mathbb{R}^d$ we mean its euclidean norm.

We examine whether the operations deriving bridges and radial parts commute starting from the same Markov process. In case of a multidimensional standard Wiener process and in case of certain multidimensional Ornstein-Uhlenbeck processes we show that the answer is yes if we consider bridges with endpoints zero. We emphasize that Yor and Zambotti in [11] have already proved this for a multidimensional standard Wiener process. Moreover, they showed that the law of the radial part of the multidimensional Wiener bridge with endpoints different from zero is only equivalent and not equal to the law of the corresponding multidimensional Bessel bridge.

We proceed as follows. In Section 2 we give a construction of a bridge derived from a general Markov process using its transition densities. We give sufficient conditions for its

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existence and uniqueness (in law). In Sections 3 and 5 we prove that the operations deriving bridges and radial parts commute starting from multidimensional standard Wiener processes and from certain multidimensional Ornstein-Uhlenbeck processes, respectively. In Section 4 we study bridges derived from general multidimensional Ornstein-Uhlenbeck processes.

2 Construction of bridges

In what follows, let $(E, \mathcal{E})$ be a complete separable metric space endowed with the σ-algebra of its Borel subsets, let $T > 0$, let $(Z_t)_{0 \leqslant t \leqslant T}$ be a time-homogeneous Markov process with state space $(E, \mathcal{E})$ admitting transition densities $(p_t^Z)_{0 < t \leqslant T}$ with respect to a fixed σ-finite measure λ on $\mathcal{E}$ (i.e., $\mathsf{P}(Z_t \in A \mid Z_s) = \int_A p_{t-s}^Z(Z_s, y) \, \lambda(\mathsf{d}y)$ P-a.s. for all $A \in \mathcal{E}$, $0 \leqslant s < t \leqslant T$), and let $a, b \in E$.

If $p_t^Z(x, b) > 0$ for all $x \in E$, $0 < t \leqslant T$, and

$$p_{s,t}(x, y) := \frac{p_{t-s}^Z(x, y) p_{T-t}^Z(y, b)}{p_{T-s}^Z(x, b)}, \qquad x, y \in E, \quad 0 \leqslant s < t < T, \tag{2.1}$$

then by a bridge from a to b over $[0, T]$ derived from Z we could understand a Markov process $(Y_t)_{0 \leqslant t \leqslant T}$ with initial distribution $\mathsf{P}(Y_0 = a) = 1$ and with transition densities $(p_{s,t})_{0 \leqslant s < t < T}$, provided that such a process exists (see, e.g., Fitzsimmons, Pitman and Yor [3, Proposition 1], Fitzsimmons [2, Proposition 2.2]). But this definition does not apply for example in case of a d-dimensional Bessel bridge with $d > 1$ and with $b = 0$, since for the transition densities $(p_t^R)_{t > 0}$ of the d-dimensional Bessel process $(R_t)_{t \geqslant 0}$ we have $p_t^R(x, 0) = 0$ for all $x \geqslant 0$, $t > 0$ (see, e.g., Revuz and Yor [10, p. 446] or Section 3).

The motivation how to modify (2.1) is inspired by Karlin and Taylor [8, p. 267] and Revuz and Yor [10, Chapter XI, §3]. For $\varepsilon > 0$, denote by $B(b, \varepsilon)$ the open ball in E with centre at b and radius ε. Let $(Y_t^\varepsilon)_{0 \leqslant t \leqslant T}$ denote $(Z_t)_{0 \leqslant t \leqslant T}$ conditioned that $Z_T \in B(b, \varepsilon)$. In virtue of Karlin and Taylor [8, (9.17)], for $x, y \in E$, $0 \leqslant s < t < T$, the transition densities of Y^ε are given by

$$p_{s,t}^{Y^\varepsilon}(x, y) = p_{t-s}^Z(x, y) \frac{\int_{B(b,\varepsilon)} p_{T-t}^Z(y, z) \, \lambda(\mathsf{d}z)}{\int_{B(b,\varepsilon)} p_{T-s}^Z(x, z) \, \lambda(\mathsf{d}z)},$$

provided that $\int_{B(b,\varepsilon)} p_{T-s}^Z(x, z) \, \lambda(\mathsf{d}z) \neq 0$. Indeed, by Proposition 7.2 in Kallenberg [5],

$$\begin{aligned}
\mathsf{P}(Y_t^\varepsilon \in A \mid Y_s^\varepsilon = x) = \mathsf{P}(Z_t \in A \mid Z_s = x, \, Z_T \in B(b, \varepsilon)) &= \frac{\mathsf{P}(Z_t \in A, \, Z_T \in B(b, \varepsilon) \mid Z_s = x)}{\mathsf{P}(Z_T \in B(b, \varepsilon) \mid Z_s = x)} \\
&= \frac{\int_A \int_{B(b,\varepsilon)} p_{t-s}^Z(x, y) p_{T-t}^Z(y, z) \, \lambda(\mathsf{d}y) \lambda(\mathsf{d}z)}{\int_{B(b,\varepsilon)} p_{T-s}^Z(x, z) \, \lambda(\mathsf{d}z)} = \int_A p_{s,t}^{Y^\varepsilon}(x, y) \, \lambda(\mathsf{d}y)
\end{aligned}$$

for all $A \in \mathcal{E}$. We can think of the desired bridge as the limit of Y^ε as $\varepsilon \downarrow 0$, hence our definition is the following.

2.1 Definition. *For* $x, y \in E$ *and* $0 \leqslant s < t < T$, *let*

$$p_{s,t}(x,y) := p^Z_{t-s}(x,y) \lim_{\varepsilon \downarrow 0} \frac{\int_{B(b,\varepsilon)} p^Z_{T-t}(y,z)\,\lambda(\mathsf{d}z)}{\int_{B(b,\varepsilon)} p^Z_{T-s}(x,z)\,\lambda(\mathsf{d}z)} \tag{2.2}$$

if the right hand side exists, and $p_{s,t}(x,y) := 0$ *otherwise.*

By a bridge from a *to* b *over* $[0,T]$ *derived from* Z *we mean a Markov process* $(Y_t)_{0 \leqslant t \leqslant T}$ *with initial distribution* $\mathsf{P}(Y_0 = a) = 1$, *with* $\mathsf{P}(Y_T = b) = 1$ *and with transition densities* $(p_{s,t})_{0 \leqslant s < t < T}$ *provided that such a process exists.*

Note that the Markov process $(Y_t)_{0 \leqslant t \leqslant T}$ (if it exists) is in general not time-homogeneous. Moreover, additional conditions on $(p^Z_t)_{0 < t \leqslant T}$ are needed to assure that $(Y_t)_{0 \leqslant t \leqslant T}$ admits a version having sample paths with some regularity properties such as continuity.

2.2 Lemma. *Suppose that* $(p_{s,t})_{0 \leqslant s < t < T}$ *defined by* (2.2) *satisfy the following properties:*

(i) *for all* $0 \leqslant s < t < T$, *the function* $(x,y) \mapsto p_{s,t}(x,y)$ *is measurable,*

(ii) *for all* $x \in E$ *and* $0 \leqslant s < t < T$, *the function* $y \mapsto p_{s,t}(x,y)$ *is a probability density,*

(iii) *for all* $x, z \in E$ *and* $0 \leqslant s < t < u < T$, *the Kolmogorov-Chapman equation* $p_{s,u}(x,z) = \int_E p_{s,t}(x,y) p_{t,u}(y,z)\,\lambda(\mathsf{d}y)$ *holds.*

Then there exists a unique probability measure $\mathsf{P}^Z_{a,b,T}$ *on* $\big(E^{[0,T]}, \mathcal{E}^{[0,T]}\big)$ *such that the coordinate process* $(X_t)_{0 \leqslant t \leqslant T}$ *on* $\big(E^{[0,T]}, \mathcal{E}^{[0,T]}\big)$ *under* $\mathsf{P}^Z_{a,b,T}$ *is a bridge from* a *to* b *over* $[0,T]$ *derived from* Z.

Consequently, if $(Y_t)_{0 \leqslant t \leqslant T}$ *is a bridge from* a *to* b *over* $[0,T]$ *derived from* Z *then its law on* $\big(E^{[0,T]}, \mathcal{E}^{[0,T]}\big)$ *is* $\mathsf{P}^Z_{a,b,T}$.

Proof. For $x \in E$, $A \in \mathcal{E}$ and $0 \leqslant s < t < T$, let $\mu_{s,t}(x,A) := \int_A p_{s,t}(x,y)\,\lambda(\mathsf{d}y)$, $\mu_{s,T}(x,A) := \mathbb{1}_A(b)$, where $\mathbb{1}_A$ denotes the indicator function of the set A. Then $\mu_{s,t}$ is a transition probability for all $0 \leqslant s < t \leqslant T$, and one can check easily that the Kolmogorov-Chapman equation $\mu_{s,u}(x,A) = \int_E \mu_{s,t}(x,\mathsf{d}y)\mu_{t,u}(y,A)$ holds for all $x \in E$, $A \in \mathcal{E}$, $0 \leqslant s < t < u \leqslant T$. By Revuz and Yor [10, Chapter III, Theorem 1.5], there exists a unique probability measure $\mathsf{P}^Z_{a,b,T}$ on $\big(E^{[0,T]}, \mathcal{E}^{[0,T]}\big)$ such that the coordinate process $(X_t)_{0 \leqslant t \leqslant T}$ is Markov under $\mathsf{P}^Z_{a,b,T}$ with transition probabilities $(\mu_{s,t})_{0 \leqslant s < t \leqslant T}$ and with initial distribution $\mathsf{P}^Z_{a,b,T}(X_0 = a) = 1$.

Moreover, $\mathsf{P}^Z_{a,b,T}(X_T = b) = \mathsf{P}^Z_{a,b,T}(X_T = b \,|\, X_0 = a) = \mu_{0,T}(a, \{b\}) = 1$. □

The proof of the next lemma is trivial.

2.3 Lemma. *If* $f : E \to \mathbb{R}$ *is a continuous function then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\lambda(B(x,\varepsilon))} \int_{B(x,\varepsilon)} f(z)\,\lambda(\mathsf{d}z) = f(x) \tag{2.3}$$

for all $x \in E$. *Consequently, if* $f, g : E \to \mathbb{R}$ *are continuous functions such that* $f(z) = g(z)$ λ*-a.e.* $z \in E$ *then* $f(z) = g(z)$ *for all* $z \in E$.

2.4 Lemma. *If for each* $0 < t \leqslant T$, *the probability density* p_t^Z *satisfies the properties*

(i) *the function* $(x, y) \mapsto p_t^Z(x, y)$ *is continuous,*

(ii) *for all* $x_0 \in E$, *there is a* $\delta > 0$ *such that* $\sup\limits_{x \in B(x_0, \delta)} \sup\limits_{y \in E} p_t^Z(x, y) < \infty$,

(iii) *for all* $y_0 \in E$, *there is a* $\delta > 0$ *such that* $\sup\limits_{x \in E} \sup\limits_{y \in B(y_0, \delta)} p_t^Z(x, y) < \infty$,

(iv) *for all* $y \in E$, *we have* $\int_E p_t^Z(x, y)\, \lambda(\mathsf{d}x) < \infty$,

then the Kolmogorov-Chapman equation

$$p_{s+t}^Z(x, z) = \int_E p_s^Z(x, y) p_t^Z(y, z)\, \lambda(\mathsf{d}y) \tag{2.4}$$

holds for all $x, z \in E$ *and all* $s, t > 0$ *with* $s + t \leqslant T$. (Compare with Fitzsimmons, Pitman and Yor [3, (2.3)], Fitzsimmons [2, (1.9)].)

Proof. For $x \in E$, $A \in \mathcal{E}$ and $0 < t \leqslant T$, let $\mu_t^Z(x, A) := \int_A p_t^Z(x, y)\, \lambda(\mathsf{d}y)$. Let us fix $s, t > 0$ with $s + t \leqslant T$. Then for all $A \in \mathcal{E}$ the Kolmogorov-Chapman equation $\mu_{s+t}^Z(x, A) = \int_E \mu_s^Z(x, \mathsf{d}y) \mu_t^Z(y, A)$ holds for P_{Z_s}-a.e. $x \in E$, where P_{Z_s} denotes the distribution of Z_s (see, e.g., Kallenberg [5, Corollary 7.3]). Thus for all $A \in \mathcal{E}$

$$\begin{aligned}
\int_A p_{s+t}^Z(x, z)\, \lambda(\mathsf{d}z) &= \int_E p_s^Z(x, y) \left(\int_A p_t^Z(y, z) \lambda(\mathsf{d}z) \right) \lambda(\mathsf{d}y) \\
&= \int_A \left(\int_E p_s^Z(x, y) p_t^Z(y, z) \lambda(\mathsf{d}y) \right) \lambda(\mathsf{d}z) \qquad \mathsf{P}_{Z_s}\text{-a.e. } x \in E.
\end{aligned}$$

Hence we obtain that for P_{Z_s}-a.e. $x \in E$, equation (2.4) holds for λ-a.e. $z \in E$. By assumptions (i) and (iii) and the dominated convergence theorem, both sides of equation (2.4) are continuous in $z \in E$ for every fixed $x \in E$. By Lemma 2.3, if $x \in E$ such that (2.4) holds for λ-a.e. $z \in E$ then it holds for all $z \in E$. By assumptions (i), (ii) and (iv) and the dominated convergence theorem, both sides of equation (2.4) are continuous in $x \in E$ for every fixed $z \in E$. The measure P_{Z_s} is clearly σ-finite, hence, again by Lemma 2.3, we conclude that (2.4) holds for all $x, z \in E$ and all $s, t > 0$ with $s + t \leqslant T$. $\square$

2.5 Lemma. *Suppose that the densities* $(p_t^Z)_{0 < t \leqslant T}$ *satisfy the following properties:*

(i) *for all* $0 < t \leqslant T$, *the function* $(x, y) \mapsto p_t^Z(x, y)$ *is continuous,*

(ii) *for all* $x, z \in E$ *and all* $s, t > 0$ *with* $s + t \leqslant T$, *the Kolmogorov-Chapman equation* (2.4) *holds,*

(iii) *for all* $x \in E$ *and all* $0 < t \leqslant T$, *we have* $p_t^Z(x, b) > 0$.

Then (2.1) *holds, and the functions* $(p_{s,t})_{0 \leqslant s < t < T}$ *satisfy conditions of Lemma 2.2.*

Proof. Clearly, assumptions (i), (iii) and Lemma 2.3 imply (2.1). Using (i) and (ii), it is easy to check that the functions $(p_{s,t})_{0 \leqslant s < t < T}$ satisfy conditions of Lemma 2.2. $\square$

2.6 Lemma. *Let* $E = [0, \infty)$, *let* λ *be the Lebesgue measure on* $[0, \infty)$, *and let* $b = 0$. *Suppose that the densities* $(p_t^Z)_{0<t\leqslant T}$ *satisfy the following properties:*

(i) *for all* $0 < t \leqslant T$, *the function* $(x, y) \mapsto p_t^Z(x, y)$ *is continuous,*

(ii) *for all* $x, z \in [0, \infty)$ *and all* $s, t > 0$ *with* $s + t \leqslant T$, *the Kolmogorov-Chapman equation* (2.4) *holds,*

(iii) *for all* $x, y \in [0, \infty)$ *and all* $0 \leqslant s < t < T$, *the limit* $\lim_{\varepsilon\downarrow 0} \frac{p_{T-t}^Z(y,\varepsilon)}{p_{T-s}^Z(x,\varepsilon)}$ *exists,*

(iv) *for all* $0 \leqslant s < t < T$ *and all* $x \in [0, \infty)$, *there is a* $\delta > 0$ *such that*

$$\sup_{y\in[0,\infty)} \sup_{0<\varepsilon<\delta} \frac{p_{T-t}^Z(y, \varepsilon)}{p_{T-s}^Z(x, \varepsilon)} < \infty.$$

Then for all $x, y \in [0, \infty)$, $0 \leqslant s < t < T$, *we have*

$$p_{s,t}(x, y) = p_{t-s}^Z(x, y) \lim_{\varepsilon\downarrow 0} \frac{p_{T-t}^Z(y, \varepsilon)}{p_{T-s}^Z(x, \varepsilon)}, \tag{2.5}$$

and the functions $(p_{s,t})_{0\leqslant s<t<T}$ *satisfy conditions of Lemma 2.2.*

Proof. Assumptions (i), (iii) and $\mathcal{L}$'Hospital's rule yield (2.5). For every $0 \leqslant s < t < T$, measurability of $(x, y) \mapsto p_{s,t}(x, y)$ follows from (2.5) and assumptions (i) and (iii). For every $0 \leqslant s < t < T$ and $x \in [0, \infty)$, the function $y \mapsto p_{s,t}(x, y)$ is a probability density, since by the assumptions and the dominated convergence theorem,

$$\begin{aligned}\int_0^\infty p_{s,t}(x, y)\, \mathsf{d}y &= \int_0^\infty \lim_{\varepsilon\downarrow 0} \frac{p_{t-s}^Z(x, y)p_{T-t}^Z(y, \varepsilon)}{p_{T-s}^Z(x, \varepsilon)}\, \mathsf{d}y \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{p_{T-s}^Z(x, \varepsilon)} \int_0^\infty p_{t-s}^Z(x, y)p_{T-t}^Z(y, \varepsilon)\, \mathsf{d}y = 1.\end{aligned}$$

For every $0 \leqslant s < t < u < T$ and $x, z \in [0, \infty)$, the Kolmogorov-Chapman equation $p_{s,u}(x, z) = \int_0^\infty p_{s,t}(x, y)p_{t,u}(y, z)\, \mathsf{d}y$ follows from (2.5) and assumptions (i)–(iii). □

3 The case of a standard d-dimensional Wiener process

Let $(B_t)_{t\geqslant 0}$ be a standard d-dimensional Wiener process and $T > 0$ be fixed. Let $(X_t)_{0\leqslant t\leqslant T}$ be the bridge with endpoints zero over $[0, T]$ derived from $(B_t)_{t\geqslant 0}$ (called the d-dimensional Wiener bridge between 0 and 0 over $[0, T]$). Let $R_t = \|B_t\|$, $t \geqslant 0$ be the radial part of $(B_t)_{t\geqslant 0}$ (called the d-dimensional Bessel process), where $\|\cdot\|$ denotes the euclidean norm. Let $(Y_t)_{0\leqslant t\leqslant T}$ be the bridge with endpoints zero over $[0, T]$ derived from $(R_t)_{t\geqslant 0}$ (called the d-dimensional Bessel bridge between 0 and 0 over $[0, T]$). As it is explained by Yor and Zambotti in [11], a simple invariance by rotation argument implies that the laws of $(\|X_t\|)_{0\leqslant t\leqslant T}$ and $(Y_t)_{0\leqslant t\leqslant T}$ coincide. Intuitively, taking bridges with endpoints zero and taking radial parts commutate in case of a standard Wiener process, or, in other words, the radial part of a Wiener bridge with endpoints zero is the Bessel bridge with endpoints zero. We want to show this result by computing the transition densities of the processes $(\|X_t\|)_{0\leqslant t\leqslant T}$ and $(Y_t)_{0\leqslant t\leqslant T}$ to demonstrate our method which will also work for certain multidimensional Ornstein-Uhlenbeck processes.

It is well known that the transition densities of the process $(B_t)_{t\geqslant 0}$ is

$$p_t^B(x,y)=\frac{1}{(2\pi t)^{d/2}}\exp\left\{-\frac{\|x-y\|^2}{2t}\right\},\qquad t>0,\quad x,y\in\mathbb{R}^d.$$

To demonstrate how to prove Markov property for the radial part of certain Markov processes and how to calculate their transition densities, we consider the radial part of $(B_t)_{t\geqslant 0}$ for $d\geqslant 2$. We use the ideas due to Karlin and Taylor [7, Chapter 7, Section 6] and Revuz and Yor [10, Chapter VI, Proposition 3.1]. Taking $t>0$, $0<t_1<\cdots<t_n$, $b>0$ and $x^{(1)},\dots,x^{(n-1)},x\in\mathbb{R}^d$, we have

$$\begin{aligned}
&P(R_{t_n+t}<b\,|\,B_{t_1}=x^{(1)},\dots,B_{t_{n-1}}=x^{(n-1)},\,B_{t_n}=x)=P(R_{t_n+t}<b\,|\,B_{t_n}=x)\\
&=P(R_t<b\,|\,B_0=x)=\int_{\|y\|<b}\frac{1}{(2\pi t)^{d/2}}\exp\left\{-\frac{\|x-y\|^2}{2t}\right\}\mathrm{d}y
\end{aligned}$$

for almost every $x\in\mathbb{R}^d$ (with respect to the Lebesgue measure). Introducing polar coordinates $y=(y_1,\dots,y_d)$ by

$$\begin{aligned}
y_1&=r\sin\theta_1\cdots\sin\theta_{d-3}\sin\theta_{d-2}\sin\theta_{d-1},\\
y_2&=r\sin\theta_1\cdots\sin\theta_{d-3}\sin\theta_{d-2}\cos\theta_{d-1},\\
y_3&=r\sin\theta_1\cdots\sin\theta_{d-3}\cos\theta_{d-2},\\
&\vdots\\
y_{d-1}&=r\sin\theta_1\cos\theta_2,\\
y_d&=r\cos\theta_1,
\end{aligned}$$

we obtain

$$P(R_t<b\,|\,B_0=x)=\int_0^b\frac{r^{d-1}}{(2\pi t)^{d/2}}\exp\left\{-\frac{\|x\|^2+r^2}{2t}\right\}G_d(r,x)\,\mathrm{d}r$$

for almost every $x\in\mathbb{R}^d$, where

$$G_d(r,x)=\int_{[0,\pi]^{d-2}\times[0,2\pi]}(\sin\theta_1)^{d-2}\cdots(\sin\theta_{d-2})\exp\left\{\frac{1}{t}\sum_{k=1}^d x_ky_k\right\}\mathrm{d}\theta_1\dots\mathrm{d}\theta_{d-1}$$

with $x=(x_1,\dots,x_d)$. Clearly, the integral $\int_{\|y\|<b}\exp\left\{-\frac{\|x-y\|^2}{2t}\right\}\mathrm{d}y$ as a function of x depends only on $\|x\|$, hence we may put $x=(0,\dots,0,\|x\|)$, and so we obtain

$$P(R_t<b\,|\,B_0=x)=\int_0^b\frac{r^{d-1}}{(2\pi t)^{d/2}}\exp\left\{-\frac{\|x\|^2+r^2}{2t}\right\}H_d(r,x)\,\mathrm{d}r$$

for almost every $x \in \mathbb{R}^d$, where

$$H_d(r,x) = 2\pi \int_0^\pi (\sin\theta_1)^{d-2} \exp\left\{ \frac{r\|x\|}{t} \cos\theta_1 \right\} \mathsf{d}\theta_1 \prod_{k=2}^{d-2} \int_0^\pi (\sin\theta_k)^{d-k-1} \mathsf{d}\theta_k.$$

By Gradstein and Ryzhik [4, 8.431], for $x \neq 0$ we have

$$\int_0^\pi (\sin\theta_1)^{d-2} \exp\left\{ \frac{r\|x\|}{t} \cos\theta_1 \right\} \mathsf{d}\theta_1 = \frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{r\|x\|}{2t}\right)^\nu} I_\nu\left(\frac{r\|x\|}{t}\right),$$

where $\nu = \frac{d}{2} - 1$ and I_ν denotes the modified Bessel function of index ν defined by

$$I_\nu(z) = \sum_{m=0}^\infty \frac{(z/2)^{2m+\nu}}{m!\Gamma(\nu+m+1)}, \qquad z > 0.$$

Moreover, if k is a positive integer then $\int_0^\pi (\sin\theta)^k \,\mathsf{d}\theta = c_k \frac{(k-1)!!}{k!!}$, where $c_k = \pi$ if k is even and $c_k = 2$ if k is odd. Consequently,

$$P(R_{t_n+t} < b \,|\, B_{t_1} = x^{(1)}, \dots, B_{t_{n-1}} = x^{(n-1)},\, B_{t_n} = x) = \int_0^b p_t^R(\|x\|, r)\, \mathsf{d}r$$

for almost every $x \in \mathbb{R}^d$, where

$$(3.1) \qquad p_t^R(x,y) = \begin{cases} \dfrac{y^{\nu+1}}{tx^\nu} \exp\left\{ -\dfrac{x^2+y^2}{2t} \right\} I_\nu\left(\dfrac{xy}{t}\right) & \text{if } x, y > 0, \\[2ex] \dfrac{y^{2\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)} \exp\left\{ -\dfrac{y^2}{2t} \right\} & \text{if } x = 0,\ y > 0, \end{cases}$$

and $p_t^R(x,0) := \lim_{y \downarrow 0} p_t^R(x,y) = 0$ if $x \geqslant 0$. Hence

$$P(R_{t_n+t} < b \,|\, B_{t_1}, \dots, B_{t_n}) = \int_0^b p_t^R(R_{t_n}, r)\, \mathsf{d}r \qquad \mathsf{P}\text{-a.s.}$$

Clearly, the process $(R_t)_{t\geqslant 0}$ is adapted to the filtration $(\mathcal{F}_t^B)_{t\geqslant 0}$, where $\mathcal{F}_t^B := \sigma(B_s,\, 0 \leqslant s \leqslant t)$, hence we conclude that $(R_t)_{t\geqslant 0}$ is a time-homogeneous Markov process with transition densities $(p_t^R)_{t>0}$. Note that formula (3.1) is valid also for $d = 1$ with $p_t^R(x,0) := \lim_{y\downarrow 0} p_t^R(x,y) = \sqrt{\frac{2}{\pi t}} \exp\left\{ -\frac{x^2}{2t} \right\}$ if $x \geqslant 0$ (see, e.g., Revuz and Yor [10, p. 446]).

Obviously, for all $t > 0$ and $z \in \mathbb{R}^d$, we have

$$\sup_{x,\, y \in \mathbb{R}^d} p_t^B(x,y) = (2\pi t)^{-d/2}, \qquad \int_{\mathbb{R}^d} p_t^B(x,z)\, \mathsf{d}x = 1,$$

hence by Lemmas 2.2, 2.4 and 2.5 we obtain the existence of the Wiener bridge $(X_t)_{0\leqslant t\leqslant T}$ and its transition densities

$$p_{s,t}^X(x,y) = \left(\frac{T-s}{2\pi(t-s)(T-t)} \right)^{d/2} \exp\left\{ -\frac{\|x-y\|^2}{2(t-s)} - \frac{\|y\|^2}{2(T-t)} + \frac{\|x\|^2}{2(T-s)} \right\}$$

for all $x, y \in \mathbb{R}^d$ and all $0 \leqslant s < t < T$.

As in case of the Bessel process, one can prove that $(\|X_t\|)_{0\leqslant t\leqslant T}$ is again a Markov process and obtain its transition densities :

$$p_{s,t}^{\|X\|}(x,y) = \frac{y^{\nu+1}}{(t-s)x^\nu}\left(\frac{T-s}{T-t}\right)^{\nu+1}\exp\left\{-\frac{x^2+y^2}{2(t-s)} - \frac{y^2}{2(T-t)} + \frac{x^2}{2(T-s)}\right\} I_\nu\left(\frac{xy}{t-s}\right)$$

for all $0 \leqslant s < t < T$ and all $x, y > 0$, and

$$p_{s,t}^{\|X\|}(0,y) = \frac{y^{2\nu+1}}{2^\nu (t-s)^{\nu+1}\Gamma(\nu+1)}\left(\frac{T-s}{T-t}\right)^{\nu+1}\exp\left\{-\frac{y^2}{2(t-s)} - \frac{y^2}{2(T-t)}\right\}$$

for all $0 \leqslant s < t < T$ and all $y > 0$.

The aim of the following discussion is to prove that the densities $(p_t^R)_{t>0}$ satisfy conditions of Lemmas 2.4 and 2.6. It is known that

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}[1 + O(z^2)] \quad \text{as} \quad z \downarrow 0, \qquad I_\nu(z) = \frac{\mathsf{e}^z}{\sqrt{2\pi z}}[1 + O(z^{-1})] \quad \text{as} \quad z \to \infty.$$

(For the second statement see Gradstein and Ryzhik [4, 8.451].) Hence

$$c_1\left[z^\nu \mathbb{1}_{(0,1)}(z) + z^{-1/2}\mathsf{e}^z \mathbb{1}_{[1,\infty)}(z)\right] \leqslant I_\nu(z) \leqslant c_2\left[z^\nu \mathbb{1}_{(0,1)}(z) + z^{-1/2}\mathsf{e}^z \mathbb{1}_{[1,\infty)}(z)\right]$$

with some $0 < c_1 < c_2$ for all $z > 0$. Thus

$$c_1 f_t(x,y) \leqslant p_t^R(x,y) \leqslant c_2 f_t(x,y) \tag{3.2}$$

for all $x, y, t > 0$, where

$$f_t(x,y) := t^{-d/2}y^{d-1}\mathsf{e}^{-(x^2+y^2)/(2t)}\mathbb{1}_{(0,1)}(xy/t) + t^{-1/2}(y/x)^{(d-1)/2}\mathsf{e}^{-(x-y)^2/(2t)}\mathbb{1}_{[1,\infty)}(xy/t).$$

Using (3.2) we obtain $\sup\limits_{x\geqslant 0}\sup\limits_{y\geqslant 0} p_t^R(x,y) < \infty$ for all $t > 0$. Indeed, for all $t > 0$ we have

$$\sup_{0<xy<t} p_t^R(x,y) \leqslant c_2 t^{-d/2}\sup_{y>0} y^{d-1}\mathsf{e}^{-y^2/(2t)} < \infty,$$

$$\sup_{xy\geqslant t,\, y<x} p_t^R(x,y) \leqslant c_2 t^{-1/2},$$

$$\sup_{xy\geqslant t,\, y\geqslant x} p_t^R(x,y) = \sup_{\alpha\geqslant 1}\sup_{xy\geqslant t,\, y=\alpha x} p_t^R(x,y) = \sup_{\alpha\geqslant 1}\sup_{x\geqslant\sqrt{t/\alpha}} p_t^R(x,\alpha x)$$

$$\leqslant \sup_{\alpha\geqslant 1}\sup_{x\geqslant\sqrt{t/\alpha}} c_2 t^{-1/2}\alpha^{(d-1)/2}\mathsf{e}^{-(\alpha-1)^2x^2/(2t)} = c_2 t^{-1/2}\sup_{\alpha\geqslant 1}\alpha^{(d-1)/2}\mathsf{e}^{-(\alpha-1)^2/(2\alpha)} < \infty.$$

Moreover, for all $y, t > 0$, we have

$$\int_0^\infty p_t^R(x,y)\,\mathsf{d}x \leqslant c_2 t^{-d/2}y^{d-1}\left(\int_0^{t/y}\mathsf{e}^{-x^2/(2t)}\,\mathsf{d}x + \int_{t/y}^\infty \mathsf{e}^{-(x-y)^2/(2t)}\,\mathsf{d}x\right) < \infty.$$

Furthermore, by (3.2), for all $x > 0$ and all $0 \leqslant s < t < T$, we have

$$\sup_{y>0}\sup_{0<\varepsilon<(T-s)/x}\frac{p_{T-t}^R(y,\varepsilon)}{p_{T-s}^R(x,\varepsilon)} \leqslant \frac{c_2}{c_1}\left(\frac{T-s}{T-t}\right)^{d/2}\exp\left\{\frac{x^2}{2(T-s)} + \frac{T-s}{2x^2}\right\}.$$

Using $\lim_{z\to 0} z^{-\nu} I_\nu(z) = 1/(2^\nu \Gamma(\nu+1))$ and Lemmas 2.2, 2.4, 2.6, one can prove the existence of the Bessel bridge $(Y_t)_{0\leqslant t\leqslant T}$ and calculate its transition densities. It turns out that the transition densities of the processes $(\|X_t\|)_{0\leqslant t\leqslant T}$ and $(Y_t)_{0\leqslant t\leqslant T}$ coincide. By Lemma 2.2, their laws on $\big([0,\infty)^{[0,T]}, (\mathcal{B}([0,\infty)))^{[0,T]}\big)$ coincide.

Note that, as a by-product, we proved that the Kolmogorov-Chapman equation $\int_0^\infty p_s^R(x,y)p_t^R(y,z)\,\mathsf{d}y = p_{s+t}^R(x,z)$ holds for all $x,z\geqslant 0$ and all $s,t>0$, hence

$$\int_0^\infty y\mathsf{e}^{-\gamma y^2} I_\nu(\alpha y) I_\nu(\beta y)\,\mathsf{d}y = \frac{1}{2\gamma}\exp\left\{\frac{\alpha^2+\beta^2}{4\gamma}\right\} I_\nu\left(\frac{\alpha\beta}{2\gamma}\right) \tag{3.3}$$

for all $\alpha,\beta,\gamma>0$. (Compare with Gradstein and Ryzhik [4, 8.663].) In other words, we obtained a probabilistic proof of (3.3).

4 Bridges derived from general multidimensional Ornstein-Uhlenbeck processes

Let us consider the d-dimensional stochastic differential equation (SDE)

$$\begin{cases} \mathsf{d}Z_t = AZ_t\,\mathsf{d}t + \Sigma\,\mathsf{d}W_t, & t\geqslant 0,\\ Z_0 = 0, \end{cases} \tag{4.1}$$

where $A\in\mathbb{R}^{d\times d}$, $\Sigma\in\mathbb{R}^{d\times r}$ and $(W_t)_{t\geqslant 0}$ is a standard r-dimensional Wiener process. It is known that there exists a strong solution of equation (4.1), namely

$$Z_t = \int_0^t e^{(t-s)A}\Sigma\,\mathsf{d}W_s, \qquad t\geqslant 0, \tag{4.2}$$

and pathwise uniqueness for (4.1) holds. (See, e.g., Karatzas and Shreve [6, 5.6].) The process $(Z_t)_{t\geqslant 0}$ is a time-homogeneous Gauss-Markov process, which is called a general d-dimensional Ornstein-Uhlenbeck (OU) process. From (4.2) we obtain

$$Z_t = \mathsf{e}^{(t-s)A}Z_s + \int_s^t \mathsf{e}^{(t-u)A}\Sigma\,\mathsf{d}W_u$$

for all $0\leqslant s<t$, thus the conditional distribution of Z_t with respect to $Z_s=x$ is a normal distribution with mean $\mathsf{e}^{(t-s)A}x$ and variance matrix

$$\int_s^t \mathsf{e}^{(t-u)A}\Sigma\Sigma^\top\mathsf{e}^{(t-u)A^\top}\,\mathsf{d}u = \int_0^{t-s}\mathsf{e}^{(t-s-v)A}\Sigma\Sigma^\top\mathsf{e}^{(t-s-v)A^\top}\,\mathsf{d}v.$$

Hence if $\Sigma\Sigma^\top$ is a (strictly) positive definite matrix (necessarily $r\geqslant d$) then $(Z_t)_{t\geqslant 0}$ has transition densities $(p_t^Z)_{t>0}$ given by

$$p_t^Z(x,y) = \frac{1}{\sqrt{(2\pi)^d\det(V_t)}}\exp\left\{-\frac{1}{2}(y-\mathsf{e}^{tA}x)^\top V_t^{-1}(y-\mathsf{e}^{tA}x)\right\} \tag{4.3}$$

for all $x,y\in\mathbb{R}^d$ and all $t>0$, where

$$V_t := \int_0^t \mathsf{e}^{(t-v)A}\Sigma\Sigma^\top\mathsf{e}^{(t-v)A^\top}\,\mathsf{d}v, \qquad t>0.$$

We also have

$$p_t^Z(x,y) = \frac{1}{\sqrt{(2\pi)^d \det(V_t)}} \exp\left\{-\frac{1}{2}(x - \mathrm{e}^{-tA}y)^\top \widetilde{V}_t^{-1}(x - \mathrm{e}^{-tA}y)\right\}$$

for all $x, y \in \mathbb{R}^d$ and all $t > 0$, where

$$\widetilde{V}_t := \int_0^t \mathrm{e}^{-vA}\Sigma\Sigma^\top \mathrm{e}^{-vA^\top}\,\mathrm{d}v, \qquad t > 0.$$

If all the eigenvalues of A have negative real parts then $V_t = V - \mathrm{e}^{tA}V\mathrm{e}^{tA^\top}$, $t > 0$, where V is the unique solution of the algebraic matrix equation $AV + VA^\top = -\Sigma\Sigma^\top$ given by $V = \int_0^\infty \mathrm{e}^{uA}\Sigma\Sigma^\top \mathrm{e}^{uA^\top}\,\mathrm{d}u$. (See, e.g., Karatzas and Shreve [6, 5.6 A].)

Obviously, for all $t > 0$ and $z \in \mathbb{R}^d$, we have

$$\sup_{x,\,y\in\mathbb{R}^d} p_t^Z(x,y) = \frac{1}{\sqrt{(2\pi)^d \det(V_t)}}, \qquad \int_{\mathbb{R}^d} p_t^Z(x,z)\,\mathrm{d}x = \det(\mathrm{e}^{-tA}).$$

Hence by Lemmas 2.2, 2.4 and 2.5 we obtain the existence of the general Ornstein-Uhlenbeck bridge $(X_t)_{0\leqslant t\leqslant T}$ over $[0,T]$ with endpoints zero and its transition densities

$$\begin{aligned}p_{s,t}^X(x,y) &= \sqrt{\frac{\det(\widetilde{V}_{T-s})}{(2\pi)^d \det(\widetilde{V}_{t-s}\widetilde{V}_{T-t})}} \\ &\quad\times \exp\left\{-\frac{1}{2}(x - \mathrm{e}^{-(t-s)A}y)^\top \widetilde{V}_{t-s}^{-1}(x - \mathrm{e}^{-(t-s)A}y) - \frac{1}{2}y^\top \widetilde{V}_{T-t}^{-1}y + \frac{1}{2}x^\top \widetilde{V}_{T-s}^{-1}x\right\}\end{aligned} \tag{4.4}$$

for all $x, y \in \mathbb{R}^d$ and all $0 \leqslant s < t < T$.

5 The case of certain Ornstein-Uhlenbeck processes

Let us consider the d-dimensional SDE

$$\begin{cases}\mathrm{d}Z_t = aZ_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, & t \geqslant 0,\\ Z_0 = 0,\end{cases} \tag{5.1}$$

where $a, \sigma \in \mathbb{R}$ such that $\sigma \neq 0$, and $(W_t)_{t\geqslant 0}$ is a standard d-dimensional Wiener process. By (4.2), the SDE (5.1) has a strong solution given by

$$Z_t = \sigma \int_0^t e^{a(t-s)}\,\mathrm{d}W_s, \qquad t \geqslant 0, \tag{5.2}$$

and pathwise uniqueness for (5.1) holds.

Let $T > 0$ be fixed. Let $(X_t)_{0\leqslant t\leqslant T}$ be the bridge with endpoints zero over $[0,T]$ derived from $(Z_t)_{t\geqslant 0}$. Let $R_t := \|Z_t\|$, $t \geqslant 0$ be the radial part of $(Z_t)_{t\geqslant 0}$. Let $(Y_t)_{0\leqslant t\leqslant T}$ be the bridge with endpoints zero over $[0,T]$ derived from $(R_t)_{t\geqslant 0}$. Our aim is to show that the transition densities of the processes $(\|X_t\|)_{0\leqslant t\leqslant T}$ and $(Y_t)_{0\leqslant t\leqslant T}$ coincide. In fact, we obtain the result of Section 3 as a special case with $a = 0$ and $\sigma = 1$.

From (4.3) we obtain the transition densities of the OU process $(Z_t)_{t\geqslant 0}$:

$$p_t^Z(x,y)=\frac{1}{(2\pi\sigma^2\kappa(a,t))^{d/2}}\exp\left\{-\frac{\|y-\mathrm{e}^{at}x\|^2}{2\sigma^2\kappa(a,t)}\right\},\qquad t>0,\quad x,y\in\mathbb{R}^d,$$

where $\kappa(a,t)=\frac{\mathrm{e}^{2at}-1}{2a}$ for $a\neq 0$, and $\kappa(0,t)=t$.

As in case of the d-dimensional Bessel process, one can prove that $(R_t)_{t\geqslant 0}$ is a time-homogeneous Markov process with transition densities

$$p_t^R(x,y)=\begin{cases}\dfrac{\mathrm{e}^{-a\nu t}y^{\nu+1}}{\sigma^2\kappa(a,t)x^\nu}\exp\left\{-\dfrac{\mathrm{e}^{2at}x^2+y^2}{2\sigma^2\kappa(a,t)}\right\}I_\nu\left(\dfrac{\mathrm{e}^{at}xy}{\sigma^2\kappa(a,t)}\right) & \text{if } x,y>0,\\[2ex] \dfrac{y^{2\nu+1}}{2^\nu(\sigma^2\kappa(a,t))^{\nu+1}\Gamma(\nu+1)}\exp\left\{-\dfrac{y^2}{2\sigma^2\kappa(a,t)}\right\} & \text{if } x=0,\ y>0,\end{cases}$$

where $\nu=\frac{d}{2}-1$, and $p_t^R(x,0):=\lim\limits_{y\downarrow 0}p_t^R(x,y)=0$ if $d\geqslant 2$, $x\geqslant 0$, and $p_t^R(x,0):=\lim\limits_{y\downarrow 0}p_t^R(x,y)=\sqrt{\frac{2}{\pi\sigma^2\kappa(a,t)}}\exp\left\{-\frac{\mathrm{e}^{2at}x^2}{2\sigma^2\kappa(a,t)}\right\}$ if $d=1$, $x\geqslant 0$.

By (4.4), the transition densities of the OU bridge $(X_t)_{0\leqslant t\leqslant T}$ is

$$p_{s,t}^X(x,y)=\left(\frac{\kappa(a,T-s)}{2\pi\sigma^2\kappa(a,t-s)\kappa(a,T-t)}\right)^{d/2}$$

$$\times\exp\left\{-\frac{\|y-\mathrm{e}^{a(t-s)}x\|^2}{2\sigma^2\kappa(a,t-s)}-\frac{\mathrm{e}^{2a(T-t)}\|y\|^2}{2\sigma^2\kappa(a,T-t)}+\frac{\mathrm{e}^{2a(T-s)}\|x\|^2}{2\sigma^2\kappa(a,T-s)}\right\}$$

for all $0\leqslant s<t<T$ and all $x,y\in\mathbb{R}^d$.

As in case of the d-dimensional Bessel process, one can prove that $(\|X_t\|)_{0\leqslant t\leqslant T}$ is again a Markov process and one can calculate its transition densities :

$$p_{s,t}^{\|X\|}(x,y)=\frac{\mathrm{e}^{-a\nu(t-s)}y^{\nu+1}}{\sigma^2\kappa(a,t-s)x^\nu}\left(\frac{\kappa(a,T-s)}{\kappa(a,T-t)}\right)^{\nu+1}I_\nu\left(\frac{\mathrm{e}^{a(t-s)}xy}{\sigma^2\kappa(a,t-s)}\right)$$

$$\times\exp\left\{-\frac{\mathrm{e}^{2a(t-s)}x^2+y^2}{2\sigma^2\kappa(a,t-s)}-\frac{\mathrm{e}^{2a(T-t)}y^2}{2\sigma^2\kappa(a,T-t)}+\frac{\mathrm{e}^{2a(T-s)}x^2}{2\sigma^2\kappa(a,T-s)}\right\}$$

for all $0\leqslant s<t<T$ and all $x,y>0$, and

$$p_{s,t}^{\|X\|}(0,y)=\frac{y^{2\nu+1}}{2^\nu(\sigma^2\kappa(a,t-s))^{\nu+1}\Gamma(\nu+1)}\left(\frac{\kappa(a,T-s)}{\kappa(a,T-t)}\right)^{\nu+1}$$

$$\times\exp\left\{-\frac{y^2}{2\sigma^2\kappa(a,t-s)}-\frac{y^2}{2\sigma^2\kappa(a,T-t)}\right\}$$

for all $0\leqslant s<t<T$ and all $y>0$.

As in Section 3, one can check that the densities $(p_t^R)_{t>0}$ satisfy conditions of Lemmas 2.4 and 2.6 and one obtains the existence of the bridge $(Y_t)_{0\leqslant t\leqslant T}$ and its transition densities. It turns out that the transition densities of the processes $(\|X_t\|)_{0\leqslant t\leqslant T}$ and $(Y_t)_{0\leqslant t\leqslant T}$ coincide. By Lemma 2.2, their laws on $\left([0,\infty)^{[0,T]},(\mathcal{B}([0,\infty)))^{[0,T]}\right)$ coincide.

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