Geometric Quantities and Their Meanings in Finsler Geometry

PhD dissertation

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A jelölt aláírása

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A témavezető aláírása

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Chapter 1

Introduction

A Finsler metric on a manifold is a family of Minkowski norms on tangent spaces. There are several geometric quantities in Finsler geometry. The flag curvature $\mathbf{K}$ is an analogue of the sectional curvature in Riemannian geometry. The Cartan torsion $\mathbf{C}$ is a primary quantity. There is another quantity which is determined by the Busemann-Hausdorff volume form, that is the so-called distortion τ. The vertical differential of τ on each tangent space gives rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^k} dx^k$. $\mathbf{C}, \tau$ and $\mathbf{I}$ are the basic geometric quantities which characterize Riemannian metrics among Finslers metrics. Differentiating $\mathbf{C}$ along geodesics gives rise to the Landsberg curvature $\mathbf{L}$. The horizontal derivative of τ along geodesics is the so-called S-curvature $\mathbf{S} := \tau_{|k} y^k$. The horizontal derivative of $\mathbf{I}$ along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|k} y^k$. Besides, from the geodesic coefficients $G^i(x, y)$, we can define the Berwald curvature $\mathbf{B} := B^i_{jkl} dx^j \otimes dx^k \otimes dx^l \otimes \partial_i$ and the mean Berwald curvature $\mathbf{E} := E_{ij} dx^i \otimes dx^j$, which are defined by

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{ij} := \frac{1}{2} B_m{}^m{}_{ij}.$$

Furthermore, we can define the Douglas curvature $\mathbf{D}$ by $\mathbf{B}$ and $\mathbf{E}$. Obviously, τ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$, $\mathbf{C}$, $\mathbf{L}$ and $\mathbf{B}$, $\mathbf{E}$, $\mathbf{D}$ all vanish for Riemannian metrics. Thus they are said to be non-Riemannian. The Riemann curvature measures the shape of the space while the non-Riemannian quantities describe the change of the "color" on the space. It is found that the flag curvature is closely related to these non-Riemannian quantities [AIM][MoSh][Sh2].

Finsler projective geometry is an important part of Finsler geometry. Given two Finsler metrics F and $\tilde{F}$ on an n-dimensional manifold M. We say F and $\tilde{F}$ to be *pointwise projectively related* (or the change $F \to \tilde{F}$ is a *projective change*) if any geodesic of F is also a geodesic of $\tilde{F}$ as a point set and the inverse is also true. Two regular Finsler metric spaces are said to be *projectively related* if there is a diffeomorphism between them such that the pull-back metric is pointwise projectively related to another one. In general, given a Finsler metric F on a manifold M, we would like to determine all Finsler metrics on M which are pointwise projectively related to F. Particularly, it is interesting and meaningful to determine all Finsler metrics on M which are pointwise projectively related to a locally Minkowski metric on M. Such Finsler metrics are said to be locally projectively flat. The problem of characterizing and studying locally projectively flat Finsler metrics is known as *Hilbert's fourth problem*.

The Ricci curvature plays an important role in the Finsler projective geometry. It is proved [Sh1] that for two pointwise projectively related Einstein metrics g and $\tilde{g}$ on an n-dimension compact manifold M, their Einstein constants have the same sign. In addition, if their Einstein constants are negative and equal, then $g = \tilde{g}$. In section 3, we will continue to study pointwise projectively related Finsler metrics and give a comparison theorem on the Ricci curvatures. At the same time, we will take a look at role that S-curvature plays in Finsler projective geometry. Besides, we will also discuss the projectively flat Finsler metrics with some special cuevature properties in sections 6 and 7. One of the important problems in Finsler geometry is to study and characterize locally projectively flat Finsler metrics.

Another important problem in Finsler geometry is to study and characterize Finsler metrics of scalar curvature. This problem has not been solved yet, even for Finsler metrics of constant flag curvature. In section 4, we discuss the Finsler metrics of scalar curvature and partially determine the flag curvature when F is of isotropic S-curvature or relatively isotropic mean Landsberg curvature. In fact, all known Randers metrics $F = \alpha + \beta$ of scalar curvature (in dimension $n > 2$) satisfy $\mathbf{S} = (n+1)c(x)F$ or $\mathbf{J} + c(x)F\mathbf{I} = 0$, where $c(x)$ is a function on M. Motivated by such phenomena, in section 5, we study Randers metrics satisfying $\mathbf{J} + c(x)F\mathbf{I} = 0$ and classify Randers metrics with flag curvature $\mathbf{K} = \lambda(x)$ and $\mathbf{J} + c(x)F\mathbf{I} = 0$. Furthermore, we study Randers metrics with isotropic S-curvature in section 6. It is known that every locally projectively flat Finsler metric is of scalar curvature. Using the obtained formula for the flag curvature in section 4, we classify locally projectively flat Randers metrics with isotropic S-curvature in section 6. And then, we study and characterize locally projectively flat Finsler with isotropic S-curvature in section 7.

The Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics. The class of Douglas metrics is also much larger than that of Berwald metrics. The study on Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities. In section 8, we discuss Douglas metrics with relatively isotropic Landsberg curvature or isotropic mean Berwald curvature. Then we introduce the Finsler metrics of isotropic Berwald curvaure. We prove an equivalence among the above metrics.

Chapter 2

Preliminaries

A Finsler metric on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties:

(a) F is C^∞ on $TM \setminus \{0\}$;

(b) $F(x, \lambda y) = \lambda F(x, y), \quad \forall \lambda > 0$;

(c) For any tangent vector $y \in T_xM \setminus \{0\}$, the following bilinear symmetric form $g_y : T_xM \times T_xM \to R$ is positive definite:

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[F^2(x, y + su + tv) \right] |_{s=t=0}.$$

Let

$$g_{ij}(x, y) := \frac{1}{2} \left[F^2 \right]_{y^i y^j} (x, y).$$

By the homogeneity of F, we have

$$g_y(u, v) = g_{ij}(x, y) u^i v^j, \quad F(x, y) = \sqrt{g_{ij}(x, y) y^i y^j}.$$

Let F be a Finsler metric on an n-dimensional manifold. The geodesics of F are characterized by the following equations:

$$\ddot{c}^i(t) + 2G^i \left(c(t), \dot{c}(t) \right) = 0, \tag{2.1}$$

where

$$G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}$$

and $(g^{ij}(x,y)) := (g_{ij}(x,y))^{-1}$. $G^i(x,y)$ are called the *geodesic coefficients* of F.

The *Riemann curvature* $\mathbf{R}_y := R^i_k dx^k \bigotimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.2}$$

The *Ricci curvature* **Ric** is defined to be the trace of $\mathbf{R_y}$ on each tangent space T_xM,

$$\mathbf{Ric}(y) := R^i_i(x,y).$$

The Ricci curvature **Ric** is a positively homogeneous function of degree two on TM, i.e., $\mathbf{Ric}(\lambda y) = \lambda^2 \mathbf{Ric}(y), \lambda > 0$. If F is a Riemannian metric, then

$$R^i_k(x,y) = R^i_{jkl}(x) y^j y^l,$$

where $R^i_{jkl}(x)$ denote the coefficients of the Riemannian curvature tensor on M. In this case, $\mathbf{Ric}(y) = R^k_{jkl}(x) y^j y^l$ is quadratic in $y \in T_xM$.

For a flag $P = span\{y, u\} \subset T_xM$ with flagpole y, the *flag curvature* $\mathbf{K}(P,y)$ is defined by

$$\mathbf{K}(P,y) := \frac{g_y(u, \mathbf{R}_y(u))}{g_y(y,y)g_y(u,u) - g_y(y,u)^2}. \tag{2.3}$$

When F is Riemannian, $\mathbf{K}(P,y) = \mathbf{K}(P)$ is independent of $y \in P(flagpole)$. It is just the sectional curvature of P in Riemannian geometry. We say that F is *of scalar curvature* if for any $y \in T_xM$, the flag curvature $\mathbf{K}(P,y) = \mathbf{K}(x,y)$ is independent of P containing $y \in T_xM$, or equivalently,

$$R^i_k = \mathbf{K}(x,y) F^2 h^i_k, \tag{2.4}$$

where $h^i_k := g^{ij} h_{jk}$ and $h_{jk} := g_{jk} - F_{y^j} F_{y^k}$. F is said to be of *constant flag curvature* if $\mathbf{K}(P,y) = constant$. h_{jk} define a tensor field on TM called the *angular metric tensor* of F.

To characterize Riemannian metrics among Finsler metrics, we introduce the quantity

$$\tau(x,y) := \ln\left[\frac{\sqrt{det(g_{ij}(x,y))}}{\sigma(x)}\right],$$

where

$$\sigma(x) := \frac{Vol(B^n)}{Vol\,\{(y^i) \in R^n | F(x,y) < 1\}}.$$

τ is called the *distortion.* F is Riemannian if and only if $\tau = constant$ [Sh2]. Let

$$C_{ijk}(x,y) := \frac{1}{4}[F^2]_{y^iy^jy^k}(x,y), \qquad I_i(x,y) := g^{jk}(x,y)C_{ijk}(x,y),$$

A direct computation yields

$$I_i(x,y) = \tau_{y^i}(x,y).$$

For $y \in T_xM \setminus \{0\}$, set

$$\mathbf{C}_y(u,v,w) := C_{ijk}(x,y)u^iv^jw^k, \qquad \mathbf{I}_y(u) := I_i(x,y)u^i,$$

where $u = u^i\frac{\partial}{\partial x^i}|_x, v = v^j\frac{\partial}{\partial x^j}|_x, w = w^k\frac{\partial}{\partial x^k}|_x \in T_xM$. The family $\mathbf{C} := \{\mathbf{C}_y | y \in TM \setminus \{0\}\}$ is called the *Cartan torsion* and the family $\mathbf{I} := \{\mathbf{I}_y | y \in TM \setminus \{0\}\}$ is called the *mean Cartan torsion.* A trivial fact is that a Finsler metric F is Riemannian if and only if $\mathbf{I} = 0$ (Deicke, 1953, cf. [Sh1]).

To find the relationship between the Riemann curvature and non-Riemannian quantities, we employ the Chern connection on the pull-back tangent bundle π^*TM where $\pi : TM\setminus\{0\} \to M$ is the natural projection. Let $\omega^i := \pi^*\theta^i$, where $\{\theta^i := dx^i\}$ is the local coframe for TM dual to $\{\frac{\partial}{\partial x^i}\}$. The Chern connection forms are the unique local 1-forms $\omega_j^{\ i}$ satisfying

$$d\omega^i = \omega^j \wedge \omega_j^{\ i},$$

$$dg_{ij} = g_{ik}\omega_j^{\ k} + g_{kj}\omega_i^{\ k} + 2C_{ijk}\{dy^k + y^j\omega_j^{\ k}\},$$

Let

$$\omega^{n+k} := dy^k + y^j\omega_j^{\ k}.$$

We obtain a local coframe $\{\omega^i, \omega^{n+i}\}$ for $T^*(TM \setminus \{0\})$. Let

$$\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega_j^{\ i}.$$

We can express Ω^i in the following form

$$\Omega^i = \frac{1}{2}R^i_{\ kl}\omega^k \wedge \omega^l - L^i_{\ kl}\omega^k \wedge \omega^l, \tag{2.5}$$

where $R^i_{kl} + R^i_{lk} = 0$. R^i_k in (2.2) and R^i_{kl} in (2.5) are related by

$$R^i_k = R^i_{kl} y^l.$$

With the Chern connection, we define the covariant derivatives of quantities on TM in the usual way. For example, for a scalar function f, we define $f_{|i}$ and $f_{.i}$ by

$$df = f_{|i}\omega^i + f_{.i}\omega^{n+i}.$$

For the mean Cartan torsion $\mathbf{I} = I_i\omega^i$, define $I_{i|j}$ and $I_{i.j}$ by

$$dI_i - I_k\omega_i^k = I_{i|j}\omega^j + I_{i.j}\omega^{n+j}.$$

Without much difficulty, one can show that

$$R^i_{kl} = \frac{1}{3}\left\{R^i_{k.l} - R^i_{l.k}\right\}$$

and

$$I_i = \tau_{.i}, \quad L_{ijk} = C_{ijk|m}y^m, \quad J_i = I_{i|m}y^m, \tag{2.6}$$

where $L_{ijk} := g_{im}L^m_{kl}$ and $J_i := g^{jk}L_{ijk}$ [Sh2]. We obtain the *Landsberg curvature* $\mathbf{L} := L_{ijk}dx^i \otimes dx^j \otimes dx^k$ and the *mean Landsberg curvature* $\mathbf{J}_y = J_i dx^i$. Let

$$\mathbf{S}(x, y) := \frac{d}{dt}[\tau(\sigma(t), \dot\sigma(t))]_{t=0},$$

or equivalently

$$\mathbf{S} = \tau_{|m}y^m.$$

We call $\mathbf{S}$ the *S-curvature* [Sh3]. We say S-curvature is *isotropic* if there exists a scalar function $c(x)$ on M such that $\mathbf{S}(x, y) = (n + 1)c(x)F(x, y)$. If $c(x) =$ *constant*, we say that F has constant S-curvature. S-curvature $\mathbf{S}(x, y)$ is the rate of change of τ along geodesics and measures the averages rate of change of (T_xM, F_x) in the direction $y \in T_xM$. If (M, F) is modeled on a single Minkowski space, then $\mathbf{S} = 0$ ([Sh2][Sh3]). Many known Finsler metrics of constant (scalar) flag curvature actually have constant (isotropic) S-curvature ([CMS][Sh6]).

Let

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big). \tag{2.7}$$

It is easy to verify that $\mathbf{D} := D^i_{jkl}dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on $TM \setminus \{0\}$. We call $\mathbf{D}$ the *Douglas tensor*. By a direct computation, one can express D^i_{jkl} as follows.

$$D^i_{jkl} := B_j{}^i{}_{kl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l}\, y^i\Big\}. \tag{2.8}$$

A Finsler metric is called a Douglas metric if $\mathbf{D} = 0$. By (2.8), one can easily see that every Berwald metric is a Douglas metric. There are many non-Berwaldian Douglas metrics. For example, a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed but $F = \alpha + \beta$ is a Berwald metric if and only if β is parallel with respect to α [BaMa1].

Consider two pointwise projectively related Finsler metrics F and $\tilde{F}$. We have the following important lemma:

Lemma 2.1.([Ra]) Let F and $\tilde{F}$ be two Finsler metrics on a manifold M. F and $\tilde{F}$ are pointwise projectively related if and only if there is a scalar function P on TM such that

$$\tilde{G}^i = G^i + Py^i \tag{2.9}$$

with $P = \tilde{F}_{;k}y^k/(2\tilde{F})$, where lower " ; " denotes the horizontal covariant derivative with respect to the Berwald connection of F and this P is called the *projective factor*.

Plugging (2.9) into (2.2), one obtains

$$\tilde{R}^i_k = R^i_k + \Xi\delta^i_k + \tau_k y^i, \tag{2.10}$$

where

$$\Xi := P^2 - P_{;k}y^k, \qquad \tau_k := 3(P_{;k} - PP_{y^k}) + \Xi_{y^k}.$$

Furthermore, we have

$$\widetilde{\mathbf{Ric}}(y) = \mathbf{Ric}(y) + (n-1)\Xi(y). \tag{2.11}$$

Now, let us consider a projectively flat Finsler metric $F = F(x, y)$. By (2.9), its geodesic coefficients are in the form $G^i = Py^i$. Then, we have

$$R^i_k = \Xi\delta^i_k + \tau_k y^i,$$

where

$$\Xi := P^2 - P_{x^k}y^k, \qquad \tau_k := 3(P_{x^k} - PP_{y^k}) + \Xi_{y^k}.$$

Using the following facts (cf. [Sh2][Sh3])

$$\mathbf{R}_y(y) = 0, \qquad g_y(\mathbf{R}_y(u), v) = g_y(u, \mathbf{R}_y(v)),$$

we can show that $\tau_k = -\Xi F^{-1} F_{y^k}$ and

$$R^i_k = \Xi \left\{ \delta^i_k - F^{-1} F_{y^k} y^i \right\}. \tag{2.12}$$

Thus F is of scalar curvature with

$$\mathbf{K} = \frac{\Xi}{F^2} = \frac{P^2 - P_{x^k} y^k}{F^2}. \tag{2.13}$$

Hence one immediately obtains the following

Proposition 2.2. Every locally projectively flat Finsler metric is of scalar curvature.

This fact is due to L. Berwald.

Chapter 3

Curvature Properties in Finsler Projective Geometry

In this section, we will first discuss an interesting result given by Rapcsák on pointwise projectively related Finsler metrics. Given two Finsler metrics F and $\tilde{F}$ on M. Let $g := F^2 = g_{ij}(x,y)y^i y^j$ and $\tilde{g} := \tilde{F}^2 = \tilde{g}_{ij}(x,y)y^i y^j$. One can easily verify that the geodesic coefficients $\tilde{G}^i = \tilde{G}^i(x,y)$ of $\tilde{g}$ are related to that of g by

$$\tilde{G}^i = G^i + \frac{1}{4}\tilde{g}^{il}\Big\{\frac{\partial \tilde{g}_{;k}}{\partial y^l}y^k - \tilde{g}_{;l}\Big\},$$

where $\tilde{g}_{;k} := \tilde{g}_{ij;k}y^i y^j$ denote the covariant derivatives of $\tilde{g}$ with respect to g,

$$\tilde{g}_{;k} := \frac{\partial \tilde{g}}{\partial x^k} - \frac{\partial G^l}{\partial y^k}\frac{\partial \tilde{g}}{\partial y^l}.$$

We simply denote $\nabla \tilde{g} := \tilde{g}_{;k}dx^k$ which is a 1-form on $TM\setminus\{0\}$. We immediately conclude that $\tilde{g}$ is pointwise projective equivalent to g if and only if there is a scalar function P on TM such that

$$\frac{\partial \tilde{g}_{;k}}{\partial y^l}y^k - \tilde{g}_{;l} = 2P\frac{\partial \tilde{g}}{\partial y^l}. \tag{3.1}$$

Lemma 3.1.([ChSh1]) $\tilde{g}$ is pointwise projectively related to g if and only if there is a scalar function P on TM such that

$$\tilde{g}_{;k} = P\frac{\partial \tilde{g}}{\partial y^k} + 2\frac{\partial P}{\partial y^k}\tilde{g}. \tag{3.2}$$

In this case

$$P = \frac{\tilde{g}_{;k} y^k}{4\tilde{g}}. \tag{3.3}$$

Suppose that the projective equivalence is trivial, $P = 0$, then $\tilde{g}$ is horizontally parallel with respect to g, $\nabla\tilde{g} = 0$.

Proof. First we assume that $\tilde{g}$ is pointwise projective to g. Then (3.1) holds for some scalar function P on TM. Contracting (3.1) with y^l yields

$$\tilde{g}_{;k} y^k = 4P\tilde{g}. \tag{3.4}$$

By (3.1) and (3.4), we obtain

$$\begin{aligned} 2P\frac{\partial \tilde{g}}{\partial y^l} &= \frac{\partial}{\partial y^l}\Big[g_{;k} y^k\Big] - 2\tilde{g}_{;l} \\ &= \frac{\partial}{\partial y^l}\Big[4P\tilde{g}\Big] - 2\tilde{g}_{;l}. \end{aligned}$$

This gives (3.2). Conversely, if (3.2) holds, then

$$\begin{aligned} \frac{\partial \tilde{g}_{;k}}{\partial y^l} y^k - \tilde{g}_{;l} &= y^k \frac{\partial}{\partial y^l}\Big\{P\frac{\partial \tilde{g}}{\partial y^k} + 2\frac{\partial P}{\partial y^k}\tilde{g}\Big\} - \Big\{P\frac{\partial \tilde{g}}{\partial y^l} + 2\frac{\partial P}{\partial y^l}\tilde{g}\Big\} \\ &= 2\frac{\partial P}{\partial y^l}\tilde{g} + P\frac{\partial \tilde{g}}{\partial y^l} + 2P\frac{\partial \tilde{g}}{\partial y^l} - P\frac{\partial \tilde{g}}{\partial y^l} - 2\frac{\partial P}{\partial y^l}\tilde{g} \\ &= 2P\frac{\partial \tilde{g}}{\partial y^l}. \end{aligned}$$

This gives (3.1). Q.E.D.

By Lemma 3.1, we obtain the following

Theorem 3.2.([ChSh1]) Let (M, g) be a complete Finsler manifold and $\tilde{g}$ another Finsler metric on M, which is pointwise projectively related to g. Suppose that

$$\widetilde{\mathbf{Ric}} \le \mathbf{Ric}. \tag{3.5}$$

Then the projective equivalence is trivial. Further, $\tilde{g}$ is horizontally parallel with respect to g, $\nabla\tilde{g} = 0$ and the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$.

Proof. By Lemma 2.1, there is a scalar function P on TM, such that (2.9) holds. Fix an arbitrary vector $\mathbf{y} \in T_xM \setminus \{0\}$ and let $c(t)$ denote the geodesic of g with $\dot{c}(0) = \mathbf{y}$. By assumption, g is complete, hence $c(t)$ is defined for $-\infty < t < \infty$. Let

$$P(t) := P(\dot{c}(t)).$$

Observe that

$$P'(t) = P_{;k}(\dot{c}(t))\dot{c}^k(t).$$

By assumption and (2.11),

$$P_{;k}y^k - P^2 = \frac{1}{n-1}\Big(\mathbf{Ric} - \widetilde{\mathbf{Ric}}\Big) \geq 0.$$

Thus

$$P'(t) - P(t)^2 \geq 0.$$

Let

$$P_0(t) := \frac{P(\mathbf{y})}{1 - P(\mathbf{y})t}.$$

$P_0(t)$ satisfies

$$P_0'(t) - P_0(t)^2 = 0.$$

To compare $P(t)$ with $P_0(t)$, define

$$h(t) := \exp\Big\{ - \int_0^t [P(s) + P_0(s)]ds\Big\}\Big\{P(t) - P_0(t)\Big\}.$$

Observe that

$$h'(t) = \exp\Big\{ - \int_0^t [P(s) + P_0(s)]ds\Big\}\Big\{P'(t) - P_0'(t) + P_0(t)^2 - P(t)^2\Big\} \geq 0.$$

Note that $h(0) = 0$. Thus $h(t) \geq 0$ for $t > 0$ and $h(t) < 0$ for $t < 0$. This implies that

$$P(t) \geq P_0(t), \qquad t > 0,$$

$$P(t) \leq P_0(t), \qquad t < 0.$$

Assume that $P(\mathbf{y}) \neq 0$. Let $t_o = 1/P(\mathbf{y})$. If $P(\mathbf{y}) > 0$, then $t_o > 0$ and

$$P(\dot{c}(t_o)) \geq \lim_{t\to t_o^-} P_0(t) = \infty.$$

If $P(\mathbf{y}) < 0$, then $t_o < 0$ and

$$P(\dot{c}(t_o)) \leq \lim_{t\to t_o^+} P_0(t) = -\infty.$$

Both are impossible. Therefore, $P(\mathbf{y}) = 0$ for any $\mathbf{y} \in TM$ and $\tilde{G}^i = G^i$. By (2.10), we conclude that the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$. Q.E.D.

Example 3.3. Theorem 3.2 is false if the completeness of g is weakened to the positive completeness. Let $|\cdot|$ and $\langle\ ,\ \rangle$ denote the standard Euclidean norm and inner product in R^n. Define

$$\begin{aligned}
\varphi(\mathbf{y}) : &= \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)} + \langle \mathbf{x}, \mathbf{y}\rangle}{1 - |\mathbf{x}|^2},\\
\bar{\varphi}(\mathbf{y}) : &= \varphi(-\mathbf{y}),\\
\tilde{\varphi}(\mathbf{y}) : &= \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1 - |\mathbf{x}|^2} = \frac{1}{2}\Big\{\varphi(\mathbf{y}) + \bar{\varphi}(\mathbf{y})\Big\},
\end{aligned}$$

where $\mathbf{y} \in T_x\mathrm{B}^n(1) = \mathrm{R}^n$. $\varphi(\mathbf{y}) > 0$ is determined by the following identity,

$$\mathbf{x} + \frac{\mathbf{y}}{\varphi(\mathbf{y})} \in \partial \mathrm{B}^n(1).$$

$\varphi, \bar{\varphi}$ and $\tilde{\varphi}$ are Finsler metrics on the unit ball $\mathrm{B}^n(1) \subset \mathrm{R}^n$. φ and $\tilde{\varphi}$ are the Funk metric and the Klein metric on the unit ball $\mathrm{B}^n(1)$, respectively. We have

$$\frac{1}{2}\varphi \leq \tilde{\varphi}. \tag{3.6}$$

φ and $\tilde{\varphi}$ have the following geodesic coefficients G^i and $\tilde{G}^i$, respectively,

$$G^i = \frac{1}{2}\varphi y^i,$$

$$\tilde{G}^i = \frac{1}{2}(\varphi - \bar{\varphi}) y^i.$$

Note that φ is only positively complete. By a direction computation, we obtain the Ricci curvatures of $\tilde{\varphi}$ and φ,

$$\widetilde{\mathbf{Ric}} = -(n-1)\tilde{\varphi}^2, \qquad \mathbf{Ric} = -(n-1)\frac{1}{4}\varphi^2.$$

By (3.6), we see that

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}.$$

Thus the Ricci curvature condition (3.5) holds. But $\tilde{G}^i \neq G^i$, even $\tilde{\varphi}$ is complete. The above example also shows that if the inequality in (3.5) is reversed, then the conclusion in Theorem 3.2 is false.

According to Theorem 3.2, if two Ricci-flat Finsler metrics are pointwise projectively related and one of them is complete, then the projective equivalence is trivial and the Riemann curvatures are equal. One can show if two negative Ricci-constant Finsler metrics are pointwise projectively related and one of them is complete, then they are isometric up to a scaling. These facts are proved in [Sh1]. In the positive Ricci-constant case, we have the following

Corollary 3.4.([ChSh1]) Let g and $\tilde{g}$ be pointwise projectively related Einstein metrics on a compact n-manifold with $\mathbf{Ric} = (n-1)g$ and $\widetilde{\mathbf{Ric}} = (n-1)\tilde{g}$. Suppose that $\tilde{g} \leq g$, then $\tilde{g} = g$.

Now we take a look at the role that S-curvature plays in the projective geometry of Finsler manifolds. From the definition, for a vector $y \in T_xM \setminus \{0\}$, the S-curvature $\mathbf{S}(x,y)$ is given by

$$\mathbf{S}(x,y) = \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big(\ln \sigma(x) \Big), \tag{3.7}$$

See [Sh3][Sh9] for detailed discussion on the S-curvature. We first have the following

Lemma 3.5.([ChSh1]) Let g and $\tilde{g}$ be Finsler metrics on an n-manifold M. Suppose that $\tilde{g}$ is pointwise projectively related to g. Then the projective factor P is given by

$$P = \frac{1}{n+1}\Big(\tilde{\mathbf{S}} - \mathbf{S} \Big) - y^i \frac{\partial}{\partial x^i}\Big[\ln f \Big], \tag{3.8}$$

where $f = f(x)$ is a scalar function on M determined by $dV_{\tilde{g}} = (1/f^{n+1})\, dV_g$.

Proof. By assumption, the geodesic coefficients of g and $\tilde{g}$ satisfy

$$\tilde{G}^i = G^i + Py^i.$$

This implies that

$$\frac{\partial \tilde{G}^i}{\partial y^i} = \frac{\partial G^i}{\partial y^i} + (n+1)P. \tag{3.9}$$

Express $dV_{\tilde{g}} = \tilde{\sigma}(x) dx^1 \cdots dx^n$. The S-curvature of $\tilde{g}$ is given by

$$\tilde{\mathbf{S}} = \frac{\partial \tilde{G}^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i}\Big(\ln \tilde{\sigma}(x) \Big). \tag{3.10}$$

Let

$$f(x) := \Big(\frac{\sigma(x)}{\tilde{\sigma}(x)}\Big)^{\frac{1}{n+1}}. \tag{3.11}$$

$f(x)$ is a well-defined function on M, although $\sigma(x)$ and $\tilde{\sigma}(x)$ depend on the local coordinates. The volume forms of g and $\tilde{g}$ are related by

$$dV_{\tilde{g}} = \tilde{\sigma}(x)dx^1 \cdots dx^n = \frac{1}{f(x)^{n+1}}\sigma(x)dx^1 \cdots dx^n = \frac{1}{f(x)^{n+1}}\, dV_g.$$

It follows from (3.7), (3.9) and (3.10) that

$$\begin{aligned} P &= \frac{1}{n+1}\Big[\tilde{\mathbf{S}} - \mathbf{S} + y^i \frac{\partial}{\partial x^i}\Big(\ln \tilde{\sigma}\Big) - y^i \frac{\partial}{\partial x^i}\Big(\ln \sigma\Big)\Big] \\ &= \frac{1}{n+1}\Big(\tilde{\mathbf{S}} - \mathbf{S}\Big) - y^i \frac{\partial}{\partial x^i}\Big(\ln f\Big). \end{aligned}$$

This proves Lemma 3.5. Q.E.D.

By Lemma 3.5, we obtain an additional conclusion to Theorem 3.2 for projectively related Finsler metrics with the same S-curvatures.

Theorem 3.6.([ChSh1]) Let (M, g) be a complete Finsler manifold and $\tilde{g}$ another Finsler metric on M, which is pointwise projectively related to g. Suppose that both g and $\tilde{g}$ satisfy

$$\widetilde{\mathbf{Ric}} \le \mathbf{Ric}, \qquad \tilde{\mathbf{S}} = \mathbf{S}.$$

Then the projective equivalence between g and $\tilde{g}$ is trivial. Further, $\tilde{g}$ is horizontally parallel with respect to g, the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$, and $dV_{\tilde{g}}$ is proportional to dV_g.

Now suppose that $\tilde{g}$ and g be pointwise projectively related Riemannian metrics on an n-dimensional manifold M. We know that the S-curvature of any Riemannian metric always vanishes. Thus by Lemma 3.5, the projective factor P is given by

$$P = -y^i \frac{\partial}{\partial x^i}\Big[\ln f\Big],$$

where f is a positive function on M which is defined in Lemma 3.5. In this case, we can get

$$\widetilde{\mathbf{Ric}}(y) = \mathbf{Ric}(y) + \frac{n-1}{f} D_g^2 f, \tag{3.12}$$

where $D_g^2 f$ denotes the Hessian of f with respect to g. From (3.12), we can prove the following theorem.

Theorem 3.7.([ChSh1]) Let g and $\tilde{g}$ be pointwise projectively related Riemannian metrics on a compact manifold M. Assume that one of the following conditions is satisfied,

(a) $tr_g\widetilde{\mathbf{Ric}} \le \mathbf{s}_g$,

(b) $tr_g\widetilde{\mathbf{Ric}} \ge \mathbf{s}_g$,

then the projective equivalence is trivial. Further, $\tilde{g}$ is parallel with respect to g, the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$, and $dV_{\tilde{g}}$ is proportional to dV_g. Here $tr_g\widetilde{\mathbf{Ric}}$ denotes the trace of the Ricci curvature $\widetilde{\mathbf{Ric}}$ of $\tilde{g}$ with respect to g and $\mathbf{s}_g := tr_g\mathbf{Ric}$ denotes the trace of the Ricci curvature $\mathbf{Ric}$ of g with respect to g. The function $\mathbf{s}_g$ is called the scalar curvature of g.

Proof. For Riemann metric, the Ricci curvature becomes a quadratic form on each tangent space T_xM. Thus at each point $x \in M$, there is an orthonormal basis $\{e_i\}_{i=1}^n$ for (T_xM, g_x) such that

$$\widetilde{\mathbf{Ric}}(\mathbf{y}) = \sum_{i=1}^n \lambda_i \, (y^i)^2, \qquad y = y^i e_i.$$

The trace of $\widetilde{\mathbf{Ric}}$ with respect to g is given by

$$\mathrm{tr}_g\widetilde{\mathbf{Ric}} = \sum_{i=1}^n \lambda_i, \tag{3.13}$$

and the trace of $\mathbf{Ric}$ with respect to g is just the scalar curvature $\mathbf{s}_g$ of g. Taking the trace on both sides of (3.12) with respect to g, we obtain

$$\mathrm{tr}_g\widetilde{\mathbf{Ric}} - \mathbf{s}_g = \frac{n-1}{f}\Delta_g f. \tag{3.14}$$

Let $\mathbf{r} := \frac{1}{n-1}(\mathrm{tr}_g\widetilde{\mathbf{Ric}} - \mathbf{s}_g)$. Equation (3.14) becomes

$$\Delta_g f = \mathbf{r}\, f. \tag{3.15}$$

We assume that M is compact. Integrating (3.15) over M, we obtain

$$\int_M \mathbf{r}\, f \, dV_g = 0.$$

Now we assume that

$$\mathbf{r} \leq 0, \qquad \text{or} \qquad \mathbf{r} \geq 0,$$

then $\mathbf{r} = 0$. Thus the function f determined by $dV_{\tilde{g}} = \frac{1}{f^{n+1}} dV_g$ satisfies

$$\Delta_g f = \mathbf{r}\, f \equiv 0.$$

Since M is compact, we conclude that $f = constant$. Therefore, the projective factor $P = 0$ by Lemma 3.5 and $dV_{\tilde{g}}$ is proportional to dV_g. By Lemma 3.1, $\tilde{g}$ is horizontally parallel with respect to g too. Q.E.D.

Because the S-curvature of any Berwald metric vanishes and its spray coefficients can be induced by a Riemannian metric, the equation (3.12) still holds for the projective equivalence from a Berwald space to a Riemann space. Hence, Theorem 3.7 can be generalized as follows.

Theorem 3.8([Ch1]) Let g be a Riemann metric on a compact manifold M and $\tilde{g}$ a Berwald metric on M which is pointwise projectively related to g. Assume that one of the following conditions is satisfied,

(a) $tr_g \widetilde{\mathbf{Ric}} \leq \mathbf{s}_g$,

(b) $tr_g \widetilde{\mathbf{Ric}} \geq \mathbf{s}_g$,

then the conclusion in Theorem 3.7 holds.

If we modify the inequality (3.5) in Theorem 3.2 into equality, we have the following theorem.

Theorem 3.9.([Ch1]) Let F be a Finsler metric on a manifold M and $\tilde{F}$ a another Finsler metric on M which is pointwise projectively related to F. Suppose that both F and $\tilde{F}$ satisfy

$$\widetilde{\mathbf{Ric}} = \mathbf{Ric}.$$

Then F is complete if and only if $\tilde{F}$ is complete. In this case, along any geodesic $c(t)$ of F or $\tilde{F}$,

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = constant.$$

Proof. Let $c(t)$ be an arbitrary unit speed geodesic in (M, F) and

$$\tilde{F}(t) := \tilde{F}(\dot{c}(t)), \quad P(t) := P(\dot{c}(t)).$$

Observe that

$$\tilde{F}'(t) = \tilde{F}_{;k}(\dot{c}(t))\dot{c}^k(t), \quad P'(t) = P_{;k}(\dot{c}(t))\dot{c}^k(t).$$

From Lemma 2.1, we have

$$P(t) = \tilde{F}'(t)/[2\tilde{F}(t)]. \tag{3.16}$$

Let

$$f(t) := \frac{1}{\sqrt{\tilde{F}(t)}}.$$

(3.16) becomes

$$P(t) = -f'(t)/f(t). \tag{3.17}$$

Now, we assume that $\widetilde{\mathbf{Ric}} = \mathbf{Ric}$. From (2.11), we have $\Xi := P^2 - P_{;j}y^j = 0$. Thus for any unit speed geodesic $c(t)$ of F, we have

$$P'(t) - P^2(t) = 0. \tag{3.18}$$

Let $f(0) := a > 0, f^{'}(0) := b$. Then, from (3.18), we get

$$P(t) = -\frac{b}{a+bt}.$$

Thus, by (3.17), we obtain

$$f(t) = a + bt, \quad i.e., \quad \tilde{F}(\dot{c}(t)) = \frac{1}{(a+bt)^2}.$$

(i) If $b = 0$, then $f(t) = a$. Thus $f(t)$ is defined on $I = (-\infty, +\infty)$ and $\tilde{F}(\dot{c}(t)) = 1/a^2$.

(ii) If $b > 0$, then $f(t)$ is defined on $I = (-\delta, +\infty)$ and

$$\int_{-\delta}^{0} \tilde{F}(\dot{c}(t)) = +\infty \quad and \quad \int_{0}^{+\infty} \tilde{F}(\dot{c}(t))dt < +\infty.$$

The case when $b < 0$ is similar, so is omitted.

According to the discussion as above, we can conclude that, if F is complete, then any unit speed geodesic $c(t)$ of F is defined on $(-\infty, +\infty)$ and $\tilde{F}(\dot{c}(t)) =$

$1/a^2$. In this case, $\tilde{F}$ must be complete. The inverse is also obviously true. Hence, F is complete if and only if $\tilde{F}$ is complete and

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = constant.$$

Q.E.D.

We note that, the condition that the projective change preserves the Ricci curvature can not be cancelled.

Example 3.10. Suppose that (M, F) and $(M, \tilde{F})$ are pointwise projectively related and the projective change is characterized by

$$\tilde{G}^i = G^i + Fy^i. \tag{3.19}$$

That is, the projective factor P is just F. Thus we get $P_i = y_i/F$, where $y_i = g_{ij}(x, y)y^j$. Since $F_{;i} = 0$, we get $\Xi = F^2 \neq 0$. Thus the projective change (3.19) does not preserve the Ricci curvature. In this case, $P(t) = F(\dot{c}(t)) = 1$. Then, by (3.17), we obtain

$$f(t) = ae^{-t}.$$

Hence, if F is complete, $f(t)$ is defined on $I = (-\infty, +\infty)$, and

$$\int_{-\infty}^{0} \tilde{F}(\dot{c}(t))dt < +\infty \quad and \quad \int_{0}^{+\infty} \tilde{F}(\dot{c}(t))dt = +\infty. \tag{3.20}$$

Furthermore, for any unit speed geodesic $c(t)$ of F,

$$\tilde{F}(\dot{c}(t)) = e^{2t}/a^2. \tag{3.21}$$

From (3.20) and (3.21), we see that $\tilde{F}$ is just positively complete and

$$F(\dot{c}(t))/\tilde{F}(\dot{c}(t)) \neq constant.$$

Chapter 4

Finsler Metrics of Scalar Curvature

It has been proved that the flag curvature in Finsler geometry is closely related to some non-Riemannian geometric quantities, such as $\mathbf{C}, \mathbf{L}, \mathbf{J}, \mathbf{I}$ and $\mathbf{S}$. Firstly, the Riemann curvature satisfies the following Bianchi identity [Sh2][AIM]

$$R^i_{k|l} - R^i_{l|k} - R^i_{kl|m} y^m = L^i_{km} R^m_l - L^i_{lm} R^m_k. \tag{4.1}$$

Furthermore, we can prove the following important equations [MoSh]

$$\begin{aligned} L_{ijk|m} y^m + C_{ijm} R^m_k = & -\frac{1}{3} g_{im} R^m_{k.j} - \frac{1}{3} g_{jm} R^m_{k.i} \\ & -\frac{1}{6} g_{im} R^m_{j.k} - \frac{1}{6} g_{jm} R^m_{i.k}. \end{aligned} \tag{4.2}$$

Contracting (4.2) with g^{ij} gives

$$J_{k|m} y^m + I_m R^m_k = -\frac{1}{3}\{2R^m_{k.m} + R^m_{m.k}\} \tag{4.3}$$

Further, we obtain

$$\mathbf{S}_{.k|m} y^m - \mathbf{S}_{|k} = -\frac{1}{3}\{2R^m_{k.m} + R^m_{m.k}\}. \tag{4.4}$$

It is a difficult task to classify Finsler metrics of scalar curvature. All known Randers metrics of scalar curvature (in dimension $n > 2$) satisfy $\mathbf{S} = (n +$

$1)c(x)F$ or $\mathbf{J} + c(x)F\mathbf{I} = 0$, where $c(x)$ is a scalar function on M. Thus it is a natural idea to investigate firstly Finsler metrics of scalar curvature with isotropic S-curvature.

Theorem 4.1.([CMS][ChSh3]) Let (M, F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\mathbf{K}(x, y)$. Suppose that the S-curvature is isotropic,

$$\mathbf{S} = (n+1)c(x)F(x, y), \tag{4.5}$$

where $c(x)$ is a scalar function on M. Then there is a scalar function $\sigma(x)$ on M such that

$$\mathbf{K} = 3\frac{c_{x^m} y^m}{F(x, y)} + \sigma(x). \tag{4.6}$$

In particular, $c = constant$ if and only if $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on M.

Proof. Plugging (2.4) into (4.4), we obtain

$$\mathbf{S}_{\cdot k|l} y^l - \mathbf{S}_{|k} = -\frac{n+1}{3}\mathbf{K}_{\cdot k} F^2. \tag{4.7}$$

Plugging (4.5) into (4.7) yields

$$c_{|l}(x) y^l F_{\cdot k} - c_{|k}(x)F = -\frac{1}{3}\mathbf{K}_{\cdot k} F^2. \tag{4.8}$$

It follows from (4.8) that

$$\Big[\frac{1}{3}\mathbf{K} - \frac{c_{|m}(x) y^m}{F(x, y)}\Big]_{y^k} = 0. \tag{4.9}$$

Thus

$$\sigma := \mathbf{K} - \frac{3c_{|m} y^m}{F}$$

is a scalar function on M. This proves the theorem. Q.E.D.

In Theorem 4.1, we partially determine the flag curvature when the S-curvature is isotropic. This is a generalization of a theorem in [Mo] where Mo shows that the flag curvature is isotropic, $\mathbf{K} = \mathbf{K}(x)$ if $\mathbf{S} = (n+1)cF$ for c =constant. In this case, $\mathbf{K}$ = constant when $n \geq 3$ by the Schur theorem.

Theorem 4.2([CMS][ChSh3]) Let (M, F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\mathbf{K}(x, y)$. Suppose that F has relatively isotropic mean Landsberg curvature,

$$\mathbf{J} + c(x)F\mathbf{I} = 0, \tag{4.10}$$

where $c = c(x)$ is a C^∞ scalar function on M. Then the flag curvature $\mathbf{K}$ and the distortion τ satisfy

$$\frac{n+1}{3}\mathbf{K}_{y^k} + \Big(\mathbf{K} + c(x)^2 - \frac{c_{x^m} y^m}{F(x,y)}\Big)\tau_{y^k} = 0. \tag{4.11}$$

(a) If $c(x) = constant$, then there is a scalar function $\rho(x)$ on M such that

$$\mathbf{K} = -c^2 + \rho(x)e^{-\frac{3\tau(x,y)}{n+1}}, \qquad y \in T_xM \setminus \{0\}.$$

(b) Suppose that F is non-Riemannian on any open subset of M. Then $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{K} = -c^2$ is a nonpositive constant. In this case, $\rho(x) = 0$.

Proof. By (4.3) and (2.4), we obtain

$$J_{k|m}y^m = -\frac{1}{3}F^2\Big\{(n+1)\mathbf{K}_{\cdot k} + 3\mathbf{K}I_k\Big\}. \tag{4.12}$$

By assumption, $J_k = -cFI_k$ and $J_k = I_{k|m}y^m$, we obtain

$$J_{k|m}y^m = -c_{|m}y^m FI_k - cFI_{k|m}y^m = -c_{|m}y^m FI_k + c^2F^2I_k.$$

It follows from (4.12) that

$$\frac{n+1}{3}\mathbf{K}_{\cdot k} + \Big(\mathbf{K} + c^2 - \frac{c_{x^m}y^m}{F}\Big)I_k = 0. \tag{4.13}$$

By (2.6), $I_k = \tau_{\cdot k}$. We obtain (4.11).

(a) Suppose that $c_{x^m}(x) = 0$ at some point $x \in M$. Then equation (4.11) simplifies to

$$\frac{n+1}{3}\mathbf{K}_{y^k} + \Big(\mathbf{K} + c^2\Big)\tau_{y^k} = 0.$$

This implies that

$$\Big[\Big(\mathbf{K} + c^2\Big)^{\frac{n+1}{3}} e^\tau\Big]_{y^k} = \Big(\mathbf{K} + c^2\Big)^{\frac{(n-2)}{3}} e^\tau \Big\{\frac{n+1}{3}\mathbf{K}_{y^k} + \Big(\mathbf{K} + c^2\Big)\tau_{y^k}\Big\} = 0.$$

Thus the function $(\mathbf{K} + c^2)^{\frac{(n+1)}{3}} e^\tau$ is independent of $y \in T_xM$. There is a number $\rho(x)$ such that

$$\mathbf{K} = -c(x)^2 + \rho(x)e^{-\frac{3\tau(x,y)}{n+1}}. \tag{4.14}$$

When $c(x) = constant$, from (4.14), we obtain

$$\mathbf{K} = -c^2 + \rho(x)e^{-\frac{3\tau(x,y)}{n+1}}, \qquad y \in T_xM \setminus \{0\}.$$

Note that $\rho(x)$ is not necessarily a constant.

(b) Suppose that $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on M. Then (4.11) simplifies to

$$\Big(\mathbf{K} + c^2 - \frac{c_{x^m}y^m}{F}\Big)\tau_{y^k} = 0. \tag{4.15}$$

We claim that $c(x) = constant$. Suppose this is false. Then there is an open subset $\mathcal{U}$ such that $dc(x) \neq 0$ for any $x \in \mathcal{U}$. Clearly, at any $x \in \mathcal{U}$, $\mathbf{K}(x) \neq -c(x)^2 + c_{x^m}(x)y^m/F(x,y)$ for almost all $y \in T_xM$. By (4.15), $\tau_{\cdot k} = I_k = 0$. Thus F is Riemannian on $\mathcal{U}$ by Deicke's theorem (cf. [Sh2]). This contradicts our assumption in the theorem. This proves the claim. By (4.14) and (4.15), we obtain

$$\rho(x)\ \tau_{y^k} = 0. \tag{4.16}$$

We claim that $\rho(x) \equiv 0$. If this is false, then there is an open subset $\mathcal{U}$ such that $\rho(x) \neq 0$ for any $x \in \mathcal{U}$. By (4.16), we obtain that $\tau_{y^k} = I_k = 0$ on $\mathcal{U}$. Thus F is Riemannian on $\mathcal{U}$. This again contradicts the assumption in the theorem. Therefore $\rho(x) \equiv 0$. We conclude that $\mathbf{K} = -c^2$ by (4.14). Q.E.D.

Finsler metrics with $\mathbf{J} = 0$ are said to be *weakly Landsbergian*. From Theorem 4.2 (b), we have the following

Corollary 4.3([ChSh3]) For a non-Riemannian weak Landsberg metric, $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{K} = 0$.

According to [ChSh2], for any Randers metric $F = \alpha + \beta$, (4.10) holds if and only if (4.5) holds and β is closed. For a general Finsler metric, (4.10) does not imply (4.5). Now we combine two conditions (4.5) and (4.10) and give the following

Theorem 4.4([CMS][ChSh3]) Let (M, F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\mathbf{K}(x,y)$. Suppose that

$$\mathbf{S} = (n+1)c(x)F, \qquad \mathbf{J} + c(x)F\mathbf{I} = 0, \tag{4.17}$$

where $c = c(x)$ is a scalar function on M. Then there are scalar functions $\sigma(x)$ and $\mu(x)$ on M, such that the flag curvature is given by

$$\mathbf{K} = 3\frac{c_{x^m}y^m}{F(x,y)} + \sigma(x) = -\frac{3c(x)^2 + \sigma(x)}{2} + \mu(x)e^{-\frac{2\tau(x,y)}{n+1}}. \tag{4.18}$$

(a) Suppose that F is non-Riemannian on any open subset of M. Then $c(x) =$ *constant* if and only if $\mathbf{K} = -c^2, \sigma(x) = -c^2$ and $\mu(x) = 0$.

(b) If $c(x) \neq$ *constant*, then the distortion is given by

$$\tau = \ln \left[\frac{2\mu(x)F(x,y)}{6c_{x^m}y^m + 3[\sigma(x) + c(x)^2]F(x,y)} \right]^{\frac{n+1}{2}}. \tag{4.19}$$

Proof. By the above argument, $\mathbf{K}$ is given by (4.6) and it satisfies (4.11). By (4.6), we obtain

$$\frac{c_{x^m}(x)y^m}{F(x,y)} = \frac{1}{3}\Big(\mathbf{K} - \sigma(x)\Big).$$

Plugging it into (4.11) yields

$$\frac{n+1}{3}\mathbf{K}_{y^k} + \Big(\frac{2}{3}\mathbf{K} + c(x)^2 + \frac{1}{3}\sigma(x)\Big)\tau_{y^k} = 0.$$

We obtain

$$\Big[\Big(2\mathbf{K} + 3c(x)^2 + \sigma(x)\Big)^{\frac{n+1}{2}} e^{\tau}\Big]_{y^k} = 0.$$

Thus there is a scalar function $\mu(x)$ on M such that

$$\mathbf{K} = -\frac{3c(x)^2 + \sigma(x)}{2} + \mu(x)e^{-\frac{2\tau(x,y)}{n+1}}. \tag{4.20}$$

Comparing (4.20) with (4.6), we obtain

$$\frac{c_{x^m}(x)y^m}{F(x,y)} = -\frac{c(x)^2 + \sigma(x)}{2} + \frac{\mu(x)}{3}e^{-\frac{2\tau(x,y)}{n+1}}. \tag{4.21}$$

(a) If $c(x) = c$ is a constant, we claim that $\mu(x) = 0$. If this is false, then $\mathcal{U} := \Big\{x \in M, \mu \neq 0\Big\} \neq \emptyset$. From (4.21), one can see that $\tau = \tau(x)$ is a scalar function on $\mathcal{U}$, hence F is Riemannian on $\mathcal{U}$ by Deicke's theorem. This contradicts our assumption. Now (4.21) is reduced to $\sigma(x) = -c(x)^2$ and (4.20) is reduced to $\mathbf{K} = -c^2$. The inverse is also true by (4.18).

(b) If $c(x) \neq$ *constant*, then $\mu(x) \neq 0$ by (4.21). In this case, we can solve (4.21) for τ and obtain (4.19).

Q.E.D.

It follows from Theorem 4.4 that if a Finsler metric of scalar curvature satisfies $\mathbf{S} = (n+1)cF$ and $\mathbf{J} + cF\mathbf{I} = 0$ for some constant c, then the flag curvature is given by $\mathbf{K} = -c^2$. One would like to know whether or not there are non-Riemannian, non-locally Minkowskian Finsler metrics with these properties. If a Randers metric has these properties, then it is, up to a scaling, locally isometric to the generalized Funk metric on the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ [ChSh2] (cf. section 5). In dimension two, any Finsler metric with $\mathbf{S} = 0$, $\mathbf{J} = 0$ and $\mathbf{K} = 0$ is locally Minkowskian.

Example 4.5. For an arbitrary number ε with $0 < \varepsilon \leq 1$, define

$$\alpha: \;=\; \frac{\sqrt{(1-\varepsilon^2)(xu+yv)^2 + \varepsilon(u^2+v^2)(1+\varepsilon(x^2+y^2))}}{1+\varepsilon(x^2+y^2)}$$

$$\beta: \;=\; \frac{\sqrt{1-\varepsilon^2}(xu+yv)}{1+\varepsilon(x^2+y^2)}.$$

We have

$$\|\beta\|_\alpha = \sqrt{1-\varepsilon^2}\sqrt{\frac{x^2+y^2}{\varepsilon+x^2+y^2}} < 1.$$

Thus $F := \alpha + \beta$ is a Randers metric on R^2. In [ChSh2], we have verified that

$$\mathbf{S} = 3cF, \qquad \mathbf{J}_y + cF\,\mathbf{I}_y = 0$$

where

$$c = \frac{\sqrt{1-\varepsilon^2}}{2(\varepsilon+x^2+y^2)},$$

and obtained a formula for the Gauss curvature

$$\mathbf{K} \;=\; \frac{-3\sqrt{1-\varepsilon^2}\;(xu+yv)/(1+\varepsilon(x^2+y^2))}{\sqrt{(1-\varepsilon^2)(xu+yv)^2+\varepsilon(u^2+v^2)(1+\varepsilon(x^2+y^2))}+\sqrt{1-\varepsilon^2}(xu+yv)} + \frac{7(1-\varepsilon^2)+8\varepsilon(\varepsilon+x^2+y^2)}{4(\varepsilon+x^2+y^2)^2}.$$

Here we are going to compute σ and μ in Theorem 4.4. By a direct computation we can express the function $\sigma := \mathbf{K} - \frac{3(c_x u + c_y v)}{F}$ in (4.18) by

$$\sigma = \frac{7(1-\varepsilon^2)}{4(\varepsilon+x^2+y^2)^2} + \frac{2\varepsilon}{\varepsilon+x^2+y^2}.$$

That is, the Gauss curvature is given by

$$\begin{aligned}\mathbf{K} &= 3\frac{c_x u + c_y v}{F} + \sigma \\ &= -\frac{3\sqrt{1-\varepsilon^2}(xu+yv)}{(\varepsilon+x^2+y^2)^2 F} + \frac{7(1-\varepsilon^2)}{4(\varepsilon+x^2+y^2)^2} + \frac{2\varepsilon}{\varepsilon+x^2+y^2}.\end{aligned}$$

For any Randers metric $F = \alpha + \beta$, the distortion is given by

$$\tau = \ln\left[\frac{F}{\alpha}\cdot\frac{1}{1-\|\beta\|_\alpha^2}\right]^{\frac{3}{2}}.$$

A direct computation yields

$$1 - \|\beta\|_\alpha^2 = \frac{\varepsilon(1+\varepsilon(x^2+y^2))}{\varepsilon+x^2+y^2}.$$

Then the function $\mu := \Big(\mathbf{K} + \frac{3c^2+\sigma}{2}\Big)e^{\frac{2\tau}{3}}$ in (4.18) is given by

$$\mu = \frac{3}{\varepsilon(\varepsilon+x^2+y^2)}.$$

That is, the Gauss curvature can also be given by

$$\begin{aligned}\mathbf{K} &= -\frac{3c^2+\sigma}{2} + \mu\, e^{-\frac{2\tau}{3}} \\ &= -\frac{5-\varepsilon^2+4\varepsilon(x^2+y^2)}{2(\varepsilon+x^2+y^2)^2} + \frac{3\Big(1+\varepsilon(x^2+y^2)\Big)\alpha}{(\varepsilon+x^2+y^2)^2 F}.\end{aligned}$$

Chapter 5

Randers Metrics with $\mathbf{L} + c(x)F\mathbf{C} = 0$

Finsler metrics with $\mathbf{L} = 0$ (i.e. Landsberg metrics) can be generalized as follows. Let F be a Finsler metric on an n-dimensional manifold M. F is said to have relatively isotropic Landsberg curvature (resp. relatively isotropic mean Landsberg curvature) if

$$\mathbf{L} + cF\mathbf{C} = 0, \qquad (resp. \quad \mathbf{J} + cF\mathbf{I} = 0),$$

where $c = c(x)$ is a scalar funcrion on M. We note that $\mathbf{L}/\mathbf{C}$ (resp. $\mathbf{J}/\mathbf{I}$) characterizes the relative growth rate of the Cartan torsion (resp. the mean Cartan torsion) along geodesics.

Many interesting Finsler metrics have relatively isotropic L-curvature (J-curvature) or isotropic S-curvature. For example, the shortest time problem on a Riemannian manifold (or Zermelo's problem of navigation on Riemannian manifolds) gives rise to a Randers metric. By choosing appropriate Riemann metric h and an external force field W, we can obtain many Randers metrics with many special non-Rienannian curvature properties as above [Sh4][Sh5]. In particular, on the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$, taking W as the position vector field, we obtain the well-known Funk metric on $\mathbf{B}^n$ with the following curvature properties: (i) $\mathbf{S} = (1/2)(n+1)F$; (ii) $\mathbf{E} = (1/4)(n+1)F^{-1}h$; (iii) $\mathbf{J} + (1/2)F\mathbf{I} = 0$ and (iv) $\mathbf{K} = -1/4$. Motivated by the properties of Funk metrics, we study Randers metrics satisfying (i), (ii) or (iii). In this section, we mainly study Randers metrics with relatively isotropic L-curvature.

From definition, the mean Landsberg curvature is the mean value of the Landsberg curvature. Thus if a Finsler metric has relatively isotropic Landsberg curvature, then it must have relatively isotropic mean Landsberg curvature. We don't know whether or not the converse is true too. So far no counter-example has been found yet. Nevertheless, for Randers metrics, we have the following

Lemma 5.1. For any Randers metric $F = \alpha + \beta$, the following are equivalent:

(a) $\mathbf{J} + c(x)F\mathbf{I} = 0$;

(b) $\mathbf{L} + c(x)F\mathbf{C} = 0$,

where $c(x)$ is a scalar function on M.

Proof. By [Ma], Randers metric $F = \alpha + \beta$ is C-reducible, that is,

$$C_{ijk} = \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\Big\}. \tag{5.1}$$

From (5.1) and using $h_{ij|m} = 0$, we obtain

$$C_{ijk|m}y^m = \frac{1}{n+1}\Big\{I_{i|m}y^m h_{jk} + I_{j|m}y^m h_{ik} + I_{k|m}y^m h_{ij}\Big\},$$

that is,

$$L_{ijk} = \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ik} + J_k h_{ij}\Big\}. \tag{5.2}$$

From (5.1) and (5.2), we get

$$L_{ijk} - cFC_{ijk} = \frac{1}{n+1}\Big\{(J_i - cFI_i)h_{jk} + (J_j - cFI_j)h_{ik} + (J_k - cFI_k)h_{ij}\Big\}.$$

Hence, $\mathbf{J} + cF\mathbf{I} = 0$ implies that $\mathbf{L} + cF\mathbf{C} = 0$. Q.E.D.

Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. An easy computation yields

$$g_{ij} = \frac{F}{\alpha}\Big(a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha}\Big) + \Big(\frac{y_i}{\alpha} + b_i\Big)\Big(\frac{y_j}{\alpha} + b_j\Big), \tag{5.3}$$

where $y_i := a_{ij}y^j$. By an elementary argument in linear algebra, we obtain

$$\det(g_{jk}) = \Big(\frac{F}{\alpha}\Big)^{n+1}\det(a_{ij}). \tag{5.4}$$

Define $b_{i;j}$ by

$$b_{i;j}\theta^j := db_i - b_j\theta_i^j,$$

where θ_i^j denotes the Levi-Civita connection forms of α.

Let

$$r_{ij} := \frac{1}{2}\left(b_{i;j} + b_{j;i}\right), \quad s_{ij} = \frac{1}{2}\left(b_{i;j} - b_{j;i}\right), \quad s^i_j := a^{ih}s_{hj},$$

$$s_j := b_i s^i_j, \quad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then the geodesic coefficients G^i are given by

$$G^i = \bar{G}^i + \frac{e_{00}}{2F}y^i - s_0 y^i + \alpha s^i_0, \tag{5.5}$$

where $\bar{G}^i$ denote the geodesic coefficients of α, $e_{00} := e_{ij}y^iy^j$, $s_0 := s_i y^i$ and $s^i_0 := s^i_j y^j$. See [AIM].

Lemma 5.2.([ChSh2]) For a Randers metric $F = \alpha + \beta$, the mean Cartan torsion $\mathbf{I} = I_i dx^i$ and the mean Landsberg curvature $\mathbf{J} = J_i dx^i$ are given by

$$\begin{aligned} I_i &= \frac{1}{2}(n+1)F^{-1}\alpha^{-2}\Big\{\alpha^2 b_i - \beta y_i\Big\} && (5.6)\\ J_i &= \frac{1}{4}(n+1)F^{-2}\alpha^{-2}\Big\{2\alpha\Big[(e_{i0}\alpha^2 - y_i e_{00}) - 2\beta(s_i\alpha^2 - y_i s_0) + s_{i0}(\alpha^2+\beta^2)\Big] \\ &\quad +\alpha^2(e_{i0}\beta - b_i e_{00}) + \beta(e_{i0}\alpha^2 - y_i e_{00}) \\ &\quad -2(s_i\alpha^2 - y_i s_0)(\alpha^2+\beta^2) + 4s_{i0}\alpha^2\beta\Big\}. && (5.7)\end{aligned}$$

Proof. By (2.6), we have $I_i = \tau_{.i}$. From (5.4) and the definition of τ, we get (5.6). Now we are going to compute J_i. From (2.6), we can get the following

$$J_i = y^j\frac{\partial I_i}{\partial x^j} - I_j\frac{\partial G^j}{\partial y^i} - 2G^j\frac{\partial I_i}{\partial y^j}.$$

Let

$$H^i := \frac{e_{00}}{2F}y^i - s_0 y^i + \alpha {s^i}_0.$$

We can rewrite the expression above on J_i as follows

$$J_i = y^j I_{i;j} - I_j {H^j}_{\cdot i} - 2H^j I_{i\cdot j}, \tag{5.8}$$

where $H^j{}_{\cdot i} := \frac{\partial H^j}{\partial y^i}$, $I_{i\cdot j} = \frac{\partial I_i}{\partial y^j}$ and $I_{i;j}$ are defined by

$$dI_i - I_j \frac{\partial \bar{G}^j}{\partial y^i \partial y^k} dx^k = I_{i;j} dx^j + I_{i\cdot j}\Big(dy^j + \frac{\partial \bar{G}^j}{\partial y^k} dx^k\Big).$$

By a direct computation, we obtain

$$\begin{aligned}
I_{i\cdot j} &= -\frac{n+1}{2} F^{-2}\alpha^{-2}\Big(y_j\alpha^{-1} + b_j\Big)\Big(\alpha^2 b_i - \beta y_i\Big) \\
&\quad -(n+1)F^{-1}\alpha^{-4} y_j\Big(\alpha^2 b_i - \beta y_i\Big) \\
&\quad +\frac{n+1}{2} F^{-1}\alpha^{-2}\Big(2y_j b_i - b_j y_i - \beta a_{ij}\Big) \\
H^j{}_{\cdot i} &= \frac{e_{00}}{2F}\delta^j_i + \frac{e_{i0}}{F} y^j - \frac{e_{00}}{2F^2}\Big(y_i\alpha^{-1} + b_i\Big) y^j \\
&\quad -s_0\delta^j_i - s_i y^j + y_i\alpha^{-1} s^j{}_0 + \alpha s^j{}_i.
\end{aligned}$$

where $b_{i;0} = b_{i;j}y^j$ and $b_{0;0} = b_{i;j}y^iy^j$. Observe that

$$b_{i;j} = r_{ij} + s_{ij} = e_{ij} - b_i s_j - b_j s_i + s_{ij}.$$

We have

$$b_{i;0} = e_{i0} - b_i s_0 - s_i\beta + s_{i0}, \qquad b_{0;0} = e_{00} - 2s_0\beta.$$

By these identities, we obtain

$$\begin{aligned}
y^j I_{i;j} &= -\frac{n+1}{2} F^{-2}\alpha^{-2} b_{0;0}\Big(\alpha^2 b_i - \beta y_i\Big) + \frac{n+1}{2} F^{-1}\alpha^{-2}\Big(\alpha^2 b_{i;0} - b_{0;0} y_i\Big) \\
&= -\frac{n+1}{2} F^{-2}\alpha^{-2}\Big(e_{00} - 2s_0\beta\Big)\Big(\alpha^2 b_i - \beta y_i\Big) \\
&\quad +\frac{n+1}{2} F^{-1}\alpha^{-2}\Big((\alpha^2 e_{i0} - e_{00} y_i) - \alpha^2(b_i s_0 + s_i\beta) + 2s_0\beta y_i + \alpha^2 s_{i0}\Big).
\end{aligned}$$

Plugging them into (5.8) yields (5.7). Q.E.D.

As mentioned in Lemma 5.2, the mean Landsberg curvature $\mathbf{J}$ can be expressed in term of α and β. But the formula is very complicated. We find a simpler necessary and sufficient condition for $\mathbf{J} + cF\mathbf{I} = 0$.

Theorem 5.3.([ChSh2][ChSh3]) Let $F = \alpha + \beta$ be a Randers metric on a manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent

(a) $\mathbf{J} + c(x)F\mathbf{I} = 0$;

(b) $e_{00} = 2c(\alpha^2 - \beta^2)$ and β is closed.

Proof. Let

$$f_{ij} := e_{ij} - 2c(a_{ij} - b_i b_j)$$

and $f_{i0} := f_{ij}y^j$, $f_{00} := f_{ij}y^iy^j$. We have

$$2\alpha(e_{i0}\alpha^2 - y_i e_{00}) + \alpha^2(e_{i0}\beta - b_i e_{00}) + \beta(e_{i0}\alpha^2 - y_i e_{00}) =$$
$$2\alpha(f_{i0}\alpha^2 - y_i f_{00}) + \alpha^2(f_{i0}\beta - b_i f_{00}) + \beta(f_{i0}\alpha^2 - y_i f_{00}) - 2c(b_i\alpha^2 - y_i\beta)F^2.$$

Plugging it into (5.7), we see that $\mathbf{J} + cF\,\mathbf{I} = 0$ if and only if

$$(f_{i0}\beta - b_i f_{00})\alpha^2 + (f_{i0}\alpha^2 - y_i f_{00})\beta + 4s_{i0}\alpha^2\beta - 2(s_i\alpha^2 - y_i s_0)(\alpha^2 + \beta^2) = 0, \quad (5.9)$$

$$(f_{i0}\alpha^2 - y_i f_{00}) + s_{i0}(\alpha^2 + \beta^2) - 2(s_i\alpha^2 - y_i s_0)\beta = 0. \quad (5.10)$$

Differentiating (5.10) with respect to y^j, y^k and y^l, we obtain

$$\begin{aligned} 0 \;\; = \;\; & f_{ij}a_{kl} + f_{ik}a_{jl} + f_{il}a_{jk} - a_{ij}f_{kl} - a_{ik}f_{jl} - a_{il}f_{jk} \\ & + s_{ij}(a_{kl} + b_k b_l) + s_{ik}(a_{jl} + b_j b_l) + s_{il}(a_{jk} + b_j b_k) \\ & -(2a_{kl}s_i - a_{ik}s_l - a_{il}s_k)b_j \\ & -(2a_{jl}s_i - a_{ij}s_l - a_{il}s_j)b_k \\ & -(2a_{jk}s_i - a_{ij}s_k - a_{ik}s_j)b_l. \end{aligned} \quad (5.11)$$

Contracting (5.11) with a^{kl} yields

$$nf_{ij} - \lambda a_{ij} + s_{ij}(n + 2 + \|\beta\|^2) - 2(n+1)s_i b_j + 2(b_i s_j - b_j s_i) = 0, \quad (5.12)$$

where $\lambda := a^{kl}f_{kl}$. Here we have made the use of the identity $b_k a^{kl} s_{il} = -s_i$. It follows from (5.12) that

$$f_{ij} = \frac{\lambda}{n}\, a_{ij} + \frac{n+1}{n}(s_i b_j + s_j b_i), \quad (5.13)$$

$$s_{ij}(n + 2 + \|\beta\|^2) = (n-1)(s_i b_j - s_j b_i). \quad (5.14)$$

Contracting (5.14) with $b^i := b_r a^{ri}$ yields

$$s_j = 0.$$

Plugging it into (5.14) we obtain that

$$s_{ij} = 0$$

and

$$f_{ij} = \frac{\lambda}{n} a_{ij}. \tag{5.15}$$

Now equation (5.9) simplifies to

$$\lambda(b_i \alpha^2 - y_i \beta) = 0. \tag{5.16}$$

Taking $y_i = b_i$ in (5.16) we obtain

$$\lambda(\|\beta\|^2 - 1) b_i = 0. \tag{5.17}$$

Assume that $\beta \not\equiv 0$. It follows from (5.17) that $\lambda = 0$. From (5.15), we conclude that $f_{ij} = 0$.

Conversely, we suppose that $e_{00} = 2c(\alpha^2 - \beta^2)$. Then

$$e_{i0} = 2c(y_i - b_i \beta), \qquad e_{00} = 2c(\alpha^2 - \beta^2).$$

We obtain

$$e_{i0}\alpha^2 - y_i e_{00} = -2c(b_i\alpha^2 - y_i\beta)\beta, \tag{5.18}$$

$$e_{i0}\beta - b_i e_{00} = -2c(b_i\alpha^2 - y_i\beta). \tag{5.19}$$

Plugging (5.18) and (5.19) into (5.7) yields

$$J_i = \frac{1}{2}(n+1)\alpha^{-2}\Big\{ -c\Big[(b_i\alpha^2 - y_i\beta) + (s_i\alpha^2 - y_i s_0)\Big] + s_{i0}\alpha\Big\}. \tag{5.20}$$

Further, suppose that β is closed, hence $s_{ij} = 0$. From (5.6) and (5.20), we obtain

$$J_i = -\frac{1}{2}(n+1)c\alpha^{-2}\Big\{b_i\alpha^2 - y_i\beta\Big\} = -cF\, I_i.$$

This proves the theorem. Q.E.D.

In 2003, Z. Shen has classified all locally projectively flat Randers metrics with constant flag curvature [Sh7]. He proves that a locally projectively flat Randers metric with constant flag curvature $\mathbf{K} = \lambda$ is either locally Minkowskian or after a scaling, isometric to the a Finsler metric on the unit ball $\mathbf{B}^n$ in the following form

$$F_{\mathbf{a}} = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1 - |\mathbf{x}|^2} \pm \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1 - |\mathbf{x}|^2} \pm \frac{\langle \mathbf{a}, \mathbf{y}\rangle}{1 + \langle \mathbf{a}, \mathbf{x}\rangle}, \qquad \mathbf{y} \in T_{\mathbf{x}}\mathrm{R}^n,$$

where $\mathbf{a} \in \mathrm{R}^n$ is a constant vector with $|\mathbf{a}| < 1$. One can directly verify that $F_{\mathbf{a}}$, $\mathbf{a} \neq 0$, are locally projectively flat Finsler metrics with negative constant flag curvature. Moreover, they have the following properties of the Funk metrics: (i) $\mathbf{S} = \pm(1/2)(n+1)F$, (ii) $\mathbf{E} = \pm(1/4)(n+1)F^{-1}h$ and (iii)$\mathbf{J} \pm (1/2)F\mathbf{I} = 0$ and (iv) $\mathbf{K} = -\frac{1}{4}$. In fact, there are lots of Randers metrics satisfying

$$\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}h, \qquad \mathbf{J} + cF\,\mathbf{I} = 0.$$

Besides, the Randers metrics stated as above with $c = \pm\frac{1}{2}$, we have the following example with $c = c(x) \neq constant$.

Example 5.4. For an arbitrary number ε with $0 < \varepsilon \leq 1$, define

$$\begin{aligned}\alpha : &= \frac{\sqrt{(1-\varepsilon^2)(xu+yv)^2 + \varepsilon(u^2+v^2)(1+\varepsilon(x^2+y^2))}}{1+\varepsilon(x^2+y^2)}\\ \beta : &= \frac{\sqrt{1-\varepsilon^2}(xu+yv)}{1+\varepsilon(x^2+y^2)}.\end{aligned}$$

We have

$$\|\beta\|_\alpha = \sqrt{1-\varepsilon^2}\sqrt{\frac{x^2+y^2}{\varepsilon+x^2+y^2}} < 1.$$

Thus $F := \alpha+\beta$ is a Randers metric on R^2. By a direct computation, we obtain

$$\mathbf{J} + cF\,\mathbf{I} = 0,$$

where

$$c = \frac{\sqrt{1-\varepsilon^2}}{2(\varepsilon+x^2+y^2)}.$$

Moreover, the Gauss curvature of F is given by

$$\mathbf{K} = -\frac{3\sqrt{1-\varepsilon^2}(xu+yv)}{(\varepsilon+x^2+y^2)^2F} + \frac{7(1-\varepsilon^2)}{4(\varepsilon+x^2+y^2)^2} + \frac{2\varepsilon}{\varepsilon+x^2+y^2}.$$

Thus F does not have constant Gauss curvature.

Based on Theorem 5.3, we can classify Randers metrics with scalar flag curvature $\mathbf{K} = \lambda(x)$ and $\mathbf{J} + c(x)F\mathbf{I} = 0$.

Theorem 5.5.([ChSh2][ChSh3]) Let $F = \alpha + \beta$ ba Randers metric on an n-dimensional manifold M satisfying

1. $\mathbf{K} = \lambda(x)$ is independent of $y \in T_xM$;
2. $\mathbf{J} + c(x)F\mathbf{I} = 0$ for some scalar function $c(x)$ on M.

Then $\mathbf{K} = constant = -c^2 \leq 0$. Further, F is either locally Minkowskian ($\mathbf{K} = -c^2 = 0$) or in the form

$$F = \Theta \pm \frac{< \mathbf{a}, y >}{1+ < \mathbf{a}, x >}$$

($\mathbf{K} = -c^2 = -1/4$) after a scaling, where Θ denotes the Funk metric on the unit ball $\mathbf{B}^n$ and $\mathbf{a} \in \mathbf{R}^n$ is a constant vector with $|\mathbf{a}| < 1$.

Proof. By assumption and Theorem 4.2, we know that $\lambda = -c^2$ is a non-positive constant. Further, by assumption that $\mathbf{J} + cF\mathbf{I} = 0$ and Theorem 5.3, we know that

$$e_{ij} = 2c(a_{ij} - b_i b_j), \qquad s_{ij} = 0.$$

Plugging them into (5.5) yields

$$G^i = \bar{G}^i + c(\alpha - \beta)y^i. \tag{5.21}$$

Thus $F = \alpha + \beta$ is pointwise projectively equivalent to α. By assumption,

$$R^i{}_k = \lambda F^2 \Big\{ \delta^i_k - \frac{F_{y^k}}{F} y^i \Big\}.$$

Thus α is of scalar curvature $\mu := \mu(x)$ and $\mu = \mu(x)$ must be a constant when $n = \dim M > 2$. Using (5.21), (2.10) and

$$b_{i;j} y^i y^j = e_{00} = 2c(\alpha^2 - \beta^2), \tag{5.22}$$

we obtain

$$R^i{}_k = \bar{R}^i{}_k + \Xi \ \delta^i_k + \tau_k y^i, \tag{5.23}$$

where

$$\Xi = 3c^2\alpha^2 - 2c^2\alpha\beta - c^2\beta^2.$$

Then

$$\bar{R}^i{}_k = R^i{}_k - \Xi\delta^i_k - \tau_k y^i = (\lambda - 3c^2)\alpha^2\delta^i_k + \tilde{\tau}_k y^i.$$

This implies that

$$\frac{(\lambda - 3c^2)\alpha^2}{\alpha^2} = \mu.$$

It follows that

$$\lambda - 3c^2 - \mu = 0, \qquad \lambda + c^2 = 0.$$

Thus $\mu = -4c^2 = constant$, i.e.,

$$\bar{R}^i{}_k = -4c^2\alpha^2 \Big\{ \delta^i_k - \frac{\alpha_{y^k}}{\alpha} y^i \Big\}.$$

First, we suppose that $c = 0$. It follows from (5.21) that $G^i = \bar{G}^i(x, y)$ are quadratic in $y \in \mathrm{R}^n$ for any x. Hence F is a Berwald metric. Moreover,

$$R^i{}_k = \bar{R}^i{}_k.$$

On the other hand, $\mu = -4c^2 = 0$ implies that α is flat, $\bar{R}^i{}_k = 0$. Thus $F = \alpha + \beta$ is flat. We conclude that F is locally Minkowskian.

Now suppose that $c \neq 0$. After an appropriate scaling, we may assume that $c = \pm 1/2$. We can express α in the following Klein form

$$\alpha = \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}{1 - |\mathbf{x}|^2}.$$

Since β is closed, we can express it in the following form

$$\beta = \pm \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{1 - |\mathbf{x}|^2} \pm d\varphi(\mathbf{y}), \qquad \mathbf{y} = (y^i) \in T_{\mathbf{x}}\mathbf{B}^n.$$

It follows from (5.22) that

$$b_{i;j} = \pm (a_{ij} - b_i b_j). \tag{5.24}$$

The Christoffel symbols of α are given by

$$\bar{\Gamma}^i_{jk} = \frac{x^k \delta^i_j + x^j \delta^i_k}{1 - |\mathbf{x}|^2}.$$

The covariant derivatives of β with respect to α are given by

$$b_{i;j} = \pm \Big\{ \frac{\partial^2 \varphi}{\partial x^i \partial x^j} + \frac{1}{1 - |\mathbf{x}|^2} \Big(\delta_{ij} - x^i \frac{\partial \varphi}{\partial x^j} - x^j \frac{\partial \varphi}{\partial x^i} \Big) \Big\},$$

and

$$a_{ij} - b_i b_j = \frac{1}{1 - |\mathbf{x}|^2} \Big(\delta_{ij} - x^i \frac{\partial \varphi}{\partial x^j} - x^j \frac{\partial \varphi}{\partial x^i} \Big) - \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j}.$$

Plugging them into (5.24) yields

$$\frac{\partial^2 \varphi}{\partial x^i \partial x^j} + \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = 0. \tag{5.25}$$

Let $f = \exp(\varphi)$. Then (5.25) simplifies to

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = 0. \tag{5.26}$$

Thus f is a linear function

$$f = k(1 + \langle \mathbf{a}, \mathbf{x} \rangle), \qquad k > 0.$$

We obtain that

$$\varphi = \ln k + \ln(1 + \langle \mathbf{a}, \mathbf{x} \rangle).$$

Finally, we find the most general solution for β,

$$\beta = \pm \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{1 - |\mathbf{x}|^2} \pm \frac{\langle \mathbf{a}, \mathbf{y} \rangle}{1 + \langle \mathbf{a}, \mathbf{x} \rangle}, \qquad \mathbf{y} \in T_{\mathbf{x}} \mathbf{B}^n. \tag{5.27}$$

Q.E.D.

Chapter 6

Randers Metrics with Isotropic S-Curvature

In this section, we study Randers metrics with isotropic S-curvature. It is shown that, if a Randers metric is of constant curvature, then it has constant S-curvature [BaoRo].

Consider a Randers metric $F = \alpha + \beta$ on a manifold M. Let

$$\rho := \ln \sqrt{1 - \|\beta\|_\alpha^2}$$

and $d\rho = \rho_i dx^i$. According to [Sh3], the S-curvature of $F = \alpha + \beta$ is given by

$$\mathbf{S} = (n+1)\left\{\frac{e_{00}}{2F} - (s_0 + \rho_0)\right\}, \tag{6.1}$$

where $e_{00} := e_{ij}y^iy^j, s_0 := s_iy^i$ and $\rho_0 := \rho_iy^i$ (cf. section 5). We have the following

Lemma 6.1.([ChSh2][ChSh3]) Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent

(a) $\mathbf{S} = (n+1)cF$;

(b) $e_{00} = 2c(\alpha^2 - \beta^2)$.

Proof. From (6.1), we see that $\mathbf{S} = (n+1)cF$ if and only if

$$e_{ij} = (s_i + \rho_i)b_j + (s_j + \rho_j)b_i + 2c(a_{ij} + b_ib_j) \tag{6.2}$$

$$s_i + \rho_i + 2cb_i = 0. \tag{6.3}$$

On the other hand, $e_{00} = 2c(\alpha^2 - \beta^2)$ is equivalent to the following identity,

$$e_{ij} = 2c(a_{ij} - b_i b_j). \tag{6.4}$$

First suppose that $\mathbf{S} = (n+1)cF$. Then (6.2) and (6.3) hold. Plugging (6.3) into (6.2) gives (6.4).

Conversely, suppose that (6.4) holds. Contracting (6.4) with b^j yields

$$r_{ij}b^j + \|\beta\|^2 s_i = 2c(1 - \|\beta\|^2)b_i, \tag{6.5}$$

where we have used the fact $s_j b^j = 0$. Note that

$$-b^j b_{j;i} = (1 - \|\beta\|^2)\rho_i. \tag{6.6}$$

Adding (6.6) to (6.5) gives

$$-(1 - \|\beta\|^2)s_i = 2c(1 - \|\beta\|^2)b_i + (1 - \|\beta\|^2)\rho_i. \tag{6.7}$$

This is equivalent to (6.3) since $1 - \|\beta\|^2 \neq 0$. From (6.4) and (6.3), one immediately obtains (6.2). This proves the lemma. Q.E.D.

By the definitions and (3.7), $E_{ij} = (1/2)\mathbf{S}_{y^i y^j}$. Hence, if a Finsler metric F has isotropic S-curvature, $\mathbf{S} = (n+1)c(x)F$, then F must have isotropic mean Berwald curvature, $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$. But the converse does not hold in general. However, for Randers metrics, we have the following

Lemma 6.2.([ChSh2]) Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent

(a) $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$;

(b) $e_{00} = 2c(\alpha^2 - \beta^2)$.

Proof. It follows from $E_{ij} = (1/2)\mathbf{S}_{y^i y^j}$ and (6.1) that

$$E_{ij} = \frac{1}{4}(n+1)\Big[\frac{e_{00}}{F}\Big]_{y^i y^j}. \tag{6.8}$$

Suppose that $e_{00} = 2c(\alpha^2 - \beta^2)$. Then

$$\frac{e_{00}}{F} = 2c(\alpha - \beta).$$

Plugging it into (6.8) we obtain

$$E_{ij} = \frac{1}{2}(n+1)c\ \alpha_{y^i y^j} = \frac{1}{2}(n+1)c\ F_{y^i y^j}. \tag{6.9}$$

That is, $\mathbf{E} = \frac{1}{2}(n+1)c\, F^{-1}h$.

Conversely, suppose that (6.9) holds. It follows from (6.8) and (6.9) that

$$\left[\frac{e_{00}}{F}\right]_{y^iy^j} = 2c\, F_{y^iy^j}.$$

Thus at each point $p \in M$, the following holds on $T_pM \setminus \{0\}$,

$$\frac{e_{00}}{F} = 2cF + \eta + \tau,$$

where $\eta \in T_p^*M$ and τ is a constant. By the homogeneity, we conclude that $\tau = 0$. Then

$$e_{00} = 2cF^2 + \eta F. \tag{6.10}$$

Equation (6.10) is equivalent to the following equations,

$$e_{00} = 2c(\alpha^2 + \beta^2) + \eta\beta \tag{6.11}$$

$$0 = 4c\beta + \eta. \tag{6.12}$$

By (6.12), we obtain $\eta = -4c\beta$. Plugging it into (6.11), we obtain

$$e_{00} = 2c(\alpha^2 - \beta^2).$$

This completes the proof. Q.E.D.

From Lemma 6.1 and Lemma 6.2, we have the following

Theorem 6.3.([ChSh2]) Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent

(a) $\mathbf{S} = (n+1)cF$;

(b) $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$;

(c) $e_{00} = 2c(\alpha^2 - \beta^2)$.

From Theorem 5.3 and Theorem 6.3, we have following

Theorem 6.4.([ChSh2]) Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent,

(a) $\mathbf{L} + c(x)F\mathbf{C} = 0$ (or $\mathbf{J} + cF\mathbf{I} = 0$);

(b) $\mathbf{S} = (n+1)cF$ and β is closed.

(c) $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$ and β is closed.

In 1997, Bácsó and Matsumoto proved that a Randers metric is a Douglas metric if and only if β is closed [BaMa1]. From Theorem 6.4, we have the following result.

Corollary 6.5. Let $F = \alpha + \beta$ be a Randers metric of Douglas type on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent,

(a) $\mathbf{L} + c(x)F\mathbf{C} = 0$;

(b) $\mathbf{S} = (n+1)cF$;

(c) $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$.

Now, we consider a projectively flat Randers metrics with isotropic S-curvature $\mathbf{S} = (n+1)c(x)F$. First, we know that α is locally projectively flat and β is closed. According to the Beltrami theorem in Riemann geometry, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Thus α is of constant curvature μ. It is locally isometric to the following standard metric α_μ on the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ or the whole $\mathbf{R}^n$ for $\mu = -1, 0, +1$:

$$\alpha_{-1}(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)}}{1-|x|^2}, \qquad y \in T_x\mathbf{B}^n \cong \mathbf{R}^n,$$

$$\alpha_0(x,y) = |y|, \qquad y \in T_x\mathbf{R}^n \cong \mathbf{R}^n,$$

$$\alpha_{+1}(x,y) = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - <x,y>^2)}}{1+|x|^2}, \qquad y \in T_x\mathbf{R}^n \cong \mathbf{R}^n.$$

Theorem 6.6.([CMS][ChSh3])) Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M and μ denote the constant sectional curvature of α. Suppose that the S-curvature is isotropic, $\mathbf{S} = (n+1)c(x)F$. Then F can be classified as follows.

(A) If $\mu + 4c(x)^2 \equiv 0$, then $c(x) = constant$ and the flag curvature $K = -c^2 \leq 0$.

$(A1)$ If $c = 0$, then F is locally Minkowskian with $K = 0$;

$(A2)$ if $c \neq 0$, then after a scaling, F is locally isometric to the following Randers metric on the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$,

$$F(x,y) = \Theta \pm \frac{< a, y >}{1+ < a, x >}, \tag{6.13}$$

where $a \in \mathbf{R}^n$ with $|a| < 1$, and F has negative constant flag curvature $K = -\frac{1}{4}$.

(B) If $\mu + 4c(x)^2 \neq 0$, then F is given by

$$F(x,y) = \alpha(x,y) - \frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2} \tag{6.14}$$

and the flag curvature of F is given by

$$\begin{aligned} K &= 3\left\{\frac{c_{x^k}(x)y^k}{F(x,y)} + c(x)^2\right\} + \mu \\ &= \frac{3}{4}\left\{\mu + 4c(x)^2\right\}\frac{F(x,-y)}{F(x,y)} + \frac{\mu}{4}. \end{aligned}$$

$(B1)$ when $\mu = -1$, we can express $\alpha = \alpha_{-1}$. In this case,

$$c(x) = \frac{\lambda + < a, x >}{2\sqrt{(\lambda + < a, x >)^2 \pm (1 - |x|^2)}},$$

where $\lambda \in \mathbf{R}$ and $a \in \mathbf{R}^n$ with $|a|^2 < \lambda^2 \pm 1$.

$(B2)$ when $\mu = 0$, we can express $\alpha = \alpha_0$. In this case,

$$c(x) = \frac{\pm 1}{2\sqrt{\kappa + 2 < a, x > + |x|^2}},$$

where $\kappa > 0$ and $a \in \mathbf{R}^n$ with $|a|^2 < \kappa$.

$(B3)$ when $\mu = 1$, we can express $\alpha = \alpha_{+1}$. In this case,

$$c(x) = \frac{\varepsilon + < a, x >}{2\sqrt{1 + |x|^2 - (\varepsilon + < a, x >)^2}},$$

where $\varepsilon \in \mathbf{R}$ and $a \in \mathbf{R}^n$ with $|\varepsilon|^2 + |a|^2 < 1$.

Proof. Assume that α is of constant sectional curvature and β is closed (hence $s_{ij} = 0$ and $s_i = 0$). Let

$$\Phi := b_{i;j} y^i y^j, \qquad \Psi := b_{i;j;k} y^i y^j y^k.$$

By (8.56) in [Sh3], we have

$$\mathbf{K} F^2 = \mu\alpha^2 + 3\Big[\frac{\Phi}{2F}\Big]^2 - \frac{\Psi}{2F}. \tag{6.15}$$

Further we assume that $\mathbf{S} = (n+1)c(x)F$, which is equivalent to

$$e_{ij} = 2c(a_{ij} - b_i b_j)$$

by Lemma 6.1. Since $s_{ij} = 0$, $e_{ij} = r_{ij} = b_{i;j}$ and the above equation simplifies to

$$b_{i;j} = 2c(a_{ij} - b_i b_j).$$

We obtain

$$\begin{aligned} \Phi &= 2c(\alpha^2 - \beta^2) \\ \Psi &= 2c_{x^k} y^k (\alpha^2 - \beta^2) - 8c^2(\alpha^2 - \beta^2)\beta. \end{aligned}$$

By Theorem 4.1, we know that the flag curvature is in the following form

$$\mathbf{K} = \frac{3c_{x^k}(x) y^k}{F(x,y)} + \sigma(x), \tag{6.16}$$

where $\sigma(x)$ is a scalar function on M. It follows from (6.16) and (6.15) that

$$3c_{x^k} y^k F + \sigma F^2 = \mathbf{K} F^2 = \mu\alpha^2 + 3\Big[\frac{\Phi}{2F}\Big]^2 - \frac{\Psi}{2F}. \tag{6.17}$$

Using the above formulas for Φ and Ψ, we obtain

$$2\Big\{2c_{x^k} y^k + (\sigma + c^2)\beta\Big\}\alpha + \Big\{2c_{x^k} y^k + (\sigma + c^2)\beta\Big\}\beta + \Big\{\sigma - 3c^2 - \mu\Big\}\alpha^2 = 0.$$

This gives

$$2c_{x^k} y^k + (\sigma + c^2)\beta = 0, \tag{6.18}$$

$$\sigma - 3c^2 - \mu = 0. \tag{6.19}$$

Plugging (6.19) into (6.16) and (6.18) yields

$$\mathbf{K} = 3\Big\{\frac{c_{x^k}(x)y^k}{F(x,y)} + c(x)^2\Big\} + \mu. \tag{6.20}$$

$$2c_{x^k}y^k + (\mu + 4c^2)\beta = 0. \tag{6.21}$$

Now we are ready to determine β and c.

Case 1: Suppose that $\mu + 4c(x)^2 \equiv 0$. Then $c(x) = constant$. It follows from (6.20) that

$$\mathbf{K} = 3c^2 + \mu = -c^2.$$

Then Theorem 6.6(A) follows from the classification theorem for projectively flat Randers of constant curvature [Sh7].

Case 2: Suppose that $\mu + 4c(x)^2 \neq 0$ on an open subset $\mathcal{U} \subset M$. It follows from (6.21) that

$$\beta = -\frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2}. \tag{6.22}$$

Note that β is exact. Let $c_i dx^i := dc$ and $c_{i;j}dx^j := dc_i - c_k\bar{\Gamma}^k_{ij}dx^j$ denote the covariant derivative of dc with respect to α, were $\bar{\Gamma}^k_{ij}$ denote the Christoffel symbols of α. We have

$$c_i = c_{x^i}(x), \qquad c_{i;j} = c_{x^i x^j}(x) - c_{x^k}(x)\bar{\Gamma}^k_{ij}(x).$$

Similarly, we can define $b_{i;j}$ and $b_{i;j;k}$. Since β is closed, $b_{i;j} = b_{j;i}$. In this case, $\mathbf{S} = (n+1)c(x)F$ is equivalent to

$$b_{i;j} = 2c(a_{ij} - b_i b_j). \tag{6.23}$$

From (6.22), we have

$$b_i = -\frac{2c_i}{\mu + 4c^2}. \tag{6.24}$$

Plugging (6.24) into (6.23) yields

$$c_{i;j} = -c(\mu + 4c^2)a_{ij} + \frac{12cc_ic_j}{\mu + 4c^2}. \tag{6.25}$$

Next we are going solve (6.25) for $c(x)$ in three cases when $\mu = -1, 0, 1$.

(B1): $\mu = -1$. We can express that $\alpha = \alpha_{-1}$. We have

$$a_{ij} = \frac{\delta_{ij}}{1-|x|^2} + \frac{x^i x^j}{(1-|x|^2)^2}.$$

The Christoffel symbols of α are given by

$$\bar{\Gamma}^k_{ij} = \frac{x^i \delta^k_j + x^j \delta^k_i}{1-|x|^2}.$$

Equation (6.25) becomes

$$c_{x^i x^j} - \frac{x^i c_{x^j} + x^j c_{x^i}}{1-|x|^2} = -c(-1+4c^2)\Big\{\frac{\delta_{ij}}{1-|x|^2} + \frac{x^i x^j}{(1-|x|^2)^2}\Big\} + \frac{12 c c_{x^i} c_{x^j}}{-1+4c^2}. \tag{6.26}$$

Let

$$f := \frac{2c\sqrt{1-|x|^2}}{\sqrt{\pm(-1+4c^2)}},$$

where the sign depends on the value of c such that $\pm(-1+4c^2) > 0$. Equation (6.26) simplifies to

$$f_{x^i x^j} = 0.$$

We obtain that $f = \langle a, x\rangle + \lambda$, where $\lambda \in \mathbf{R}$ and $a \in \mathbf{R}^n$. Then we obtain

$$c = \frac{\lambda + \langle a, x\rangle}{2\sqrt{(\lambda + \langle a, x\rangle)^2 \pm (1-|x|^2)}}. \tag{6.27}$$

Plugging (6.27) into (6.22) yields

$$\beta = \frac{(\lambda + \langle a, x\rangle)\langle x, y\rangle + (1-|x|^2)\langle a, y\rangle}{(1-|x|^2)\sqrt{(\lambda + \langle a, x\rangle)^2 \pm (1-|x|^2)}}$$

and

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2} + \frac{(\lambda + \langle a, x\rangle)\langle x, y\rangle + (1-|x|^2)\langle a, y\rangle}{(1-|x|^2)\sqrt{(\lambda + \langle a, x\rangle)^2 \pm (1-|x|^2)}}. \tag{6.28}$$

By a direct computation,

$$1-\|\beta\|_\alpha^2=\frac{(1-|x|^2)\Big\{\pm 1-(|a|^2-\lambda^2)\Big\}}{(\lambda+\langle a,x\rangle)^2\pm(1-|x|^2)}.$$

Clearly, $F=\alpha+\beta$ is a Randers metric on an open subset of $\mathbf{B}^n$ if and only if $|a|^2-\lambda^2<\pm 1$. In this case, $(\lambda+\langle a,x\rangle)^2\pm(1-|x|^2)>0$ for any $x\in\mathbf{B}^n$. Thus F can be extended to the whole $\mathbf{B}^n$. By (6.20), (6.27) and (6.28), we obtain

$$\mathbf{K}=-\frac{3}{4}\frac{\pm(1-|x|^2)}{(\lambda+\langle a,x\rangle)^2\pm(1-|x|^2)}\cdot\frac{F(x,-y)}{F(x,y)}-\frac{1}{4}.$$

(B2): $\mu=0$. We can express that $\alpha=\alpha_0$. Equation (6.25) becomes

$$c_{x^ix^j}=-4c^3\delta_{ij}+\frac{3c_{x^i}c_{x^j}}{c}. \tag{6.29}$$

Let $\mathcal{U}:=\{x\in\mathrm{R}^n\mid c(x)\neq 0\}$ and let

$$f=\frac{1}{c^2}.$$

Equation (6.29) simplifies to

$$f_{x^ix^j}=8\delta_{ij}. \tag{6.30}$$

We obtain

$$f=4(\kappa+2\langle a,x\rangle+|x|^2),$$

where $\kappa\in\mathbf{R}$ and $a\in\mathbf{R}^n$ such that $f(x)>0$ for $x\in\mathcal{U}$. Then $c=\pm 1/\sqrt{f}$ is given by

$$c=\frac{\pm 1}{2\sqrt{\kappa+2\langle a,x\rangle+|x|^2}}. \tag{6.31}$$

Plugging (6.31) into (6.22) yields

$$\beta=\pm\frac{\langle a,y\rangle+\langle x,y\rangle}{\sqrt{\kappa+2\langle a,x\rangle+|x|^2}},$$

and

$$F=|y|\pm\frac{\langle a,y\rangle+\langle x,y\rangle}{\sqrt{\kappa+2\langle a,x\rangle+|x|^2}}. \tag{6.32}$$

Note that

$$1-\|\beta\|_\alpha^2=\frac{\kappa-|a|^2}{\kappa+2\langle a,x\rangle+|x|^2}.$$

Clearly, $F=\alpha+\beta$ is a Randers metric on an open subset of $\mathbf{R}^n$ if and only if $|a|^2<\kappa$. In this case,

$$\kappa+2\langle a,x\rangle+|x|^2\geq\kappa-|a|^2+(|a|-|x|)^2>0,\qquad\forall x\in\mathbf{R}^n.$$

Thus F can be extended to the whole $\mathbf{R}^n$. By (6.20), (6.31) and (6.32), we obtain

$$\mathbf{K}=\frac{3}{4(\kappa+2\langle a,x\rangle+|x|^2)}\cdot\frac{F(x,-y)}{F(x,y)}>0.$$

(B3): $\mu=+1$. We can express that $\alpha=\alpha_{+1}$. We have

$$a_{ij}=\frac{\delta_{ij}}{1+|x|^2}-\frac{x^ix^j}{(1+|x|^2)^2}.$$

The Christoffel symbols of α are given by

$$\bar{\Gamma}^k_{ij}=-\frac{x^i\delta^k_j+x^j\delta^k_i}{1+|x|^2}.$$

Equation (6.25) becomes

$$c_{x^ix^j}+\frac{x^ic_{x^j}+x^jc_{x^i}}{1+|x|^2}=-c(1+4c^2)\Big\{\frac{\delta_{ij}}{1+|x|^2}-\frac{x^ix^j}{(1+|x|^2)^2}\Big\}+\frac{12cc_{x^i}c_{x^j}}{1+4c^2}. \quad (6.33)$$

Let

$$f:=\frac{2c\sqrt{1+|x|^2}}{\sqrt{1+4c^2}}.$$

Equation (6.33) simplifies to $f_{x^ix^j}=0$. We obtain that $f=\varepsilon+\langle a,x\rangle$. Then we obtain

$$c=\frac{\varepsilon+\langle a,x\rangle}{2\sqrt{1+|x|^2-(\varepsilon+\langle a,x\rangle)^2}}. \quad (6.34)$$

Thus

$$\beta=\frac{(\varepsilon+\langle a,x\rangle)\langle x,y\rangle-(1+|x|^2)\langle a,y\rangle}{(1+|x|^2)\sqrt{1+|x|^2-(\varepsilon+\langle a,x\rangle)^2}}.$$

and

$$F = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+|x|^2} + \frac{(\varepsilon + \langle a, x\rangle)\langle x, y\rangle - (1+|x|^2)\langle a, y\rangle}{(1+|x|^2)\sqrt{(1+|x|^2) - (\varepsilon + \langle a, x\rangle)^2}}.$$

By a direct computation,

$$1 - \|\beta\|_\alpha^2 = \frac{(1+|x|^2)\Big\{1 - \varepsilon^2 - |a|^2\Big\}}{1+|x|^2 - (\varepsilon + \langle a, x\rangle)^2}.$$

Thus $F = \alpha + \beta$ is a Randers metric on some open subset of $\mathbf{R}^n$ if and only if $\varepsilon^2 + |a|^2 < 1$. In this case, $1 + |x|^2 - (\varepsilon + \langle a, x\rangle)^2 > 0$ for all $x \in \mathbf{R}^n$. Thus F can extended to the whole $\mathbf{R}^n$. By (6.20), we obtain

$$\mathbf{K} = \frac{3(1+|x|^2)}{4\{1+|x|^2 - (\varepsilon + \langle a, x\rangle)^2\}} \cdot \frac{F(x,-y)}{F(x,y)} + \frac{1}{4} > \frac{1}{4}.$$

Q.E.D.

From Theorem 6.6, we obtain some imteresting projectively flat Randers metrics with isotropic S-curvature.

Example 6.7. Let

$$F_-(x,y) := \frac{\sqrt{(1-|x|^2)|y|^2 + < x, y >^2}\sqrt{(1-|x|^2) + \lambda^2} + \lambda < x, y >}{(1-|x|^2)\sqrt{(1-|x|^2)+\lambda^2}},$$

$$y \in T_x\mathbf{B}^n,$$

where $\lambda \in \mathbf{R}$ is an arbitrary constant. The geodesic of F_- are straight lines in $\mathbf{B}^n$. Thus F_- is of scalar curvature. One can easily verify that F_- is complete in the sense that every unit speed geodesic of F_- is defined on $(-\infty, \infty)$. Moreover, F_- has strictly negative flag curvature $\mathbf{K} \le -\frac{1}{4}$.

Example 6.8. Let

$$F_0(x,y) := \frac{|y|\sqrt{1+|x|^2} + < x, y >}{\sqrt{1+|x|^2}}, \qquad y \in T_x\mathbf{R}^n.$$

The geodesics of F_0 are straight lines in $\mathbf{R}^n$. Thus F_0 is of scalar curvature. One can easily vertify that F_0 is positively complete in the sense that every unit speed geodesic of F_0 is defined on (a, ∞) for some $a \in \mathbf{R}$. Moreover, F_0 has positive flag curvature $\mathbf{K} > 0$.

Theorem 6.6 is a local classification theorem. If we assume that the manifold is closed (compact without boundary), the the scalar function $c(x)$ takes much more special values.

Theorem 6.9.([CMS][ChSh3])) Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional closed manifold M. Let μ denote the constant sectional curvature of α. Suppose that $\mathbf{S} = (n+1)c(x)F$.

(a) If $\mu = -1$, then $F = \alpha$ is Riemannian.

(b) If $\mu = 0$, then F is locally Minkowskian.

(c) If $\mu = 1$, then $c(x) = f(x)/2\sqrt{1 - f(x)^2}$ so that

$$F(x, y) = \alpha(x, y) - \frac{f_{x^k}(x)y^k}{\sqrt{1 - f(x)^2}},$$

where $f(x)$ is an eigenfunction of the standard Laplacian of (M, α) corresponding to the eigenvalue $\lambda = n$ with $max_{x\in M}|f|(x) < 1$. Moreover, the flag curvature and the S-curvature of F are given by

$$K(x, y) = \frac{3}{4(1 - f(x)^2)} \frac{F(x, -y)}{F(x, y)} + \frac{1}{4},$$

$$\mathbf{S}(x, y) = \frac{(n+1)f(x)}{2\sqrt{1 - f(x)^2}} F(x, y).$$

Proof. By assumption, the manifold M is closed. Assume that $\mu + 4c^2(x) \neq 0$ on some open subset of M.

When $\mu \neq 0$, let

$$f(x) := \frac{2c(x)}{\sqrt{\pm(\mu + 4c(x)^2)}},$$

where the sign is chosen so that $\pm(\mu + 4c^2) > 0$. By (6.25), we have

$$f_{;i;j} = -\mu f a_{ij}.$$

This gives

$$\Delta f = -n\mu\, f. \tag{6.35}$$

When $\mu = 0$, we take

$$f(x) := \frac{1}{c(x)^2}.$$

We have

$$f_{;i;j} = 8a_{ij}.$$

This gives

$$\Delta f = 8n. \tag{6.36}$$

Case 1: $\mu = -1$. Suppose that $1 - 4c(x)^2 \neq 0$ on M. Integrating (6.35) yields

$$\int_M |\nabla f|^2 dV_\alpha = -\int_M f\Delta f dV_\alpha = -n\int_M f^2 \, dV_\alpha$$

Thus $f = 0$. This implies that $c = 0$ and $F = \alpha$ is Riemannian.

Suppose that $1 - 4c(x_o)^2 \neq 0$ at some point $x_o \in M$. Let $(\tilde{M}, \tilde{x}_o)$ be the universal cover of (M, x_o). We may assume that $\tilde{M}$ is isometric to $(\mathbf{B}^n, \alpha_{-1})$ with $\tilde{x}_o$ corresponding to the origin. The Randers metric F is lifted to a complete Randers metric $\tilde{F}$ on $\tilde{M} = \mathrm{B}^n$. $\tilde{F}$ is given by (6.28). Let $\tilde{c}(\tilde{x})$ be the lift of $c(x)$, which is given by (6.27). Thus $1 - 4\tilde{c}(\tilde{x})^2 \neq 0$ for all $\tilde{x} \in \mathbf{B}^n$. This implies that $1 - 4c(x)^2 \neq 0$ for all $x \in M$. By the above argument, we see that $c = 0$. Hence $F = \alpha$ is Riemannian by (6.14).

Suppose that $1 - 4c(x)^2 \equiv 0$. Then the lift $\tilde{F}$ of F to the universal cover $\tilde{M} = \mathbf{B}^n$ is given by (6.13), hence it is incomplete. This is impossible because of the compactness of M. We also see that F has negative constant flag curvature and bounded Cartan torsion, hence it is Riemannian according to Akbar-Zadeh's theorem [Sh4][ShSh5]. Then $c(x) = 0$. This is a contradiction again.

Case 2: $\mu = 0$. Suppose that $c(x_o) \neq 0$. Let $\tilde{M}$ denote the universal cover of M. We may assume that $\tilde{M} = \mathrm{R}^n$ with the origin corresponding to x_o. The Randers metric F lifted to $\tilde{M} = \mathrm{R}^n$ is given by (6.32). Thus $c(x) \neq 0$ for all $x \in M$. Integrating (6.36) over M yields

$$0 = \int_M \Delta f dV_\alpha = 8n\mathrm{Vol}(M, \alpha).$$

This is impossible. Therefore $c(x) \equiv 0$. In this case, F is a locally projectively flat Randers metric with flag curvature $\mathbf{K} = 0$, hence it is locally Minkowskian by [Sh2].

Case 3: $\mu = 1$. Note that $1 + 4c(x)^2 \neq 0$ on M. Let

$$f(x) := \frac{2c(x)}{\sqrt{1+4c(x)^2}}. \tag{6.37}$$

It follows from (6.35) that

$$f_{;i;j} = -f a_{ij}. \tag{6.38}$$

This gives

$$\Delta f = -nf.$$

Thus f is an eigenfunction of (M, α) with $\max_{x \in M} |f|(x) < 1$. We can express

$$F(x,y) = \alpha(x,y) - \frac{2c_{x^k}(x)y^k}{1+4c(x)^2} = \alpha(x,y) - \frac{f_{x^k}(x)y^k}{\sqrt{1-f(x)^2}}. \tag{6.39}$$

$$\mathbf{K}(x,y) = 3\Big\{\frac{c_{x^k}(x)y^k}{F(x,y)} + c(x)^2\Big\} + 1 = \frac{3}{4(1-f(x)^2)}\frac{F(x,-y)}{F(x,y)} + \frac{1}{4}. \tag{6.40}$$

Q.E.D.

Assume that $(M, \alpha) = S^n$ is the standard unit sphere. Let $F = \alpha + \beta$ be a Randers metric. From Theorem 6.9, we obtain

Theorem 6.10.([CMS][ChSh3]) Let $S^n = (M, \alpha)$ is the standard unit sphere and $F = \alpha + \beta$ be a locally projectively flat Randers metric on S^n. Suppose that $\mathbf{S} = (n+1)c(x)F$. Then

$$F(x,y) = \alpha(x,y) - \frac{f_{x^k}(x)y^k}{\sqrt{1-f(x)^2}},$$

where $f(x)$ is an eigenfunction of S^n corresponding to the first eigenvalue. Moreover,

(a)

$$\frac{2-\delta}{2(1+\delta)} \leq K \leq \frac{2+\delta}{2(1-\delta)},$$

where $\delta := \sqrt{|\nabla f|^2_\alpha(x) + f(x)^2} < 1$ is a constant.

(b) The geodesics of F are the great circles on S^n with F-length 2π.

Proof. Using (6.38), one can verify that

$$\delta := \sqrt{|\nabla f|^2_\alpha(x) + f(x)^2}$$

is a constant. Since F is positive definite, $\delta < 1$.

Let

$$\lambda(x) := \sup_{y \in T_xM} \frac{F(x,-y)}{F(x,y)}.$$

Using $|\nabla f|^2_\alpha(x) = \delta^2 - f(x)^2$, we obtain

$$\lambda(x) = \frac{\sqrt{1-f(x)^2} + \sqrt{\delta^2 - f(x)^2}}{\sqrt{1-f(x)^2} - \sqrt{\delta^2 - f(x)^2}}.$$

Let $\lambda := \max_{x\in M} \lambda(x)$. We have

$$1 \le \lambda(x) \le \lambda = \frac{1+\delta}{1-\delta}$$

and

$$1 - f(x)^2 = \frac{(1-\delta^2)(\lambda(x)+1)^2}{4\lambda(x)}.$$

Note that $\lambda(x) = \lambda$ if and only if $f(x) = 0$. It follows from (6.40) that

$$\frac{2-\delta}{2(1+\delta)} = \frac{3+\lambda}{4\lambda} \le \mathbf{K} \le \frac{3\lambda+1}{4} = \frac{2+\delta}{2(1-\delta)}. \tag{6.41}$$

Let

$$h(x) := \arctan\Big(2c(x)\Big).$$

The Randers metric $F(x,y)$ in (6.39) can be expressed by

$$F(x,y) = \alpha(x,y) - h_{x^k}(x)y^k.$$

Clearly F is pointwise projectively equivalent to α, namely the geodesics of F are geodesics of α as point sets. Let $\sigma(t)$ be a closed geodesic of α. Observe that

$$F\Big(\sigma(t),\dot\sigma(t)\Big) = \alpha\Big(\sigma(t),\dot\sigma(t)\Big) - \frac{d}{dt}\Big[h(\sigma(t))\Big].$$

By the above equation we obtain

$$\mathrm{Length}_F(\sigma) = \int F\Big(\sigma(t),\dot\sigma(t)\Big)dt = \int \alpha\Big(\sigma(t),\dot\sigma(t)\Big)dt = \mathrm{Length}_\alpha(\sigma). \tag{6.42}$$

Assume that M is simply connected. Then $(M, \alpha) = \mathrm{S}^n$. Let σ be an arbitrary great circle on S^n. By (6.42),

$$\mathrm{Length}_F(\sigma) = 2\pi.$$

Q.E.D.

Chapter 7

Projectively Flat Finsler Metrics with Isotropic S-Curvature

As we stated in section 6, we have classified locally projectively flat Randers metrics with isotropic S-curvature. It is a natural problem to study and characterize locally projectively flat Finsler with isotropic S-curvature.

Theorem 7.1.([ChSh4]) Let $F = F(x,y)$ be a locally projectively flat Finsler metric on an open subset $\Omega \subset \mathbf{R}^n$. Suppose that F has almost isotropic S-curvature satisfying

$$\mathbf{S} = (n+1)\{c(x)F + \eta\}, \tag{7.1}$$

where $c = c(x)$ is a scalar function and $\eta = \eta(x,y)$ is a closed 1-form on M. Then the flag curvature is in the form

$$\mathbf{K} = 3\frac{c_{x^m}y^m}{F} + \sigma, \tag{7.2}$$

where $\sigma = \sigma(x)$ is a scalar function on Ω.

(a) If $\mathbf{K} \neq -c^2 + \frac{c_{x^m}y^m}{F}$ on Ω, then $F = \alpha + \beta$ is a projectively flat Randers metric with isotropic S-curvature $\mathbf{S} = (n+1)cF$;

(b) If $\mathbf{K} \equiv -c^2 + \frac{c_{x^m}y^m}{F}$ on Ω, then $c = constant$ and F is either locally

Minkowskian ($c = 0$) or, up to a scaling, locally isometric to the metric

$$\Theta_a := \Theta(x, y) + \frac{< a, y >}{1+, a, x >} \qquad (c = \frac{1}{2})$$

or its reverse

$$\bar{\Theta}_a := \Theta(x, -y) - \frac{< a, y >}{1+ < a, x >} \qquad (c = -\frac{1}{2}),$$

where $a \in \mathbf{R}^n$ is a constant vector and $\Theta(x, y)$ is Funk metric on Ω.

Proof. By assumption, $\mathbf{S}$ is in the form (7.1). Since every closed 1-form on an connected open subset in R^n is exact, we may assume that

$$\mathbf{S} = (n+1)\Big\{cF + dh\Big\},$$

where $h = h(x)$ is a scalar function on Ω.

On the other hand, F is projectively flat, hence the spray coefficients are in the form $G^i = Py^i$, where

$$P = \frac{F_{x^k} y^k}{2F}. \tag{7.3}$$

By (3.7), one obtains

$$\mathbf{S} = (n+1)P - y^m \frac{\partial(\ln \sigma_F)}{\partial x^m}.$$

Thus

$$P = cF + d\varphi, \tag{7.4}$$

where $\varphi = \ln[\sigma_F(x)]^{\frac{1}{n+1}} + h(x)$. It follows from (7.3) and (7.4) that

$$F_{x^i} y^i = 2FP = 2F\Big\{cF + \varphi_{x^i} y^i\Big\} \tag{7.5}$$

Plugging (7.4) into (2.13) and using (7.5), one obtains

$$\begin{aligned} \mathbf{K} &= \frac{\Big\{cF + \varphi_{x^i} y^i\Big\}^2 - \Big\{c_{x^i} y^i F + cF_{x^i} y^i + \varphi_{x^i x^j} y^i y^j\Big\}}{F^2} \\ &= \frac{-c^2 F^2 - c_{x^m} y^m F + [\varphi_{x^i}\varphi_{x^j} - \varphi_{x^i x^j}] y^i y^j}{F^2}. \end{aligned} \tag{7.6}$$

Comparing (7.6) with (7.2) yields

$$[\sigma + c^2]F^2 + 4c_{x^m}y^m F + [\varphi_{x^ix^j} - \varphi_{x^i}\varphi_{x^j}]y^iy^j = 0. \tag{7.7}$$

Assume that $\mathbf{K} \neq -c^2 + \frac{c_{x^m}y^m}{F}$. By (7.2), this is equivalent to the following inequality:

$$\sigma + c^2 + \frac{2c_{x^m}y^m}{F} \neq 0. \tag{7.8}$$

We claim that $\sigma + c^2 \neq 0$ on Ω. If this is not true at some point $x_o \in \Omega$, i.e., $\sigma(x_o) + c(x_o)^2 = 0$. By (7.8), $dc_{x_o} \neq 0$. From (7.7), one obtains

$$F = \frac{[\varphi_{x^i}(x_o)\varphi_{x^j}(x_o) - \varphi_{x^ix^j}(x_o)]y^iy^j}{4c_{x^m}(x_o)y^m}.$$

Namely, F is a so-called *Kropina metric* which is not a regular Finsler metric under our consideration. Therefore the above claim holds on Ω. Now, one can solve the quadratic equation (7.7) for F,

$$F = \frac{\sqrt{[\sigma + c^2][\varphi_{x^i}\varphi_{x^j} - \varphi_{x^ix^j}]y^iy^j + 4[c_{x^m}y^m]^2} - 2c_{x^m}y^m}{\sigma + c^2}.$$

That is, $F = \alpha + \beta$ is a Randers metric. We have classified projectively flat Randers metrics with almost isotropic S-curvature (cf. Theorem 6.6).

We now assume that $\mathbf{K} \equiv -c^2 + \frac{c_{x^m}y^m}{F}$. It follows from (7.2) that

$$\sigma + c^2 + \frac{2c_{x^m}y^m}{F} \equiv 0.$$

This implies that $c = constant$, hence $\sigma = -c^2$ is a constant too. In this case, the flag curvature is given by $\mathbf{K} = -c^2$. The equation (7.7) is reduced to

$$\varphi_{x^ix^j} - \varphi_{x^i}\varphi_{x^j} = 0.$$

It is easy to solve this equation,

$$\varphi = -\ln\Big(1 + \langle a, x\rangle\Big) + C,$$

where $a \in \mathbf{R}^n$ is a constant vector and C is a constant.

When $c = 0$, $\mathbf{K} = -c^2 = 0$. It follows from (7.4) that the projective factor $P = d\varphi$ is a 1-form, hence the spray coefficients $G^i = Py^i$ are quadratic in

$y \in T_x\Omega$. By definition, F is a Berwald metric. It is known that every Berwald metric with vanishing flag curvature is locally Minkowskian.

When $c \neq 0$, we may assume that $c = \pm\frac{1}{2}$ after a suitable scaling. Let

$$\Psi := P + cF.$$

Since F is projectively flat and P is the projective factor, by (7.3) and

$$F_{x^k} = F_{x^m y^k} y^m, \tag{7.9}$$

one obtains

$$F_{x^k} = P_{y^k} F + P F_{y^k}. \tag{7.10}$$

It follows from (2.13) that

$$P_{x^k} - P P_{y^k} = -\frac{1}{3F}(\Xi F)_{y^k}.$$

In particular, if $\mathbf{K} = \lambda$ is a constant,

$$P_{x^k} - P P_{y^k} = -\lambda F F_{y^k}. \tag{7.11}$$

The above identities can be found in [Ber][Sh8]. By (7.10) and (7.11), one can easily verify that

$$\Psi_{x^i} = \Psi \Psi_{y^i}.$$

Let

$$\Theta := \begin{cases} \Psi(x,y) & \text{if } c = \frac{1}{2} \\ -\Psi(x,-y) & \text{if } c = -\frac{1}{2} \end{cases}.$$

Then $\Theta = \Theta(x,y)$ satisfies $\Theta_{x^k} = \Theta\Theta_{y^k}$. Thus by definition it is a Funk metric. By (7.4), $\Psi = 2cF + d\varphi$. Thus

$$F = \frac{1}{2c}\Big\{\Psi(x,y) - d\varphi_x\Big\}.$$

When $c = \frac{1}{2}$, $\Psi(x,y) = \Theta(x,y)$. Thus

$$F = \Theta(x,y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle} =: \Theta_a(x,y).$$

When $c < 0$, $\Psi(x,y) = -\Theta(x,-y)$. Thus

$$F = \Theta(x,-y) - \frac{\langle a, y\rangle}{1 + \langle a, x\rangle} =: \bar{\Theta}_a(x,y),$$

where $\bar{\Theta}_a(x,y) := \Theta_a(x,-y)$. Q.E.D.

In Theorem 7.1(a), the local structure of F has been completely determined in Theorem 6.6. Namely, if a Randers metric $F = \alpha + \beta$ is locally projectively flat with (7.1), then α is locally isometric to the standard projectively metric

$$\alpha_\mu := \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu < x, y >^2}}{1+\mu|x|^2}, \qquad y \in T_x\mathbf{B}^n(r) \cong \mathbf{R}^n,$$

the scalar function $\sigma = \sigma(x)$ in (7.2) is given by $\sigma = \mu + 3c^2$ and β satisfies $2c_{x^k}y^k + (\mu+4c^2)\beta = 0$. Suppose that $dc = 0$ at a point $x \in \Omega$, then at the point x, either $\beta = 0$ or $\mu + 4c^2 = 0$. In the later case, $\mathbf{K} = \mu + 3c^2 = -c^2 + \frac{c_{x^m}y^m}{F}$. This contradicts the assumption (a). We may assume that $dc \neq 0$ on Ω. Then $\mu + 4c^2 \neq 0$ and β is given by

$$\beta = -\frac{2c_{x^k}y^k}{\mu + 4c^2}.$$

In this case, we can completely determine the scalar function $c = c(x)$ as follows.

$$c = \begin{cases} \frac{(\lambda + \langle a,x\rangle)}{2}\sqrt{\frac{\mu}{\pm(1+\mu|x|^2) - (\lambda+\langle a,x\rangle)^2}} & \text{if } \mu \neq 0 \\ \frac{\pm 1}{2\sqrt{\kappa + 2\langle a,x\rangle + |x|^2}} & \text{if } \mu = 0. \end{cases}$$

where $a \in \mathbf{R}^n$ is a constant vector and $\kappa \in \mathbf{R}$ is a constant number. See Theorem 6.6 or [CMS] for more details.

Chapter 8

Douglas Metrics with Special Non-Riemannian Curvature Properties

In [ChSh2], we prove that a Randers metric $F = \alpha + \beta$ of Douglas type has isotropic mean Berwald curvature if and only if it has relatively isotropic Landsberg curvature (Corollary 6.5). In this section, we will generalize this result.

We say that F has *isotropic Berwald curvature* [ChSh5] if

$$B_j{}^i{}_{kl} = c\left\{F_{jk}\delta^i_l + F_{jl}\delta^i_k + F_{kl}\delta^i_j + F_{jkl}y^i\right\}, \tag{8.1}$$

where $F_{ij} := F_{y^iy^j}$, $F_{ijk} := F_{y^iy^jy^k}$ and $c = c(x)$ is a scalar function on M.

Since $h_{ij} = FF_{y^iy^j}$ and $h^i_j := g^{ik}h_{jk} = \delta^i_j - F^{-2}g_{js}y^sy^i$, (8.1) can expressed as

$$B_j{}^i{}_{kl} = cF^{-1}\Big\{h_{jk}h^i_l + h_{jl}h^i_k + h_{kl}h^i_j + 2C_{jkl}y^i\Big\}. \tag{8.2}$$

It is easy to vertify the following

Lemma 8.1.([ChSh5]) F has isotropic Berwald curvature if and only if

$$\begin{aligned} B_j{}^i{}_{kl} &= \frac{c}{F}\left\{h_{jk}\delta^i_l + h_{jl}\delta^i_k + h_{kl}\delta^i_j\right\} \\ &+ \frac{c}{F^3}\left\{2F^2C_{jkl} - (h_{jk}g_{lm} + h_{jl}g_{km} + h_{kl}g_{jm})y^m\right\}y^i. \end{aligned} \tag{8.3}$$

From the definition, we obtain

Lemma 8.2.([ChSh5]) If F has isotropic Berwald curvature with scalar function $c = c(x)$, then F is a Douglas metric and satisfies

$$E_{ij} = \frac{n+1}{2} cF_{ij}, \qquad L_{ijk} + cFC_{ijk} = 0. \tag{8.4}$$

Proof. Assume that (8.2) holds. Then

$$E_{jk} = \frac{1}{2} B_j{}^m{}_{km} = \frac{1}{2}(n+1)cF^{-1}h_{jk} = \frac{1}{2}(n+1)cF_{y^j y^k}.$$

Note that

$$C_{ijk} = \frac{1}{2} g_{ij\cdot k} = \{F_{ik}F_j + F_{jk}F_i + F_{ij}F_k + FF_{ijk}\}/2.$$

From (8.1), we get

$$\begin{aligned} L_{jkl} &= -\frac{1}{2} y_i B_j{}^i{}_{kl} = -\frac{c}{2}\{F_{jk}F_l + F_{jl}F_k + F_{kl}F_j + F_{jkl}F\}F \\ &= -cFC_{jkl}. \end{aligned}$$

Finally, plugging $cF_{jk} = \frac{2}{n+1} E_{jk}$ into (8.1), one obtains

$$B_j{}^i{}_{kl} = \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l}\ y^i\Big\}. \tag{8.5}$$

By (2.8), it means that $\mathbf{D} = 0$. Q.E.D.

Furthermore, we have the following

Lemma 8.3.([ChSh5]) For a Douglas metric F, if

$$E_{ij} = \frac{n+1}{2} cF^{-1} h_{ij},$$

then F has isotropic Berwald curvature with scalar $c = c(x)$.

Proof. F is Douglas metric if and only if (8.5) holds. Plugging $E_{jk} = \frac{1}{2}(n+1)cF_{y^j y^k}$ into (8.5), one obtains (8.1). Q.E.D.

By Lemma 8.2 and Lemma 8.3, we obtain the following theorem.

Theorem 8.4.([ChSh5]) F has isotropic Berwald curvature with scalar function $c(x)$ if and only if F is a Douglas metric satisfying

$$\mathbf{E} = \frac{n+1}{2} cF^{-1} h.$$

Further, from Theorem 8.4 and Lemma 8.2, we obtain the following interesting corollary.

Corollary 8.5.([ChSh5]) Let F be a Douglas metric on an n-dimensional manifold M. If $\mathbf{E} = \frac{n+1}{2} cF^{-1}h$, then $\mathbf{L} + cF\mathbf{C} = 0$.

On the other hand, we have the following

Lemma 8.6.([BaMa2][Sh3]) For a Douglas metric F, the following are equivalent,

(a) $\mathbf{L} = 0$;

(b) $\mathbf{B} = 0$;

(c) $\mathbf{E} = 0$.

In particular, when F is a Randers metric of Douglas type, the Corollary 6.5 and Theorem 8.4 say the following are equivalent,

(i) $\mathbf{L} + c(x)F\mathbf{C} = 0$;

(ii) $B_j{}^i{}_{kl} = c(x)\left\{F_{jk}\delta^i_l + F_{jl}\delta^i_k + F_{kl}\delta^i_j + F_{jkl}y^i\right\}$;

(iii) $\mathbf{E} = \frac{n+1}{2} c(x)F^{-1}h$.

From Corollary 8.5 and the results as above, we have the following natual question: for a general Douglas metric F, is $\mathbf{L} + cF\mathbf{C} = 0$ equivalent to $\mathbf{E} = \frac{n+1}{2} cF^{-1}h$? We will discuss this question in the following. As a basis, we first give the following important Bianchi identity:

$$\frac{\partial L_{jkl}}{\partial y^m} - \frac{\partial L_{jkm}}{\partial y^l} = \frac{1}{2} g_{il} B_m{}^i{}_{kj} - \frac{1}{2} g_{im} B_l{}^i{}_{kj}. \tag{8.6}$$

See (10.12) in [Sh3] for a proof.

Lemma 8.7.([ChSh5]) Let (M, F) be a non-Riemannian Douglas manifold of dimension $n \geq 3$. Suppose that F has relatively isotropic Landsberg curvature, $\mathbf{L} + cF\mathbf{C} = 0$, then $\mathbf{E} = \frac{n+1}{2} c(x)F^{-1}h$, where $c = c(x)$ is a scalar function on M.

Proof By assumption, (8.5) holds and

$$L_{jkl} = -cFC_{jkl}. \tag{8.7}$$

Contracting $B_j{}^i{}_{kl}$ with $h^m_i := \delta^m_i - F^{-2} g_{is} y^s y^m$ and using (8.7) and

$$L_{ijk} = C_{ijk|m} y^m = -\frac{1}{2} y^m g_{ml} B^l_{ijk} \tag{8.8}$$

(see section 6.2 in [Sh3]), one obtains

$$h_i^m B_j{}^i{}_{kl} = B_j{}^m{}_{kl} + 2F^{-2}L_{jkl}y^m = B_j{}^m{}_{kl} - 2cF^{-1}C_{jkl}y^m. \tag{8.9}$$

Contracting (8.5) with h_i^m and using (8.9), one obtains

$$B_j{}^m{}_{kl} = \frac{2}{n+1}\Big\{E_{jk}h_l^m + E_{jl}h_k^m + E_{kl}h_j^m\Big\} + 2cF^{-1}C_{jkl}y^m. \tag{8.10}$$

Plugging (8.7) and (8.10) into (8.6), one obtains

$$E_{km}h_{jl} + E_{jm}h_{kl} - E_{kl}h_{jm} - E_{jl}h_{km} = 0. \tag{8.11}$$

Contracting (8.11) with g^{jm} yields

$$E_{kl} = \frac{1}{2}(n+1)\lambda F^{-1}h_{kl}, \tag{8.12}$$

where

$$\lambda := \frac{2}{n^2-1}Fg^{jm}E_{jm}.$$

Next we are going to show that $\lambda = \lambda(x,y)$ is independent of $y \in T_xM$ at any point $x \in M$. Plugging (8.12) into (8.5) and (8.10) respectively, one obtains

$$\begin{aligned} B_j{}^i{}_{kl} &= \frac{\lambda}{F}\Big\{h_{jk}\delta_l^i + h_{jl}\delta_k^i + h_{kl}\delta_j^i\Big\} + \Big[\lambda F^{-1}h_{jk}\Big]_{y^l} y^i \\ &= \frac{\lambda}{F}\Big\{h_{jk}h_l^i + h_{jl}h_k^i + h_{kl}h_j^i\Big\} + 2cF^{-1}C_{jkl}y^i. \end{aligned}$$

Comparing the above two identities yields

$$\lambda_{y^l}h_{jk} = 2(c-\lambda)C_{jkl}. \tag{8.13}$$

Contracting (8.13) with g^{jk} yields

$$\lambda_{y^l} = \frac{2}{n-1}(c-\lambda)I_l. \tag{8.14}$$

Plugging (8.14) into (8.13), one obtains

$$(c-\lambda)\Big\{(n-1)C_{jkl} - I_l h_{jk}\Big\} = 0. \tag{8.15}$$

Contracting the above identity with g^{jl} yields

$$(n-2)(c-\lambda)I_k = 0.$$

Since $n \geq 3$, the above equation becomes

$$(c - \lambda)I_k = 0.$$

Then it follows from (8.14) that $\lambda_{y^k} = 0$.

Thus $\lambda = \lambda(x)$ is independent of $y \in T_xM$.

Now it follows from (8.13) that

$$(c - \lambda)C_{jkl} = 0.$$

At any point $x \in M$ where $F_x = F|_{T_xM}$ is not Euclidean, $C_{jkl}(x, y) \neq 0$ for some $y \in T_xM \setminus \{0\}$. Then $\lambda(x) = c(x)$. By (8.12), This completes the proof. Q.E.D.

Combining Corollary 8.5 with Lemma 8.7, we have the following theorem.

Theorem 8.8.([ChSh5]) Let (M, F) be a non-Riemannian Douglas manifold of dimension $n \geq 3$. Then the following are equivalent,

(a) F has relatively isotropic Landsberg curvature, $\mathbf{L} + cF\mathbf{C} = 0$;

(b) F has isotropic mean Berwald curvature, $\mathbf{E} = \frac{n+1}{2}cF^{-1}h$,
where $c = c(x)$ is a scalar function on M.

Furthermore, by Theorem 8.4 and Theorem 8.8, we obtain the following

Theorem 8.9.[ChSh5]) Let F be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 3$. The following are equivalent.

(a) F is of isotropic Berwald curvature;

(b) F is a Douglas metric with isotropic mean Berwald curvature;

(c) F is a Douglas metric with relatively isotropic Landsberg curvature.

From Theorem 8.9 and Theorem 8.4, one gets the following

Corollary 8.10. For a non-Riemannian Douglas metric F on a manifold of dimension $n \geq 3$. The following are equivalent.

(i) $\mathbf{L} + c(x)F\mathbf{C} = 0$;

(ii) $B_j{}^i{}_{kl} = c(x)\left\{F_{jk}\delta^i_l + F_{jl}\delta^i_k + F_{kl}\delta^i_j + F_{jkl}y^i\right\}$;

(iii) $\mathbf{E} = \frac{n+1}{2}c(x)F^{-1}h$.
Here $c = c(x)$ is a scalar function on M.

It is clear that Corollary 8.10 is the generalization of Corollary 6.5 and It generalizes Bácsó and Matsumoto's result (cf. Lemma 8.6) which says that, for a Douglas metric F, $\mathbf{L} = 0$ if and only if $\mathbf{B} = 0$ ([BaMa2]).

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Chapter 9

Summary

There are several important geometric quantities in Finsler geometry. The flag curvature $\mathbf{K}$ is an analogue of the sectional curvature in Riemannian geometry. The Ricci curvature $\mathbf{Ric}$ is another kind of Riemannian geometric quantity. Besides, we have some so-called non-Riemannian geometric quantities. The Cartan torsion $\mathbf{C}$ is a primary quantity. There is another quantity which is determined by the Busemann-Hausdorff volume form, that is the so-called distortion τ. The vertical differential of τ on each tangent space gives rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^k} dx^k$. $\mathbf{C}, \tau$ and $\mathbf{I}$ are the basic geometric quantities which characterize the Riemannian metrics among Finslers metrics. Differentiating $\mathbf{C}$ along geodesics gives rise to the Landsberg curvature $\mathbf{L}$. The horizontal derivative of τ along geodesics is the so-called S-curvature $\mathbf{S} := \tau_{|k} y^k$. The horizontal derivative of $\mathbf{I}$ along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|k} y^k$. From the geodesic coefficients $G^i(x, y)$, we can define the Berwald curvature $\mathbf{B}$ and the mean Berwald curvature $\mathbf{E}$ which are defined by

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{ij} := \frac{1}{2} B_m{}^m{}_{ij}.$$

Furthermore, we can define the Douglas curvature $\mathbf{D}$ by $\mathbf{B}$ and $\mathbf{E}$. Obviously, τ, $\mathbf{I}$, $\mathbf{S}$, $\mathbf{J}$, $\mathbf{C}$, $\mathbf{L}$ and $\mathbf{B}$, $\mathbf{E}$, $\mathbf{D}$ all vanish for Riemannian metrics. The Riemann curvature measures the shape of the space while the non-Riemannian quantities describe the change of the "color" on the space. It is found that the flag curvature is closely related to these non-Riemannian quantities.

Finsler projective geometry is an important part of Finsler geometry and the Ricci curvature plays an important role in the Finsler projective geometry.

Z. Shen proved that for two pointwise projectively related Einstein metrics g and $\tilde{g}$ on an n-dimension compact manifold M, their Einstein constants have the same sign. In addition, if their Einstein constants are negative and equal, then $g = \tilde{g}$. In section 3, we continue to study pointwise projectively related Finsler metrics and give a comparison theorem on the Ricci curvatures.

Theorem 3.2.([ChSh1]) Let (M, g) be a complete Finsler manifold and $\tilde{g}$ another Finsler metric on M, which is pointwise projectively related to g. Suppose that

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}.$$

Then the projective equivalence is trivial. Further, $\tilde{g}$ is horizontally parallel with respect to g, $\nabla\tilde{g} = 0$ and the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$.

Furthermore, we obtain an additional conclusion to Theorem 3.2 for projectively related Finsler metrics with the same S-curvatures.

Theorem 3.6.([ChSh1]) Let (M, g) be a complete Finsler manifold and $\tilde{g}$ another Finsler metric on M, which is pointwise projectively related to g. Suppose that both g and $\tilde{g}$ satisfy

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}, \qquad \tilde{\mathbf{S}} = \mathbf{S}.$$

Then the projective equivalence between g and $\tilde{g}$ is trivial. Further, $\tilde{g}$ is horizontally parallel with respect to g, the Riemann curvatures are equal, $\tilde{\mathbf{R}} = \mathbf{R}$, and $dV_{\tilde{g}}$ is proportional to dV_g.

If we modify the inequality in Theorem 3.2 into equality, we obtain the following theorem.

Theorem 3.9.([Ch1]) Let F be a Finsler metric on a manifold M and $\tilde{F}$ a another Finsler metric on M which is pointwise projectively related to F. Suppose that both F and $\tilde{F}$ satisfy

$$\widetilde{\mathbf{Ric}} = \mathbf{Ric}.$$

Then F is complete if and only if $\tilde{F}$ is complete. In this case, along any geodesic $c(t)$ of F or $\tilde{F}$,

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = constant.$$

Besides, we study pointwise projectively related Riemannian metrics. We also discuss the projectively flat Finsler metrics with some special curvature properties in sections 6 and 7. One of the important problems in Finsler geometry is to study and characterize locally projectively flat Finsler metrics.

Another important problem in Finsler geometry is to study and characterize Finsler metrics of scalar curvature. This problem has not been solved yet, even for Finsler metrics of constant flag curvature. In section 4, we disscuss the Finsler metrics of scalar curvature and partially determine the flag curvature when F is of isotropic S-curvature or relatively isotropic mean Landsberg curvature.

Theorem 4.1.([CMS][ChSh3]) Let (M, F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\mathbf{K}(x, y)$. Suppose that

$$\mathbf{S} = (n+1)c(x)F(x, y).$$

Then there is a scalar function $\sigma(x)$ on M such that

$$\mathbf{K} = 3\frac{c_{x^m}y^m}{F(x, y)} + \sigma(x).$$

In particular, $c = constant$ if and only if $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on M.

Theorem 4.2.([CMS][ChSh3]) Let (M, F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\mathbf{K}(x, y)$. Suppose that

$$\mathbf{J} + c(x)F\mathbf{I} = 0.$$

Then the flag curvature $\mathbf{K}$ and the distortion τ satisfy

$$\frac{n+1}{3}\mathbf{K}_{y^k} + \left(\mathbf{K} + c(x)^2 - \frac{c_{x^m}y^m}{F(x, y)}\right)\tau_{y^k} = 0.$$

(a) If $c(x) = constant$, then there is a scalar function $\rho(x)$ on M such that

$$\mathbf{K} = -c^2 + \rho(x)e^{-\frac{3\tau(x,y)}{n+1}}, \qquad y \in T_xM \setminus \{0\}.$$

(b) Suppose that F is non-Riemannian on any open subset of M. Then $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{K} = -c^2$ is a nonpositive constant. In this case, $\rho(x) = 0$.

In fact, all known Randers metrics $F = \alpha + \beta$ of scalar curvature (in dimension $n > 2$) satisfy $\mathbf{S} = (n+1)c(x)F$ or $\mathbf{J} + c(x)F\mathbf{I} = 0$, where $c(x)$ is a function on M. Motivated by such phenomena, in section 5, we study Randers metrics satisfying $\mathbf{J} + c(x)F\mathbf{I} = 0$ and classify Randers metrics with flag curvature $\mathbf{K} = \lambda(x)$ and $\mathbf{J} + c(x)F\mathbf{I} = 0$.

Theorem 5.3.([ChSh2][ChSh3]) Let $F = \alpha + \beta$ be a Randers metric on a manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent

(a) $\mathbf{J} + c(x)F\mathbf{I} = 0$;

(b) $e_{00} = 2c(\alpha^2 - \beta^2)$ and β is closed.

Theorem 5.5.([ChSh2][ChSh3]) Let $F = \alpha + \beta$ be Randers metric on an n-dimensional manifold M satisfying

1. $\mathbf{K} = \lambda(x)$ is independent of $y \in T_xM$;
2. $\mathbf{J} + c(x)F\mathbf{I} = 0$ for some scalar function $c(x)$ on M.
   Then $\mathbf{K} = constant = -c^2 \leq 0$. Further, F is either locally Minkowskian ($\mathbf{K} = -c^2 = 0$) or in the form

$$F = \Theta \pm \frac{< \mathbf{a}, y >}{1 + < \mathbf{a}, x >}$$

($\mathbf{K} = -c^2 = -1/4$) after a scaling, where Θ denotes the Funk metric on the unit ball $\mathbf{B}^n$ and $\mathbf{a} \in \mathbf{R}^n$ is a constant vector with $|\mathbf{a}| < 1$.

Furthermore, we study Randers metrics with isotropic S-curvature in section 6. We first obtain the following theorems.

Theorem 6.3.([ChSh2]) Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent

(a) $\mathbf{S} = (n+1)cF$;

(b) $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$;

(c) $e_{00} = 2c(\alpha^2 - \beta^2)$.

Theorem 6.4.([ChSh2]) Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent,

(a) $\mathbf{L} + c(x)F\mathbf{C} = 0$ (or $\mathbf{J} + cF\mathbf{I} = 0$);

(b) $\mathbf{S} = (n+1)cF$ and β is closed.

(c) $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$ and β is closed.

It is known that every locally projectively flat Finsler metric is of scalar curvature. Using the obtained formula for the flag curvature in Theorem 4.1, we classify locally projectively flat Randers metrics with isotropic S-curvature.

Theorem 6.6.([CMS][ChSh3]) Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M and μ denote the constant sectional curvature of α. Suppose that the S-curvature is isotropic, $\mathbf{S} = (n+1)c(x)F$. Then F can be classified as follows.

(A) If $\mu + 4c(x)^2 \equiv 0$, then $c(x) = \textit{constant}$ and $K = -c^2 \leq 0$.

(A1) if $c = 0$, then F is locally Minkowskian with $K = 0$;

(A2) if $c \neq 0$, then after a scaling, F is locally isometric to the following Randers metric on the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$,

$$F(x,y) = \Theta \pm \frac{<\mathbf{a}, y>}{1+ <\mathbf{a}, x>},$$

where $\mathbf{a} \in \mathbf{R}^n$ with $|\mathbf{a}| < 1$, and F has negative constant flag curvature $K = -\frac{1}{4}$.

(B) If $\mu + 4c(x)^2 \neq 0$, then F is given by

$$F(x,y) = \alpha(x,y) - \frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2}$$

and the flag curvature of F is given by

$$\begin{aligned} K &= 3\left\{\frac{c_{x^k}(x)y^k}{F(x,y)} + c(x)^2\right\} + \mu \\ &= \frac{3}{4}\left\{\mu + 4c(x)^2\right\}\frac{F(x,-y)}{F(x,y)} + \frac{\mu}{4}. \end{aligned}$$

(B1) when $\mu = -1$, we can express $\alpha = \alpha_{-1}$. In this case,

$$c(x) = \frac{\lambda + <a, x>}{2\sqrt{(\lambda + <a,x>)^2 \pm (1 - |x|^2)}},$$

where $\lambda \in \mathbf{R}$ and $a \in \mathbf{R}^n$ with $|a|^2 < \lambda^2 \pm 1$.

(B2) when $\mu = 0$, we can express $\alpha = \alpha_0$. In this case,

$$c(x) = \frac{\pm 1}{2\sqrt{\kappa + 2 <a, x> + |x|^2}},$$

where $\kappa > 0$ and $a \in \mathbf{R}^n$ with $|a|^2 < \kappa$.

(B3) when $\mu = 1$, we can express $\alpha = \alpha_{+1}$. In this case,

$$c(x) = \frac{\varepsilon + < a, x >}{2\sqrt{1 + |x|^2 - (\varepsilon + < a, x >)^2}},$$

where $\varepsilon \in \mathbf{R}$ and $a \in \mathbf{R}^n$ with $|\varepsilon|^2 + |a|^2 < 1$.

We also study projectively flat Randers metrics with isotropic S-curvature in the case when the manifold M is closed. And then, in section 7, we study and characterize locally projectively flat Finsler with isotropic S-curvature and obtain the following

Theorem 7.1.([ChSh4]) Let $F = F(x, y)$ be a locally projectively flat Finsler metric on an open subset $\Omega \subset \mathbf{R}^n$. Suppose that F has isotropic S-curvature, $\mathbf{S} = (n + 1)c(x)F$. Then the flag curvature is in the form

$$\mathbf{K} = 3\frac{c_{x^m} y^m}{F} + \sigma,$$

where $\sigma = \sigma(x)$ is a scalar function on Ω.

(a) If $\mathbf{K} \neq -c^2 + \frac{c_{x^m} y^m}{F}$ on Ω, then $F = \alpha + \beta$ is a projectively flat Randers metric with isotropic S-curvature $\mathbf{S} = (n + 1)cF$;

(b) If $\mathbf{K} \equiv -c^2 + \frac{c_{x^m} y^m}{F}$ on Ω, then $c = constant$ and F is either locally Minkowskian $(c = 0)$ or, up to a scaling, locally isometric to the metric

$$\Theta_a := \Theta(x, y) + \frac{< a, y >}{1 +, a, x >} \qquad (c = \frac{1}{2})$$

or its reverse

$$\bar{\Theta}_a := \Theta(x, -y) - \frac{< a, y >}{1 + < a, x >} \qquad (c = -\frac{1}{2}),$$

where $a \in \mathbf{R}^n$ is a constant vector and $\Theta(x, y)$ is Funk metric on Ω.

The Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics. The class of Douglas metrics is also much larger than that of Berwald metrics. The study on Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities. In section 8, we discuss Douglas metrics with relatively isotropic Landsberg curvature or isotropic mean Berwald curvature. Then we introduce the Finsler metrics of isotropic Berwald curvature. We prove an equivalence among the above metrics.

Theorem 8.8.([ChSh5]) Let (M, F) be a non-Riemannian Douglas manifold of dimension $n \geq 3$. Then the following are equivalent,

(a) F has relatively isotropic Landsberg curvature, $\mathbf{L} + cF\mathbf{C} = 0$;

(b) F has isotropic mean Berwald curvature, $\mathbf{E} = \frac{n+1}{2} cF^{-1}h$,
where $c = c(x)$ is a scalar function on M.

Furthermore, we have the following

Theorem 8.9.([ChSh5]) Let F be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 3$. The following are equivalent.

(a) F is of isotropic Berwald curvature;

(b) F is a Douglas metric with isotropic mean Berwald curvature;

(c) F is a Douglas metric with relatively isotropic Landsberg curvature.

Corollary 8.10. For a non-Riemannian Douglas metric F on a manifold of dimension $n \geq 3$. The following are equivalent.

(i) $\mathbf{L} + c(x)F\mathbf{C} = 0$;

(ii) $B^i_{jkl} = c(x)\left\{F_{jk}\delta^i_l + F_{jl}\delta^i_k + F_{kl}\delta^i_j + F_{jkl}y^i\right\}$;

(iii) $\mathbf{E} = \frac{n+1}{2} c(x)F^{-1}h$.

It is clear that Corollary 8.10 generalizes Bácsó and Matsumoto's result which says that, for a Douglas metric F, $\mathbf{L} = 0$ if and only if $\mathbf{B} = 0$ ([BaMa2]).

Chapter 10

Összefoglalás

A Finsler geometriában több igen fontos geometriai objektum található. A **K** zászlógörbület a Riemann geometria szekcionális görbületének analógja. A **Ric** Ricci görbület egy másik fontos Riemann geometriai objektum. Ezen kívül több, nem Riemann geometriai objektummal rendelkezünk. A **C** Cartan torzió az elsődleges geometriai objektum. Több objektum származtatható a Busemann-Hausdorff térfogat formából, mint például a τ torzítás. A τ vertikális deffierenciálja a tangens téren megadja az $\mathbf{I} := \tau_{y^k} dx^k$ Cartan torziót. $\mathbf{C}, \tau$ és az **I** alapvető geometriai objektumok, amelyek meghatározzák a Riemann metrikákat a Finsler metrikák között. A **C** geodetikusok mentén vett differenciálja meghatározza a **L** Landsberg görbületet. A τ geodetikusok mentén vett horizontális deriváltja előállítja az úgynevezett $\mathbf{S} := \tau_{|k} y^k$ S-görbületet. Az **I** geodetikusok mentén vett horizontális deriváltját a $\mathbf{J} := \mathbf{I}_{|k} y^k$ Landsberg görbületnek hívjuk. A $G^i(x, y)$ geodetikus együtthatókból definiálhatjuk a **B** Berwald görbületet és az **E** fő Berwald görbületet, amelyek a következő képletekkel vannak megadva

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{ij} := \frac{1}{2} B_m{}^m{}_{ij}.$$

Továbbá, definiálhatjuk a **D** Douglas görbületet **B**-ből és **E**-ből. Látható, hogy τ, **I**, **S** , **J**, **C**, **L** és **B**, **E**, **D** mind eltűnik Riemann metrikák esetén. A zászlógörbület szoros kapcsolatban van a nem Riemann tulajdonságokkal.

A projektív Finsler geometria nagyon fontos része a Finsler geometriának és a Ricci görbület egy fontos szerepet játszik a projektív Finsler geometriában. Z. Shen bebizonyította, hogy két egymáshoz projektív vonatkozásban lévő g

és $\tilde{g}$ Einstein metrika egy n-dimenziós kompakt M sokaságon a következő tulajdonsággal rendelkezik: az Einstein konstansok ugyanazzal az előjellel rendelkeznek. Mindazonáltal, ha az Einstein konstansok negatívak és egyenlők, akkor a $g = \tilde{g}$. A 3-as fejezetben folytatjuk a Finsler terek projektív vonatkozásainak tanulmányozását és megadunk a Ricci görbületekre vonatkozóan egy összehasonlító tételt.

3.2. Tétel([ChSh1]) Legyen (M, g) egy teljes Finsler sokaság és $\tilde{g}$ egy másik Finsler metrika M-en, amelyik projektív vonatkozásban van g-vel. Tegyük fel, hogy

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}.$$

Ekkor a projektív megfeleltetés triviális. Továbbá, a $\tilde{g}$ horizontálisan párhuzamos g-vel ($\nabla\tilde{g} = 0$), és a Riemann görbületek egyenlők $\tilde{\mathbf{R}} = \mathbf{R}$.

Továbbá, a 3.2 Tételből Finsler metrikák prejektív vonatkozásaira az S-görbületek kapcsolatára tudunk rámutatni.

3.6. Tétel([ChSh1]) Legyen (M, g) egy teljes Finsler sokaság és $\tilde{g}$ egy másik Finsler metrika M-en, amely projektív vonatkozásban van g-vel. Tegyük fel, hogy a g és $\tilde{g}$ kielégítik a

$$\widetilde{\mathbf{Ric}} \leq \mathbf{Ric}, \qquad \tilde{\mathbf{S}} = \mathbf{S}$$

feltételeket. Ekkor a projektív vonatkozás g és $\tilde{g}$ között triviális. Továbbá, $\tilde{g}$ horizontálisan párhuzamos g-vel, és a Riemann görbületek megegyeznek $\tilde{\mathbf{R}} = \mathbf{R}$, és $dV_{\tilde{g}}$ skalárszorosa a dV_g-hez.

Ha a 3.2. Tételben az egyenlőtlenséget egyenlőségre cseréljük, akkor a következő tételt kapjuk.

3.9. Tétel([Ch1]) Legyen F egy olyan Finsler metrika az M sokaságon és $\tilde{F}$ egy másik Finsler metrika M-en, amelyek projektívek egymáshoz. Tegyük fel, hogy F és $\tilde{F}$ kielégítik

$$\widetilde{\mathbf{Ric}} = \mathbf{Ric}.$$

Ekkor F akkor és csak akkor teljes, ha $\tilde{F}$ is teljes. Ebben az esetben bármely közös $c(t)$ geodetikus mentén

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = konstans.$$

Továbbá, a projektív vonatkozásban lévő Riemann metrikákat is tanulmányozzuk, és vizsgáljuk a több speciális görbületi tulajdonságokkal rendelkező síkprojektív Finsler metrikákat a 6-ik és 7-ik fejezetben. Az egyik fontos probléma a Finsler geometriában a lokálisan síkprojektív Finsler metrikák tanulmányozása.

A másik fontos probléma a Finsler geometriában a skalárgörbületű Finsler metrikák jellemzése. Ez a probléma eddig még nem megoldott, sőt nem megoldott a konstans zászlógörbületekre sem. A 4-ik fejezetben azokat a skalárgörbületű Finsler metrikákat tanulmányozzuk, amikoris a F izotróp S-görbülettel rendelkezik vagy relatíve izotrópikus Landsberg görbülettel.

4.1. Tétel([CMS][ChSh3]) Legyen (M, F) egy n-dimenziós skalárgörbületű Finsler sokaság $\mathbf{K}(x, y)$ zászlógörbülettel. Tegyük fel, hogy

$$\mathbf{S} = (n+1)c(x)F(x, y).$$

Ekkor létezik egy $\sigma(x)$ skalárfüggvény M-en úgy, hogy

$$\mathbf{K} = 3\frac{c_{x^m}y^m}{F(x, y)} + \sigma(x).$$

Speciális esetként, $c = konstans$ akkor és csak akkor ha $\mathbf{K} = \mathbf{K}(x)$ skalárfüggvény M-en.

4.2. Tétel([CMS][ChSh3]) Legyen (M, F) egy n-dimenziós skalárgörbületű Finsler sokaság $\mathbf{K}(x, y)$ zászlógörbülettel. Tegyük fel, hogy

$$\mathbf{J} + c(x)F\mathbf{I} = 0.$$

Ekkor a $\mathbf{K}$ zászlógörbület és τ torzió eleget tesz a következő egyenlőségnek

$$\frac{n+1}{3}\mathbf{K}_{y^k} + \left(\mathbf{K} + c(x)^2 - \frac{c_{x^m}y^m}{F(x, y)}\right)\tau_{y^k} = 0.$$

(a) Ha $c(x) = konstans$, akkor létezik egy olyan $\rho(x)$ skalárfüggvény M-en úgy, hogy

$$\mathbf{K} = -c^2 + \rho(x)e^{-\frac{3\tau(x,y)}{n+1}}, \qquad y \in T_xM \setminus \{0\}.$$

(b) Tegyük fel, hogy F nem Riemann az M egy nyílt részhalmazán. Ekkor $\mathbf{K} = \mathbf{K}(x)$ akkor és csak akkor ha $\mathbf{K} = -c^2$ egy nempozitív konstans. Ebben az esetben $\rho(x) = 0$.

Tulajdonképpen, minden $F = \alpha + \beta$ skalárgörbületű Randers metrika (dimenzió $n > 2$) eleget tesz az $\mathbf{S} = (n+1)c(x)F$ vagy a $\mathbf{J} + c(x)F\mathbf{I} = 0$

egyenlőségeknek, ahol a $c(x)$ egy M-en lévő függvény. Ezáltal érdekesnek gondoljuk az 5-ik fejezetben azon Randers metrikák tanulmányozását, amelyek eleget tesznek a $\mathbf{J}+c(x)F\mathbf{I}=0$ egyenlőségnek, továbbá ismertetjük a $\mathbf{K}=\lambda(x)$ és a $\mathbf{J}+c(x)F\mathbf{I}=0$ zászlógörbületű Randers metrikák osztályozását.

5.3. Tétel([ChSh2][ChSh3]) Legyen $F=\alpha+\beta$ egy Randers metrika az M sokaságon. Egy $c=c(x)$ skalárfüggvényre M-en a következők ekvivalensek

(a) $\mathbf{J}+c(x)F\mathbf{I}=0$;

(b) $e_{00}=2c(\alpha^2-\beta^2)$ és β zárt forma.

5.5. Tétel([ChSh2][ChSh3]) Legyn $F=\alpha+\beta$ egy Randers metrika az M n-dimenziós sokaságon, amely eleget tesz

1. $\mathbf{K}=\lambda(x)$, azaz $y\in T_xM$ független;
2. $\mathbf{J}+c(x)F\mathbf{I}=0$ valamilyen $c(x)$ M-en lévő skalárfüggvényre.
   Ekkor $\mathbf{K}=constant=-c^2\leq 0$. Továbbá, F vagy lokálisan Minkowski ($\mathbf{K}=-c^2=0$) vagy F a következő formulával rendelkezik

$$F=\Theta\pm\frac{<\mathbf{a},y>}{1+<\mathbf{a},x>}$$

($\mathbf{K}=-c^2=-1/4$) ahol Θ egy Funk metrikát jelent a $\mathbf{B}^n$ egységgömbbön és minden $\mathbf{a}\in\mathbf{R}^n$ esetén $|\mathbf{a}|<1$.

Továbbá, tanulmányozni fogjuk az izotróp S-görbületű Randers metrikákat a 6-ik fejezetben. Először a következő tételeket nyerjük.

6.3. Tétel([ChSh2]) Legyen $F=\alpha+\beta$ egy Randers metrika az n-dimenziós M sokaságon. Egy M sokaságon megadott $c=c(x)$ skalárfüggvényre a következők ekvivalensek

(a) $\mathbf{S}=(n+1)cF$;

(b) $\mathbf{E}=(1/2)(n+1)c(x)F^{-1}h$;

(c) $e_{00}=2c(\alpha^2-\beta^2)$.

6.4. Tétel([ChSh2]) Legyen $F=\alpha+\beta$ egy Randers metrika a M n-dimenziós sokaságon. M-en megadott $c=c(x)$ skalárfüggvényre a következők ekvivalensek,

(a) $\mathbf{L}+c(x)F\mathbf{C}=0$ (vagy $\mathbf{J}+cF\mathbf{I}=0$);

(b) $\mathbf{S} = (n+1)cF$ és β zárt.

(c) $\mathbf{E} = (1/2)(n+1)c(x)F^{-1}h$ és β zárt.

Az jól ismert, hogy minden lokálisan síkprojektív Finsler metrika skalárgörbületű. Felhasználva a zászlógörbületre vonatkozó formulát osztályozzuk a lokálisan síkprojektív izotróp S-görbülettel rendelkező Randers metrikákat.

6.6. Tétel([CMS][ChSh3]) Legyen $F = \alpha+\beta$ lokálisan síkprojektív Randers metrika az n-dimenziós M sokaságon és μ jelölje az α metrika szekcionális konstans görbületét. Tegyük fel, hogy az S-görbület izotróp és $\mathbf{S} = (n+1)c(x)F$. Ekkor F a következőképpen osztályozható.

(A) Ha $\mu + 4c(x)^2 \equiv 0$, akkor $c(x) = \mathit{konstans}$ és $K = -c^2 \leq 0$.

(A1) ha $c = 0$, akkor F lokálisan Minkowski, ahol $K = 0$;

(A2) ha $c \neq 0$, akkor az F lokálisan izometrikus a következő Randers metrikához $\mathbf{B}^n \subset \mathbf{R}^n$ egységgömbön, ahol

$$F(x,y) = \Theta \pm \frac{< \mathbf{a}, y >}{1+ < \mathbf{a}, x >},$$

miközben $\mathbf{a} \in \mathbf{R}^n$ ahol $|\mathbf{a}| < 1$, és az F negatív konstans $K = -\frac{1}{4}$ zászlógörbülettel rendelkezik.

(B) Ha $\mu + 4c(x)^2 \neq 0$, akkor az F a következőképpen adott

$$F(x,y) = \alpha(x,y) - \frac{2c_{x^k}(x)y^k}{\mu + 4c(x)^2}$$

és az F zászlógörbülete a következőképpen adható meg

$$\begin{aligned} K &= 3\left\{\frac{c_{x^k}(x)y^k}{F(x,y)} + c(x)^2\right\} + \mu \\ &= \frac{3}{4}\left\{\mu + 4c(x)^2\right\}\frac{F(x,-y)}{F(x,y)} + \frac{\mu}{4}. \end{aligned}$$

(B1) ha $\mu = -1$, akkor az α-t az $\alpha = \alpha_{-1}$ fejezhetjük ki. Ebben az esetben a

$$c(x) = \frac{\lambda+ < a, x >}{2\sqrt{(\lambda+ < a, x >)^2 \pm (1 - |x|^2)}},$$

ahol $\lambda \in \mathbf{R}$ és $a \in \mathbf{R}^n$ miközben $|a|^2 < \lambda^2 \pm 1$.

(B2) ha $\mu = 0$, akkor az α-t az $\alpha = \alpha_0$ fejezhetjük ki. Ebben az esetben a

$$c(x) = \frac{\pm 1}{2\sqrt{\kappa + 2 < a, x > + |x|^2}},$$

ahol $\kappa > 0$ és $a \in \mathbf{R}^n$ miközben $|a|^2 < \kappa$.

(B3) ha $\mu = 1$, akkor az α-t az $\alpha = \alpha_{+1}$ fejezhetjük ki. Ebben az esetben a

$$c(x) = \frac{\varepsilon + < a, x >}{2\sqrt{1 + |x|^2 - (\varepsilon + < a, x >)^2}},$$

ahol $\varepsilon \in \mathbf{R}$ és $a \in \mathbf{R}^n$ miközben $|\varepsilon|^2 + |a|^2 < 1$.

A továbbiakban tanulmányozzuk az izotróp S-görbülettel rendelkező síkprojektív Randers metrikákat amikor az M sokaság zárt. Ekkor a 7-ik fejezetben tanulmányozzuk és jellemezzük az izotróp S-görbülettel rendelkező lokálisan síkprojektív Finsler tereket és a következőket kapjuk

7.1. Tétel([ChSh4]) Legyen $F = F(x, y)$ egy lokálisan síkprojektív Finsler metrika az $\Omega \subset \mathbf{R}^n$ nyílt részhalmazon. Tegyük fel, hogy F izotróp S-görbülettel rendelkezik, ahol $\mathbf{S} = (n + 1)c(x)F$. Ekkor a zászlógörbület a következő formulával rendelkezik

$$\mathbf{K} = 3\frac{c_{x^m} y^m}{F} + \sigma,$$

ahol $\sigma = \sigma(x)$ az Ω-n definiált skalárfüggvény.

(a) Ha $\mathbf{K} \neq -c^2 + \frac{c_{x^m} y^m}{F}$ az Ω-n, akkor $F = \alpha + \beta$ egy síkprojektív Randers metrika izotróp S-görbülettel, ahol $\mathbf{S} = (n + 1)cF$;

(b) Ha $\mathbf{K} \equiv -c^2 + \frac{c_{x^m} y^m}{F}$ az Ω-n, akkor $c = konstans$ és F vagy lokálisan Minkowski ($c = 0$) vagy lokálisan izometrikus

$$\Theta_a := \Theta(x, y) + \frac{< a, y >}{1 +, a, x >} \qquad (c = \frac{1}{2})$$

metrikához vagy pedig a következő metrikához

$$\bar{\Theta}_a := \Theta(x, -y) - \frac{< a, y >}{1 + < a, x >} \qquad (c = -\frac{1}{2}),$$

ahol $a \in \mathbf{R}^n$ egy konstans vektor és $\Theta(x, y)$ egy Funk metrika Ω-n.

A Douglas metrikák a Finsler metrikák egy igen gazdag osztálya, amely magában foglalja a síkprojektív Finsler metrikákat. A Douglas metrikák osztálya sokkal gazdagabb, mint a Berwald metrikáké. A Douglas metrikák tanulmányozása lehetőséget ad a nem Riemannian geometriai objektumok jellemzésére. A 8-ik fejezetben tanulmányozzuk azokat a Douglas metrikákat, amelyek relatív izotróp Landsberg görbülettel vagy izotróp Berwald görbülettel rendelkeznek. Vizsgálatainkban a következő tételeket kapjuk.

8.8. Tétel([ChSh5]) Legyen (M, F) egy nem Riemann Douglas sokaság, amelynek dimenziója $n \geq 3$. Ekkor a következők ekvivalensek,

(a) F relatív izotróp Landsberg görbülettel rendelkezik, ahol $\mathbf{L} + cF\mathbf{C} = 0$;

(b) F izotróp Berwald görbülettel rendelkezik, ahol $\mathbf{E} = \frac{n+1}{2} cF^{-1}h$, miközben $c = c(x)$ egy skalár függvény M-en.

Továbbá a következőket kapjuk

8.9. Tétel([ChSh5]) Legyen F egy nem Riemann Finsler metrika az M sokaságon, ahol a dimenzió $n \geq 3$. Ekkor a következők ekvivalensek.

(a) F egy izotróp Berwald görbülettel rendelkezik;

(b) F egy izotróp Berwald görbülettel rendelkező Douglas metrika;

(c) F egy relatív izotróp Landsberg görbülettel rendelkező Douglas metrika.

8.10. Következmény Egy nem Riemann F Douglas metrikára $n \geq 3$ dimenzió esetén a következők ekvivalensek.

(i) $\mathbf{L} + c(x)F\mathbf{C} = 0$;

(ii) $B^i_{jkl} = c(x)\left\{F_{jk}\delta^i_l + F_{jl}\delta^i_k + F_{kl}\delta^i_j + F_{jkl}y^i\right\}$;

(iii) $\mathbf{E} = \frac{n+1}{2} c(x)F^{-1}h$.

Világos, hogy a 8.10 következmény Bácsó és Matsumoto eredményének általánosítása, amely azt mondja, hogy egy F Douglas metrika esetén $\mathbf{L} = 0$ akkor és csak akkor, ha $\mathbf{B} = 0$ ([BaMa2]).

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[ChSh2] X. Chen and Z. Shen, Randers metrics with special curvature properties, Osaka J. of Math. **40**(2003), 87-101.

[ChSh3] X. Chen and Z. Shen, Finsler metrics with special curvature properties, Periodica Math. Hungarica, **48**(1-2)(2004), 33-47.

[ChSh4] X. Chen and Z. Shen, Projectively flat Finsler metrics with almost isotropic S-curvature, Acta Mathematica Scientia (to appear).

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[Sh9] Z. Shen, Riemann-Finsler Geometry, World Scientific, Singapore, 2005.

Appendix A

Publications

A.1 All Publications

1. X. Cheng, Z. Shen:
   Randers metrics of scalar flag curvature
   preprint, 2005.

2. X. Cheng, Z. Shen:
   On (α, β)-metrics with almost isotropic S-curvature
   preprint, 2005.

3. X. Cheng:
   The Ricci curvature in Finsler projective geometry
   J. of Math. (PRC) (to appear).

4. X. Cheng, Z. Shen:
   On Douglas metrics
   Publ. Math. Debrecen, 66(3-4)(2005), 503-512.

5. X. Cheng, Z. Shen:
   Projectively flat Finsler metrics with almost isotropic S-curvature
   Acta Mathematica Scientia (to appear).

6. X. Cheng, Z. Shen:
   Finsler metrics with special curvature properties
   Periodica Mathematica Hungarica, 48(1-2)(2004), 33-47.

7. X. Cheng, Z. Shen:
A comparison theorem on the Ricci curvature in projective geometry
Annals of Global Analysis and Geometry, 23(2)(2003), 141-156. Math. Reviews 2003k: 53089

8. X. Cheng, Z. Shen:
Randers metrics with special curvature properties
Osaka Journal of Mathematics, 40(2003), 87-101. Math. Reviews 2004a: 53096

9. X. Cheng, X. Mo and Z. Shen:
On the flag curvature of Finsler metrics of scalar curvature
Journal of the London Mathematical Society, 68(2)(2003), 762-780.

10. X. Cheng:
On projectively flat Finsler spaces
Advances in Mathematics (China), 31(4)(2002), 337 -342. Math. Reviews 2003h: 53026

11. X. Cheng, W. Yang:
The characteristics of hyperspheres in Minkowski spaces
Acta Mathematica Scientia 23B(1)(2003), 139-144. Math. Reviews 2003j: 53114

12. X. Cheng:
Finsler spaces of scalar curvature and projective changes of Finsler metrics
J. of Math. (PRC), 23(4)(2003), 455-462.

13. X. Cheng:
Randers metrics and their curvature properties
the Workshop on Finsler Geometry at MSRI, 2002.
www.msri.org/publications/ln/msri/2002/finsler/chen/1/index.html

14. X. Cheng, W. Yang:
On the Chern connection of Finsler submanifolds
Acta Mathematica Scientia, 20B(4)(2000), 551-557. Zentralblatt MATH 992: 53043

15. X. Cheng, J. Yan:
Some properties of submanifolds in a Minkowski space
J. of Math. (PRC), 20(1)(2000), 17-22. Math. Reviews 2001e: 53056

16. X. Cheng, W. Yang:
On a property of quasi-Kahler submanifolds
J. of Math. (PRC), 18(4)(1998), 369-372. Math. Reviews 99k:53051

17. X. Cheng, J. Yan:
The infinitesimal Φ_γ*-isometry of surfaces in E (Chinese)*
J. of Southwest Normal Univ. of China (N. S. Edition) 23(4)(1998), 394-400. Math. Review 2000d:53007

18. X. Cheng, W. Yang:
On the infinitesimal isometric deformations of submanifolds
Acta Mathematica Scientia, 17(4)(1997), 392-404.
Math. Reviews 99i:53056

19. X. Cheng, W. Yang and D. Qiu:
A theorem on infinitesimal I-isometry of surfaces immersed in a space with constant curvature
Chinese Quarterly J. of Mathematics, 12(4)(1997), 63-69.
Math. Reviews 2000d:53086

20. X. Cheng:
On compact hypersurfaces with a conformal variation
Math. Applicata, 9 (suppl.)(1996), 130-135. Math. Reviews 97k:53057

21. X. Cheng, J. Zhou and H. Li:
Analogues of Hadwiger's theorem in space R^n*-sufficient conditions for a convex domain to enclose another*
Acta Math. Sinica (new series), 11(1)(1995), 12-22. Math. Reviews 97j:52006

22. X. Cheng:
On compact submanifolds with a conformal variation (Chinese)
J. of Southwest Normal Univ. of China (N. S. Edition), 20(6)(1995), 632-637.

23. X. Cheng, G. Zhang:
The theory of the infinitesimal isometry of hypersurfaces in the space with constant curvature (Chinese)
Acta Math. Sinica 36(3)(1993), 306-320. Math. Reviews 95d:53017

24. X. Cheng:
The characterization of totally umbilical submanifolds (Chinese)
J. of Math. (PRC), 13(2)(1993), 229-231. Math. Review97k:53067

25. X. Cheng:
The characterization of hyperspheres in Euclidean space (Chinese)
J. of Sichuan Normal Univ. (N. S. Edition), 14(5)(1991), 27-30. Math. Reviews 93:53082

26. X. Cheng:
A theorem on the surfaces with constant mean curvature (Chinese)
Math. Applicata 3(1)(1990), 100-103. Math. Reviews 91:53076

27. X. Cheng:
Some theorems on the minimal submanifolds (Chinese)
J. of Sichuan Normal Univ. (N. S. Edition), 13(2)(1990), 39-43.

A.2 Publications That Are Joint with the Dissertation

1. X. Chen:
The Ricci curvature in Finsler projective geometry
J. of Math. (PRC) (to appear)

2. X. Cheng:
On projectively flat Finsler spaces
Advances in Math. (China), **31**(4)(2002), 337-342.

3. X. Chen and Z. Shen:
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5. X. Chen and Z. Shen:
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Geometric Quantities and Their Meanings in Finsler Geometry

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a Matematika tudományágban

Írta: Xinyue Cheng
matematikus (differenciál geometria)

Készült a Debreceni Egyetem Matematika és Számítástudományi
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