ON A CONJECTURE OF POMERANCE

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Dedicated to Professor Schinzel on the occasion of his 75th birthday

ABSTRACT. We say that k is a P-integer if the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k. In 1980 Pomerance proved the finiteness of the set of P-integers and conjectured that 30 is the largest P-integer. We prove the conjecture assuming the Riemann Hypothesis. We further prove that there is no P-integer between 30 and 10^{11} and above 10^{3500}.

1. INTRODUCTION

Let $k > 1$ be an integer. We denote Euler's totient function by $\varphi(k)$ and the number of distinct prime divisors of k by $\omega(k)$. We say that k is a P-integer if the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k. In 1980, Pomerance [8] proved the finiteness of the set of P-integers. The following conjecture was proposed by him in [8].

Conjecture of Pomerance. If k is a P-integer, then $k \leq 30$.

This conjecture is still open. Recently, Hajdu and Saradha [3] and Saradha [12] have given simple conditions under which an integer k is not a P-integer. By their results, it follows that

- *no prime is a P-integer except* 2*;*
- *no square or a cube of a prime is a P-integer except* 4*;*
- *no integer k with its least odd prime divisor* $> \log k$ *is a P-integer except when* $k \in \{2, 4, 6, 12, 18, 30\}$.

It is easy to check that the only P-integers ≤ 30 are $2, 4, 6, 12, 18, 30$. It was checked by computation in [3] that if k is another P-integer, then $k \geq 5.5 \cdot 10^5$. In Theorem 4.1 we improve this bound to 10^{11}.

2010 *Mathematics Subject Classification.* Primary 11N13.

Key words and phrases. primes in residue classes, Riemann Hypothesis.

[1] Research supported in part by the OTKA grants K67580 and K75566, and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project. The project is implemented through the New Hungary Development Plan, cofinanced by the European Social Fund and the European Regional Development Fund.

In this paper, we give a quantitative version of the finiteness result of Pomerance and prove the conjecture of Pomerance under the Riemann Hypothesis. We have

Theorem 1.1. *If k is a P-integer, then* $k < 10^{3500}$.

Theorem 1.2. *Suppose the Riemann Hypothesis holds. Then the only P-integers are* $2, 4, 6, 12, 18, 30$.

Pomerance's conjecture is closely related to the classical problem about the least primes in arithmetic progressions. Let ℓ be a positive integer with $\gcd(k,\ell) = 1$. Denote by $p(k,\ell)$ the least prime $p \equiv \ell \pmod{k}$. Let $P(k)$ be the maximum value of $p(k,\ell)$ for all ℓ. Linnik [7] has shown that

$$P(k) \ll k^L$$

for some constant L which is known as Linnik's constant. A huge literature is available on finding good values for L (see [4, 15]). In the other direction, Prachar [9] and Schinzel [13] have shown that there is an absolute constant c such that for each ℓ there are infinitely many k with

$$p'(k,\ell) > \frac{ck \log k \cdot \log\log k \cdot \log\log\log\log k}{(\log\log\log k)^2}$$

where $p'(k,\ell)$ is the first prime q with $q \equiv \ell \pmod{k}$. In his proof of the finiteness of P-integers Pomerance [8] used the Jacobsthal function to show that

$$P(k) \geq (e^\gamma + o(1))\varphi(k) \log k$$

where γ is Euler's constant.

In our proofs we applied different tools. We use that the primitive residues modulo k between 0 and k are symmetric around $k/2$. Our arguments are based on results about the zeros of the Riemann zeta function and estimates for the number of primes in intervals.

2. Lemmas

Throughout the paper, let $p_1 < p_2 < \dots$ be the increasing sequence of prime numbers. For any $x > 1$, let $\pi(x)$ denote the number of prime numbers not exceeding x, and $\mathrm{Li}(x) = \lim_{\epsilon \to 0^+} \int_{t=0}^{1-\epsilon} \frac{dt}{\log t} + \int_{t=1+\epsilon}^{x} \frac{dt}{\log t}$. We put $\pi(x) = 0$ for $0 \leq x \leq 1$.

Lemma 2.1. *For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have*
(i) $\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{1.8x}{\log^3 x}$ *for* $x > 32299$;
(ii) $\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2.51x}{\log^3 x}$ *for* $x > 355991$;
(iii) $|\pi(x) - \mathrm{Li}(x)| < .4394 \frac{x}{(\log x)^{3/4}} \exp\left(-\sqrt{\frac{\log x}{9.646}}\right)$ *for* $x \geq 58$;

(iv) *if the Riemann Hypothesis holds, then* $|\pi(x) - \mathrm{Li}(x)| < \frac{1}{8\pi}\sqrt{x}\log x$ *for* $x > 2656$;
(v) $\mathrm{Li}(x) > \pi(x)$ *for* $x \leq 10^{14}$;
(vi) $p_n < n(\log n + \log\log n)$ *for* $n \geq 6$;
(vii) $p_n > n\log n$ *for* $n \geq 1$;
(viii) $\frac{n}{\varphi(n)} < 1.7811\log\log n + \frac{2.51}{\log\log n}$ *for* $n \geq 3$.

Proof. We mention the references where the estimates from Prime Number Theory given in the lemma can be found.
(i), (ii) Dusart [2], p. 36.
(iii) Dusart [2], p. 41.
(iv) Schoenfeld [14], p. 339.
(v) Kotnik [6], p. 59.
(vi), (vii) Rosser and Schoenfeld [10], p. 69.
(viii) Rosser and Schoenfeld [10], p. 72. □

Lemma 2.2. *Let x be a real number with $x > 712000$. Then we have*

$$2\pi\left(\frac{x}{2}\right) - \pi(x) > \frac{.693x}{\log^2 x}.$$

Proof. We have, by Lemma 2.1 (i), (ii), for $x > 712000$,

$$2\pi(x/2) - \pi(x) >$$

$$\frac{x}{\log(x/2)} + \frac{x}{\log^2(x/2)} + \frac{1.8x}{\log^3(x/2)} - \frac{x}{\log x} - \frac{x}{\log^2 x} - \frac{2.51x}{\log^3 x} >$$

$$\frac{x}{\log x\left(1 - \frac{\log 2}{\log x}\right)} - \frac{x}{\log x} + \frac{x}{\log^2 x\left(1 - \frac{\log 2}{\log x}\right)^2} - \frac{x}{\log^2 x} - \frac{.71x}{\log^3 x} >$$

$$\frac{x}{\log x} \cdot \frac{\log 2}{\log x} + \frac{x}{\log^2 x} \cdot \frac{2\log 2}{\log x} - \frac{.71x}{\log^3 x} > \frac{.693x}{\log^2 x}.$$

□

Lemma 2.3. *Let x and y be positive real numbers with $x > y$, $x \geq 59$. Then*

$$2\pi(x+y) - \pi(x) - \pi(x+2y) >$$

$$\frac{y^2}{(x+2y)\log^2(x+2y)} - \frac{1.7576(x+2y)}{(\log x)^{3/4}} e^{-\sqrt{\frac{\log x}{9.646}}}.$$

Proof. By Lemma 2.1 (iii),

$$2\pi(x+y) - \pi(x) - \pi(x+2y) >$$

$$2\mathrm{Li}(x+y) - \mathrm{Li}(x) - \mathrm{Li}(x+2y) - 1.7576\frac{x+2y}{(\log x)^{3/4}}\exp\left(-\sqrt{\frac{\log x}{9.646}}\right).$$

Observe that

$$2\mathrm{Li}(x+y)-\mathrm{Li}(x)-\mathrm{Li}(x+2y)=\int_x^{x+y}\frac{dt}{\log t}-\int_{x+y}^{x+2y}\frac{dt}{\log t}$$
$$=\int_x^{x+y}\left(\frac{1}{\log t}-\frac{1}{\log(t+y)}\right)dt=\frac{y^2}{\xi\log^2\xi}$$

for some ξ with $x<\xi<x+2y$, by the mean value theorem applied twice. Thus

$$2\pi(x+y)-\pi(x)-\pi(x+2y)>$$
$$\frac{y^2}{(x+2y)\log^2(x+2y)}-1.7576\frac{x+2y}{(\log x)^{3/4}}\exp\left(-\sqrt{\frac{\log x}{9.646}}\right).$$

□

Lemma 2.4. *Suppose the Riemann Hypothesis holds true. Let $x>y>0$, $x\geq 2657$. Then*

$$2\pi(x+y)-\pi(x)-\pi(x+2y)>$$
$$\frac{y^2}{(x+2y)\log^2(x+2y)}-\frac{\log(x+2y)}{\theta}\sqrt{x+2y}$$

where

$$\theta=\begin{cases}2\pi \ if \ x+2y>10^{14}\\ 4\pi \ if \ x+2y\leq 10^{14}.\end{cases}$$

Proof. By Lemma 2.1 (iv), (v),

$$2\pi(x+y)-\pi(x)-\pi(x+2y)>$$
$$2\mathrm{Li}(x+y)-\mathrm{Li}(x)-\mathrm{Li}(x+2y)-\frac{\log(x+2y)}{\theta}\sqrt{x+2y}.$$

The lemma follows in the same way as in the proof of Lemma 2.3. □

3. A criterion for an integer k to be not a P-integer

Suppose k is a P-integer >30. Further, due to results from [3] and [12] mentioned in the introduction, we may also assume that neither k nor $k/2$ is prime. Let $\varphi(k)+\omega(k)=T$. Then there are exactly $\varphi(k)$ primes belonging to the set $\{p_1,\cdots,p_T\}$ which are coprime to k and form a reduced residue system mod k. The remaining $\omega(k)$ primes in this set divide k. Let

$$D'_k=\left\{i\leq T: p_i \pmod{k}<\frac{k}{2}\right\},$$
$$D''_k=\left\{i\leq T: p_i \pmod{k}\geq\frac{k}{2}\right\}$$

and

$$D_k''' = \{i \le T : p_i | k\}.$$

Note that $|D_k'''| = \omega(k)$ where $|A|$ denotes the number of elements of a set A. By the symmetry of the primitive residues about $k/2$, we get

$$|D_k' \setminus D_k'''| = |D_k'' \setminus D_k'''|$$

which implies

$$|D_k'| - |D_k''| \le |D_k'''| = \omega(k). \tag{1}$$

Let t be an integer such that $tk < p_T < (t+1)k$. We observe that if $p_T \in (tk, tk + \frac{k}{2})$ we have

$$|D_k'| = \sum_{n=0}^{t-1} \left(\pi \left(nk + \frac{k}{2} \right) - \pi(nk) \right) + T - \pi(tk),$$

$$|D_k''| = \sum_{n=0}^{t-1} \left(\pi(nk+k) - \pi \left(nk + \frac{k}{2} \right) \right)$$

and if $p_T \in \left(tk + \frac{k}{2}, tk + k\right)$, then

$$|D_k'| = \sum_{n=0}^{t} \left(\pi \left(nk + \frac{k}{2} \right) - \pi(nk) \right),$$

$$|D_k''| = \sum_{n=0}^{t-1} \left(\pi(nk+k) - \pi \left(nk + \frac{k}{2} \right) \right) + T - \pi \left(tk + \frac{k}{2} \right).$$

Thus we get

$$|D_k'| - |D_k''| = \sum_{n=0}^{t-1} \left(2\pi \left(nk + \frac{k}{2} \right) - \pi(nk) - \pi(nk+k) \right) + T - \pi(tk)$$

in the former case, and in the latter case

$$|D_k'| - |D_k''| = \sum_{n=0}^{t} \left(2\pi \left(nk + \frac{k}{2} \right) - \pi(nk) - \pi(nk+k) \right) + \pi(tk+k) - T.$$

Let $L(k) = t - 1$ in the former case and $L(k) = t$ in the latter. Let $L := L(k)$. We shall use this parameter L later on without any further mentioning. Noting that $T - \pi(tk)$ and $\pi(tk+k) - T$ are both non-negative and that $\omega(k) < \log k$, we find by (1) the following criterion.

Lemma 3.1. *The integer k is not a P-integer, if*

$$S_L := \sum_{n=0}^{L} \left(2\pi \left(nk + \frac{k}{2} \right) - \pi(nk) - \pi(nk+k) \right) - \log k > 0.$$

We note that

$$tk < p_T \le p_k \le k \log(k \log k)$$

by Lemma 2.1 (vi). Thus

$$L \le t < \log(k \log k). \tag{2}$$

On the other hand, using Lemma 2.1 (vii), (viii), putting $h(k) = 1.7811 \log\log k + \frac{2.51}{\log\log k}$, we get

$$L + 2 \ge t + 1 > \frac{p_T}{k} \ge \frac{p_{\varphi(k)}}{k} > \frac{\log k - \log h(k)}{h(k)}. \tag{3}$$

4. A computational result

Theorem 4.1. *If* $30 < k \le 10^{11}$*, then* k *is not a* P*-integer. Further, if* k *is even with* $30 < k \le 2 \cdot 10^{11}$ *then* k *is not a* P*-integer.*

Proof. In [3] it has been computationally verified that no integer k with $30 < k < 5.5 \cdot 10^5$ is a P-integer. Hence we may assume henceforth that

$$5.5 \cdot 10^5 \le k \le 2 \cdot 10^{11}.$$

To cover this interval, we apply a modified version of the algorithm used in [3].

To prove the statement for a given k we apply the following strategy. We find a prime p such that $k < p < p_{\varphi(k)}$ and $p \pmod k$ is also a prime. Then k is not a P-integer. To make this strategy work on the whole range for k under consideration, we shall make use of the following two properties. Let k be an integer with $k \ge 5.5 \cdot 10^5$. Then we have

$$\pi(k+1) + 100 < \varphi(k) \tag{4}$$

and

$$p_{\pi(k+1)+100} < 1.5k. \tag{5}$$

These assertions can be easily checked e.g. by Magma [1], using parts (ii), (vi), (viii) of Lemma 2.1.

First we prove the statement for the even values of k. This is done by the algorithm below, which is based on the strategy indicated above.

Initialization. Let $k_0 = 5.5 \cdot 10^5$. Let H be the list of the first 100 primes larger than $k_0 + 1$, i.e. $H = [p_{\pi(k_0+1)+1}, \dots, p_{\pi(k_0+1)+100}]$.

Step 1. Check successively for the primes $p \in H$ whether $p \pmod{k_0}$ is also a prime. When such a p is found then, by (4), k_0 is not a P-integer – proceed to the next step.

Step 2. Check if k_0+3 is a prime. If not, then proceed to Step 3. If so, this is the first element of H. Remove this prime from H, and append to H the prime $p_{\pi(k_0+1)+101}$ which is the next prime to the last element of H.

Step 3. If $k_0 < 2\cdot 10^{11}$ then put $k_0 := k_0+2$, and go to Step 1.

Using this procedure we could check by a Magma program that there is no even P-integer in the interval $[5.5\cdot 10^5, 2\cdot 10^{11}]$.

Let now k be odd with $5.5\cdot 10^5 < k < 10^{11}$. Then by our algorithm above, using (4) and (5), we know that there exists a prime p satisfying $2k < p < \min\{3k, p_{\varphi(2k)}\}$ such that $q := p \pmod{2k}$ is also a prime. Observe that $q < k$. Thus, as $\varphi(k) = \varphi(2k)$, p is a prime such that $k < p < p_{\varphi(k)}$ and $q = p \pmod{k}$ is also a prime. Hence k is not a P-integer and the theorem follows. $\square$

5. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let k be an integer with $k \geq 10^{3500}$. Then by (3), $L > 500$. We apply Lemma 2.1 (i), (ii) to get

$$2\pi(k/2) - \pi(k) >$$
$$\frac{k}{\log(k/2)} + \frac{k}{\log^2(k/2)} + \frac{1.8k}{\log^3(k/2)} - \frac{k}{\log k} - \frac{k}{\log^2 k} - \frac{2.51k}{\log^3 k}.$$

For $n \geq 1$ we apply Lemma 2.3 with $x = nk$, $y = k/2$ to find

$$2\pi(nk+k/2) - \pi(nk) - \pi(nk+k) >$$
$$\frac{k}{4(n+1)\log^2(nk+k)} - 1.7576\frac{nk+k}{(\log nk)^{3/4}}\exp\left(-\sqrt{\frac{\log(nk)}{9.646}}\right)$$

Put

$$f_0(k) := \frac{k}{\log\frac{k}{2}} + \frac{k}{\log^2\frac{k}{2}} + \frac{1.8k}{\log^3\frac{k}{2}} - \frac{k}{\log k} - \frac{k}{\log^2 k} - \frac{2.51k}{\log^3 k} - \log k,$$

$$f_n(k) := \frac{k}{4(n+1)\log^2(nk+k)} - 1.7576\frac{nk+k}{(\log nk)^{3/4}}\exp\left(-\sqrt{\frac{\log(nk)}{9.646}}\right)$$

for $n \geq 1$. A simple calculation shows that S_L, defined in Lemma 3.1, satisfies

$$S_L \geq f_0(k) + \sum_{n=1}^{L} f_n(k) > 0$$

for $L \leq 1500$. This shows that k is not a P-integer for such L. Hence we may assume that $L > 1500$.

We first check by Maple that $f_n(k)$ is a strictly monotone decreasing function of n. By (2) it is therefore enough to show that

$$f_0(k)+\sum_{i=1}^{1500} f_i(k)+(L-1500)f_n(k)>0$$

for $k=10^{3500}$ and $n=\lfloor\log(k\log k)\rfloor$. We check this again with Maple to get the final contradiction. □

Remark. The constant 9.646 which occurs in Lemma 2.1 (iii) originates from a zero-free region of the Riemann-zeta function derived by Rosser and Schoenfeld ([11] Theorem 11), where the constant appears as R. The zero-free region has been widened by Kadiri [5] where the corresponding constant R is 5.69693. If this constant would be substituted into Lemma 2.1 (iii) instead of the constant 9.646 and we follow our argument, we obtain that if k is a P-integer, then $k<10^{1000}$. However, we do not know if this substitution is justified.

Proof of Theorem 1.2. Suppose the Riemann Hypothesis is true. Let k be an integer with $k\geq 3\cdot 10^{13}$. By Lemma 2.2, we get

$$2\pi\left(\frac{k}{2}\right)-\pi(k)>\frac{.693k}{\log^2 k}>\log k>\omega(k).$$

For $n=1,2,\dots,\lfloor\log(k\log k)\rfloor-1$ we apply Lemma 2.4 with $x=nk$, $y=k/2$ to find

$$2\pi\left(nk+\frac{k}{2}\right)-\pi(nk)-\pi(nk+k)>$$

$$\frac{k}{4(n+1)\log^2(nk+k)}-\frac{\log(nk+k)}{2\pi}\sqrt{nk+k}.$$

The term on the right hand side of the above inequality is positive if

$$\pi\sqrt{k}>2(n+1)^{1.5}\log^3(nk+k).$$

This is satisfied, since $n<\log(k\log(k))-1$ and $k\geq 3\cdot 10^{13}$. Hence by Lemma 3.1, we find that k is not a P-integer.

Next we take $k<3\cdot 10^{13}$. By Theorem 4.1, we may assume $k>10^{11}$. Note that $L<\log(k\log k)\leq 34$. Further

$$L<\log k+\log\log k<1.13\log k$$

giving

$$k>e^{.88L}>10^{.38L}.$$

Define

$$k_L=[10^{\{.38L\}}]10^{[.38L]}.$$

where $[x]$ and $\{x\}$ denote the integral and fractional part of any real number x. Note that for any fixed L with $L \le 34$ if $L(k) \ge L$, then $k \in [k_L, 3 \cdot 10^{13})$. Applying Lemma 2.4 with $x = nk$, $y = k/2$ we find

$$S_L > 2\pi(k/2) - \pi(k) +$$

$$+\sum_{n=1}^{L}\left(\frac{k}{4(n+1)\log^2(nk+k)} - \frac{\log(nk+k)}{4\pi}\sqrt{nk+k}\right).$$

For $n = 1, \dots, L$, put

$$F_n(k) := \frac{1}{L}\left(\frac{k}{\log(k/2)} + \frac{k}{\log^2(k/2)} + \frac{1.8k}{\log^3(k/2)}\right)$$

$$-\frac{1}{L}\left(\frac{k}{\log k} + \frac{k}{\log^2 k} + \frac{2.51k}{\log^3 k} + \log k\right)$$

$$+\frac{k}{4(n+1)\log^2(nk+k)} - \frac{\log(nk+k)}{4\pi}\sqrt{nk+k}.$$

We have, by Lemma 2.1 (i), (ii),

$$S_L - \log k > \sum_{n=1}^{L} F_n(k).$$

So it is sufficient to show that the right hand side is positive. For this, we proceed as follows. First, let $29 \le L \le 34$. We calculate the value k_L from its definition above. Thus (L, k_L) is one of the pairs from

$$\{(29, 10^{11}), (30, 2\cdot 10^{11}), (31, 6\cdot 10^{11}), (32, 10^{12}), (33, 3\cdot 10^{12}), (34, 8\cdot 10^{12})\}.$$

We check by Maple that all functions $F_n(k)$ are strictly monotone increasing on $[k_L, 3 \cdot 10^{13}]$, and further

$$\sum_{n=1}^{L} F_n(k_L) > 0.$$

Hence by Lemma 3.1, there is no P-integer k with $L(k) \in [29, 34]$. Now we consider $k \in [10^{11}, 3 \cdot 10^{13}]$. Then obviously $L(k) > 0$. We may assume $1 \le L \le 28$. We check that all functions $F_n(k)$ are strictly monotone increasing and the preceding inequality also holds. Hence we conclude that no integer $k \in [10^{11}, 3 \cdot 10^{13}]$ is a P-integer. □

6. Acknowledgements

The authors are grateful to the referee for her/his helpful and useful remarks.

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