ON ADDITIVE SOLUTIONS OF A LINEAR EQUATION

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Abstract. In this paper we investigate the functional equation

$$\sum_{i=1}^{n} \alpha_i A(\beta_i x) = 0$$

which holds for all $x \in \mathbb{R}$ with an unknown additive function $A\colon \mathbb{R} \to \mathbb{R}$ and fixed real parameters α_i, β_i, where $i = 1, \dots, n$. Here we give sufficient and necessary conditions for the existence of non-trivial additive solutions of equation above in some cases depending on the algebraic properties of the parameters.

1. Introduction and preliminaries

Consider the functional equation

$$\sum_{i=1}^{n} \alpha_i A(\beta_i x) = 0 \tag{1.1}$$

which holds for all $x \in \mathbb{R}$ (the reals) with an unknown additive function $A\colon \mathbb{R} \to \mathbb{R}$ and fixed real parameters α_i, β_i, where $i = 1, \dots, n$. Since for any additive function vanishes at $x = 0$, without loss of generality we can suppose that none of the parameters equals to zero.

The theory of functional equations containing weighted arithmetic means gives motivations to investigate (1.1) (see [4]).

(1.1) has been investigated for the case $n = 2$ by Daróczy [2]. His fundamental result states that the functional equation

$$\alpha_1 A(\beta_1 x) + \alpha_2 A(\beta_2 x) = 0 \tag{1.2}$$

has a non-trivial additive solution A if and only if both the parameters

$$\alpha := -\frac{\alpha_2}{\alpha_1} \quad \text{and} \quad \beta := \frac{\beta_1}{\beta_2}$$

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are transcendent or they are algebraic with the same defining polynomial over $\mathbb{Q}$ (the rationals). Equation (1.2) is equivalent to

$$A(\beta x) = \alpha A(x) \quad (x \in \mathbb{R}),$$

i.e. the solutions must be semi-homogeneous. This is a motivation to find conditions for the existence of non-trivial semi-homogeneous solutions of equation (1.1) for the case $n \geq 3$. It is easy to see that equation (1.1) is equivalent to the equation

$$\sum_{i=1}^{n} \frac{\alpha_i}{\alpha_n} A(\frac{\beta_i}{\beta_n} x) = 0 \quad (x \in \mathbb{R}).$$

In terms of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ and $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ $(n \geq 3)$ sufficient conditions can be found for the existence of non-trivial semi-homogeneous additive solutions in [3]. In connection with these results we need the following notions.

Definition 1.1. Let m be a positive integer and consider the element $\vec{\lambda} := (\lambda_1, \dots, \lambda_m)$ of the coordinate space $\mathbb{R}^m$. If the ideal

$$\mathcal{I}(\vec{\lambda}) := \{\ p \in \mathbb{Q}[x_1, \dots, x_m] \mid p(\lambda_1, \dots, \lambda_m) = 0\ \}$$

of the polynomial ring $\mathbb{Q}[x_1, \dots, x_m]$ contains only the zero polynomial we say that the coordinates $\lambda_1, \dots, \lambda_m$ are algebraically independent. Otherwise they are algebraically dependent.

Theorem 1.2. *Suppose that $n \geq 3$. If the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraically independent and at least one of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ is transcendent then equation (1.1) has non-trivial additive solutions which are semi-homogeneous in the sense that*

$$A\left(\frac{\beta_i}{\beta_n} x\right) = \delta_i A(x)$$

for some δ_i's, for all $x \in \mathbb{R}$ and $i = 1, \dots n-1$.

Theorem 1.3. *Suppose that $n \geq 3$. If the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraically independent and at least one of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ is transcendent then equation (1.1) has non-trivial additive solution which is semi-homogeneous in the sense that*

$$A(\delta_i x) = \frac{\alpha_i}{\alpha_n} A(x)$$

for some δ_i's, for all $x \in \mathbb{R}$ and $i = 1, \dots n-1$.

For example the coordinates of $(\sqrt{\pi}, 2\pi+1)$ are algebraically dependent over $\mathbb{Q}$, since the non-zero polynomial $P(x_1, x_2) = 2x_1^2 - x_2 + 1$ vanishes at $(\sqrt{\pi}, 2\pi + 1)$.
The Lindemann-Weierstrass theorem gives a method to construct algebraically independent systems (see [1], Theorem 1.4. p.6.). It says that if $\lambda_1, \ldots, \lambda_n$ are algebraic numbers such that they are linearly independent over $\mathbb{Q}$, then

$$e^{\lambda_1}, \ldots, e^{\lambda_n}$$

are algebraically independent over $\mathbb{Q}$.

In this paper we give sufficient and necessary conditions for the existence of non-trivial additive solutions of (1.1) under some conditions like in Theorem 1.1 and Theorem 1.2.

2. Gauss elimination process for a system of equations containing an unknown additive function

The main tool of our investigations is the following

Theorem 2.1. *Let k be a natural number such that $k \geq 2$. Furthermore, let u, a_{ij} be fixed real numbers $(i, j = 1, \ldots, k)$. If the matrix $M_1 := (a_{ij})_{k\times k}$ is regular, then the only additive function $A\colon \mathbb{R} \to \mathbb{R}$ that satisfies the system of equations*

$$u^{k-1}A(a_{i1}x) + u^{k-2}A(a_{i2}x) + \ldots + uA(a_{ik-1}x) + A(a_{ik}x) = 0 \tag{2.1}$$

$(x \in \mathbb{R}; i = 1, \ldots, k)$ is the identically zero function.

Proof. For the simplicity M_1 will be called the matrix of the system (2.1), and the equation

$$u^{k-1}A(\mu a_{i1}x) + u^{k-2}A(\mu a_{i2}x) + \ldots + uA(\mu a_{ik-1}x) + A(\mu a_{ik}x) = 0$$

will be denoted by $E^{k-1}_{i(\mu x)}$ for any indeces $i = 1, \ldots, k$ and $\mu \in \mathbb{R}$.

In the proof we imitate the steps of the Gauss elimination process. Without loss of generality we may assume that $a_{11} \neq 0$ because the rank of M_1 is maximal. Taking the difference $E^{k-1}_{i(a_{11}x)} - E^{k-1}_{1(a_{i1}x)}$ $(i = 2, \ldots, k)$ and using the additivity of A we get that

$$u^{k-2}A((a_{11}a_{i2} - a_{12}a_{i1})x) + \ldots + uA((a_{11}a_{ik-1} - a_{1k-1}a_{i1})x) + \\ + A((a_{11}a_{ik} - a_{1k}a_{i1})x) = 0 \tag{2.2}$$

holds for all $x \in \mathbb{R}$ and for all indeces $i = 2, \ldots, k$. Equations (2.2) and $E^{k-1}_{1(x)}$ form a system of equations which has the matrix

$$M_2 = \begin{pmatrix} a_{11} & a_{12} & . & . & a_{1k} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & a_{11}a_{k2} - a_{12}a_{k1} & . & . & a_{11}a_{kk} - a_{1k}a_{k1} \end{pmatrix}.$$

We may suppose that $a_{11}a_{22} - a_{12}a_{21} \neq 0$ because of the regularity of M_1. Continuing this process, in the $(k-1)^{st}$ step the matrix M_1 becomes of triangular form $M_k := (\lambda_{ij})_{k\times k}$ for some $\lambda_{ij} \in \mathbb{R}$ with $\lambda_{ij} = 0$ if $i > j$. Since $0 \neq \det M_1 = \det M_k = \prod_{i=1}^{k} \lambda_{ii}$ and the last equation of the system obtained in the $(k-1)^{st}$ step is $A(\lambda_{kk}x) = 0$ for all $x \in \mathbb{R}$ we get that A is the identically zero function. □

3. The main result

The new part of the main result of this paper is the following

Theorem 3.1. *Suppose that $n \geq 3$. If the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraically independent and all of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraic then the only additive solution of equation (1.1) is the identically zero function.*

Proof. For the simplicity we use the notations

$$\omega_i := \frac{\alpha_i}{\alpha_n}, \quad \xi_i := \frac{\beta_i}{\beta_n} \quad (i = 1, \dots, n-1).$$

As ω_i $(i = 1, \dots, n-1)$ is algebraic over $\mathbb{Q}$, therefore $\mathbb{Q}(\omega_1, \dots, \omega_{n-1})$ is a finite extension of $\mathbb{Q}$. Moreover $\mathbb{Q}$ is a field with zero characteristic, thus there exist $u \in \mathbb{R}$ such that $\mathbb{Q}(u) = \mathbb{Q}(\omega_1, \dots, \omega_{n-1})$. Let k be the degree of extension. As $1, u, \dots, u^{k-1}$ is a basis of the vector space $\mathbb{Q}(u)$, thus $\omega_i = \sum_{j=0}^{k-1} r_{ij}u^j$ with some $r_{ij} \in \mathbb{Q}$ $(i = 1, \dots, n-1;\ \ j = 0, \dots, k-1)$. Using this form of ω_i, equation (1.1) goes over into the equation

$$u^{k-1}A(p_{k-1}x) + u^{k-2}A(p_{k-2}x) + \dots + uA(p_1x) + A(p_0x) = 0 \tag{3.1}$$

where $p_j = \sum_{i=1}^{n-1} \xi_i r_{ij}$ $(j = 1, \dots, k-1)$ and $p_0 = 1 + \sum_{i=1}^{n-1} \xi_i r_{i0}$. It is easy to see that $p_0 \neq 0$. (Otherwise $\xi_1, \dots, \xi_{n-1}$ can not be algebraically independent.) Let

$$f(t) := t^k + q_{k-1}t^{k-1} + \dots + q_1t + q_0 \quad (q_i \in \mathbb{Q},\ \ i = 0, \dots, k-1)$$

be the defining polynomial of u. Then we have that

$$u^k = -(q_{k-1}u^{k-1} + \dots + q_1u + q_0). \tag{3.2}$$

Multiplying both sides of (3.1) by u and using (3.2), the additivity and rational homogeneity of A we get that

$$(3.3)\qquad u^{k-1}A((p_{k-2}-q_{k-1}p_{k-1})x)+\ldots+u^2A((p_1-q_2p_{k-1})x)+$$
$$+uA((p_0-q_1p_{k-1})x)+A(-q_0p_{k-1}x)=0.$$

Multiplying both sides of (3.3) by u and using the process above $k-1$ times we get a system of equations of type (2.1). So, in what follows, it is enough to prove that the matrix $M := (b_{ij})_{k\times k}$ of this system is regular, since by Theorem (2.1), $A=0$.

Suppose, in contrary, that $\det M = 0$. One can easily check that $b_{ij} = z^{(ij)}(\xi_1,\ldots,\xi_{n-1})$ for some $z^{(ij)} \in \mathbb{Q}[x_1,\ldots,x_{n-1}]$ $(i,j=1,\ldots,k)$ therefore

$$\det M = z(\xi_1,\ldots,\xi_{n-1})$$

where $z \in \mathbb{Q}[x_1,\ldots,x_{n-1}]$ is defined by

$$z(x_1,\ldots,x_{n-1}) =$$

$$= det\begin{pmatrix} z^{(11)}(x_1,\ldots,x_{n-1}) & . & . & . & z^{(1k)}(x_1,\ldots,x_{n-1}) \\ . & & & . & . \\ . & & z^{(i\ k-i+1)}(x_1,\ldots,x_{n-1}) & & . \\ . & . & & & . \\ z^{(k1)}(x_1,\ldots,x_{n-1}) & . & . & . & z^{(kk)}(x_1,\ldots,x_{n-1}) \end{pmatrix}.$$

We show that z is not the identically zero polynomial which contradicts to the algebraic independence of $\xi_1,\ldots,\xi_{n-1}$.

To see this we show that $z^{(ij)}$ does not vanish at the zero vector if $j=k-i+1$ $(i=1,\ldots,k)$, moreover in this case $z^{(ij)}(0,\ldots,0)=1$, otherwise it vanishes. The proof of this statement goes by induction on i. If $i=1$ then the statement is true, because

$$z^{(1j)}(x_1,\ldots,x_{n-1}) = \sum_{i=1}^{n-1} r_{ij}x_i \text{ thus } z^{(1j)}(0,\ldots,0)=0\ (j=1,\ldots,k-1),$$

$$z^{(1k)}(x_1,\ldots,x_{n-1}) = 1+\sum_{i=1}^{n-1} r_{i0}x_i \text{ thus } z^{(1k)}(0,\ldots,0)=1.$$

Assume that, for $2 \le i \le k-1$,

$$z^{(1j)}(0,\ldots,0)=0 \text{ if } j\neq k-i+1\ \ (j=1,\ldots,k) \text{ and } z^{(i\ k-i+1)}(0,\ldots,0)=1$$

hold. Now we look for connection between the coefficients $b_{i+1\ j}$ and b_{il} $(j,l=1,\ldots,k)$.

Since the i^{th} equation of the system is

$$u^{k-1}A(b_{i1}x)+\ldots+u^{i-1}A(b_{i\ k-i+1}x)+\ldots+A(b_{ik}x)=0$$

while the $(i+1)^{st}$ one is

$$-(q_{k-1}u^{k-1}+\ldots+q_iu^i+\ldots+q_0)A(b_{i1}x)+\ldots+u^iA(b_{i\ k-i+1}x)+\ldots+uA(b_{ik}x)=0,$$

that is,

$$u^{k-1}A((b_{i2}-q_{k-1}b_{i1})x)+\ldots+u^iA((b_{i\ k-i+1}-q_ib_{i1})x)+\ldots+A(q_0b_{i1}x)=0,$$

we get that $b_{i+1\ j}=b_{i\ j+1}-q_{k-j}b_{i1}$ $(j=1,\ldots,k)$. So, we have that $z^{(i+1\ j)}=z^{(i\ j+1)}-q_{k-j}z^{(i1)}$ $(j=1,\ldots,k)$.
If $j=k-i$ then $z^{(i+1\ k-i)}=z^{(i\ k-i+1)}-q_iz^{(i1)}$. Since $z^{(i1)}(0,\ldots,0)=0$ and $z^{(i\ k-i+1)}(0,\ldots,0)=1$ it follows that $z^{(i+1\ k-i)}(0,\ldots,0)=1$.
If $j\neq k-i$ then $z^{(i\ j+1)}(0,\ldots,0)=0$ thus $z^{(i+1\ j)}(0,\ldots,0)=0$.

According to the facts above it follows that

$$z(0,\ldots,0)=det\begin{pmatrix} 0 & . & . & . & 0 & 1 \\ . & & & & 1 & 0 \\ . & & & . & & . \\ . & & . & & & . \\ 0 & 1 & & & & . \\ 1 & 0 & . & . & . & 0 \end{pmatrix}=(-1)^{\frac{k(k+3)}{2}}$$

which implies that z is not the identically zero polynomial. □

The following theorem says that the role of the inner and outher parameters can be interchanged in Theorem 3.1.

Theorem 3.2. *Suppose that $n\geq 3$. If the parameters $\frac{\alpha_1}{\alpha_n},\ldots,\frac{\alpha_{n-1}}{\alpha_n}$ are algebraically independent and all of the parameters $\frac{\beta_1}{\beta_n},\ldots,\frac{\beta_{n-1}}{\beta_n}$ are algebraic then the only additive solution of equation (1.1) is the identically zero function.*

Proof. The proof is similar to that of the previous theorem. With the notations $\omega_i:=\frac{\alpha_i}{\alpha_n}$, $\xi_i:=\frac{\beta_i}{\beta_n}$ $(i=1,\ldots,n-1)$ we get that there exists $u\in\mathbb{R}$ such that $\mathbb{Q}(u)=\mathbb{Q}(\xi_1,\ldots,\xi_{n-1})$. If the degree of the extension is equal to k then $1,u,\ldots,u^{k-1}$ is a basis of $\mathbb{Q}(u)$, thus $\xi_i=\sum_{j=0}^{k-1}r_{ij}u^j$ with some $r_{ij}\in\mathbb{Q}$ $(i=1,\ldots,n-1;\ \ j=0,\ldots,k-1)$. Using this form of ξ_i equation (1.1) goes over into the equation

$$p_{k-1}A(u^{k-1}x)+p_{k-2}A(u^{k-2}x)+\ldots+p_1A(ux)+p_0A(x)=0 \tag{3.4}$$

where $p_j=\sum_{i=1}^{n-1}\omega_ir_{ij}$ $(j=1,\ldots,k-1)$ and $p_0=1+\sum_{i=1}^{n-1}\omega_ir_{i0}$. It is easy to see, that $p_0\neq 0$. Using again the notation

$$f(t):=t^k+q_{k-1}t^{k-1}+\ldots+q_1t+q_0\ \ (q_i\in\mathbb{Q},\ \ i=0,\ldots,k-1)$$

for the defining polynomial of u we can use again the equality (3.2). Replace x by ux in (3.4). Using (3.2), the additivity and rational homogeneity of A we get that

$$(3.5)\qquad (p_{k-2}-q_{k-1}p_{k-1})A(u^{k-1}x)+\ldots+(p_1-q_2p_{k-1})A(u^2x)+ \\ +(p_0-q_1p_{k-1})A(ux)+(-q_0p_{k-1})A(x)=0.$$

Replace x by $u^2x,\ldots,u^{k-1}x$, respectively in equation (3.4). Then we get a system of equations with k unknowns and k equations which has the matrix form

$$L\begin{pmatrix} A(u^{k-1}x) \\ \cdot \\ \cdot \\ A(ux) \\ A(x) \end{pmatrix}=0$$

where $L:=(c_{ij})_{k\times k}$ consists of the coefficients of the system. We are going to prove that $\det L\neq 0$ which means that the linear transformation represented by L is regular and the kernel contains only the zero vector. Hence $A(x)=0$ for any $x\in\mathbb{R}$.
Suppose, in contrary that $\det L=0$. It is easy to see that

$$c_{ij}=z^{(ij)}(\omega_1,\ldots,\omega_{n-1})\ \text{ for some }\ z^{(ij)}\in\mathbb{Q}[x_1,\ldots,x_{n-1}]\ \ (i,j=1,\ldots,k).$$

Therefore $\det L=z(\omega_1,\ldots,\omega_{n-1})$ where $z\in\mathbb{Q}[x_1,\ldots,x_{n-1}]$ is defined as in the proof of Theorem 3.1. Like in the proof of Theorem 3.1, we can show that

$$(3.6)\qquad z^{(ij)}(0,\ldots,0)=0 \text{ if } j\neq k-i+1\ \ (i,j=1,\ldots,k) \text{ and } z^{(i\ k-i+1)}(0,\ldots,0)=1,$$

which implies that $z(0,\ldots,0)=(-1)^{\frac{k(k+3)}{2}}$. This means that z is not the identically zero polynomial which contradicts to the algebraic independence of $\omega_1,\ldots,\omega_{n-1}$. (3.6) can be proved by induction on i using the connection of coefficients $c_{i+1\ j}$ and c_{il}. As it was in the proof of Theorem 3.1 this connection can easily be found if we consider the i^{th} equation

$$c_{i1}A(u^{k-1}x)+\ldots+c_{i\ k-i+1}A(u^{i-1}x)+\ldots+c_{ik}A(x)=0$$

and the $(i+1)^{st}$ one

$$(c_{i2}-q_{k-1}c_{i1})A(u^{k-1}x)+\ldots+(c_{i\ k-i+1}-q_ic_{i1})A(u^ix)+\ldots+q_0c_{i1}A(x)=0$$

from the system of equations. □

Combining Theorems 1.1 and 3.1, Theorems 1.2 and 3.2, respectively we have the following main results.

Theorem 3.3. *Suppose that $n\geq 3$ and the parameters $\frac{\beta_1}{\beta_n},\ldots,\frac{\beta_{n-1}}{\beta_n}$ are algebraically independent. There exist a not identically zero additive solution of equation (1.1) if and only if at least one of the parameters $\frac{\alpha_1}{\alpha_n},\ldots,\frac{\alpha_{n-1}}{\alpha_n}$ is transcendent.*

Theorem 3.4. *Suppose that $n \geq 3$ and the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraically independent. There exist a not identically zero additive solution of equation (1.1) if and only if at least one of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ is transcendent.*

4. About the remaining case

To complete the discussion we need to investigate the case when both of the collections of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ and $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraically dependent. In case of a special functional equation of the form (1.1) we give necessary and sufficient conditions for the existence of non-trivial solutions.

Theorem 4.1. *Let $n = 3$, $\beta_1 = \sqrt{d_1}$, $\beta_2 = \sqrt{d_2}$, $\beta_3 = 1$ in (1.1) where d_1 and d_2 are positive rational numbers such that $\sqrt{d_1}$ and $\sqrt{d_2}$ are irrationals. Furthermore,suppose that $\alpha_1\alpha_2\alpha_3 \neq 0$. The functional equation*

$$\alpha_1 A(\sqrt{d_1}x) + \alpha_2 A(\sqrt{d_2}x) + \alpha_3 A(x) = 0 \tag{4.1}$$

has non-trivial additive solutions if and only if one of the conditions

$$\text{(i) } 1 + \frac{\alpha_1}{\alpha_3}\sqrt{d_1} + \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0,$$

$$\text{(ii) } 1 + \frac{\alpha_1}{\alpha_3}\sqrt{d_1} - \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0,$$

$$\text{(iii) } 1 - \frac{\alpha_1}{\alpha_3}\sqrt{d_1} + \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0,$$

$$\text{(iv) } 1 - \frac{\alpha_1}{\alpha_3}\sqrt{d_1} - \frac{\alpha_2}{\alpha_3}\sqrt{d_2} = 0.$$

is satisfied.

Proof. Suppose that A is not the identically zero function. Equation (4.1) is equivalent to the equation

$$\frac{\alpha_1}{\alpha_3} A(\sqrt{d_1}x) + \frac{\alpha_2}{\alpha_3} A(\sqrt{d_2}x) + A(x) = 0. \tag{4.2}$$

Substituting $\sqrt{d_1}x$, $\sqrt{d_2}x$ and $\sqrt{d_1 d_2}x$ in (4.2), respectively we get the system of equations which has the matrix form

$$M \begin{pmatrix} A(x) \\ A(\sqrt{d_1}x) \\ A(\sqrt{d_2}x) \\ A(\sqrt{d_1 d_2}x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M := \begin{pmatrix} 1 & \frac{\alpha_1}{\alpha_3} & \frac{\alpha_2}{\alpha_3} & 0 \\ d_1\frac{\alpha_1}{\alpha_3} & 1 & 0 & \frac{\alpha_2}{\alpha_3} \\ d_2\frac{\alpha_2}{\alpha_3} & 0 & 1 & \frac{\alpha_1}{\alpha_3} \\ 0 & d_2\frac{\alpha_2}{\alpha_3} & d_1\frac{\alpha_1}{\alpha_3} & 1 \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} \frac{\alpha_1}{\alpha_3} & \frac{\alpha_2}{\alpha_3} & 0 \\ 1 & 0 & \frac{\alpha_2}{\alpha_3} \\ 0 & 1 & \frac{\alpha_1}{\alpha_3} \end{pmatrix} = -2\frac{\alpha_1\alpha_2}{\alpha_3^2} \neq 0$$

we have that the rank of M is at least 3. As A is not the identically zero function the rank of M cannot be maximal. So we may suppose that the rank of M is 3 which implies that the kernel of the transformation represented by M is one-dimensional, i.e. it is generated by a vector $\vec{v} = (v_0, v_1, v_2, v_3)$. This means that

$$\begin{pmatrix} A(x) \\ A(\sqrt{d_1}x) \\ A(\sqrt{d_2}x) \\ A(\sqrt{d_1 d_2}x) \end{pmatrix} = \lambda(x) \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

for some additive function $\lambda\colon \mathbb{R} \to \mathbb{R}$ and $v_0 \neq 0$. Thus we get that

$$A(\sqrt{d_1}x) = v_1\lambda(x) = \frac{v_1}{v_0}A(x) \tag{4.3}$$

and, in a similar way

$$A(\sqrt{d_2}x) = v_2\lambda(x) = \frac{v_2}{v_0}A(x). \tag{4.4}$$

Both (4.3) and (4.4) has the form equation (1.2), therefore

$$\frac{v_1}{v_0} = \pm\sqrt{d_1} \quad \text{and} \quad \frac{v_2}{v_0} = \pm\sqrt{d_2}.$$

Thus one of the conditions (i), (ii), (iii) and (iv) is satisfied.

Conversely, if (i) is satisfied then, for example, let $A(x) := x$ or $A(x) := -x$. As an illustration consider the case when (ii) is satisfied. Now we prove that

$$\vec{\mu} = (\sqrt{d_1}, \sqrt{d_2}) \quad \text{and} \quad \vec{\nu} = (\sqrt{d_1}, -\sqrt{d_2})$$

have the same defining ideal, i.e. the ideals

$$\mathcal{I}(\vec{\mu}) := \{\ P \in \mathbb{Q}[x_1, x_2] \mid P(\sqrt{d_1}, \sqrt{d_2}) = 0\ \}$$

and

$$\mathcal{I}(\vec{\nu}) := \{\ P \in \mathbb{Q}[x_1, x_2] \mid P(\sqrt{d_1}, -\sqrt{d_2}) = 0\ \}$$

are the same. Suppose that $P(\sqrt{d_1},\sqrt{d_2}) = 0$. Then we can write $P(\sqrt{d_1},x_2) = f(x_2)(x_2^2 - d_2)$ because the degree of $\sqrt{d_2}$ is 2. Therefore $P(\sqrt{d_1},-\sqrt{d_2}) = 0$. The method is similar for any elements $Q[x_1,x_2]$ of the defining ideal of $\vec{\nu}$. Therefore there exists a field isomorphism $\delta\colon \mathbb{Q}(\sqrt{d_1},\sqrt{d_2}) \to \mathbb{Q}(\sqrt{d_1},-\sqrt{d_2})$ such that $\delta(\sqrt{d_1}) = \sqrt{d_1}$ and $\delta(\sqrt{d_2}) = -\sqrt{d_2}$. For the proof see Lemma 3.1. in [4]. Now we can use the procedure of semilinear extension to construct a not identically zero additive function A such that $A(\sqrt{d_1}x) = \sqrt{d_1}A(x)$ and $A(\sqrt{d_2}x) = -\sqrt{d_2}A(x)$. (See Theorem 3.2. in [4].) According to (ii) the function A is obviously satisfies (4.1). The case of (iii) and (iv) is similar. □

Example 4.2. The coordinates of $(e, \frac{\sqrt{d_1}}{\sqrt{d_2}}e + \frac{1}{\sqrt{d_2}})$ are algebraically dependent over $\mathbb{Q}$, since the non-zero polynomial

$$P(x_1,x_2) = (d_1x_1^2 - d_2x_2^2 - 1)^2 - 4d_2x_2^2$$

vanishes at $(e, \frac{\sqrt{d_1}}{\sqrt{d_2}}e + \frac{1}{\sqrt{d_2}})$. Theorem 4.1. implies that if $\frac{\alpha_1}{\alpha_3} = e$ and $\frac{\alpha_2}{\alpha_3} = \frac{\sqrt{d_1}}{\sqrt{d_2}}e + \frac{1}{\sqrt{d_2}}$, there exist non-trivial additive solutions of equation (4.1).

The coordinates of $(e, \frac{\sqrt{d_1}}{\sqrt{d_2}}e)$ are algebraically dependent over $\mathbb{Q}$, since the non-zero polynomial $P(x_1,x_2) = \frac{d_1}{d_2}x_1^2 - x_2^2$ vanishes at $(e, \frac{\sqrt{d_1}}{\sqrt{d_2}}e)$. Theorem 4.1. implies that if $\frac{\alpha_1}{\alpha_3} = e$ and $\frac{\alpha_2}{\alpha_3} = \frac{\sqrt{d_1}}{\sqrt{d_2}}e$ then the only additive solution of (4.1) is the identically zero function.

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