



A refinement of Faulhaber's theorem concerning sums of powers of natural numbers

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ARTICLE INFO

Article history:

Received 17 March 2011

Received in revised form 7 July 2011

Accepted 21 September 2011

Keywords:

Sums of powers of natural numbers

Faulhaber's polynomials

Bernoulli polynomials and Bernoulli numbers

Laguerre polynomials

Decomposition

Ring of polynomials

ABSTRACT

In an attempt to present a refinement of Faulhaber's theorem concerning sums of powers of natural numbers, the authors investigate and derive all the possible decompositions of the polynomial $\mathcal{S}_{a,b}^k(x)$ which is given by

$$\mathcal{S}_{a,b}^k(x) = b^k + (a+b)^k + (2a+b)^k + \cdots + (a(x-1)+b)^k.$$

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1. Introduction, definitions and preliminaries

Throughout this work, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \dots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Also, as usual, \mathbb{Z} denotes the set of integers, \mathbb{R} denotes the set of real numbers and \mathbb{C} denotes the set of complex numbers. We denote by $\mathbb{C}[x]$ the ring of polynomials in the variable x with complex coefficients.

A decomposition of a polynomial $F(x) \in \mathbb{C}[x]$ is an equality of the following form:

$$F(x) = G_1(G_2(x)) \quad (G_1(x), G_2(x) \in \mathbb{C}[x]). \quad (1)$$

The decomposition in (1) is *nontrivial* if

$$\deg\{G_1(x)\} > 1 \quad \text{and} \quad \deg\{G_2(x)\} > 1.$$

Two decompositions

$$F(x) = G_1(G_2(x)) \quad \text{and} \quad F(x) = H_1(H_2(x))$$

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are said to be equivalent if there exists a linear polynomial $\ell(x) \in \mathbb{C}[x]$ such that

$$G_1(x) = \ell(H_1(x)) \quad \text{and} \quad H_2(x) = \ell(G_2(x)).$$

The polynomial $F(x)$ is called *decomposable* if it has at least one nontrivial decomposition; otherwise, it is said to be *indecomposable*.

In his monumental book [1], Johann Faulhaber (1580–1635) discovered that the sums of the first n odd powers can be expressed as the polynomials of the simple sum N given by

$$N = 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1).$$

He also conjectured that similar representation exists for the sum of every odd power. The first correct proof of this conjecture was published by Jacobi [2] (see also [3]). The following sums of the powers of natural numbers:

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + (n-1)^k \quad (n \in \mathbb{N} \setminus \{1\}; k \in \mathbb{N}_0) \quad (2)$$

are closely related to the classical Bernoulli polynomials $B_k(x)$, namely, for positive integers n we have

$$S_k(n) = \frac{1}{k+1} [B_{k+1}(n) - B_{k+1}] \quad (n \in \mathbb{N} \setminus \{1\}; k \in \mathbb{N}_0), \quad (3)$$

where the classical Bernoulli polynomials $B_n(x)$ are usually defined by means of the following generating function (see, for details, [4, p. 59 et seq.]; see also [5,6] and the references cited in each of these recent investigations on the subject):

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \quad (|z| < 2\pi) \quad (4)$$

with, of course, the classical Bernoulli numbers B_n given by

$$B_n := B_n(0) \quad (n \in \mathbb{N}_0).$$

By using the connection exhibited in (3), we can extend $S_k(n)$ appropriately to $\mathcal{S}_k(x)$ for every real value of $x \in \mathbb{R}$. We thus have

$$\mathcal{S}_k(x) := \frac{1}{k+1} [B_{k+1}(x) - B_{k+1}] \quad (x \in \mathbb{R} \setminus \{1\}; k \in \mathbb{N}_0). \quad (5)$$

Such sums as $S_k(n)$ in (2) of powers of natural numbers, but with real or complex exponents, have also been investigated in the existing literature. For example, Srivastava et al. [7] made use of certain operators of fractional calculus to derive, among other results, the following summation identity:

$$\begin{aligned} Z(n, \lambda) &:= 1^\lambda + 2^\lambda + 3^\lambda + \cdots + n^\lambda \quad (\lambda \in \mathbb{C}; n \in \mathbb{N}; 1^\lambda := 1) \\ &= \frac{1}{\Gamma(-\lambda)} \sum_{k=1}^n \sum_{j=0}^{\infty} (j-\lambda)^{-1} L_j^{(-\lambda)}(k) \quad (\lambda \in \mathbb{C} \setminus \mathbb{N}_0), \end{aligned} \quad (6)$$

where $L_n^{(\alpha)}(z)$ denotes the classical Laguerre polynomial of order (or index) α and degree n in z , defined by (see [4, p. 55, Equation 1.4(72)])

$$L_n^{(\alpha)}(z) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-z)^k}{k!} \quad (z, \alpha \in \mathbb{C}; n \in \mathbb{N}_0). \quad (7)$$

On the other hand, the following functional relation for the sum $Z(n, m)$ defined by (6) with $\lambda = m$ ($m \in \mathbb{N}_0$) was proven by Nishimoto and Srivastava [8, p. 130, Equation 2.3]:

$$Z(n, m) = m! \sum_{j=0}^m (-1)^j \binom{m+\alpha}{m-j} \sum_{k=0}^j \frac{(-1)^k}{k!} \binom{j+\alpha}{j-k} Z(n, k) \quad (m \in \mathbb{N}_0; n \in \mathbb{N}). \quad (8)$$

Rakaczki [9] proved that the polynomial $\mathcal{S}_k(x)$, defined by (5), is indecomposable for even values of $k \in \mathbb{N}_0$. Furthermore, for odd $k \in \mathbb{N}_0$, Rakaczki [9] observed that all the decompositions of $S_k(x)$ are equivalent to the following decomposition:

$$\mathcal{S}_k(x) = \tilde{S}_k \left(\left(x - \frac{1}{2} \right)^2 \right). \quad (9)$$

His result is a consequence of Theorem 1 below, which is due to Bilu et al. [10].

Theorem 1. The polynomial $B_n(x)$ is indecomposable for odd $n \in \mathbb{N}_0$. If $n = 2m$ ($m, n \in \mathbb{N}_0$ is even), then any nontrivial decomposition of $B_n(x)$ is equivalent to the following decomposition:

$$B_n(x) = \tilde{B}_m \left(\left(x - \frac{1}{2} \right)^2 \right).$$

In particular, the polynomial $\tilde{B}_m(x)$ is indecomposable for any $m \in \mathbb{N}_0$.

Chen et al. [11] formulated a generalization of the classical Faulhaber theorem. For a given arithmetic progression

$$a + b, 2a + b, \dots, a(n-1) + b,$$

Faulhaber's result implies that odd power sums of this arithmetic progression are polynomials in

$$(n-1)b + \frac{1}{2} n(n-1)a.$$

For example,

$$1^{2m-1} + 3^{2m-1} + 5^{2m-1} + \dots + (2n-1)^{2m-1}$$

is a polynomial in n^2 . Furthermore,

$$1^{2m-1} + 4^{2m-1} + 7^{2m-1} + \dots + (3n-2)^{2m-1}$$

is a polynomial in the pentagonal number $\frac{1}{2} n(3n-1)$.

For a positive integer $n \in \mathbb{N} \setminus \{1\}$, let

$$S_{a,b}^k(n) := b^k + (a+b)^k + (2a+b)^k + \dots + (a(n-1)+b)^k. \quad (10)$$

It is easy to see that

$$S_{a,b}^k(n) = \frac{a^k}{k+1} \left(\left[B_{k+1} \left(n + \frac{b}{a} \right) - B_{k+1} \right] - \left[B_{k+1} \left(\frac{b}{a} \right) - B_{k+1} \right] \right). \quad (11)$$

We can thus extend the definition (10) to hold true for every real value of $x \in \mathbb{R}$ as follows:

$$\mathcal{S}_{a,b}^k(x) := \frac{a^k}{k+1} \left[B_{k+1} \left(x + \frac{b}{a} \right) - B_{k+1} \left(\frac{b}{a} \right) \right]. \quad (12)$$

The main object of this work is to prove a generalization and refinement of the above-cited results of Rakaczki [9] and Chen et al. [11].

2. The main result

In this section, we apply Theorem 1 in order to derive the following generalization and refinement of the results of Rakaczki [9] and Chen et al. [11], which we referred to in the preceding section.

Theorem 2. The polynomial $\mathcal{S}_{a,b}^k(x)$ is indecomposable for even $k \in \mathbb{N}_0$. If $k = 2v - 1$ is odd, then any nontrivial decomposition of $\mathcal{S}_{a,b}^k(x)$ is equivalent to the following decomposition:

$$\mathcal{S}_{a,b}^k(x) = \hat{S}_v \left(\left(x + \frac{b}{a} - \frac{1}{2} \right)^2 \right) \quad (k = 2v - 1). \quad (13)$$

Proof. We prove Theorem 2 by suitably applying Theorem 1. First of all, we let $k \in \mathbb{N}_0$ be an even integer,

$$t = x + \frac{b}{a}$$

and suppose that there exist polynomials f_1 and f_2 such that

$$\deg\{f_1(t)\} > 1 \quad \text{and} \quad \deg\{f_2(t)\} > 1 \quad (14)$$

and

$$\frac{a^k}{k+1} \left[B_{k+1}(t) - B_{k+1} \left(\frac{b}{a} \right) \right] = f_1(f_2(t)). \quad (15)$$

From this last expression, Eq. (15), we have a decomposition for the $(k+1)$ th Bernoulli polynomial, which is a contradiction.

We now let $k \in \mathbb{N} \setminus \{1\}$ be an odd integer. Then we have

$$B_{k+1}(t) = \frac{k+1}{a^k} f_1(f_2(t)) + B_{k+1}\left(\frac{b}{a}\right)$$

and upon choosing

$$f(t) = \frac{k+1}{a^k} f_1(t) + B_{k+1}\left(\frac{b}{a}\right),$$

we obtain

$$B_{k+1}(t) = f(f_2(t)).$$

We thus find from Theorem 1 that

$$f_2(t) = \left(t - \frac{1}{2}\right)^2 = \left(x + \frac{b}{a} - \frac{1}{2}\right)^2. \quad (16)$$

This evidently completes our proof of Theorem 2. \square

3. Remarks and observations

By considering the following power sum:

$$\sum_{j=0}^{n-1} (5j+3)^3 = \frac{125}{4}n^4 + \frac{25}{2}n^3 - \frac{55}{4}n^2 - 3n, \quad (17)$$

we can easily observe that the polynomial in (17) is given by

$$\sum_{j=0}^{n-1} (5j+3)^3 = \frac{125}{4} \left(n + \frac{1}{10}\right)^4 - \frac{125}{8} \left(n + \frac{1}{10}\right)^2 + \frac{49}{320}. \quad (18)$$

We conclude our investigation by remarking that the decomposition properties of a polynomial with rational coefficients play an important rôle in the theory of such separable diophantine equations as follows (see [12]):

$$f(x) = g(y).$$

Acknowledgements

This research was supported, in part, by the Hungarian Academy of Sciences, the OTKA Grants T67580 and K75566, the János Bolyai Fellowship and the TÁMOP Project 4.2.1./B-09/1/KONV-2010-0007 implemented through the New Hungary Development Plan co-financed by the European Social Fund, and the European Regional Development Fund.

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