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Almost Everywhere Summability of Vilenkin-Fourier Series

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Debrecen, 2022.

Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at the University of Debrecen.
The results published in the thesis are not reported in any other PhD theses.

Debrecen, March 8, 2022.

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Anteneh Tilahun Adimasu
signature of the candidate

Hereby I confirm that Anteneh Tilahun Adimasu candidate conducted his studies with my supervision within the Functional Analysis Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen between 2017 and 2022. The independent studies and research work of the candidate significantly contributed to the results published in this thesis. I also declare that the results published in the thesis are not reported in any other PhD theses.
I support the acceptance of the thesis.

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Almost Everywhere Summability of Vilenkin-Fourier Series

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ABSTRACT

The classical theory of Fourier series deals with decomposition of a function into sinusoidal waves. Unlike these continuous waves the Vilenkin (Walsh) functions are rectangular waves. These waves have already been used frequently in the theory of signal transmission, multiplexing, filtering, image enhancement, coding theory, digital signal processing and pattern recognition.

The development of the theory of Vilenkin-Fourier series has been strongly influenced by the classical theory of trigonometric series. Because of this it is inevitable to compare results on Vilenkin series to those on trigonometric series. There are many similarities between these theories, but there exist differences too. Much of these can be explained by modern abstract harmonic analysis, which studies orthonormal systems from the point of view of the structure of a topological group.

In this PhD dissertation we discuss, develop and apply this fascinating theory connected to modern harmonic analysis. In particular, we make new estimations of Vilenkin-Fourier Dirichlet kernel, Cesàro kernel and prove some new results concerning boundedness of maximal operators of Cesàro means in the variable parameter setting. Moreover, we prove the almost everywhere convergence of Cesàro means with respect to the Vilenkin system in the variable parameter setting. Besides, we also do a similar investigation for the generalized Marcinkiewicz means with respect to Vilenkin-like systems.

This PhD dissertation consists of four Chapters: Preliminaries, almost everywhere convergence of Cesàro means in the variable parameter setting with respect to one dimensional Vilenkin systems, almost everywhere convergence of generalized Marcinkiewicz means with respect to two dimensional Vilenkin-like systems and Summary. It is based on two published papers with the candidate as author.

In Chapter 1, we first present some basic definitions and notions, which are crucial for the next chapters. After that we also define some summability methods and recall some classical facts and results.

In Chapter 2, we devote to present and prove new results about almost everywhere convergence of Cesàro means in the variable parameter setting with respect to one dimensional Vilenekin systems. First, we show the boundedness estimation of Cesàro kernel. After that we investigate the more general case of Cesàro kernel, we prove the boundedness of the maximal Cesàro means operator and finally we prove the almost everywhere convergence of Cesàro means

in the variable parameter setting with respect to one dimensionals Vilenkin systems.

In Chapter 3, we devote to present and prove new results about the generalized Marcinkiewicz means with respect to Vilenkin-like systems. First, we show the boundedness estimation of the generalized Marcinkiewicz kernel and deal with its maximal operator. After that we investigate the generalized Marcinkiewicz means and their maximal operator. Finally, we prove the almost everywhere convergence of generalized Marcinkiewicz means with respect to two dimensional Vilenkin-like systems.

In Chapter 4, we include Summary of Chapter 1, 2 and 3.

PREFACE

Fourier analysis, in mathematics, is a method in which general functions can be represented or approximated by sums of simpler trigonometric functions. Fourier analysis was originated from the study of Fourier series. Its name was obtained after Joseph Fourier who showed that representing a function as a sum of trigonometric functions wich greatly simplifies the study of heat transfer.

Now a days, the topic Fourier analysis covers a wide area of mathematics. Besides, it has different applications in sciences and engineering. The process of decomposing a function into oscillatory components is often said to be Fourier analysis while the operation of reconstructing the function from these pieces is called Fourier synthesis. For example, determining which component frequencies exist in a musical note would involve calculating the Fourier transform of a sampled musical note. Therefore, the same sound can be synthesized again by including the frequency components as revealed in Fourier analysis. In mathematics, the term Fourier analysis refers to the study of both operations.

Fourier transformation is the decomposition process and its result is called the Fourier transform. Based on the domain and other properties of the function being transformed, Fourier transform is usually gotten a more particular name. The original concept of Fourier analysis has become grown rapidly and applied broadly to different abstract and general situations. The general field is often mentioned as harmonic analysis.

The term ”harmonics” originated from the ancient Greek word ”harmonikos” and means ”experienced” in music [15]. Harmonic analysis is a branch of mathematics and it is an expanded form of Fourier analysis that has dealt with representing functions or signals as the superposition of fundamental waves, studying and generalizing the terms Fourier series and Fourier transforms. In the last two centuries it has been developed into different fields with wide range of applications. For instance, some of its applications are related to number theory, representation theory, signal processing, quantum mechanics, tidal analysis and neuroscience.

One of the most modern branches of harmonic analysis, which is introduced in the mid-20th century, is analysis on topological groups. The fundamental motivational ideas are the different Fourier transforms which can be generalized to a transform of functions that are defined on locally compact topological Hausdorff groups.

The theory of locally compact Abelian groups is called Pontryagin duality. Harmonic analysis investigates the properties of this duality and the Fourier transform. Moreover, it tries to extend these properties to different settings, for instance to the case of non-Abelian Lie groups.

Many applications of harmonic analysis in science and technology are based on the idea or hypothesis that a phenomenon or signal consists of a sum of individual vibration components. Oceans, tides and vibrating strings are common and simple examples. The theoretical approach is often to describe the system by a differential equation or system of equations in order to predict the important characteristics including the amplitude, frequency and phase of the vibration components.

This doctoral dissertation is written as a monograph and is based on the following two published papers in peer reviewed journals:

György Gát and Anteneh Tilahun[6], Multi-parameter setting Cesàro means with respect to one dimensional Vilenkin system, FILOMAT, Vol.35 (2021), No.12, pp. 4121–4133.

György Gát and Anteneh Tilahun[5], On almost everywhere convergence of the generalized Marcienkiwicz means with respect to two dimensional Vilenkin-like systems, Miskolc Math. Notes, Vol.21 (2020), No. 2, pp. 823-840.

Contents

Chapter 1

Preliminaries

In this chapter, we mainly introduce the basic concepts of the three orthonomal systems: Walsh systems, Vilenkin systems and Vilenkin-like systems. Besides, we also include lemmas and definitions which are essential for discussing our main results in Chapter 2 and 3.

1.1 Walsh systems

Walsh system is an orthonormal system which is formed from Walsh functions. It is the representation of dyadic group ordered in the Paley sense.

From the practical and theoretical point of view, Walsh system can be applied in many situations. All the usual applications of orthogonal systems (e.g. data transmission, multiplexing, filtering, image enhancement, and pattern recognition) can be performed in the Walsh system more efficiently. Due to the fact that Walsh functions take only the values +1 and -1, they are not difficult to be applicable on high speed computers and can be used with very little storage space. Moreover, as early as the late 1800's, transposition of conductors in open wire lines used Walsh functions. It is also interesting from a theoretical point of view since it is the simplest non-trivial model for harmonic analysis.

In 1923 Walsh[45] introduced the original Walsh system. He also showed that the Walsh and Haar systems are Hadamard transforms of each other. Although Rademacher functions were introduced in 1922 by Rademacher[43], they were probably unknown to Walsh. In 1932 R.E.A.C.Paley[42] introduced Walsh system which is often referred to as the Walsh-Paley system. He was also the first to recognize that Walsh functions are products of Rademacher functions.

Moreover, important observation were made by Fine [24] in 1949 that the Walsh functions can be viewed as characters of the dyadic group ([44]). Vilenkin in 1947[18] extended this fact in a more general form.

One of the three orthonormal systems (Walsh-Paley system, the original Walsh system, or the Walsh-Kaczmarz system) are referred to "Walsh functions" and differed only in enumeration. Besides, they are complete orthonormal systems on $[0,1)$ and share many properties with classical trigonometric, Sturm-Liouville, and Legendre systems[44].

In this section, we follow the standard notions of dyadic analysis introduced in the book of F. Schipp, P. Simon, W. R. Wade (see eg.[14]).

Denote the set of non-negative integers by $\mathbb{N}$, the set of positive integers by $\mathbb{P}$, the set of real numbers by $\mathbb{R}$, the complex plane by $\mathbb{C}$, and the set of dyadic rationals in the unit interval $[0,1)$ by $\mathbb{Q}$. Particularly, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \leq p < 2^n$.

For $x \in [0,1)$ denote r_n the n^{th} Rademacher function:

$$r_n(x) := r(2^n x) \quad (x \in \mathbb{N})$$

where

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$

extended to $\mathbb{R}$ by periodicity of period 1.
The n^{th} Walsh-Paley function is defined as

$$\omega_n := \prod_{k=0}^{\infty} r_k^{n_k}$$

for any $n \in \mathbb{N}$, $n = \sum_{k=0}^{\infty} n_k 2^k \quad (n_k = 0 \text{ or } 1 \text{ for } k \in \mathbb{N})$.

The system $\omega := (\omega_n, n \in \mathbb{N})$ is called a Walsh-Paley system.

1.2 Vilenkin systems

A natural generalization of the Walsh Paley system is called Vilenkin system. These are orthonormal systems which were introduced by N.Ya. Vilenkin in 1947 (see [18]). First we give a brief introduction to the theory of Vilenkin orthonormal systems.

Denote by $m := (m_k : k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and Z_{m_k} the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, 2, ..., m_k - 1\}$, with the group operation being the $\mod m_k$ addition. Since the group is discrete, every subset is open.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, for $k \in \mathbb{N}$ be the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. This allows one to say that the sequence $(n_0, n_1, ...)$ is the expansion of n with respect to m. We often use the following notation.

Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$, for any $n > 0$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$.

The normalized Haar measure μ_k on Z_{m_k} is defined by $\mu_k(\{j\}) := \frac{1}{m_k}(j \in \{0, 1, ..., m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then, every $x \in G_m$ can be represented by a sequence $x = (x_i,\, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$.

The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), is the normalized Haar measure, and the topology is the product topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this dissertation we discuss bounded Vilenkin groups, only.

The Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods in G_m can be given by the intervals: $I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \, \text{for} \, i < n\}$ for $x \in G_m$, $n \in \mathbb{P}$. Let $0 = (0, \, i \in \mathbb{N}) \in G_m$ denote the null element of G_m and $I_n = I_n(0)$, $\bar{I}_n = G_m \backslash I_n$.

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) $(1 \leq p \leq \infty)$, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x) \, (x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$. We say that an operator $T : L^1 \to L^0$ ($L^0(G_m)$ is the space of measurable functions on G_m) is of type (L^p, L^p) (for $1 \leq p \leq \infty$) if $\|Tf\|_p \leq C_p \|f\|_p$ for all $f \in L^p(G_m)$ and the constant C_p depends only on p. We say that T is of weak type (L^1, L^1) if $\mu(|Tf| > \lambda) \leq C\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$.

Next, we introduce an orthonormal system that we call the Vilenkin system on G_m.

Definition

For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp(2\pi\imath \frac{x_k}{m_k}) \quad (x \in G_m, \ \imath := \sqrt{-1}, \ k \in \mathbb{N}).$$

The n^{th} Vilenkin function is defined as

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} (n \in \mathbb{N}).$$

The system $\psi := (\psi_n, \, n \in \mathbb{N})$ is called a Vilenkin system.

Each ψ_n is a character of G_m and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then $\psi_{k \oplus n} = \psi_k \psi_n$, $\psi_n(x + y) = \psi_n(x)\psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1 \quad (k, \, n \in \mathbb{N}, \, x, \, y \in G_m)$.

Definition

The Dirichlet and the Fejér or (C,1) kernels on the Vilenkin system are defined and denoted as

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad \text{and} \quad K_n := \frac{1}{n+1} \sum_{k=0}^{n} D_k, \text{ respectively.}$$

Definition

For $f \in L^1(G_m),$ the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the (C, α) kernels and means with respect to the Vilenkin system ψ are defined as follows

$$\begin{aligned}
\hat{f}(n) &:= \int_{G_m} f\bar{\psi}_n d\mu, \\
S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k)\psi_k, \\
\sigma_n^\alpha f &:= \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \\
K_n^\alpha &:= \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k, \\
\sigma_n f &:= \sigma_n^1 f,\ K_n := K_n^1.
\end{aligned}$$

It is known [20] that,

$$A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1},\ A_k^\alpha - A_{k+1}^\alpha = \frac{-\alpha A_k^\alpha}{k+1} \tag{1.1}$$

where A_k^α is defined for all possible values of $\alpha \in \mathbb{R} \setminus \{-1,\ -2,\ ...,\ -k\}$ as

$$A_k^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+k)}{k!},$$

α may also be a sequence $\alpha = (\alpha_n)$.
It is known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x) \quad (n \in \mathbb{N},\ f \in L^1(G_m)).$$

It is also well-known that(see [4], [7])

$$\begin{aligned}
D_{M_n}(y, x) &= \begin{cases} M_n, & \text{if } y \in I_n(x), \\ 0, & \text{if } y \notin I_n(x) \end{cases} \\
S_{M_n} f(x) &= M_n \int_{I_n(x)} f d\mu = E_n f(x)
\end{aligned} \tag{1.2}$$

$$f \in L^1(G_m),\, n \in \mathbb{N},$$

$$\begin{aligned} D_{sM_n} &= D_{M_n} \sum_{k=0}^{s-1} \psi_{kM_n} \\ &= D_{M_n} \sum_{k=0}^{s-1} r_n^k, \quad \text{for} \quad s \leq m_n. \end{aligned} \tag{1.3}$$

1.3 Vilenkin-like systems

Next on a Vilenkin space G_m we introduce an orthonormal system called a Vilenkin-like system (or $\psi\alpha$ system).

Vilenkin-like orthonormal systems were introduced by György Gát in 1991 (see [30]) and they are defined as follows.

Definition
Let the functions

$$\psi_n, \alpha_n, \alpha_k^j : G_m \to \mathbb{C} \ (n, j, k \in \mathbb{N})$$

satisfy :

$$\alpha_k^j \text{ is measurable with respect to } \mathcal{A}_k \ (j, k \in \mathbb{N}), \tag{1.4}$$

$$|\alpha_k^j| = \alpha_k^j(0) = \alpha_0^j = \alpha_k^0 = 1 \ (j, k \in \mathbb{N}), \tag{1.5}$$

$$\alpha_n := \prod_{k=0}^{\infty} \alpha_k^{n^{(k)}}, \psi_n := \prod_{k=0}^{\infty} r_k^{n_k}, \ n^{(k)} := \sum_{i=k}^{\infty} n_i M_i \ (n \in \mathbb{N}). \tag{1.6}$$

Let $\chi_n := \psi_n \alpha_n \, (n \in \mathbb{N})$. The system $\chi := \{\chi_n : n \in \mathbb{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system (see [30], [22]).

We also introduce the two-variable functions:

$$\chi_n(y, x) := \chi_n(y)\bar{\chi}_n(x), \quad r_n(y, x) := r_n(y)\bar{r}_n(x) \quad (n \in \mathbb{N}, y, x \in G_m).$$

This will not cause misunderstanding by clearly making a difference between $\chi_n(x)$ and $\chi_n(y, x)$.

Example A: the Vilenkin and the Walsh system

Let $\alpha_k^j(x) := 1$, where $j, k \in \mathbb{N}, x \in G_m$ where G_m is the Vilenkin group. The system $\chi := (\chi_n, \, n \in \mathbb{N})$ is the Vilenkin system, where $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k} \alpha_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k}$. In the case of the Vilenkin group, if $m_k = 2$ for all $k \in \mathbb{N}$, we get the Walsh-Paley system. Properties (1.4), (1.5), (1.6) are trivially fulfilled. For more on Vilenkin and Walsh systems and groups see e.g. [21] and [27].

Example B: the group of 2-adic (m-adic) integers

Let $G_{m_k} := \{0,1,...,m_k-1\}$ for all $k \in \mathbb{N}$. On G_m define the following (commutative) addition: Let $x, y \in G_m$. Then $x+y=z \in G_m$ is defined in a recursive way. $x_0+y_0=t_0m_0+z_0$, where (of course) $z_0 \in \{0,1,...,m_0-1\}$ and $t_0 \in \mathbb{N}$. Suppose that $z_0,...,z_k$ and $t_0,...,t_k$ have been defined. Then write $x_{k+1}+y_{k+1}+t_k = t_{k+1}m_{k+1}+z_{k+1}$, where $z_{k+1} \in \{0,1,...,m_{k+1}-1\}$ and $t_{k+1} \in \mathbb{N}$. Then G_m is called the group of m-adic integers (if $m_k=2$ for all $k \in \mathbb{N}$, then 2-adic integers). In this case let

$$\alpha_k^j(x) := \left(\exp\left(2\pi\imath\left(\frac{x_{k-1}}{m_km_{k-1}}+\cdots+\frac{x_0}{m_km_{k-1}...m_0}\right)\right)\right)^j.$$

Let $\chi_n := \prod_{k=0}^{\infty} r_k^{n_k}\alpha_k^{n^{(k)}}$. Then the system $\chi := (\chi_n,\, n \in \mathbb{N})$ is the character system of the group of m-adic (if $m_k=2$ for each $k \in \mathbb{N}$ then 2-adic) integers. Conditions (1.4), (1.5), (1.6) are trivially fulfilled. For more on the group of m-adic (if $m_k=2$ for each $k \in \mathbb{N}$ then 2-adic) integers see e.g. [22] or [12]. For the case when $m_k=2\,(k \in \mathbb{N})$ the a.e. convergence of the ordinary Marcinkiewicz means were discussed by Blahota and Gát in [22]. That is, the results included in Chapter 3 of this Dissertation are new on the two-dimensional group of m-adic integers. Not only with respect to the general case $\alpha : \mathbb{N}^2 \to \mathbb{N}^2$ but also for $\alpha_1(n)=\alpha_2(n)=n$. Besides, the same can be said in the situation of Example C below.

Example C: a system in the field of number theory

Let

$$\alpha_n(x) := \exp\left(2\pi\imath\sum_{j=0}^{\infty}\frac{n_j}{M_{j+1}}\sum_{i=0}^{\infty}x_iM_i\right)$$

for $n \in \mathbb{N}$ and $x \in G_m$. Then

$$\chi_n(x) = \exp\left(2\pi\imath\left(\sum_{k=0}^{\infty}\frac{n_kx_k}{m_k}+\sum_{k=0}^{\infty}\frac{n_k}{M_{k+1}}\sum_{i=0}^{k-1}x_iM_i\right)\right) = \psi_n(x)\alpha_n(x),$$

where $\alpha_k^{n^{(k)}}(x) = \exp\left(2\pi\imath\frac{n_k}{M_{k+1}}\sum_{i=0}^{k-1}x_iM_i\right)$. Then, $\chi := (\chi_n,\, n \in \mathbb{N})$ is a Vilenkin-like system (introduced in [30]) which is a useful tool in the approximation theory of limit periodic, almost even arithmetical functions [30] and [31]. Again, properties (1.4), (1.5), (1.6) are trivially fulfilled. This system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions. For the definition of these arithmetical functions see also the book of Mauclaire [39].

Definition

For $n \in \mathbb{N}, \quad y, x \in G_m, f \in L^1(G_m),$ the Fourier coefficients, the partial sums of the Fourier series and the Dirichlet kernels with respect to the Vilenkin-like system χ are defined respectively as follows

$$\begin{aligned}
\hat{f}(n) &:= \int_{G_m} f \bar{\chi}_n d\mu, \\
S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k) \chi_k, \\
D_n(y,x) &:= \sum_{k=0}^{n-1} \chi_k(y) \bar{\chi}_k(x) = \sum_{k=0}^{n-1} \chi_k(y,x).
\end{aligned}$$

For $n \in \mathbb{N}, \quad y \in G_m, f \in L^1(G_m),$ it is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y,x) d\mu(x).$$

It is also well-known [30] that

$$\begin{aligned}
&D_{M_n}(y,x) = \begin{cases} M_n, & \text{if } y \in I_n(x), \\ 0, & \text{if } y \notin I_n(x) \end{cases} \\
&D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \sum_{j=0}^{\infty} D_{M_j}(y,x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(y) \bar{r}_j^p(x), \\
&S_{M_n} f(x) = M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m),\, n \in \mathbb{N}), \\
&D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t-n_t}^{m_t-1} r_t^i(y) \bar{r}_t^i(x) \right), \\
&y \in I_t(x) \setminus I_{t+1}(x),\, t \in \mathbb{N}.
\end{aligned} \tag{1.7}$$

Next, we introduce some notation used in the theory of two-dimensional Vilenkin-like systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group and $\chi_{k,l}(x,y) = \chi_k(x)\chi_l(y) \quad (k,l \in \mathbb{N},\, x \in G_m,\, y \in G_{\tilde{m}})$ is called a two dimensional Vilenkin-like system. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstanding. In this dissertation we also suppose that $m = \tilde{m}$.

Definition

For $y = (y^1, y^2),\ x = (x^1, x^2) \in G_m \times G_m, n \in \mathbb{N},$ the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Marcinkiewicz means, and the Marcinkiewicz kernels with respect to a two-dimensional Vilenkin-like system are defined respectively as follows:

$$\hat{f}(n_1, n_2) := \int_{G_m \times G_{\tilde{m}}} f(x^1, x^2)\bar{\chi}_{n_1}(x^1)\bar{\chi}_{n_2}(x^2)d\mu(x^1, x^2),$$

$$S_{n_1,n_2} f(y^1, y^2) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2)\chi_{k_1}(y^1)\chi_{k_2}(y^2),$$

$$D_{n_1,n_2}(y, x) = D_{n_1}(y^1, x^1)D_{n_2}(y^2, x^2)$$

$$:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \chi_{k_1}(y^1)\chi_{k_2}(y^2)\bar{\chi}_{k_1}(x^1)\bar{\chi}_{k_2}(x^2),$$

$$\sigma_n f := \frac{1}{n} \sum_{j=0}^{n-1} S_{j,j} f,$$

$$K_n(y, x) := \frac{1}{n+1} \sum_{j=0}^{n} D_{j,j}(y, x).$$

It is also well-known that

$$\sigma_n f(y) = \int_{G_m \times G_m} f(x)K_n(y, x)d\mu(x) =: f * K_n(y).$$

The next well-known Lemmas concerning Dirichlet kernels will be used many times in the proofs of our main results.

Lemma 1.1. *[3] If k and n are natural numbers, then*

$$\begin{aligned}
&a). \quad C_1(1+\alpha_n)(2+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < C_2(1+\alpha_n)(2+\alpha_n)k^{\alpha_n},\\
&-2 < \alpha_n < -1,\\
&b). \quad C_1(1+\alpha_n)k^{\alpha_n} < A_k^{\alpha_n} < C_2(1+\alpha_n)k^{\alpha_n},\\
&-1 < \alpha_n < 0,\\
&c). \quad C_1(d)k^{\alpha_n} < A_k^{\alpha_n} < C_2(d)k^{\alpha_n},\\
&0 < \alpha_n \leq d.
\end{aligned}$$

where C_1, C_2 are positive absolute constants(though in case (c) they depend on d).

Lemma 1.2. *[7] Let $0 \leq j < n_t M_t$ and $0 \leq n_t < m_t$. Then,*

$$D_{n_t M_t - j} = D_{n_t M_t} - \psi_{n_t M_t - 1}\bar{D}_j.$$

Proof. We know that this result is not a new one, but in order to give some introduction to the methods of Vilenkin systems we give here the proof (see also [7]).
It is clear that

$$D_{n_t M_t} = D_{n_t M_t - j} + \sum_{k=n_t M_t - j}^{n_t M_t - 1} \psi_k = D_{n_t M_t - j} + \sum_{k=0}^{j-1} \psi_{n_t M_t - k - 1}.$$

Consequently,

$$\begin{aligned}
&\psi_{n_t M_t - k - 1}(x) = \psi_{(n_t-1)M_t + (m_{t-1}-1)M_{t-1} + \ldots + (m_0-1)M_0 - k}(x)\\
&= \psi_{(n_t-k_t-1)M_t + (m_{t-1}-k_{t-1}-1)M_{t-1} + \ldots + (m_0-k_0-1)M_0}(x)\\
&= \psi_{(n_t-1)M_t + (m_{t-1}-1)M_{t-1} + \ldots + (m_0-1)M_0}(x)\bar{\psi}_k(x)\\
&= \psi_{n_t M_t - 1}(x)\bar{\psi}_k(x).
\end{aligned}$$

Hence, the Lemma follows. □

Lemma 1.3. *[18]*

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n := I_n(0),\\ 0, & \text{if } x \notin I_n. \end{cases}$$

Lemma 1.4. *[18]*

$$D_n(x) = \psi_n(x) \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(x).$$

Lemma 1.5. *[18]*

$$D_n(x) = D_n(z) = \psi_n(z)(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=_{m_t-n_t}}^{m_t-1} r_t^i(z));$$

$$z \in I_t \setminus I_{t+1},\, t \in \mathbb{N}.$$

The following Lemmas concerning to Vilenkin-like systems are also well-known [30]

Lemma 1.6. *[30] Let* $t, n, l \in \mathbb{N}, u \in G_m$. *Then we have that*

$$\int_{I_{t+1}(u)} \chi_n(x)\bar{\chi}_l(x)d\mu(x) \neq 0$$

implies $n^{(t+1)} = l^{(t+1)}$.

Lemma 1.7. *[30]*

$$D_{M_n}(y,x) = \begin{cases} M_n, & \textit{if } y \in I_n(x), \\ 0, & \textit{if } y \notin I_n(x). \end{cases}$$

Lemma 1.8. *[30]*

$$D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \sum_{j=0}^{\infty} D_{M_j}(y,x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p(y)\bar{r}_j^p(x).$$

Lemma 1.9. *[30]*

$$D_n(y,x) = \chi_n(y)\bar{\chi}_n(x) \left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=_{m_t-n_t}}^{m_t-1} r_t^i(y)\bar{r}_t^i(x) \right),$$

$$y \in I_t(x) \setminus I_{t+1}(x),\, t \in \mathbb{N}.$$

Besides to the above lemmas, the following lemma and inequalities are also played the vital role to prove our main results.

Lemma 1.10. *(Calderon-Zygmund decomposition lemma)*
Let $f \in L^1(I)$, $\lambda > \|f\|_1$. *Then there exists a decomposition*

$$f = \sum_{j=0}^{\infty} f_j \text{ and disjoint intervals } I^j := I_{k_j}(u^j)$$

of I for which

$$\operatorname{supp} f_j \subset I^j, \int_{I^j} f_j = o, \lambda < \left|I^{j^{-1}}\right| \int_{I^j} |f_j| \leq c\lambda,$$
$$(u^j \in I, k_j \in \mathbb{N}, j \in \mathbb{P}), \|f_0\|_\infty \leq c\lambda, |F| \leq \frac{c\|f\|_1}{\lambda},$$

where $F = \cup_{j \in \mathbb{P}} I^j$.

We can get this lemma in [44] and in the dyadic case it is proved by Gát[32]. It is well-known that the Calderon-Zygmund decomposition lemma plays a prominent role in the theory of harmonic analysis. This famous lemma is mainly used to prove weak type (L^1, L^1) estimations for the maximal operators of the summability methods.

Chapter 2

Cesàro means in the variable parameter setting

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [13]. In 2007 Akhobadze [3] introduced the notion of (C, α) means of trigonometric Fourier series with variable parameter setting. Fine [24] introduced this for Walsh-Paley system for constant sequences. On the rate of convergence of (C, α) means in the constant sequences case see the paper of Fridli [8]. For the two dimensional case see the papers of [48] and Goginava [11]. The almost everywhere convergence of this summability method for a constant parameter in the quadrilateral partial sums of double Vilenkin-Fourier series was proved by Gát and Goginiva in 2006 [7]. In 2008 Abu Joudeh and Gát [1] proved for varying-parameter setting in the case of Walsh-Paley system. The a.e. divergence of Cesàro means with varying-parameter of Walsh-Fourier series was investigated by Tetunashvili [17].
In [10] Lemma 8 about the Fejér kernel with a constant $\alpha = 1$ is proved with respect to Vilenkin system.

The aim of this Chapter is to prove the almost everywhere convergence of (C, α) means in the variable parameter setting with respect to the one dimensional bounded Vilenkin system. That is,

$$\sigma_n^{\alpha_n} f \to f \quad \text{when} \quad n \to \infty,$$

where $\alpha = (\alpha_n)$ is not constant but it is varying in the open interval $(0, 1)$ for all $n \in \mathbb{N}_{\alpha,q}$, where $\mathbb{N}_{\alpha,q}$ will be defined later.

In the first part of this chapter, we proof some lemmas on (C, α) kernel functions which are important tools. The first and important lemma, which is proved, is about the kernel function, when $n, a \in \mathbb{N}$, $M_B \leq n < M_{B+1}$, $|n| = B$, $\alpha_a \in (0, 1)$. Then, it is showed that the inequality

$$|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}$$

holds true. When $0 < \alpha_n < 1$, $M_B \leq n < M_{B+1}$, $|n| = B$, $n \in \mathbb{N}$, it is also proved that the following inequality holds true,

$$|K_n^{\alpha_n}| \leq \tilde{K}_n^{\alpha_n}.$$

In the second part of this Chapter, we focused on the maximal operators of (C, α) means. We proved the quasi-locality property of the maximal operators $\tilde{t}_*^{\alpha_a}$ and $\tilde{\sigma}_{*,q}^{\alpha}$ (for the notion of quasi-locality see [14]). The other very important lemma states that $\sigma_{*,q}^{\alpha}$ is of weak type (L^1, L^1). We prove this after we give the proof of the statement that the maximal operator $\tilde{\sigma}_{*,q}^{\alpha}$ is of type (L^∞, L^∞) and of weak type (L^1, L^1). Having all these important tools proved, in the third part of the chapter we finally give the proof of the main result.

$T_n^{\alpha_a}$, $\tilde{T}_n^{\alpha_a}$, $K_n^{\alpha_n}$, $\tilde{K}_n^{\alpha_n}$, $\tilde{t}_*^{\alpha_a}$, $\tilde{\sigma}_{*,q}^{\alpha}$ and $\sigma_{*,q}^{\alpha}$ will be defined latter

2.1 Cesàro kernel in the variable parameter setting

The following notations as well as definitions of functions and operators are used through the proofs of this chapter.
For $a,\ s,\ n \in \mathbb{N}$ let $n_{(s)} := \sum_{j=0}^{s-1} n_j M_j,$ that is, $n_{(0)} = 0,\ n_{(1)} = n_0$ and for $M_B \leq n < M_{B+1},\ |n| := B,\ n_{(B+1)} = n.$
Define two variable function $P(n, \alpha) := \sum_{i=0}^{\infty} n_i M_i^{\alpha}$ for $n \in \mathbb{N},\ \alpha \in \mathbb{R}$. For example $P(n, 1) = n$. Besides, set for sequences $\alpha = (\alpha_n)$ and positive reals $q,$ the subset of natural numbers

$$\mathbb{N}_{\alpha,q} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq q \right\}. \tag{2.1}$$

For sequence α such that $0 < \alpha_0 \leq \alpha_n < 1$ we have $\mathbb{N}_{\alpha,\,q} = \mathbb{N}$ for some q depending only on α_0. We remark that $M_n \in \mathbb{N}_{\alpha,\,q}$ for every $\alpha = (\alpha_n),\ 0 < \alpha_n < 1$ and $q \geq 1$.
In this dissertation, C denotes an absolute constant and C_q another one which may depend only on q.
We also used lemma 2.1 (see[10]) in the proof of our results. Thus, it is included as follows without its proof.

Lemma 2.1. $\int_{\bar{I}_k} \sup_{j \geq M_k} \Big| K_j \Big| d\mu \leq C.$

Next, we introduce the following functions and operators in the variable parameter setting $(n \in \mathbb{N},\ 0 < \alpha_a < 1)$.

$$T_n^{\alpha_a} = \frac{1}{A_n^{\alpha_a}} \sum_{k=0}^{n_B M_B - 1} A_{n-k}^{\alpha_a - 1} D_k,$$

$$\begin{aligned}
\tilde{T}_n^{\alpha_a} :=& \frac{n_B D_{M_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} \\
&+ \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \Big| K_j \Big| \\
&+ \alpha_a \Big| K_{n_B M_B - 1} \Big|,
\end{aligned}$$

$$t_n^{\alpha_a} f(y) := \int_{G_m} f(x) T_n^{\alpha_a}(y-x) d\mu(x),$$

$$\tilde{t}_n^{\alpha_a} f(y) := \int_{G_m} f(x) \tilde{T}_n^{\alpha_a}(y-x) d\mu(x).$$

Lemma 2.2. *[6] For $n, a \in \mathbb{N}$, $M_B \leq n < M_{B+1}$, $|n| = B$, $\alpha_a \in (0,1)$. Then,*

$$|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}.$$

Proof. Since $|n| = B$. Then,

$$\begin{aligned}
A_n^{\alpha_a}T_n^{\alpha_a} &= \sum_{j=0}^{n_BM_B-1} A_{n-j}^{\alpha_a-1}D_j = \sum_{j=0}^{n_BM_B-1} A_{n_BM_B+n_{(B)}-j}^{\alpha_a-1}D_j \\
&= \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1}D_{n_BM_B-j}.
\end{aligned}$$

By Lemma 1.2 and 1.3 we have

$$\begin{aligned}
T_n^{\alpha_a} &= \frac{D_{n_BM_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1} \\
&- \frac{\psi_{n_BM_B-1}}{A_n^{\alpha_a}} \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1}\bar{D}_j \\
&= \frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B-1} r_n^k \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1} \\
&- \frac{\psi_{n_BM_B-1}}{A_n^{\alpha_a}} \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1}\bar{D}_j \\
&=: \frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B-1} r_n^k \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1} + I.
\end{aligned}$$

This implies that

$$\begin{aligned}
\left|T_n^{\alpha_a}\right| &\leq \left|\frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B-1} r_n^k \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1}\right| + \left|I\right| \\
&\leq \frac{D_{M_B}}{A_n^{\alpha_a}} \sum_{k=0}^{n_B-1} \left|r_n^k\right| \sum_{j=0}^{n_BM_B-1} A_{n_{(B)}+j}^{\alpha_a-1} + \left|I\right| \\
&= \frac{n_BD_{M_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_BM_B-1} A_{n^{(B)}+j}^{\alpha_a-1} + \left|I\right|.
\end{aligned}$$

By the help of Abel's transformation and (2) we get

$$
\begin{aligned}
|I| &= \Bigg| -\frac{\psi_{n_B M_B-1}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B-1} A_{n_{(B)}+j}^{\alpha_a-1} \bar{D}_j \Bigg| \\
&= \frac{1}{A_n^{\alpha_a}} \Bigg| \sum_{j=0}^{n_B M_B-2} \left[A_{n_{(B)}+j}^{\alpha_a-1} - A_{n_{(B)}+j+1}^{\alpha_a-1} \right] \sum_{i=0}^{j} \bar{D}_i \\
&+ A_{n_{(B)}+n_B M_B}^{\alpha_a-1} \sum_{i=0}^{n_B M_B-1} \bar{D}_i \Bigg| \\
&\leq \sum_{j=0}^{n_B M_B-2} \frac{(1-\alpha_a) A_{n_{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \frac{j+1}{n_{(B)}+j+1} \Big| K_j \Big| \\
&+ \frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}} \Bigg| \sum_{i=0}^{n_B M_B-1} D_i \Bigg| =: h_1 + h_2.
\end{aligned}
$$

It is known from lemma 1.1 that

$$
\frac{A_{n_{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \leq \frac{\alpha_a (n_{(B)}+j)^{\alpha_a-1}}{n^{\alpha_a}}.
$$

So, the situation for h_1 becomes

$$
\begin{aligned}
&\sum_{j=0}^{n_B M_B-2} \left| \frac{(1-\alpha_a) A_{n_{(B)}+j}^{\alpha_a-1}}{A_n^{\alpha_a}} \frac{j+1}{n_{(B)}+j+1} K_j \right| \\
&\leq \sum_{j=0}^{n_B M_B-2} \left| \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}(n_{(B)}+j)^{1-\alpha_a}} \frac{j+1}{n_{(B)}+j+1} K_j \right| \\
&\leq \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B-2} \frac{j+1}{(n_{(B)}+j)^{1-\alpha_a}(n_{(B)}+j+1)} \Big| K_j \Big| \\
&\leq \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B-2} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \Big| K_j \Big|.
\end{aligned}
$$

The case for h_2 becomes

$$\begin{aligned} h_2 &= \frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}}\left|\sum_{i=0}^{n_BM_B-1} D_i\right| = \frac{A_n^{\alpha_a-1}}{A_n^{\alpha_a}}(n_BM_B)\Big|K_{n_BM_B-1}\Big| \\ &\leq \frac{\alpha_a(n_BM_B)}{n}\Big|K_{n_BM_B-1}\Big| \leq \alpha_a\Big|K_{n_BM_B-1}\Big|. \end{aligned}$$

Thus,

$$\begin{aligned} |I| \leq & \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=0}^{n_BM_B-2}\frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}}\Big|K_j\Big| \\ & + \alpha_a\Big|K_{n_BM_B-1}\Big|. \end{aligned}$$

The proof completed. □

The following Lemma plays a central role in the proof of the next lemmas and the main theorem too.
For $f \in L^1(G_m)$ and for all real number $\alpha_n \neq -1, -2, -3, ...$, define the Kernel of the (C, α_n) summability method as follows

$$K_n^{\alpha_n} := \frac{1}{A_n^{\alpha_n}}\sum_{t=0}^{n} A_{n-t}^{\alpha_n-1}D_t \tag{2.2}$$

and where $A_k^{\alpha_n}$ is defined in (1.1) for the case where $\alpha = (\alpha_n)$. Besides, introduce the following Kernel functions and operators where $0 < \alpha_n < 1$.

$$\begin{aligned} \tilde{K}_n^{\alpha_n} &:= \Big|\tilde{T}_n^{\alpha_n}\Big| + \sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}n_lD_{M_l} + \sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}|T_{n_{(l-1)}}^{\alpha_n}|, \\ \tilde{\sigma}_n^{\alpha_n}f(y) &:= \int_{G_m} f(x)\tilde{K}_n^{\alpha_n}(y-x)d\mu(x). \end{aligned}$$

Lemma 2.3. *[6] Let* $0 < \alpha_n < 1$, $n \in \mathbb{N}$, $M_B \leq n < M_{B+1}$, $|n| = B$. *Then,*

$$|K_n^{\alpha_n}| \leq \tilde{K}_n^{\alpha_n}.$$

Proof. By definition, we have

$$
\begin{aligned}
K_n^{\alpha_n} &= \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-j}^{\alpha_n-1} D_j \\
&= \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n_B M_B-1} A_{n-j}^{\alpha_n-1} D_j + \frac{1}{A_n^{\alpha_n}} \sum_{j=n_B M_B}^{n-1} A_{n-j}^{\alpha_n-1} D_j \\
&= T_n^{\alpha_n} + \frac{1}{A_n^{\alpha_n}} \sum_{j=n_B M_B}^{n_B M_B+n_{(B)}-1} A_{n_{(B)}+n_B M_B-j}^{\alpha_n-1} D_j .
\end{aligned}
$$

By Lemma 1.2 we have

$$
\begin{aligned}
&\frac{1}{A_n^{\alpha_n}} \sum_{j=n_B M_B}^{n_B M_B+n_{(B)}-1} A_{n_{(B)}+n_B M_B-j}^{\alpha_n-1} D_j \\
&= \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n-1} A_{n-t}^{\alpha_n-1} D_{t+n_B M_B} \\
&= \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1} \left(D_{n_B M_B} + \psi_{n_B M_B-1} \overline{D_t} \right) \\
&= \frac{D_{n_B M_B}}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1} + \frac{\psi_{n_B M_B-1}}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1} \overline{D_t} . \\
&= \frac{A_{n_{(B)}}^{\alpha_n}}{A_n^{\alpha_n}} \left(D_{n_B M_B} + \psi_{n_B M_B-1} \overline{K_{n_{(B)}}^{\alpha_n}} \right) .
\end{aligned}
$$

Then,

$$
K_n^{\alpha_n} = T_n^{\alpha_n} + \frac{A_{n_{(B)}}^{\alpha_n}}{A_n^{\alpha_n}} \left(D_{n_B M_B} + \psi_{n_B M_B-1} \overline{K_{n_{(B)}}^{\alpha_n}} \right) .
$$

In general, for $j = 1, \ldots, B+1,$ we get

$$
K_{n_{(j)}}^{\alpha_n} = T_{n_{(j)}}^{\alpha_n} + \frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}} \left(D_{n_{(j-1)} M_{(j-1)}} + \psi_{n_{(j-1)} M_{(j-1)}-1} \overline{K_{n_{(j-1)}}^{\alpha_n}} \right) .
$$

Recursively applying this formula and considering that $n_{(-1)} = 0$, $T_0^{\alpha_n} =$

$K_0^{\alpha_n} = 0,\ A_0^{\alpha_n} = 1$, we get

$$|K_n^{\alpha_n}| \leq |T_n^{\alpha_n}| + \sum_{l=0}^{B} \left(\prod_{j=l}^{B} \frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}} D_{M_l} \sum_{k=0}^{n_l - 1} |r_n^k| + \prod_{j=l}^{B} \frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}| \right)$$

$$= \left|\tilde{T}_n^{\alpha_n}\right| + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} n_l D_{M_l} + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}| = \tilde{K}_n^{\alpha_n}.$$

Hence, the lemma follows.

□

2.2 Maximal operators of Cesàro means in the variable parameter setting

In the variable parameter setting α, the following results about the maximal operator are investigated. We proved lemma 2.4 below, which means that the maximal operator $\tilde{t}_*^{\alpha_a} := \sup_{n,\, a\in\mathbb{N}} |\tilde{t}_n^{\alpha_a}|$ is quasi-local.

Lemma 2.4. *[6] Let* $1 > \alpha_a > 0$, $a \in \mathbb{N}$, $f \in L^1(G_m)$ *such that supp*$f \subset I_k(u)$, $\int_{I_k(u)} f d\mu(x) = 0$ *for some m-adic interval* $I_k(u)$*. Then, we have*

$$\int_{\bar{I}_k(u)} \sup_{n,\, a\in N} |\tilde{t}_n^{\alpha_a} f| d\mu(x) \leq C\|f\|_1.$$

Proof. We can easily show that for $n < M_k$ and $x \in I_k(u)$, $y \in \bar{I}_k(u)$ we have

$$\tilde{T}_n^{\alpha_a}(y-x) = \tilde{T}_n^{\alpha_a}(y-u),$$

$$\int_{I_k(u)} f(x)\tilde{T}_n^{\alpha_a}(y-x)d\mu(x)$$

$$= \tilde{T}_n^{\alpha_a}(y-u)\int_{I_k(u)} f(x)d\mu(x) = 0.$$

Consequently,

$$\int_{\bar{I}_k(u)} \sup_{n,\, a\in N} \left|\tilde{t}_n^{\alpha_a} f\right| d\mu$$

$$= \int_{\bar{I}_k(u)} \sup_{n\geq M_k,\, a\in N} \left|\tilde{t}_n^{\alpha_a} f\right| d\mu.$$

By the shift invariance of the Haar measure it can be supposed that $u = 0$. That is, $I_k(u) = I_k$. Thus,

$$\int_{\bar{I}_k(u)} \sup_{n\geq M_k,\, a\in N} |\tilde{t}_n^{\alpha_a} f| d\mu$$

$$= \int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N} \left| \int_{I_k} \tilde{T}_n^{\alpha_a}(y-x)f(x)d\mu(x)\right| d\mu(y).$$

By Lemma 2.2 we have,

$$
\begin{aligned}
&\int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N} \left| \int_{I_k} \tilde{T}_n^{\alpha_a}(y-x) f(x) d\mu(x) \right| d\mu(y) \\
&= \int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N} \left| \int_{I_k} f(x) \left[\frac{n_B D_{M_B}(y-x)}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} \right.\right. \\
&+ \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n^{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \\
&\left.\left. + \alpha_a \left| K_{n_B M_B - 1}(y-x) \right| \right] d\mu(x) \right| d\mu(y) \\
&= \int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N} \left| \int_{I_k} f(x) \left[\frac{n_B D_{M_B}(y-x)}{A_n^{\alpha_a}} \right.\right. \\
&\left.\left. \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} \right] d\mu(x) \right| d\mu(y) \\
&+ \int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \right.\right. \\
&\left.\left. \frac{j+1}{(n^{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y) \\
&+ \int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N} \left| \int_{I_k} f(x) \left[\alpha_a \left| K_{n_B M_B - 1}(y-x) \right| \right] d\mu(x) \right| d\mu(y) \\
&:= \phi_1 + \phi_2 + \phi_3.
\end{aligned}
$$

It is simple to find that

$$
\frac{n_B D_{M_B}(y-x)}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} = 0,
$$

for any $y - x \in \bar{I}_k$. This holds because $D_{M_B}(y-x) = 0$ for $B = |n| \geq k$

and $y - x \in \bar{I}_k$. Hence, $\phi_1 = 0$. Besides,

$$\phi_2 = \int_{\bar{I}_k} \sup_{n \geq M_k,\, a \in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \right.\right.$$

$$\left.\left. \frac{j+1}{(n^{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$= \int_{\bar{I}_k} \sup_{n \geq M_k,\, a \in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{M_k - 1} \right.\right.$$

$$\frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right|$$

$$+ \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=M_k}^{n_B M_B - 2} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}}$$

$$\left.\left. \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$\leq \int_{\bar{I}_k} \sup_{n \geq M_k,\, a \in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{M_k - 1} \right.\right.$$

$$\left.\left. \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$+ \int_{\bar{I}_k} \sup_{n \geq M_k,\, a \in N} \left| \int_{I_k} f(x) \left[\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=M_k}^{n_B M_B - 2} \right.\right.$$

$$\left.\left. \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j(y-x) \right| \right] d\mu(x) \right| d\mu(y)$$

$$:= \int_{\bar{I}_k} \sup_{n \geq M_k,\, a \in N} \left| \int_{I_k} f(x) H_1(y-x) d\mu(x) \right| d\mu(y)$$

$$+ \int_{\bar{I}_k} \sup_{n \geq M_k,\, a \in N} \left| \int_{I_k} f(x) H_2(y-x) d\mu(x) \right| d\mu(y).$$

However, since for any $j < M_k$ we have that the Fejér kernel $K_j(y-x)$

depends with respect to x only on coordinates $x_0 = 0,\ ...,\ x_{k-1} = 0,$ then

$$\int_{I_k} f(x)\Big|K_j(y-x)\Big|d\mu(x) = |K_j(y)| \int_{I_k} f(x) = 0$$

gives

$$\int_{I_k} f(x)H_1(y-x)d\mu(x) = H_1(y)\int_{I_k} f(x)d\mu(x) = 0.$$

On the other hand,

$$\begin{aligned}
&\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=M_k}^{n_B M_B - 1}\frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}}\Big|K_j\Big| \\
&\leq \sup_{j\geq M_k}\Big|K_j\Big|\frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=1}^{n}\frac{j+1}{j^{2-\alpha_a}} \\
&\leq \sup_{j\geq M_k}\Big|K_j\Big|\frac{2\alpha_a(1-\alpha_a)}{n^{\alpha_a}}\sum_{j=1}^{n} j^{\alpha_a-1} \\
&\leq 2(1-\alpha_a)\sup_{j\geq M_k}\Big|K_j\Big|.
\end{aligned}$$

By lemma 3 in [10], this implies

$$\begin{aligned}
&\int_{\bar{I}_k}\sup_{n\geq M_k,\, a\in N}\left|\int_{I_k} f(x)H_2(y-x)d\mu(x)\right|d\mu(y) \\
&\leq \int_{I_k}|f(x)|\Big(\int_{\bar{I}_k}\sup_{n\geq M_k,\, a\in N}\Big|H_2(y-x)\Big|d\mu(y)\Big)d\mu(x) \\
&\leq C\int_{I_k}|f(x)|\Big(\int_{\bar{I}_k}\sup_{j\geq M_k}\Big|K_j(y-x)\Big|d\mu(y)\Big)d\mu(x) \\
&\leq C\int_{I_k}|f(x)|d\mu(x) = C\left\|f\right\|_1.
\end{aligned}$$

Thus, $\phi_2 \leq C\left\|f\right\|_1.$

Similarly, for the case ϕ_3 we apply Lemma 3 in [10]

$$\begin{aligned}
\phi_3 &= \int_{\bar{I}_k} \sup_{n\geq M_k,\, a\in N} \left| \int_{I_k} f(x) \left[\left| K_{n_B M_B - 1}(y-x) \right| \right] d\mu(x) \right| d\mu(y) \\
&\leq \int_{I_k} |f(x)| \Big(\int_{\bar{I}_k} \sup_{n\geq M_k} \left| K_{n_B M_B - 1}(y-x) \right| d\mu(y) \Big) d\mu(x) \\
&\leq C \int_{I_k} |f(x)| d\mu(x) = C \left\| f \right\|_1 .
\end{aligned}$$

Hence, the lemma follows. □

In the following corollary, it is also proved that operators $t_n^{\alpha_a}$, $\tilde{t}_n^{\alpha_a}$ are of type $(L^1,\, L^1)$ and $(L^\infty,\, L^\infty)$ uniformly in n.

Corollary 2.5. *[6] Let* $1 > \alpha_a > 0,\ a \in \mathbb{N}$. *Then, we have*

$$\begin{aligned}
&\|T_n^{\alpha_a}\|_1 \leq \|\tilde{T}_n^{\alpha_a}\|_1 \leq C, \\
&\|t_n^{\alpha_a} f\|_1,\ \|\tilde{t}_n^{\alpha_a} f\|_1 \leq C\|f\|_1
\end{aligned}$$

and

$$\|t_n^{\alpha_a} g\|_\infty,\ \|\tilde{t}_n^{\alpha_a} g\|_\infty \leq C\|g\|_\infty$$

for all natural numbers a, n *and where* C *is some absolute constant and* $f \in L^1(G_m)$, $g \in L^\infty(G_m)$. *That is, operator* $t_n^{\alpha_a}$, $\tilde{t}_n^{\alpha_a}$ *are of type* $(L^1,\, L^1)$ *and* $(L^\infty,\, L^\infty)$ *uniformly in* n.

Proof. The proof is a direct consequence of lemma 2.4. Then

$$\begin{aligned}
\|\tilde{T}_n^{\alpha_a}\|_1 \leq\ & C \frac{n_B \|D_{M_B}\|_1}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n-j}^{\alpha_a - 1} \\
&+ \frac{(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n_{(B)} + j)^{2-\alpha_a}} \left\| K_j \right\|_1 \\
&+ \|K_{n_B M_B - 1}\|_1 .
\end{aligned}$$

Consequently, by $\|D_{M_B}\|_1,\ \|K_j\|_1 \leq C,$ the proof of the Corollary follows. □

For the general case by considering $n \in \mathbb{N}_{\alpha,q}$, where $\mathbb{N}_{\alpha,q}$ is defined in (2.1), we proved the following lemma.

That is, the maximal operator

$$\tilde{\sigma}^{\alpha}_{*,\,q} := \sup_{n\in\mathbb{N}_{\alpha,\,q}} |\tilde{\sigma}^{\alpha_n}_n|$$

is quasi-local. We get this by the investigation of kernel functions, its maximal function on the Vilenkin group by making a hole around zero and some quasi-locality issues.

Lemma 2.6. *[6] Let $0 < \alpha_n < 1$, $f \in L^1(G_m)$ such that supp$f \subset I_k(u)$, $\int_{I_k(u)} f d\mu = 0$ for some m-adic interval $I_k(u)$. Then we have*

$$\int_{G_m\setminus I_k(u)} \tilde{\sigma}^{\alpha}_{*,\,q} f d\mu \leq C_q \|f\|_1.$$

Where constants C_q can depend only on q.

Proof. From the formula of the kernel function $\tilde{K}^{\alpha_n}_n$ we have

$$\begin{aligned}\tilde{K}^{\alpha_n}_n &= \left|T^{\alpha_n}_n\right| + \sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} n_l D_{M_l} \\ &+ \sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} |T^{\alpha_n}_{n_{(l-1)}}| \\ &=: N_1 + N_2 + N_3.\end{aligned}$$

The integral,

$$\int_{G_m\setminus I_k(u)} \sup_{n\in N} \left| \int_{I_k(u)} f(x) \bigg(N_2(y-x) \bigg) d\mu(x) \right| d\mu(y) = 0$$

since $f * D_{M_l} = 0$ for $l < s \leq k$ because of the $\mathcal{A}_k$ measurablity of D_{M_l} and $\int f = 0$. Besides, $D_{M_l}(y-x) = 0$; for $s > k$, $y - x \notin I_k$.
Since from lemma 1.1 we have

$$\frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} \leq \frac{(n_{(l-1)})^{\alpha_n}}{n^{\alpha_n}} \leq C \frac{M^{\alpha_n}_l}{n^{\alpha_n}}.$$

Besides, by the help of lemma 2.4 and by the fact that $n \in \mathbb{N}_{\alpha,\, q}$ implies

$$\sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \leq C \sum_{l=0}^{B} \frac{M_l^{\alpha_n}}{n^{\alpha_n}} \leq C_q$$

we get

$$\begin{aligned}
&\int_{G_m \backslash I_k(u)} \sup_{n \in \mathbb{N}_{\alpha,\, q}} \left| \int_{I_k(u)} f(x) \Bigg(N_1(y-x) + N_3(y-x) \Bigg) d\mu(x) \right| d\mu(y) \\
&\leq \int_{G_m \backslash I_k(u)} \sup_{n \in \mathbb{N}_{\alpha,\, q}} \left| \int_{I_k(u)} f(x) \Bigg(\left| \tilde{T}_n^{\alpha_n}(y-x) \right| \right. \\
&\left. + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \left| \tilde{T}_{n_{(l-1)}}^{\alpha_a}(y-x) \right| \Bigg) d\mu(x) \right| d\mu(y) \\
&\leq C_q \int_{G_m \backslash I_k(u)} \sup_{n \in \mathbb{N}_{\alpha,\, q}} \left| \int_{I_k(u)} f(x) \left| \tilde{T}_n^{\alpha_n}(y-x) \right| d\mu(x) \right| d\mu(y) \\
&\leq C_q \|f\|_1.
\end{aligned}$$

Hence, the lemma follows. $\square$

For the general case of $n \in \mathbb{N}_{\alpha,q}$. Define (C, α_n) mean as follows

$$\sigma_n^{\alpha_n} f(x) := \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k(x) = \int_{G_m} f(y) K_n^{\alpha_n}(x-y) d\mu(y), \qquad (2.3)$$

where $K_n^{\alpha_n}$ is the kernel function defined in (2.2).

Considering the definition in (2.3), we define maximal operators as follows.

$$\sigma_{*,\, q}^{\alpha} f := \sup_{n \in N_{\alpha,\, q}} |\sigma_n^{\alpha_n} f|.$$

Lemma 2.7. *[6] The operator* $\tilde{\sigma}_{*,q}^{\alpha}$ *is of type* $(L^{\infty},\, L^{\infty})$ *and weak type* $(L^1,\, L^1)$; $\sigma_{*,\, q}^{\alpha}$ *is of weak type* $(L^1,\, L^1)$.

Proof. By the help of the method of lemma 2.4 and corollary 2.5 we get

$$\|\tilde{K}_n^{\alpha_n}\|_1 \leq \|T_n^{\alpha_n}\|_1 + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} n_l \|D_{M_l}\|_1$$

$$+ \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \|T_{n_{(l-1)}}^{\alpha_n}\|_1$$

$$\leq C + C \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \leq C_q$$

since $n \in \mathbb{N}_{\alpha, q}$. Thus, $\tilde{\sigma}_{*,q}^{\alpha}$ is of type (L^{∞}, L^{∞}).
To proof the weak type (L^1, L^1) case we apply the Calderon-Zygmund decomposition lemma [10].
Let $f \in L^1(G_m)$ and $\|f\|_1 < \delta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that

$$\|f_0\|_{\infty} \leq C\delta, \ \|f_0\|_1 \leq C\|f\|_1$$

and $G_m^j = I_{k_j}(u^j)$ are disjoint m-adic intervals for which

$$supp f_j \subset G_m^j, \ \int_{G_m^j} f_j d\mu = 0, \ |F| \leq \frac{C\|f\|_1}{\delta}$$

$(u^j \in G_m, \ k_j \in \mathbb{N}, \ j \in \mathbb{P})$, where $F = \bigcup_{i=1}^{\infty} G_m^j$.
By the σ-sublinearity of the maximal operator with an appropriate constant C_q we have

$$\mu(\tilde{\sigma}_{*,q}^{\alpha} f > 2C_q\delta) \leq \mu(\tilde{\sigma}_{*,q}^{\alpha} f_0 > C_q\delta) + \mu(\tilde{\sigma}_{*,q}^{\alpha} \sum_{j=1}^{\infty} f_j > C_q\delta)$$

$$=: W + M.$$

Since $\tilde{\sigma}_{*,q}^{\alpha}$ is of type (L^{∞}, L^{∞}), we have that

$$\|\tilde{\sigma}_{*,q} f_0\|_{\infty} \leq C_q \|f_0\|_{\infty} \leq C_q\delta.$$

Then we have $W = 0$. The situation for M becomes,

$$\begin{aligned} M &= \mu(\tilde{\sigma}^{\alpha}_{*,\,q}\sum_{j=1}^{\infty} f_j > C_q\delta) \leq |F| \\ &+ \mu(\bar{F} \cap [\tilde{\sigma}^{\alpha}_{*,\,q}\sum_{j=1}^{\infty} f_j > C_q\delta]) \\ &\leq \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta}\sum_{j=1}^{\infty}\int_{G_m\backslash G_m^j} \tilde{\sigma}^{\alpha}_{*,\,q} f_j d\mu \\ &=: \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta}\sum_{j=1}^{\infty} N_j, \end{aligned}$$

in which

$$\begin{aligned} N_j &= \int_{G_m\backslash G_m^j} \tilde{\sigma}^{\alpha}_{*,\,q} f_j d\mu \\ &\leq \int_{G_m\backslash I_{k_j}(u^j)} \sup_{n\in\mathbb{N}_{\alpha,\,q}} \left| \int_{I_{k_j}(u^j)} f_j(x)\tilde{K}_n^{\alpha_n}(y-x)d\mu(x)\right| d\mu(y). \end{aligned}$$

The next estimation for N_j is given by lemma 2.6. Then,

$$N_j \leq C_q\|f_j\|_1.$$

That is, operator $\tilde{\sigma}^{\alpha}_{*,\,q}$ is of weak type $(L^1,\, L^1)$.
By lemma 2.3 and since

$$\begin{aligned} &\mu(\sigma^{\alpha}_{*,\,q} f > 2C_q\delta) \leq \mu(\tilde{\sigma}^{\alpha}_{*,\,q}|f| > 2C_q\delta) \\ &\leq C_q\frac{\|f\|_1}{\delta}. \end{aligned}$$

We concluded that the maximal operator $\sigma^{\alpha}_{*,\,q}$ is of weak type $(L^1,\, L^1)$.
Hence, the lemma follows. □

2.3 Almost everywhere convergence of the Cesàro means

Following the proofs of lemmas and corollaries in section 2.1 and 2.2, the almost everywhere convergence of the (C, α_n) means is proved in the following theorem.

Theorem 2.8. *[6] Let* $0 < \alpha_n < 1$. *Let* $f \in L^1(G_m)$. *Then* $\sigma_n^{\alpha_n} f \longrightarrow f \quad$ *a.e. if* $n \longrightarrow \infty$, $n \in \mathbb{N}_{\alpha,\, q}$.

Proof. Let us consider a Vilenkin Polynomial P such that $P(x) = \sum_{i=0}^{M_k-1} c_i \psi_i(x)$. Then for all natural number $n \geq M_k$, $n \in \mathbb{N}_{\alpha,\, q}$ we have that $S_n P \equiv P$. Thus, the statement $\sigma_n^{\alpha_n} P \longrightarrow P$ holds everywhere which is not only for $n \in \mathbb{N}_{\alpha,\, q}$, but for arbitrary $n \to \infty$.

Now, let ϵ, $\delta > 0$, $f \in L^1(G_m)$. Let P be a Vilenkin polynomial such that $\|f - P\|_1 < \delta$. Then,

$$
\begin{aligned}
&\mu(\overline{\lim_{n \in \mathbb{N}_{\alpha,\, q}}} |\sigma_n^{\alpha_n} f - f| > \epsilon) \\
&\leq \mu(\overline{\lim_{n \in \mathbb{N}_{\alpha,\, q}}} |\sigma_n^{\alpha_n} (f - P)| > \frac{\epsilon}{3}) \\
&+ \mu(\overline{\lim_{n \in \mathbb{N}_{\alpha,\, q}}} |\sigma_n^{\alpha_n} P - P| > \frac{\epsilon}{3}) \\
&+ \mu(\overline{\lim_{n \in \mathbb{N}_{\alpha,\, q}}} |P - f| > \frac{\epsilon}{3}) \\
&\leq \mu(\overline{\lim_{n \in \mathbb{N}_{\alpha,\, q}}} |\sigma_n^{\alpha_n} (f - P)| > \frac{\epsilon}{3}) \\
&+ 0 + \frac{3}{\epsilon} \|P - f\|_1 \\
&\leq C_q \|P - f\|_1 \frac{3}{\epsilon} \leq \frac{C_q}{\epsilon} \delta
\end{aligned}
$$

since (from lemma 2.7) $\sigma_{*,\, q}^{\alpha}$ is of weak type (L^1, L^1) with any fixed $q > 0$. This holds for all $\delta > 0$.
That is, for an arbitrary $\epsilon > 0$

$$\mu(\overline{\lim_{n \in \mathbb{N}_{\alpha,\, q}}} |\sigma_n^{\alpha_n} f - f| > \epsilon) = 0$$

and as a result we also have

$$\mu(\overline{\lim_{n \in \mathbb{N}_{\alpha,\, q}}} |\sigma_n^{\alpha_n} f - f| > 0) = 0.$$

This finally gives

$$\overline{\lim_{n\in\mathbb{N}_{\alpha,\, q}}} \left|\sigma_n^{\alpha_n} f - f\right| = 0 \quad \text{a.e.}$$

Consequently,

$$\sigma_n^{\alpha_n} f \longrightarrow f \text{a.e as} \quad n \longrightarrow \infty,\, n \in \mathbb{N}_{\alpha,\, q}.$$

Hence, the theorem follows. □

Chapter 3

Generalized Marcinkiewicz means

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz[38] proved that for all $f \in L \log L([0, 2\pi]^2)$ the a.e. relation

$$\frac{1}{n} \sum_{j=1}^{n} S_{j,j} f \to f \tag{3.1}$$

holds as $n \to \infty$. Zhizhiashvili [49] improved this result for $f \in L([0, 2\pi]^2)$. Dyachenko [23] proved this result for dimensions greater than 2.
In 2003 Goginava [33] proved this result with respect to the multiple Walsh-Paley system. The case $d = 2$ is due to Weisz [19]. This result for bounded Vilenkin systems due to Gát [27]. In 2012 Gát [28] proved this result for generalized Marcinkiewicz means with respect to the two dimensional Walsh system and in 2016 [29] for bounded two-dimensional Vilenkin systems.
We generalized the result of Gát [29] with respect to two-dimensional generalized Vilenkin-like systems. Besides, we give an application of the main result. That is, theorem 3.9 with respect to triangular summability of Vilenkin-like-Fourier series.
The two-dimensional generalized Marcinkiewicz kernels and Marcinkiewicz means, with respect to the two-dimensional Vilenkin-like system are defined as follows: Let $\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$ be a function. From now functions α_1, α_2 play the role of indices. We know that in the preliminary part of the dissertation the function α_n appeared in the definition of the Vilenkin-like (or $\psi\alpha$ system), but this will not cause any misunderstanding. Define the following

generalized Marcinkiewicz kernels and means respectively:

$$M_n^\alpha(y,x) := \frac{1}{n}\sum_{k=0}^{n-1} D_{\alpha_1(|n|,k)}(y^1,x^1)D_{\alpha_2(|n|,k)}(y^2,x^2),$$
$$t_n^\alpha f := f * M_n^\alpha \quad (f \in L^1(G_m^2),\, n \in \mathbb{P}).$$

This concept of Marcinkiewicz-like kernels and means is due to Gát [28].

The main aim of this Chapter is to give a class of functions α for which we have the a.e. convergence relation $t_n^\alpha f \to f$ for each integrable two variable function with respect to two dimensional bounded Vilenkin-like systems.

To investigate this the following properties play a prominent role ($Car(B)$ denotes cardinality of the set B),

$$Car\{l \in \mathbb{N} : \alpha_j(|n|,l) = \alpha_j(|n|,k),\, l < n\} \leq C \tag{3.2}$$
$$(k < n,\, n \in \mathbb{P},\, j = 1,2),$$
$$\max\{\alpha_j(|n|,k) :\ k < n\} \leq Cn \quad (n \in \mathbb{P},\, j = 1,2). \tag{3.3}$$

Our first aim is to prove that the operator $t_*^\alpha := \sup_{n\in\mathbb{P}} \mid t_n^\alpha \mid$ is of weak type (L^1, L^1). In order to do this we need a sequence of Lemmas. Lemma 3.2 is the base of the proof of Theorems 3.6 and 3.7. The Walsh-Paley version of Theorems 3.6 and 3.7 are due to Gát [28]. That is, we generalize a result of Gát. Moreover, techniques of papers [28] and [29] will also be used in the proof of the forthcoming lemmas. Denote for $k \in \mathbb{N}, x \in G_m$, $J_k(x) = I_k(x) \setminus I_{k+1}(x)$ and recall also that

$$n^{(s)} = \sum_{k=s}^{\infty} n_k M_k,\ (n,\, s \in \mathbb{N}),\ n^{(0)} = n,\ n^{(|n|+1)} = 0.$$

3.1 Generalized Marcinkiewicz kernels and their maximal operators

For $x, y \in G_m^2, A, n, s, j, k \in \mathbb{N}$ let

$$\Phi(A, n^{(s+1)} + jM_s + k,\, y, x) = D_{\alpha_1(A,\, n^{(s+1)}+jM_s+k)}(y^1, x^1)$$
$$D_{\alpha_2(A,\, n^{(s+1)}+jM_s+k)}(y^2, x^2).$$

Lemma 3.1. *[5] Let $t^1, t^2, A, s \in \mathbb{N}, s \le A$ and $y \in G_m^2$. Then,*

$$\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A,\, n^{(s+1)} + jM_s + k,\, y, x)\Big| d\mu(x)$$
$$\le C(M_A M_{t^1})^{\frac{1}{2}}.$$

Proof. We discuss the integral

$$\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A,\, n^{(s+1)} + jM_s + k,\, y, x)\Big| d\mu(x)$$

for fixed $t = (t^1,\, t^2),\ s,\ A$. Check the function

$$\sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A,\, n^{(s+1)} + jM_s + k,\, y, x)$$

on the set $J_{t^1}(y^1) \times J_{t^2}(y^2)$.
For each $j \in \mathbb{N}$ and since we have $x^2 \in J_{t^2}(y^2)$, then by (1.7) we get

$$|D_j(y^2, x^2)| \le CM_{t^2}$$

(since the sequence m is bounded this inequality holds true). Thus by (1.7), we have

$$|D_{\alpha_2(A,\, n^{(s+1)}+jM_s+k)}(y^2, x^2)| \le CM_{t^2}.$$

On the other hand, again by (1.7) for $x^1 \in J_{t^1}(y^1)$ we have

$$D_{\alpha_1(A,\, n^{(s+1)}+jM_s+k)}(y^1, x^1) = \chi_{\left[\alpha_1(A, n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1)$$

$$\bar{\chi}_{\left[\alpha_1(A, n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(x^1) \times \left[\sum_{w=0}^{t^1-1} [\alpha_1(A,\, n^{(s+1)} + jM_s + k)]_w M_w\right.$$

$$\left. + \sum_{i=m_{t^1}-(\alpha_1(A,\, n^{(s+1)}+jM_s+k))_{t^1}}^{m_{t^1}-1} r_{t^1}^i(y^1)\bar{r}_{t^1}^i(x^1)M_{t^1}\right].$$

As x varies the function

$$\sum_{w=0}^{t^1-1}[\alpha_1(A,n^{(s+1)}+jM_s+k)]_w M_w + \sum_{i=m_{t^1}-(\alpha_1(A,n^{(s+1)}+jM_s+k))_{t^1}}^{m_{t^1}-1} r^i_{t^1}(y^1)\bar{r}^i_{t^1}(x^1)M_{t^1}$$

depends only on $x^1_{t^1}$ (and not on the other coordinates of x^1).
Consequently,

$$\left|\sum_{w=0}^{t^1-1}[\alpha_1(A,n^{(s+1)}+jM_s+k)]_w M_w + \sum_{i=m_{t^1}-(\alpha_1(A,n^{(s+1)}+jM_s+k))_{t^1}}^{m_{t^1}-1} r^i_{t^1}(y^1)\bar{r}^i_{t^1}(x^1)M_{t^1}\right|$$
$$\leq CM_{t^1}.$$

Furthermore,

$$\left|\sum_{j=0}^{n_s-1}\sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)\right|^2$$

$$= \sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1} D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)\overline{D}_{\alpha_1(A,n^{(s+1)}+hM_s+l)}(y^1,x^1)$$
$$\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2).$$

Applying Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\int_{J_{t^2}(y^2)}\left[\int_{J_{t^1}(y^1)}\sup_{|n|=A}\left|\sum_{j=0}^{n_s-1}\sum_{k=0}^{M_s-1}\right.\right.$$

$$\left.\left.\Phi(A,n^{(s+1)}+jM_s+k,y^1,y^2,x^1,x^2)\right|d\mu(x^1)\right]d\mu(x^2)$$

$$\leq \int_{J_{t^2}(y^2)} M_{t^1}^{\frac{-1}{2}}\left[\int_{J_{t_1}(y^1)}\sup_{|n|=A}\left|\sum_{j=0}^{n_s-1}\sum_{k=0}^{M_s-1}\right.\right.$$

$$\left.\left.\Phi(A,n^{(s+1)}+jM_s+k,y^1,y^2,x^1,x^2)\right|^2 d\mu(x^1)\right]^{\frac{1}{2}} d\mu(x^2)$$

$$\leq \int_{J_{t^2}(y^2)} M_{t^1}^{\frac{-1}{2}} \Bigg[\int_{J_{t_1}(y^1)} \sup_{|n|=A} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1}$$

$$D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)\overline{D}_{\alpha_1(A,n^{(s+1)}+hM_s+l)}(y^1,x^1)$$

$$\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)$$

$$d\mu(x^1)\Bigg] d\mu(x^2)$$

$$= \int_{J_{t^2}(y^2)} M_{t^1}^{\frac{-1}{2}} \Bigg[\int_{J_{t^1}(y^1)} \sup_{|n|=A} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1}$$

$$\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)$$

$$\times \left(\sum_{w=0}^{t^1-1} \alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w \right.$$

$$\left. + \sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1} r_{t^1}^{a}(y^1,x^1) \right)$$

$$\times \left(\sum_{v=0}^{t^1-1} \alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v \right.$$

$$\left. + \sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1} \overline{r_{t^1}^{b}}(y^1,x^1) \right)$$

$$D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(x^2)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)d\mu(x^1)\Bigg]^{\frac{1}{2}} d\mu(x^2)$$

$$=: B^1.$$

As $n^{(s+1)}$ relies only on natural numbers n_{s+1}, ..., n_{A-1}, n_A, thus the supreme $\sup_{n\in\mathbb{N},\,|n|=A}$ above also relies only on n_s, n_{s+1}, ..., n_{A-1}, n_A. Hence, by $|n|=A$

$$B^1 \leq \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \Bigg[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}-1} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} \int_{J_{t^1}(y^1)} \sum_{j,h=0}^{n_s-1} \sum_{k,l=0}^{M_s-1}$$

$$\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)$$

$$\times\left(\sum_{w=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w+\sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1}\right.$$

$$\left.r^a_{t^1}(y^1,x^1)\right)\times\left(\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v\right.$$

$$\left.+\sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}\overline{r^b_{t^1}}(y^1,x^1)\right)$$

$$\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)$$

$$\left.d\mu(x^1)\right]^{\frac{1}{2}}d\mu(x^2)$$

$$=\int_{J_{t^2}(y^2)}M_{t^1}^{-\frac{1}{2}}\left[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}\sum_{n_s=0}^{m_s-1}\sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1}\right.$$

$$\int_{J_{t^1}(y^1)}\left(\sum_{w=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w\right.$$

$$\left.+\sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1}r^a_{t_1}(y^1,x^1)\right)$$

$$\times\left(\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v+\sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}\right.$$

$$\left.\overline{r^b_{t^1}}(y^1,x^1)\right)\times\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)$$

$$\overline{\chi}_{\left[\alpha_1(A,\,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)d\mu(x^1)\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)$$

$$\left.\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)\right]^{\frac{1}{2}}d\mu(x^2)$$

$$=\int_{J_{t^2}(y^2)}M_{t^1}^{-\frac{1}{2}}\left[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}\sum_{n_s=0}^{m_s-1}\sum_{j,h=0}^{n_s-1}\sum_{k,l=0}^{M_s-1}\right.$$

$$
\Bigg(\sum_{w=0}^{t^1-1}\sum_{v=0}^{t^1-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w \alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v
$$

$$
\times\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,\,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)
$$

$$
d\mu(x^1)+\sum_{w=0}^{t^1-1}\ \sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}\alpha_1(A,n^{(s+1)}+jM_s+k)_w M_w
$$

$$
\times\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(x^1)\overline{\chi}_{\left[\alpha_1(A,\,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(x^1)
$$

$$
\overline{r^b_{t^1}}(y^1,x^1)d\mu(x^1)+\sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1}\ \sum_{v=0}^{t^1-1}
$$

$$
\alpha_1(A,n^{(s+1)}+hM_s+l)_v M_v\times\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)
$$

$$
\overline{\chi}_{\left[\alpha_1(A,\,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(x^1)r^a_{t_1}(y^1,x^1)d\mu(x^1)
$$

$$
+\sum_{a=m_{t^1}-[\alpha_1(A,n^{(s+1)})+jM_s+k]_{t^1}}^{m_{t^1}-1}\ \sum_{b=m_{t^1}-[\alpha_1(A,n^{(s+1)})+hM_s+l]_{t^1}}^{m_{t^1}-1}
$$

$$
\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,\,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)
$$

$$
r^a_{t_1}(y^1,x^1)\overline{r^b_{t^1}}(y^1,x^1)d\mu(x^1)\Bigg)\times D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)
$$

$$
\left.\overline{D}_{\alpha_2(A,n^{(s+1)}+hM_s+l)}(y^2,x^2)\right]^{\frac{1}{2}}d\mu(x^2)=:\sum_{q=1}^{4}B^{2,q}=:B^2.
$$

After we determine the possible number of k, $l's$ in which the integral

$$
\int_{J_{t^1}(y^1)}\chi_{\left[\alpha_1(A,n^{(s+1)}+jM_s+k)\right]^{(t^1)}}(y^1,x^1)\overline{\chi}_{\left[\alpha_1(A,\,n^{(s+1)}+hM_s+l)\right]^{(t^1)}}(y^1,x^1)
$$

$$
\times\, r^a_{t_1}(y^1,x^1)\overline{r^b_{t^1}}(y^1,x^1)d\mu(x^1)\neq 0
$$

we estimate B^2, (that is, any of $B^{2,q}$). We have $a = b = 0$ in the case of $B^{2,1}$, we have $a = 0$ in the case of $B^{2,2}$ and we have $b = 0$ in the case of $B^{2,3}$. For some $a, b \in \{0,\ 1,\ ...,\ m_{t_1} - 1\}$, assume the integral

$$\int_{J_{t^1}(y^1)} \chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1)\overline{\chi}_{[\alpha_1(A,\,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$

$$\times\, r_{t_1}^{a}(y^1,x^1)\overline{r_{t^1}^{b}}(y^1,x^1)d\mu(x^1) \neq 0.$$

By using Lemma 1.6 and (1.4)-(1.6), we get $(t^1 + 1)^{th}$, $(t^1 + 2)^{th}$, ..., coordinates of $\alpha_1(A,\ n^{(s+1)} + jM_s + k)$ and $\alpha_1(A,\ n^{(s+1)} + hM_s + l)$ equal. We have,

$$\Bigg| \int_{J_{t^1}(y^1)} \chi_{[\alpha_1(A,n^{(s+1)}+jM_s+k)]^{(t^1)}}(y^1,x^1)\overline{\chi}_{[\alpha_1(A,\,n^{(s+1)}+hM_s+l)]^{(t^1)}}(y^1,x^1)$$

$$\times\, r_{t_1}^{a}(y^1,x^1)\overline{r_{t^1}^{b}}(y^1,x^1)d\mu(x^1)\Bigg| \leq 1/M_{t^1}.$$

Since by (4.6) we have that for every k, there exist $l's$ for which $\alpha_1(A,\ n^{(s+1)} + jM_s + k) = \alpha_1(A,\ n^{(s+1)} + hM_s + l)$. This gives that for every k, there exist at most CM_{t^1} number of $l's$ for which this integral is not zero. Consequently, by

$$\sum_{w=0}^{t^1-1}[\alpha_1(A,\ n^{(s+1)} + jM_s + k)]_w M_w \leq CM_{t^1},$$

$$|D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \leq CM_{t^2}$$

we have

$$B^{2,1} \leq C \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \Bigg[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1}$$

$$\sum_{n_s=0}^{m_s-1} \sum_{j,h=0}^{n_s-1} \sum_{k=0}^{M_s-1} M_{t^1}M_{t^1}^2 \frac{1}{M_{t^1}} M_{t^2}^2 \Bigg]^{\frac{1}{2}} d\mu(x^2).$$

(Keep in mind that the Vilenkin space is bounded. That is, $m_j \leq C$ for all $j \in \mathbb{N}$.) By the very same steps we get the identical upper bound also for $B^{2,2}$, $B^{2,3}$ and $B^{2,4}$. So does for their sum. That is,

$$\begin{aligned}
B^2 &\leq C \int_{J_{t^2}(y^2)} M_{t^1}^{-\frac{1}{2}} \left[\sum_{n_A=1}^{m_A-1} \sum_{n_{A-1}=0}^{m_{A-1}-1} \right. \\
&\left. \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{n_s=0}^{m_s-1} ((M_{t^1} M_{t^2})^2 M_s M_{t^1} M_{t^1}^{-1}) \right]^{\frac{1}{2}} d\mu \\
&\leq C M_{t^2}^{-1} M_{t^1}^{\frac{-1}{2}} \left[\frac{M_A}{M_s} M_{t^1}^2 M_{t^2}^2 M_s \right]^{\frac{1}{2}} \\
&\leq C (M_A M_{t^1})^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{|n|=A} \left| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A,\, n^{(s+1)} + jM_s + k,\, y, x) \right| d\mu(x) \\
&\leq C (M_A M_{t^1})^{\frac{1}{2}}.
\end{aligned}$$

Hence, the lemma follows. □

Lemma 3.2. *[5] Let $a \in \mathbb{N}, y \in G_m^2$. Then,*

$$\begin{aligned}
&\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1) \times J_{t^2}(y^2)} \sup_{A \geq a} \sup_{|n|=A} \frac{1}{M_A} \times \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1} \\
&| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \mid d\mu(x) \leq C.
\end{aligned}$$

Proof. Using Lemma 3.1, the system (i.e. the Vilenkin space) is bounded and due to the fact that $M_{\max(a,t^2)} \leq C M_{\max(a,t^2-c)}$ we get the following estimation:

$$\begin{aligned}
&\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)}\sup_{A\geq\max(a,t^2-c)}\ \sup_{|n|=A}\frac{1}{M_A}\sum_{s=t^1}^{A}\sum_{j=0}^{n_s-1}\\
&\left|\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s+k,y,x)\right|d\mu(x)\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\ \sum_{A=\max(a,t^2-c)}^{\infty}\ \sum_{s=t^1}^{A}(M_{t^1}M_A^{-1})^{\frac{1}{2}}\\
&= C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\ \sum_{A=\max(a,t^2-c)}^{\infty}(A-t^1+1)(M_{t^1}M_A^{-1})^{\frac{1}{2}}\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}(\max(a,t^2)-t^1)(M_{t^1}M^{-1}_{\max(a,t^2-c)})^{\frac{1}{2}}\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}(\max(a,t^2)-t^1)(M_{t^1}M^{-1}_{\max(a,t^2)})^{\frac{1}{2}}\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{a-1}(a-t^1)(M_{t^1}M_a^{-1})^{\frac{1}{2}}\\
&+C\sum_{t^1=0}^{a-1}\sum_{t^2=a+1}^{\infty}(t^2-t^1)(M_{t^1}M_{t^2}^{-1})^{\frac{1}{2}}\leq C.
\end{aligned}$$

This inequality shows that if we want to complete the proof of this Lemma, then we have to discuss also the case when $\sup_{\{A:t^2-c>A\geq a\}}$. This follows that t^2 should be at least $a+c$. That is, we have to prove that the following integral is bounded

$$\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)}\sup_{\{A:t^2-c>A\geq a\}}\ \sup_{|n|=A}\frac{1}{M_A}\sum_{s=t^1}^{A}\sum_{j=0}^{n_s-1}$$

$$\left|\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s+k,x^1,x^2)\right|d\mu(x):=B^3.$$

The method we are going to use in order to discuss B^3 is the same as we used for the investigation of B^1. The only difference is that in the situation of B^1 we used the estimation $|D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \le CM_{t^2}$ and in the case of B^3 we use the formula of the Dirichlet kernel D_n (1.7) and the estimation $|D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \le CM_A$. The other steps of this process are the same. That is,

$$
\begin{aligned}
B^3 &\le C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\int_{J_{t^2}(y^2)}\sum_{A=a}^{t^2-c}\frac{1}{M_A}\sum_{s=t^1}^{A}M_{t^1}^{-\frac{1}{2}} \\
&\times\left[\sum_{n_A=1}^{m_A-1}\sum_{n_{A-1}=0}^{m_{A-1}-1}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}(M_{t^1}M_A)^2M_sM_{t^1}^{-1}\right]^{\frac{1}{2}}d\mu(x^2) \\
&= C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}\sum_{s=t^1}^{A}M_{t^2}^{-1}M_A^{-1}M_{t^1}^{-\frac{1}{2}}(M_AM_s^{-1}M_{t^1}^2M_A^2M_s)^{\frac{1}{2}} \\
&\le C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}\sum_{s=t^1}^{A}\frac{(M_AM_{t^1})^{\frac{1}{2}}}{M_{t^2}} \\
&\le C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}\sum_{A=a}^{t^2-c}(A-t^1+1)M_A^{\frac{1}{2}}M_{t^1}^{\frac{1}{2}}M_{t^2}^{-1} \\
&\le C\sum_{t^1=0}^{a-1}\sum_{t^2=a+c}^{\infty}(t^2-t^1+1)M_{t^1}^{\frac{1}{2}}M_{t^2}^{-\frac{1}{2}} \le C.
\end{aligned}
$$

□

In the sequel we step further and with the application of Lemma 3.2, we prove the main tool with respect to the maximal generalized Marcinkiewicz kernel in order to prove that the maximal operator $t_*^\alpha := \sup_{n\in\mathbb{P}}|t_n^\alpha f|$ is quasi-local and then it is of weak type (L^1, L^1).

Lemma 3.3. *[5] Let $u \in G_m^2, a \in \mathbb{N}, y \in I_a(u^1)\times I_a(u^2)$. Then we have*

$$\int_{G_m^2\setminus(I_a(u^1)\times I_a(u^2))}\sup_{n\ge M_{a-c}}|M_n^\alpha(y,x)|d\mu(x) \le C.$$

Proof. For $t^1 \le a-1, t^2 \ge t^1$ and $x \in J_{t^1}(y^1)\times J_{t^2}(y^2)$ by (1.7) and (4.7) it is clear that

$$
\begin{aligned}
&|\Phi(A, n^{(s+1)}+jM_s+k, y, x)| \\
&= |D_{\alpha_1(A,n^{(s+1)}+jM_s+k)}(y^1,x^1)D_{\alpha_2(A,n^{(s+1)}+jM_s+k)}(y^2,x^2)| \\
&\le CM_{t^1}M_{\min(t^2,A)}.
\end{aligned}
$$

This gives,

$$\begin{aligned}
&\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)}\sup_{A\geq a}\sup_{|n|=A}\frac{1}{M_A}\\
&\times\sum_{s=0}^{t^1}\sum_{j=0}^{n_s-1}\left|\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s+k,y,x)\right|d\mu(x)\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)}\sup_{A\geq a-c}\frac{1}{M_A}\\
&\times\sum_{s=0}^{t^1}M_sM_{t^1}M_{\min(t^2,A)}d\mu(x)\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{a-c}\frac{1}{M_{t^1}M_{t^2}}\sup_{A\geq a-c}\frac{1}{M_A}M_{t^1}^2M_{t^2}\\
&+C\sum_{t^1=0}^{a-1}\sum_{t^2=a-c}^{\infty}\frac{1}{M_{t^1}M_{t^2}}M_{t^1}^2\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{a-c}M_a^{-1}M_{t^1}+C\sum_{t^1=0}^{a-1}\sum_{t^2=a-c}^{\infty}M_{t^1}M_{t^2}^{-1}\\
&\leq C.
\end{aligned}$$

This by equality

$$M_n^\alpha(y,x)=\frac{1}{n}\sum_{s=0}^{A}\sum_{j=0}^{n_s-1}\sum_{k=0}^{M_s-1}\Phi(A,n^{(s+1)}+jM_s+k,y,x)$$

and by Lemma 3.2 immediately follows that

$$\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)}\sup_{\{n:|n|=A\geq a-c\}}|M_n^\alpha(y,x)|d\mu(x)\leq C.$$

Now, we prove for each $y\in I_a(u^1)\times I_a(u^2)$ the almost everywhere relation

$$\begin{aligned}
&G_m^2\setminus\big(I_a(u^1)\times I_a(u^2)\big)\\
&\subset\left(\bigcup_{t^1=0}^{a-1}\bigcup_{t^2=t^1}^{\infty}J_{t^1}(y^1)\times J_{t^2}(y^2)\right)\cup\left(\bigcup_{t^1=0}^{a-1}\bigcup_{t^1=t^2}^{\infty}J_{t^1}(y^1)\times J_{t^2}(y^2)\right)\\
&=:J^1(y)\cup J^2(y).
\end{aligned}$$

Let $x = (x^1, x^2) \in G_m^2 \setminus \big(I_a(u^1) \times I_a(u^2)\big)$. Then, either $x^1 \notin I_a(u^1)$ or $x^2 \notin I_a(u^2)$ (or both). Say, x^1 is not element of $I_a(u^1)$. Then, $x^1 \in J_{t^1}(u^1) = J_{t^1}(y^1)$ for some $t^1 < a$.

If $x^2 \in I_a(u^2) = I_a(y^2)$ and $x^2 \neq y^2$, then $x \in J^1(y)$. If $x^1 \in J_{t^1}(u^1) = J_{t^1}(y^1)$ and x^2 is not element of $I_a(u^2) = I_a(y^2)$, then, $x^1 \in J_{t^1}(u^1) = J_{t^1}(y^1)$ and $x^2 \in J_{t^2}(u^2) = J_{t^2}(y^2)$ for some $t^1, t^2 < a$.

For $t^2 \geq t^1$ we have $x \in J^1(y)$ and for $t^1 \geq t^2$ we have $x \in J^2(y)$. This procedure can be done if x^1, x^2 different from y^1 and y^2 respectively. The set of the points $x = (x^1, x^2)$, where either $x^1 = y^1$ or $x^2 = y^2$ is a zero measure set, so this can be supposed and the a.e. relation $G_m^2 \setminus \big(I_a(u^1) \times I_a(u^2)\big) \subset J^1(y) \cup J^2(y)$ is proved for each $y \in I_a(u^1) \times I_a(u^2)$. Therefore, the proof of Lemma 3.3 is complete. □

Corollary 3.4. *[5] Let $y \in G_m, n \in \mathbb{P}$. Then,*

$$||M_n^\alpha(y, \cdot)||_1 \leq C.$$

Proof. Using Lemma 3.3, we have

$$\int_{G_m^2 \setminus \left(I_{|n|}(y^1) \times I_{|n|}(y^2)\right)} |M_n^\alpha(y, x)| d\mu(x) \leq C.$$

Further more, by the conditions (4.6) and (4.7)

$$\begin{aligned} |M_n^\alpha(y, x)| &\leq \frac{1}{n} \sum_{k=0}^{n-1} D_{\alpha_1(|n|,k)}(y^1, x^1) D_{\alpha_2(|n|,k)}(y^2, x^2) \\ &\leq C \frac{1}{n} \sum_{k=0}^{n-1} M_{|n|} M_{|n|} \leq C M_{|n|}^2. \end{aligned}$$

Consequently,

$$\int_{I_{|n|}(y^1) \times I_{|n|}(y^2)} |M_n^\alpha(y, x)| d\mu(x) \leq C.$$

Hence, $||M_n^\alpha(y, \cdot)||_1 \leq C$.

□

3.2 Generalized Marcinkiewicz means and their maximal operators

In this section of the chapter, we check the quasi-locality of the maximal operator t_*^α is quasi-local .

Lemma 3.5. *[5] Let* $f \in L^1(G_m^2)$ *such that* $\operatorname{supp} f \subset I_a(u^1) \times I_a(u^2), \int f d\mu(x) = 0$ *for some* $u \in G_m^2$ *and* $a \in \mathbb{N}$. *Then,*

$$\int_{G_m^2 \backslash (I_a(u^1) \times (I_a(u^2))} t_*^\alpha f(x) d\mu(x) \leq C ||f||_1.$$

Proof. If $|n| \leq a - c$ for some fixed constant $c > 0$ depending only on α_1 and α_2, then we have by (4.7) that $\alpha_1(|n|, k), \alpha_2(|n|, k) < M_a$ for every $k < n$. Consequently, the kernel $M_n^\alpha(y, x)$, which is a linear combination of the product of Vilenkin-like functions χ_k with $k < M_a$, which is $\mathcal{A}_a$ measurable. This implies that

$$\begin{aligned} t_n^\alpha f(y) &= \int_{I_a^2(u)} f(x) M_n^\alpha(x, y) d\mu(x) \\ &= M_n^\alpha(y) \int_{I_a(u^1) \times I_a(u^2)} f(x) d\mu = 0. \end{aligned}$$

That is, $|n| \geq a - c$ can be supposed. By the Theorem of Fubini, by Lemma 3.3 and by the fact that for kernel $|M_n^\alpha(y, x)| = |M_n^\alpha(x, y)|$ we get,

$$\begin{aligned} & \int_{G_m^2 \backslash I_a^2(u)} t_*^\alpha f \\ & = \int_{G_m^2 \backslash I_a^2(u)} \sup_{|n| \geq a-c} |t_n^\alpha f| d\mu(x) \\ & = \int_{G_m^2 \backslash I_a^2(u)} \sup_{|n| \geq a-c} \left| \int_{I_a^2(u)} f(x) M_n^\alpha(y, x) d\mu(x) \right| d\mu(y) \\ & \leq \int_{I_a^2(u)} |f(x)| \int_{G_m^2 \backslash I_a^2(u)} \sup_{|n| \geq a-c} |M_n^\alpha(y, x)| d\mu(y) | d\mu(x) \\ & \leq C \int_{I_a^2(u)} |f(x)| d\mu(x) = C ||f||_1. \end{aligned}$$

This completes the proof of Lemma 3.5. ☐

Theorem 3.6. *[5] The operator* t_*^α *is of weak type* (L^1, L^1) *and it is also of type* (L^p, L^p) *for all* $1 < p \leq \infty$.

Proof. Now, we know that the operator t_*^α is of type (L^∞, L^∞) which is given by Corollary 3.4 and it is quasi-local by Lemma 3.5. Consequently, to prove that operator t_*^α is of weak type (L^1, L^1) is nothing else but to follow the standard argument (see e.g. [44]). Finally, the interpolation Lemma of the Marcinkiewicz gives that it is also of type (L^p, L^p) for all $1 < p \leq \infty$. □

3.3 Almost everywhere convergence of the generalized Marcinkiewicz means

Using the Lemmas in section 3.1 and section 3.2, we get the following almost everywhere convergence result.

Theorem 3.7. *[5] Let* α *satisfy conditions* (4.6) *and* (4.7)*. Then, we have* $t_n^{\alpha} f \to f$ *for each* $f \in L^1(G_m^2)$ *a.e. with respect to every bounded Vilenkin-like system.*

Proof. The proof of Theorem 3.7 is just a standard consequence of the fact that the maximal operator t_*^{α} is of weak type (L^1, L^1), the fact that it holds for each two-dimensional Vilenkin-like polynomial (linear combinations of $\chi_k(x^1)\chi_n(x^2)$) and the fact that the set of two-dimensional Vilenkin-like polynomials is dense in $L^1(G_m^2)$. This density property comes from the behavior of the one dimensional kernel function D_{M_n}. That is, it is either M_n or zero. $\square$

Finally, we give an application of Theorem 3.7. Before this, the following Corollary is given.

Corollary 3.8. *[5] Let* (a_n) *be a lacunary sequence of positive reals, i.e.* $a_{n+1} \geq a_n q$ *for some* $q > 1$ *(*$n \in \mathbb{N}$ *and* $\alpha_j(n,k) \leq Ca_n$ *(*$k < a_n$, $j = 1,2$*)) (modified version of condition (4.7). Then for every integrable function* $f \in L^1(G_m^2)$ *we have*

$$\frac{1}{a_n} \sum_{k=0}^{a_n - 1} S_{\alpha_1(n,k),\alpha_2(n,k)} f(x) \to f(x)$$

for a.e. $x \in G_m^2$.

Proof. Let b_n be defined as $M_{b_n - 1} \leq a_n < M_{b_n}$ (that is, $b_n = |a_n| + 1$) and

$$\tilde{\alpha}_j(b_n, k) = \begin{cases} \alpha(n,k), & \text{for } 0 \leq k < a_n, \\ k, & \text{if } a_n \leq k < M_{b_n} \end{cases} \qquad (j = 1,2).$$

Then, $\tilde{\alpha}$ satisfies conditions (4.6) (trivially) and (4.7) since for $k < a_n$, $\tilde{\alpha}_j(b_n, k) = \alpha_j(n,k) \leq Ca_n \leq CM_{b_n}$. By Theorem 3.7 it follows that for the maximal operator $t_*^{\tilde{\alpha}} f := \sup |t_n^{\tilde{\alpha}} f|$ we have $\mu\left\{t_*^{\tilde{\alpha}} f \geq \lambda\right\} \leq C\|f\|_1/\lambda$

for all $f \in L^1(G_m^2)$ and $\lambda > 0$. Since

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)}f = \frac{M_{b_n}}{a_n}\frac{1}{M_{b_n}}\sum_{k=0}^{M_{b_n}-1} S_{\tilde\alpha_1(b_n,k),\tilde\alpha_2(b_n,k)}f$$
$$-\frac{M_{b_n}}{a_n}\frac{1}{M_{b_n}}\sum_{k=a_n}^{M_{b_n}-1} S_{k,k}f$$

and consequently,

$$|t^{\alpha}_{a_n}f| \le C|t^{\tilde\alpha}_{M_{b_n}}f| + C|t^{\text{id}}_{M_{b_n}}f| + C|t^{\text{id}}_{a_n}f|,$$

then (id denotes the "identical function") that is

$$\text{id}(n,k) = (k,k), \quad t^{\alpha}_* f \le Ct^{\tilde\alpha}_* f + Ct_* f.$$

The ordinary maximal Marcinkiewicz operator is of weak type (L^1, L^1) (see e.g. [27]) and this by standard argument [44] completes the proof of this corollary. □

In the sequel we give an application of the Corollary above. The triangular partial sums of the 2-dimensional Fourier series and the triangular Dirichlet kernels (with respect to the Vilenkin-like system χ) are defined as

$$S^{\triangle}_k f(x^1,x^2) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \hat f(i,j)\chi_i(x^1)\chi_j(x^2),$$
$$D^{\triangle}_k(x^1,x^2) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \chi_i(x^1)\chi_j(x^2).$$

The Fejér means of the triangular partial sums of the two-dimensional integrable function f (see e.g. [34]) are

$$\sigma^{\triangle}_n f := \frac{1}{n}\sum_{k=0}^{n-1} S^{\triangle}_k f.$$

For the trigonometric system Herriot proved [36] the a.e. (and norm) convergence $\sigma^{\triangle}_n f \to f$ ($f \in L^1$). His method can not be adopted for the Vilenkin system, since for the time being there is no kernel formula available for these

systems. The first result in this a.e. convergence issue of triangular means is due to Goginava and Weisz [34]. They proved for the Walsh-Paley system and each integrable function the a.e. convergence relation $\sigma_{2^n}^{\triangle} f \to f$. This result for the whole sequence of the triangular mean operators in the Walsh case is given by Gát [25].
In the Vilenkin-like situation there is nothing proved yet. By the Corollary above, (Corollary 3.8) we prove for bounded Vilenkin-like systems:

Theorem 3.9. *[5] For every lacunary sequence* (a_n) *(that is,* $a_{n+1} \geq q a_n$, $q > 1$*) we have the a.e. convergence* $\sigma_{a_n}^{\triangle} f \to f$ *for each* $f \in L^1(G_m^2)$.

To demonstrate the proof of this, see some calculations below [25] relationship between the triangle Kernel and the one dimensional Dirichlet kernels.

$$K_n^{\triangle}(y^1, y^2, x^1, x^2) = \frac{1}{n} \sum_{k=0}^{n-1} D_k^{\triangle}(y^1, y^2, x^1, x^2)$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \chi_i(y^1, x^1) \chi_j(y^2, x^2)$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \sum_{i=0}^{k-1} \chi_i(y^1, x^1) D_{k-i}(y^2, x^2)$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \sum_{i=1}^{k} \chi_{k-i}(y^1, x^1) D_i(y^2, x^2)$$

$$= \frac{1}{n} \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \chi_{k-i}(y^1, x^1) D_i(y^2, x^2)$$

$$= \frac{1}{n} \sum_{i=1}^{n-1} D_{n-i}(y^1, x^1) D_i(y^2, x^2)$$

$$= \frac{1}{n} \sum_{i=1}^{n-1} D_i(y^1, x^1) D_{n-i}(y^2, x^2)$$

which is a generalized Marcinkiewicz kernel discussed (see Corollary 3.8: $\alpha_1(|n|, k) = k$, $\alpha_2(|n|, k) = a_n - k$).

Chapter 4

Summary

In this part, we included the results obtained in [5] and [6] and this dissertation is mainly constructed based on them.

The idea of Cesàro means with variable parameters of numerical sequences is due to Kaplan [13]. In 2007 Akhobadze [3] introduced the notion of (C, α) means of trigonometric Fourier series with variable parameter setting. Fine [24] proved this for Walsh-Paley system for constant sequences. On the rate of convergence of (C, α) means in the constant sequences case see the paper of Fridli [8]. For the two dimensional case see the paper of Goginava [11]. The almost everywhere convergence of this summability method for a constant parameter in the quadrilateral partial sums of double Vilenkin-Fourier series was proved by Gát and Goginiva in 2006 [7]. In 2008 Abu Joudeh and Gát [1] proved for varying-parameter setting in the case of Walsh-Paley system. The a.e. divergence of Cesàro means in the variable parameter setting of Walsh-Fourier series was investigated by Tetunashvili [17].
In [10] Lemma 8 about the fejér kernel with a constant $\alpha = 1$ is proved with respect to Vilenkin system.

The aim of **Chapter 2** was to prove the almost everywhere convergence of (C, α) means in the variable parameter setting with respect to the one dimensional bounded Vilenkin system. That is,

$$\sigma_n^{\alpha_n} f \to f \quad \text{when} \quad n \to \infty$$

where α_n is not constant but it is varying in the open interval $(0, 1)$ for all $n \in \mathbb{N}_{\alpha, q}$.

In the first part of this chapter, we proved Lemmas on (C, α) kernel functions which are important tools. The first and important Lemma was proved that the inequality

$$|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}$$

holds true for $\alpha_a \in (0,1)$, $n, a \in \mathbb{N}$. The other very important Lemma is proved that,

$$|K_n^{\alpha_n}| \leq \tilde{K}_n^{\alpha_n}$$

holds true where $0 < \alpha_n < 1$, $n \in \mathbb{N}$.

In the second part of this Chapter, we focused on the maximal operators of (C, α) means. We proved the quasi-locality of the maximal operators $\tilde{t}_*^{\alpha_a} := \sup_{n,\,a \in \mathbb{N}} |\tilde{t}_n^{\alpha_a}|$. Similarly, we also proved the quasi-locality of the maximal operator $\tilde{\sigma}_{*,\,q}^{\alpha} := \sup_{n \in \mathbb{N}_{\alpha,\,q}} |\tilde{\sigma}_n^{\alpha}|$. The other very important and proved tool is the Lemma which states that $\sigma_{*,\,q}^{\alpha}$ is of weak type (L^1, L^1). We proved this after we gave the proof of the statement that maximal operator $\tilde{\sigma}_{*,\,q}^{\alpha}$ is of type (L^∞, L^∞) and of weak type (L^1, L^1). Having all these important tools proved, we finally gave the proof of the main result.

The following notations as well as definitions of functions and operators are basic notions of Vilenkin system.
A natural generalization on the Walsh Paley system is called Vilenkin system. These are orthonormal systems which were introduced by N.Ya. Vilenkin in 1947 (see [18]). First we give a brief introduction to the theory of Vilenkin orthonormal systems.

Denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{P}$ the set of positive integers, respectively. Denote $m := (m_k,\ k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and Z_{m_k} the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, 2, ..., m_k - 1\}$, with the group operation $\mod m_k$ addition. Since the group is discrete, every subset is open.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, for $k \in \mathbb{N}$ be the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. This allows one to say that the sequence $(n_0, n_1, ...)$ is the expansion of n with respect to m. We often use the following notations.

Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$, for any $n > 0$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$.

The normalized Haar measure μ_k on Z_{m_k} is defined by $\mu_k(\{j\}) := \frac{1}{m_k} (j \in \{0, 1, ..., m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then, every $x \in G_m$ can be represented by a sequence $x = (x_i,\ i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$.

The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this dissertation we discuss bounded Vilenkin groups, only.

The Vilenkin group is metrizable in the following way:

$$d(x,y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals: $I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n\}$ for $x \in G_m$, $n \in \mathbb{P}$. Let $0 = (0,\, i \in \mathbb{N}) \in G_m$ denote the null element of G_m and $I_n(0) := I_n$, $\bar{I}_n = G_m \backslash I_n$.

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) $(1 \leq p \leq \infty)$, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\ (n \in \mathbb{N})$. We say that an operator $T : L^1 \to L^0$ ($L^0(G_m)$ is the space of measurable functions on G_m) is of type (L^p, L^p) (for $1 \leq p \leq \infty$) if $\|Tf\|_p \leq C_p\|f\|_p$ for all $f \in L^p(G_m)$ and the constant C_p depends only on p. We say that T is of weak type (L^1, L^1) if $\mu(|Tf| > \lambda) \leq C\|f\|_1/\lambda$ for all $f \in L^1(G_m)$ and $\lambda > 0$.

Next, we introduce an orthonormal system we call Vilenkin system on G_m.

For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp(2\pi\imath\frac{x_k}{m_k}) \quad (x \in G_m,\ \imath := \sqrt{-1},\ k \in \mathbb{N}).$$

The n^{th} Vilenkin function is defined as

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} (n \in \mathbb{N}).$$

The system $\psi := (\psi_n,\, n \in \mathbb{N})$ is called a Vilenkin system.

Each ψ_n is a character of G_m and all the characters of G_m are of the this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then $\psi_{k\oplus n} = \psi_k \psi_n$, $\psi_n(x+y) = \psi_n(x)\psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1 \quad (k,\, n \in \mathbb{N},\, x,\, y \in G_m)$.

The Dirichlet and the Fejér or (C,1) kernels on the Vilenkin system are defined and denoted as follows respectively as,

$$D_n := \sum_{k=0}^{n-1} \psi_k,\; K_n := \frac{1}{n+1} \sum_{k=0}^{n} D_k.$$

For $f \in L^1(G_m),$ the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the (C, α) kernels and means with respect to the Vilenkin system ψ are defined as follows

$$\begin{aligned}
\hat{f}(n) &:= \int_{G_m} f \bar{\psi}_n d\mu, \\
S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \\
\sigma_n^{\alpha} f &:= \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \\
K_n^{\alpha} &:= \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k, \\
\sigma_n f &:= \sigma_n^1 f,\; K_n := K_n^1.
\end{aligned}$$

It is known in [20] that,

$$A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1},\; A_k^{\alpha} - A_{k+1}^{\alpha} = \frac{-\alpha A_k^{\alpha}}{k+1}$$

where A_k^{α} is defined for all possible values of $\alpha \in \mathbb{R} \setminus \{-1,\, -2,\, ...,\, -k\}$ as

$$A_k^{\alpha} = \frac{(\alpha+1)(\alpha+2)...(\alpha+k)}{k!},$$

α may also be a sequence $\alpha = (\alpha_n)$.
It is known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x) \quad (n \in \mathbb{N},\, f \in L^1(G_m)).$$

It is also well-known that (see [4], [7])

$$D_{M_n}(y,x) = \begin{cases} M_n, & \text{if } y \in I_n(x) \\ 0, & \text{if } y \notin I_n(x), \end{cases}$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f d\mu = E_n f(x)$$

$$(f \in L^1(G_m),\, n \in \mathbb{N}),$$

$$D_{sM_n} = D_{M_n} \sum_{k=0}^{s-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s-1} r_n^k$$

$$(\text{for} \quad s \le m_n).$$

Define the two variable function $P(n,\alpha) := \sum_{i=0}^{\infty} n_i M_i^{\alpha}$ for $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$. For example $P(n,1) = n$. Besides, set for sequences $\alpha = (\alpha_n)$ and positive reals $q,$ the subset of natural numbers

$$\mathbb{N}_{\alpha,q} := \left\{ n \in \mathbb{N} : \frac{P(n,\alpha_n)}{n^{\alpha_n}} \le q \right\}. \tag{4.1}$$

For a sequence α such that $0 < \alpha_0 \le \alpha_n < 1$ we have $\mathbb{N}_{\alpha,\,q} = \mathbb{N}$ for some q depending only on α_0. We remark that $M_n \in \mathbb{N}_{\alpha,\,q}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $q \ge 1$.
In the sequel, C denotes an absolute constant and C_q another one which may depend only on q.
We used the following Lemma in the proof of our results and is put here without proof.

Lemma 4.1. *[10]*

$$\int_{\bar{I}_k} \sup_{j \ge M_k} \Big| K_j \Big| d\mu \le C.$$

Next, we introduced the following functions and operators in the variable parameter setting ($n \in \mathbb{N}$, $0 < \alpha_a < 1$).

$$
\begin{aligned}
T_n^{\alpha_a} &= \frac{1}{A_n^{\alpha_a}} \sum_{k=0}^{n_B M_B - 1} A_{n-k}^{\alpha_a - 1} D_k, \\
\tilde{T}_n^{\alpha_a} &:= \frac{n_B D_{M_B}}{A_n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 1} A_{n_{(B)}+j}^{\alpha_a - 1} \\
&+ \frac{\alpha_a(1-\alpha_a)}{n^{\alpha_a}} \sum_{j=0}^{n_B M_B - 2} \frac{j+1}{(n_{(B)}+j)^{2-\alpha_a}} \left| K_j \right| \\
&+ \alpha_a \left| K_{n_B M_B - 1} \right|, \\
t_n^{\alpha_a} f(y) &:= \int_{G_m} f(x) T_n^{\alpha_a}(y-x) d\mu(x), \\
\tilde{t}_n^{\alpha_a} f(y) &:= \int_{G_m} f(x) \tilde{T}_n^{\alpha_a}(y-x) d\mu(x).
\end{aligned}
$$

We used the following Lemma played the vital role to proof the next Lemma.

Lemma 4.2. *[6] For* $n, a \in \mathbb{N}$, $M_B \leq n < M_{B+1}$, $|n| = B$, $\alpha_a \in (0,1)$. *Then,*

$$|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}.$$

In the variable parameter setting, the following results about the maximal operator were investigated. We proved the following Lemma which means that the maximal operator $\tilde{t}_*^{\alpha_a} := \sup_{n,\, a \in \mathbb{N}} |\tilde{t}_n^{\alpha_a}|$ is quasi-local.

Lemma 4.3. *[6] Let* $1 > \alpha_a > 0 (a \in \mathbb{N})$, $f \in L^1(G_m)$ *such that*

$$\mathit{supp} f \subset I_k(u), \int_{I_k(u)} f d\mu(x) = 0$$

for some m-adic interval $I_k(u)$. *Then, we have*

$$\int_{\bar{I}_k(u)} \sup_{n,\, a \in \mathbb{N}} |\tilde{t}_n^{\alpha_a} f| d\mu(x) \leq C \|f\|_1.$$

In the following Corollary it is proved that operators $t_n^{\alpha_a}$, $\tilde{t}_n^{\alpha_a}$ are of type (L^1, L^1) and (L^∞, L^∞) uniformly in n.

Corollary 4.4. *[6] Let* $1 > \alpha_a > 0$*.Then, we have*

$$\|T_n^{\alpha_a}\|_1 \leq \|\tilde{T}_n^{\alpha_a}\|_1 \leq C;$$
$$\|t_n^{\alpha_a} f\|_1,\ \|\tilde{t}_n^{\alpha_a} f\|_1 \leq C\|f\|_1$$

and

$$\|t_n^{\alpha_a} g\|_\infty,\ \|\tilde{t}_n^{\alpha_a} g\|_\infty \leq C\|g\|_\infty$$

for all natural numbers a, n *where* C *is some absolute constant and* $f \in L^1$, $g \in L^\infty$. *That is, operator* $t_n^{\alpha_a}$, $\tilde{t}_n^{\alpha_a}$ *are of type* $(L^1,\, L^1)$ *and* $(L^\infty,\, L^\infty)$ *uniformly in* n.

For the general case by considering $n \in \mathbb{N}_{\alpha,q}$, where $\mathbb{N}_{\alpha,q}$ is defined in 4.1, we proved in the next Lemma. That is, the maximal operator $\tilde{\sigma}_{*,\,q}^{\alpha} := \sup_{n \in \mathbb{N}_{\alpha,\,q}} |\tilde{\sigma}_n^{\alpha_n}|$ is quasi-local. We got this by the investigation of kernel functions, its maximal function on the Vilenkin group by making a hole around zero and some quasi-locality issues.

Lemma 4.5. *[6] Let* $0 < \alpha_n < 1$, $f \in L^1(G_m)$ *such that* $suppf \subset I_k(u)$, $\int_{I_k(u)} f d\mu = 0$ *for some m-adic interval* $I_k(u)$. *Then we have*

$$\int_{G_m \setminus I_k(u)} \tilde{\sigma}_{*,\,q}^{\alpha} f d\mu \leq C_q \|f\|_1$$

where constants C_q *can depend only on* q.

For the general case of $n \in \mathbb{N}_{\alpha,q}$ we proved the Lemma in the sequel that operator $\tilde{\sigma}_{*,\,q}^{\alpha}$ is of type $(L^\infty,\, L^\infty)$ and weak type $(L^1,\, L^1)$; $\sigma_{*,\,q}^{\alpha}$ is of weak type $(L^1,\, L^1)$.
Define (C, α_n) mean as follows

$$\begin{aligned}\sigma_n^{\alpha_n} f(x) &:= \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k(x)\\ &= \int_{G_m} f(y) K_n^{\alpha_n}(x-y) d\mu(y)\end{aligned}$$

where $K_n^{\alpha_n}$ is the n^{th} kernel function. Besides, introduce the following Kernel functions and operators

$$\begin{aligned}\tilde{K}_n^{\alpha_n} &:= \left|\tilde{T}_n^{\alpha_n}\right| + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} n_l D_{M_l} + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}|,\\ \tilde{\sigma}_n^{\alpha_n} f(y) &:= \int_{G_m} f(x) \tilde{K}_n^{\alpha_n}(y-x) d\mu(x).\end{aligned}$$

We define maximal operators as follows.

$$\sigma_{*,\,q}^{\alpha} f := \sup_{n \in \mathbb{N}_{\alpha,\,q}} |\sigma_n^{\alpha_n} f|.$$

Lemma 4.6. *[6] The operator $\tilde{\sigma}_{*,q}^{\alpha}$ is of type $(L^{\infty},\, L^{\infty})$ and weak type $(L^1,\, L^1)$; $\sigma_{*,q}^{\alpha}$ is of weak type $(L^1,\, L^1)$.*

Following the above Lemmas and Corollaries, the almost everywhere convergence of the (C, α_n) means was proved.

Theorem 4.7. *[6] Let $0 < \alpha_n < 1$. Let $f \in L^1(G_m)$. Then*

$$\sigma_n^{\alpha_n} f \longrightarrow f \quad a.e. \quad if \quad n \longrightarrow \infty,\, n \in \mathbb{N}_{\alpha,\,q}.$$

The main aim of **Chapter 3** was to give a class of functions α for which the a.e. convergence relation $t_n^\alpha f \to f$ for each integrable two variable function with respect to two dimensional bounded Vilenkin-like systems holds. In this chapter α is a class of function defined below and it is different from α which was defined in chapter two.

Besides, we gave an application of the main result, that is, Theorem 4.17 with respect to triangular summability of Vilenkin-like-Fourier series. This concept of Marcinkiewicz-like kernels and means is due to Gát [28].

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [38] proved that for all $f \in L\log L([0,2\pi]^2)$ the a.e. relation

$$\frac{1}{n}\sum_{j=1}^{n} S_{j,j}f \to f. \tag{4.2}$$

holds as $n \to \infty$.

Zhizhiashvili [49] improved this result for $f \in L([0,2\pi]^2)$. Dyachenko [23] proved this result for dimensions greater than 2.

In 2003 Goginava [33] proved this result with respect to the multiple Walsh-Paley system. The case $d = 2$ is due to Weisz [19]. This result for bounded Vilenkin systems due to Gát [27]. In 2012 Gát [28] proved this result for generalized Marcinkiewicz means with respect to the two dimensional Walsh system and in 2016 [29] for bounded two-dimensional Vilenkin systems. We generalized the result of Gát [29] with respect to two-dimensional Vilenkin-like systems.

Next, we introduced on a Vilenkin group G_m an orthonormal system called a Vilenkin-like system (or $\psi\alpha$ system) . Vilenkin-like orthonormal systems were introduced by György Gát in 1991 (see [30]) and it is defined as follows.

Let functions

$$\psi_n, \alpha_n, \alpha_k^j : G_m \to \mathbb{C}\ (n, j, k \in \mathbb{N})$$

satisfy :

$$\alpha_k^j \text{ is measurable with respect to } \mathcal{A}_k\ (j, k \in \mathbb{N}), \tag{4.3}$$

$$|\alpha_k^j| = \alpha_k^j(0) = \alpha_0^j = \alpha_k^0 = 1\ (j, k \in \mathbb{N}), \tag{4.4}$$

$$\alpha_n := \prod_{k=0}^{\infty} \alpha_k^{n^{(k)}}, \psi_n := \prod_{k=0}^{\infty} r_k^{n_k},\ n^{(k)} := \sum_{i=k}^{\infty} n_i M_i\ (n \in \mathbb{N}). \tag{4.5}$$

Let $\chi_n := \psi_n \alpha_n$ $(n \in \mathbb{N})$. The system $\chi := \{\chi_n,\ n \in \mathbb{N}\}$ is called a Vilenkin-like (or $\psi\alpha$ system) (see [30], [22]).
We also introduced the two-variable functions:

$$\chi_n(y,x) := \chi_n(y)\bar{\chi}_n(x), \quad r_n(y,x) := r_n(y)\bar{r}_n(x) \quad (n \in \mathbb{N}, y, x \in G_m).$$

This will not cause misunderstand by clearly making a difference between $\chi_n(x)$ and $\chi_n(y,x)$.
The two-dimensional generalized Marcinkiewicz kernels and Marcinkiewicz means, with respect to the two-dimensional Vilenkin-like system are defined as follows: Let

$$\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$$

be a function. (From now functions α_1, α_2 play the role of indices. We know that the function α_n appeared in the definition of the Vilenkin-like (or $\psi\alpha$ system), but this will not cause any misunderstanding.)
Define the following generalized Marcinkiewicz kernels and means respectively:

$$M_n^{\alpha}(y,x) := \frac{1}{n}\sum_{k=0}^{n-1} D_{\alpha 1(|n|,k)}(y^1,x^1) D_{\alpha 2(|n|,k)}(y^2,x^2),$$
$$t_n^{\alpha} f := f * M_n^{\alpha} \quad (f \in L^1(G_m^2),\ n \in \mathbb{P}).$$

To investigate this the following properties played a prominent role ($Car(B)$ denotes cardinality of the set B),

$$Car\{l \in \mathbb{N} : \alpha_j(|n|,l) = \alpha_j \quad (|n|,k),\ l < n\} \le C \tag{4.6}$$

$$(k < n,\ n \in \mathbb{P},\ j = 1,2),$$

$$\max\{\alpha_j(|n|,k) :\ k < n\} \le Cn \quad (n \in \mathbb{P},\ j = 1,2). \tag{4.7}$$

Our next aim was to prove that the operator $t_*^{\alpha} := \sup_{n \in \mathbb{P}} \mid t_n^{\alpha} \mid$ is of weak type (L^1, L^1). In order to do this we needed a sequence of Lemmas. All those Lemmas are base for the proof of the Theorems. The Walsh-Paley version of those Theorems are due to Gát [28]. That is, we generalized a result of Gát. Moreover, techniques of papers [28] and [29] have also been used in the proof of the forthcoming Lemmas.

Denote for $k \in \mathbb{N}, x \in G_m,\ J_k(x) = I_k(x) \setminus I_{k+1}(x)$ and recall also that

$$n^{(s)} = \sum_{k=s}^{\infty} n_k M_k,\ (n,\ s \in \mathbb{N}),\ n^{(0)} = n,\ n^{(|n|+1)} = 0.$$

Lemma 4.8. *[30] Let $t, n, l \in \mathbb{N}, u \in G_m$. Then we have that*

$$\int_{I_{t+1}(u)} \chi_n(x)\bar{\chi}_l(x) d\mu(x) \neq 0$$

implies $n^{(t+1)} = l^{(t+1)}$.

For $x, y \in G_m^2, A, n, s, j, k \in \mathbb{N}$ let

$$\Phi(A, n^{(s+1)} + jM_s + k,\ y, x) = D_{\alpha_1(A,\, n^{(s+1)}+jM_s+k)}(y^1, x^1)$$
$$\times D_{\alpha_2(A,\, n^{(s+1)}+jM_s+k)}(y^2, x^2).$$

The next Lemma is the base for the subsequent one.

Lemma 4.9. *[5] Let $t^1, t^2, A, s \in \mathbb{N}, s \leq A$ and $y \in G_m^2$. Then,*

$$\int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{|n|=A} \Big| \sum_{j=0}^{n_s-1} \sum_{k=0}^{M_s-1} \Phi(A,\ n^{(s+1)} + jM_s + k,\ y, x) \Big| d\mu(x)$$
$$\leq C(M_A M_{t^1})^{\frac{1}{2}}.$$

Lemma 4.10. *[5] Let $a \in \mathbb{N}, y \in G_m^2$. Then,*

$$\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}(y^1)\times J_{t^2}(y^2)} \sup_{A\geq a} \sup_{|n|=A} \frac{1}{M_A} \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1}$$
$$\Big| \sum_{k=0}^{M_s-1} \Phi(A, n^{(s+1)} + jM_s + k, y, x) \Big| \, d\mu(x) \leq C.$$

Using these two Lemmas we stepped further and proved the Lemma which is the main tool with respect to the maximal generalized Marcinkiewicz kernel in order to prove that the maximal operator $t_*^{\alpha} := \sup_{n\in\mathbb{P}} |t_n^{\alpha} f|$ is quasi-local and then it is of weak type (L^1, L^1).

Lemma 4.11. *[5] Let* $u \in G_m^2, a \in \mathbb{N}, y \in I_a(u^1) \times I_a(u^2)$*. Then we have*

$$\int_{G_m^2 \setminus (I_a(u^1) \times I_a(u^2))} \sup_{n \geq M_{a-c}} |M_n^\alpha(y,x)| d\mu(x) \leq C.$$

Corollary 4.12. *[5] Let* $y \in G_m, n \in \mathbb{P}$*. Then,*

$$||M_n^\alpha(y, \cdot)||_1 \leq C.$$

We established the following lemma that maximal operator t_*^α is quasi-local.

Lemma 4.13. *[5] Let* $f \in L^1(G_m^2)$ *such that*

$$\operatorname{supp} f \subset I_a(u^1) \times I_a(u^2), \int f d\mu(x) = 0$$

for some $u \in G_m^2$ *and* $a \in \mathbb{N}$*. Then,*

$$\int_{G_m^2 \setminus (I_a(u^1) \times (I_a(u^2)))} t_*^\alpha f(x) d\mu(x) \leq C||f||_1.$$

Theorem 4.14. *[5] The operator* t_*^α *is of weak type* (L^1, L^1) *and it is also of type* (L^p, L^p) *for all* $1 < p \leq \infty$*.*

Theorem 4.15. *[5] Let* α *satisfy conditions* 4.6 *and* 4.7*. Then, we have*

$$t_n^\alpha f \to f$$

for each $f \in L^1(G_m^2)$ *a.e with respect to every bounded Vilenkin-like system.*

Finally, we gave an application of this Theorem. Before this the following Corollary was given:

Corollary 4.16. *[5] Let* (a_n) *be a lacunary sequence of positive reals, i.e.* $a_{n+1} \geq a_n q$ *for some* $q > 1$ *(*$n \in \mathbb{N}$*) and* α *satisfy conditions* 4.4 *and* $\alpha_j(n,k) \leq C a_n$ *(*$k < a_n$, $j = 1, 2$*) (modified version of condition* 4.7*). Then for every integrable function* $f \in L^1(G_m^2)$ *we have*

$$\frac{1}{a_n} \sum_{k=0}^{a_n - 1} S_{\alpha_1(n,k), \alpha_2(n,k)} f(x) \to f(x)$$

a.e. for $x \in G_m^2$*.*

In the sequel we gave an application of this Corollary. First we defined the triangular partial sums of the 2-dimensional Fourier series and the triangular Dirichlet kernels (with respect to the Vilenkin-like system χ) as

$$S_k^{\triangle} f(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i,j) \chi_i(x^1) \chi_j(x^2),$$

$$D_k^{\triangle}(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \chi_i(x^1) \chi_j(x^2).$$

The Fejér means of the triangular partial sums of the two-dimensional integrable function f (see e.g. [34]) are

$$\sigma_n^{\triangle} f := \frac{1}{n} \sum_{k=0}^{n-1} S_k^{\triangle} f.$$

For the trigonometric system Herriot proved [36] the a.e. (and norm) convergence $\sigma_n^{\triangle} f \to f$ $(f \in L^1)$. His method can not be adopted for the Vilenkin system, since for the time being there is no kernel formula available for these systems. The first result in this a.e. convergence issue of triangular means is due to Goginava and Weisz [34]. They proved for the Walsh-Paley system and each integrable function the a.e. convergence relation $\sigma_{2^n}^{\triangle} f \to f$. This result for the whole sequence of the triangular mean operators in the Walsh case is given by the first author [25]. In the Vilenkin-like situation there is nothing proved yet. By the Corollary above, we proved for bounded Vilenkin-like systems:

Theorem 4.17. *[5] For every lacunary sequence* (a_n) *(that is,* $a_{n+1} \geq q a_n$, $q > 1$*) we have the a.e. convergence*

$$\sigma_{a_n}^{\triangle} f \to f$$

for each $f \in L^1(G_m^2)$*.*

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7. The 14th International Student Conference on Analysis: in Sikfőkút, Hungary, 3-6 of February, 2018.
8. Regular Department Research Seminars: Debrecen, Hungary, 2017-2021.
9. The 19th International Conference on Functional Equations and Inequalities, Pedagogical University of Krakow and Stefan Banach International Mathematical Center, Poland, September 12-18, 2021(online).

LIST OF SYMBOLS

A_n- σ algebra generated by the sets of I_n 5
D_n- n^{th} Dirichlet kernel 5
E_n- Conditional expectation operator with respect to A_n 55
$\hat{f}(n_1, n_2)-$Two dimensional Fourier coefficients 11
G_m- Vilenkin group 4
$G_m \times G_{\tilde{m}}-$ Two dimensional Vilenkin group 11
K_n- n^{th} Fejér kernel 5
$m := (m_k : k \in \mathbb{N})-$ Sequence of positive integers such that $m_k \geq 2$ 32
$k \oplus n-$ m-adic addition of k, $n \in \mathbb{N}$ 1
$Z_{m_k}-$ Discrete cyclic group of order m_k 50
$\hat{f}(n)-$ n^{th} Fourier coefficient 6,10
$I_n(x)-$ n^{th} Interval of the Vilenkin Group G_m 32
$L^p(G_m)-$ usual Lebesgue space on the Vilenkin group G_m 5
$M_n^{\alpha}-$ Marcinkiewicz kernel 11
r_k- k^{th} Rademacher functions 2,5
$S_n f-$ n^{th} Partial sums of Fourier series 6,10
$S_{n_1,n_2}-$ Rectangular partial sums of the Fourier series 11
$t_n^{\alpha}-$ Marcinkiewicz means 11
μ_k- Normalized Haar Measure on Z_{m_k} 5
ψ_n- n^{th} Vilenkin function (character of G_m) 5
$\psi-$ Vilenkin system 4
$\sigma_n^{\alpha}-$ n^{th} (C, α) mean 6,10
$\sigma_\star^{\alpha} f-$ Maximal operator of (C, α) mean 2,5
$\Psi\psi-$ Vilenkin-like system 8

LIST OF NOTIONS

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   Miskolc Math. Notes. 21 (2), 823-840, 2020. ISSN: 1787-2405.
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1. Gát, G., **Tilahun, A.**: On almost everywhere convergence of the generalized Marcienkiwicz means with respect to two dimensional Vilenkin-like systems.
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