Investigations in the Theory of Functional Equations

doktori (PhD) értekezés

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Tanúsítom, hogy Gődényné Hajdu Gabriella doktorjelölt 1996–2000 között a fent megnevezett Doktori Iskola Matematika programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységgel meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

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Contents

Abstract

This PhD dissertation contains new results in the theory of functional equations. This field of mathematics is so extended that we make an attempt to investigate only small but interesting parts of this ramifying theory. The following two parts quite differ from each other, showing that functional equations turn up almost everywhere.

In the first half of the dissertation we start out from an identity of the brilliant mathematician Ramanujan. Among the many, sometimes amazing results in number theory, Z. Daróczy found one he thought interesting and set up a functional equation generalising the original identity. He asked what were the general solutions of this equation on the set $\mathbb{Z}$ of the integers. Giving the solutions of the equation on $\mathbb{Z}$ (Section 2), we examined another abstract algebraic structure and also found the answer to his question (Section 3).

Part I belongs rather to number theory and algebra than to analysis. As it happens so often in number theory, the problem presented in Section 2 is formulated in a way that the reader can understand it without having deep knowledge of mathematics.

The problem examined in Section 3 is connected to abstract algebra and the theory of polynomials on groups. Since the reader might not be familiar with this topic, a brief summary of the theory is also given.

Part II is more "real" analysis, in both senses of meaning. Firstly, it deals with problems on the set $\mathbb{R}$ of the real numbers. Secondly, it is really analysis, with functions, monotonity, continuity, etc. But similarly to the first half, the basic problem of the second part can be easily understood.

These sections mainly deal with extension theorems for Matkowski–Sutô type problems and material connected with them. This problem has a history of almost a century long. Sutô started the investigations in 1914, then it was re-discovered by Matkowski in 1999. The original problem for quasi–arithmetic means was solved completely by Daróczy and Páles in 2001. In the proof they employ an extension theorem of Daróczy, Maksa, and Páles.

Since Matkowski–Sutô type problems can be formulated for other classes of means as well, extension theorems in these cases are also important and interesting. Moreover, they immediately generalise the existing results.

In Section 4 we give the basic definitions and preliminaries of the problem and also the original extension theorem. We also show another aspect, the Gauss–composition.

The extension theorems for weighted quasi–arithmetic means and quasi–arithmetic means of order α with their proofs are presented in Section 5.

Section 6 provides a tool to prove the extension theorems, the complementary means. We think that these theorems and proofs are interesting not only because of their use in the proofs of the previous section but also in their own right.

The material is arranged in the conventional mathematical form (definitions, theorems, lemmas, proofs, remarks, etc.). The first section of each part (Section 1 and 4) contains an introduction to the topic. Here we give further references as well.

Throughout the dissertation with every result we give (usually in the text) the source it is taken from, unless it is the first appearance.

Part I

On an Identity of Ramanujan

1. Introduction

If you have built castles in the air,
your work need not be lost;
that is where they should be.
Now put the foundations under them.
H. D. Thoreau

There are such castles in the history of mathematics. Some of them are smaller and humbler, some bigger and fairer. But there are only few (if any) that can be compared to the vastness and beauty of the one that was built by a poor young Indian. Since his death many mathematicians have been constructing the foundations for this magnificent structure.

Srinivasa Ramanujan was born on December 22, 1887 in southern India. At the age of 12 he borrowed Loney's Trigonometry and completely mastered its contents. Three years later he read Carr's Synopsis of Elementary Results in Pure Mathematics and it had the greatest influence on him. This book contained about 6000 results mainly on calculus and geometry but nothing on functions of a complex variable or elliptic functions. For the most part Carr ignored the proofs, and if a proof is given it is usually very brief and sketchy.

In 1903 Ramanujan took the matriculation examination of the University of Madras. However, by this time he was completely absorbed in mathematics and would not study any other subject. No wonder he failed his examinations at the end of the first year. Four years later he tried again and failed again his exams.

He got married and feeling it necessary to have a job, Ramanujan accepted a clerical position, which turned out to be fortunate in his career. There he met an English engineer and a mathematician and they encouraged Ramanujan to communicate his mathematical discoveries to England. He had tried thrice before at last in 1913 he wrote the famed English mathematician G. H. Hardy.

When Hardy received Ramanujan's letter, he and J. E. Littlewood spent two and a half hours studying its contents. Beforehand Hardy exclaimed that this Hindu was either a crank or a genius. In the end they decided that Ramanujan was indeed a genius. Some results contained in the letter were false, others well known, but many were undoubtedly new and true. Hardy replied without delay and invited Ramanujan to go to Cambridge.

In 1914 Ramanujan arrived in England. The next three years were happy and fruitful, though he had difficulties with the English climate and getting proper vegetarian food. Both Hardy and Ramanujan profited from each other's ideas and knowledge, but after three years Ramanujan fell seriously ill. The war prevented him from returning to India, so he stayed in England and continued his work. Finally, in 1919 he departed for home. However, the friendlier climate and diet did not improve his condition. He died young, on 26 April, 1920, at the age of 32.

Ramanujan collected his results in three notebooks and other manuscripts. The variety of these results is amazing. Ramanujan seems to have dealt with most branches of mathematics. In the Notebooks the reader can find theorems, examples concerning magic squares, harmonic series, combinatorial analysis, Eulerian polynomials and numbers, Bernoulli numbers, the Riemann zeta-function, divergent series, the gamma function, transformations and evaluations of infinite series, hypergeometric series, continued fractions, asymptotic expansions, elementary algebra, number theory, prime numbers, theta-functions, q-series, integrals, partial fractions, elementary analysis.

It is generally believed that Ramanujan made many errors, gave no proofs and his proofs are sometimes incorrect. However, one must be careful with this opinion. Ramanujan intended the notebooks for his personal use and not for publication. Thus, notation is sometimes not explained and hypotheses are rarely given with theorems and formulae. In fact the notebooks contain some minor errors and misprints but very few serious errors. As for the proofs, indeed, they are usually ignored and those proofs that are given are only sketches. It is not surprising if we take into account that first, the notebooks were a collection of results for Ramanujan himself and he no doubt could reproduce any of his proofs. Secondly, a poor uneducated Hindu could not afford much paper, which was expensive. And thirdly, Carr's Synopsis served as a model for Ramanujan how to write mathematics.

Ramanujan's results are often rediscoveries and are sometimes rediscovered. Several of his theorems and formulae are so surprising and peculiar that the reader does not understand how someone could think of them at all. As Hardy wrote about a few continued fraction formulae in Ramanujan's first letter, "if they were not true, no one would have had the imagination to invent them." The same applies to some of his identities in number theory.

(The above brief account on Ramanujan's life and work was mainly taken from Berndt's book [Ber94].)

In Ramanujan's third notebook [Ram57], on page 385, the following entry can be found (Entry 44).

If $ad = bc$ then

$$(a+b+c)^n+(b+c+d)^n+(a-d)^n = (a+b+d)^n+(a+c+d)^n+(b-c)^n, \tag{1.1}$$

where $n = 2$ or 4.

Expanding both sides in each case one can easily verify the equality.

It seems probable from the context and the examples following the entry that Ramanujan thought identities (1.1) important in the (commutative) ring $\mathbb{Z}$ of the integers. We note that Ramanujan's statement is true in any commutative ring.

It would be interesting to know if Ramanujan found all the identities of type (1.1). This means the following. Let $f : \mathbb{Z} \to \mathbb{R}$ be an unknown function, and according to the Ramanujan identities, suppose that for all $a, b, c, d \in \mathbb{Z}$ such that $ad = bc$ (in the following we shall denote it by $[a, b, c, d] \in \Gamma$) the functional equation

$$f(a+b+c)+f(b+c+d)+f(a-d) = f(a+b+d)+f(a+c+d)+f(b-c) \tag{1.2}$$

holds. Let $S(\mathbb{Z}, \mathbb{R})$ denote the set of the solutions of the functional equation (1.2), i.e., the class of all the functions $f : \mathbb{Z} \to \mathbb{R}$ for which (1.2) holds for all $[a, b, c, d] \in \Gamma$. It is clear from the remarks above that the functions

$$f_1(x) := 1, \quad f_2(x) := x^2, \quad f_3(x) := x^4 \qquad (x \in \mathbb{Z}) \tag{1.3}$$

are elements of $S(\mathbb{Z}, \mathbb{R})$, which trivially implies that the linear combinations of the functions f_1, f_2, f_3 (over $\mathbb{R}$) are also solutions, that is, they belong to $S(\mathbb{Z}, \mathbb{R})$. The question is whether we have all the solutions of the functional equation (1.2). The answer is negative, since the function

$$f_4(x) := \begin{cases} 1 & \text{if } 2 \mid x \\ 0 & \text{otherwise} \end{cases} \qquad (x \in \mathbb{Z}) \tag{1.4}$$

given by T. Farkas belongs to $S(\mathbb{Z}, \mathbb{R})$ and f_4 is linearly independent of the system $\{f_1, f_2, f_3\}$.

In order to check that f_4 is a solution of (1.2) we have to examine every possible case.

(i) If $2 \mid a$, $2 \mid b$, $2 \mid c$, and $2 \mid d$ then $2 \mid (a+b+c)$, $2 \mid (b+c+d)$, $2 \mid (a-d)$, $2 \mid (a+b+d)$, $2 \mid (a+c+d)$, $2 \mid (b-c)$, and so (1.2) holds.

(ii) If $2 \mid a$, $2 \mid b$, $2 \nmid c$, and $2 \nmid d$ then $2 \nmid (a+b+c)$, $2 \mid (b+c+d)$, $2 \nmid (a-d)$, $2 \nmid (a+b+d)$, $2 \mid (a+c+d)$, $2 \nmid (b-c)$, and so (1.2) holds again.

(iii) If $2 \nmid a$, $2 \nmid b$, $2 \nmid c$, and $2 \nmid d$ then $2 \nmid (a+b+c)$, $2 \nmid (b+c+d)$, $2 \mid (a-d)$, $2 \nmid (a+b+d)$, $2 \nmid (a+c+d)$, $2 \mid (b-c)$, which implies (1.2).

It is easy to verify that the remaining cases are either not possible ($ad \neq bc$) or, by the symmetry of the equation, are not essentially different from the above ones.

After this it is a natural problem to characterise the set $S(\mathbb{Z}, \mathbb{R})$. Section 2 contains the solution given by Daróczy and Hajdu ([DH99]). A question of this kind was earlier formulated by Daróczy ([Dar94]) for commutative rings. According to this paper, in the following we shall denote by $S(R, G)$ the set of all the solutions of the functional equation (1.2), where $(R, +, \cdot)$ is an arbitrary commutative ring and $(G, +)$ is an Abelian group.

2. The solutions of the Ramanujan equation on $\mathbb{Z}$

As it was mentioned in the introduction, the results of this section can be found in [DH99].

2.1. A recursive formula.

If $f \in S(\mathbb{Z}, \mathbb{R})$ then from equation (1.2) by interchanging b and c we have

$$f(b-c) = f(c-b),$$

that is, with the notation $x := b - c$

$$f(x) = f(-x) \qquad (x \in \mathbb{Z}). \tag{2.1}$$

(Equation (2.1) is true not only on $\mathbb{Z}$ but on any commutative ring.) This means that it is sufficient to determine the unknown function f on the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $I := \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 12\} \subset \mathbb{Z}$. Then card $I = 11$. The following theorem gives a formula for the elements of $S(\mathbb{Z}, \mathbb{R})$.

Theorem 2.1. *Let $f \in S(\mathbb{Z}, \mathbb{R})$. Then for any $x \in \mathbb{N}_0 \setminus I$ there exist integers $k_i(x)$ $(i \in I)$ such that*

$$f(x) = \sum_{i \in I} k_i(x) f(i). \tag{2.2}$$

Proof. Let us define the following sets:

$$\begin{aligned}
A_2 &:= \{x = 3k + 2 \mid \ k \geq 2,\ k \in \mathbb{N}\} = \{8, 11, 14, \dots\} \\
A_1 &:= \{x = 3k + 1 \mid \ k \geq 4,\ k \in \mathbb{N}\} = \{13, 16, 19, \dots\} \\
A_0 &:= \{x = 3k \qquad \mid \ k \geq 6,\ k \in \mathbb{N}\} = \{18, 21, 24, \dots\}.
\end{aligned}$$

Then

$$\mathbb{N}_0 = I \cup A_2 \cup A_1 \cup A_0 \cup \{15\},$$

where the sets in the union are pairwise disjoint. Now let $f \in S(\mathbb{Z}, \mathbb{R})$ be arbitrary, which, as a consequence of our previous remarks, satisfies $f(x) = f(-x)$ $(x \in \mathbb{Z})$.

If $x \in \mathbb{N}_0 \setminus I$ we shall show that $f(x)$ can be represented as a linear combination of the values $f(k) \quad (k \in \{0, 1, \dots, x-1\})$ with integer coefficients. There are four possible cases:

(i) $x \in A_2$. Then $x = 3k+2 \quad (k \geq 2)$ and from (1.2) by substituting $[2k, k, 2, 1] \in \Gamma$ we get

$$\begin{aligned} f(x) =& f(3k+2) = \\ & f(3k+1) + f(2k+3) + f(k-2) - f(k+3) - f(2k-1), \end{aligned}$$

where the arguments of the function f on the right hand side are less than x, and nonnegative.

(ii) $x \in A_1$. Then $x = 3k+1 \quad (k \geq 4)$ and from (1.2) by substituting $[2(k-1), k-1, 4, 2] \in \Gamma$ we get

$$\begin{aligned} f(x) =& f(3k+1) = \\ & f(3k-1) + f(2k+4) + f(k-5) - f(k+5) - f(2k-4), \end{aligned}$$

where the arguments of the function f on the right hand side are less than x. If $k = 4$ then $k - 5 = -1$, and since $f(-1) = f(1)$, the arguments on the right hand side can be regarded as nonnegative.

(iii) $x \in A_0$. Then $x = 3k \quad (k \geq 6)$ and from (1.2) by substituting $[2(k-2), k-2, 6, 3] \in \Gamma$ we get

$$\begin{aligned} f(x) =& f(3k) = \\ & f(3k-3) + f(2k+5) + f(k-8) - f(k+7) - f(2k-7), \end{aligned}$$

where the arguments of f on the right hand side are less than x. If $k = 6$ then $k - 8 = -2$, and since $f(-2) = f(2)$, the arguments on the right hand side can be regarded as nonnegative again.

(iv) $x \in \{15\}$. Then from (2) by substituting $[9, 3, 3, 1] \in \Gamma$ we get

$$f(x) = f(15) = f(13) + f(13) + f(0) - f(8) - f(7),$$

that is, the arguments on the right hand side are less than 15, and nonnegative.

Since we have proved for any $x \in \mathbb{N}_0 \setminus I$ that $f(x)$ can be written as a linear combination of the values $f(k) \quad (0 \leq k \leq x-1)$ with integer coefficients, the continuation of the procedure while it is possible gives (2.2). $\square$

Theorem 2.2. *Let $f \in S(\mathbb{Z}, \mathbb{R})$. Then for any $x \in \mathbb{Z}$ there exist integers $k_i(x)$ $(i \in I)$ such that*

$$f(x) = \sum_{i \in I} k_i(x) f(i). \tag{2.3}$$

Proof. If $x \in \mathbb{N}_0 \setminus I$ then the assertion follows from Theorem 2.1. If $x \in I$ then let

$$k_i(x) := \begin{cases} 1 & \text{if} \quad i = x \\ 0 & \text{if} \quad i \in I \text{ and } i \neq x. \end{cases}$$

Obviously, x satisfies (2.3). If $x \in \mathbb{Z} \setminus \mathbb{N}_0$ then, since $-x \in \mathbb{N}_0$,

$$f(x) = f(-x) = \sum_{i \in I} k_i(-x) f(i),$$

so (2.3) holds. □

Remark 2.1. As an example, we note that some calculations show that according to the order of the elements of I

$$\{k_i(15) \mid i \in I\} = \{0, 4, 0, 1, -2, -1, -2, -3, 0, 2, 2\}.$$

2.2. Further solutions.

According to the above results any solution $f \in S(\mathbb{Z}, \mathbb{R})$ can be obtained using solely the elements of $\{f(i) \mid i \in I\}$. The cardinality of I implies 11 degrees of freedom, therefore it is natural to ask if there exist seven more solutions, which together with the already known four linearly independent ones, form an 11 element set of linearly independent solutions. The theorems below answer this question.

Theorem 2.3. *The following functions, defined on $\mathbb{Z}$, are elements of $S(\mathbb{Z},\mathbb{R})$:*

$$\begin{aligned}
f_5(x) &:= \begin{cases} 1 & \text{if } 3 \mid x \\ 0 & \text{otherwise,} \end{cases} \\
f_6(x) &:= \begin{cases} 1 & \text{if } 5 \mid x \\ 0 & \text{otherwise,} \end{cases} \\
f_7(x) &:= \begin{cases} 1 & \text{if } 5 \mid (x+1)(x-1) \\ 0 & \text{otherwise,} \end{cases} \\
f_8(x) &:= \begin{cases} 1 & \text{if } 7 \mid x(x+3)(x-3) \\ 2 & \text{if } 7 \mid (x+1)(x-1) \\ 0 & \text{otherwise,} \end{cases} \\
f_9(x) &:= \begin{cases} 1 & \text{if } 7 \mid x(x+2)(x-2) \\ 0 & \text{if } 7 \mid (x+1)(x-1) \\ 2 & \text{otherwise,} \end{cases} \\
f_{10}(x) &:= \begin{cases} 1 & \text{if } 4 \mid x \\ 0 & \text{otherwise,} \end{cases} \\
f_{11}(x) &:= \begin{cases} 1 & \text{if } 8 \mid x \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.4}$$

Proof. Functions f_i, $i = 5, \dots, 9$ are periodic with respect to some m and so can naturally be obtained from functions φ_i defined on the ring $Z_m := \{\bar{0}, \bar{1}, \dots, \overline{m-1}\}$ of the integers modulo m. Define $\varphi_i : Z_m \to Z_m$, $i = 5, \dots, 9$ by

$$\begin{aligned}
\varphi_5(x) &:= 2x^2 + 1 && (m = 3), \\
\varphi_6(x) &:= 4x^4 + 1 && (m = 5), \\
\varphi_7(x) &:= 3x^4 + 3x^2 && (m = 5), \\
\varphi_8(x) &:= 6x^4 + 2x^2 + 1 && (m = 7), \\
\varphi_9(x) &:= 5x^4 + x^2 + 1 && (m = 7).
\end{aligned}$$

One can readily check that $f_i(x) = \underline{\varphi_i(\bar{x})}$, where underlining is understood as the map from Z_m into $\mathbb{Z}$ defined by the following rule: if $\bar{x} \in Z_m$ then $\underline{\bar{x}} := x \in \{0, 1, \ldots, m-1\} \subset \mathbb{Z}$. Functions φ_i are linear combinations of polynomials $1, x^2, x^4$ over the field Z_m, hence it is guaranteed that they belong to $S(Z_m, Z_m)$. For φ_i, if $i = 6, 7, 8, 9$, we have

$$\underline{\varphi_i(\bar{x}) + \varphi_i(\bar{y}) + \varphi_i(\bar{z})} = \underline{\varphi_i(\bar{x})} + \underline{\varphi_i(\bar{y})} + \underline{\varphi_i(\bar{z})}$$

for any $\bar{x}, \bar{y}, \bar{z} \in Z_m$, which shows that the functions f_i, $i = 5, \ldots, 9$ are elements of $S(\mathbb{Z}, \mathbb{R})$.

For $i = 5$ the above identity may not hold only if $\varphi_i(\bar{x}) = \varphi_i(\bar{y}) = \varphi_i(\bar{z}) = 1$. This, however, causes no significant difficulty since it can be directly checked that (1.2) holds and f_5 also belongs to $S(\mathbb{Z}, \mathbb{R})$.

Unfortunately, the above method breaks down for f_{10} and f_{11}, since they cannot be obtained from polynomials of the right form. For them one has to examine every possible case and check directly that they are indeed solutions of (1.2). (The computations can be carried out similarly to the case of f_4 in (1.4).) □

Theorem 2.4. *The 11 element subset* $\{f_k : \mathbb{Z} \to \mathbb{R} \mid 1 \leq k \leq 11\}$ *of* $S(\mathbb{Z}, \mathbb{R})$, *consisting of the functions defined in* (1.3), (1.4), *and* (2.4) *is linearly independent over* $\mathbb{R}$.

Proof. Suppose that there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{11}$ for which

$$\sum_{l=1}^{11} \alpha_l f_l(x) = 0 \quad \text{for all } x \in \mathbb{Z}.$$

Substitutions $x = i$, where i runs through the elements of I, result in the system

$$\sum_{l=1}^{11} f_l(i)\alpha_l = 0 \qquad\qquad (i \in I)$$

of linear equations for the α_l. Since functions f_l are all known, one can compute the entries of the matrix

$$A := \big(f_l(i)\big)_{1 \leq l \leq 11,\ i \in I}$$

of the system. The result of the computations is given in the following table:

$i \in I$ / $1 \le l \le 11$	0	1	2	3	4	5	6	7	9	10	12
1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	4	9	16	25	36	49	81	100	144
3	0	1	16	81	256	625	1296	2401	6561	10000	20736
4	1	0	1	0	1	0	1	0	0	1	0
5	1	0	0	1	0	0	1	0	1	0	1
6	1	0	0	0	0	1	0	0	0	1	0
7	0	1	0	0	1	0	1	0	1	0	0
8	1	2	0	1	1	0	2	1	0	1	0
9	1	0	1	2	2	1	0	1	1	2	1
10	1	0	0	0	1	0	0	0	0	0	1
11	1	0	0	0	0	0	0	0	0	0	0

The determinant[1] of A is not zero, hence the system has only a trivial solution, i.e., solutions f_l $(1 \le l \le 11)$ are linearly independent. $\square$

2.3. The solutions of the equation.

Employing the above statements an exact and complete characterization of the solution set $S(\mathbb{Z}, \mathbb{R})$ can be given.

Theorem 2.5. *For any* $f \in S(\mathbb{Z}, \mathbb{R})$ *there exist constants* $a_l \in \mathbb{R}$ $(1 \le l \le 11)$ *such that*

$$f(x) = \sum_{l=1}^{11} a_l f_l(x) \qquad (x \in \mathbb{Z}), \tag{2.5}$$

where $f_l : \mathbb{Z} \to \mathbb{R}$ $(1 \le l \le 11)$ *were defined in* (1.3), (1.4), *and* (2.4). *Conversely for any* $a_l \in \mathbb{R}$ $(1 \le l \le 11)$, *the function defined by* (2.5) *belongs to* $S(\mathbb{Z}, \mathbb{R})$.

Proof. Since linear combinations of functions belonging to $S(\mathbb{Z}, \mathbb{R})$ are also in $S(\mathbb{Z}, \mathbb{R})$, the second assertion of the theorem is trivial.

[1]The computation of the determinant was done by Maple.

Now let $f \in S(\mathbb{Z}, \mathbb{R})$ be arbitrary. Then the system of linear equations (for the unknowns a_l $(1 \le l \le 11)$)

$$\sum_{l=1}^{11} f_l(i) a_l = f(i) \qquad (i \in I)$$

has a unique solution, because, as we saw it in the proof of Theorem 2.4, its matrix is regular.

On the other hand, it is clear (from Theorem 2.2) that for any $x \in \mathbb{Z}$

$$\begin{aligned} f(x) = \sum_{i \in I} k_i(x) f(i) = \sum_{i \in I} k_i(x) \left(\sum_{l=1}^{11} f_l(i) a_l \right) = \\ \sum_{l=1}^{11} a_l \left(\sum_{i \in I} k_i(x) f_l(i) \right) = \sum_{l=1}^{11} a_l f_l(x), \end{aligned}$$

which proves the first assertion. □

Remark 2.2. The above results imply the uniqueness of the integers $x \mapsto k_i(x)$ $(i \in I)$ in Theorem 2.2. Indeed, if we assume that there exists another $x \mapsto k_i^*(x)$ $(i \in I,\ x \in \mathbb{Z})$ with the same properties then for any $f \in S(\mathbb{Z}, \mathbb{R})$

$$\sum_{i \in I} \left(k_i(x) - k_i^*(x) \right) f(i) = 0 \qquad (x \in \mathbb{Z})$$

would hold. Replacing f by the solutions f_l $(1 \le l \le 11)$, and using the fact that $\det\bigl(f_l(i)\bigr) \ne 0$, equations

$$\sum_{i \in I} \left(k_i(x) - k_i^*(x) \right) f_l(i) = 0 \qquad (1 \le l \le 11)$$

imply that

$$k_i(x) = k_i^*(x)$$

holds for all $x \in \mathbb{Z}$, which proves the uniqueness.

3. The Ramanujan differences

3.1. General investigations.

In the previous section we solved the Ramanujan equation on the commutative ring of the integers. In the following we replace the integers with another structure and give the general solutions. These results were achieved by Daróczy and Hajdu [DH98].

Let $R(+,\cdot)$ be a commutative ring with identity. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, R)$$

then let

$$\det A := ad - bc, \qquad A^o := \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

Denote by $\mathrm{Mat}^*(2, R)$ the set of the matrices $A \in \mathrm{Mat}(2, R)$ for which $\det A = 0$. Let $G(+)$ be an Abelian group. If $f : R \to G$ is a function then let

$$C_f(A) := f(a+b+c) + f(b+c+d) + f(a-d) \tag{3.1}$$

for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, R)$ (see Daróczy's paper [Dar94]). Obviously, $C_f : \mathrm{Mat}(2, R) \to G$. In this case the Ramanujan difference of the generating function $f : R \to G$ is defined by the equation

$$D_f(A) := C_f(A) - C_f(A^o)$$

for any $A \in \mathrm{Mat}(2, R)$. Now, the Ramanujan equation (1.2) can be written in the following form

$$D_f(A) = 0 \quad \text{if } A \in \mathrm{Mat}^*(2, R). \tag{3.2}$$

If

$$a_i : R \to G \qquad (i = 1, 2)$$

are additive functions (i.e., $a_i(x + y) = a_i(x) + a_i(y)$ for any $x, y \in R$), and $a_0 \in G$ then the function given by

$$f(x) := a_2(x^4) + a_1(x^2) + a_0 \qquad (x \in R)$$

$(f : R \to G)$ solves equation (3.2), that is, an element of $S(R, G)$.

Theorem 3.1. *Let $R(+, \cdot)$ be a commutative ring with identity $(e := 1)$. If f belongs to $S(R, G)$ then*

$$\begin{aligned} f(tz) + f\big(tz(y+1) + t(y^2+y+1)\big) + f\big(tzy + t(y^2+y+1)\big) \\ - f\big(tz(y+1) + t(y^2+2y)\big) - f\left(tz + t(2y+1)\right) \\ - f\big(tzy + t(y^2-1)\big) = 0 \end{aligned} \tag{3.3}$$

for any $t, z, y \in R$.

Proof. For any $t, x, y \in R$ let

$$A := \begin{pmatrix} txy & tx \\ ty & t \end{pmatrix} \in \mathrm{Mat}^*(2, R).$$

Then from (1.2)

$$\begin{aligned} f(txy + tx + ty) + f(tx + ty + t) + f(txy - t) = \\ f(txy + tx + t) + f(txy + ty + t) + f(tx - ty) \end{aligned}$$

follows, which implies, with the notation $z := x - y$ (i.e., $x = z + y$), functional equation (3.3). □

Theorem 3.2. *If R is a field and $f \in S(R, G)$ then for any $x, y, t \in R$*

$$\begin{aligned} f(x) + f\big(x(y+1) + t(y^2+y+1)\big) \\ + f\big(xy + t(y^2+y+1)\big) - f\big(x(y+1) + t(y^2+2y)\big) \\ - f\left(x + t(2y+1)\right) - f\big(xy + t(y^2-1)\big) = 0. \end{aligned} \tag{3.4}$$

Proof. If $t \neq 0$ then let $z := t^{-1}x$ in (3.3), where $x \in R$ is arbitrary. This implies the validity of (3.4) for any $x, y \in R$ and $t \neq 0$ $(t \in R)$. It is easy to verify that (3.4) also holds for $t = 0$, and this completes the proof of the theorem. □

The above theorem shows that another type of functional equations can be used to solve our problem. Functional equation (3.4) can also be written in the following form. Let $f_1 = f_2 = f$, $f_3 = f_4 = f_5 = -f$ and for any $y \in R$ fixed

$$\begin{aligned} \varphi_{1,y}(x) = \varphi_{3,y}(x) &:= (y+1)x \\ \varphi_{2,y}(x) = \varphi_{5,y}(x) &:= yx \qquad\qquad (x \in R) \\ \varphi_{4,y}(x) &:= x \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \psi_{1,y}(t) = \psi_{2,y}(t) &:= (y^2+y+1)t \\ \psi_{3,y}(t) &:= (y^2+2y)t \\ \psi_{4,y}(t) &:= (2y+1)t \\ \psi_{5,y}(t) &:= (y^2-1)t \end{aligned} \qquad (t \in R). \tag{3.6}$$

With the above notation (3.4) implies

$$f(x) + \sum_{i=1}^{5} f_i\left(\varphi_{i,y}(x) + \psi_{i,y}(t)\right) = 0 \tag{3.7}$$

for any $x, t \in R$ and $y \in R$, where $f, f_i : R \to G$ $(i = 1,2,3,4,5)$ are unknown functions and for a fix $y \in R$ the functions

$$\varphi_{i,y}, \psi_{i,y} : R \to R \qquad (i = 1,2,3,4,5)$$

are additive. The type of functional equations of form (3.7) is known for a fix $y \in R$, this is the so–called "linear" functional equation, which can be solved generally under certain conditions as we shall see later. Since this equation is strictly connected with the Fréchet equation and the theory of polynomials on groups, in the next subsection we give a short summary of the general theory.

3.2. Polynomials, the Fréchet and the linear functional equation.

The notion of polynomials on groups was introduced by S. Mazur and W. Orlicz [MO34], M. Fréchet [Fré29], and G. Van der Lijn [VdL45]. A thorough description of the theory of polynomials on groups and their applications can be found in Székelyhidi's book [Szé91].

Definition 3.1. Let S, G be commutative semigroups, n a positive integer. A function $A : S^n \to G$ is called *n–additive* if it is a homomorphism of S into G in each variable. If $n = 0$ then let $S^0 = S$ and we call any constant function from S to G *0–additive*. A is said to be *multi–additive* if there exists $n \in \mathbb{N}$ such that A is n–additive. The *diagonalisation* of an n–additive function $A : S^n \to G$ is denoted by A^* and means the following.

For any $x \in S$ let

$$A^*(x) := A(x, x, \dots, x).$$

Definition 3.2. Let S, G be commutative topological semigroups. A continuous function $p : S \to G$ is called a *polynomial* if it has a representation as the sum of diagonalisations of multi–additive functions, that is,

$$p = \sum_{k=0}^{n} A_k^*,$$

where $n \in \mathbb{N}$ and $A_k : S^k \to G$ are k–additive functions $(k = 0, 1, \dots n)$. In this case we call p a *polynomial of degree at most n.*

Remark 3.1. In our case the topology on S is the discrete topology. Consequently, every function is continuous, and we need not examine the continuity of the polynomials.

The following theorem shows that the Fréchet equation characterises the polynomials.

Theorem 3.3. *Let S be a commutative semigroup, G a commutative group and $n \in \mathbb{N}$, and let the multiplication by $n!$ be injective in G. A function $f : S \to G$ solves the Fréchet equation*

$$\Delta_{y_1, y_2, \dots, y_{n+1}} f(x) = 0 \tag{3.8}$$

for all $x, y_1, y_2, \dots, y_{n+1} \in S$ if and only if f is a polynomial of degree at most n. Here Δ_y is the difference operator, i.e., for $f : S \to G$ and $y \in S$

$$\Delta_y f(x) := f(x + y) - f(x)$$

for any $x \in S$.

Remark 3.2. Theorem 3.3 shows that we should suppose that G is an Abelian group in which multiplication by any positive integer is bijective, that is, G is a divisible, torsion–free Abelian group. Since these groups are linear spaces over the field $\mathbb{Q}$ of the rationals, in the following we assume that G is linear space over a field of characteristic zero.

Now consider the following "linear" functional equation, which is a common generalisation of classical functional equations like the Pexider equation, the Jensen equation and the square–norm equation:

$$f(x)+\sum_{i=1}^{n+1} f_i\left(\varphi_i(x)+\psi_i(y)\right)=0 \tag{3.9}$$

for all $x,y\in S$, where φ_i,ψ_i are homomorphisms of S and $f,f_i:S\to G$ $(i=1,2,\dots,n+1)$ are the unknown functions (cf. [Szé82], [Szé91]).

Theorem 3.4. *Let S,G be Abelian groups, $n\in\mathbb{N}$, $\varphi_i,\psi_i:S\to S$ additive functions, and let* $\mathrm{rg}(\varphi_i)\subseteq\mathrm{rg}(\psi_i)$ $(i=1,2,\dots,n+1)$. *If the functions $f,f_i:S\to G$ $(i=1,2,\dots,n+1)$ satisfy* (3.9) *then f satisfies* (3.8).

3.3. The polynomial solutions.

Theorem 3.5. *If R is a field of characteristic zero, G is a linear space over a field of characteristic zero, and $f\in S(R,G)$ then there exist k–additive, symmetric functions $A_k:R^k\to G$ $(k=4,2,0,\ G^0:=G)$ such that*

$$f(x)=A_4^*(x)+A_2^*(x)+A_0^* \tag{3.10}$$

holds for any $x\in R$.

Proof. In this case f satisfies the functional equation (3.4) for all $x,t,y\in R$. Let $y=2$ in (3.4). Then

$$f(x)+f(3x+7t)+f(2x+7t)-f(3x+8t)-f(x+5t)-f(2x+3t)=0$$

for any $x,t\in R$, and with the notation of (3.5) and (3.6) the functional equation

$$f(x)+\sum_{i=1}^{5} f_i\left(\varphi_{i,2}(x)+\psi_{i,2}(t)\right)=0 \tag{3.11}$$

holds for all $x,t\in R$. Since R is a field of characteristic zero,

$$\mathrm{rg}(\psi_{i,2})=R \qquad (i=1,2,3,4,5) \tag{3.12}$$

holds, therefore $\mathrm{rg}(\varphi_{i,2})\subseteq\mathrm{rg}(\psi_{i,2})$ $(i=1,2,3,4,5)$, and thus, by Theorem 3.4 and Theorem 3.3, f is a polynomial of degree at most 4. That is,

$$f(x) = \sum_{k=0}^{4} A_k^*(x) \qquad (x \in R), \tag{3.13}$$

where $A_k : R^k \to G$ are k–additive, symmetric functions. On the other hand, from (3.13) we get

$$f(-x) = A_4^*(x) - A_3^*(x) + A_2^*(x) - A_1^*(x) + A_0^* \tag{3.14}$$

for all $x \in R$, which implies, as a consequence of equations (2.1), (3.13), and (3.14),

$$f(x) = \frac{f(x) + f(-x)}{2} = A_4^*(x) + A_2^*(x) + A_0^*$$

for any $x \in R$. □

Remark 3.3. The assumption that R should be a field of characteristic zero guarantees (3.12). However, a weaker condition is sufficient. If we suppose that R is a commutative ring with identity in which the elements 2, 3, 5, 7 are invertible then in each case the range of $\psi_{i,2}$ is the whole R, and (3.12) holds.

Theorem 3.5 does not state that the final form of f is (3.10), only that f has the representation (3.10). This however does not imply that the functions of the form (3.10) satisfy the functional equation (1.2). Therefore in the following we shall examine under which (other) conditions a function of the form (3.10) belongs to $S(R, G)$.

Let $A_k : R^k \to G$ $(k = 4, 2, 0)$ be k–additive and symmetric functions given and, according to (3.10), let

$$f = A_4^* + A_2^* + A_0^* \qquad (f : R \to G).$$

We shall examine what conditions are necessary and sufficient for the above defined function f to belong to $S(R, G)$.

Theorem 3.6. *If the function f given in* (3.10) *is in $S(R, G)$ then $A_k^* \in S(R, G)$ if $k = 4, 2, 0$.*

Proof. The assertion is trivial for A_0^*. So if $f \in S(R, G)$ then $g := (f - A_0^*) \in S(R, G)$. On the other hand, note that if $g \in S(R, G)$ then the function

$$g_2(x) := g(2x) \qquad (x \in R)$$

is an element of $S(R,G)$, too. This implies that the function

$$\frac{1}{12}\left(g(2x)-4g(x)\right)=\frac{1}{12}\left(A_4^*(2x)+A_2^*(2x)-4A_4^*(x)-4A_2^*(x)\right)=$$
$$\frac{1}{12}\left(16A_4^*(x)+4A_2^*(x)-4A_4^*(x)-4A_2^*(x)\right(=A_4^*(x)\qquad(x\in R)$$

also belongs to $S(R,G)$, from which $A_2^*\in S(R,G)$ follows, as well. □

Theorem 3.7. *$A_2^*\in S(R,G)$ if and only if*

$$A_2^*(x)=A_2(x^2,1)\qquad(x\in R).\tag{3.15}$$

Proof. Let $x\in R$ be arbitrary, and let

$$A=\begin{pmatrix}x^2 & x\\ x & 1\end{pmatrix}.$$

Then $D_{A_2^*}(A)=0$ holds if and only if

$$A_2^*(x^2+2x)+A_2^*(2x+1)+A_2^*(x^2-1)=2A_2^*(x^2+x+1).$$

By the binomial theorem for multiadditive functions from this

$$\begin{aligned}A_2^*(x^2)&+2A_2(x^2,2x)+A_2^*(2x)+A_2^*(2x)+2A_2(2x,1)+A_2^*(1)\\&+A_2^*(x^2)+2A_2(x^2,-1)+A_2^*(-1)=\\&2A_2^*(x^2)+4A_2(x^2,x)+4A_2(x^2,1)+2A_2^*(x)+4A_2(x,1)+2A_2^*(1)\end{aligned}$$

follows, whence

$$A_2^*(x)=A_2(x^2,1),$$

so (3.15) holds. On the other hand, the Ramanujan identity implies that the function $x\mapsto A_2(x^2,1)$ $(x\in R)$ belongs to $S(R,G)$. □

Theorem 3.8. *$A_4^*\in S(R,G)$ if and only if*

$$A_4^*(x)=A_4(x^4,1,1,1)\qquad(x\in R).\tag{3.16}$$

Proof. Let $x\in R$ be arbitrary, and let

$$A:=\begin{pmatrix}x^2 & x\\ x & 1\end{pmatrix},\qquad\text{and}\qquad A':=\begin{pmatrix}x^2 & -x\\ -x & 1\end{pmatrix}.$$

Then $D_{A_4^*}(A)=0$ and $D_{A_4^*}(A')=0$ hold if and only if

$$A_4^*(x^2+2x)+A_4^*(2x+1)+A_4^*(x^2-1)=2A_4^*(x^2+x+1),\tag{3.17}$$

and

$$A_4^*(x^2-2x)+A_4^*(-2x+1)+A_4^*(x^2-1)=2A_4^*(x^2-x+1). \tag{3.18}$$

Adding the two equations then using the binomial theorem, and the fact that

$$A_4^*(a+b)+A_4^*(a-b)=2A_4(a,a,a,a)+12A_4(a,a,b,b)+2A_4(b,b,b,b),$$

we have

$$\begin{aligned}5A_4(x,x,x,x)=&\,2A_4(x^2,x^2,x^2,1)+2A_4(x^2,1,1,1)\\&-2A_4(x^2,x^2,x,x)+A_4(x^2,x^2,1,1)\\&-2A_4(x,x,1,1)+4A_4(x^2,x,x,1).\end{aligned} \tag{3.19}$$

If in (3.19) we put $2x$ for x, then subtract the equation from (3.19) multiplied by 2^6 we obtain

$$\begin{aligned}10A_4(x,x,x,x)=&5A_4(x^2,1,1,1)-5A_4(x,x,1,1)\\&+2A_4(x^2,x^2,1,1)+8A_4(x^2,x,x,1).\end{aligned} \tag{3.20}$$

Now replacing x by $2x$ in (3.20) again, then subtracting the equation from (3.20) multiplied by 2^2, we get

$$5A_4(x,x,x,x)=A_4(x^2,x^2,1,1)+4A_4(x^2,x,x,1), \tag{3.21}$$

and similarly we get that expressions of the same degree must be equal. The equalities involving odd degrees follow from (3.17) and (3.18) similarly:

$$\begin{aligned}A_4(x^2,1,1,1)&=A_4(x,x,1,1),\\A_4(x^2,x,1,1)&=A_4(x,x,x,1),\end{aligned} \tag{3.22}$$

$$\begin{aligned}A_4(x^2,x^2,x,1)&=A_4(x^2,x,x,x),\\A_4(x^2,x^2,x^2,1)&=A_4(x^2,x^2,x,x).\end{aligned} \tag{3.23}$$

Putting $x+1$ instead of x in (3.23), we have

$$A_4(x^2+2x+1,x^2+2x+1,x+1,1)=A_4(x^2+2x+1,x+1,x+1,x+1).$$

Using the binomial theorem and the above equations, we obtain

$$2A_4(x,x,x,x)=A_4(x^2,x^2,1,1)+A_4(x^2,x,x,1),$$

which together with (3.21) gives

$$A_4(x,x,x,x)=A_4(x^2,x^2,1,1).$$

Now, using equation (3.22), we have

$$A_4(x,x,x,x) = A_4(x^4,1,1,1),$$

which was to be proved. Again, the Ramanujan identity implies that the function $x \mapsto A_4^*(x^4,1,1,1)$ $(x \in R)$ is in $S(R,G)$. □

Now we can give the complete characterisation of the solution set $S(R,G)$.

Theorem 3.9. *Let R be a field of characteristic zero, and G a linear space over a field of characteristic zero. Then $f \in S(R,G)$ if and only if there exist additive functions $a_i : R \to G$ $(i = 1,2)$, and $a_0 \in G$ such that*

$$f(x) = a_2(x^4) + a_1(x^2) + a_0 \tag{3.24}$$

for all $x \in R$.

Proof. Theorem 3.5, Theorem 3.6, Theorem 3.7, and Theorem 3.8 imply that if $f \in S(R,G)$ then there exist k–additive and symmetric functions $A_k : R^k \to G$ $(k = 4,2,0)$ for which

$$f(x) = A_4^*(x) + A_2^*(x) + A_0^* = A_4(x^4,1,1,1) + A_2(x^2,1) + A_0$$

for all $x \in R$. With the notations $a_2(x) := A_4(x,1,1,1)$, $a_1(x) := A_2(x,1)$ and $a_0 := A_0$ $(x \in R)$ we have (3.24). On the other hand, earlier we proved that functions of the form (3.24) are elements of $S(R,G)$ indeed. □

Part II

Extension Theorems for Matkowski–Sutô type Problems and Complementary Means

4. Quasi–arithmetic means and the origins of the Matkowski–Sutô problem

4.1. Means and quasi–arithmetic means.

The history of the quasi–arithmetic means dates back to the turn of the 19th and 20th centuries. Most probably it was Jensen who first mentioned this class of means, but not under this name. Even the notion of means was not well defined. In 1929 Chisni [Chi29] gave an exact definition of mean values, later Kolmogorov [Kol30], Nagumo [Nag30] and mainly de Finetti [dF31] started the investigation of the theory of mean values. The following definitions are the result of their work.

Definition 4.1. Let $I \subset \mathbb{R}$ be an open interval. We say that a function $M : I^2 \to I$ is a *mean* on I if it satisfies the following conditions.

(i) If $x \neq y$ and $x, y \in I$ then $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$;
(ii) M is continuous on I^2.

A mean is called *strict* if the inequalities in (i) are strict. If $M(x, y) = M(y, x)$ for all $x, y \in I$ then we say that M is *symmetric*.

The best known mean is the arithmetic mean

$$A(x, y) := \frac{x + y}{2} \qquad (x, y \in I),$$

which is defined on any interval I.

The other two most widely known means are the geometric mean

$$G(x, y) := \sqrt{xy}$$

and the harmonic mean

$$H(x, y) := \frac{2}{\frac{1}{x} + \frac{1}{y}},$$

which are defined on any $I \subset \mathbb{R}_+$.

A common generalisation of the above means are the *power means* or the *Hölder–means*

$$H_p(x,y) := \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \sqrt{xy} & \text{if } p = 0 \end{cases} \qquad (x, y \in \mathbb{R}_+,\ p \in \mathbb{R}).$$

An important property of the power means is that they are *homogenous*, that is, for all $p \in \mathbb{R}$, $x, y, t \in \mathbb{R}_+$

$$H_p(tx, ty) = tH_p(x, y).$$

Also they are the only homogenous quasi–arithmetic means (see below).

A possible generalisation of the power means are the *quasi–arithmetic means*, where the power function is replaced by a strictly monotone continuous function.

Definition 4.2. Let $CM(I)$ denote the set of all continuous and strictly monotone real functions on I. A mean M is called a *quasi–arithmetic mean* if there exists $\varphi \in CM(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) =: A_\varphi(x,y) \qquad (x, y \in I).$$

In this case, $\varphi \in CM(I)$ is called the *generating function* of the quasi–arithmetic mean M.

Quasi–arithmetic means have been investigated by many authors and the reader can find extensive literature concerning this class of means ([Acz66], [HLP34], [Kuc85], [Dar99]).

Definition 4.3. Let $\varphi, \psi \in CM(I)$. If there exist real constants $\alpha \neq 0$ and β such that

$$\varphi(x) = \alpha\psi(x) + \beta \qquad (x \in I)$$

then we say that φ is *equivalent* to ψ on I; and, in this case, we write: $\varphi \sim \psi$ on I or $\varphi(x) \sim \psi(x) \quad (x \in I)$. If $\varphi, \psi, \Phi, \Psi \in CM(I)$ and $\varphi \sim \Phi$ and $\psi \sim \Psi$ then we say that the pair (φ, ψ) is *equivalent* to the pair (Φ, Ψ); in notation $(\varphi, \psi) \sim (\Phi, \Psi)$.

The following well–known result deals with the equality of two quasi–arithmetic means (see [Dar99], [HLP34]).

Theorem 4.1. *If $\varphi, \psi \in CM(I)$ then the equation $A_\varphi \equiv A_\psi$ holds on I^2 if and only if φ is equivalent to ψ on I.*

4.2. The history of the Matkowski–Sutô problem and the original extension theorem.

The Matkowski–Sutô problem originates in a paper by Sutô from 1914. He examined when the sum of two quasi–arithmetic means equals the double of the arithmetic mean, that is, which quasi–arithmetic means A_φ and A_ψ satisfy equation

$$A_\varphi(x, y) + A_\psi(x, y) = x + y \qquad (x, y \in I). \tag{4.1}$$

Using the generating functions of A_φ and A_ψ (4.1) can be written as

$$\varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) = x + y \tag{4.2}$$

for all $x, y \in I$, where $\varphi, \psi \in CM(I)$.

Sutô [Sut14] proved the following

Theorem 4.2. *If $\varphi, \psi \in CM(I)$ satisfy* (4.2) *and φ, ψ are analytic functions then there exists $p \in \mathbb{R}$ for which*

$$\varphi(x) \sim \chi_p(x), \qquad \psi(x) \sim \chi_{-p}(x) \qquad (x \in I), \tag{4.3}$$

where

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I). \tag{4.4}$$

The problem was independently re-discovered by Matkowski in 1999 ([Mat99]). He proved the following more general result.

Theorem 4.3. *If $\varphi, \psi \in CM(I)$ satisfy* (4.2) *and φ, ψ are twice continuously differentiable on I then there exists $p \in \mathbb{R}$ such that* (4.3) *holds.*

The next improvement was gained by Daróczy and Páles [DP01].

Theorem 4.4. *If $\varphi, \psi \in CM(I)$ satisfy* (4.2) *and either φ or ψ is continuously differentiable on I then there exists $p \in \mathbb{R}$ such that* (4.3) *holds.*

The latest and complete answer was given by Daróczy and Páles. They proved the following general theorem in 2001 ([DP02a]).

Theorem 4.5. *If $\varphi, \psi \in CM(I)$ satisfy* (4.2) *then there exists $p \in \mathbb{R}$ such that* (4.3) *holds.*

In the proof they show that there exists a nonvoid open subinterval $K \subset I$ on which (4.3) holds. Then they apply the following extension theorem of Daróczy, Maksa, and Páles [DMP00].

Theorem 4.6. *If $\varphi, \psi \in CM(I)$ satisfy* (4.2) *and there exists a nonvoid open interval $K \subset I$ such that $\varphi(x) \sim \chi_p(x), \quad \psi(x) \sim \chi_{-p}(x) \quad (x \in K)$ for some $p \in \mathbb{R}$ then $\varphi(x) \sim \chi_p(x), \quad \psi(x) \sim \chi_{-p}(x) \quad (x \in I)$, that is,* (4.3) *holds.*

Theorem 4.6 is vital in the proof of the general theorem and also it is interesting itself. Therefore it seems worthwhile to try and prove extension theorems for other classes of means, this can be find in Section 5.

4.3. Another aspect: the Gauss–composition.

The very young Gauss, taking the arithmetic and the geometric mean as a starting point, defined the following iteration.

Definition 4.4. Let $x, y \in \mathbb{R}_+$ be arbitrary and let

$$\begin{aligned} & x_1 := x, && y_1 := y, \\ & x_{n+1} := \frac{x_n + y_n}{2}, && y_{n+1} := \sqrt{x_n y_n} \qquad (n \in \mathbb{N}). \end{aligned} \tag{4.5}$$

These iterations converge to a common limit

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n =: A \otimes G(x, y),$$

which is a mean, called the *arithmetic–geometric mean* on $\mathbb{R}_+$.

Later it turned out that the arithmetic–geometric mean plays an important role in mathematics, in the theory of elliptic integrals and functions and also in numerical analysis. The topic has a wide literature (e.g., [Gau27], [AB88], [BB87], [Sch82], [Tod79]).

Gauss noticed that the arithmetic–geometric mean satisfies an invariance equation

$$A \otimes G\left(\frac{x+y}{2}, \sqrt{xy}\right) = A \otimes G(x,y) \tag{4.6}$$

for all $x, y \in \mathbb{R}_+$.

In Gauss' original iteration the arithmetic and geometric mean can be naturally replaced by any two means. This generalisation was examined for instance in [BB87] and [DP02a].

Definition 4.5. Let $M_i : I^2 \to I$ $(i = 1, 2)$ be given means on I and let $(x, y) \in I^2$ be arbitrary. Then the iteration sequence

$$\begin{aligned} x_1 &:= x, & y_1 &:= y, \\ x_{n+1} &:= M_1(x_n, y_n), & y_{n+1} &:= M_2(x_n, y_n) \qquad (n \in \mathbb{N}) \end{aligned} \tag{4.7}$$

is called the *Gauss–iteration* determined by the pair (M_1, M_2) with the initial values $(x, y) \in I^2$.

Let I_n be the closed interval determined by x_n and y_n. Then, because of property (i) of means, we have

$$I_{n+1} \subseteq I_n \qquad (n \in \mathbb{N}).$$

The Gauss–iteration (4.7) is said to be *convergent* if the set $\bigcap\limits_{n=1}^{\infty} I_n$ is a singleton for any initial value $(x, y) \in I^2$. By Cantor's theorem, this is true if and only if

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n =: M_1 \otimes M_2(x, y), \tag{4.8}$$

where $M_1 \otimes M_2 : I^2 \to I$ is a function.

Theorem 4.7. *If M_1 and M_2 are given means on I and the Gauss–iteration determined by the pair (M_1, M_2) is convergent, then the function $M_1 \otimes M_2 : I^2 \to I$ is a mean on I.*

Definition 4.6. If M_1 and M_2 are given means on I and the Gauss–iteration determined by the pair (M_1, M_2) is convergent, then the uniquely defined mean $M_1 \otimes M_2 : I^2 \to I$ is called the *Gauss–composition* of M_1 and M_2 on I.

Theorem 4.8. *Let M_1 and M_2 be means on I and suppose that either of them is strict. Then the Gauss–iteration determined by the pair (M_1, M_2) is convergent.*

The following theorem shows why the Gauss–composition is strongly connected with the Matkowski–Sutô problem.

Theorem 4.9. *If M_1 and M_2 are means on I and the Gauss–iteration determined by the pair (M_1, M_2) is convergent then the Gauss–composition $M_1 \otimes M_2$ satisfies the invariance equation*

$$M_1 \otimes M_2(M_1(x,y), M_2(x,y)) = M_1 \otimes M_2(x,y) \tag{4.9}$$

for all $x, y \in I$. Furthermore, if $F : I^2 \to \mathbb{R}$ is a continuous function satisfying $F(x,x) = x \quad (x \in I)$ and

$$F(M_1(x,y), M_2(x,y)) = F(x,y) \qquad (x, y \in I)$$

then

$$F(x,y) = M_1 \otimes M_2(x,y) \qquad (x, y \in I).$$

In the light of Theorem 4.9, the Matkowski–Sutô problem (4.1) can be formulated as:

Find all those quasi–arithmetic means whose Gauss–composition is the arithmetic mean.

The problem in this generality was answered by Daróczy and Páles [DP02a].

5. Extension theorems for various Matkowski–Sutô type problems

5.1. Weighted quasi–arithmetic means.

Definition 5.1. Let $I \subset \mathbb{R}$ be an open interval and $0 < \lambda < 1$. A mean M on I is called a *weighted quasi–arithmetic mean* if there exists $\varphi \in CM(I)$ such that

$$M(x,y) = \varphi^{-1}(\lambda\varphi(x) + (1-\lambda)\varphi(y)) =: A_\varphi(x,y;\lambda) \qquad (x,y \in I).$$

In this case φ is called the *generating function* of the weighted quasi–arithmetic mean with *weight* λ.

The equivalence of generating functions can be defined just as in Definition 4.3, and Theorem 4.1 applies to weighted quasi–arithmetic means as well. Weighted quasi–arithmetic means are strict.

If φ is equivalent to the identity map id on I, $A_\varphi(x,y;\lambda)$ is simply denoted by $A(x,y;\lambda)$ and is the well-known weighted arithmetic mean

$$A(x,y;\lambda) := \lambda x + (1-\lambda)y \qquad (x,y \in I).$$

A generalised Matkowski–Sutô problem can be formulated for weighted quasi–arithmetic means in the following way.
Let $0 < \lambda < 1,\ \mu \neq 0, 1$. Find all $\varphi, \psi \in CM(I)$ satisfying

$$\mu A_\varphi(x,y;\lambda) + (1-\mu)A_\psi(x,y;\lambda) = A(x,y;\lambda), \tag{5.1}$$

that is,

$$\begin{aligned}\mu\varphi^{-1}(\lambda\varphi(x) + (1-\lambda)\varphi(y)) +& \\ (1-\mu)\psi^{-1}(\lambda\psi(x) + (1-\lambda)\psi(y)) &= \lambda x + (1-\lambda)y\end{aligned} \tag{5.2}$$

for all $x, y \in I$.

The case $\lambda = \mu = \frac{1}{2}$ is the original Matkowski–Sutô problem.

The complete description of the solution has not been given yet. The best result was obtained by Daróczy and Páles [DP].

Theorem 5.1. *Let $0 < \lambda < 1$ and $\mu > 0$ $(\mu \neq 1)$. If $\varphi, \psi \in CM(I)$ solve the generalised Matkowski–Sutô problem* (5.2) *and φ, ψ are continuously* *differentiable with nonvanishing derivatives on I then the following cases are possible:*

(i) *If $\lambda \neq \frac{1}{2}$ then $(\varphi,\psi) \sim (\chi_0,\chi_0)$ on I;*
(ii) *If $\lambda = \frac{1}{2}$ and $\mu \notin \{\frac{1}{2}, 2\}$ then $(\varphi,\psi) \sim (\chi_0,\chi_0)$ on I;*
(iii) *If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then there exists $(s_1, s_2) \in S(I)$ such that $(\varphi,\psi) \sim (s_1, s_2)$ on I;*
(iv) *If $\lambda = \frac{1}{2}$ and $\mu = 2$ then there exists $(t_1, t_2) \in T(I)$ such that $(\varphi,\psi) \sim (t_1, t_2)$ on I.*

The above notation means:

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I),$$

$$\begin{aligned} P_+(I) &:= \{p \in \mathbb{R} \mid I + p \subset \mathbb{R}_+\} \\ P_-(I) &:= \{p \in \mathbb{R} \mid -I + p \subset \mathbb{R}_+\}, \end{aligned}$$

$$\begin{aligned} \gamma_p(x) &:= \sqrt{x+p} \qquad \text{if } p \in P_+(I) \qquad (x \in I) \\ \delta_p(x) &:= \sqrt{-x+p} \qquad \text{if } p \in P_-(I) \qquad (x \in I), \end{aligned}$$

and

$$\begin{aligned} S(I) &:= \{(\chi_p, \chi_{-p}) \mid p \in \mathbb{R}\}, \\ T(I) &:= \{(\chi_0,\chi_0)\} \cup \{(\gamma_p, \log\gamma_p) \mid p \in P_+(I)\} \cup \\ &\qquad \{(\delta_p, \log\delta_p) \mid p \in P_-(I)\}. \end{aligned}$$

The functions given in (i)–(iv) are solutions of (5.2).

To prove the extension theorem for weighted quasi–arithmetic means we need the following lemma, which is a generalisation of a result of Daróczy and Páles [DP02a].

Lemma 5.1. *Let $\varphi, \psi \in CM(I)$ satisfy (5.2) on I and let $J \subset I$ be a proper subinterval on which $\varphi \sim f_1$ and $\psi \sim f_2$, where $(f_1, f_2) \in S(I) \cup T(I)$. Then there exist $\widetilde{\varphi}, \widetilde{\psi} \in CM(I)$ satisfying (5.2) such that $\varphi \sim \widetilde{\varphi}$ and $\psi \sim \widetilde{\psi}$ on I and*

$$\widetilde{\varphi}(x) = f_1(x), \qquad \widetilde{\psi}(x) = f_2(x) \qquad (x \in J).$$

Proof. There exist constants $\alpha_i \neq 0$ and β_i $(i = 1,2)$ such that

$$\alpha_1\varphi(x) + \beta_1 = f_1(x), \qquad \alpha_2\psi(x) + \beta_2 = f_2(x)$$

for all $x \in K$. Then $\widetilde{\varphi} := \alpha_1\varphi + \beta_1$ and $\widetilde{\psi}(x) := \alpha_2\psi + \beta_2$ have the asserted properties. □

The extension theorem we prove for weighted quasi–arithmetic means is the following.

Theorem 5.2. *Let $\varphi, \psi \in CM(I)$ satisfy* (5.2) *for all $x, y \in I$ and let $(f_1, f_2) \in S(I) \cup T(I)$. Suppose that J is a proper subinterval of I on which $\varphi \sim f_1$ and $\psi \sim f_2$. Then $\varphi \sim f_1$ and $\psi \sim f_2$ on I.*

Proof. The proof is long and consists of several steps. Therefore we interrupt its course and insert some remarks and prove some claims within the proof. Hopefully, this makes the proof easier to follow.

According to Lemma 5.1, without loss of generality we can suppose that

$$\varphi(x) = f_1(x), \qquad \psi(x) = f_2(x) \qquad (x \in J).$$

We need to show that $\varphi = f_1$ and $\psi = f_2$ on the full interval I. For this purpose let $K \subset I$ be the maximal interval containing J such that

$$\varphi(x) = f_1(x), \qquad \psi(x) = f_2(x) \qquad (x \in K).$$

We are going to argue that $K = I$. By the continuity of φ and ψ, K is closed in I. Suppose to the contrary that $K \neq I$, then either $\inf K$ or $\sup K$ is an interior point of I. Say, that $a := \inf K$ is an interior point of I.

Choose another element $b \in K$ which is above a, i.e. $a < b$. Then $]a, b[$ is an open neighbourhood of $A_\varphi(a, b; \lambda)$ and $A_\psi(a, b; \lambda)$ because the two means are strict. Since φ, ψ are continuous and strictly monotone functions, there exists $\delta > 0$ such that for all $x \in [a - \delta, a] \subset I$ and $y \in]b - \delta, b[\subset K$

$$\begin{aligned} \lambda\varphi(x) + (1-\lambda)\varphi(y) &\in \varphi(K), \\ \lambda\psi(x) + (1-\lambda)\psi(y) &\in \psi(K) \end{aligned}$$

hold.

Now according to Theorem 5.1, there are the following possible cases.

If *(i)* $\lambda \neq \frac{1}{2}$ or *(ii)* $\lambda = \frac{1}{2}$ and $\mu \notin \{\frac{1}{2}, 2\}$ then $\varphi(x) = \psi(x) = x$ for all $x \in [a, b]$.

Now let $x \in [a-\delta, a]$, $y \in [b-\delta, b]$. Then

$$\begin{aligned} &\varphi(y) = y, \qquad \varphi^{-1}(t) = t \quad (t \in \varphi(K)), \\ &\psi(y) = y, \qquad \psi^{-1}(t) = t \quad (t \in \psi(K)). \end{aligned} \tag{5.3}$$

Then (5.2) implies

$$\mu\varphi(x) + (1-\mu)\psi(x) = x \qquad (x \in [a-\delta, a]),$$

that is,

$$\psi(x) = \frac{x - \mu\varphi(x)}{1-\mu} \qquad (x \in [a-\delta, a]). \tag{5.4}$$

Let $x, y \in [a-\delta, a]$ and substitute (5.4) into (5.2). Then we have

$$\psi^{-1}\left(\lambda\frac{x-\mu\varphi(x)}{1-\mu} + (1-\lambda)\frac{y-\mu\varphi(y)}{1-\mu}\right) = \frac{\lambda x + (1-\lambda)y - \mu A_\varphi(x,y;\lambda)}{1-\mu}.$$

Applying (5.4) again and arranging the equation in a more suitable form we get

$$\begin{aligned} \lambda\varphi(x) + (1-\lambda)\varphi(y) + \frac{\lambda x + (1-\lambda)y - A_\varphi(x,y;\lambda)}{1-\mu} &= \\ \varphi\left(\frac{\lambda x + (1-\lambda)y - \mu A_\varphi(x,y;\lambda)}{1-\mu}\right) &\qquad (x, y \in [a-\delta, a]). \end{aligned} \tag{5.5}$$

With the notation $f(t) := \varphi(t) - t$, equation (5.5) can also be written as

$$f(A_\varphi(x,y;\lambda)) = f\left(\frac{\lambda x + (1-\lambda)y - \mu A_\varphi(x,y;\lambda)}{1-\mu}\right) \quad (x, y \in [a-\delta, a]),$$

and using (5.1) we have

$$f(A_\varphi(x,y;\lambda)) = f(A_\psi(x,y;\lambda)) \qquad (x, y \in [a-\delta, a]). \tag{5.6}$$

In (5.5) put

$$\begin{aligned} &[\alpha, \beta] := \varphi[a-\delta, a], \qquad \varphi^{-1} =: g : [\alpha, \beta] \to [a-\delta, a], \\ &x := \varphi^{-1}(u), \qquad y := \varphi^{-1}(v) \qquad (u, v \in [\alpha, \beta]). \end{aligned}$$

Then (5.5) takes the form

$$g\left(\lambda u+(1-\lambda)v+\frac{\lambda g(u)+(1-\lambda)g(v)-g(\lambda u+(1-\lambda)v)}{1-\mu}\right) = \frac{\lambda g(u)+(1-\lambda)g(v)-\mu g(\lambda u+(1-\lambda)v)}{1-\mu} \tag{5.7}$$

for all $u, v \in [\alpha, \beta]$.

Now define the following function b:

$$b(u) := \tau u + \frac{(a-\delta)\beta - a\alpha}{\beta-\alpha} - g(u) \qquad (u \in [\alpha, \beta]),$$

where

$$\tau := \frac{\delta}{\beta-\alpha}.$$

Then

$$b(\alpha) = b(\beta) = 0, \tag{5.8}$$

and substituting function b into (5.7) we obtain

$$b\left(\lambda u+(1-\lambda)v+\frac{b(\lambda u+(1-\lambda)v)}{1-\mu}-\lambda\frac{b(u)}{1-\mu}-(1-\lambda)\frac{b(v)}{1-\mu}\right) = \frac{(\tau-\mu)b(\lambda u+(1-\lambda)v)}{1-\mu}+\frac{\lambda(1-\tau)b(u)}{1-\mu}+ \frac{(1-\lambda)(1-\tau)b(v)}{1-\mu} \qquad (u, v \in [\alpha, \beta]). \tag{5.9}$$

Let $B := \dfrac{b}{1-\mu}$ and $p := \dfrac{\tau-\mu}{1-\mu}$. Then

$$B\left(\lambda u+(1-\lambda)v+B(\lambda u+(1-\lambda)v)-\lambda B(u)-(1-\lambda)B(v)\right) = pB(\lambda u+(1-\lambda)v)+ (1-p)(\lambda B(u)+(1-\lambda)B(v)) \qquad (u, v \in [\alpha, \beta]). \tag{5.10}$$

By (5.8), also

$$B(\alpha) = B(\beta) = 0. \tag{5.11}$$

Remark 5.1. If function B is defined as above then the left hand side of (5.10) makes sense, that is, the argument belongs to the domain of B. More precisely, the argument is a strict mean; if $u < v$ then

$$u < \lambda u + (1-\lambda)v + B(\lambda u + (1-\lambda)v) - \lambda B(u) - (1-\lambda)B(v) < v.$$

Indeed, tracing back the definition of B, b, g, u, v we obtain

$$\varphi(x) < \lambda\varphi(x) + (1-\lambda)\varphi(y) + \frac{\lambda x + (1-\lambda)y - A_\varphi(x,y;\lambda)}{1-\mu} < \varphi(v)$$

if $\varphi(x) < \varphi(y)$, and this contains the expression on the left hand side of (5.5). Since the right hand side of (5.5) equals $\varphi(A_\psi(x,y;\lambda))$, this inequality holds.

Now we show that $B(x) = 0$ for all $x \in [\alpha,\beta]$. It shall be proved in a separate theorem and claims.

Theorem 5.3. *If $B : [\alpha,\beta] \to \mathbb{R}$ is a continuous function satisfying* (5.10) *and* (5.11) *then $B(x) = 0$ for all $x \in]\alpha,\beta[$.*

Proof. Suppose to the contrary that there exists $x \in]\alpha,\beta[$ for which $B(x) \neq 0$, let $B(x) > 0$, say. Then B has a maximum M on $[\alpha,\beta]$. Let $\xi := \max\{x \in [\alpha,\beta] \mid B(x) = M\}$. Then $\alpha < \xi < \beta$.

Claim 5.1. Let $\eta \in]\alpha,\beta[$. Then either
(i) B is linear in a (one-sided) neighbourhood of η with slope $p-1$, or
(ii) for all $u, v \in]\alpha,\beta[$ with $\eta = \lambda u + (1-\lambda)v$

$$B(\eta) = \lambda B(u) + (1-\lambda)B(v). \tag{5.12}$$

Proof. Let $\eta \in]\alpha,\beta[$ and $u, v \in]\alpha,\beta[$ such that $\eta = \lambda u + (1-\lambda)v$. Then by (5.10),

$$\begin{aligned} B(\eta + B(\eta) - {}&\lambda B(u) - (1-\lambda)B(v)) = \\ &pB(\eta) + (1-p)(\lambda B(u) + (1-\lambda)B(v)). \end{aligned} \tag{5.13}$$

Put $x := \eta + B(\eta) - \lambda B(u) - (1-\lambda)B(v)$. Then (5.13) can be written as

$$B(x) = (p-1)(x-\eta) + B(\eta). \tag{5.14}$$

Let $I := \{x \in [\alpha,\beta] \mid \exists u, v \in [\alpha,\beta]$ such that $\eta = \lambda u + (1-\lambda)v$ and $x = \eta + B(\eta) - \lambda B(u) - (1-\lambda)B(v)\}$. If there exist $u, v \in [\alpha,\beta]$ with $\eta = \lambda u + (1-\lambda)v$ such that *(ii)* does not hold, then I is not a singleton.

Since B is continuous, I contains an interval and, by (5.14), B is linear on a (one-sided) neighbourhood of η, i.e., *(i)* holds. $\square$

Remark 5.2. If there exist $u, v \in [\alpha, \beta]$ such that $\eta = \lambda u + (1-\lambda)v$ and $B(\eta) > \lambda B(u) + (1-\lambda)B(v)$ then there exists $x \in I$ for which $x > \eta$ and B is linear in a right-sided neighbourhood of η.

Claim 5.2. There exists $\delta_\xi > 0$ such that B is linear on $[\xi, \xi + \delta_\xi]$ with slope $p - 1$.

Proof. Since $B(\xi) = M$ and for all $\beta \geq v > \xi$ $B(v) < B(\xi)$, taking any $\alpha \leq u < \xi < v \leq \beta$ with $\xi = \lambda u + (1-\lambda)v$, we have $B(\xi) > \lambda B(u) + (1-\lambda)B(v)$. Therefore, by Claim 5.1, and Remark 5.2, B is linear in a right-sided neighbourhood of ξ with slope $p-1$. $\square$

Corollary. We have $p < 1$, for the slope has to be negative, the linear function B must decrease on $[\xi, \xi + \delta_\xi]$.

Claim 5.3. Let v_0 with $\xi < v_0 < \min\left\{\xi + \delta_\xi, \dfrac{\xi - \lambda\alpha}{1-\lambda}\right\}$ be fixed and let $0 < \delta_0 < \xi - (\lambda\alpha + (1-\lambda)v_0)$. Then for all $\eta \in [\xi - \delta_0, \xi]$ there exists $\alpha < u < \eta$ for which $\lambda u + (1-\lambda)v_0 = \eta$.

Proof. Let $u := \dfrac{\eta - (1-\lambda)v_0}{\lambda}$. We only need to show that $\alpha < u < \eta$. Since $\eta < v_0$, $u < \eta$ immediately follows. On the other hand, by the choice of δ_0, $\lambda\alpha + (1-\lambda)v_0 < \xi - \delta_0 < \eta$, which implies $\alpha < u$. $\square$

Now we return to the proof of Theorem 5.3.

B is continuous at ξ, therefore to $(M - B(v_0))(1-\lambda) > 0$ there exists $\delta \in]0, \delta_0[$ such that for all $\eta \in [\xi - \delta, \xi]$

$$0 \leq M - B(\eta) < (M - B(v_0))(1-\lambda).$$

This implies

$$\lambda M + (1-\lambda)B(v_0) < B(\eta). \tag{5.15}$$

Let $\eta \in [\xi - \delta, \xi]$ be arbitrary. Then, by Claim 5.3, there exists $u \in]\alpha, v[$ such that $\lambda u + (1-\lambda)v_0 = \eta$. Then (5.15) yields

$$\lambda B(u) + (1-\lambda)B(v_0) \leq \lambda M + (1-\lambda)B(v_0) < B(\eta).$$

By Claim 5.1 and Remark 5.2, there exists $\delta_\eta > 0$ such that B is linear on the closed interval $[\eta, \eta+\delta_\eta]$ with slope $p-1<0$.
Now let $m := \min_{u\in[\xi-\delta,\xi]} B(u)$. Then for all $\eta \in [\xi-\delta, \xi[$, $B(\eta) > m$ because B is monotone decreasing in a right-sided neighbourhood of η. Necessarily then $m = B(\xi) = M$, consequently $B(\eta) = M$ for all $\eta \in [\xi-\delta, \xi[$, which contradicts the fact that B is linearly decreasing on the interval $[\eta, \eta+\delta_\eta]$ for all $\eta \in [\xi-\delta, \xi]$.

Thus $B(x) = 0$ for all $x \in [\alpha, \beta]$.

The proof in the case when $B(x) < 0$ for some $x \in [\alpha, \beta]$ is analogous. We only take the minimum m of B on $[\alpha, \beta]$ and $\xi := \min\{x \in [\alpha,\beta] \mid B(x) = m\}$.

This completes the proof of Theorem 5.3. $\square$

Now we can continue the proof of Theorem 5.2. By Theorem 5.3, B (and consequently b) equals 0 on $[\alpha, \beta]$, i.e.,

$$\varphi(x) = \frac{1}{\tau}x + \sigma \qquad (x \in [a-\delta, a]),$$

where $\sigma = \beta - \dfrac{a}{\tau}$. From (5.4) we have

$$\psi(x) = \frac{\tau-\mu}{\tau(1-\mu)}x - \frac{\mu}{1-\mu}\sigma \qquad (x \in [a-\delta, a]).$$

Now let $x \in [a-\delta, a]$ and $y \in [a, b]$ satisfying $\lambda\varphi(x) + (1-\lambda)\varphi(y) \in \varphi([a-\delta, a])$ and $\lambda\psi(x) + (1-\lambda)\psi(y) \in \psi([a-\delta, a])$. Since

$$\varphi^{-1}(t) = \tau(t-\sigma) \qquad \text{and} \qquad \psi^{-1}(t) = \frac{\tau(1-\mu)}{\tau-\mu}\left(t + \frac{\mu}{1-\mu}\sigma\right),$$

(5.2) implies

$$\frac{\mu(1-\lambda)(\tau-1)^2}{\tau-\mu}y + \frac{\mu\tau(1-\lambda)(1-\tau)}{\tau-\mu}\sigma = 0.$$

In this equation y can take values from an interval, therefore $\tau = 1$. Since $\varphi(a) = a$ and φ is continuous, necessarily $\sigma = 0$. Thus

$$\varphi(x) = \psi(x) = x \qquad (x \in [a-\delta, a]),$$

and cases *(i)* and *(ii)* of Theorem 5.1 are proved.

Now we examine the next possibilities from Theorem 5.1.

(iii) If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then there exists $(s_1, s_2) \in S(I)$ such that $(\varphi, \psi) \sim (s_1, s_2)$ on I. This case was solved by Daróczy, Maksa and Páles [DMP00].

(iv) If $\lambda = \frac{1}{2}$ and $\mu = 2$ then we have three possibilities.
(a) $\varphi(x) = \psi(x) = x$ for all $x \in [a, b]$, which was studied in *(i)*.
(b) $\varphi(x) = \sqrt{x+p}$, $\psi(x) = \log\sqrt{x+p}$ $(x \in [a, b])$ for some $p \in P_+(I)$. Let $x \in [a - \delta, a]$, $y \in]b - \delta, b[$. Then

$$\begin{aligned} \varphi(y) &= \sqrt{y+p}, & \varphi^{-1}(t) &= t^2 - p \quad (t \in \varphi(K)), \\ \psi(y) &= \log\sqrt{y+p}, & \psi^{-1}(t) &= e^{2t} - p \quad (t \in \psi(K)). \end{aligned}$$

Now (5.2) implies

$$2\left(\frac{\varphi(x) + \sqrt{y+p}}{2}\right)^2 - 2p - e^{2\frac{\psi(x)+\sqrt{y+p}}{2}} + p = \frac{x+y}{2}.$$

From this equation we have

$$\varphi^2(x) + 2\varphi(x)\sqrt{y+p} - 2e^{\psi(x)}\sqrt{y+p} - p = x,$$

that is,

$$2\sqrt{y+p}\left(\varphi(x) - e^{\psi(x)}\right) = x + p - \varphi^2(x). \tag{5.16}$$

Since y can take values from an interval, (5.16) implies

$$\varphi^2(x) = x + p,$$

i.e.,

$$\varphi(x) = \sqrt{x+p}.$$

Again, from (5.16) we have

$$e^{\psi(x)} = \varphi(x),$$

so

$$\psi(x) = \log\sqrt{x+p},$$

which proves (b).

(c) $\varphi(x) = \sqrt{-x+p}$, $\psi(x) = \log\sqrt{-x+p}$ $(x \in [a,b])$ for some $p \in P_-(I)$. This case is similar to (b), so we ignore the proof. □

5.2. Quasi–arithmetic means of order α.

Definition 5.2. ([DP02b]) Let $\alpha \geq -1$. A function $M : I^2 \to I$ is called a *quasi–arithmetic mean of order* α on I if there exists $\varphi \in CM(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)+\alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right) =: A_\varphi^{(\alpha)}(x,y) \tag{5.17}$$

for all $x, y \in I$. Then the function $\varphi \in CM(I)$ is called the *generating function* of the quasi–arithmetic mean of order α on I.

Remark 5.3. (1) It is easy to see that $A_\varphi^{(\alpha)} : I^2 \to I$ is a *mean* on I if $\alpha \geq -1$ and $\varphi \in CM(I)$.

(2) If $\alpha = 0$ then $A_\varphi^{(0)}(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)$ is the well–known quasi–arithmetic mean on I with generating function $\varphi \in CM(I)$.

(3) If $\alpha = -1$ then $A_\varphi^{(-1)}(x,y) = \varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right)$ is the known *conjugate arithmetic mean* on I with generating function $\varphi \in CM(I)$ (cf. [Dar99], [Dar00], [DP01]).

The equivalence of generating functions can be defined just as in Definition 4.3, and Theorem 4.1 applies to quasi–arithmetic means of order α as well.

The Matkowski–Sutô type problem for quasi–arithmetic means of order α is the following: Find all $\varphi, \psi \in CM(I)$ for which

$$A_\varphi^{(\alpha)}(x,y) + A_\psi^{(\alpha)}(x,y) = x + y \qquad (x, y \in I). \tag{5.18}$$

Daróczy and Páles ([DP02b]) proved the following

Theorem 5.4. *If $\varphi, \psi \in CM(I)$ satisfy* (5.18) *and either φ or ψ is continuously differentiable on I then there exists $p \in \mathbb{R}$ for which* (4.3) *holds.*

Knowing the solutions of the Matkowski–Sutô type problem (5.18), we state and prove the extension theorem. The special case when $\alpha = -1$ was studied by Hajdu [Haj02], here we examine the general case.

Theorem 5.5. *Let $\varphi, \psi \in CM(I)$ satisfy* (5.18) *on I and let J be a proper subinterval of I on which $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ for some $p \in \mathbb{R}$. Then $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on I.*

Proof. The beginning proof is similar to that of Theorem 5.2. Again, according to Lemma 5.1, without loss of generality we can suppose that

$$\varphi(x) = \chi_p(x), \qquad \psi(x) = \chi_{-p}(x) \qquad (x \in J).$$

We need to show that $\varphi = \chi_p$ and $\psi = \chi_{-p}$ on the full interval I. Let $K \subset I$ be the maximal interval containing J such that

$$\varphi(x) = \chi_p(x), \qquad \psi(x) = \chi_{-p}(x) \qquad (x \in K).$$

We are going to prove that $K = I$. Since φ and ψ are continuous, K is closed in I. Suppose to the contrary that $K \neq I$, then either $\inf K$ or $\sup K$ is an interior point of I. Say, that $a := \inf K$ is an interior point of I.

Choose another element $b \in K$ with $a < b$. Then $]a, b[$ is an open neighbourhood of $A_\varphi^{(\alpha)}(a, b)$ and $A_\psi^{(\alpha)}(a, b)$ because the two means are strict. Since φ, ψ are continuous and strictly monotone functions, there exists $\delta > 0$ such that for all $x \in [a - \delta, a] \subset I$ and $y \in]b - \delta, b[\subset K$

$$\begin{aligned} \frac{x+y}{2} &\in K \\ \frac{\varphi(x) + \varphi(y) + \alpha\varphi(\frac{x+y}{2})}{2+\alpha} &\in \varphi(K), \\ \frac{\psi(x) + \psi(y) + \alpha\psi(\frac{x+y}{2})}{2+\alpha} &\in \psi(K) \end{aligned}$$

hold.

Now there are two possible cases. Either *(i)* $p \neq 0$ or *(ii)* $p = 0$.

(i) In this case $\varphi(x) = e^{px}$ and $\psi(x) = e^{-px}$ for all $x \in]a, b[$. Let $x \in [a - \delta, a], y \in]b - \delta, b[$. Then

$$\varphi(y) = e^{py}, \qquad \varphi\left(\tfrac{x+y}{2}\right) = e^{p\frac{x+y}{2}}, \qquad \varphi^{-1}(t) = \frac{1}{p}\log t \quad (t \in \varphi(K)),$$

$$\psi(y) = e^{-py}, \qquad \psi\left(\tfrac{x+y}{2}\right) = e^{-p\frac{x+y}{2}}, \qquad \psi^{-1}(t) = -\frac{1}{p}\log t \quad (t \in \psi(K)).$$

Now (5.18) can be written as

$$\frac{1}{p}\log\frac{\varphi(x)+e^{py}+\alpha e^{p\frac{x+y}{2}}}{2+\alpha}-\frac{1}{p}\log\frac{\psi(x)+e^{-py}+\alpha e^{-p\frac{x+y}{2}}}{2+\alpha}=x+y,$$

from which

$$\frac{\varphi(x)+e^{py}+\alpha e^{p\frac{x+y}{2}}}{\psi(x)+e^{-py}+\alpha e^{-p\frac{x+y}{2}}}=e^{px}e^{py}$$

follows, and we have

$$\varphi(x)-e^{px}=e^{py}\left(\psi(x)e^{px}-1\right).$$

Since y takes values from an interval, this yields

$$\psi(x)e^{px}-1=0,$$

whence

$$\psi(x)=e^{px},\qquad \varphi(x)=e^{px}\qquad (x\in[a-\delta,a]).$$

Thus $\psi(x)=e^{-px}$, $\varphi(x)=e^{px}$ on the whole of $]a-\delta,b[$, and *(i)*.

(ii) In this case $\varphi(x)=\psi(x)=x$ for all $x\in]a,b[$. Again, let $x\in[a-\delta,a], y\in]b-\delta,b[$ then

$$\begin{aligned}&\varphi(y)=y, &&\varphi\left(\tfrac{x+y}{2}\right)=\tfrac{x+y}{2}, &&\varphi^{-1}(t)=t\quad (t\in\varphi(K)),\\ &\psi(y)=y, &&\psi\left(\tfrac{x+y}{2}\right)=\tfrac{x+y}{2}, &&\psi^{-1}(t)=t\quad (t\in\psi(K)).\end{aligned}$$

Now (5.18) can be written as

$$\frac{\varphi(x)+y+\alpha\frac{x+y}{2}}{2+\alpha}+\frac{\psi(x)+y+\alpha\frac{x+y}{2}}{2+\alpha}=x+y,$$

from which

$$\varphi(x)+\psi(x)=2x,$$

that is,

$$\psi(x)=2x-\varphi(x)\qquad (x\in[a-\delta,a]). \tag{5.19}$$

Now let $x,y\in[a-\delta,a]$. Then, by (5.19), (5.18) yields

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)+\alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right)+$$
$$\psi^{-1}\left(\frac{2x-\varphi(x)+2y-\varphi(y)+\alpha\left(x+y-\varphi\left(\frac{x+y}{2}\right)\right)}{2+\alpha}\right)=x+y.$$

This implies

$$\psi^{-1}\left(\frac{2x-\varphi(x)+2y-\varphi(y)+\alpha\left(x+y-\varphi\left(\frac{x+y}{2}\right)\right)}{2+\alpha}\right)=$$
$$x+y-A_\varphi^{(\alpha)}(x,y),$$

from which, applying ψ to both sides, then (5.19) again,

$$\varphi(x+y-A_\varphi^{(\alpha)}(x,y))-(x+y-A_\varphi^{(\alpha)}(x,y))=\varphi(A_\varphi^{(\alpha)}(x,y))-A_\varphi^{(\alpha)}(x,y)$$

follows for all $x,y\in[a-\delta,a]$. Introducing the notation $f(t):=\varphi(t)-t$, we have

$$f(x+y-A_\varphi^{(\alpha)}(x,y))=f(A_\varphi^{(\alpha)}(x,y))\qquad(x,y\in[a-\delta,a]),$$

that is, using (5.18)

$$f(A_\varphi^{(\alpha)}(x,y))=f(A_\psi^{(\alpha)}(x,y))\qquad(x,y\in[a-\delta,a]). \tag{5.20}$$

By a result of Daróczy and Ng [DN00] (see Section 6, Corollary 6.1), there exists $\sigma\neq 0$ and τ such that $\varphi(x)=\sigma x+\tau$ for all $x\in[a-\delta,a]$. We shall show that $\sigma=1$ and $\tau=0$. By (5.19), then also $\psi(x)=x$ ($x\in[a-\delta,a]$).

From (5.19) we have

$$\psi(x)=2x-\varphi(x)=(2-\sigma)x-\tau\qquad(x\in[a-\delta,a]).$$

Now let $x\in[a-\delta,a]$ and let $y\in[a,b]$ such that

$$\begin{aligned}\frac{x+y}{2}&\in[a-\delta,a],\\ \frac{\varphi(x)+\varphi(y)+\alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}&\in\varphi([a-\delta,a])\\ \frac{\psi(x)+\psi(y)+\alpha\psi\left(\frac{x+y}{2}\right)}{2+\alpha}&\in\psi([a-\delta,a]).\end{aligned}$$

Then

$$\begin{aligned}
&\varphi(x) = \sigma x + \tau, && \varphi(y) = y,\\
&\varphi\left(\frac{x+y}{2}\right) = \sigma\frac{x+y}{2} + \tau, && \varphi^{-1}(t) = \frac{t-\tau}{\sigma},\\
&\psi(x) = (2-\sigma)x - \tau, && \psi(y) = y,\\
&\psi\left(\frac{x+y}{2}\right) = (2-\sigma)\frac{x+y}{2} - \tau, && \psi^{-1}(t) = \frac{t+\tau}{2-\sigma}.
\end{aligned}$$

Now (5.18) implies

$$\frac{1}{\sigma}\left(\frac{\sigma x + \tau + y + \alpha\sigma\frac{x+y}{2} + \alpha\tau}{2+\alpha} - \tau\right) + \\
\frac{1}{2-\sigma}\left(\frac{(2-\sigma)x - \tau + y + \alpha(2-\sigma)\frac{x+y}{2} - \alpha\tau}{2+\alpha} + \tau\right) = x + y,$$

from which

$$\left(\frac{1}{\sigma} + \frac{1}{2-\sigma} - 2\right) y = \frac{\tau}{\sigma} - \frac{\tau}{2-\sigma}$$

follows. Since this equation for y holds on an interval, this yields

$$\frac{1}{\sigma} + \frac{1}{2-\sigma} - 2 = \frac{\tau}{\sigma} - \frac{\tau}{2-\sigma} = 0,$$

and necessarily $\sigma = 1$. Since $\varphi(a) = a$ and φ is continuous, $\tau = 0$.

This completes the proof of the Theorem 5.5. □

Remark 5.4. Equations (5.6) and (5.20) are interesting themselves, Section 6 concerns this topic.

6. Complementary means

When we proved the extension theorem for quasi–arithmetic means of order α (Theorem 5.5, equation (5.20)) we used a general result of Daróczy and Ng [DN00].

Definition 6.1. Let $I := [a, b] \subset \mathbb{R}$ be a nonvoid interval and let M be a mean on I. Then

$$\hat{M}(x, y) := x + y - M(x, y) \qquad (x, y \in I)$$

is also a mean on I, and $\hat{M}$ is called *complementary* to M.

The pair $(M, \hat{M})$ satisfies $A(M, \hat{M}) = A$, where A is the arithmetic mean.

Definition 6.2. Let M be a mean on $[a, b]$. A function $f : [a, b] \to \mathbb{R}$ is said to be *M–associate* if it possesses the following property.
If $x, y \in [a, b]$ satisfy $M(x, y) = \dfrac{x + y}{2}$ and $f(x) = f\left(\dfrac{x + y}{2}\right)$ then $f(y) = f(x)$.

The main result of Daróczy and Ng is the following

Theorem 6.1. *Let M be a mean on $[a, b]$ and let $f : [a, b] \to \mathbb{R}$ be a function satisfying the functional equation*

$$f(M(x, y)) = f(x + y - M(x, y)) \tag{6.1}$$

for all $x, y \in [a, b]$. Then

(a) *For each $x, y \in [a, b]$ with $M(x, y) \neq A(x, y)$, f is locally constant at $A(x, y)$.*
(b) *If f is continuous and M–associate then either*
   (i) f is constant on $[a, b]$, or
   (ii) f is injective on $[a, b]$ and $M = A$.

The following corollary of Theorem 6.1 gives the solutions of equation (5.20).
Corollary 6.1. If $\varphi \in CM(I)$, and function f defined by

$$f(x) := \varphi(x) - x \qquad (x \in I)$$

satisfies (5.20) (i.e., f satisfies (6.1) with $M = A_\varphi^{(\alpha)}$) then there exist constants $\alpha \neq 0$, β such that

$$\varphi(x) = \alpha x + \beta \qquad (x \in I).$$

Proof. First we show that f is $A_\varphi^{(\alpha)}$–associate. For this purpose let $x, y \in I$ with $A_\varphi^{(\alpha)}(x,y) = \frac{x+y}{2}$ and $f(x) = f\left(\frac{x+y}{2}\right)$. Then

$$\varphi(x) + \varphi(y) - \varphi\left(\frac{x+y}{2}\right) = \varphi\left(\frac{x+y}{2}\right)$$

and

$$\varphi(x) - x = \varphi\left(\frac{x+y}{2}\right) - \frac{x+y}{2}.$$

Using these equations we easily obtain $f(y) = \varphi(y) - y = \varphi(x) - x = f(x)$.

There are two possible cases:

(i) If f is constant on I, then with the notation $f(x) := \beta$ $(x \in I)$ and $\alpha := 1$ we have the assertion.

(ii) If f is not constant, then by Theorem 6.1, $A_\varphi^{(\alpha)} = A$, thus $\varphi(x) = \alpha x + \beta$ $(x \in I)$ for some $\alpha \neq 0$, β. □

Theorem 6.1 was generalised by Daróczy, Hajdu, and Ng [DHN]. Now we give an even more general theorem, which contains the mentioned result when $\mu = \lambda$.

Definition 6.3. Let M, N be strict means on I and let $0 < \lambda < 1$, $\mu \neq 0, 1$. N is called (μ, λ)–*complementary* to M if

$$\mu M(x,y) + (1-\mu) N(x,y) = \lambda x + (1-\lambda) y \qquad (x, y \in I). \tag{6.2}$$

Remark 6.1. (1) Without loss of generality we may assume $\mu > 0$, otherwise we interchange the terms on the left hand side of (6.2).

(2) If M is a strict mean on I, and N is defined according to (6.2), that is,

$$N(x,y) := \frac{\lambda x + (1-\lambda) y - \mu M(x,y)}{1-\mu} \qquad (x, y \in I)$$

then N is a strict mean if $0 < \mu < \min\{\lambda, 1-\lambda\}$ or $\mu < 0$.

Definition 6.4. A function $f : I \to \mathbb{R}$ is called (M, μ, λ)–*associate* if it possesses the following property:

If $x, y \in I$ satisfy $M(x,y) = N(x,y)$ and $f(x) = f(M(x,y))$ then $f(y) = f(x)$.

Theorem 6.2. *Let $0 < \lambda < 1$, $\mu > 0$, $\mu \neq 1$, let M, N be strict means on I such that N is (μ, λ)-complementary to M, and let $f : I \to \mathbb{R}$ be a continuous function satisfying the functional equation*

$$f(M(x,y)) = f(N(x,y)) \tag{6.3}$$

for all $x, y \in I$. Then

(a) *For each $x, y \in I$ where $M(x,y) \neq N(x,y)$, f is locally constant at $A(x,y;\lambda)$ on one side.*
(b) *If f is (M, μ, λ)–associate then either*
  (i) f is constant on I, or
  (ii) $M(x,y) = N(x,y) = A(x,y;\lambda)$ for all $x, y \in I$.

Proof. Since the proof is long and consists of several steps, we interrupt its course to insert and prove separate claims.

Let us denote by I_{xy} the closed interval joining $M(x,y)$ and $N(x,y)$ and recall that $A(x,y;\lambda) := \lambda x + (1-\lambda)y$ is the weighted arithmetic mean on I. We also recall that $\mu M(x,y) + (1-\mu)N(x,y) = A(x,y;\lambda)$.

First we examine the case when $\mu > 1$.

Claim 6.1. For each $x_0, y_0 \in I$ there are two possible cases:

(I) If $M(x_0,y_0) \leq N(x_0,y_0)$ then

$$f(A(x_0,y_0;\lambda) + s) = f\left(A(x_0,y_0;\lambda) + \frac{\mu}{\mu-1}s\right)$$

for all $0 \leq s \leq M(x_0,y_0) - A(x_0,y_0;\lambda)$.

(II) If $N(x_0,y_0) < M(x_0,y_0)$ then

$$f(A(x_0,y_0;\lambda) - s) = f\left(A(x_0,y_0;\lambda) - \frac{\mu}{\mu-1}s\right)$$

for all $0 \leq s \leq A(x_0,y_0;\lambda) - M(x_0,y_0)$.

Proof. The assertion is trivial when $I_{x_0y_0}$ is a singleton. Suppose $I_{x_0y_0}$ is proper. There are two cases: either $x_0 < y_0$ or $y_0 < x_0$. First let $x_0 < y_0$.

Consider $x_t := x_0 + t, y_t := y_0 - \frac{\lambda}{1-\lambda}t$ for $0 \le t \le A(x_0, y_0; \lambda) - x_0$. We note that for all $t \in [0, A(x_0, y_0; \lambda) - x_0]$ we have

$$\lambda x_t + (1-\lambda)y_t = A(x_0, y_0; \lambda),$$

and consequently

$$\mu M(x_t, y_t) + (1-\mu)N(x_t, y_t) = A(x_0, y_0; \lambda).$$

Now suppose $M(x_0, y_0) < N(x_0, y_0)$. This immediately implies $M(x_0, y_0) > A(x_0, y_0; \lambda)$. The function $t \mapsto M(x_t, y_t)$ is a continuous function taking the values $M(x_0, y_0)$ and $A(x_0, y_0; \lambda)$. By the Intermediate Value Theorem, to each $0 \le s \le M(x_0, y_0) - A(x_0, y_0; \lambda)$, there exists $t \in [0, A(x_0, y_0; \lambda) - x_0]$ such that

$$M(x_t, y_t) = A(x_0, y_0; \lambda) + s \quad \text{and} \quad N(x_t, y_t) = A(x_0, y_0; \lambda) - \frac{\mu}{1-\mu}s.$$

Thus by equation (6.3),

$$f(A(x_0, y_0; \lambda) + s) = f\left(A(x_0, y_0; \lambda) + \frac{\mu}{\mu - 1}s\right).$$

A similar argument proves that if $N(x_0, y_0) < M(x_0, y_0)$ then to each $0 \le s \le A(x_0, y_0; \lambda) - M(x_0, y_0)$, there exists $t \in [0, A(x_0, y_0; \lambda) - x_0]$ such that $M(x_t, y_t) = A(x_0, y_0; \lambda) - s,\quad N(x_t, y_t) = A(x_0, y_0; \lambda) + \frac{\mu}{1-\mu}s$. Then again, by equation (6.3),

$$f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) - \frac{\mu}{\mu - 1}s\right).$$

If $y_0 < x_0$ then let $x_t := x_0 - \frac{1-\lambda}{\lambda}t, y_t := y_0 + t$ for $0 \le t \le A(x_0, y_0; \lambda) - y_0$. The rest of the proof goes in the same way as above. □

Claim 6.2. Suppose f is continuous and $I_{x_0y_0}$ is proper. Then f is constant on the closed interval joining $A(x_0, y_0; \lambda)$ and $N(x_0, y_0)$.

Proof. Since the other case is analogous, we only examine $M(x_0, y_0) < N(x_0, y_0)$.

Let $c := f(A(x_0, y_0; \lambda))$. Suppose that there exists $u_0 \in [A(x_0, y_0; \lambda), N(x_0, y_0)]$ such that $f(u_0) \neq c$. Now let

$$u_1 := A(x_0, y_0; \lambda) + \frac{\mu - 1}{\mu}(u_0 - A(x_0, y_0; \lambda)).$$

By Claim 6.1 (I), $f(u_1) = f(u_0)$. Create a sequence

$$u_n := A(x_0, y_0; \lambda) + \frac{\mu - 1}{\mu}(u_{n-1} - A(x_0, y_0; \lambda)) = A(x_0, y_0; \lambda) + \left(\frac{\mu - 1}{\mu}\right)^n (u_0 - A(x_0, y_0; \lambda)) \qquad (n \in \mathbb{N}).$$

Since $0 < \frac{\mu-1}{\mu} < 1$, $u_n \to A(x_0, y_0; \lambda)$ $(n \to \infty)$. On the other hand, $f(u_n) = f(u_{n-1}) = \cdots = f(u_0) \neq c$ $(n \in \mathbb{N})$, which contradicts the continuity of f. $\square$

The above proves (a) for $\mu > 1$. To prove (b), in what follows we assume that f is continuous and (M, μ, λ)–associate.

Claim 6.3. Suppose there exist $x^* < y^*$ such that $I_{x^* y^*}$ is proper. Then f is constant on I.

Proof. Let $J \subset I$ be the *maximal* interval containing $A(x^*, y^*; \lambda)$ on which f is constant, i.e.,

$$J := \{x \in I \mid f(y) = c \text{ for all } y \text{ in the closed interval joining } x \text{ and } A(x^*, y^*; \lambda)\},$$

where $c := f(A(x^*, y^*; \lambda))$. By the continuity of f, J is closed relative to I; and by Claim 6.2, it is a proper interval. We shall argue that $J = I$; thus f is constant on I.

Suppose to the contrary that $\beta := \sup J$ is an interior point of I. Then there exists $\varepsilon > 0$ such that $\beta - \varepsilon \in J$ and $\beta + \frac{\lambda}{1-\lambda}\varepsilon \in I$. Let $y_0 \in]\beta, \beta + \frac{\lambda}{1-\lambda}\varepsilon]$ for which $f(y_0) \neq c$. Then there exists a unique $x_0 \in [\beta - \varepsilon, \beta[$ such that $A(x_0, y_0; \lambda) = \beta$. Now there are three possible cases:

(a) If $M(x_0, y_0) < N(x_0, y_0)$ then by Claim 6.1 (I), f would be constant in a (right-sided) neighbourhood of β and so J would not be maximal.

(b) If $M(x_0, y_0) = N(x_0, y_0) = \beta$, i.e., $I_{x_0 y_0}$ is a singleton then since x_0 and β belong to J,

$$f(x_0) = f(\beta) = c.$$

As f is (M, λ, μ)–associate, we get $f(y_0) = c$, a contradiction.

(c) If $N(x_0, y_0) < M(x_0, y_0) < \beta$ then there exists $x < x' < \beta$ for which $N(x', y_0) < M(x', y_0) < \beta$. Let $\beta' := \lambda x' + (1 - \lambda) y_0 > \beta$. By Claim 6.1 (II), f is constant on $[N(x', y_0), \beta']$ and this interval contains β. Again, in this case J would not be maximal.

Thus $\sup J = \sup I$. One can similarly prove that $\inf J = \inf I$. J being closed in I, we have $J = I$ and this completes the proof of Claim 6.3. □

Now let $0 < \mu < 1$.

The following claims and their proofs are the equivalent of the previous ones.

Claim 6.4. For each $x_0, y_0 \in I$ there are two possible cases:

(I) If $M(x_0, y_0) \le N(x_0, y_0)$ then
$$f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) + \frac{\mu}{1-\mu}s\right)$$
for all $0 \le s \le A(x_0, y_0; \lambda) - M(x_0, y_0)$.

(II) If $N(x_0, y_0) < M(x_0, y_0)$ then
$$f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) + \frac{1-\mu}{\mu}s\right)$$
for all $0 \le s \le A(x_0, y_0; \lambda) - N(x_0, y_0)$.

Proof. The proof of Claim 6.1 can be applied here. □

Claim 6.5. Suppose $I_{x_0 y_0}$ is proper. Then f is locally constant at $A(x_0, y_0; \lambda)$; i.e., there exists a neighbourhood of $A(x_0, y_0; \lambda)$ on which f is constant.

Proof. The other case is similar, so we only examine the case when $M(x_0, y_0) < N(x_0, y_0)$.

Let $x_0 < y_0$, say. For some sufficiently small $\delta > 0$, we have $[A(x_0, y_0; \lambda) - \lambda\delta, A(x_0, y_0; \lambda) + \frac{\mu}{1-\mu}\lambda\delta] \subset I_{x y_0}$ for all $x \in [x_0, x_0 + \delta]$.

Now for all $x \in [x_0, x_0 + \delta]$, $I_{x y_0}$ is proper, and by Claim 6.4,
$$f(A(x, y_0; \lambda) - s) = f(A(x, y_0; \lambda) + \frac{\mu}{1-\mu}s)$$
whenever both arguments are in $[A(x_0, y_0; \lambda) - \lambda\delta, A(x_0, y_0; \lambda) + \frac{\mu}{1-\mu}\lambda\delta]$. The point $A(x, y_0; \lambda)$ being arbitrary in $[A(x_0, y_0; \lambda), A(x_0, y_0; \lambda) + \lambda\delta]$, this gives the constancy of f on $[A(x_0, y_0; \lambda) - \lambda\delta, A(x_0, y_0; \lambda) + \frac{\mu}{1-\mu}\lambda\delta]$.

The other cases, when $y_0 < x_0$ and (II) can be proved similarly. □

The above proves (a) for $0 < \mu < 1$. To prove (b), in what follows we assume that f is continuous and (M, μ, λ)–associate.

Claim 6.6. Suppose there exist $x_0 < y_0$ such that $I_{x_0 y_0}$ is proper. Then f is constant on I.

Proof. Like in the proof of Claim 6.3, let $J \subset I$ be the *maximal* interval containing $A(x_0, y_0; \lambda)$ on which f is constant, i.e.,

$$J := \{x \in I \mid f(y) = c \text{ for all } y \text{ in the closed interval joining } x \text{ and } A(x_0, y_0; \lambda)\},$$

where $c := f(A(x_0, y_0; \lambda))$. By the continuity of f, J is closed relative to I; and by Claim 6.5, it is a proper interval neighbourhood of $A(x_0, y_0; \lambda)$. We shall show that $J = I$; thus f is constant on I.

Suppose to the contrary that $\beta := \sup J$ is an interior point of I. Then there exists $\varepsilon > 0$ such that $\beta - \varepsilon \in J$ and $\beta + \frac{\lambda}{1-\lambda}\varepsilon \in I$. Let $y \in]\beta, \beta + \frac{\lambda}{1-\lambda}\varepsilon]$ for which $f(y) \neq c$. Then there exists a unique $x \in [\beta - \varepsilon, \beta[$ such that $A(x, y; \lambda) = \beta$. If the interval I_{xy} were proper then f would be constant in a neighbourhood of β by Claim 6.5, and so J would not be maximal. Therefore I_{xy} is a singleton, that is, $M(x, y) = N(x, y)$. So

$$M(x, y) = A(x, y; \lambda) = \beta.$$

Because x and β belong to J,

$$f(x) = f(\beta) = c,$$

and since f is (M, μ, λ)–associate, we get $f(y) = c$, a contradiction. Thus $\sup J = \sup I$. One can similarly prove that $\inf J = \inf I$. J being closed in I, we have $J = I$ and this completes the proof of Claim 6.6. $\square$

Now we return to the general case when $\mu > 0,\ \mu \neq 1$.

Suppose that f is nonconstant on I. Then by Claim 6.3 and Claim 6.6, I_{xy} is a singleton for all $x, y \in I$, that is, $M(x, y) = N(x, y)$. As $A(x, y; \lambda) = \mu M(x, y) + (1 - \mu) N(x, y)$, we get $A(x, y; \lambda) = M(x, y) = N(x, y)$, proving (b) of Theorem 6.2. $\square$

Summary

This PhD dissertation contains new results in the theory of functional equations. It consists of two parts, which are quite different from each other.

In the first half of the dissertation we start out from an identity of the brilliant Hindu mathematician Ramanujan. Among the many, sometimes amazing results in number theory, Z. Daróczy found one he thought interesting and set up a functional equation generalising the original identity. He asked whether Ramanujan found all the identities of that kind or not. In other words, what are the general solutions of the equation on the set $\mathbb{Z}$ of the integers.

We found that every solution on $\mathbb{Z}$ can be written as a linear combination of eleven linearly independent functions (Section 2, Theorem 2.5).

Next we replaced the set of integers by another abstract algebraic structure and also determined the solutions (Section 3, Theorem 3.9).

Part II is more "real" analysis, in both senses of meaning. Firstly, it deals with problems on the set $\mathbb{R}$ of the real numbers. Secondly, it is really analysis, with functions, monotonity, continuity, etc.

These sections mainly deal with extension theorems for Matkowski–Sutô type problems and material connected with them. In Section 4 we give the basic definitions (means, quasi–arithmetic means) and preliminaries of the problem and also the original extension theorem (Theorem 4.6). We also show another aspect, the Gauss–composition.

Briefly speaking, the original Matkowski–Sutô problem is when the sum of two quasi–arithmetic means equals the double of the arithmetic mean. When proving the theorem that gives the solution, it is necessary to extend the solutions from a subinterval to the whole. That is why the original extension theorem was proved. Since this theorem is crucial in the proof, and since Matkowski–Sutô type problems are formulated for other classes of means, it is important to state and prove the extension theorem in other cases as well.

The extension theorems for weighted quasi–arithmetic means and quasi–arithmetic means of order α and their proofs are presented in Section 5 (Theorem 5.2, Theorem 5.5).

Section 6 provides another tool to prove the extension theorems, the complementary means. We think that these results and proofs are interesting not only because of their use in the proofs of the previous section but also in their own right.

Összefoglaló

Ezen doktori értekezés a függvényegyenletek témakörében tartalmaz új eredményeket. Mivel a matematikának ez a területe oly gazdag és szerteágazó, egy ilyen dolgozatban csak egy–egy kisebb, de érdekes terület vizsgálatát tűzhetjük ki célul. Az értekezés két fejezete meglehetősen különbözik egymástól, jelezve, hogy függvényegyenletek lépten–nyomon felbukkannak a matematika majd minden ágában.

Az első fejezetben a fiatalon elhunyt zseniális indiai matematikus, Ramanudzsan egy számelméleti azonosságából indulunk ki. Ramanudzsan (1887–1920) a matematika szinte minden területén alkotott maradandót; jegyzetfüzeteiben (a híres Notebooks-ban) például a következő témakörökkel kapcsolatos tételek és példák maradtak fenn: mágikus négyzetek, harmonikus sorok, kombinatorika, Euler–polinomok és –számok, Bernoulli–számok, divergens sorok, végtelen sorok transzformációi és kiértékelése, hipergeometrikus sorok, lánctörtek, elemi algebra, számelmélet, prímszámelmélet, integrálok, parciális törtek, elemi analízis. És a felsorolást lehetne még folytatni.

Ramanudzsan néhány lánctörtekkel kapcsolatos tétele és formulája olyan meglepő és különös, hogy mint Hardy fogalmazott: "Ha nem lennének igazak, nincs ember, aki kitalálta volna őket". Ugyanez vonatkozik néhány számelméleti összefüggésére is.

A számos számelméleti azonosság között fedezte fel Daróczy Zoltán a következőt, melyet érdekesnek talált.

Ha $ad = bc$, akkor

$$(a+b+c)^n+(b+c+d)^n+(a-d)^n = (a+b+d)^n+(a+c+d)^n+(b-c)^n,$$

ahol $n = 2$ vagy 4.

Mindkét oldalon elvégezve a hatványozást, az állítás könnyen ellenőrizhető.

A szövegkörnyezetből valószínűnek tűnik, hogy Ramanudzsan a fenti azonosságot az egész számok $\mathbb{Z}$ halmazán gondolta érdekesnek. Ezt az egyenletet kezdte vizsgálni Daróczy, és feltette a kérdést, hogy vajon Ramanudzsan megtalálta-e az összes ilyen típusú azonosságot. Pontosan fogalmazva, keressük meg az összes olyan $f : \mathbb{Z} \to \mathbb{R}$ függvényt, melyre

$$f(a+b+c)+f(b+c+d)+f(a-d) = \\ f(a+b+d)+f(a+c+d)+f(b-c)$$

teljesül minden $a, b, c, d \in \mathbb{Z}$ esetén, melyek kielégítik az $ad = bc$ feltételt.

Az alábbi függvények nyilván megoldásai az egyenletnek:

$$\begin{array}{rcl} f_1(x) & := & 1, \\ f_2(x) & := & x^2, \\ f_3(x) & := & x^4 \end{array} \qquad (x \in \mathbb{Z}).$$

A kérdés az, hogy vannak-e a fentiektől különböző megoldások.

Az első példát Farkas Tibor adta: az

$$f_4(x) := \begin{cases} 1 & \text{ha } 2 \mid x \\ 0 & \text{különben} \end{cases} \qquad (x \in \mathbb{Z})$$

függvény megoldása az egyenletünknek. Ez esetszétválasztással könnyen ellenőrizhető.

Célunk ezután az volt, hogy meghatározzuk az egyenlet összes megoldását az egész számok halmaza felett.

Ehhez először azt kell észrevenni, hogy minden megoldásfüggvény egyértelműen meg van határozva az $I := \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 12\}$ halmazon felvett értékeivel (Theorem 2.2). Mivel az I halmaz 11 elemű, a már meglévő lineárisan független f_1, f_2, f_3, f_4 függvényhez még további hét lineárisan független megoldást keresve megkapjuk a megoldások egy bázisát. Valóban, a (2.4)-gyel jelölt függvények megoldásai az egyenletnek (Theorem 2.3).

Már csak azt kellett megmutatni, hogy az $f_1, f_2, \ldots, f_{11}$ függvények lineárisan független rendszert alkotnak. Az I halmazon felvett értékeik segítségével ez is bizonyítható (Theorem 2.4).

Most már kimondhatjuk a 2. szakasz fő állítását (Theorem 2.5):
Az egyenlet bármely megoldása előállítható az $f_1, f_2, \ldots, f_{11}$ függvények lineáris kombinációjaként; és fordítva, minden ilyen lineáris kombináció megoldása az egyenletnek.

A 3. szakaszban az egyenletet nem az egész számok halmaza felett, hanem egy algebrai struktúrán, kommutatív, egységelemes gyűrűn vizsgáljuk.

Legyen $R(+,\cdot)$ kommutatív, egységelemes gyűrű és $G(+)$ Abel–csoport. Ha egy $f : R \to G$ függvény megoldása az egyenletnek, akkor ennek jelölése: $f \in S(R, G)$.

Ha $a_1, a_2 : R \to G$ additív függvények és $a_0 \in G$, akkor az

$$f(x) := a_2(x^4) + a_1(x^2) + a_0 \qquad (x \in R)$$

függvény nyilván megoldása az egyenletnek.

Alkalmas helyettesítésekkel az egyenletünk a következő alakra hozható:

$$f(x) + \sum_{i=1}^{5} f_i[\varphi_{i,y}(x) + \psi_{i,y}(t)] = 0,$$

ahol $x, t \in R$ és $y \in R$, $f, f_i : R \to G$ $(i = 1, 2, 3, 4, 5)$ ismeretlen függvények, valamint egy rögzített $y \in R$ esetén a

$$\varphi_{i,y}, \psi_{i,y} : R \to R \qquad (i = 1, 2, 3, 4, 5)$$

függvények additívak. Ez az úgynevezett “lineáris” függvényegyenlet, melynek megoldása ismeretes. A megoldás a csoporton értelmezett polinomok elméletéhez illetve a Fréchet–egyenlethez kapcsolódik. Ekkor bizonyos további feltételek mellett a “lineáris” függvényegyenletünk megoldásai a legfeljebb negyedfokú polinomok.

A következőkben azt mutatjuk meg, hogy ebből a polinomból hiányoznak a páratlan fokszámú tagok, azaz (Theorem 3.5):

Ha R nullkarakterisztikájú test, G lineáris tér egy nullkarakterisztikájú test felett és $f \in S(R, G)$, akkor léteznek olyan $A_k : R^k \to G$ $(k = 4, 2, 0,\ R^0 := R)$ k–additív, szimmetrikus függvények, melyekre

$$f(x) = A_4^*(x) + A_2^*(x) + A_0^*$$

teljesül bármely $x \in R$ esetén. Itt

$$A_k^*(x) := A_k(x, x, \ldots, x)$$

az A_k függvény diagonalizáltja.

Ez a tétel azt állítja, hogy csak a fenti alakú függvények lehetnek az egyenlet megoldásai. Meg kell azt is vizsgálnunk, hogy milyen feltételek mellett lesz egy ilyen függvény valóban megoldása az egyenletnek. Egy

kicsit hosszadalmasabb számolás végén megkaphatjuk ennek a résznek a fő eredményét (Theorem 3.9):
Legyen R nullkarakterisztikájú test és G lineáris tér egy nullkarakterisztikájú test felett. Ekkor $f \in S(R,G)$ pontosan akkor áll fenn, ha léteznek $a_i : R \to G$ $(i = 1,2)$ additív függvények és $a_0 \in G$, hogy

$$f(x) = a_2(x^4) + a_1(x^2) + a_0$$

teljesül minden $x \in R$ esetén.

Az értekezés második része – az első résszel ellentétben, mely inkább algebra és számelmélet – "igazi" analízis. Értjük ezalatt azt, hogy valós függvényekkel, folytonossággal, monotonitással és más, az első féléves Analízis tárgyból jól ismert fogalmakkal és tételekkel találkozhatunk. Ez a fejezet az úgynevezett Matkowski–Sutô problémával, ilyen típusú problémákra vonatkozó kiterjesztési tételekkel, továbbá a témakörhöz még kapcsolódó komplementer közepekkel foglalkozik.

A dolgozat 4. szakaszában áttekintjük a kváziaritmetikai közepekre vonatkozó, eredeti Matkowski–Sutô probléma történetét, valamint ismertetjük a megértéshez szükséges definíciókat és korábbi eredményeket.

Legyen $I \subset \mathbb{R}$ nyílt intervallum, és jelölje $CM(I)$ az I-n értelmezett, folytonos és monoton valós függvények osztályát. Egy M közepet *kváziaritmetikai középnek* nevezünk, ha létezik olyan $\varphi \in CM(I)$ függvény, hogy az alábbi egyenlőség teljesül:

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) =: A_\varphi(x,y) \qquad (x,y \in I).$$

Ebben az esetben a $\varphi \in CM(I)$ függvényt az M kváziaritmetikai közép *generáló függvényének* nevezzük.

A kváziaritmetikai közepek osztálya jól ismert, a rájuk vonatkozó irodalom igen kiterjedt.

Fontos a kváziaritmetikai közepek egyenlőségéről szóló alábbi tétel (Theorem 4.1).
Ha $\varphi, \psi \in CM(I)$, akkor az $A_\varphi \equiv A_\psi$ egyenlőség pontosan akkor áll fenn I^2-en, ha φ és ψ ekvivalens I-n. A két függvény ekvivalenciája olyan $\alpha \neq 0$ és β konstansok létezését jelenti, melyekkel teljesül az alábbi egyenlőség:

$$\varphi(x) = \alpha\psi(x) + \beta \qquad (x \in I).$$

Jelölésben: $\varphi \sim \psi$ I-n, vagy $\varphi(x) \sim \psi(x) \quad (x \in I)$.

A Matkowski–Sutô problémát Sutô vetette fel egy 1914-es dolgozatában: mikor lesz két kváziaritmetikai közép számtani átlaga a számtani közép? Azaz mely kváziaritmetikai közepekre teljesül az alábbi egyenlet:

$$A_\varphi(x,y) + A_\psi(x,y) = x + y \qquad (x, y \in I),$$

vagy a közepek generáló függvényét használva

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = x + y \qquad (x, y \in I).$$

Sutô eredménye a következő (Theorem 4.2):
Ha $\varphi, \psi \in CM(I)$ kielégítik a Matkowski–Sutô egyenletet, és φ, ψ analitikus függvények, akkor létezik $p \in \mathbb{R}$, melyre

$$\varphi(x) \sim \chi_p(x), \qquad \psi(x) \sim \chi_{-p}(x) \qquad (x \in I)$$

teljesül, ahol

$$\chi_p(x) := \begin{cases} x & \text{ha } \ p = 0 \\ e^{px} & \text{ha } \ p \neq 0 \end{cases} \qquad (x \in I).$$

Ezt a problémát Sutôtól függetlenül 1999-ben Matkowski újra felfedezte, aki a fenti tételt általánosabb feltételek mellett bizonyította, a φ és ψ függvényről kétszeres folytonos differenciálhatóságot tételezve fel (Theorem 4.3).

A végleges választ végül Daróczy és Páles adta meg 2001-ben, amikor ugyanezt a tételt igazolták, anélkül hogy a φ és ψ függvénytől bármilyen regularitási tulajdonságot megköveteltek volna (Theorem 4.5).

A bizonyítás során megmutatják, hogy létezik olyan $K \subset I$ részintervallum, amelyen teljesül a tétel állítása, azaz

$$\varphi(x) \sim \chi_p(x), \qquad \psi(x) \sim \chi_{-p}(x) \qquad (x \in K).$$

Ezután a Daróczy, Maksa és Páles által bizonyított alábbi kiterjesztési tételt használják (Theorem 4.6):
Ha $\varphi, \psi \in CM(I)$ kielégítik a Matkowski–Sutô egyenletet, és létezik olyan nemüres nyílt $K \subset I$ részintervallum, amelyen $\varphi(x) \sim \chi_p(x)$, $\psi(x) \sim \chi_{-p}(x)$ valamely $p \in \mathbb{R}$ esetén, akkor $\varphi(x) \sim \chi_p(x)$, $\psi(x) \sim \chi_{-p}(x)$ $(x \in I)$.

Ez a kiterjesztési tétel kulcsfontosságú a Matkowski–Sutô probléma megoldása során, és önmagában is érdekes. Ezért a továbbiakban hasonló kiterjesztési tételeket mondunk ki és bizonyítunk, más középértékosztályok esetén.

Az első középértékosztály, melyet vizsgálunk, a súlyozott kváziaritmetikai közepek osztálya.
Legyen $0 < \lambda < 1$. Egy $M : I^2 \to \mathbb{R}$ közepet *súlyozott kváziaritmetikai középnek* nevezünk, ha létezik $\varphi \in CM(I)$, hogy

$$M(x, y) = \varphi^{-1}(\lambda\varphi(x) + (1 - \lambda)\varphi(y)) =: A_\varphi(x, y; \lambda) \qquad (x, y \in I).$$

A φ függvényt a λ *súllyal* súlyozott kváziaritmetikai közép *generáló függvényének* nevezzük. $A(x, y; \lambda)$ jelöli a közismert súlyozott aritmetikai közepet.

A Matkowski–Sutô probléma súlyozott kváziaritmetikai közepek esetén az alábbi formában írható:

Legyen $0 < \lambda < 1$, $\mu \neq 0, 1$. Keressük meg az összes $\varphi, \psi \in CM(I)$ függvényt, melyek kielégítik a

$$\mu A_\varphi(x, y; \lambda) + (1 - \mu)A_\psi(x, y; \lambda) = A(x, y; \lambda),$$

azaz a

$$\mu\varphi^{-1}(\lambda\varphi(x) + (1 - \lambda)\varphi(y)) + \\ (1 - \mu)\psi^{-1}(\lambda\psi(x) + (1 - \lambda)\psi(y)) = \lambda x + (1 - \lambda)y.$$

egyenletet minden $x, y \in I$ mellett.

A $\lambda = \mu = \frac{1}{2}$ eset az eredeti Matkowski–Sutô probléma.

A probléma teljes megoldása még nem ismert, a legáltalánosabb eredmény a következő (Theorem 5.1):
Legyen $0 < \lambda < 1$, $\mu > 0$ ($\mu \neq 1$). Tegyük fel, hogy a $\varphi, \psi \in CM(I)$ függvények kielégítik a súlyozott kváziaritmetikai közepekre vonatkozó Matkowski–Sutô problémát, és φ, ψ folytonosan differenciálható el nem tűnő deriváltakkal I-n. Ekkor a következő esetek lehetségesek:

(i) Ha $\lambda \neq \frac{1}{2}$, akkor $(\varphi, \psi) \sim (\chi_0, \chi_0)$ I-n;
(ii) Ha $\lambda = \frac{1}{2}$ és $\mu \notin \{\frac{1}{2}, 2\}$, akkor $(\varphi, \psi) \sim (\chi_0, \chi_0)$ I-n;
(iii) Ha $\lambda = \frac{1}{2}$ és $\mu = \frac{1}{2}$, akkor létezik $(s_1, s_2) \in S(I)$, hogy $(\varphi, \psi) \sim (s_1, s_2)$ I-n;
(iv) Ha $\lambda = \frac{1}{2}$ és $\mu = 2$, akkor létezik $(t_1, t_2) \in T(I)$, hogy $(\varphi, \psi) \sim (t_1, t_2)$ I-n.

A fenti jelölések jelentése:

$$\begin{aligned} P_+(I) &:= \{p \in \mathbb{R} \mid I + p \subset \mathbb{R}_+\} \\ P_-(I) &:= \{p \in \mathbb{R} \mid -I + p \subset \mathbb{R}_+\}, \end{aligned}$$

$$\begin{aligned} \gamma_p(x) &:= \sqrt{x+p} \qquad \text{ha } p \in P_+(I) \qquad (x \in I) \\ \delta_p(x) &:= \sqrt{-x+p} \qquad \text{ha } p \in P_-(I) \qquad (x \in I), \end{aligned}$$

$$\begin{aligned} S(I) &:= \{(\chi_p, \chi_{-p}) \mid p \in \mathbb{R}\}, \\ T(I) &:= \{(\chi_0, \chi_0)\} \cup \{(\gamma_p, \log\gamma_p) \mid p \in P_+(I)\} \cup \\ &\qquad \{(\delta_p, \log\delta_p) \mid p \in P_-(I)\}. \end{aligned}$$

A bizonyítani kívánt kiterjesztési tétel ebben az esetben (Theorem 5.2): Tegyük fel, hogy a $\varphi, \psi \in CM(I)$ függvények kielégítik a súlyozott kváziaritmetikai közepekre vonatkozó Matkowski–Sutô egyenletet, és legyen $(f_1, f_2) \in S(I) \cup T(I)$. Tegyük fel továbbá, hogy J valódi részintervalluma I-nek, melyen $\varphi \sim f_1$ és $\psi \sim f_2$ teljesül. Ekkor $\varphi \sim f_1$ és $\psi \sim f_2$ I-n.

A bizonyítás meglehetősen terjedelmes, meghaladja ezen összefoglaló kereteit. Csak egyetlen lépést emelünk itt ki, ez ugyanis érthetővé teszi, hogyan kapcsolódik a témához a 6. szakasz. A kiindulási egyenlet különböző átalakításai után kapjuk az alábbi összefüggést (5.6):

$$f(A_\varphi(x, y; \lambda)) = f(A_\psi(x, y; \lambda))$$

valamely zárt intervallumon, ahol $f(t) := \varphi(t) - t$. Azt igazoljuk, hogy ennek az egyenletnek a fennállásából következik az állításunk. Erre két módszert is ismertetünk. Az egyik konkrét számolás, ez található az 5. szakaszban; a másik a komplementer közepekre vonatkozó eredmények felhasználása, ez a témája a 6. szakasznak.

Az 5. szakasz 5.2-vel jelölt része az α–rendű kváziaritmetikai közepekre vonatkozó Matkowski–Sutô problémát tárgyalja, illetve a rájuk vonatkozó kiterjesztési tételt bizonyítja.

Legyen $\alpha \geq -1$. Egy $M : I^2 \to \mathbb{R}$ közepet *α–rendű kváziaritmetikai középnek* nevezünk, ha létezik $\varphi \in CM(I)$, hogy

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y) + \alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right) =: A_\varphi^{(\alpha)}(x,y)$$

minden $x, y \in I$ esetén.

Az α–rendű kváziaritmetikai közepekre vonatkozó Matkowski–Sutô probléma a következő:

Keressük meg azon $\varphi, \psi \in CM(I)$ függvényeket, melyekre

$$A_\varphi^{(\alpha)}(x,y) + A_\psi^{(\alpha)}(x,y) = x + y \qquad (x, y \in I)$$

teljesül.

A megoldásokról az alábbi eredmény ismert (Theorem 5.4):
Tegyük fel, hogy a $\varphi, \psi \in CM(I)$ függvények kielégítik az α–rendű kváziaritmetikai közepekre vonatkozó Matkowski–Sutô egyenletet, és φ vagy ψ folytonosan differenciálható. Akkor létezik $p \in \mathbb{R}$, melyre $\varphi(x) \sim \chi_p(x)$ és $\psi(x) \sim \chi_{-p}(x)$ $(x \in I)$ teljesül.

Most az alábbi kiterjesztési tételt mondhatjuk ki (Theorem 5.5):
Tegyük fel, hogy a $\varphi, \psi \in CM(I)$ függvények kielégítik az α–rendű kváziaritmetikai közepekre vonatkozó Matkowski–Sutô egyenletet. Tegyük fel továbbá, hogy J olyan valódi részintervalluma I-nek, melyen $\varphi \sim \chi_p$ és $\psi \sim \chi_{-p}$ teljesül valamely $p \in \mathbb{R}$ esetén. Akkor $\varphi \sim \chi_p$ és $\psi \sim \chi_{-p}$ I-n.

A bizonyítás hasonló az előző esethez. Ismét csak azt az egyenletet közöljük, mely kapcsolatot teremt a komplementer közepekkel (5.20):

$$f(A_\varphi^{(\alpha)}(x,y)) = f(A_\psi^{(\alpha)}(x,y))$$

valamely zárt intervallumon. Itt is $f(t) := \varphi(t) - t$.

A 6. szakasz a már említett komplementer közepekkel foglalkozik.
Legyen M, N szigorú közép I-n, és legyen $0 < \lambda < 1$, $\mu \neq 0, 1$. N-et az M *(μ, λ)–komplementerének* nevezzük, ha az alábbi egyenlet teljesül

$$\mu M(x,y) + (1-\mu)N(x,y) = \lambda x + (1-\lambda)y \qquad (x, y \in I).$$

Egy $f : I \to \mathbb{R}$ függvényt *(M, μ, λ)–asszociáltnak* nevezünk, ha rendelkezik az alábbi tulajdonsággal:
Ha $x, y \in I$ kielégíti az $M(x,y) = N(x,y)$ és az $f(x) = f(M(x,y))$ feltételt, akkor $f(y) = f(x)$.

Ennek a szakasznak a fő eredménye (Theorem 6.2):
Legyen $0 < \lambda < 1$, $\mu > 0$, $\mu \neq 1$, és legyen M, N olyan szigorú közép I-n, hogy N (μ, λ)–komplementere M-nek. Legyen továbbá $f : I \to \mathbb{R}$ folytonos függvény, mely kielégíti az

$$f(M(x,y)) = f(N(x,y))$$

függvényegyenletet minden $x, y \in I$-re. Ekkor

(a) Minden olyan $x, y \in I$ esetén, melyre $M(x,y) \neq N(x,y)$, f valamely oldalon lokálisan konstans az $A(x,y;\lambda)$ pontban.

(b) Ha f (M, μ, λ)–asszociált, akkor

(i) f konstans I-n, vagy

(ii) $M(x,y) = N(x,y) = A(x,y;\lambda)$ bármely $x, y \in I$ esetén.

Látható, hogy ez a tétel jól használható az előbb említett két egyenlet kezelésére.

Bibliography

[AB88] G. Almkvist and B. Berndt, *Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, π, and the Ladies diary*, Amer. Math. Monthly **95** (1988), no. 7, 585–608.

[Acz66] J. Aczél, *Lectures on Functional Equations and Their Applications*, Mathematics in Science and Engineering, vol. 19, Academic Press, New York–London, 1966.

[BB87] J. M. Borwein and P. B. Borwein, *Pi and the AGM, (A study in analytic number theory and computational complexity)*, John Wiley & Sons Inc., New York, 1987.

[Ber94] B. C. Berndt, *Ramanujan's Notebooks Part IV*, Springer-Verlag, New York, 1994.

[Chi29] O. Chisni, *Sul concetto di media*, Periodico di Mathematiche n. 2. (1929).

[Dar94] Z. Daróczy, *On a functional equation connected with an identity of Ramanujan*, Ann. Univ. Sci. Budapest. Sect. Comput. **14** (1994), 23–31.

[Dar99] ———, *On a class of means of two variables*, Publ. Math. Debrecen **55** (1999), no. 1-2, 177–197.

[Dar00] ———, *Matkowski-Sutô type problem for conjugate arithmetic means*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 89–100.

[dF31] B. de Finetti, *Sul concetto di media*, Giornale dell' Instituto, Italiano degli Attuarii **2** (1931), 369–396.

[DH98] Z. Daróczy and G. Hajdu, *On the Ramanujan differences*, Publ. Math. Debrecen **53** (1998), no. 3-4, 281–291.

[DH99] ———, *On an identity of Ramanujan*, Acta Math. Hungar. **82** (1999), no. 1-2, 1–9.

[DHN] Z. Daróczy, G. Hajdu, and C. T. Ng, *An extension theorem for a Matkowski-Sutô problem*, Colloq. Math., to appear.

[DMP00] Z. Daróczy, Gy. Maksa, and Zs. Páles, *Extension theorems for the Matkowski-Sutô problem*, Demonstratio Math. **33** (2000), no. 3, 547–556.

[DN00] Z. Daróczy and C. T. Ng, *A functional equation on complementary means*, Acta Sci. Math. (Szeged) **66** (2000), 603–611.

[DP] Z. Daróczy and Zs. Páles, *On functional equations involving means*, Publ. Math. Debrecen, submitted.

[DP01] ______, *On means that are both quasi-arithmetic and conjugate arithmetic*, Acta Math. Hungar. **90** (2001), no. 4, 271–282.

[DP02a] ______, *Gauss-type composition of means and the solution of the Matkowski-Sutô type problem*, Publ. Math. Debrecen **61** (2002), no. 1-2, 157–218.

[DP02b] ______, *A Matkowski-Sutô type problem for quasi-arithmetic means of order α*, Functional Equations — Results and Advances (Z. Daróczy and Zs. Páles, eds.), Advances in Mathematics, vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, pp. 189–200.

[Fré29] M. Fréchet, *Les polynômes abstraits*, Journal de Math. **8** (1929), 179–189.

[Gau27] C. F. Gauss, *Bestimmung der Anziehung eines elliptischen Ringen*, Akademische Verlagsgesellschaft M. B. H., Leipzig, 1927, Nachlass zur Theorie des arithmetisch-geometrischen Mittels und der Modulfunktion.

[Haj02] G. Hajdu, *An extension theorem for the Matkowski–Sutô problem for conjugate arithmetic means*, Functional Equations — Results and Advances (Z. Daróczy and Zs. Páles, eds.), Advances in Mathematics, vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, pp. 201–208.

[HLP34] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934, (first edition), 1952 (second edition).

[Kol30] A. Kolmogorov, *Sur la notion de la moyenne*, Rend. Accad. dei Lincei (6) **12** (1930), 388–391.

[Kuc85] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe — Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985.

[Mat99] J. Matkowski, *Invariant and complementary quasi-arithmetic means*, Aequationes Math. **57** (1999), 87–107.

[MO34] S. Mazur and W. Orlicz, *Grundlegende Eigenschaften der polynomischen Operationen, I., II.*, Studia Math. **5** (1934), no. 56–68, 179–189.

[Nag30] M. Nagumo, *Über eine Klasse der Mittelwerte*, Japanese Journal of Mathematics VII (1930), 71–79.

[Ram57] S. Ramanujan, *Notebooks*, Tata Institute of Fundamental Research, Bombay, 1957.

[Sch82] I. J. Schoenberg, *Mathematical time exposures*, Mathematical Association of America, Washington DC, 1982.

[Sut14] O. Sutô, *Studies on some functional equations I-II*, Tôhoku Math. J. **6** (1914), 1–15 and 82–101.

[Szé82] L. Székelyhidi, *On a class of linear functional equations*, Publ. Math. Debrecen **29** (1982), no. 1-2, 19–28.

[Szé91] ______, *Convolution type functional equations on topological abelian groups*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1991.

[Tod79] J. Todd, *Basic Numerical Mathematics, Vol. 1, Numerical analysis*, Birkhäuser Verlag, Basel, 1979.

[VdL45] G. Van der Lijn, *La définition fonctionelle des polynômes dans les groupes abéliens*, Fund Math. **33** (1945), 43–50.

[Ver36] P. Veres, *A középérték fogalmáról*, Matematikai és Fizikai Lapok **43** (1936), 46–60.