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Some Relational and Sequential Results, and a Relational Modification of a False Lemma of Paweł Pasteczka on the Constancy of the Composition of Certain Set-Valued Functions †

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† Dedicatedto the Memory of our Teacher and Leader Professor Zoltán Daróczy.

Abstract: After establishing some basic facts on binary relations and sequential convergences, we prove a relational modification of a false, but interesting lemma of Paweł Pasteczka on the constancy of the composition of certain set-valued functions [*There is at most one continuous invariant mean*, Aequat. Math. 96 (2022), 833–841.]. In particular, we prove that if F is an inclusion-increasing, compact-valued, closed relation of the half line $X = [0, +\infty[$ to a sequential convergence space $Y = Y(\lim)$, and G is an inclusion-continuous relation of Y to X such that their composition relation $\Phi = G \circ F$ is inclusion-left-continuous, then Φ is a constant relation.

Keywords: relational and sequential convergence spaces; closed and compact sets; closed, closed-valued, inclusion increasing and continuous relations

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1. Introduction

In a recent paper [1], by using some sequential methods, Paweł Pasteczka has tried to prove the following strange, but interesting lemma which is then used to prove the uniqueness of some continuous, invariant means.

Lemma 1. *Let D be a Hausdorff, σ–compact topological space and $F : D \to [0, \infty)$ be a continuous function.*

If $T : [0, \infty) \to 2^D$ is nondecreasing, right-continuous, and such that

- *each $T(x)$ is closed;*
- *$\overrightarrow{F} \circ T : [0, \infty) \to 2^{[0,\infty)}$ is left-continuous,*

then $\overrightarrow{F} \circ T$ is constant.

Here, Pasteczka used the associated set-to-set function $\overrightarrow{F} : 2^D \mapsto 2^{[0,\infty)}$, defined by $\overrightarrow{F}(A) = \{F(x) : x \in A\}$ for all $A \subseteq D$. Thus, we actually have $\overrightarrow{F}(A) = F[A]$ for all $A \subseteq X$. Therefore, instead of not only the set-valued function T, but also the ordinary function F too, it is better to consider a relation. And, thus the auxiliary function $\overrightarrow{F}$ need not be introduced.

Reading of the proof of the above lemma, the third author could not understand the arguments of Pasteczka in the last two paragraphs on page 835 of [1]. Therefore, he asked his colleagues Zoltán Boros and Rezső Lovas, and later also Paweł Pasteczka, to clarify these obscure arguments. Namely, the remaining parts of the proof were correct.

Each of the above mentioned mathematicians confirmed that this lemma, and thus Theorem 1 of [1] may be completely false. Pasteczka even expressed his feeling that "To be honest, I wonder if the proof of Theorem 1 will survive in the present form".

Therefore, to preserve most of the clever arguments of Pasteczka, it seems reasonable to provide a true modification of this lemma. For this, we shall use sequential convergence spaces instead of the topological ones, and appropriate relation-theoretic notations and terminology instead of the somewhat confused ones of Pasteczka.

In particular, before establishing a counter-example to Lemma 1 of Pasteczka, we shall prove the following

Theorem 1. *Suppose that*
(1) $X = \mathbb{R}_+ = [0, +\infty[$;
(2) $Y = Y\,(\lim)$ *is a sequential convergence space;*
(3) F *is an inclusion-increasing, compact-valued, closed relation of* X *to* Y*;*
(4) G *is an inclusion-continuous relation of* Y *to* X *such that the relation* $\Phi = G \circ F$ *is inclusion-left-continuous.*

Then, Φ *is a constant relation of* X *to itself.*

The necessary prerequisites, which are probably unfamiliar to the reader, will be systematically worked out in several preparatory sections. These sections will certainly prove to be more useful for the reader than the theorem itself.

2. A Few Basic Facts on Relations

Concerning terminology and notations about relations and set-valued functions, there is a much confusion in the existing literature. Therefore, the reader is advised to carefully read the following well-established definitions.

A subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation F on X to itself is called a *relation on* X. And, $\Delta_X = \{(x,x) :\ x \in X\}$ and $X^2 = X \times X$ are called the *identity* and *universal relations* on X, respectively.

If F is a relation on X to Y, then for any point $x \in X$ and set $A \subseteq X$, the sets $F\,(x) = \{y \in Y : (x,y) \in F\}$ and $F\,[A] = \bigcup_{a \in A} F\,(a)$ are called the *images* or *neighbourhoods* of x and A under F, respectively.

If $(x,y) \in F$, then instead of $y \in F\,(x)$, we may also write $x\,F\,y$. However, instead of $F\,[A]$, we cannot write $F\,(A)$. Namely, in some extreme cases it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

The sets $D_F = \{x \in X : F\,(x) \neq \emptyset\}$ and $R_F = F\,[X]$ are called the *domain* and *range* of F, respectively. And, if $D_F = X$, then we say that F is a *relation of* X *to* Y, or that F is a *non-partial relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star\,(x)$ and $*\,\big((x,y)\big)$, respectively.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathcal{P}(Y)$ and Φ_F of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_F(x) = F\,(x)$ for all $x \in X$ and $\Phi_F(A) = F\,[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. They were briefly called *corelations* on X to Y in [2,3].

However, if U is a relation on $\mathcal{P}(X)$ to Y and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that U is a *super relation* and V is a *hyper relation on X to Y* [4]. Thus, closures (proximities) [5] are super (hyper) relations.

Note that a super relation on X to Y is an arbitrary subset of $\mathcal{P}(X)\times Y$. While, a corelation on X to Y is a particular subset of $\mathcal{P}(X)\times\mathcal{P}(Y)$. Thus, set inclusion is a natural partial order for super relations, but not for corelations.

For a relation F on X to Y, the relations $F^c = (X\times Y)\setminus F$ and $F^{-1} = \{(y,x) : (x,y)\in F\}$ are called the *complement* and the *inverse* of F, respectively. Thus, c is *inversion compatible* in the sense that $\left(F^c\right)^{-1} = \left(F^{-1}\right)^c$.

Now, we can also show that $F^c(x) = F(x)^c = Y\setminus F(x)$ for all $x\in X$, and $F^c[A]^c = \bigcap_{a\in A} F(a)$ for all $A\subseteq X$. Moreover, $F^{-1}[B] = \{x\in X : F(x)\cap B\neq\emptyset\}$ for all $B\subseteq Y$, and thus, in particular, $D_F = F^{-1}[Y]$.

If F is a relation on X to Y, then we have $F = \bigcup_{x\in X}\left(\{x\}\times F(x)\right)$. Therefore, the values $F(x)$, where $x\in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x\in X$.

For instance, if G is a relation on Y to Z, then the *composition product relation* $G\circ F$ can be naturally defined such that $(G\circ F)(x) = G[F(x)]$ for all $x\in X$. Thus, it can be shown that $(G\circ F)[A] = G\left[F[A]\right]$ also holds for all $A\subseteq X$.

While, if G is a relation on Z to W, then the *box product relation* $F\boxtimes G$ can be defined such that $(F\boxtimes G)(x,z) = F(x)\times G(z)$ for all $x\in X$ and $z\in Z$. Thus, it can be shown that $(F\boxtimes G)[A] = G\circ A\circ F^{-1}$ for all $A\subseteq X\times Z$ [6].

Hence, by taking $A = \{(x,z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

3. Some Important Relational Properties

Now, a relation R on X, i.e., a subset R of X^2, may be briefly defined to be *reflexive* and *transitive* if under the plausible notations $R^0 = \Delta_X$ and $R^2 = R\circ R$, we have $R^0\subseteq R$ and $R^2\subseteq R$, respectively.

Moreover, R may be briefly defined to be *symmetric* and *antisymmetric* if $R^{-1}\subseteq R$ and $R\cap R^{-1}\subseteq R^0$, respectively. And, R may be briefly defined to be *total* and *directive* if $X^2\subseteq R\cup R^{-1}$ and $X^2\subseteq R^{-1}\circ R$, respectively.

Several further relational properties were also studied in [7]. For instance, R was called *non-mingled-valued* if $R(x)\cap R(y)\neq\emptyset$ implies $R(x) = R(y)$ for all $x,y\in X$. Thus, equivalence and linear relations [8] are non-mingled-valued.

By using the reasonable notations $R^- = R^{-1}\circ R$ and $R^\circ = \left(R^{-1}\circ R^c\right)^c$, the above property can be reformulated in the form $R\circ R^-\subseteq R$. Moreover, if R is non-partial, then we can already see that $R\subseteq R\circ R^-$.

In the sequel, as is usual, a reflexive and transitive (symmetric) relation will be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation will be called an *equivalence (partial order) relation*.

For a relation R on X, by using $R^0 = \Delta_X$, we may also define $R^n = R\circ R^{n-1}$ for $n\in\mathbb{N}$. Moreover, we may also define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, it can be shown that R^∞ is the smallest preorder relation on X containing R [9].

Now, in contrast to $(R^c)^c = R$ and $(R^{-1})^{-1} = R$, we have $(R^\infty)^\infty = R^\infty$. And, analogously to $(R^c)^{-1} = (R^{-1})^c$, we also have $(R^\infty)^{-1} = (R^{-1})^\infty$. Moreover, R may be briefly defined to be *well-chained* if $X^2\subseteq R^\infty$ [10,11].

For $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup (A^c \times X)$ is an important preorder on X [12]. While, for a *pseudometric* d on X, the *Weil surrounding* $B_r^d = \{(x, y) \in X^2 : d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [13].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cup (A^c)^2$ is already an equivalence relation on X. And, more generally, if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

Moreover, as a straightforward generalisation of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker–Lindgren relation* $R_{(A, B)} = (A \times B) \cup (A^c \times Y)$ [14].

However, it is now more important to note that if $\mathcal{A} = (A_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{P}(X)$, then the *Cantor relation* $R_{\mathcal{A}} = \Delta_X \cup \bigcup_{n=1}^{\infty} (A_n \times A_n^c)$ is also a preorder on X [15,16].

Moreover, for a real function φ of X and a quasi-pseudo-metric d on X [17], the *Brøndsted relation* $R_{(\varphi, d)} = \{(x,y) \in X^2 : d(x,y) \leq \varphi(y) - \varphi(x)\}$ is also an important preorder on X [18].

From this relation, by letting φ and d to be the zero functions, we can obtain the *specialisation* and *preference relations* $R_d = \{(x,y) \in X^2 : d(x,y) = 0\}$ and $R_\varphi = \{(x,y) \in X^2 : \varphi(x) \leq \varphi(y)\}$, respectively (see [19,20]).

In this respect, it is also worth mentioning that the *divisibility relation* on $\mathbb{Z}$, the *subsequence relation* on $X^{\mathbb{N}}$, and the *refines* and *divides relations* for covers, relations and relators are also, in general, only preorder relations [21].

If R is a relation on X to Y, then the ordered pair $(X, Y)(R) = ((X, Y), R)$ is usually called a *formal context* or *context space* [22]. However, it is better to call it a *relational space* or a *properly simple relator space* [23].

If, in particular, R is a relation on X, then analogously to the abbreviation *poset* of Birkhoff [24], the ordered pair $X(R) = (X, R)$ may be called a *goset* (generalised ordered set) [25], instead of a *relational system* [26,27].

If P is a relational property, then the goset $X(R)$ will be said to have property P if the relation R has this property. For instance, the goset $X(R)$ will be called *reflexive* if R is a reflexive relation on X.

In particular, the goset $X(R)$ will be called a *proset* (preordered set) if R is a preorder on X [25]. Moreover, the abbreviations *toset* (totally ordered set) and *woset* (well-ordered set) of Rudeanu [28] can also be used well.

Thus, every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the universal relation X^2. And, the power set $\mathcal{P}(X) = \{A : A \subseteq X\}$ of X is a poset with the ordinary set inclusion $\subseteq$, or its inverse $\supseteq$.

Several definitions on posets can as well be applied to gosets. For instance, if $X(R)$ is a goset, then for any $Y \subseteq X$, the goset $Y(R \cap Y^2)$ is called a *subgoset* of $X(R)$. While, the goset $X'(R') = X(R^{-1})$ is called the *dual* of $X(R)$.

4. Some Algebraic Tools in Relational Spaces

Notation 1. ***In this section, we shall assume that*** R ***is a relation on*** X ***to*** Y.

Remark 1. *The subsequent definitions can be easily extended to the more general case when R is replaced by a* relator *(family of relations) $\mathcal{R}$ on X to Y [29].*

However, the most important case when $\mathcal{R}$ is a preorder relator on X in the sense that each element of $\mathcal{R}$ is a preorder relation on X [30,31]. That is, $X(\mathcal{R})$ is a multi-preordered set.

Definition 1. *For any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$, we define*

(1) $A \in \operatorname{Lb}_R(B)$ *and* $B \in \operatorname{Ub}_R(A)$ *if* $A \times B \subseteq R$;

(2) $x \in \operatorname{lb}_R(B)$ *if* $\{x\} \in \operatorname{Lb}_R(B)$; (3) $y \in \operatorname{ub}_R(A)$ *if* $\{y\} \in \operatorname{Ub}_R(A)$;

(4) $B \in \mathfrak{L}_R$ *if* $\mathrm{lb}_R\ (B) \neq \emptyset$; (5) $A \in \mathfrak{U}_R$ *if* $\mathrm{ub}_R\ (A) \neq \emptyset$.

Remark 2. *Now, for any relator* $\mathcal{R}$ *on* X *to* Y*, we may also naturally define* $\mathrm{Lb}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathrm{Lb}_R$ *and* $\mathrm{Ub}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathrm{Ub}_R$.

Thus, most of the forthcoming theorems can also be easily proved for relators. However, in the present paper, we shall be mainly interested in relations.

Theorem 2. *We have*
(1) $\mathrm{Ub}_R = \mathrm{Lb}_{R^{-1}} = \mathrm{Lb}_R^{-1}$; (2) $\mathrm{ub}_R = \mathrm{lb}_{R^{-1}}$; (3) $\mathfrak{U}_R = \mathfrak{L}_{R^{-1}}$.

Theorem 3. *For any* $A \subseteq X$ *and* $B \subseteq Y$*, we have*
(1) $\mathrm{Lb}_R(B) = \mathcal{P}\left(\mathrm{lb}_R(B)\right)$; (2) $\mathrm{Ub}_R(A) = \mathcal{P}\left(\mathrm{ub}_R(A)\right)$.

Proof. For instance, by Definition 1, we have

$$A \in \mathrm{Lb}_R(B) \iff A \times B \subseteq R \iff \forall x \in A : \{x\} \times B \subseteq R \iff$$
$$\forall x \in A : \{x\} \in \mathrm{Lb}_R(B) \iff \forall x \in A : x \in \mathrm{lb}_R(B) \iff$$
$$A \subseteq \mathrm{lb}_R(B) \iff A \in \mathcal{P}\left(\mathrm{lb}_R(B)\right).$$

□

Remark 3. *The above two theorems show that the lower and upper bound relations are actually equivalent tools in the relational space* $(X, Y)(R)$.

However, in a relator space $(X, Y)(\mathcal{R})$*, the hyper relation* $\mathrm{Lb}_{\mathcal{R}}$ *is, in general, a much stronger tool than the super relation* $\mathrm{lb}_{\mathcal{R}}$.

Thus, we may naturally look for some necessary or sufficient conditions, on the relator $\mathcal{R}$*, in order that a counterpart of Theorem 3 could be true for* $\mathcal{R}$.

Now, as an immediate consequence of Theorems 2 and 3, we can also state

Corollary 1. *For any* $A \subseteq X$ *and* $B \subseteq Y$*, we have*

$$A \subseteq \mathrm{lb}_R(B) \iff B \subseteq \mathrm{ub}_R(A).$$

Proof. By Theorems 3 and 2, it is clear that

$$A \subseteq \mathrm{lb}_R(B) \iff A \in \mathrm{Lb}_R(B) \iff B \in \mathrm{Lb}_R^{-1}(A) \iff$$
$$B \in \mathrm{Ub}_R(A) \iff B \subseteq \mathrm{ub}_R(A).$$

□

Hence, by identifying singletons with their elements, we can immediately derive

Corollary 2. *For any* $A \subseteq X$ *and* $B \subseteq Y$*, we have*
(1) $\mathrm{lb}_R(B) = \{x \in X : B \subseteq \mathrm{ub}_R(x)\}$; (2) $\mathrm{ub}_R(A) = \{y \in X : A \subseteq \mathrm{lb}_R(y)\}$.

Remark 4. *However, it is now more important to note that by defining*

$$F(A) = \mathrm{ub}_R(A) \qquad \textit{and} \qquad G(B) = \mathrm{lb}_R(B),$$

for all $A \subseteq X$ *and* $B \subseteq Y$, *we can at once see that*

$$F(A) \supseteq B \iff B \subseteq \mathrm{ub}_R(A) \iff A \subseteq \mathrm{lb}_R(B) \iff A \subseteq G(B)$$

for all $A \subseteq X$ *and* $B \subseteq Y$.

Thus, the functions F and G establish a Galois connection *[32], (p. 155) between the posets* $\mathcal{P}(X)\ (\subseteq)$ *and* $\mathcal{P}(Y)\ (\supseteq)$.

Therefore, several properties of the super relations ub_R *and* lb_R *can be derived from the extensive theory of Galois connections [22,32–34].*

Thus, for instance, from Corollary 1, we can already derive the following theorem. However, it is frequently more convenient to apply some direct proofs.

Theorem 4. *If* $B \subseteq Y$, *then*
(1) $\mathrm{lb}_R(B) \subseteq \mathrm{lb}_R(C)$ *for all* $C \subseteq B$;
(2) $B \subseteq \mathrm{ub}_R\left(\mathrm{lb}_R(B)\right)$; (3) $\mathrm{lb}_R(B) = \mathrm{lb}_R\left(\mathrm{ub}_R\left(\mathrm{lb}_R(B)\right)\right)$.

In addition to Corollary 2, it is also worth proving the following:

Theorem 5. *For any* $A \subseteq X$ *and* $B \subseteq Y$, *we have*
(1) $\mathrm{ub}_R(A) = \bigcap_{x \in A} \mathrm{ub}_R(x)$; (2) $\mathrm{lb}_R(B) = \bigcap_{y \in B} \mathrm{lb}_R(y)$.

Remark 5. *Assertion (1) can be generalised by showing that the relation* $F = \mathrm{ub}_R$ *is* union-reversing *in the sense that, for any* $\mathcal{A} \subseteq \mathcal{P}(X)$, *we have* $F(\bigcup \mathcal{A}) = \bigcap_{A \in \mathcal{A}} F(A)$.

Now, by Theorem 5 and Corollary 2, we can also state the following

Corollary 3. *For any* $A \subseteq X$ *and* $B \subseteq Y$, *we have*
(1) $\mathrm{ub}_R(A) = \bigcap_{x \in A} R(x)$; (2) $\mathrm{lb}_R(B) = \{x \in X : B \subseteq R(x)\}$.

Remark 6. *Assertion (1) can be briefly reformulated by stating that*

$$\mathrm{ub}_R(A) = R^c[A]^c$$

for all $A \subseteq X$.

5. Some Further Algebraic Tools in Gosets

Notation 2. ***In this and the next section, we shall assume that*** R ***is a relation on*** X.

Now, by using Definition 1, we may also naturally introduce the following definition which can also be immediately generalised to relators.

Definition 2. *For any* $A \subseteq X$, *we define*
(1) $\min_R(A) = A \cap \mathrm{lb}_R(A)$; (2) $\max_R(A) = A \cap \mathrm{ub}_R(A)$;
(3) $\mathrm{Min}_R(A) = \mathcal{P}(A) \cap \mathrm{Lb}_R(A)$; (4) $\mathrm{Max}_R(A) = \mathcal{P}(A) \cap \mathrm{Ub}_R(A)$;
(5) $\inf_R(A) = \max_R\left(\mathrm{lb}_R(A)\right)$; (6) $\sup_R(A) = \min_R\left(\mathrm{ub}_R(A)\right)$;
(7) $\mathrm{Inf}_R(A) = \mathrm{Max}_R\left[\mathrm{Lb}_R(A)\right]$; (8) $\mathrm{Sup}_R(A) = \mathrm{Min}_R\left[\mathrm{Ub}_R(A)\right]$;
(9) $A \in \ell_R$ *if* $A \in \mathrm{Lb}_R(A)$; (10) $A \in \mathcal{L}_R$ *if* $A \subseteq \mathrm{lb}_R(A)$.

By using this definition, for instance, we can prove the following theorems.

Theorem 6. *We have*
(1) $\mathrm{Max}_R = \mathrm{Min}_{R^{-1}}$; (2) $\mathrm{Sup}_R = \mathrm{Inf}_{R^{-1}}$; (3) $\ell_R = \ell_{R^{-1}}$
(4) $\max_R = \min_{R^{-1}}$; (5) $\sup_R = \inf_{R^{-1}}$; (6) $\ell_R = \mathcal{L}_R$.

Theorem 7. *For any* $A \subseteq X$, *we have*
(1) $\mathrm{Min}_R(A) = \mathcal{P}\left(\min_R(A)\right)$; (2) $\mathrm{Max}_R(A) = \mathcal{P}\left(\max_R(A)\right)$.

Proof. By Definition 2 and Theorem 3, for any $B \subseteq X$, we have

$$B \in \mathrm{Min}_R(B) \iff B \in \mathcal{P}(A) \cap \mathrm{Lb}_R(A) \iff B \in \mathcal{P}(A) \cap \mathcal{P}\left(\mathrm{lb}_R(A)\right) \iff B \in \mathcal{P}\left(A \cap \mathrm{lb}_R(A)\right) \iff B \in \mathcal{P}\left(\min_R(A)\right).$$

Thus, assertion (1) is true. □

Theorem 8. *For any* $A \subseteq X$, *we have*
(1) $\max_R(A) = \bigcap_{x \in A}\left(A \cap \mathrm{ub}_R(x)\right)$;
(2) $\max_R(A) = \left\{x \in A : A \subseteq \mathrm{lb}_R(x)\right\}$.

Theorem 9. *For any* $A \subseteq X$, *we have*
(1) $\sup_R(A) = \mathrm{ub}_R(A) \cap \mathrm{lb}_R\left(\mathrm{ub}_R(A)\right)$;
(2) $\max_R(A) = A \cap \sup_R(A)$; (3) $\sup_R(A) = \inf_R\left(\mathrm{ub}_R(A)\right)$.

Proof. To prove assertion (3), note that by assertion (1) and Theorem 4, and their duals, we have

$$\sup_R(A) = \mathrm{lb}_R\left(\mathrm{ub}_R(A)\right) \cap \mathrm{ub}_R(A) = \mathrm{lb}_R\left(\mathrm{ub}_R(A)\right) \cap \mathrm{ub}_R\left(\mathrm{lb}_R\left(\mathrm{ub}_R(A)\right)\right) = \inf_R\left(\mathrm{ub}_R(A)\right).$$

□

Theorem 10. *For any* $A \subseteq X$, *we have*
(1) $\sup_R(A) = \left\{x \in X : \mathrm{ub}_R(x) = \mathrm{ub}_R(A)\right\}$;
(2) $\sup_R(A) = \left\{x \in \mathrm{ub}_R(A) : \mathrm{ub}_R(A) \subseteq \mathrm{ub}_R(x)\right\}$.

Theorem 11. *For any* $A \subseteq X$ *the following assertions are equivalent:*
(1) $A \in \ell_R$; (2) $A \in \mathrm{Ub}_R(A)$;
(3) $A \in \mathrm{Min}_R(A)$; (4) $A \in \mathrm{Max}_R(A)$.

Corollary 4. *For any* $A \subseteq X$ *the following assertions are equivalent:*
(1) $\mathrm{ub}_R(A) \in \mathcal{L}_R$;
(2) $\mathrm{ub}_R(A) = \sup_R(A)$; (3) $\mathrm{ub}_R(A) \subseteq \mathrm{lb}_R\left(\mathrm{ub}_R(A)\right)$.

Theorem 12. *We have*
(1) $\mathcal{L}_R = \left\{\min_R(A) : A \subseteq X\right\}$; (2) $\mathcal{L}_R = \left\{\max_R(A) : A \subseteq X\right\}$.

Proof. If $A \in \mathcal{L}_R$, then by Definition 2, we have $A \subseteq \mathrm{lb}_R(A)$. Hence, we can infer that $A \subseteq A \cap \mathrm{lb}_R(A) = \min_R(A)$. Thus, since $\min_R(A) = A \cap \mathrm{lb}_R(A) \subseteq A$ always holds, we necessarily have $A = \min_R(A)$.

On the other hand, if $A \subseteq X$ and $V = \min_R(A)$, then we can see that $V = A \cap \mathrm{lb}_R(A)$, and thus $V \subseteq A$ and $V \subseteq \mathrm{lb}_R(A)$. Hence, we can infer that $V \subseteq \mathrm{lb}_R(A) \subseteq \mathrm{lb}_R(V)$. Therefore, $V \in \mathcal{L}_R$ also holds.

The above arguments show that assertion (1) is true. Assertion (2) can be derived from assertion (1) by using Theorem 6. □

Theorem 13. *If R is reflexive, then the following assertions are equivalent:*
(1) *R is antisymmetric;* (2) $\operatorname{card}(A) \leq 1$ *if* $A \in \mathcal{L}_R$*;*
(3) $\max_R$ *is a function;* (4) $\inf_R$ *is a function.*

Proof. If assertion (2) does not hold, then there exists $A \in \mathcal{L}_R$ such that $\operatorname{card}(A) > 1$. Therefore, there exist $x, y \in A$ such that $x \neq y$. Moreover, since $A \in \mathcal{L}_R$, we have $A \subseteq \operatorname{lb}_R(A)$. Therefore, $x\,R\,y$ and $y\,R\,x$, and thus assertion (1) also cannot hold. Consequently, (1) $\Longrightarrow$ (2).

While, if assertion (3) does not hold, then there exists $A \subseteq X$ such that $\operatorname{card}(\max_R(A)) > 1$. Moreover, by Theorem 12, we have $\max_R(A) \in \mathcal{L}_R$. Therefore, assertion (2) also cannot hold. Thus, (2) $\Longrightarrow$ (3) is also true.

On the other hand, if assertion (3) holds, then from Definition 2, we can see that assertion (4) also holds. While, if assertion (4), holds, then from Theorem 9, we can see that $\sup_R$ is a function, and thus by Theorem 9, assertion (3) also holds.

Therefore, to complete the proof, we need only show that assertion (3) also implies assertion (1). For this, note that if $x, y \in X$ such that $x\,R\,y$ and $y\,R\,x$, then for the set $A = \{x, y\}$, we have both $x \in \max_R(A)$ and $y \in \max_R(A)$. Namely, by the reflexivity of R, the properties $x\,R\,x$ and $y\,R\,y$ also hold. Thus, since $\max_R$ is a function, we necessarily have $x = y$. □

Remark 7. *Note that the practically important implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) do not require the relation R to be reflexive.*

Moreover, the relation R is reflexive (antisymmetric) if, and only if, its inverse R^{-1} has the same property. Therefore, by using Theorem 6, a dual of Theorem 13 can also be easily established.

Remark 8. *Finally, we note that by Definition 2 and Remark 6, for instance, we can also state that*

$$\max_R(A) = A \setminus R^c[A]$$

for all $A \subseteq X$.

6. Some Important Completeness Properties of Gosets

Definition 3. *The relation R on X, or the goset $X(R)$, will be called*
(1) inf-complete *if* $\inf_R(A) \neq \emptyset$ *for all* $A \subseteq X$*;*
(2) min-complete *if* $\min_R(A) \neq \emptyset$ *for all* $A \subseteq X$ *with* $A \neq \emptyset$*;*
(3) conditionally inf-complete *if* $\inf_R(A) \neq \emptyset$ *for all* $A \subseteq X$ *with* $A \neq \emptyset$ *and* $\operatorname{lb}_R(A) \neq \emptyset$*;*
(4) conditionally min-complete *if* $\min_R(A) \neq \emptyset$ *for all* $A \subseteq X$ *with* $A \neq \emptyset$ *and* $\operatorname{lb}_R(A) \neq \emptyset$.

Remark 9. *The corresponding maximum and supremum completeness properties of R can be defined quite similarly.*

Moreover, for easy illustrations of the various completeness properties, we may naturally use the number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Note that on the set $\mathbb{C} = \mathbb{R}^2$ of complex numbers, instead of the coordinatewise order, *we may also naturally consider the* lexicographic order *[32], (p. 18).*

Now, by letting A be a singleton, and then a doubleton, we can obtain

Theorem 14. *If R is min-complete, then R is reflexive and total.*

Moreover, by using Theorem 9 and its dual, we can also easily prove

Theorem 15. *The following assertions hold:*
(1) *R is inf-complete if, and only if, it is sup-complete;*
(2) *R is conditionally inf-complete if, and only if, it is conditionally sup-complete.*

Proof. To prove the "only if part of (2)", suppose that $A \subseteq X$ such that $A \neq \emptyset$ and $\mathrm{ub}_R(A) \neq \emptyset$. Then, by the dual of Theorem 4, we have $A \subseteq \mathrm{lb}_R(\mathrm{ub}_R(A))$, and thus $\mathrm{lb}_R(\mathrm{ub}_R(A)) \neq \emptyset$. Therefore, if R is conditionally inf-complete, then $\inf_R(\mathrm{ub}_R(A)) \neq \emptyset$. However, by Theorem 9, we have $\sup_R(A) = \inf_R(\mathrm{ub}_R(A))$. Therefore, $\sup_R(A) \neq \emptyset$, and thus R is conditionally sup-complete. □

Remark 10. *Moreover, it is also worth noting that R is inf-complete if, and only if, it is conditionally inf-complete and $\mathrm{lb}_R(X) \neq \emptyset$ and $\mathrm{ub}_R(X) \neq \emptyset$.*

For several other reasonable completeness properties of gosets, see [35] and [36], where conditionally completeness was actually called semi-inf-completeness.

Concerning conditional complete relations, we can also prove the following

Theorem 16. *If the relation $\leq = R$ is a conditionally complete partial order, then for any $A, B \subseteq X$, we have*
(1) $\inf(A) \leq \sup(A)$ *if $A \neq \emptyset$, $\mathrm{lb}_R(A) \neq \emptyset$ and $\mathrm{ub}_R(A) \neq \emptyset$;*
(2) $\inf_R(A) \leq \inf_R(B)$ *if $\mathrm{lb}_R(A) \neq \emptyset$, $B \neq \emptyset$ and $B \subseteq R[A]$;*
(3) $\sup_R(A) \leq \sup_R(B)$ *if $A \neq \emptyset$, $\mathrm{ub}_R(B) \neq \emptyset$ and $A \subseteq R^{-1}[B]$.*

Proof. Suppose that the conditions of assertion (2) hold. Then, for each $y \in B$, there exists $x_y \in A$ such that $y \in R(x_y)$, and thus $x_y \leq y$. Hence, we can see that $A \neq \emptyset$ and $\mathrm{lb}_R(B) \neq \emptyset$. Thus, by Theorems 13 and 15, $\inf_R(A)$ and $\sup_R(A)$ uniquely exist.

Moreover, we can also easily note that

$$\inf_R(A) \leq x_y \leq y$$

for all $y \in B$. Therefore, $\inf_R(A) \in \mathrm{lb}_R(B)$, and thus $\inf_R(A) \leq \inf_R(B)$ also holds.

Assertion (3) can be proved quite similarly, Moreover, in principle, it can be derived from assertion (2) by dualisation. □

Remark 11. *Note that by Definition 2, for instance, we have*

$$\inf(\emptyset) = \max(\mathrm{lb}(\emptyset)) = \mathrm{lb}(\emptyset) \cap \mathrm{ub}(\mathrm{lb}(\emptyset)) = X \cap \mathrm{ub}(X) = \max(X),$$

which is now either the empty set or a singleton.

However, it is now more important to note that, as an immediate consequence of Theorem 16, we can also state

Corollary 5. *If the relation $\leq = R$ is a conditionally complete partial order, then for any $A, B \subset X$, with $A \neq \emptyset$ and $A \subseteq B$, we have*
(1) $\inf_R(B) \leq \inf_R(A)$ *if* $\mathrm{lb}_{\mathcal{R}}(B) \neq \emptyset$;
(2) $\sup_R(A) \leq \sup_R(B)$ *if* $\mathrm{ub}_R(B) \neq \emptyset$.

Proof. By the reflexivity of R, we have $B \subseteq R[B]$. Therefore, by $A \subseteq B$, we also have $A \subseteq R[B]$. Therefore, assertion (1) of Theorem 16, by changing the roles of A and B, can be applied to obtain the present assertion (1). □

7. Some Topological Tools in Relational Spaces

Notation 3. ***In this section, we shall assume that*** R ***is a relation on*** X ***to*** Y.

Remark 12. *The subsequent definitions can also be easily extended to the more general case when* R *is replaced by a relator* $\mathcal{R}$*, or even a super relator* $\mathcal{U}$ *on* X *to* Y *[4,37].*

However, despite this and the forthcoming Theorem 20, Definition 1 cannot be immediately generalised to super relators.

Definition 4. *For any* $A \subseteq X$, $B \subseteq Y$ *and* $x \in X$*, we define:*
(1) $A \in \operatorname{Int}_R(B)$ *if* $R[A] \subseteq B$; (2) $A \in \operatorname{Cl}_R(B)$ *if* $R[A] \cap B \neq \emptyset$;
(3) $x \in \operatorname{int}_R(B)$ *if* $\{x\} \in \operatorname{Int}_R(B)$; (4) $x \in \operatorname{cl}_R(B)$ *if* $\{x\} \in \operatorname{Cl}_R(B)$;
(5) $B \in \mathcal{E}_R$ *if* $\operatorname{int}_R(B) \neq \emptyset$; (6) $B \in \mathcal{D}_R$ *if* $\operatorname{cl}_R(B) = X$.

Remark 13. *The relations* Int_R *and* int_R *are called* the proximal *and* topological interiors *generated by* R*, respectively. While, the members of the families,* $\mathcal{E}_R$ *and* $\mathcal{D}_R$ *are called the* fat and dense subsets *of the simple relator space* $(X, Y)(R)$*, respectively.*

The origins of the relations Cl_R *and* Int_R *go back to Efremović's proximity* δ *[38] and Smirnov's strong inclusion* $\Subset$ *[39], respectively. While, the convenient notations* Cl_R *and* Int_R*, and the family* $\mathcal{E}_R$*, together with its dual* $\mathcal{D}_R$*, were first explicitly used by the third author in [40–42].*

Now, for any relator $\mathcal{R}$ *on* X *to* Y*, we may naturally define* $\operatorname{Int}_{\mathcal{R}} = \bigcup_{R\in\mathcal{R}} \operatorname{Int}_R$ *and* $\operatorname{Cl}_{\mathcal{R}} = \bigcap_{R\in\mathcal{R}} \operatorname{Cl}_R$.

The following theorem shows that, in a relational space, and thus also a relator space, the closure of a set can be more directly described than in a topological one. Moreover, the corresponding closure and interior relations are equivalent tools.

Theorem 17. *For any* $B \subseteq X$*, we have*
(1) $\operatorname{cl}_R(B) = R^{-1}[B]$;
(2) $\operatorname{cl}_R(B) = \left(\operatorname{int}_R \circ \mathcal{C}_Y\right)^c(B) = X \setminus \operatorname{int}_R(Y \setminus B)$;
(3) $\operatorname{Cl}_R(B) = \left(\operatorname{Int}_R \circ \mathcal{C}_Y\right)^c(B) = \mathcal{P}(X) \setminus \operatorname{Int}_R(Y \setminus B)$.

Remark 14. *From assertions (2) and (3), we can see easily that the equalities* $\operatorname{cl}_R = \left(\operatorname{int}_R\right)^c \circ \mathcal{C}_Y$ *and* $\operatorname{Cl}_R = \left(\operatorname{Int}_R\right)^c \circ \mathcal{C}_Y$ *also hold.*

The following theorem shows that, in contrast to their equivalence, the big closure relation is usually a more convenient tool than the big interior one.

Theorem 18. *We have*
(1) $\operatorname{Cl}_{R^{-1}} = \operatorname{Cl}_R^{-1}$; (2) $\operatorname{Int}_{R^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_R^{-1} \circ \mathcal{C}_X$.

In an arbitrary relator space, the small closure and interior relations are usually much weaker tools than the big ones. However, now we can also prove the following

Theorem 19. *For any* $B \subseteq Y$*, we have*
(1) $\operatorname{Int}_R(B) = \mathcal{P}\left(\operatorname{int}_R(B)\right)$; (2) $\operatorname{Cl}_R(B) = \mathcal{P}\left(\operatorname{cl}_R(B)^c\right)^c$.

Proof. By Definition 4, for any $A \subseteq X$, we have

$$A \in \mathrm{Int}_R(B) \iff R[A] \subseteq B \iff \forall x \in A: R(x) \subseteq B \iff \forall x \in A: R[\{x\}] \subseteq B \iff \forall x \in A: \{x\} \in \mathrm{Int}_R(B) \iff \forall x \in A: x \in \mathrm{int}_R(B) \iff A \subseteq \mathrm{int}_R(B) \iff A \in \mathcal{P}(\mathrm{int}_R(B)).$$

Thus, assertion (1) is true.

Now, by using Theorem 17 and assertion (1), we can also see that

$$A \in \mathrm{Cl}_R(B) \iff A \in \mathrm{Int}_R(B^c)^c \iff A \notin \mathrm{Int}_R(B^c) \iff A \notin \mathcal{P}(\mathrm{int}_R(B^c)) \iff A \notin \mathcal{P}(\mathrm{cl}_R(B)^c)) \iff A \notin \mathcal{P}(\mathrm{cl}_R(B)^c)^c.$$

Thus, assertion (2) is also true. □

Now, analogously to Corollary 1, we can also prove the following:

Corollary 6. *For any $A \subseteq X$ and $B \subseteq Y$, we have*

$$\mathrm{cl}_{R^{-1}}(A) \subseteq B \iff A \subseteq \mathrm{int}_R(B).$$

Proof. By Theorems 19 and 18, it is clear that

$$\mathrm{cl}_{R^{-1}}(A) \subseteq B \iff B^c \cap \mathrm{cl}_{R^{-1}}(A) = \emptyset \iff B^c \notin Cl_{R^{-1}}(A) \iff B^c \notin Cl_R^{-1}(A) \iff A \notin \mathrm{Cl}_R(B^c) \iff A \in \mathrm{Cl}_R(B^c)^c \iff A \in \mathrm{Int}_{\mathcal{R}}(B) \iff A \subseteq \mathrm{int}_R(B).$$

□

Remark 15. *This corollary shows that the functions F and G, defined by*

$$F(A) = \mathrm{cl}_{R^{-1}}(A) \qquad \textit{and} \qquad G(B) = \mathrm{int}_R(B)$$

for all $A \subseteq X$ and $B \subseteq Y$, establish a Galois connection *between the posets $\mathcal{P}(X)$ $(\subseteq)$ and $\mathcal{P}(Y)$ $(\subseteq)$ [43].*

Some further useful examples for Galois connections were also given in [2,44–48]. Galois connections generate Pataki connections which generate closure operations.

Unfortunately, the important facts established in Remarks 4 and 15 can only be generalised to some very particular relators [48]. However, Galois and Pataki connections can be nicely generalised to arbitrary relators [49].

Actually, Corollaries 1 and 6 can be more easily proved directly. Moreover, they can be derived from each other. Namely, we can also prove the following theorem, which can be easily extended to arbitrary relators.

Theorem 20. *We have*
(1) $\mathrm{Lb}_R = (\mathrm{Cl}_{R^c})^c = \mathrm{Int}_{R^c} \circ \mathcal{C}_Y$; (2) $\mathrm{lb}_R = (\mathrm{cl}_{R^c})^c = \mathrm{int}_{R^c} \circ \mathcal{C}_Y$.

Proof. By the corresponding definitions, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$A \in \mathrm{Lb}_R(B) \iff A \times B \subseteq R \iff \forall (a,b) \in A \times B: (a,b) \notin R^c \iff \forall a \in A: \forall b \in B: b \notin R^c(a) \iff R^c[A] \cap B = \emptyset \iff A \notin \mathrm{Cl}_{R^c}(B) \iff A \in \mathrm{Cl}_{R^c}(B)^c \iff A \in (\mathrm{Cl}_{R^c})^c(B).$$

Therefore, $\mathrm{Lb}_R(B) = \mathrm{Cl}^c_{R^c}(B)$ for all $B \subseteq Y$, and thus the first part of assertion (1) is true. The second part of it is now immediate by Theorem 17. □

Now, by using Theorem 17 and Definition 4, we can also easily establish

Theorem 21. *We have*
(1) $\mathcal{D}_R = \{B \subseteq Y: X = R^{-1}[B]\}$;
(2) $\mathcal{E}_R = \bigcup_{x \in X} \mathcal{U}_R(x)$, *where* $\mathcal{U}_R(x) = \mathrm{int}_R^{-1}(x)$.

Remark 16. *Note that thus*

$$\mathcal{U}_R(x) = \mathrm{int}_R^{-1}(x) = \{B \subseteq Y: x \in \mathrm{int}_R(B)\}$$

is just the family of all neighbourhoods *of the point* x *of* X *in* Y.

Quite similarly, we may also naturally define the family

$$\mathcal{U}_R(A) = \mathrm{Int}_R^{-1}(A) = \{B \subseteq Y: A \in \mathrm{Int}_R(B)\}.$$

of all neighbourhoods of a subset A *of* X *in* Y.

The neighbourhoods of sets, belonging to a given family containing all singletons, was a remarkable starting point of a planned unifying theory of Doičinov [50].

Another similar, more powerful unifying theory was formerly completely worked out by Császár [51] by using some inconvenient notation and terminology.

The following theorem, which holds also for an arbitrary relator, shows that the families of fat and dense sets are also equivalent tools.

Theorem 22. *We have*
(1) $\mathcal{D}_R = \{D \subseteq Y: D^c \notin \mathcal{E}_R\}$;
(2) $\mathcal{D}_R = \{D \subseteq Y: \forall E \in \mathcal{E}_R: E \cap D \neq \emptyset\}$.

Remark 17. *By using Theorem 20, we can also see that* $\mathfrak{L}_R = \mathcal{P}(Y) \setminus \mathcal{D}_{R^c}$.

8. Some Further Topological Tools in Gosets

Notation 4. ***In this section, we shall assume that*** R ***is a relation on*** X.

Now, by using Definition 4, we may also naturally introduce the following definition which can also be immediately generalised to relators.

Definition 5. *For any* $A \subseteq X$, *we define:*
(1) $A \in \tau_R$ *if* $A \in \mathrm{Int}_R(A)$; (2) $A \in \tau_R$ *if* $A^c \notin \mathrm{Cl}_R(A)$;
(3) $A \in \mathcal{T}_R$ *if* $A \subseteq \mathrm{int}_R(A)$; (4) $A \in \mathcal{F}_R$ *if* $\mathrm{cl}_R(A) \subseteq A$;
(5) $A \in \mathcal{N}_R$ *if* $\mathrm{cl}_R(A) \notin \mathcal{E}_R$; (6) $A \in \mathcal{M}_R$ *if* $\mathrm{int}_R(A) \in \mathcal{D}_R$.

Remark 18. *The members of the families, τ_R, $\mathcal{T}_R$, and $\mathcal{N}_R$ are called the* proximally open, topologically open, *and* rare (or nowhere dense) subsets *of the relational space $X(R)$, respectively.*

The families τ_R and τ_R were first explicitly used by the third author in [41]. While, the practical notation τ_R has been suggested by János Kurdics who first noticed that connectedness is a particular case of well-chainedness [10,52].

By using Definition 5 and the corresponding results of Section 7, we can easily establish the following two theorems.

Theorem 23. *We have*
(1) $\tau_R = \tau_{R^{-1}}$; (2) $\tau_R = \{A \subseteq X : A^c \in \tau_R\}$;
(3) $\mathcal{F}_R = \{A \subseteq X : A^c \in \mathcal{T}_R\}$; (4) $\mathcal{M}_R = \{A \subseteq X : A^c \in \mathcal{N}_R\}$.

Theorem 24. *We have*
(1) $\tau_R = \mathcal{T}_R$; (2) $\mathcal{T}_R \setminus \{\emptyset\} \subseteq \mathcal{E}_R$; (3) $\mathcal{D}_R \cap \mathcal{F}_R \subseteq \{X\}$.

Proof. By Theorem 19, for any $A \subseteq X$, we have

$$A \in \tau_R \iff A \in \mathrm{Int}_R(A) \iff A \subseteq \mathrm{int}_R(A) \iff A \in \mathcal{T}_R.$$

Thus, assertion (1) is true. □

Remark 19. *However, if $\mathcal{R}$ is a relator on X, then we can only prove that*

$$\tau_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \tau_R = \bigcup_{R \in \mathcal{R}} \mathcal{T}_R \subseteq \mathcal{T}_{\mathcal{R}}.$$

Moreover, if $\mathcal{R} \neq \emptyset$, then $\mathcal{T}_R = \bigcup_{R \in \mathcal{R}^\wedge} \mathcal{T}_R$, where

$$\mathcal{R}^\wedge = \{S \subseteq X^2 : \ \forall x \in X : \ x \in \ \mathrm{int}_{\mathcal{R}} \ (S(x))\}.$$

Thus, in contrast to τ, the structure $\mathcal{T}$ is already not union-preserving.

Remark 20. *From assertion (3) of Theorem 24, by using global complementations, we can infer that $\mathcal{F}_{\mathcal{R}} \subseteq (\mathcal{D}_{\mathcal{R}})^c \cup \{X\}$ and $\mathcal{D}_{\mathcal{R}} \subseteq (\mathcal{F}_{\mathcal{R}})^c \cup \{X\}$.*

However, it is now more important to note that we also have the following

Theorem 25. *For any $A \subseteq X$, we have*
(1) $\mathcal{P}(A) \cap (\mathcal{T}_R \setminus \{\emptyset\}) \neq \emptyset$ *implies* $A \in \mathcal{E}_R$;
(2) $\bigcup \mathcal{T}_R \cap \mathcal{P}(A) \subseteq \mathrm{int}_R(A)$; (3) $\mathcal{P}\,[\tau_R \cap \mathcal{P}(A)] \subseteq \mathrm{Int}_R(A)$.

Remark 21. *The fat sets are frequently more important tools than the open ones. Namely, for any $A \subseteq X$, we have*
(1) *$A \in \mathcal{D}_R$ if, and only if, for each $x \in X$, there exists $y \in A$ such that $x\,R\,y$;*
(2) *$A \in \mathcal{E}_R$ if, and only if, there exists $x \in X$ such that for any $y \in X$, with $x\,R\,y$, we have $y \in A$.*
Thus, $\mathcal{D}_R$ and $\mathcal{E}_R$ are just the families of all cofinal *and* residual *subsets of the goset $X(R)$, respectively.*

This fact, stressed first by Száz [Abstracts of the SeventhPrague Topological Symposium, Inst. Math., Czech. Acad. Sci., 1991, p. 106], can also be well seen from the following

Example 1. *If, in particular,* $X = \mathbb{R}$ *and*

$$R(x) = \{x-1\} \cup [x, +\infty[$$

for all $x \in X$*, then* R *is a reflexive relation on* X *such that* $\mathcal{T}_R = \{\emptyset, \ X\}$*, but* $\mathcal{E}_R$ *is quite a large family.*

Remark 22. *However, if* R *is a preorder relation on* X*, then the converses of the assertions (1)–(3) of Theorem 25 can also be proved. Therefore, in this case, the family* $\mathcal{T}_R$ *is also a quite powerful tool in the proset* $X(R)$*.*

9. A Few Basic Facts on Sequences

Definition 6. *As is usual, a function* x *of the set* $\mathbb{N}$ *of all natural numbers to a set* X *will be called a* sequence *in* X*.*

Remark 23. *Thus,* $x \in X^{\mathbb{N}} = \prod_{n=1}^{\infty} X$*. Therefore, we may also naturally use the notation* $x = (x_n)_{n=1}^{\infty}$ *with* $x_n = x(n)$*.*

By using the usual definition of increasing functions [25], we can easily prove the following:

Theorem 26. *If* x *is a sequence in a proset* $X = X(\leq)$*, then the following assertions are equivalent:*
(1) *x is increasing;* (2) $x_n \leq x_{n+1}$ *for all* $n \in \mathbb{N}$*.*

Proof. Since (1) evidently implies (2), we need only prove the converse implication. That is, if (2) holds and $n, m \in \mathbb{N}$ such that $n \leq m$, then $x_n \leq x_m$.

For this, note that if $m = n$ or $m = n+1$, then by the reflexivity of $\leq$ and assertion (2), respectively, we have $x_n \leq x_m$.

Moreover, if $m \in \mathbb{N}$ such that $x_n \leq x_m$, then by assertion (2) and the transitivity of $\leq$, we also have $x_n \leq x_{m+1}$.

Hence, by induction, it is clear that if $n \in \mathbb{N}$, then $x_n \leq x_m$ holds for all $m \in \mathbb{N}$ with $n \leq m$. Therefore, assertion (1) also holds. □

Now, by using induction, we can also prove the following two theorems.

Theorem 27. *If* x *is a sequence in a proset* $X = X(\leq)$*, then the following assertions are equivalent:*
(1) *x is strictly increasing;*
(2) *x is injective and increasing;* (3) $x_n < x_{n+1}$ *for all* $n \in \mathbb{N}$*.*

Proof. To prove the implication (3) $\Longrightarrow$ (1), a similar argument as in the proof of Theorem 27 can be applied. □

Theorem 28. *If* k *is a strictly increasing sequence in* $\mathbb{N}$*, then* k *is extensive in the sense that* $n \leq k_n$ *for all* $n \in \mathbb{N}$*.*

Proof. Instead of using induction, it is better to note that if x is not extensive, then $A = \{n \in \mathbb{N} : n \not\leq k_n\} \neq \emptyset$. Thus, by the min-completenes of $\mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m \in \min(A)$, and thus $m \in A$ and $m \in \operatorname{lb}(A)$. Hence, we can infer that $m \not\leq k_m$, and thus by the totality of $\leq$, we have $k_m < m$. Now, since k is strictly increasing, we can also state that $k_{k_m} < k_m$. Thus, by the antisymmetry of $\leq$, we also have $k_m \not\leq k_{k_m}$. Therefore, $k_m \in A$, and thus $m \leq k_m$ also holds. This is already a contradiction by the antisymmetry of $\leq$. □

Remark 24. *Note that most of the implications stated in the above theorems do not require all of the basic properties of the inequality relations in* $\mathbb{N}$ *and* X. *Therefore, they can be greatly generalised. However, by Theorem 14, the min-completeness of a relation already implies its totality and reflexivity.*

The importance of strictly increasing sequences in $\mathbb{N}$ lies mainly in the following:

Definition 7. *If* x *and* y *are sequences in* X, *then* y *will be called a* subsequence *of* x, *if there exists a strictly increasing sequence* k *in* $\mathbb{N}$ *such that* $y = x \circ k$.

Remark 25. *In detailed form, this means only that*

$$y_n = y(n) = (x \circ k)(n) = x\big(k(n)\big) = x(k_n) = x_{k_n}$$

for all $n \in \mathbb{N}$.

Thus, if y *is a subsequence of* x, *then, in particular, the range* $y[\mathbb{N}]$ *of* y *is a subset of the range* $x[\mathbb{N}]$ *of* x.

Concerning subsequences, we can also at once state the following:

Theorem 29. *For a sequence* x *in* X *the following assertions hold:*
(1) x *is a subsequence of itself;*
(2) *if* y *is a subsequence of* x *and* z *is a subsequence of* y, *then* z *is also a subsequence of* x.

Proof. To prove (2), note that if k and ℓ are strictly increasing sequences in $\mathbb{N}$, then $\ell \circ k$ is also such a sequence in $\mathbb{N}$. □

Remark 26. *This theorem shows that the 'subsequence relation' is a preorder on* $X^{\mathbb{N}}$.

While, the following two examples, constructed by the second author, show that this preorder is not, in general, either total or antisymmetric.

Example 2. *If* x *and* y *are sequences in* $\mathbb{Z}$ *such that*

$$x[\mathbb{N}] = \{0, 1\} \qquad \text{and} \qquad y[\mathbb{N}] = \{-1, 1\},$$

then none of these sequences is a subsequence of the other one.

Namely, if this is not the case, then by Remark 25, we should have either $x[\mathbb{N}] \subseteq y[\mathbb{N}]$ *or* $y[\mathbb{N}] \subseteq x[\mathbb{N}]$.

Example 3. *If*

$$x_n = (-1)^n \qquad \text{and} \qquad y_n = (-1)^{n+1},$$

or

$$x_n = 2^{-1}\left(1 + (-1)^n\right) \qquad \text{and} \qquad y_n = 2^{-1}\left(1 + (-1)^{n+1}\right)$$

for all $n \in \mathbb{N}$, *then* $x = (x_n)_{n=1}^{\infty}$ *and* $y = (y_n)_{n=1}^{\infty}$ *are subsequences of each other such that* $x_n \neq y_n$ *for all* $n \in \mathbb{N}$.

Namely, by defining $k_n = n + 1$ *for all* $n \in \mathbb{N}$, *we can obtain a strictly increasing sequence* k *in* $\mathbb{N}$ *such that* $y = x \circ k$ *and* $x = y \circ k$.

Remark 27. *However, if* x *and* y *are sequences in* X *such that both of them is a subsequence of the other one, then we can only state that* x *and* y *have the same ranges.*

10. Some Further Theorems on Sequences

Notation 5. ***Because of Definition 4, in the woset*** $\mathbb{N} = \mathbb{N}(\leq)$**, *we shall use the practical notations*** $R = \leq$,

$$\mathcal{E} = \{A \subseteq \mathbb{N} : \exists\ n \in \mathbb{N} : R(n) \subseteq A\}$$

and

$$\mathcal{D} = \{A \subseteq \mathbb{N} : \forall\ n \in \mathbb{N} : R(n) \cap A \neq \emptyset\}.$$

Remark 28. *Thus, the members of the families* $\mathcal{E}$ *and* $\mathcal{D}$ *are just the* residual (fat) *and* cofinal (dense) *subsets of* $\mathbb{N}$*, respectively.*

Moreover, by the above definitions and the corresponding results of Section 7, we can at once state the following

Theorem 30. $\mathcal{E}$ *and* $\mathcal{D}$ *are proper, nonvoid, stacks in* $\mathcal{P}(\mathbb{N})$ *such that:*
(1) $\mathcal{E}$ *is a proper subset of* $\mathcal{D}$*;* (2) $\mathcal{E}$ *is closed under finite intersections;*
(3) $\mathcal{D} = \{A \subseteq \mathbb{N} : A^c \notin \mathcal{E}\}$ *;* (4) $\mathcal{D} = \{A \subseteq \mathbb{N} : \forall B \in \mathcal{E} : A \cap B \neq \emptyset\}$.

Remark 29. *Since ascending families are, in particular, closed under nonvoid unions, assertion (2) shows that* $\mathcal{E} \cup \{\emptyset\}$ *is a topology on* $\mathbb{N}$.

While, assertion (1) shows that $\mathbb{N}$ *is both* hyperconnected *and* resolvable. *These very particular properties have several useful reformulations in [53].*

Important topologies on the sets $\mathbb{Z}$ *and* $\mathbb{N}$ *were introduced by Furstenberg [54], Golomb [55] and Kirch [56] (see also [57–65]).*

In addition to Theorem 30, it is also worth proving the following

Theorem 31. *If k is a strictly increasing sequence in* $\mathbb{N}$*, then*
(1) *k is denseness preverving in the sense that* $A \in \mathcal{D}$ *implies* $k[A] \in D$*;*
(2) k^{-1} *is fatness preverving in the sense that* $B \in \mathcal{E}$ *implies* $k^{-1}[B] \in \mathcal{E}$.

Proof. If $A \in \mathcal{D}$ and $n \in \mathbb{N}$, then by Notation 5 $R(n) \cap A \neq \emptyset$. Thus, there exists $m \in A$ such that $m \in R(n)$. Hence, we can infer that $k_m \in k[A]$. Moreover, from Theorem 28, we can see that $k_m \in R(m)$. Hence, by using the transitivity of R, we can infer that $k_m \in R(n)$. Therefore, $k_m \in R(n) \cap k[A]$, and thus $R(n) \cap k[A] \neq \emptyset$. Hence, by Notation 5, we can see that $k[A] \in D$, and thus assertion (1) is true.

Now, assertion (2) can be easily derived from assertion (1). Namely, if assertion (1) holds, then for any $A \in \mathcal{D}$, we have $k[A] \in D$. Therefore, by assertion (4) of Theorem 30, for any $B \in E$, we have $k[A] \cap B \neq \emptyset$. Hence, we can infer that, $A \cap k^{-1}[B] \neq \emptyset$. Therefore, by the dual of assertion (4) of Theorem 30, we can state that $k^{-1}[B] \in \mathcal{E}$, and thus assertion (2) is also true. □

Remark 30. *Note that the equivalence of assertion (1) and (2) does not require any particular property of the function k. It can be easily generalised to any relation F on one relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$*, and to some even more general situations [66,67].*

The importance of fat and dense sets lies mainly in the following definition and its straightforward generalisation to *nets* [68,69].

Tukey [70], (p. 30) wrote that: "It is unfortunate that the work of Moore and Smith [71] was neglected by topologists for so many years".

Definition 8. *If $A \subseteq X$ and x is a sequence in X, then we shall say that:*
(1) x *is* eventually (residually or fatly) *in A if $x^{-1}[A] \in \mathcal{E}$;*
(2) x *is* frequently (cofinally or densely) *in A if $x^{-1}[A] \in \mathcal{D}$.*

Thus, by using Theorem 30, we can easily establish the following

Theorem 32. *If $A \subseteq X$ and x is a sequence in X, then the following assertions hold:*
(1) *if x is eventually in A, then x is also frequently in A;*
(2) *x is eventually in A if, and only if, x is not frequently in A^c.*

Proof. To prove (2), note that $x^{-1}[A^c] = x^{-1}[A]^c$, and thus by Theorem 30, we have

$$x^{-1}[A] \in \mathcal{E} \iff x^{-1}[A]^c \notin \mathcal{D} \iff x^{-1}[A^c] \notin \mathcal{D}.$$

□

Theorem 33. *If $A_i \subseteq X$ for $i = 1, 2, \ldots, k$, and x is a sequence in X, then the following assertions hold:*
(1) *if x is eventually in each A_i, then x is also eventually in $\bigcap_{i=1}^{k} A_i$;*
(2) *x is frequently in $\bigcup_{i=1}^{k} A_i$, then x is also frequently in some A_i.*

Proof. To prove assertion (2), note that if x is not frequently in each A_i, then by Theorem 32 x is eventually in each A_i^c. Thus, by assertion (1), x is also eventually in $\bigcap_{i=1}^{k} A_i^c$. Hence, since $\bigcap_{i=1}^{k} A_i^c = \left(\bigcup_{i=1}^{k} A_i\right)^c$, we can infer that x is also eventually in $\left(\bigcup_{i=1}^{k} A_i\right)^c$. Thus, by Theorem 32, x is not frequently in $\bigcup_{i=1}^{k} A_i$. □

Theorem 34. *If $A \subseteq X$ and x is a sequence in X, then the following assertions hold:*
(1) *if x is eventually in A, then each subsequence y of x is also eventually in A;*
(2) *if x has a subsequence y such that y is frequently in A, then x is also frequently in A.*

Proof. If x is eventually in A, then by Definition 8, we have $x^{-1}[A] \in \mathcal{E}$. Moreover, if y is a subsequence of x, then by Definition 7, there exists a strictly increasing sequence k in $\mathbb{N}$ such that $y = x \circ k$. Hence, by using Theorem 31, we can already infer that

$$y^{-1}[A] = (x \circ k)^{-1}[A] = \left(k^{-1} \circ x^{-1}\right)[A] = k^{-1}\left[x^{-1}[A]\right] \in \mathcal{E}.$$

Therefore, by Definition 8, the sequence y is also eventually in A, and thus assertion (1) is true.

To prove assertion (2), note that if x is not frequently in A, then by Theorem 32 the sequence x is eventually in A^c. Hence, by using assertion (1), we can infer that y is also eventually in A^c. Therefore, by Theorem 32, the sequence y is also not frequently in A. Thus, assertion (2) is also true. □

Theorem 35. *If $A = (A_i)_{i=1}^{\infty}$ is a decreasing sequence in $\mathcal{P}(X)$ and $x = (x_n)_{n=1}^{\infty}$ is a sequence in X such that x is frequently in each A_i, then there exists a subsequence y of x such that y is eventually in each A_i.*

Proof. Since x is frequently in A_1, there exists $k_1 \in \mathbb{N}$ such that $x_{k_1} \in A_1$. Moreover, since x is frequently in A_2, there exists $k_2 \in \mathbb{N}$ such that $k_2 \geq k_1 + 1$ and $x_{k_2} \in A_2$.

Hence, by induction, it is clear that there exists a strictly increasing sequence k in $\mathbb{N}$ such that $x_{k_n} \in A_n$ for all $n \in \mathbb{N}$. Hence, by using the decreasingness of A, we can easily see that the subsequence $y = x \circ k$ is eventually in each A_i. □

Remark 31. *Note that if* $A = (A_i)_{i=1}^{\infty}$ *is a directed family in* $\mathcal{P}(X)(\supseteq)$, *then the above theorem can be applied to the decreasing sequence* $B = (B_i)_{i=1}^{\infty}$ *defined such that* $B_i = \bigcap_{j=1}^{i} A_j$ *for all* $i \in \mathbb{N}$.

Now, as an immediate consequence of Theorem 35, we can also state

Corollary 7. *If* $A \subseteq X$ *and* x *is a sequence in* X *such that* x *is frequently in* A, *then there exists a subsequence* y *of* x *such that* y *is eventually in* A.

Finally, we note that the following theorem is also true.

Theorem 36. *If* $\mathcal{A} \subseteq \mathcal{P}(X)$ *and* x *is a sequence in* X *such that each subsequence* y *of* x *has a subsequence* z *such that* z *is frequently in each member of* $\mathcal{A}$, *then* x *is eventually in each member of* $\mathcal{A}$.

Proof. Assume on the contrary that there exists $A \in \mathcal{A}$ such that x is not eventually in A. Then, by Theorem 32, the sequence x is frequently in A^c. Thus, by Corollary 7, there exists a subsequence y of x such that y is eventually in A^c. Hence, by using Theorem 34, we can infer that each subsequence z of y is also eventually in A^c. Thus, by Theorem 32, the sequence z not frequently in A. This contradiction proves the required assertion. □

Remark 32. *This theorem can certainly be sharpened by using some convenient modifications of Definitions 7 and 8.*

11. Sequential Convergence Spaces

Instead of closures and open sets, it is sometimes more convenient to start with convergences of sequences or nets. They can be very naturally derived from metrics and neighbourhoods [72].

Definition 9. *If* X *is a set and* lim *is a relation on* $X^{\mathbb{N}}$ *to* X *such that, for any* $x \in X^{\mathbb{N}}$, *we have:*
(a) $a \in \lim(x)$ *if* $x_n = a$ *for all* $n \in \mathbb{N}$;
(b) $\lim(x) \subseteq \lim(y)$ *for any subsequence* y *of* x;
then the relation lim *will be called a* sequential convergence *on* X, *and the ordered pair* $X(\lim) = (x, \lim)$ *will be called a* sequential convergence space.

Remark 33. *Another useful property which the relation* lim *may have:*
(c) *if* $a \in X$ *such that each subsequence* y *of* x *has a subsequence* z *such that* $a \in \lim(z)$, *then* $a \in \lim(x)$ *also holds.*

Remark 34. *Moreover, the relation* lim *may, in particular, be a function. In this case, we say that the relation* lim, *or the space* $X(\lim)$, *is* Hausdorff. *Fréchet [73], Urysohn [74] and most of their followers considered only such convergences.*

If lim *is a function and properties (a) and (b) hold, then the convergence space* $X(\lim)$ *is called an* $\mathcal{L}$*–space. Moreover, if in addition (c) also holds, then* $X(\lim)$ *is called an* $\mathcal{L}^*$*-space. For a rapid overview of the subject, see Frič [75].*

Remark 35. *Some further important properties which the convergence* lim *may have are about* double sequences *[76–78]. These have been mainly motivated by the famous* iterated limit property *of Kelley [79], (p. 74). (See also [80,81].*

Double sequences are usually called infinite matrices. *Convergence theorems for them have been studied by Mikusiński, Antosik and Swartz [82,83]. For a nice recent paper on the subject, see Wenbo and Junde [84].*

Remark 36. *Multi-valued sequential convergences were studied by Dolcher [85], Novák [86], Bednarek and Mikusiński [87], Antosik [88] and Kamiński [89]. In addition to (a)–(c), two remarkable properties were introduced in [88].*

Definition 10. *For any point a and sequence x in a sequential convergence space $X(\lim)$, we define:*

$$a \in \operatorname{adh}(x) \text{ if there exists a subsequence } y \text{ of } x \text{ such that } a \in \lim(y).$$

Remark 37. *Thus, for any sequence x in $X(\lim)$, we have $\lim(x) \subseteq \operatorname{adh}(x)$. Note that, if the convergence $\lim$ is Hausdorff, i. e., $\lim$ is a function, then the relation adh still need not be a function.*

Now, the Urysohn property (c) of the convergence $\lim$ can be reformulated in the shorter form that:

(d) *if a is a point and x is a sequence in X such that for each subsequence y of x, we have $a \in \operatorname{adh}(y)$, then $a \in \lim(x)$ also holds.*

Thus, if the convergence $\lim$ has property (c), then the relations $\lim$ and adh are equivalent tools in the convergence space $X(\lim)$.

Remark 38. *If the convergence relation $\lim$ does not have property (c), then following Urysohn [74], for any $a \in X$ and $x \in X^{\mathbb{N}}$, one may also define:*

$$a \in \lim^*(x) \text{ if for any subsequence } y \text{ of } x, \text{ we have } a \in \operatorname{adh}(y).$$

However, it is now more important to list some natural examples for sequential convergence relations with property (c). For this, instead of metric and topological spaces, and their immediate generalisations, we shall use relator spaces.

Example 4. *Suppose that $X(\mathcal{R})$ is a reflexive relator space. That is, $\mathcal{R}$ is a family of reflexive relations on X.*

For any $a \in X$, define:

$$\mathcal{R}(a) = \{R(a) : R \in \mathcal{R}\}.$$

Moreover, for any $a \in X$ and $x \in X^{\mathbb{N}}$, define:

$$a \in \lim_{\mathcal{R}}(x) \quad \text{if} \quad x \text{ is eventually in each member of } \mathcal{R}(a).$$

Then, the relation $\lim_{\mathcal{R}}$ is a sequential convergence on X having property (c). Moreover, $\lim_{\mathcal{R}}$ is Hausdorff if $\mathcal{R}$ is so. That is, for any $a, b \in X$, with $a \neq b$, there exist $R, S \in \mathcal{R}$ such that $R(a) \cap S(b) = \emptyset$.

Hence, we can infer that

$$b \notin S^{-1}[R(a)] = \big(S^{-1} \circ R\big)(a), \qquad \text{and thus} \qquad (a,b) \notin S^{-1} \circ R.$$

Therefore, $\lim_{\mathcal{R}}$ is Hausdorff if, and only if, $\bigcap \big(\mathcal{R}^{-1} \circ \mathcal{R}\big) = \{\Delta_X\}$.

Remark 39. *By the definition $\lim_{\mathcal{R}}$, for any sequence $x = (x_n)_{n=1}^{\infty}$ in $X(\mathcal{R})$, we have*

$$\lim_{\mathcal{R}}(x) = \bigcap_{R \in \mathcal{R}} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} R^{-1}(x_i) = \bigcap_{R \in \mathcal{R}} \underline{\lim}_{n \to \infty} R^{-1}(x_n).$$

Therefore, for any sequence $A = (A_n)_{n=1}^{\infty}$ of subsets of $X(\mathcal{R})$, we may naturally define

$$\underline{\lim}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} \underline{\lim}_{n \to \infty} R^{-1}[A_n].$$

Thus, in particular, we have $\underline{\lim}_{\Delta_X}(A) = \underline{\lim}_{n\to\infty} A_n$. *Moreover, the* lower limit $\underline{\lim}_{\mathcal{R}}$ *is in accordance with those introduced by Painlevé (1902), Hausdorff [90] and Kuratowski [91], treated in [92,93].*

Remark 40. *Because of Remark 39, a sequence* $x = (x_n)_{n=1}^{\infty}$ *in* $X(\mathcal{R})$ *may be naturally called* convergence Cauchy *[94] if for any* $R \in \mathcal{R}$, *we have*

$$\lim_R(x) = \underline{\lim}_{n\to\infty} R^{-1}(x_n) \neq \emptyset.$$

Thus, each convergent sequence is obviously convergence Cauchy. However, the converse statement need not be true. Therefore, the relator space $X(\mathcal{R})$ *may be naturally called* sequentially convergence-convergence complete *if each convergence Cauchy sequence in it converges.*

Remark 41. *By Mrówka [95], Goetz [96,97] and Frič [98], instead of the convergence of sequences to points, it is usually more convenient to start with the convergence of sequences to sequences. That is, with a suitable relation* Lim *on* $X^{\mathbb{N}}$.

In such a space $X(\mathrm{Lim})$, *a sequence* x *may be naturally called* convergence Cauchy *if* $x \in \mathrm{Lim}\,(y)$ *holds for any subsequence* y *of* x, *or even if* $y \in \mathrm{Lim}\,(z)$ *holds, for any two subsequences* y *and* z *of* x.

Moreover, for any $x \in X$ *and* $y \in X^{\mathbb{N}}$, *by defining*

$$x \in \lim\,(y) \qquad \text{if} \qquad (x)_{n=1}^{\infty} \in \mathrm{Lim}\,(y),$$

we may naturally obtain an ordinary convergence lim *on* X.

For instance, for any two sequences x *and* y *in the relator space* $X(\mathcal{R})$, *we may naturally define*

$$x \in \mathrm{Lim}_{\mathcal{R}}(y) \qquad \text{if} \qquad (x,y) \text{ is eventually in each member of } \mathcal{R}.$$

Remark 42. *By Efremović and Švarc [99], even in uniform spaces, instead of sequences, we must usually work with nets [79]. Moreover, besides convergences, we must also define* adherences *by replacing the defining property "eventually" with "frequently" [40,100].*

In a lecture note [Generalized Sequences *(Hungarian), Debrecen, 1983] by Száz, following [81], a net* x *in* X *was called* universal *if it is eventually in every subset* A *of* X *in which it is frequently. Moreover, a net* y *was called a* subnet *of* x *if it is eventually in every subset* A *of* X *in which* x *is eventually.*

Thus, the net x *is universal if, and only if,* x *is a subnet of each of its subnets. By using universal nets, the various definitions of subnets can be avoided. For instance, a space may be simply called compact if each universal net in it converges.*

12. The Order Convergence of Sequences in Posets

Notation 6. ***In this section, we shall assume that*** $X = X(\leq)$ ***is a conditionally complete poset.***

Remark 43. *By Theorems 13 and 16, this assumption guarantees that each nonvoid, bounded subset of* X *has a unique infimum and supremum.*

Thus, as some straightforward generalisations of certain widely used limits for sequences of numbers and sets, we may also naturally use the following:

Definition 11. *For any bounded sequence $x = (x_n)_{n=1}^{\infty}$ in X, define*

$$\underline{\lim}(x) = \sup_{n \in \mathbb{N}} \inf_{i \geq n} x_i \qquad \text{and} \qquad \overline{\lim}\,(x) = \inf_{n \in \mathbb{N}} \sup_{i \geq n} x_i.$$

Moreover, define

$$\lim\,(x) = \underline{\lim}(x) \qquad \text{if} \qquad \underline{\lim}(x) = \overline{\lim}\,(x).$$

Remark 44. *Note that if, in particular, X is a complete poset, then every subset of X, and thus every sequence in X is bounded.*

However, if for instance $X = \overline{\mathbb{R}}$ or $\mathcal{P}(X)$, then there still may exist a great number of sequences x in X such that $\lim\,(x) = \emptyset$.

Concerning the above order-theoretic limits, we can easily prove the following generalisations of some well-known theorems.

Theorem 37. *If $a \in X$ and x is a sequence in X such that $x_n = a$ for all $n \in \mathbb{N}$, then x is bounded and $a = \lim\,(x)$.*

Proof. It can be easily seen that $\{a\}$ is a bounded subset of X such that

$$a = \inf\big(\{a\}\big) \qquad \text{and} \qquad a = \sup\big(\{a\}\big).$$

Therefore, x is a bounded sequence in X such that

$$u_n = \inf_{i \geq n} x_i = \inf\big(\{x_i\}_{i=n}^{\infty}\big) = \inf\big(\{a\}\big) = a$$

for all $n \in \mathbb{N}$, and thus

$$\underline{\lim}(x) = \sup_{n \in \mathbb{N}} \inf_{i \geq n} x_i = \sup_{n \in \mathbb{N}} u_n = \sup\big(\{u_n\}_{n=1}^{\infty}\big) = \sup\big(\{a\}\big) = a.$$

Moreover, quite similarly, we can also see that $\overline{\lim}\,(x) = a$. Therefore, the equality $\lim\,(x) = a$ also holds. □

Theorem 38. *For any bounded sequence x in X, we have*

$$\inf_{n \in \mathbb{N}} x_n \leq \underline{\lim}(x) \leq \overline{\lim}\,(x) \leq \sup_{n \in \mathbb{N}} x_n.$$

Proof. To prove the second inequality, note that by Definition 2, we have

$$\inf_{i \geq n} x_i \leq x_{n+m} \leq \sup_{j \geq m} x_j$$

for all $n, m \in \mathbb{N}$. Hence, we can already infer that

$$\underline{\lim}(x) = \sup_{n \in \mathbb{N}} \inf_{i \geq n} x_i \leq \sup_{j \geq m} x_j$$

for all $m \in \mathbb{N}$, and thus

$$\underline{\lim}(x) \leq \inf_{m \in \mathbb{N}} \sup_{j \geq m} x_j = \overline{\lim}\,(x).$$

□

Theorem 39. *If x is a bounded sequence in X, then for any subsequence y of x, we have*
(1) $\underline{\lim}(x) \leq \underline{\lim}(y)$; (2) $\overline{\lim}\,(y) \leq \overline{\lim}\,(x)$.

Proof. By Definition 7, there exists a strictly increasing sequence k in $\mathbb{N}$ such that $y = x \circ k$. Thus,

$$y\,[\mathbb{N}] = x\,\big[k\,[\mathbb{N}]\,\big] \subseteq x\,[\mathbb{N}],$$

and thus y is also a bounded sequence in X. Moreover, by Theorem 28, for any $n \in \mathbb{N}$, we have $n \leq k_n$. Therefore,

$$\{y_i\}_{i=n}^{\infty} = \{x_{k_i}\}_{i=n}^{\infty} \subseteq \{x_i\}_{i=k_n}^{\infty} \subseteq \{x_i\}_{i=n}^{\infty}.$$

Hence, by using Corollary 5, we can infer that

$$\inf_{i\geq n} x_i \leq \inf_{i\geq n} y_i \qquad \text{and} \qquad \sup_{i\geq n} y_i \leq \sup_{i\geq n} x_i.$$

Thus, by Theorem 16, we can also state that

$$\underline{\lim}(x) = \sup_{n\in\mathbb{N}} \inf_{i\geq n} x_i \leq \sup_{n\in\mathbb{N}} \inf_{i\geq n} y_i = \underline{\lim}(y)$$

and

$$\overline{\lim}\,(y) = \inf_{n\in\mathbb{N}} \sup_{i\geq n} y_i \leq \inf_{n\in\mathbb{N}} \sup_{i\geq n} x_i = \overline{\lim}\,(x).$$

□

Now, as an immediate consequence of Theorems 38 and 39, we can also state

Corollary 8. *If x is a bounded sequence in X and $a = \lim\,(x)$, then any subsequence y of x is also bounded and $a = \lim\,(y)$.*

Thus, in addition to Example 4, we can also state the following:

Example 5. *The function* lim, *introduced in Definition 11, is a Hausdorff sequential convergence on X.*

Remark 45. *Unfortunately, for the conditionally complete poset X, we could not prove the validity of property (c) for the function* lim.

However, in the next section, we shall show that, in some practically important particular cases, property (c) also holds for the function lim.

By using Definition 11, we can also easily prove the following:

Theorem 40. *For any bounded sequence x in X, we have*
(1) $\lim\,(x) = \inf_{n\in\mathbb{N}} x_n$ *if x is decreasing;*
(2) $\lim\,(x) = \sup_{n\in\mathbb{N}} x_n$ *if x is increasing.*

Proof. For instance, if x is decreasing, then for any $n, i \in \mathbb{N}$, with $n \leq i$, we have $x_i \leq x_n$. Therefore,

$$\sup_{i\geq n} x_i \leq x_n$$

for all $n \in \mathbb{N}$. Hence, by using Theorems 38 and 16, we can already infer that

$$\inf_{n\in\mathbb{N}} x_n \leq \underline{\lim}(x) \leq \overline{\lim}\,(x) = \inf_{n\in\mathbb{N}} \sup_{i\geq n} x_i \leq \inf_{n\in\mathbb{N}} x_n.$$

Therefore, the corresponding equalities are also true. □

Now, as a useful consequence of this theorem, we can also prove the following

Corollary 9. *Suppose that X is total, a is a point and x is a bounded sequence in X such that $a = \lim(x)$. Then,*
(1) *$a \leq x_n$ for all $n \in \mathbb{N}$ if x is decreasing;*
(2) *$x_n \leq a$ for all $n \in \mathbb{N}$ if x is increasing.*

Proof. For instance, if the conclusion of assertion (1) does not hold, then there exists $n_0 \in \mathbb{N}$ such that $a \not\leq x_{n_0}$. Hence, by using the totality of X, we can infer that $x_{n_0} < a$. Now, if x is decreasing, then by using Theorem 40, we can already see that

$$a = \lim(x) = \inf_{n \in \mathbb{N}} x_n \leq x_{n_0} < a.$$

This contradiction shows that assertion (1) is true. □

Thus, in particular, we can also state the following

Corollary 10. *Suppose that X is total, a is a point and x is a bounded sequence in X such that $a = \lim(x)$. Moreover, assume that at least one of the following conditions hold:*
(1) *x is decreasing and $x_n \leq a$ for all $n \in \mathbb{N}$;*
(2) *x is increasing and $a \leq x_n$ for all $n \in \mathbb{N}$.*
Then, $x_n = a$ for all $n \in \mathbb{N}$.

13. Some Further Theorems on Order Convergence

Notation 7. ***In this section, we shall assume that $X = X(\leq)$ is a total, conditionally complete poset.***

Theorem 41. *If a is a point and x is a bounded sequence in X such that $a = \lim(x)$, then x is eventually in every open interval $]\,u, v\,[$ in X with $u < a < v$.*

Proof. By Definition 11, under the notations

$$r_n = \inf_{i \geq n} x_i \qquad \text{and} \qquad s_n = \sup_{i \geq n} x_i,$$

with $n \in \mathbb{N}$, we have

$$a = \sup_{n \in \mathbb{N}} r_n \qquad \text{and} \qquad a = \inf_{n \in \mathbb{N}} s_n.$$

Hence, by using the notations

$$A = \{r_n\}_{n=1}^{\infty} \qquad \text{and} \qquad B = \{s_n\}_{n=1}^{\infty},$$

we can infer that

$$a = \sup(A) \qquad \text{and} \qquad a = \inf(B).$$

Thus, by Definition 2, we can also state that

$$\{a\} = \operatorname{ub}(A) \cap \operatorname{lb}(\operatorname{ub}(A)) \qquad \text{and} \qquad \{a\} = \operatorname{lb}(B) \cap \operatorname{ub}(\operatorname{lb}(B)).$$

Now, since $u < a$, and thus $u \leq a$ and $u \neq a$, we can see that

$$u \in \mathrm{lb}\,\big(\mathrm{ub}\,(A)\big), \qquad \text{but} \qquad u \notin \mathrm{ub}\,(A).$$

(Namely, otherwise $u = \mathrm{ub}\,(A) \cap \mathrm{lb}\,\big(\mathrm{ub}\,(A)\big) = \sup\,(A)$, and thus $u = a$ would be true.) Therefore, there exists $n \in \mathbb{N}$ such that $r_n \not\leq u$. Hence, by using the assumed totality of X, we can infer that

$$u < r_n = \inf_{i \geq n} x_i, \qquad \text{and thus} \qquad u < x_i \quad \text{for all} \quad i \geq n.$$

This shows that the sequence x is eventually in the interval $]\,u, \; +\infty\,[$.

Moreover, since $a < v$, we can quite similarly prove that the sequence is eventually also in the interval $]-\infty, \; v\,[$. Hence, by noticing that $]\,u, v\,[=]-\infty, \; v\,[\cap]\,u, \; +\infty\,[$, and using Theorem 33, we can see that the required assertion is also true. □

Remark 46. *Note that the same assertion need not be true for the closed interval* $[u\,,\,v]$.

Theorem 42. *If* $u, v \in X$ *such that* $u \leq v$, *and* x *is a sequence in* X *such that* x *is eventually in the closed interval* $[u, v]$, *then* x *is bounded and*

$$u \leq \underline{\lim}(x) \leq \overline{\lim}\,(x) \leq v.$$

Proof. By the assumption, there exists $n \in \mathbb{N}$ such that

$$u \leq x_i \leq v \qquad \text{for all} \qquad i \geq n.$$

Moreover, by induction, we can prove that there exist $r, s \in X$ such that

$$r = \min\,\big(\{x_i\}_{i=1}^{n}\big) \qquad \text{and} \qquad s = \max\,\big(\{x_i\}_{i=1}^{n}\big).$$

Hence, we can infer that

$$\min\,\big(\{u, r\}\big) \leq x_i \leq \max\,\big(\{v, s\}\big)$$

for all $i \in \mathbb{N}$, and thus the sequence x is bounded.

Now, in particular, we can also note that

$$u \in \mathrm{lb}\,\big(\{x_i\}_{i=n}^{\infty}\big).$$

Hence, we can already infer that

$$u \leq \inf_{i \geq n} x_i \leq \sup_{k \in \mathbb{N}} \inf_{i \geq k} x_i = \underline{\lim}(x).$$

Quite similarly, we can also see that $\underline{\lim}(x) \leq v$. Hence, by Theorem 38, we can see that the required assertion is true. □

Remark 47. *Note that in this theorem the closed interval* $[u, v]$ *can be replaced by the open one* $]\,u, v\,[$. *However, in this case, the strict inequality in the conclusion cannot be stated.*

Theorem 43. *Suppose that* a *is a point and* x *is a sequence in* X *such that there exists a family* $(J_i)_{i \in I}$ *of closed intervals in* X *such that:*
(1) $\{a\} = \bigcap_{i \in I} J_i$; (2) *$x$ is eventually in every* J_i.
Then, x *is bounded and* $a = \lim\,(x)$.

Proof. From Theorem 42, we can see that now x is a bounded sequence such that

$$\underline{\lim}(x), \overline{\lim}(x) \in J_i \qquad \text{for all} \qquad i \in I,$$

and thus

$$\underline{\lim}(x), \overline{\lim}(x) \in \bigcap_{i \in I} J_i = \{a\}.$$

Therefore, $a = \underline{\lim}(x) = \overline{\lim}(x)$, and thus the required assertion is also true. □

From this theorem, by using Theorem 35, we can immediately infer

Corollary 11. *Suppose that a is a point and x is a sequence in X such that there exists a decreasing sequence $(J_n)_{n=1}^{\infty}$ of closed intervals in X such that:*
(1) $\{a\} = \bigcap_{n=1}^{\infty} J_n$; (2) *$x$ is frequently in every J_n.*
Then, there exists a bounded subsequence y of x such that $a = \lim(y)$.

In the sequel, in addition to the totality and conditional completeness, we shall also need a further basic property of posets.

Definition 12. *The poset X will be called* real-like *at a point $a \in X$ if there exists a family $(\,]u_i, v_i[\,)_{i \in I}$ of open intervals in X such that*
(1) $a \in \bigcap_{i \in I}\,]u_i, v_i[\,$; (2) $\bigcap_{i \in I} [u_i, v_i] \subseteq \{a\}$.
Thus, the poset X may be naturally called real-like if it is real-like at each of its points.

Remark 48. *Note that if, in particular, $a \in \mathbb{R}$ or $\mathbb{Q}$, then by taking*

$$u_n = a - n^{-1} \qquad \text{and} \qquad v_n = a + n^{-1}$$

for all $n \in \mathbb{N}$, we can easily obtain such a family $(\,]u_n, v_n[\,)_{n \in \mathbb{N}}$ in $\mathbb{R}$ or $\mathbb{Q}$.
Thus, $\mathbb{R}$ and $\mathbb{Q}$ are real-like. However, for instance, $\mathbb{R}_+ = [0, +\infty[$ is not real-like at the point 0. While, $\mathbb{N}$ and $\mathbb{Z}$ are even worse.

The importance of Definition 12 is already apparent from the following consequence of Theorems 41 and 43.

Theorem 44. *If $a \in X$ such that X is real-like at a, then for any sequence x in X the following assertions are equivalent:*
(1) *x is bounded and $a = \lim(x)$;*
(2) *x is eventually in every open interval $]u, v[$ in X with $u < a < v$.*

From this theorem, by using Theorem 36, we can immediately derive

Corollary 12. *If the poset X is real-like, then the sequential order convergence* lim *in X has property (c).*

Remark 49. *However, the converse need not be true. Therefore, Theorem 44 should be improved.*

14. Some Basic Notions in Sequential Convergence Spaces

Notation 8. ***In this and the next section, we shall assume that** $X = X(\lim)$ **and** $Y = Y(\lim)$ **are sequential convergence spaces in the sense of Definition 9.***

Remark 50. *To be more precise, we should, for instance, use the different notations $\lim_X$ and $\lim_Y$ for the sequential convergences in the sets X and Y, respectively.*

Moreover, since the sets X and Y may have some additional topological type structures, the term "sequential" should be inserted in forthcoming definitions and statements.

However, such handy incorrectness does not, in general, cause confusions. Recall that the operations and metrics in different abelian groups and metric spaces are usually denoted by the same symbols $+$ and d, respectively.

Definition 13. *A subset A of the convergence space X will be called:*
(1) closed *if for any sequence x in A, we have* $\lim(x) \subseteq A$;
(2) compact *if for any sequence x in A, we have* $\operatorname{adh}(x) \cap A \neq \emptyset$.

Remark 51. *By using Definition 4, properties (1) and (2) can be expressed in the shorter forms that $A^{\mathbb{N}} \subseteq \operatorname{int}_{\lim}(A)$ and $A^{\mathbb{N}} \subseteq \operatorname{cl}_{\operatorname{adh}}(A)$, respectively.*

Remark 52. *However, it is now more important to note that, for any $a \in X$ and $A \subseteq X$, we may also naturally define*

$$a \in \operatorname{cl}(A) \text{ if there exists a sequence } x \text{ in A such that a} \in \lim(\text{x}).$$

Namely, thus A may be naturally called closed if $\operatorname{cl}(A) \subseteq A$, or equivalently $\operatorname{cl}(A) = A$. That is, A is a fixed point of the associated set-valued function φ_{cl}.

Remark 53. *By Definition 13 and 10, a subset A of X is*
(1) *closed if, and only if, for any sequence x in A and $a \in \lim(x)$, we have $a \in A$;*
(2) *compact if, and only if, for any sequence x in A there exist $a \in A$ and a subsequence y of x such that $a \in \lim(y)$.*

By using the latter reformulations, we can more easily prove the following two basic theorems.

Theorem 45. *If A is a compact subset of X, and* $\lim$ *is Hausdorff, then A is a closed subset of X.*

Proof. Suppose that x is a sequence in A and $a \in \lim(x)$. Then, since A is compact, there exist $b \in A$ and a subsequence y of x such that $b \in \lim(y)$. Moreover, since y is subsequence of x, we also have $a \in \lim(y)$. Hence, since lim is a function, we can infer that $a = b$, and thus $a \in A$. Therefore, A is closed. □

Theorem 46. *If A is a closed and B is a compact subset of X such that $A \subseteq B$, then A is also a compact subset of X.*

Proof. Suppose that x is a sequence in A. Then, since $A \subseteq B$, x is also a sequence in B. Thus, since B is compact, there exist $b \in B$ and a subsequence y of x such that $b \in \lim(y)$. Moreover, since y is also a sequence in A and A is closed, $b \in A$. Therefore, A isalso compact. □

Now, in addition to Definition 13, we may also naturally introduce the following:

Definition 14. *A function f of X to Y will be called* continuous *at a point a of X if for any sequence x in X*

$$a \in \lim(x) \implies f(a) \in \lim(f \circ x).$$

Remark 54. *Somewhat more generally, we may also naturally define the continuity of f at a point a of X with respect to a sequence x in X, or a family of sequences in X.*

Thus, if in particular, X is a conditionally complete poset equipped with the order convergence of sequences, then the left and right continuity properties of f at a point a of X can also be defined in the usual way.

Now, by calling the function f to be continuous if it is continuous at each point of X, i.e.,

$$f\,[\lim(x)] \subseteq \lim(f \circ x).$$

for any sequence x in X, we can also easily prove the following two basic theorems.

Theorem 47. *If f is a continuous function of X to Y, then f is compactness-preserving.*

Proof. Suppose that A is a compact subset of X and y is a sequence in $f\,[A]$. Then, there exists a sequence x in A such that $y = f \circ x$. Moreover, since A is compact, there exist $a \in A$ and a subsequence u of x such that $a \in \lim(u)$. Hence, since f is continuous at a, we can infer that $f\,(a) \in \lim(f \circ u)$. Now, by noticing that $f \circ u$ is a subsequence of y, we can see that $f\,[A]$ is also compact. □

Theorem 48. *If f is a continuous function of X to Y, then the relation f^{-1} is closedness-preserving.*

Proof. Suppose that B is a closed subset of Y, and moreover x is a sequence in $f^{-1}\,[B]$ and $a \in \lim(x)$. Then, $f \circ x$ is a sequence in B. Moreover, since f is continuous at a, we have $f\,(a) \in \lim(f \circ x)$. Hence, since B is closed, we can infer that $f\,(a) \in B$, and thus $a \in f^{-1}\,[B]$. Therefore, $f^{-1}\,[B]$ is also closed. □

Remark 55. *Some partial converses of the above theorems should also be proved.*

15. The Product of Two Sequential Convergence Spaces

Under Notation 8, we may also naturally introduce the following

Definition 15. *For any $a \in X$, $b \in Y$ and sequences x in X and y in Y, we define*

$$(a, b) \in \lim(x, y) \qquad \textit{if} \qquad a \in \lim(x) \quad \textit{and} \quad b \in \lim(y).$$

Remark 56. *Thus, the convergence in $X \times Y$ is just the box product of the convergences in X and Y.*

Moreover, we can easily establish the following

Theorem 49. *With Definition 15, the product set $X \times Y$ is a sequential convergence space such that:*
(1) *$X \times Y$ is Hausdorff if, and only if, both X and Y are Hausdorff;*
(2) *$X \times Y$ has property (c) if, and only if, both X and Y have property (c); provided that $X \times Y \neq \emptyset$.*

Now, by using Definition 14, we can also prove the following two theorems.

Theorem 50. *If f is a continuous function of X to Y and Y is Hausdorff, then f is a closed subset of $X \times Y$.*

Proof. Suppose that (x, y) is a sequence in f and

$$(a, b) \in \lim(x, y).$$

Then, $y = f \circ x$, and moreover,

$$a \in \lim(x) \qquad \text{and} \qquad b = \lim(y).$$

Hence, since f is continuous, we can infer that

$$f(a) = \lim(f \circ x) = \lim(y) = b,$$

and thus $(a, b) \in f$. Therefore, the required assertion is true. □

Remark 57. *Now, a relation F on X to Y may be naturally called* closed *if it is a closed subset of the product space $X \times Y$.*

Thus, concerning closed relations, we can easily prove the following:

Theorem 51. *If A is a compact subset of X and F is a closed relation on X to Y, then $F[A]$ is a closed subset of Y.*

Proof. Suppose that y is a sequence in $F[A]$ and $b \in \lim(y)$. Then, there exist a sequence x A such that

$$y_n \in F(x_n), \qquad \text{and thus} \qquad (x_n, y_n) \in F.$$

for all $n \in \mathbb{N}$. Moreover, since A is compact, there exist $a \in A$ and a subsequence $x \circ k$ of x such that $a \in \lim(x \circ k)$ in X. Now, since $b \in \lim(y \circ k)$ also holds, we can see that $(x \circ k,\ y \circ k)$ is a sequence in F such that

$$(a, b) \in \lim(x \circ k,\ y \circ k).$$

Hence, since F is a closed subset of $X \times Y$, we can infer that

$$(a, b) \in F, \qquad \text{and thus} \qquad b \in F(a).$$

Therefore, $b \in F[A]$ also holds, and thus $F[A]$ is closed. □

Now, having in mind the associated set-valued function φ_F, we may naturally introduce the following:

Definition 16. *A relation F on X to Y will be called* closed-valued (compact-valued) *if $F(x)$ is a closed (compact) subset of Y for all $x \in X$.*

Remark 58. *Moreover, if in addition X is poset, then the relation F may, for instance, be called cofinally closed-valued (compact-valued) if there exists a cofinal subset A of X such that $F(x)$ is a closed (compact) subset of Y for all $x \in A$.*

From Theorem 51, by using that the one-point subsets of X are compact, we can immediately derive the following:

Corollary 13. *If F is a closed relation on X to Y, then F is closed-valued.*

Remark 59. *Note that F is a closed relation on X to Y, then F^{-1} is a closed relation on Y to X. Therefore, from the above results, we can derive some similar statements for the relation F^{-1}.*

Now, having in mind the associated set-valued function φ_F, we may also naturally introduce the following:

Definition 17. *A relation F on X to Y will be called* constant *if $F(u) = F(v)$ for all $u, v \in X$.*

Remark 60. *Thus, the relation F is constant if, and only if, there exists $B \subseteq Y$ such that $F(x) = B$ for all $x \in X$. That is, $F = X \times B$.*

Thus, concerning constant relations, we can also easily prove the following:

Theorem 52. *If F is a constant relation on X to Y, then the following assertions are equivalent:*
(1) *F is closed;* (2) *F is closed-valued.*

Proof. From Corollary 13, we know that (1) implies (2). Therefore, we need only prove the converse implication (2) $\Longrightarrow$ (1).

For this, note that, by Remark 60, there exists $B \subseteq Y$ such that $F = X \times B$. Moreover, if (2) holds and $F \neq \emptyset$, then B is closed.

Therefore, if (x, y) is a sequence in F and $(a, b) \in \lim(x, y)$, then y is a sequence in B such that $b \in \lim(y)$. Hence, since B is closed, we can infer that $b \in B$, and thus $(a, b) \in X \times B = F$. Therefore, F is closed in $X \times Y$. □

16. Inclusion-Increasing Relations

Notation 9. ***In this section, we shall assume that***
(1) $X = X(\leq)$ ***is a goset and*** Y ***is a set;***
(2) F ***is a relation of*** X ***to*** Y***,*** G ***is a relation of*** Y ***to*** X ***and*** $\Phi = G \circ F$***.***

Remark 61. *Note that here $\leq$ may, in particular, be Δ_X or X^2. However, the practically important case is when $\leq$ is a proper preorder on X.*

Now, having in mind the associated set-valued function φ_F, we may also naturally introduce the following:

Definition 18. *The relation F will be called* inclusion-increasing *if for any $u, v \in X$*

$$u \leq v \implies F(u) \subseteq F(v).$$

Remark 62. *For a relation on one poset to another, we may also define some* order-increasingness *properties.*

However, in the sequel, we shall only be interested in the above very particular inclusion-increasingness property.

For instance, we can easily prove the following

Theorem 53. *If F is inclusion-increasing, then Φ is also inclusion-increasing.*

Proof. If $u, v \in X$ such that $u \leq v$, then by the inclusion-increasingness of F, we have $F(u) \subseteq F(v)$. Hence, by using two basic definitions on relations, we can infer that

$$\Phi(u) = (G \circ F)(u) = G[F(u)] \subseteq G[F(v)] = (G \circ F)(v) = \Phi(v).$$

Therefore, the relation Φ is also inclusion-increasing. □

Remark 63. *We can even more easily see that if the relation F is inclusion-increasing and φ is an increasing function of a goset to X, then the relation $F \circ \varphi$ is also increasing-increasing.*

Now, by calling the relation F^{-1} to be *ascending-valued* if for any $y \in Y$ the value $F^{-1}(y)$ is an ascending subset of X, we can also easily prove the following:

Theorem 54. *The following assertions are equivalent:*
(1) *F is inclusion-increasing;* (2) *F^{-1} is ascending-valued.*

Proof. Suppose that $y \in Y$, $x \in F^{-1}(y)$ and $u \in X$ such that $x \leq u$. Then, $y \in F(x)$. Moreover, if assertion (1) holds, then $F(x) \subseteq F(u)$. Therefore, $y \in F(u)$, and thus $u \in F^{-1}(y)$. This, shows that $F^{-1}(y)$ is an ascending subset of X, and thus assertion (2) also holds.

The converse implication (2) $\Longrightarrow$ (1) can be proved quite similarly by reversing the above argument. □

Remark 64. *If $A \subseteq X$, then under the notation $R = \leq$ the following assertions are equivalent:*
(1) A is ascending; *(2) $R[A] \subseteq A$;*
(3) $A \subseteq \operatorname{int}_R(A)$; *(4) A is R–open.*

Thus, it can be easily seen that precisely $\emptyset$ and $\mathbb{R}$, and the intervals

$$]a, \ +\infty[\qquad \text{and} \qquad [a, \ +\infty[\,,$$

with $a \in \mathbb{R}$, are the only ascending subsets of $\mathbb{R}$.

Note that to obtain the usual open subsets of $\mathbb{R}$, instead of the relation $R = \leq$, we have to consider the relation $T = S \cap S^{-1}$ with $S = R \setminus \Delta_{\mathbb{R}}$.

Now, as an immediate consequence of Theorem 54 and Remark 64, we can also state the following

Corollary 14. *Under the notation $R = \leq$, the following assertions are equivalent:*
(1) *F is inclusion-increasing;* (2) *$R \circ F^{-1} \subseteq F^{-1}$.*

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if assertion (1) holds, then by Theorem 54, we can state that $F^{-1}(y)$ is an ascending subset of X for all $y \in Y$. Hence, by Remark 64, we can infer that

$$\left(R \circ F^{-1}\right)(y) = R\left[F^{-1}(y)\right] \subseteq F^{-1}(y)$$

for all $y \in Y$. Therefore, assertion (2) also holds. □

Remark 65. *This theorem shows that, analogously to the ordinary increasingness property of functions [101], the inclusion-increasingness property of a relation is also an inclusion property for compositions.*

In this respect, it is also worth noticing that, by using the box product of relations [6], inclusion (2) can be reformulated in the form that

$$\left(F \boxtimes R\right)[\Delta_X] \subseteq \left(\Delta_Y \boxtimes F\right)^{-1}[\Delta_Y].$$

17. A Much More Particular Situation

Having in mind the Galois connections of super relations [48], we may also naturally consider the following:

Notation 10. ***In this section, we shall assume that***
(1) X ***and*** Y ***are sets;***

(2) F ***is a relation of*** X ***to*** Y, G ***is a relation of*** Y ***to*** X, ***and*** $\Phi = G \circ F$;
(3) $F[A] \subseteq B \iff A \subseteq G[B]$ ***for all*** $A \subseteq X$ ***and*** $B \subseteq Y$.

Remark 66. *By using the associated set-to-set functions and Theorem 17, assumption* **(3)** *can be reformulated in the forms that:*
(a) $\Phi_F(A) \subseteq B \iff A \subseteq \Phi_G(B)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$;
(b) $\mathrm{cl}_{F^{-1}}(A) \subseteq B \iff A \subseteq \mathrm{cl}_{G^{-1}}(B)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$.

Assertion (a) means only that the functions Φ_F and Φ_G form a Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Thus, several properties of the relations F, G, and Φ can be derived through the extensive theory of Galois connections.

However, it is now more convenient to apply some direct proofs. For instance, by using the Galois property **(3)**, we can easily prove the following

Theorem 55. *For any* $B \subseteq Y$, *we have*

$$G[B] = \{x \in X : F(x) \subseteq B\}.$$

Proof. By assumption **(3)**, for any $x \in X$, we have

$$x \in G[B] \iff \{x\} \subseteq G[B] \iff F[\{x\}] \subseteq B \iff F(x) \subseteq B.$$

Therefore, the required equality is true. □

Remark 67. *By using Definition 4, the above theorem can be reformulated in the form that* $G[B] = \mathrm{int}_F(B)$ *for all* $B \subseteq Y$.

Moreover, as an immediate consequence of Theorem 55, we can also state

Corollary 15. *We have*
(1) $G(y) = \{x \in X : F(x) \subseteq \{y\}\}$ *for all* $y \in Y$;
(2) $\Phi(x) = \{u \in X : F(u) \subseteq F(x)\}$ *for all* $x \in X$.

Remark 68. *Now, by assumption* **(2)**, *we have* $F(x) \neq \emptyset$ *and* $G(y) \neq \emptyset$ *for all* $x \in X$ *and* $y \in Y$.

Thus, in assertion (1), we may write $F(x) = \{y\}$ *instead of* $F(x) \subseteq \{y\}$. *Moreover, the relation* F *can also be only very particular.*

However, Φ *is just the inverse of the* natural preorder relation Ord_F *defined for all* $x \in X$ *by*

$$\mathrm{Ord}_F(x) = \{u \in X : F(x) \subseteq F(u)\}.$$

Thus, Ord_F *is actually the largest relation on* X *making* F *to be inclusion-increasing. Therefore, in accordance with the results of Section 16, we may naturally consider* X *to be preordered by* Ord_F.

By using the Galois property **(3)**, we can also easily prove the following:

Theorem 56. *For any* $U, V \subseteq X$, *we have*

$$U \subseteq \Phi[V] \iff F[U] \subseteq F[V].$$

Proof. By assumption **(3)**, we have

$$U \subseteq \Phi\,[V] \iff U \subseteq (G \circ F)\,[V]) \iff U \subseteq G\,[F\,[V]] \iff F\,[U] \subseteq F\,[V].$$

□

Remark 69. *By using the associated set-to-set functions, the above theorem can be reformulated in the form that*

$$U \subseteq \Phi_{\Phi}(V) \iff \Phi_F(U) \subseteq \Phi_F(V)$$

for all $U, V \subseteq X$. That is, the functions Φ_F and Φ_{Φ} form a Pataki connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ [102–106].

By using the Pataki property established in Theorem 56, we can now also prove the following:

Corollary 16. *We have*
(1) $F = F \circ \Phi$; (2) $\Phi = \Phi \circ \Phi$.

Proof. By Theorem 56, for any $A \subseteq X$,

$$\Phi\,[A] \subseteq \Phi\,[A] \implies F\,[\Phi\,[A]] \subseteq F\,[A] \implies (F \circ \Phi)\,[A] \subseteq F\,[A]$$

and

$$F\,[A] \subseteq F\,[A] \implies A \subseteq \Phi\,[A] \implies$$
$$F\,[A] \subseteq F\,[\Phi\,[A]] \implies F\,[A] \subseteq (F \circ \Phi)\,[A].$$

Therefore, $F\,[A] = (F \circ \Phi)\,[A]$ for all $A \subseteq X$, and thus, in particular,

$$F\,(x) = F\,[\{x\}] = (F \circ \Phi)\,[\{x\}] = (F \circ \Phi)\,(x)$$

for all $x \in X$. Consequently, assertion (1) is true.

On the other hand, we can note that

$$(1) \implies F \circ \Phi \subseteq F \implies (F \circ \Phi) \circ \Phi \subseteq F \circ \Phi \implies F \circ \Phi^2 \subseteq F \circ \Phi.$$

Therefore, $F \circ \Phi^2 \subseteq F$, and thus for any $A \subseteq X$, we have

$$F\,[\Phi^2\,[A]\,] \subseteq F\,[A].$$

Hence, by using Theorem 56, we can infer that

$$\Phi^2\,[A] \subseteq \Phi\,[A].$$

Moreover, by using Theorem 56, we can also see that

$$F\,[A] \subseteq F\,(A] \implies A \subseteq \Phi\,[A] \implies \Phi\,[A] \subseteq \Phi\,[\Phi\,[A]\,] \implies$$
$$\Phi\,[A] \subseteq (\Phi \circ \Phi)\,[A] \implies \Phi\,[A] \subseteq \Phi^2\,[A].$$

Therefore, $\Phi\,[A] = \Phi^2\,[A]$ for all $A \subseteq X$, and thus, in particular,

$$\Phi\,(x) = \Phi\,[\{x\}\,] = \Phi^2\,[\{x\}\,] = \Phi^2(x)$$

for all $x \in X$. Consequently, assertion (2) is also true. □

Remark 70. *Hence, it is clear that the set-to-set function Φ_{Φ} is an algebraic closure operation on the set X.*

18. Inclusion-Continuous Relations

Notation 11. ***In this and the next section, we shall assume that***
(1) $X = X(\lim)$ ***and*** $Y = Y(\lim)$ ***are sequential convergence spaces;***
(2) F ***is a relation of*** X ***to*** Y***,*** G ***is a relation of*** Y ***to*** X***, and*** $\Phi = G \circ F$***.***

Remark 71. *Note that, by Example 5, a conditionally complete poset $X(\leq)$, and thus, in particular, the complete poset $\mathcal{P}(X)\ (\subseteq)$, may also be naturally considered as a Hausdorff sequential convergence space.*

Therefore, having in mind Definition 14 and the associated set-valued function φ_F, we may naturally introduce the following:

Definition 19. *We say that the relation*
(1) *F is* inclusion-continuous *at a point a of X if for any sequence x in X*

$$a \in \lim(x) \quad \Longrightarrow \quad F(a) = \lim\big(F \circ x\big);$$

(2) *G is* inclusion-continuous *at a point b of Y if for any sequence y in Y*

$$b \in \lim(y) \quad \Longrightarrow \quad G(b) = \lim\big(G \circ y\big).$$

Remark 72. *For a relation on one relator space to another, we may naturally introduce several more important continuity properties [40,49,66,67,100,101,107–113].*

However, in the sequel, we shall only be interested in the above two very particular inclusion-continuity properties.

Now, instead of an analogue of Theorem 53, we can only prove the following two very particular theorems.

Theorem 57. *If f is a function of X to Y which is continuous at a point a of X, and G is inclusion-continuous at the point $f(a)$, then $G \circ f$ is inclusion-continuous at a.*

Proof. Suppose that x is a sequence in X such that $a \in \lim(x)$. Then, by the continuity of f at a, we have

$$f(a) \in \lim(f \circ x).$$

Hence, by the inclusion-continuity of G at $f(a)$, we can infer that

$$(G \circ f)(a) = G\big(f(a)\big) = \lim\big(G \circ (f \circ x)\big) = \lim\big((G \circ f) \circ x\big).$$

□

Theorem 58. *If F is inclusion-continuous at a point a of X and G^{-1} is a function, then Φ is also inclusion-continuous at a.*

Proof. Suppose that x is a sequence in X such that $a \in \lim(x)$. Then, by the corresponding definitions, we have

$$F(a) = \lim\big(F \circ x\big) = \underline{\lim}\big(F \circ x\big) = \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{\infty} F(x_i).$$

Hence, since the inverse of a function preserves intersections too, we can infer that

$$\Phi\,(a) = G\,[F\,(a)] = \bigcup_{n=1}^{\infty}\bigcap_{i=1}^{\infty} G\,[F\,(x_i)] = \bigcup_{n=1}^{\infty}\bigcap_{i=1}^{\infty} \Phi\,(x_i) = \underline{\lim}(\Phi \circ x).$$

Moreover, we can quite similarly see that

$$F\,(a) = \lim\,(F \circ x) = \overline{\lim}\,(F \circ x), \qquad \text{and thus} \qquad \Phi\,(a) = \overline{\lim}\,(\Phi \circ x).$$

Therefore, $\Phi\,(a) = \lim\,(\Phi \circ x)$, and thus the required assertion is also true. □

However, it is now more important to note that we can also prove

Theorem 59. *If $a \in X$ and the restriction of G to the closure of $G^{-1}(a)$ in Y is inclusion-continuous, then $G^{-1}(a)$ is a closed subset of Y.*

Proof. Suppose that y is a sequence in $G^{-1}(a)$ and $b \in \lim\,(y)$. Then,

$$y_n \in G^{-1}(a), \qquad \text{and thus} \qquad a \in G\,(y_n)$$

for all $n \in \mathbb{N}$. Moreover, since b is also in the closure of $G^{-1}(a)$, we can infer that

$$G\,(b) = \underline{\lim}(y) = \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} G\,(y_i).$$

Hence, since $a \in G\,(y_i)$ for all $i \in \mathbb{N}$, we can already see that

$$a \in G\,(b), \qquad \text{and thus} \qquad b \in G^{-1}(a).$$

This shows that $\lim\,(y) \subseteq G^{-1}(x)$ for any sequence y in $G^{-1}(a)$. Therefore, by Definition 13, $G^{-1}(a)$ is a closed subset of Y, □

Remark 73. *Note that property (2) in Definition 19 and Theorem 58 do not need the convergence in X, and the subsequential property of that in Y.*

Now, as an immediate consequence of Theorem 59, we can also state

Corollary 17. *If G is inclusion-continuous, then G^{-1} is closed-valued.*

Remark 74. *This statement can, in principle, be also proved with the help of Theorem 48.*

19. Some Further Theorems on Inclusion-Continuous Relations

To illustrate Definition 19, we can easily prove the following:

Theorem 60. *If F is constant, then F is inclusion-increasing and inclusion-continuous.*

Proof. Namely, if x is a sequence in X and $a \in \lim\,(x)$, then

$$(F \circ x)(n) = F\,\big(x\,(n)\big) = F\,(x_n) = F\,(a)$$

for all $n \in \mathbb{N}$. Therefore, by Theorem 37,

$$\lim\,(F \circ x) = \lim_{n\to\infty}\,(F \circ x)\,(n) = \lim_{n\to\infty} F\,(a) = F\,(a).$$

□

Moreover, as a certain partial converse to Theorem 50 and Corollary 13, we can also prove the following:

Theorem 61. *Suppose that, in particular, X is a total, conditionally complete poset, F is closed-valued, inclusion-increasing and inclusion-left-continuous.*

Moreover, assume that $(a, b) \in X \times Y$, $I =]-\infty, a[$ and (x, y) is a sequence in $F \mid I = F \cap (I \times Y)$ such that $(a, b) \in \lim (x, y)$.

Then, we necessarily have $(a, b) \in F$.

Proof. By the assumption, we have

$$(x_n, \ y_n) = (x, y)\,(n) \in F \cap (I \times Y),$$

and thus

$$y_n \in F\,(x_n) \qquad \text{and} \qquad x_n \leq a$$

for all $n \in \mathbb{N}$. Moreover, by Definition 15, we have

$$a = \lim\,(x) \qquad \text{and} \qquad b \in \lim\,(y).$$

Thus, by Definition 11, x is a bounded sequence in X,

$$a = \underline{\lim}(x) = \sup_{n \in \mathbb{N}} \inf_{i \geq n} x_i \qquad \text{and} \qquad a = \overline{\lim}\,(x) = \inf_{n \in \mathbb{N}} \sup_{i \geq n} x_i.$$

Moreover, by the Monotone Subsequence Theorem [114], we can also state that there exists a subsequence $u = x \circ k$ of x such that at least one of the following properties holds: (1) u is decreasing; (2) u is strictly increasing.

However, if property (1) holds, then by Corollary 10, we necessarily have $u_n = a$ for all $n \in \mathbb{N}$. Therefore, u is increasing even if (1) holds.

Now, by using the plausible notation $v = y \circ k$, we can note that

$$v_n \in F\,(u_n) \qquad \text{and} \qquad u_n \leq a$$

for all $n \in \mathbb{N}$. Moreover, by Example 5 and Definition 9, we can also state that

$$a = \lim\,(u) \qquad \text{and} \qquad b \in \lim\,(v).$$

On the other hand, since u is increasing and F is inclusion-increasing, for any $i, n \in \mathbb{N}$, with $n \leq i$, we can also note

$$u_n \leq u_i \qquad \text{and} \qquad F\,(u_n) \subseteq F\,(u_i).$$

Therefore, by Theorem 40 and the assumed inclusion-left-continuity of F, we can also state

$$a = \lim\,(u) = \sup_{n \in \mathbb{N}} u_n \qquad \text{and} \qquad F\,(a) = \lim\,(F \circ u) = \textstyle\bigcup_{n \in \mathbb{N}} F\,(u_n).$$

Hence, by using that $\{v_n\} \subseteq F\,(u_n)$ for all $n \in \mathbb{N}$, we can infer that

$$v\,[\mathbb{N}] = \{v_n\}_{n=1}^{\infty} \subseteq F\,(a).$$

Therefore, v is a sequence in $F\,(a)$. Hence, by using that $b \in \lim\,(v)$ and $F\,(a)$ is closed, we can infer that $b \in F\,(a)$, and thus $(a, b) \in F$. □

Remark 75. *Unfortunately, by assuming the inclusion-right-continuity of F, we could not prove such a restricted closedness property of F.*

The closedness of some upper and lower semicontinuous, closed-valued relations was proved in [115], (Theorems 17.10 and 17.11) and [108], (Theorems 8.1 and 8.2).

20. A True Relational Modification of Pasteczka's Lemma

Now, having all the necessary preparations, we are ready to prove the following:

Theorem 62. *Suppose that*
(1) $X = \mathbb{R}_+ = [0, +\infty[$;
(2) $Y = Y\,(\lim)$ *is a sequential convergence space;*
(3) F *is an inclusion-increasing, cofinally compact-valued and closed relation of X to Y;*
(4) G *is a relation of Y to X such that G^{-1} is closed-valued, and the relation $\Phi = G \circ F$ is inclusion-left-continuous.*

Then, Φ is a constant relation of X to itself.

Proof. From Theorem 53, we know that the relation Φ is also inclusion-increasing. Thus, if $\underline{x \in X}$, i.e., $0 \leq x < +\infty$, then we also have $\Phi(0) \subseteq \Phi(x)$. Therefore, to prove that $\Phi(x) = \Phi(0)$, we need only show that $\Phi(x) \subseteq \Phi(0)$. That is, if $\underline{v \in \Phi(x)}$, then $v \in \Phi(0)$, i.e., $0 \in \Phi^{-1}(v)$ also holds.

For this, for any $u \in X$, we can note that

$$u \in \Phi^{-1}(v) \iff u \in \left(G \circ F\right)^{-1}(v) \iff u \in \left(F^{-1} \circ G^{-1}\right)(v) \iff u \in F^{-1}\left[G^{-1}(v)\right] \iff F(u) \cap G^{-1}(v) \neq \emptyset.$$

Therefore, we have

$$\Phi^{-1}(v) = \left\{u \in X : \ F(u) \cap G^{-1}(v) \neq \emptyset\right\}.$$

Moreover, from the inclusion-increasingness of Φ, by Theorem 54, we can see that $\Phi^{-1}(v)$ is an ascending subset of X. Furthermore, since $v \in \Phi(x)$, we can also note that $x \in \Phi^{-1}(v)$. Therefore, by defining

$$a = \inf\left(\Phi^{-1}(v)\right),$$

we can state that $a \in [0, x]$.

Hence, since either $a \in \Phi^{-1}(v)$ or $a \notin \Phi^{-1}(v)$, by Remark 64, we can see that exactly one of the two cases holds:
(A) $\Phi^{-1}(v) = [a, \ +\infty[$; (B) $\Phi^{-1}(v) =]\,a, \ +\infty[$.

If $\underline{\text{(A) holds and } a = 0}$, then

$$\Phi^{-1}(v) = [0, \ +\infty[, \qquad \text{and thus, in particular,} \qquad 0 \in \Phi^{-1}(v).$$

Consequently, the required assertion is true.

Therefore, in addition to (A), we may suppose that $\underline{0 < a}$. In this case, we can construct a strictly increasing sequence $(u_n)_{n=1}^{\infty}$ in $[0, a[$ such that

$$a = \lim_{n \to \infty} u_n.$$

Hence, by using $a \in \Phi^{-1}(v)$, the left-inclusion-continuity of Φ at a, the inclusion-increasingness of Φ and Theorem 40, we can see that

$$v \in \Phi(a) = \lim_{n \to \infty} \Phi(u_n) = \bigcup_{n=1}^{\infty} \Phi(u_n).$$

Therefore, there exists $j \in \mathbb{N}$ such that $v \in \Phi(u_j)$, and thus $u_j \in \Phi^{-1}(v)$. Hence, by the definition a, we can infer that $a \leq u_j$. This contradiction shows that case (A) with $0 < a$ cannot occur.

Therefore, to complete the proof, we may suppose that (B) holds. Then, since F is cofinally compact-valued, there exists $b \geq a+1$ such that $F(b)$ is a compact subset of Y. Moreover, we can construct a strictly decreasing sequence $(u_n)_{n=1}^{\infty}$ in $]\,a, b]$ such that $u_1 = b$ and

$$a = \lim_{n \to \infty} u_n.$$

Then,

$$u_n \in]\,a, b] \subseteq]\,a, +\infty[= \Phi^{-1}(v) = \{s \in X : F(s) \cap G^{-1}(v) \neq \emptyset\},$$

and thus

$$F(u_n) \cap G^{-1}(v) \neq \emptyset$$

for all $n \in \mathbb{N}$. Therefore, by the Countable Axiom of Choice [116], there exists a sequence $(y_n)_{n=1}^{\infty}$ in Y such that

$$\underline{y_n \in F(u_n) \cap G^{-1}(v)} \subseteq F(u_1) \cap G^{-1}(v) = F(b) \cap G^{-1}(v)$$

for all $n \in \mathbb{N}$.

Since F is a closed subset of $X \times Y$, by Corollary 13, we can see that $F(b)$ is a closed subset of Y. Moreover, since G^{-1} is closed-valued, we can note that $G^{-1}(v)$ is also a closed subset Y. Hence, we can infer that $F(b) \cap G^{-1}(v)$ is also a closed subset of Y. Now, since $F(b)$ is a compact subset of Y, by Theorem 46, we can see that $F(b) \cap G^{-1}(v)$ is also a compact subset of Y. Thus, by Remark 53, there exist

$$c \in F(b) \cap G^{-1}(v)$$

and a strictly increasing sequence $(k_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that

$$c \in \lim_{n \to \infty} y_{k_n}.$$

Moreover, we can also note that

$$y_{k_n} \in F(u_{k_n}), \qquad \text{and thus} \qquad (u_{k_n}, y_{k_n}) \in F$$

for all $n \in \mathbb{N}$. And, $(u_{k_n})_{n=1}^{\infty}$ is also a strictly decreasing sequence in $]\,a, b]$ such that

$$a = \lim_{n \to \infty} u_{k_n}.$$

Thus, in particular, we can also state that

$$(a, c) \in \lim_{n \to \infty} (u_{k_n}, y_{k_n}).$$

Hence, since F closed in $X \times Y$, we can already infer that

$$(a, c) \in F, \qquad \text{and thus} \qquad c \in F(a).$$

Thus, we have

$$c \in F(a) \cap G^{-1}(v), \qquad \text{and thus} \qquad F(a) \cap G^{-1}(v) \neq \emptyset.$$

Hence, we can infer that

$$a \in \Phi^{-1}(v) = \,]\, a, +\infty\, [.$$

This contradiction shows that case (B) cannot occur, and thus the proof is complete. □

Remark 76. *If instead of the closedness of F, in accordance with [1], we assume that F is inclusion-right-continuous, then by using the corresponding properties of $(u_n)_{n=1}^{\infty}$ and the inclusion-increasingness of F, we can only infer that*

$$F(a) = \lim_{n \to \infty} F(u_{k_n}) = \bigcap_{n=1}^{\infty} F(u_{k_n}).$$

Therefore, to obtain $c \in F(a)$, some additional assumptions may be needed.

In this respect, it is also worth noticing that the present assumptions of Theorem 62 can slightly be weakened. Namely, the sequences $(u_n)_{n=1}^{\infty}$ constructed in its proof are not only convergent, but also strictly monotonic.

From Theorem 62, by letting $Y = X$ and $G = \Delta_X$, we can derive

Corollary 18. *If $X = \mathbb{R}_+$ and F is an inclusion-increasing, cofinally compact-valued, closed and inclusion-left-continuous relation of X to itself, then F is constant.*

Moreover, from Theorem 62, by using Corollary 17, we can also derive

Corollary 19. *If in addition to the hypotheses (1)–(3) of Theorem 62, we assume that*
(4*) G is an inclusion-continuous relation of Y to X such that the relation $\Phi = G \circ F$ is inclusion-left-continuous.

Then, Φ is a constant relation of X to itself.

Remark 77. *Note that if F is as in Theorem 62 or its Corollaries 18 and 19, then by Corollary 13 and Theorem 45, the relation F is actually compact-valued.*

21. An Illustrative Example

Notation 12. ***In this section, we shall assume that***
(1) $X = \mathbb{R}_+$ ***and*** $Y = X^2$***;***
(2) $F(x) = X \times [0, x]$ ***for all*** $x \in X$***;***
(3) $G(y) = y_1 y_2$ ***for all*** $y = (y_1, y_2) \in Y$.

Remark 78. *Thus, both X and Y are Hausdorff sequential convergence spaces with their usual sequential convergences.*

Moreover, F is a relation of X onto Y and G is a function of Y onto X. While, $\Phi = G \circ F$ is a relation of X onto itself.

More concretely, we can easily establish the following

Theorem 63. *For any $x \in X$, we have*

$$\Phi(x) = \{0\} \quad \text{if} \quad x = 0 \qquad \text{and} \qquad \Phi(x) = X \quad \text{if} \quad x \neq 0.$$

Proof. By the corresponding definitions, we have

$$\Phi(x) = (G \circ F)(x) = G[F(x)] = \{G(y) : y \in F(x)\} = \\ \{y_1 y_2 : y_1 \in \mathbb{R}_+, y_2 \in [0, x]\} = \mathbb{R}_+ [0, x].$$

Thus, $\Phi(0) = \mathbb{R}_+ \{0\} = \{0\}$. Moreover, if $x \neq 0$, and thus $0 < x$, then

$$\Phi(x) = \mathbb{R}_+ [0, x] \supseteq \mathbb{R}_+ \{x\} = \mathbb{R}_+ = X,$$

and thus $\Phi(x) = X$. □

Now, having in mind the assumptions of Theorem 62, we can also prove the following two theorems.

Theorem 64. *The relation*
(1) *F is sequentially closed;*
(2) *F is sequentially closed-valued;*
(3) *F is sequentially inclusion-continuous.*

Proof. By using the corresponding definitions, one can easily see that F is a sequentially closed subset of $X \times Y$, and thus assertion (1) is true. Then, assertion (2) can be derived from assertion (1) by using Corollary 13.

To prove assertion (3), suppose now that a is a point and x is a sequence in X such that $a = \lim(x)$. Now, if $a \neq 0$, and thus $0 < a$, then by defining

$$u_n = a - (n+1)^{-1} \qquad \text{and} \qquad v_n = a + (n+1)^{-1}$$

for all $n \in \mathbb{N}$, we can obtain two sequences u and v in X such that

$$u_n < u_{n+1} < a < v_{n+1} < v_n,$$

and thus

$$F(u_n) \subset F(u_{n+1}) \subset F(a) \subset F(v_{n+1}) \subset F(v_n)$$

for all $n \in \mathbb{N}$. Thus, since $a = \lim(x)$, for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$u_n < x_i < v_n, \qquad \text{and thus} \qquad F(u_n) \subset F(x_i) \subset F(v_n)$$

for all $i \geq k_n$. Hence, we can infer that

$$F(u_n) \subseteq \bigcap_{i \geq k_n} F(x_i) \subseteq \bigcup_{k=1}^{\infty} \bigcap_{i \geq k} F(x_i) = \underline{\lim}_{k \to \infty} F(x_k)$$

and

$$\overline{\lim}_{k \to \infty} F(x_k) = \bigcap_{k=1}^{\infty} \bigcup_{i \geq k} F(x_i) \subseteq \bigcup_{i \geq k_n} F(x_i) \subseteq F(v_n)$$

for all $n \in \mathbb{N}$. Now, by using the definition F, we can also see that

$$F(a) = \bigcup_{n=1}^{\infty} F(u_n) \subseteq \underline{\lim}_{k \to \infty} F(x_k)$$

and

$$\overline{\lim}_{k \to \infty} F(x_k) \subseteq \bigcap_{n=1}^{\infty} F(x_k = F(a).$$

Hence, since $\underline{\lim}_{k\to\infty} F(x_k) \subseteq \overline{\lim}_{k\to\infty} F(x_k)$, we can infer that

$$F(a) = \underline{\lim}_{k\to\infty} F(x_k) \qquad \text{and} \qquad \overline{\lim}_{k\to\infty} F(x_k) = F(a),$$

and thus $F(a) = \lim_{k\to\infty} F(x_k)$ is also true.

Thus, we have proved that the relation F is sequentially inclusion-continous at each point a of $X \setminus \{0\}$. Therefore, it remains to show only that the corresponding statement also holds for the point 0. For this, to apply a similar argument as above, the sequence u has to be defined such that $u_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 65. *The relation*
(1) *G is sequentially continuous;*
(2) *G^{-1} is sequentially closed-valued;*
(3) *Φ is sequentially inclusion-left-continuous.*

Proof. Instead of proving assertion (2) directly, we can again note that if assertion (1) holds, then by Theorem 50 G is a sequentially closed subset of $Y \times X$. Hence, by Definition 15, it is clear that G^{-1} is a sequentially closed subset of $X \times Y$. Thus, again by Corollary 13, assertion (2) is also true. □

Remark 79. *The relation F and the function G were originally constructed by the first author to show that Lemma 1 of Pasteczka is not true.*

22. Discussion, Conclusions, and Future Research

By using the convergence of sequences of points and sets, Pasteczka in [1] tried to prove the interesting Lemma 1. However, the example of Boros, treated in Section 21, showed, in particular, that this lemma is completely false.

Therefore, by using the sequential methods of Pasteczka, in Theorem 62, we have established an improvement of this lemma. In the meantime, by using our ideas, Pasteczka in [117] also corrected his lemma in the following form.

Lemma 2. *Let D be a compact, metrisable topological space and $F : D \to [0, \infty)$ be a continuous function.*

If $T : [0, \infty) \to 2^D$ is nondecreasing, right-continuous (we consider topological limit on 2^x) and such that

- *each $T(x)$ is closed;*
- *$\overrightarrow{F} \circ T : [0, \infty) \to 2^{[0,\infty)}$ is left-continuous.*

Then, $\overrightarrow{F} \circ T$ is constant.

Convergence of nets is a much stronger tool than that of sequences. It generalises not only topologies, but also closures. Therefore, it would be of some interest to establish a generalisation of Theorem 62 by using net convergences.

However, relators are much stronger tools than proximities and even the convergence of nets to nets. Therefore, one should establish a generalisation of Theorem 62 by replacing the convergence space $Y(\lim)$ with a relator space $Y(\mathcal{S})$.

The interested reader may also investigate the relationships existing between Lemma 1 and Theorem 62. Moreover, readers may also try to replace the very particular domain space $X = \mathbb{R}_+$ with a more general birelator space $X(\mathcal{R}, \leq)$ [118].

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