

ON THE LARGEST PRIME FACTOR OF NUMERATORS OF BERNOULLI NUMBERS

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ABSTRACT. We prove that for most n , the numerator of the Bernoulli number B_{2n} is divisible by a large prime.

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1. INTRODUCTION

For a positive integer n , we write $\omega(n)$ for the number of distinct prime factors of n . Let $\{B_n\}_{n \geq 0}$ be the sequence of Bernoulli numbers given by $B_0 = 1$ and

$$B_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1} \quad \text{for all } n \geq 1.$$

Then $B_1 = -1/2$ and $B_{2n+1} = 0$ for all $n \geq 0$. Furthermore, we have $(-1)^{n+1}B_{2n} > 0$. Write $B_{2n} =: (-1)^{n+1}C_n/D_n$ with coprime positive integers C_n and D_n . The denominator D_n is well-understood by the von Staudt–Clausen theorem which asserts that

$$D_n = \prod_{p-1|2n} p. \tag{1}$$

As for C_n , it was proved in [3] that the estimate

$$\omega\left(\prod_{n \leq x} C_n\right) \geq (1 + o(1)) \frac{\log x}{\log \log x} \quad \text{holds as } x \rightarrow \infty.$$

Here, we look at the largest prime factor of C_n . For a positive integer m we put $P(m)$ for the largest prime factor of m .

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Theorem 1. *The inequality*

$$P(C_n) > \frac{1}{4} \log n$$

holds for most positive integers n .

Here and in what follows, we use the symbols O and o with their usual meaning. We also use c_1, c_2, \dots for computable positive constants and x_0 for a large real number, not necessarily the same from one occurrence to the next.

Proof. We let x be large. Put

$$\mathcal{M}(x) := \{x/2 \leq n \leq x : P(C_n) \leq (1/4) \log x\}. \quad (2)$$

Put $y := x^{\log \log \log x / \log \log x}$. We let

$$\mathcal{L}_1(x) := \{n \leq x : P(n) \leq y\}. \quad (3)$$

It is known (see Chapter III.5 in [5]), that

$$\#\mathcal{L}_1(x) = x \exp(-(1 + o(1))u \log u), \quad \text{where } u := \frac{\log x}{\log y}.$$

Since for us $u = \log \log x / \log \log \log x$, we get easily that

$$\#\mathcal{L}_1(x) = O\left(\frac{x}{(\log x)^{1/2}}\right). \quad (4)$$

We let $\tau(m)$ stand for the number of divisors of m . We put

$$\mathcal{L}_2(x) := \{n \leq x : \tau(n) > (\log x)^2\}. \quad (5)$$

Since

$$\sum_{n \leq x} \tau(n) = O(x \log x),$$

(see Theorem 320 on Page 347 in [2]), it follows easily that

$$\#\mathcal{L}_2(x) = O\left(\frac{x}{\log x}\right). \quad (6)$$

Let

$$\mathcal{L}_3(x) := \{n \geq x : p-1 \mid 2n \text{ for some prime } p \text{ with } P(p-1) > y\}. \quad (7)$$

The proof of Theorem 1.1 in [1] shows that

$$\#\mathcal{L}_3(x) = O\left(\frac{x}{(\log x)^{0.05}}\right). \quad (8)$$

From now on, we look at integers n in

$$\mathcal{N}(x) := \mathcal{M}(x) \setminus \cup_{i=1}^3 \mathcal{L}_i(x). \quad (9)$$

Put $z := (\log x)^2$ and let I be an arbitrary interval in $[x/2, x]$ of length at most z . Put $T := (1/4) \log x$ and put $K := \pi(T)$. We show that for $x > x_0$, I contains less than $K + 3$ numbers from $\mathcal{N}(x)$. Assume first that we have proved this and let us see how to finish the argument. Then

$$\begin{aligned} \#\mathcal{N}(x) &\leq \left(\left\lfloor \frac{x - x/2}{(\log x)^2} \right\rfloor + 1 \right) (K + 2) = O \left(\frac{x}{(\log x)^2} \cdot \frac{T}{\log T} \right) \\ &= O \left(\frac{x}{\log x \log \log x} \right), \end{aligned} \quad (10)$$

which together with estimates (4), (6), (8) shows that

$$\#\mathcal{M}(x) \leq \#\mathcal{L}_1(x) + \#\mathcal{L}_2(x) + \#\mathcal{L}_3(x) + \#\mathcal{N}(x) = O \left(\frac{x}{(\log x)^{0.05}} \right). \quad (11)$$

The desired estimate now follows by replacing x with $x/2$, then with $x/4$, etc., and summing up the resulting estimates (11).

It remains to prove that indeed I cannot contain $K + 3$ numbers from $\mathcal{N}(x)$ for $x > x_0$. Assume that it does and let them be $n_1 < n_2 < \dots < n_{K+3}$. Put $\lambda_i := n_i - n_1$ for $i = 1, \dots, K + 3$. Then $0 = \lambda_1 < \lambda_2 < \dots < \lambda_{K+3} \leq z$. Let $n = n_i$ for some $i = 1, \dots, K + 3$. We use the formula

$$\zeta(2n) = (-1)^{n+1} B_{2n} \frac{(2\pi)^{2n}}{2(2n)!} = \frac{C_n (2\pi)^{2n}}{D_n 2(2n)!},$$

as well as the approximation

$$\zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots = 1 + O \left(\frac{1}{2^{2n}} \right),$$

to get that

$$C_n = D_n \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) = D_n \frac{2(2n)!}{(2\pi)^{2n}} \left(1 + O \left(\frac{1}{2^{2n}} \right) \right). \quad (12)$$

We take logarithms in (12) above to arrive at

$$\log C_n - \log D_n - \log(2(2n)!) + 2n \log(2\pi) = \log \left(1 + O \left(\frac{1}{2^{2n}} \right) \right) = O \left(\frac{1}{2^n} \right). \quad (13)$$

We now let p_j for $j = 1, \dots, K$ be all the primes $p \leq T$ and write

$$C_{n_i} = p_1^{\alpha_{i,1}} p_2^{\alpha_{i,2}} \dots p_K^{\alpha_{i,K}} \quad \text{for all } i = 1, \dots, K + 3.$$

Observe that since $\tau(2n) \leq 2\tau(n) \leq 2(\log x)^2$, we have that

$$D_n = \prod_{p-1|2n} p \leq (2n+1)^{\tau(2n)} \leq (2x+1)^{2(\log x)^2} < \exp(3(\log x)^3) \quad (x > x_0). \quad (14)$$

Thus, from formula (12), we have that

$$\begin{aligned} C_n &\leq D_n \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2) \leq \frac{2\zeta(2)D_n}{(2\pi)^{2n}} (2n)^{2n} < \frac{2\zeta(2)D_n}{\pi^{2n}} n^{2n} \\ &< \left(\frac{2\zeta(2) \exp(3(\log x)^3)}{\pi^x} \right) x^{2x} < x^{2x} \quad \text{for } x > x_0, \end{aligned}$$

which implies that

$$\alpha_{i,j} \leq \frac{2x \log x}{\log p_j} \leq \frac{2x \log x}{\log 2} < 3x \log x \quad \text{for all } 1 \leq i \leq K+3, 1 \leq j \leq K.$$

Let $\Delta := (\Delta_1, \dots, \Delta_{K+3})$ be a nonzero vector in the null-space of the $(K+2) \times (K+3)$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{K+3,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{K+3,2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1,K} & a_{2,K} & \cdots & a_{K+3,K} \\ 1 & 1 & \cdots & 1 \\ n_1 & n_2 & \cdots & n_{K+3} \end{pmatrix}.$$

Such a vector exists and can be computed with Cramer's rule. It's height satisfies

$$\begin{aligned} \max\{|\Delta_i|\}_{1 \leq i \leq K+3} &\leq (K+2)! \max\{|\alpha_{i,j}|, |n_\ell|, i, j, \ell\}^{K+2} \\ &< (3x(K+2) \log x)^{K+2} < (3x(\log x)^2)^{\pi(T)+2} \\ &< x^{2(\pi(T)+2)} < \exp((\log x)^2), \end{aligned} \quad (15)$$

for $x > x_0$. We now evaluate formula (13) in $n = n_i$ for $i = 1, \dots, K+3$ and take the linear combination with coefficients $\Delta_1, \dots, \Delta_{K+3}$ of the resulting relations getting

$$\begin{aligned} \left| \sum_{i=1}^{K+3} \Delta_i \log C_{n_i} - \sum_{i=1}^{K+3} \Delta_i \log D_{n_i} - \sum_{i=1}^{K+3} \Delta_i \log(2(2n_i)!) + \sum_{i=1}^{K+3} 2\Delta_i n_i \log(2\pi) \right| \\ = O\left(\frac{\sum_{i=1}^{K+3} |\Delta_i|}{2^x} \right). \end{aligned} \quad (16)$$

In the left-hand side of estimate (16) above, the first sum vanishes; i.e.,

$$\sum_{i=1}^{K+3} \Delta_i \log C_{n_i} = 0,$$

because the vector Δ is orthogonal to the first K rows of A . Similarly, the last sum also vanishes; i.e.,

$$\sum_{i=1}^{K+3} \Delta_i n_i = 0,$$

because Δ is orthogonal to the last row of A . Finally, writing

$$2(2n_i)! = 2(2n_1)!(2n_1+1)(2n_1+2) \cdots (2n_i) =: 2(2n_1)!X_i \quad (i = 1, \dots, K+3),$$

we get that

$$\log(2(2n_i)!) = \log(2(2n_1)!) + \log X_i.$$

Hence,

$$\sum_{i=1}^{K+3} \Delta_i \log(2(2n_i)!) = \sum_{i=1}^{K+3} \Delta_i \log(2(2n_1)!) + \sum_{i=1}^{K+3} \Delta_i \log X_i = \sum_{i=1}^{K+3} \Delta_i \log X_i, \quad (17)$$

where we used $\sum_{i=1}^{K+3} \Delta_i = 0$, because Δ is orthogonal to the first before last row of matrix A . Thus using also (15), estimate (16) becomes

$$\left| \sum_{i=1}^{K+3} \Delta_i \log(D_{n_i}/X_i) \right| = O\left(\frac{(K+3) \exp((\log x)^2)}{2^x}\right) = O\left(\frac{1}{2^{x/2}}\right). \quad (18)$$

In the left-hand side of estimate (18) we have a linear form in logarithms. Further,

$$X_i < (2x)^{2(n_i-n_1)} \leq (2x)^{2z} < \exp(3(\log x)^3) \quad (x > x_0), \quad (19)$$

which is the same estimate as estimate (14) with D_{n_i} replaced by X_i for all $i = 1, \dots, K+3$. For each $i = 1, \dots, K+3$, let $P_i := P(n_i)$. Then $P_i \mid X_i$. Also, P_i does not divide D_{n_j} for any $j = 1, \dots, K+3$. Indeed, otherwise there would exist $q := P_i$ such that for some j , we have that $q \mid D_{n_j}$. Thus, there exists a prime number p such that $q \mid p-1$ and $p-1 \mid 2n_j$. However, this is not possible because $n_j \notin \mathcal{L}_3(x)$. Also, P_i divides X_j for all $j \geq i$ but does not divide X_j for any $j < i$. Indeed, this last claim follows because if $P_i \mid X_j$ for some $j < i$, then there exists $m \in [2n_1, 2n_j]$ such that $P_i \mid m$. But also $P_i \mid n_i$, so $P_i \mid 2n_i - m$, and this last number is nonzero since $2n_i \notin [2n_1, 2n_j]$. However, this is not possible for large x since it would lead to $y < P_i \leq 2n_i - m \leq 2z$, which is impossible for $x > x_0$. This shows that the linear form appearing in

the left-hand side of (17) is nonzero (indeed, if i is maximal such that $\Delta_i \neq 0$, then the coefficient of $\log P_i$ in the left is exactly $\Delta_i \neq 0$).

We apply a linear form in logarithms á la Baker in the left-hand side of (18) (see [4], for example). We get that the left-hand side of (18) is at least

$$> \exp \left(-c_1 c_2^K \left(\prod_{i=1}^{K+3} \max\{\log D_{n_i} \log X_{n_i}\} \right) \log \max\{|\Delta_i|\} \right),$$

for some appropriate constants c_1 and c_2 . With the bounds (14), (19) and (15), the above expression is at least

$$> \exp \left(-c_1 c_2^K (3(\log x)^3)^{K+3} (\log x)^2 \right),$$

which compared with (18) gives

$$x(\log 2)/2 - c_3 < c_1 (3c_2(\log x)^3)^{K+3} (\log x)^2,$$

with some appropriate constant c_3 . This last estimate implies easily that the inequality $K > (1/3 - \varepsilon) \log x / \log \log x$ holds for all $\varepsilon > 0$ and $x > x_0$ (depending on ε). Taking a sufficiently small value for ε (say $\varepsilon := 1/100$), and invoking the Prime Number Theorem to estimate $K = \pi(T)$, we get a contradiction. This finishes the argument and the proof of the theorem. \square

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