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Results in Mathematics

5-Dimensional Malcev-Like Algebras

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Abstract. The five-dimensional anti-commutative algebras having an analogous family of flags of subalgebras as the solvable Malcev algebras form the class of Malcev-like algebras. Recently a classification of the binary Lie algebras in this class is achieved. Our investigation extends this result. We determine Malcev-like algebras over a field $\mathbb{K}$ of characteristic zero. These algebras are extensions of $\mathbb{K}$ by a 4-dimensional nilpotent Lie algebra and simultaneously semidirect sums of the two-dimensional non-abelian Lie algebra and an abelian algebra. We find normal forms of their multiplications and describe their isomorphism classes.

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1. Introduction

An algebra $\mathfrak{g}$ over a field $\mathbb{K}$ satisfying the identity $x^2 = 0$, $x \in \mathfrak{g}$, is called zeropotent. If the characteristic of $\mathbb{K}$ is different to two, then the above identity is equivalent to $xy + yx = 0$, $x, y \in \mathfrak{g}$ and the algebra is said to be anti-commutative. Zeropotent algebras of dimension less than four are completely classified (cf. [6,27,36]). In this paper we study nonassociative anti-commutative algebras. Prominent examples are Malcev algebras and binary Lie algebras playing important role in nonassociative generalization of Lie theory (see [14,22,31,34]). Following S. Lie's ideas A. I. Malcev initiated to associate with an analytic loop a tangent algebra at the identity and to investigate its algebraic structure derived from the manifold data and the loop operation (cf. [32]). Analogously to the relation between Lie groups and Lie algebras he

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determined successfully the correspondence between local analytic Moufang and diassociative loops and their tangent algebras, the Moufang-Lie and the binary Lie algebras. The Moufang-Lie algebras were later called Malcev algebras. A systematic study of Malcev and binary Lie algebras can be found in [11,35]. Once an algebraic object is associated with an analytic loop, then the algebraist is challenged to classify these algebras. Kuzmin [28] determined the Malcev algebras of dimension less than or equal to five. The non-Malcev binary Lie algebras have dimension greater than three and the algebras of dimension four were classified by Gainov and Kuzmin [12,30]. Since then, many authors payed a special attention for the investigation of these algebras concerning to structures, classifications, extensions, degenerations, developments of Lie loop theory (see [1–3,7,8,13,15–19,21,24–26,29,33]).

Central extensions are the key tools to determine nilpotent algebras. The classification of nilpotent algebras is proceeded by central extensions of algebras from the same variety having a smaller dimension. The nilpotent anti-commutative algebras of dimension four, five and six are determined in [5,20] and [23], respectively. In this paper we deal with a class of solvable anti-commutative algebras which can be characterized in two different manners as extensions. The studied algebras and the five-dimensional solvable Malcev algebras are close related because they have an analogous decomposition properties. These algebras are called *Malcev-like algebra*s in [10] and are extensions of a 1-dimensional algebra by a four-dimensional nilpotent Lie algebra and at the same time semidirect sums of the two-dimensional non-abelian Lie algebra by an abelian algebra. The normal forms and the isomorphism classes of the binary Lie algebras in the class of Malcev-like algebras were determined in [10]. Also the four-dimensional non-Lie binary Lie algebras have a similar family of flags of subalgebras as the five-dimensional Malcev algebras. Based on this observation the four-dimensional anti-commutative algebras having the same ideal structure as the non-Lie binary Lie algebras and their group of automorphisms were classified in [9]. Following the approach of [9,10] this paper is devoted to the investigation of the Malcev-like algebras.

In Sect. 2 we describe the theory of Malcev-like algebras and give the method of their classification improved in [10], where the authors distinguished three types $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$ of these algebras according to whether their four-dimensional nilpotent ideal is abelian $\mathfrak{a}$, or the direct sum $\mathfrak{n} = \mathfrak{n}_3 \oplus \mathbb{K}$ of the 3-dimensional Heisenberg Lie algebra $\mathfrak{n}_3$ and $\mathbb{K}$, or the filiform Lie algebra $\mathfrak{f}$. The Malcev-like algebras of abelian type are Lie algebras (see Proposition 5.3 in [10]). Therefore, we study algebras of non-abelian types and partition them into five disjoint families. These families are described by two equivalent conditions which guarantee that the isomorphic algebras belong to the same families (see Sect. 3). The normal forms of the Malcev-like algebras are an appropriate representation of their isomorphism classes. We find the normal forms and the isomorphism classes in Sect. 4 and the results are resumed in Sect. 5.

2. Preliminaries

We adapt here the useful concepts, the theory and the classification method from [10] about Malcev-like algebras.

We consider a (nonassociative) anti-commutative algebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic 0. Let $\mathbb{K}^\circ$ be the multiplicative group of the field $\mathbb{K}$, $\mathbb{K}^{\circ 2}$ be the subgroup of squares in $\mathbb{K}$, and $\sigma : \mathbb{K}^\circ/\mathbb{K}^{\circ 2} \to \mathbb{K}^\circ$ be an arbitrary section. The multiplication in $\mathfrak{g}$ is denoted by juxtaposition. The group of automorphisms of $\mathfrak{g}$ will be denoted by $\mathrm{Aut}(\mathfrak{g})$, the algebra of linear maps of $\mathfrak{g}$ by $\mathrm{End}(\mathfrak{g})$. For any $x \in \mathfrak{g}$ the left multiplication map $\lambda_x : \mathfrak{g} \to \mathfrak{g}$ and the right multiplication map $\rho_x : \mathfrak{g} \to \mathfrak{g}$ are the linear maps $\lambda_x(y) = xy$ and $\rho_x(y) = yx$, $y \in \mathfrak{g}$, respectively. The left and right multiplications differ only by sign, the left and right ideals are coincide. The commutator algebra of $\mathfrak{g}$ is the ideal $\mathfrak{g}' = \{xy|x, y \in \mathfrak{g}\}$, the second commutator algebra is the subalgebra $\mathfrak{g}'' = \{uv|u, v \in \mathfrak{g}'\}$. The Jacobian $J(x, y, z)$ of the elements $x, y, z \in \mathfrak{g}$ is defined by $J(x, y, z) = (xy)z+(zx)y+(yz)x$. $\mathfrak{g}$ is a Lie algebra if $J(x, y, z) = 0$ for all $x, y, z \in \mathfrak{g}$, a Malcev algebra if $J(x, y, xz) = J(x, y, z)x$ for all $x, y, z \in \mathfrak{g}$, a binary Lie algebra if $J(x, y, xy) = 0$ for all $x, y \in \mathfrak{g}$. Hence, every Lie algebra is a Malcev algebra, each Malcev algebra is a binary Lie algebra. The center of $\mathfrak{g}$ is the ideal $\mathcal{Z}(\mathfrak{g}) = \{z \in \mathfrak{g}|z\mathfrak{g} = \mathfrak{g}z = 0\}$. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be anti-commutative algebras. A short exact sequence

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{c} \xrightarrow{\pi} \mathfrak{b} \to 0 \tag{1}$$

such that $\iota(\mathfrak{a})$ is an ideal of $\mathfrak{c}$ and π induces an isomorphism of the factor algebra $\mathfrak{c}/\iota(\mathfrak{a})$ to $\mathfrak{b}$ is called an *extension* $\mathfrak{c}$ of $\mathfrak{b}$ by $\mathfrak{a}$. An extension (1) is split if there is a subalgebra $\widetilde{\mathfrak{b}}$ of $\mathfrak{c}$ such that $\pi(\widetilde{\mathfrak{b}})$ is isomorphic to $\mathfrak{b}$. In this case $\mathfrak{c}$ is the *semidirect sum of* $\mathfrak{a}$ *and* $\widetilde{\mathfrak{b}}$. We treat the anti-commutative semidirect sums $\mathfrak{a} \oplus_l \mathfrak{b}$ of the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ determined by the operation $(\xi, X)(\eta, Y) = \big(\xi\eta, XY + l_\xi(Y) - l_\eta(X)\big)$, $\xi, \eta \in \mathfrak{a}$, $X, Y \in \mathfrak{b}$ on $\mathfrak{a} \oplus \mathfrak{b}$, where $l : \mathfrak{a} \times \mathfrak{b} \to \mathfrak{b}$ is a given bilinear map. The algebra $\mathfrak{g}$ is called decomposable if it is the direct sum of subalgebras.

Given two bases $B = \{e_1, e_2, \dots, e_n\}$, $\hat{B} = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$ in the algebra $\mathfrak{g}$ the vector space $\mathbb{K}^n$ can be equipped with algebra structures $\mathbb{K}^n(B)$ and $\mathbb{K}^n(\hat{B})$, so that the linear coordinate maps $\phi_B : \mathfrak{g} \to \mathbb{K}^n$ and $\phi_{\hat{B}} : \mathfrak{g} \to \mathbb{K}^n$ become isomorphisms of algebras. The composition $\phi_{\hat{B}} \circ \phi_B^{-1} : \mathbb{K}^n(B) \to \mathbb{K}^n(\hat{B})$ is an isomorphism between algebras corresponding to the change of the bases B and $\hat{B}$. The 5-dimensional non-decomposable solvable Malcev algebras over $\mathbb{K}$ are determined in [28]. It turns out that they are extensions of a 1-dimensional algebra by a nilpotent Lie algebra and simultaneously the semidirect sums of the 2-dimensional non-abelian Lie algebra by an abelian algebra. We investigate anti-commutative algebras having the same ideal structure as these solvable Malcev algebras.

Definition 1. A 5-dimensional non-decomposable anti-commutative algebra $\mathfrak{c}$ is called Malcev-like algebra if

(i) the commutator algebra $\mathfrak{c}'$ is a 4-dimensional nilpotent Lie algebra,
(ii) $\mathfrak{c}$ is an anti-commutative semidirect sum $\mathfrak{l}_2 \oplus_l \mathfrak{i}$ of the 2-dimensional non-abelian Lie algebra $\mathfrak{l}_2$ and the abelian ideal $\mathfrak{i}$.

A 4-dimensional nilpotent Lie algebra is isomorphic to one of the following algebras:

- abelian $\mathfrak{a}$,
- direct sum $\mathfrak{n} = \mathfrak{n}_3 \oplus \mathbb{K}$, where $\mathfrak{n}_3$ is the 3-dimensional Heisenberg Lie algebra given by the multiplication $e_1e_2 = e_4$,
- filiform Lie algebra $\mathfrak{f}$ defined by the multiplication $e_1e_2 = e_3$, $e_1e_3 = e_4$.

Definition 2. We call an anti-commutative algebra $\mathfrak{c}$ Malcev-like algebra of type $\mathfrak{t}$ if $\mathfrak{c}' \cong \mathfrak{t}$, where $\mathfrak{t} \in \{\mathfrak{a}, \mathfrak{n}, \mathfrak{f}\}$ and denote it by $\mathfrak{c}^{\mathfrak{t}}$.
A basis $\{e_0, e_1, e_2, e_3, e_4\}$ in a Malcev-like algebra $\mathfrak{c}^{\mathfrak{t}}$ is called distinguished if $\{e_1, e_2, e_3, e_4\}$ is a basis of the commutator algebra $(\mathfrak{c}^{\mathfrak{t}})'$ with the multiplication

- $\mathfrak{c}^{\mathfrak{a}}$-algebra: abelian,
- $\mathfrak{c}^{\mathfrak{n}}$-algebra: $e_1e_2 = e_4$,
- $\mathfrak{c}^{\mathfrak{f}}$-algebra: $e_1e_2 = e_3$, $e_1e_3 = e_4$.

A distinguished basis $\{e_0, e_1, e_2, e_3, e_4\}$ in $\mathfrak{c}^{\mathfrak{t}} = \mathfrak{l}_2 \oplus_l \mathfrak{i}$ is said to be special if $e_0e_1 = e_1$ holds and the vectors e_2, e_3, e_4 generate an abelian ideal.

Proposition 1. *Every Malcev-like algebra $\mathfrak{c}^{\mathfrak{a}}$ is a Lie algebra.*

Proof. It follows from Proposition 5.3 in [10]. □

Therefore, we investigate Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$. By Propositions 3.2 and 3.3 in [10] we have

Proposition 2. *Let $\{e_0, e_1, e_2, e_3, e_4\}$ be a special distinguished basis of a Malcev-like algebra $\mathfrak{c}^{\mathfrak{t}}$, where $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$. The matrix $\mathbf{X}$ of the restriction of the map λ_{e_0} to the ideal $(\mathfrak{c}^{\mathfrak{t}})'$ has the form*

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & L_0 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0\ 0\ 0 \end{bmatrix}, \quad L_0 = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}, \tag{2}$$

where L_0 is the matrix of the map $\lambda_{e_0}|_{\mathfrak{i}} : \mathfrak{i} \to \mathfrak{i}$ induced on the abelian ideal $\mathfrak{i}$ satisfying the condition

$$\operatorname{rank} \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{bmatrix} = 2 \quad \textit{and} \quad \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

for $\mathfrak{n}$ and $\mathfrak{f}$, respectively. The matrix of the map $\lambda_{e_1}|_{\mathfrak{i}} : \mathfrak{i} \to \mathfrak{i}$ is given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4}$$

for $\mathfrak{n}$ *and* $\mathfrak{f}$, *respectively. The matrix of* $\lambda_{e_k}|_{\mathfrak{i}} : \mathfrak{i} \to \mathfrak{i}$, $k = 2, 3, 4$ *is the zero matrix.*

The $\mathfrak{c}^{\mathfrak{t}}$*-algebras,* $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$, *defined on the vector space* $\mathbb{K}^5$ *and having the canonical basis* $\{e_0, e_1, e_2, e_3, e_4\}$ *of* $\mathbb{K}^5$ *as a special distinguished basis, are parametrized by the* 3×3 *matrices* L_0 *given in* (2) *and satisfying the condition* (3).

Let $L_0 = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}$ *be an arbitrary matrix satisfying the condition* (3) *with respect to* $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$. *The multiplication determined by the matrices* λ_{e_i}, $i = 0, 1$, *given by formulas* (2) *and* (4) *defines a unique* $\mathfrak{c}^{\mathfrak{t}}$*-algebra* $\mathfrak{c}(L_0)$ *on the vector space* $\mathbb{K}^5$, *such that the canonical basis* $\{e_0, e_1, e_2, e_3, e_4\}$ *of* $\mathbb{K}^5$ *is a special distinguished basis of* $\mathfrak{c}(L_0)$.
If (2) *is the matrix of the map* $\lambda_{e_0}|_{\mathfrak{c}'} : \mathfrak{c}' \to \mathfrak{c}'$ *for a* $\mathfrak{c}^{\mathfrak{t}}$*-algebra* $\mathfrak{c}$ *having the properties given by* (3), *then the* $\mathfrak{c}^{\mathfrak{t}}$*-algebra* $\mathfrak{c}(L_0)$ *constructed on the vector space* $\mathbb{K}^5$ *is isomorphic to* $\mathfrak{c}$.

Definition 3. A 5-dimensional vector space $\mathbf{V}$ with a fixed basis $\mathcal{B}$ can be identified with the vector space $\mathbb{K}^5$. For a matrix L_0 satisfying (3) we denote by $\mathfrak{c}(L_0)$ the uniquely defined $\mathfrak{c}^{\mathfrak{t}}$-algebra on $\mathbf{V}$ with special distinguished basis $\mathcal{B}$ constructed according to Proposition 2.

Definition 4. A map $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ between Malcev-like algebras $\mathfrak{c}$ and $\hat{\mathfrak{c}}$ is called partial isomorphism if the induced map $\varphi|_{\mathfrak{c}'} : \mathfrak{c}' \to \hat{\mathfrak{c}}'$ is an isomorphism. A partial isomorphism $\varphi : \mathfrak{c} \to \mathfrak{c}$ is called partial automorphism.

We define partial isomorphic Malcev-like algebras $\mathfrak{c} = \mathfrak{c}(L_0)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(\hat{L}_0)$ given by the matrices

$$L_0 = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix} \quad \text{and} \quad \hat{L}_0 = \begin{bmatrix} \hat{x}_2^2 & \hat{x}_3^2 & \hat{x}_4^2 \\ \hat{x}_2^3 & \hat{x}_3^3 & \hat{x}_4^3 \\ \hat{x}_2^4 & \hat{x}_3^4 & \hat{x}_4^4 \end{bmatrix}. \tag{5}$$

Lemma 3. *A linear map* $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ *defines a partial isomorphism if and only if its matrix with respect to distinguished bases of* $\mathfrak{c}$ *and* $\hat{\mathfrak{c}}$ *given by*

$$\begin{bmatrix} u^0 & \mathbf{0} \\ \mathbf{u}^T & A \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u^1 \; u^2 \; u^3 \; u^4 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \; 0 \; 0 \; 0 \end{bmatrix}, \tag{6}$$

such that for an

(a) $\mathfrak{c}^{\mathfrak{n}}$*-algebra:* $A = \begin{bmatrix} p^1 & q^1 & 0 & 0 \\ p^2 & q^2 & 0 & 0 \\ p^3 & q^3 & r^3 & 0 \\ p^4 & q^4 & r^4 & p^1q^2 - p^2q^1 \end{bmatrix}$, $(p^1q^2 - p^2q^1)r^3 \neq 0$,

(b) $\mathfrak{c}^{\mathfrak{f}}$*-algebra:* $A = \begin{bmatrix} p^1 & 0 & 0 & 0 \\ p^2 & q^2 & 0 & 0 \\ p^3 & q^3 & p^1q^2 & 0 \\ p^4 & q^4 & p^1q^3 & (p^1)^2q^2 \end{bmatrix}$, $p^1q^2 \neq 0$,

$r^3, r^4, p^i, q^i \in \mathbb{K},\ i = 1, 2, 3, 4.$

Definition 5. A pair $\{\mathbf{V}, \mathfrak{p}\}$ of a 5-dimensional vector space $\mathbf{V}$ and a 4-dimensional nilpotent Lie algebra $\mathfrak{p}$ which is contained as a subspace in $\mathbf{V}$ and is isomorphic to $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ is called partial Lie algebra.
A partial Lie-automorphism of a partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$ is a linear automorphism of $\mathbf{V}$ which induces a Lie algebra automorphism of $\mathfrak{p}$. The group of partial Lie-automorphisms of a partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$ is denoted by $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$.

A Malcev-like algebra defined on $\mathbf{V}$ is said to be adapted to the partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$ if its derived algebra is $\mathfrak{p}$.

We fix a partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$. The partial automorphisms of any Malcev-like algebra adapted to $\{\mathbf{V}, \mathfrak{p}\}$ are exactly the partial Lie-automorphisms of $\{\mathbf{V}, \mathfrak{p}\}$. In particular, if $\mathfrak{p} \cong \mathfrak{t}$, then any $\mathfrak{c}^{\mathfrak{t}}$-algebra is isomorphic to some $\mathfrak{c}^{\mathfrak{t}}$-algebra adapted to $\{\mathbf{V}, \mathfrak{p}\}$. Hence, for the investigation of isomorphism classes of $\mathfrak{c}^{\mathfrak{t}}$-algebras it is sufficient to consider only $\mathfrak{c}^{\mathfrak{t}}$-algebras which are adapted to the partial Lie algebra $\{\mathbf{V}, \mathfrak{p}\}$.
Let $\mathcal{B} = \{e_0, e_1, e_2, e_3, e_4\}$ be a fixed basis of $\mathbf{V}$ such that $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{p}$ defined by multiplication $e_1 e_2 = e_4$ for $\mathfrak{p} \cong \mathfrak{n}$ and $e_1 e_2 = e_3$, $e_1 e_3 = e_4$ for $\mathfrak{p} \cong \mathfrak{f}$.
Consider the $\mathfrak{c}^{\mathfrak{t}}$-algebras on $\mathbf{V}$ having $\mathcal{B}$ as a special distinguished basis. All $\mathfrak{c}^{\mathfrak{t}}$-algebras $\mathfrak{c}(L_0)$ have the same partial automorphism group $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$. Let $\{c_{ij}^k(L_0)\}$ be the system of structure constants of $\mathfrak{c}(L_0)$, that is $e_i e_j = \sum_k c_{ij}^k e_k$. Identifying the basis $\mathcal{B}$ with the canonical basis of the vector space $\mathbb{K}^5$, we denote by γ_{L_0} the multiplication on $\mathbb{K}^5$ determined by $\{c_{ij}^k(L_0)\}$. Following [4] the group $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$ acts on the set Γ of multiplications of $\mathfrak{c}^{\mathfrak{t}}$-algebras on $\mathbb{K}^5$ by

$$\gamma_{L_0}(x, y) \mapsto M^{-1}\gamma_{L_0}(Mx, My), \quad M \in \mathrm{Aut}^{\mathfrak{p}}(\mathbf{V}), \quad x, y \in \mathbb{K}^5. \tag{7}$$

Lemma 4. *The orbits of the action* (7) *of the group* $\mathrm{Aut}^{\mathfrak{p}}(\mathbf{V})$ *on* Γ *correspond to isomorphism classes of* 5*-dimensional* $\mathfrak{c}^{\mathfrak{t}}$*-algebras.*

According to Theorem 4.4 in [10] the necessary and sufficient conditions for a partial isomorphism φ to be isomorphism between the Malcev-like algebras $\mathfrak{c} = \mathfrak{c}(L_0)$ and $\hat{\mathfrak{c}} = \mathfrak{c}(\hat{L}_0)$ are the following:

Theorem 5. *Let* $\{e_0, e_1, e_2, e_3, e_4\}$ *and* $\{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ *be special distinguished bases of Malcev-like algebras* $\mathfrak{c} = \mathfrak{c}(L_0)$ *and* $\hat{\mathfrak{c}} = \mathfrak{c}(\hat{L}_0)$*, respectively, determined by the matrices* (5)*. The partial isomorphism* $\varphi : \mathfrak{c} \to \hat{\mathfrak{c}}$ *is an isomorphism if and only if there exists a matrix* A *of the form given in Lemma* 3 *satisfying the matrix equation*

$$\begin{bmatrix} p^1 & q^1 x_2^2 & q^1 x_3^2 & q^1 x_4^2 \\ p^2 & q^2 x_2^2 & q^2 x_3^2 & q^2 x_4^2 \\ p^3 & q^3 x_2^2 + r^3 x_2^3 & q^3 x_3^2 + r^3 x_3^3 & q^3 x_4^2 + r^3 x_4^3 \\ p^4 & q^4 x_2^2 + r^4 x_2^3 + & q^4 x_3^2 + r^4 x_3^3 + & q^4 x_4^2 + r^4 x_4^3 + \\ & +(p^1 q^2 - p^2 q^1) x_2^4 & +(p^1 q^2 - p^2 q^1) x_3^4 & +(p^1 q^2 - p^2 q^1) x_4^4 \end{bmatrix} =$$

$$= \begin{bmatrix} u^0p^1 & u^0q^1 & 0 & 0 \\ u^0\hat{x}^2_jp^j & u^0\hat{x}^2_jq^j & u^0\hat{x}^2_jr^j & u^0\hat{x}^2_4(p^1q^2-p^2q^1) \\ u^0\hat{x}^3_jp^j & u^0\hat{x}^3_jq^j & u^0\hat{x}^3_jr^j & u^0\hat{x}^3_4(p^1q^2-p^2q^1) \\ u^0\hat{x}^4_jp^j+u^1p^2-u^2p^1 & u^0\hat{x}^4_jq^j+u^1q^2-u^2q^1 & u^0\hat{x}^4_jr^j & u^0\hat{x}^4_4(p^1q^2-p^2q^1) \end{bmatrix} \tag{8}$$

for $\mathfrak{c}^{\mathfrak{n}}$*-algebra and*

$$\begin{bmatrix} p^1 & 0 & 0 & 0 \\ p^2 & q^2x^2_2 & q^2x^2_3 & q^2x^2_4 \\ p^3 & q^3x^2_2+p^1q^2x^3_2 & q^3x^2_3+p^1q^2x^3_3 & q^3x^2_4+p^1q^2x^3_4 \\ p^4 & \begin{matrix} q^4x^2_2+p^1q^3x^3_2+ \\ +(p^1)^2q^2x^4_2 \end{matrix} & \begin{matrix} q^4x^2_3+p^1q^3x^3_3+ \\ +(p^1)^2q^2x^4_3 \end{matrix} & \begin{matrix} q^4x^2_4+p^1q^3x^3_4+ \\ +(p^1)^2q^2x^4_4 \end{matrix} \end{bmatrix} =$$

$$= \begin{bmatrix} p^1 & 0 & 0 & 0 \\ \hat{x}^2_jp^j & \hat{x}^2_jq^j & \hat{x}^2_3p^1q^2+\hat{x}^2_4p^1q^3 & \hat{x}^2_4(p^1)^2q^2 \\ \hat{x}^3_jp^j+u^1p^2-u^2p^1 & \hat{x}^3_jq^j+u^1q^2 & \hat{x}^3_3p^1q^2+\hat{x}^3_4p^1q^3 & \hat{x}^3_4(p^1)^2q^2 \\ \hat{x}^4_jp^j+u^1p^3-u^3p^1 & \hat{x}^4_jq^j+u^1q^3 & \hat{x}^4_3p^1q^2+\hat{x}^4_4p^1q^3+u^1p^1q^2 & \hat{x}^4_4(p^1)^2q^2 \end{bmatrix}. \tag{9}$$

for $\mathfrak{c}^{\mathfrak{f}}$*-algebra.*

3. Families of Malcev-Like Algebras

In this section we divide into five disjoint families the Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$. We consider their subspaces $(\mathfrak{c}^{\mathfrak{t}})''$ and $\mathcal{Z}((\mathfrak{c}^{\mathfrak{t}})')$. We obtain $(\mathfrak{c}^{\mathfrak{n}})'' = \mathcal{Z}((\mathfrak{c}^{\mathfrak{f}})') = \langle e_4 \rangle$ and $(\mathfrak{c}^{\mathfrak{f}})'' = \mathcal{Z}((\mathfrak{c}^{\mathfrak{n}})') = \langle e_3, e_4 \rangle$.

Proposition 6. *Every Malcev-like algebra* $\mathfrak{c}^{\mathfrak{t}}$*,* $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ *belongs to one of the following disjoint families defined by two equivalent conditions:*

$C^{(1)}$: (a) *the subspaces* $\langle e_4 \rangle$ *and* $\langle e_3, e_4 \rangle$ *are ideals in* $\mathfrak{c}^{\mathfrak{t}}$,

$$(b)\ L_0 = \begin{bmatrix} x^2_2 & 0 & 0 \\ x^3_2 & x^3_3 & 0 \\ x^4_2 & x^4_3 & x^4_4 \end{bmatrix},\ x^2_2 \neq 0 \ \textit{for}\ \mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\},$$

and additionally $x^3_3 \neq 0$ *for* $\mathfrak{t} = \mathfrak{n}$,

$C^{(2)}$: (a) *the subspace* $\langle e_3, e_4 \rangle$ *is ideal in* $\mathfrak{c}^{\mathfrak{t}}$, *but* $\langle e_4 \rangle$ *is not,*

$$(b)\ L_0 = \begin{bmatrix} x^2_2 & 0 & 0 \\ x^3_2 & x^3_3 & x^3_4 \\ x^4_2 & x^4_3 & x^4_4 \end{bmatrix},\ x^2_2 \neq 0 \ \textit{and}\ x^3_4 \neq 0 \ \textit{for}\ \mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\},$$

$C^{(3)}$: (a) *the subspace* $\langle e_4 \rangle$ *is ideal in* $\mathfrak{c}^{\mathfrak{t}}$, *but* $\langle e_3, e_4 \rangle$ *is not,*

$$(b)\ L_0 = \begin{bmatrix} x^2_2 & x^2_3 & 0 \\ x^3_2 & x^3_3 & 0 \\ x^4_2 & x^4_3 & x^4_4 \end{bmatrix},\ x^2_3 \neq 0, x^3_4 = 0 \ \textit{for}\ \mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\},$$

and in addition $x^2_2x^3_3 - x^2_3x^3_2 \neq 0$ *for* $\mathfrak{t} = \mathfrak{n}$,

$C^{(4)}$: (a) *the subspaces* $\langle e_4\rangle$ *and* $\langle e_3, e_4\rangle$ *are not ideals in* $\mathfrak{c}^{\mathfrak{t}}$ *and* $x_4^2 = 0$,

$$(b)\ L_0 = \begin{bmatrix} x_2^2 & x_3^2 & 0 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix},\ x_3^2 \neq 0, x_4^3 \neq 0 \textit{ for } \mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\},$$

$C^{(5)}$: (a) *the subspaces* $\langle e_4\rangle$ *and* $\langle e_3, e_4\rangle$ *are not ideals in* $\mathfrak{c}^{\mathfrak{t}}$ *and* $x_4^2 \neq 0$,

$$(b)\ L_0 = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix},\ x_4^2 \neq 0 \ \textit{ for } \mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$$

$$\textit{and in addition rank} \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{bmatrix} = 2 \textit{ for } \mathfrak{t} = \mathfrak{n}.$$

Proof. As a consequence of Proposition 7.1 in [10] every binary Lie Malcev-like algebra $\mathfrak{c}^{\mathfrak{n}}$ belongs to one of the disjoint families $C^{(k)}$, $k = 1, 2, 3$. By Proposition 8.1 in [10] each binary Lie Malcev-like algebra $\mathfrak{c}^{\mathfrak{f}}$ lies in the family $C^{(1)}$. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{n}}$ which are not binary Lie algebras belong to one of the disjoint families $C^{(4)}$, $C^{(5)}$. The non-binary Lie Malcev-like algebras $\mathfrak{c}^{\mathfrak{f}}$ are in one of the disjoint families $C^{(k)}$, $k = 2, 3, 4, 5$. □

Remark 1. The conditions (a) defining the families $C^{(k)}$, $k = 1, 2, 3$, $C^{(4)}$ or $C^{(5)}$ show that isomorphic Malcev-like algebras belong to the same family. The Malcev-like algebras in the family $C^{(4)}$ are not isomorphic to the algebras of the family $C^{(5)}$, which will be proved in Propositions 25, 29 in the case of $\mathfrak{t} = \mathfrak{n}$, whereas in Propositions 26, 31 in the case of $\mathfrak{t} = \mathfrak{f}$.

4. Normal Forms of Malcev-Like Algebras

In this section we prove that for any Malcev-like $\mathfrak{c}^{\mathfrak{t}}$-algebra the left translation λ_{e_0} given by (2) can be reduced to a normal form by an isomorphism. We investigate the families $C^{(n)}$, $n = 1, \dots, 5$.

4.1. $C^{(1)}$-Family

Definition 6. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ with $L_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}$ belong to the family $C^{(1)}$ if $x_2^2 \neq 0$ for $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$, and in addition $x_3^3 \neq 0$ for $\mathfrak{t} = \mathfrak{n}$.

Proposition 7. *Any* $\mathfrak{c}^{\mathfrak{n}}$*-algebra* $\mathfrak{c}(L_0)$ *in the family* $C^{(1)}$ *is isomorphic to the* $\mathfrak{c}^{\mathfrak{n}}$*-algebra* $\mathfrak{c}(\hat{L}_0)$ *with* $\hat{L}_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ \varepsilon & x_3^3 & 0 \\ 0 & \delta & x_4^4 \end{bmatrix}$ *or* $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3}{x_2^2} & 0 \\ 0 & \delta & \frac{x_4^4}{x_2^2} \end{bmatrix}$, $\varepsilon, \delta \in \{0, 1\}$. *The isomorphism class of* $\mathfrak{c}(L_0)$ *consists of the algebras given by the matrices:*

(i) $\begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ 0 & 0 & x_4^4 \end{bmatrix}$ *and* $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3}{x_2^2} & 0 \\ 0 & 0 & \frac{x_4^4}{x_2^2} \end{bmatrix}$ *if in each pair of conditions* $(x_2^3 = 0,\ x_2^2 \neq x_3^3)$ *and* $(x_3^4 = 0,\ x_3^3 \neq x_4^4)$ *one is satisfied,*

(ii) $\begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ 0 & 1 & x_3^3 \end{bmatrix}$ *and* $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3}{x_2^2} & 0 \\ 0 & 1 & \frac{x_3^3}{x_2^2} \end{bmatrix}$ *if one of the conditions* $(x_2^3 = 0,$ $x_2^2 \neq x_3^3)$ *and both conditions* $(x_3^4 \neq 0,\ x_3^3 = x_4^4)$ *are satisfied,*

(iii) $\begin{bmatrix} x_2^2 & 0 & 0 \\ 1 & x_2^2 & 0 \\ 0 & 0 & x_4^4 \end{bmatrix}$ *if both conditions* $(x_2^3 \neq 0,\ x_2^2 = x_3^3)$ *and one of the conditions* $(x_3^4 = 0,\ x_3^3 \neq x_4^4)$ *are satisfied,*

(iv) $\begin{bmatrix} x_2^2 & 0 & 0 \\ 1 & x_2^2 & 0 \\ 0 & 1 & x_2^2 \end{bmatrix}$ *if the conditions* $x_2^3 \neq 0,\ x_3^4 \neq 0,\ x_2^2 = x_3^3 = x_4^4$ *are satisfied.*

Proof. Taking into account the forms $\hat{L}_0$ and L_0 the matrix equation (8) in Theorem 5 describes the isomorphism conditions for the matrix A of partial isomorphism in Lemma 3(a). The equations given by the first two terms of the last row give

$$u^0 \begin{bmatrix} \delta p^3 + \left(\hat{x}_4^4 - \frac{1}{u^0}\right) p^4 \\ \delta q^3 + \left(\hat{x}_4^4 - \frac{x_2^2}{u^0}\right) q^4 - x_2^3 \frac{r^4}{u^0} - x_2^4 \frac{p^1 q^2 - p^2 q^1}{u^0} \end{bmatrix} = u^2 \begin{bmatrix} p^1 \\ q^1 \end{bmatrix} - u^1 \begin{bmatrix} p^2 \\ q^2 \end{bmatrix}. \quad (10)$$

Since the vectors $\begin{bmatrix} p^1 \\ q^1 \end{bmatrix}$ and $\begin{bmatrix} p^2 \\ q^2 \end{bmatrix}$ are linearly independent, the equations in (10) are solvable with appropriately chosen coefficients u^2 and u^1. Therefore, these equations do not give any restriction for the solution of the remaining equations.

Firstly, we consider the case that $p^1 \neq 0$. Hence, one has $u^0 = 1$. Comparing the diagonal elements on both sides of the matrix equation we obtain $\hat{x}_2^2 = x_2^2$, $\hat{x}_3^3 = x_3^3$, $\hat{x}_4^4 = x_4^4$. The equation $\begin{bmatrix} p^1 & q^1 x_2^2 \\ p^2 & q^2 x_2^2 \end{bmatrix} = \begin{bmatrix} p^1 & q^1 \\ x_2^2 p^2 & x_2^2 q^2 \end{bmatrix}$ gives $p^2 = q^1 = 0$ for $x_2^2 \neq 1$. Three equations remain:

$$\varepsilon = \hat{x}_2^3 = \frac{(x_2^2 - x_3^3) q^3 + x_2^3 r^3}{q^2}, \ \delta = \hat{x}_3^4 = \frac{(x_3^3 - x_4^4) r^4 + x_3^4 p^1 q^2}{r^3}, \ (1 - x_3^3) p^3 = 0.$$

Since the values of p^1, q^2 and r^3 are arbitrary non-zero numbers, we can suppose $\varepsilon, \delta \in \{0, 1\}$. The first equation yields that $\varepsilon \neq 0$ for $x_2^3 \neq 0$ and $x_2^2 = x_3^3$, otherwise $\varepsilon = 0$ is possible with a suitable choice of the parameter q^3. The second equation gives that $\delta \neq 0$ for $x_3^4 \neq 0$ and $x_3^3 = x_4^4$, otherwise we can take $\delta = 0$ by choosing suitable parameter r^4. The last equation $(1 - x_3^3) p^3 = 0$

is satisfied by the choice $p^3 = 0$. Therefore, we obtain solution with $p^1 \neq 0$ giving the matrix $\hat{L}_0$ of the form $\begin{bmatrix} x_2^2 & 0 & 0 \\ \varepsilon & x_3^3 & 0 \\ 0 & \delta & x_4^4 \end{bmatrix}$, $\varepsilon, \delta \in \{0, 1\}$, as claimed in the assertions (i), (ii), (iii) and (iv).

Secondly, we treat the case that $p^1 = 0$ and $p^2 q^1 \neq 0$. The first and the second rows imply that $u^0 = \frac{1}{\hat{x}_2^2} = x_2^2$ and for $x_2^2 \neq 1$ one gets $q^2 = 0$. Comparing the last two diagonal elements of both sides of the matrix equation we obtain $\hat{x}_3^3 = \frac{x_3^3}{x_2^2}$, $\hat{x}_4^4 = \frac{x_4^4}{x_2^2}$. Finally, we receive the equations

$$\varepsilon = \hat{x}_2^3 = \frac{(1 - x_3^3)p^3}{x_2^2 p^2}, \ \delta = \hat{x}_3^4 = \frac{(x_3^3 - x_4^4)r^4 - x_3^4 p^2 q^1}{x_2^2 r^3}, \ (x_2^2 - x_3^3)q^3 + x_2^3 r^3 = 0,$$

where $r^3 \neq 0$. Choosing $p^3 = 0$ from the first equation we get $\varepsilon = 0$. It follows from the second equation that $\delta \neq 0$ for $x_3^4 \neq 0$ and $x_3^3 = x_4^4$, otherwise $\delta = 0$ can be obtained by an appropriate choice of r^4. If $x_2^3 \neq 0$ and $x_2^2 = x_3^3$, then the third equation gives a contradiction because of $r^3 \neq 0$. Therefore, the matrix equation is solvable with $p^1 = 0$ such that the matrix $\hat{L}_0$ has the form $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3}{x_2^2} & 0 \\ 0 & \delta & \frac{x_4^4}{x_2^2} \end{bmatrix}$, $\delta \in \{0, 1\}$ precisely if $x_2^3 = 0$ or $x_2^2 \neq x_3^3$, contained in the conditions of the assertions (i), (ii). This proves the statement. □

Definition 7. In the $C^{(1)}$-family the Malcev-like algebras $\mathfrak{c}^{\mathfrak{n}}$ of normal form are the algebras $\mathfrak{c}_1^{\mathfrak{n}}(\kappa, \lambda, \mu)$, $\mathfrak{c}_2^{\mathfrak{n}}(\kappa, \lambda)$, $\mathfrak{c}_3^{\mathfrak{n}}(\kappa, \mu)$, $\mathfrak{c}_4^{\mathfrak{n}}(\kappa)$ given by the matrices L_0:

$$\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 1 & \kappa \end{bmatrix},$$

respectively, where $\kappa \neq 0$, $\lambda \neq 0$. The multiplications of these algebras are

$\mathfrak{c}_1^{\mathfrak{n}}(\kappa, \lambda, \mu)$: $e_1 e_2 = e_4, e_0 e_1 = e_1, e_0 e_2 = \kappa e_2, e_0 e_3 = \lambda e_3, e_0 e_4 = \mu e_4, \ \kappa, \lambda \neq 0$,

$\mathfrak{c}_2^{\mathfrak{n}}(\kappa, \lambda)$: $e_1 e_2 = e_4, e_0 e_1 = e_1, e_0 e_2 = \kappa e_2, e_0 e_3 = \lambda e_3 + e_4, e_0 e_4 = \lambda e_4, \ \kappa, \lambda \neq 0$,

$\mathfrak{c}_3^{\mathfrak{n}}(\kappa, \mu)$: $e_1 e_2 = e_4, e_0 e_1 = e_1, e_0 e_2 = \kappa e_2 + e_3, e_0 e_3 = \kappa e_3, e_0 e_4 = \mu e_4, \ \kappa \neq 0$,

$\mathfrak{c}_4^{\mathfrak{n}}(\kappa)$: $e_1 e_2 = e_4, e_0 e_1 = e_1, e_0 e_2 = \kappa e_2 + e_3, e_0 e_3 = \kappa e_3 + e_4, e_0 e_4 = \kappa e_4, \ \kappa \neq 0$,

with respect to a special distinguished basis.

Proposition 8. *Any Malcev-like algebra* $\mathfrak{c}^{\mathfrak{n}}$ *in the* $C^{(1)}$*-family is isomorphic to an algebra of normal form given by* $\mathfrak{c}_1^{\mathfrak{n}}(\kappa, \lambda, \mu)$, $\mathfrak{c}_2^{\mathfrak{n}}(\kappa, \lambda)$, $\mathfrak{c}_3^{\mathfrak{n}}(\kappa, \mu)$ *and* $\mathfrak{c}_4^{\mathfrak{n}}(\kappa)$, $\kappa \neq 0$, $\lambda \neq 0$. *These algebras are uniquely determined in the isomorphism class up to the following isomorphisms:* $\mathfrak{c}_1^{\mathfrak{n}}(\kappa, \lambda, \mu) \cong \mathfrak{c}_1^{\mathfrak{n}}\left(\frac{1}{\kappa}, \frac{\lambda}{\kappa}, \frac{\mu}{\kappa}\right)$, $\mathfrak{c}_2^{\mathfrak{n}}(\kappa, \lambda) \cong \mathfrak{c}_2^{\mathfrak{n}}\left(\frac{1}{\kappa}, \frac{\lambda}{\kappa}\right)$, *where* $\kappa \neq 1$.

Proof. Taking into account Proposition 7 the Malcev-like algebras $\mathfrak{c}^{\mathfrak{n}}$ fulfill

- if $x_4^4 \neq x_3^3$ and $x_2^2 \neq x_3^3$, then $\varepsilon = \delta = 0$,
- if $x_4^4 = x_3^3$ and $x_2^2 \neq x_3^3$, then $\varepsilon = 0$, $\delta \in \{0,1\}$,
- if $x_4^4 \neq x_3^3$ and $x_2^2 = x_3^3$, then $\delta = 0$, $\varepsilon \in \{0,1\}$,
- if $x_2^2 = x_3^3 = x_4^4$, then $\delta \in \{0,1\}$, $\varepsilon \in \{0,1\}$.

Let $e_0e_2 = \kappa e_2$, $e_0e_3 = \lambda e_3$, $e_0e_4 = \mu e_4$ such that $\kappa\lambda \neq 0$. For any κ, λ, μ we have the possibility $\varepsilon = \delta = 0$, hence, we obtain $\mathfrak{c}_1^{\mathfrak{n}}(\kappa, \lambda, \mu)$. One has $\varepsilon = 0$, $\delta = 1$ only if $\lambda = \mu$ giving $\mathfrak{c}_2^{\mathfrak{n}}(\kappa, \lambda)$. The case $\varepsilon = 1$, $\delta = 0$ is possible only if $\kappa = \lambda$, hence, we obtain $\mathfrak{c}_3^{\mathfrak{n}}(\kappa, \mu)$. The case $\varepsilon = \delta = 1$ yields the multiplication $\mathfrak{c}_4^{\mathfrak{n}}(\kappa)$. The inequalities for κ, λ follow from the condition that $(\mathfrak{c}^{\mathfrak{n}})'$ is 4-dimensional. By Proposition 7 the second assertion is true. □

As a consequence of Proposition 7.5 and Lemma 7.6 in [10] we receive:

Corollary 9. *Among the Malcev-like algebras* $\mathfrak{c}^{\mathfrak{n}}$ *in the family* $C^{(1)}$ *the non-Malcev binary Lie algebras are* $\mathfrak{c}_1^{\mathfrak{n}}(1, \lambda, \mu)$, $\lambda \neq 0$, $\mu \notin \{-1, 2\}$, $\mathfrak{c}_2^{\mathfrak{n}}(1, \lambda)$, $\lambda \notin \{-1, 0, 2\}$, $\mathfrak{c}_3^{\mathfrak{n}}(1, \mu)$, $\mu \notin \{-1, 2\}$, $\mathfrak{c}_4^{\mathfrak{n}}(1)$, *the non-Lie Malcev algebras are* $\mathfrak{c}_1^{\mathfrak{n}}(1, \lambda, -1)$, $\lambda \neq 0$, $\mathfrak{c}_2^{\mathfrak{n}}(1, -1)$, $\mathfrak{c}_3^{\mathfrak{n}}(1, -1)$, *and the Lie algebras are* $\mathfrak{c}_1^{\mathfrak{n}}(\kappa, \lambda, \kappa+1)$, $\kappa\lambda \neq 0$, $\mathfrak{c}_2^{\mathfrak{n}}(\kappa, \kappa+1)$, $\kappa \notin \{-1, 0\}$, $\mathfrak{c}_3^{\mathfrak{n}}(\kappa, \kappa+1)$, $\kappa \neq 0$.

Now, we deal with Malcev-like algebras $\mathfrak{c}^{\mathfrak{f}}$ in the $C^{(1)}$-family. Using the matrix equation (9) for lower triangular matrices L_0 and $\hat{L}_0$ and comparing the diagonal elements we obtain

Corollary 10. *If the* $\mathfrak{c}^{\mathfrak{f}}$*-algebras* $\mathfrak{c}(L_0)$ *and* $\mathfrak{c}(\hat{L}_0)$ *in the* $C^{(1)}$*-family are isomorphic, then* L_0 *and* $\hat{L}_0$ *have the same diagonal elements.*

Definition 8. We say that a $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ in the $C^{(1)}$-family is diagonalizable if $\mathfrak{c}(L_0)$ is isomorphic to the $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(\hat{L}_0)$ with $\hat{L}_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ 0 & 0 & x_4^4 \end{bmatrix}$. The $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(\hat{L}_0)$ is called diagonal $\mathfrak{c}^{\mathfrak{f}}$-algebra.

Proposition 11. *A* $\mathfrak{c}^{\mathfrak{f}}$*-algebra* $\mathfrak{c}(L_0)$ *in the* $C^{(1)}$*-family is diagonalizable if and only if the system of equations*

$$(x_2^2 + x_4^4 - 2x_3^3)s = (x_3^4 - x_2^3)p^1, \; (x_3^3 - x_2^2)s^2 + (x_2^2 - x_4^4)t + x_2^4(p^1)^2 = 0 \tag{11}$$

is solvable for some $s, t \in \mathbb{R}$.

Proof. If a $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ with lower triangular matrix L_0 is isomorphic to a diagonal $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(\hat{L}_0)$, then L_0 and $\hat{L}_0$ have the same diagonal elements (see Corollary 10). Applying the diagonal matrix $\hat{L}_0$ of Definition 8 the matrix equation (9) yields $(x_2^2 - 1)p^2 = 0$ and

$$\begin{aligned}
&u^1p^2 + (x_3^3 - 1)p^3 - u^2p^1 = 0, \quad u^1q^2 + (x_3^3 - x_2^2)q^3 - x_2^3p^1q^2 = 0, \\
&u^1p^3 + (x_4^4 - 1)p^4 - u^3p^1 = 0, \quad (x_4^4 - x_3^3)q^3 + u^1q^2 = x_3^4p^1q^2, \\
&u^1q^3 + (x_4^4 - x_2^2)q^4 - p^1q^3x_2^3 - (p^1)^2q^2x_2^4 = 0.
\end{aligned}$$

One can express the parameters u^1, u^2, u^3 as follows

$$u^1 =(x_2^2 - x_3^3)\frac{q^3}{q^2} + x_2^3 p^1,\ u^2 = u^1\frac{p^2}{p^1} + (x_3^3 - 1)\frac{p^3}{p^1},\ u^3 = u^1\frac{p^3}{p^1} + (x_4^4 - 1)\frac{p^4}{p^1}.$$

Putting the expression of u^1 into the remaining equations we receive

$$\begin{aligned}&(x_2^2 + x_4^4 - 2x_3^3)q^3 = (x_3^4 - x_2^3)p^1 q^2 \quad \text{and}\\ &(x_2^2 - x_4^4)q^2 q^4 + (x_3^3 - x_2^2)(q^3)^2 = -x_2^4(q^2)^2(p^1)^2.\end{aligned} \tag{12}$$

Equations (12) are independent of p^2, hence, the equation $(x_2^2 - 1)p^2 = 0$ is satisfied with the choice $p^2 = 0$. With the notations $s = \frac{q^3}{q^2}$, $t = \frac{q^4}{q^2}$, the assertion follows from the equations (12). □

Proposition 12. *A* $\mathfrak{c}^{\mathfrak{f}}$*-algebra* $\mathfrak{c}(L_0)$ *in the* $C^{(1)}$*-family is non-diagonalizable if and only if the matrix* L_0 *has one of the following forms*

(a) $L_0 = \begin{bmatrix} x_2^2 & 0 & 0\\ \beta & \frac{x_2^2+x_4^4}{2} & 0\\ \delta & \gamma & x_4^4\end{bmatrix}$, $\beta \neq \gamma$,

(b) $L_0 = \begin{bmatrix} x_2^2 & 0 & 0\\ \beta & x_3^3 & 0\\ \delta & \gamma & x_2^2\end{bmatrix}$, *either* $x_2^2 \neq x_3^3, (\gamma - \beta)^2 \neq 4\delta(x_2^2 - x_3^3)$, *or* $x_2^2 = x_3^3, \beta = \gamma, \delta \neq 0$.

Proof. The first equation in (11) cannot be solved for s precisely if $x_2^2 + x_4^4 = 2x_3^3$ and $x_2^3 \neq x_3^4$, hence, we obtain condition (a). If $x_2^2 + x_4^4 \neq 2x_3^3$ or $x_2^3 = x_3^4$, then the first equation has a solution. In fact, for $x_2^2 + x_4^4 \neq 2x_3^3$ one gets $s = \frac{(x_3^4 - x_2^3)p^1}{x_2^2 + x_4^4 - 2x_3^3}$. Putting a solution s into the second equation we receive $(x_4^4 - x_2^2)t = (x_3^3 - x_2^2)s^2 + x_2^4(p^1)^2$. It cannot be solved for t if and only if $x_4^4 = x_2^2$ and $(x_3^3 - x_2^2)s^2 + x_2^4(p^1)^2 \neq 0$. Let $x_2^2 = x_4^4$. If $x_2^2 \neq x_3^3$, then the second equation is not solvable exactly if $(x_3^3 - x_2^2)s^2 + x_2^4(p^1)^2 = \left(-\frac{(x_3^4 - x_2^3)^2}{4(x_2^2 - x_3^3)} + x_2^4\right) \neq 0$. If $x_2^2 = x_3^3$, then the condition $x_2^2 + x_4^4 = 2x_3^3$ holds. Therefore, the first equation in (11) is solvable if and only if $x_2^3 = x_3^4$ and the second equation cannot be solved precisely if $x_2^4 \neq 0$. Using the notation $\beta = x_2^3$, $\gamma = x_3^4$, $\delta = x_2^4$ we obtain either the assumption $\frac{(\gamma - \beta)^2}{4(x_2^2 - x_3^3)} \neq \delta$ or the conditions $\beta = \gamma$, $\delta \neq 0$ in (b). Hence, the statement is proved. □

Now, we determine canonical forms of the matrices belonging to non-diagonalizable $\mathfrak{c}^{\mathfrak{f}}$-algebras given in Proposition 12.

Lemma 13. *The* $\mathfrak{c}^{\mathfrak{f}}$*-algebras* $\mathfrak{c}(L_0)$ *and* $\mathfrak{c}(\hat{L}_0)$*, determined by the matrices*

$$L_0 = \begin{bmatrix} x_2^2 & 0 & 0\\ \beta & \frac{x_2^2+x_4^4}{2} & 0\\ \delta & \gamma & x_4^4\end{bmatrix},\ \beta \neq \gamma \quad \textit{and} \quad \hat{L}_0 = \begin{bmatrix} x_2^2 & 0 & 0\\ 1 & \frac{x_2^2+x_4^4}{2} & 0\\ 0 & -1 & x_4^4\end{bmatrix},$$

are isomorphic non-diagonalizable $\mathfrak{c}^{\mathfrak{f}}$*-algebras in the family* $C^{(1)}$*.*

Proof. A partial isomorphism $\varphi : \mathfrak{c}^{\mathfrak{f}} \to \hat{\mathfrak{c}}^{\mathfrak{f}}$ is an isomorphism if and only if the matrix equation (9) is satisfied with the given matrices L_0 and $\hat{L}_0$ in the assertion.

We obtain the system of equations

$$\begin{aligned}
&\frac{q^3(x_2^2-x_4^4)}{2}+p^1q^2\beta=(u^1+1)q^2,\\
&\frac{p^1q^3(x_2^2-x_4^4)}{2}+(p^1)^2q^2\gamma=(u^1-1)p^1q^2,\\
&q^4(x_2^2-x_4^4)+p^1q^3\beta+(p^1)^2q^2\delta=(u^1-1)q^3,\\
&(x_2^2-1)p^2=0,\quad (u^1+1)p^2+\frac{(x_2^2+x_4^4-2)p^3}{2}-u^2p^1=0,\\
&(u^1-1)p^3+(x_4^4-1)p^4-u^3p^1=0.
\end{aligned}\tag{13}$$

With the choice $p^2=p^3=p^4=u^2=u^3=0$ the last three equations in (13) hold. The first three equations can be formulated equivalently as follows

$$\begin{aligned}
\frac{q^3(x_2^2-x_4^4)}{2q^2}+p^1\beta=u^1+1,\quad \frac{q^3(x_2^2-x_4^4)}{2q^2}+p^1\gamma=u^1-1,\\
\frac{q^4}{q^2}(x_2^2-x_4^4)+\frac{q^3}{q^2}(p^1\beta+1-u^1)+(p^1)^2\delta=0.
\end{aligned}\tag{14}$$

Putting u^1 from the first equation into the second one we obtain $p^1(\beta-\gamma)=2$, which gives $p^1=\frac{2}{\beta-\gamma}$. Substituting the obtained expressions of u^1 and p^1 into the third equation we receive

$$\frac{q^4}{q^2}(x_2^2-x_4^4)+\frac{q^3}{q^2}\left(2-\frac{q^3}{q^2}\left(\frac{x_2^2-x_4^4}{2}\right)\right)+\frac{4}{(\beta-\gamma)^2}\delta=0.\tag{15}$$

If $x_2^2=x_4^4$, then equation (15) is satisfied with the choice $q^3=-\frac{2\delta q^2}{(\beta-\gamma)^2}$. If $x_2^2\neq x_4^4$, then choosing

$$q^4=-\frac{2q^3}{x_2^2-x_4^4}+\frac{(q^3)^2}{2q^2}-\frac{4\delta q^2}{(\beta-\gamma)^2(x_2^2-x_4^4)}$$

the equation (15) holds. Applying the formulas obtained for u^1, p^1, q^3 or q^4 the equations in (14) are valid. Hence, we get a partial isomorphism satisfying the system (13), proving the assertion. □

Lemma 14. *The* $\mathfrak{c}^{\mathfrak{f}}$*-algebras* $\mathfrak{c}(L_0)$ *and* $\mathfrak{c}(\hat{L}_0)$*, determined by the matrices*

$$L_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ \beta & x_3^3 & 0 \\ \delta & \gamma & x_2^2 \end{bmatrix},\ x_2^2 \neq x_3^3,\ \frac{(\gamma-\beta)^2}{4(x_2^2-x_3^3)} \neq \delta,\ \textit{respectively,}$$

$$x_2^2 = x_3^3, \beta = \gamma, \delta \neq 0,$$

$$\hat{L}_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ \sigma(L_0) & 0 & x_2^2 \end{bmatrix},\ x_2^2 \neq x_3^3,\ \sigma(L_0) = \sigma\left((\delta - \frac{(\gamma-\beta)^2}{4(x_2^2-x_3^3)})\mathbb{K}^{\circ 2}\right),$$

$$\textit{respectively,} x_2^2 = x_3^3,\ \sigma(L_0) = \sigma\left(\delta\mathbb{K}^{\circ 2}\right),$$

where $x_2^2 \neq 0$ *and* $\sigma : \mathbb{K}^\circ/\mathbb{K}^{\circ 2} \to \mathbb{K}^\circ$ *is a section, are isomorphic non-diagonalizable* $\mathfrak{c}^{\mathfrak{f}}$*-algebras in the family* $C^{(1)}$*. Furthermore, the* $\mathfrak{c}^{\mathfrak{f}}$*-algebras determined by the matrices*

$$\begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ s & 0 & x_2^2 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 & 0 \\ t & 0 & x_2^2 \end{bmatrix},\ s, t \in \mathbb{K}^\circ$$

are isomorphic if and only if s *and* t *belong to the same coset of* $\mathbb{K}^{\circ 2}$ *in* $\mathbb{K}^\circ$*.*

Proof. A partial isomorphism $\varphi : \mathfrak{c}^{\mathfrak{f}} \to \hat{\mathfrak{c}}^{\mathfrak{f}}$ is an isomorphism if and only if it satisfies the equation (9) with the matrices L_0 and $\hat{L}_0$ given by Lemma 14. This yields the equations

$$\begin{aligned} &(x_2^2 - x_3^3)q^3 + p^1 q^2 \beta = u^1 q^2, \\ &(x_3^3 - x_2^2)p^1 q^3 + (p^1)^2 q^2 \gamma = u^1 p^1 q^2, \\ &p^1 q^3 \beta + (p^1)^2 q^2 \delta - \sigma(L_0) q^2 = u^1 q^3, \\ &(x_2^2 - 1)p^2 = 0, \quad p^3(1 - x_3^3) + u^2 p^1 - u^1 p^2 = 0, \\ &p^4(1 - x_2^2) - \sigma(L_0) p^2 + u^3 p^1 - u^1 p^3 = 0. \end{aligned} \tag{16}$$

Choosing $p^2 = p^3 = p^4 = u^2 = u^3 = 0$ the last three equations in (16) are satisfied. The first three equations are equivalent with the following system of equations

$$\begin{aligned} &(x_2^2 - x_3^3)\frac{q^3}{q^2} + p^1\beta = u^1, \quad \frac{q^3}{q^2}(x_3^3 - x_2^2) + p^1\gamma = u^1, \\ &\frac{q^3}{q^2}(p^1\beta - u^1) + (p^1)^2\delta = \sigma(L_0). \end{aligned} \tag{17}$$

Substituting the expression of u^1 from the first equation into the second one we obtain $2\frac{q^3}{q^2}(x_3^3 - x_2^2) + p^1(\gamma - \beta) = 0$. If $x_3^3 \neq x_2^2$, then this equation gives that $q^3 = \frac{(\beta-\gamma)p^1 q^2}{2(x_3^3 - x_2^2)}$. Putting $(p^1)^2 = \frac{4\sigma(L_0)(x_2^2 - x_3^3)}{4\delta(x_2^2 - x_3^3) - (\gamma-\beta)^2}$ the third equation of

(17) is solvable if and only if

$$\sigma(L_0) = \sigma\left((\delta - \frac{(\gamma-\beta)^2}{4(x_2^2 - x_3^3)})\mathbb{K}^{\circ 2}\right) \in \left(\delta - \frac{(\gamma-\beta)^2}{4(x_2^2 - x_3^3)}\right)\mathbb{K}^{\circ 2},$$

i.e. $\delta - \frac{(\gamma-\beta)^2}{4(x_2^2-x_3^3)}$ and $\sigma(L_0) = \sigma\left((\delta - \frac{(\gamma-\beta)^2}{4(x_2^2-x_3^3)})\mathbb{K}^{\circ 2}\right)$ belong to the same coset of $\mathbb{K}^{\circ 2}$ in $\mathbb{K}^{\circ}$. If $x_2^2 = x_3^3$, then it follows from the first and the second equations in (17) that $\beta = \gamma$. Replacing $(p^1)^2 = \frac{\sigma(L_0)}{\delta}$ the third equation has a solution precisely if

$$\sigma(L_0) = \sigma\left(\delta(\mathbb{K}^{\circ})^2\right) \in \delta(\mathbb{K}^{\circ})^2,$$

i.e. δ and $\sigma(L_0)$ correspond to the same coset of $\mathbb{K}^{\circ 2}$ in $\mathbb{K}^{\circ}$.
It follows that the second claim is true in both cases. Hence, the assertions are proved. □

Definition 9. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{f}}$ of normal form in the $C^{(1)}$-family are the algebras $\mathfrak{c}_1^{\mathfrak{f}}(\kappa,\lambda,\mu)$, $\mathfrak{c}_2^{\mathfrak{f}}(\kappa,\mu)$, $\mathfrak{c}_3^{\mathfrak{f}}(\kappa,\lambda,\tau)$, given by the matrices L_0:

$$\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 1 & \frac{\kappa+\mu}{2} & 0 \\ 0 & -1 & \mu \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ \tau & 0 & \kappa \end{bmatrix},$$

and having the multiplications

$\mathfrak{c}_1^{\mathfrak{f}}(\kappa,\lambda,\mu)$: $e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2, e_0e_3 = \lambda e_3, e_0e_4 = \mu e_4,$
$\mathfrak{c}_2^{\mathfrak{f}}(\kappa,\mu)$: $e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + e_3, e_0e_3 = \frac{\kappa+\mu}{2}e_3 - e_4, e_0e_4 = \mu e_4,$
$\mathfrak{c}_3^{\mathfrak{f}}(\kappa,\lambda,\tau)$: $e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + \tau e_4, e_0e_3 = \lambda e_3, e_0e_4 = \kappa e_4,$

where $\kappa \neq 0$, $\tau \neq 0$, with respect to a special distinguished basis.

It follows from Lemmas 13, 14:

Proposition 15. *Any Malcev-like algebra $\mathfrak{c}^{\mathfrak{f}}$ in the $C^{(1)}$-family is isomorphic to a $\mathfrak{c}^{\mathfrak{f}}$-algebra of normal form. The normal forms $\mathfrak{c}_1^{\mathfrak{f}}(\kappa,\lambda,\mu)$, $\mathfrak{c}_2^{\mathfrak{f}}(\kappa,\mu)$ are uniquely determined in their isomorphism class. The algebras $\mathfrak{c}_3^{\mathfrak{f}}(\kappa,\lambda,\tau_1)$, $\mathfrak{c}_3^{\mathfrak{f}}(\kappa,\lambda,\tau_2)$, are isomorphic if and only if τ_1 and τ_2 belong to the same coset of $\mathbb{K}^{\circ 2}$ in $\mathbb{K}^{\circ}$.*

It follows from Proposition 9.6 in [10]:

Corollary 16. *Among the Malcev-like algebras $\mathfrak{c}^{\mathfrak{f}}$ in the family $C^{(1)}$ the non-Lie binary Lie algebras are $\mathfrak{c}_1^{\mathfrak{f}}(1,\lambda,\lambda+1)$, $\lambda \neq 2$, $\mathfrak{c}_2^{\mathfrak{f}}(1,3)$, $\mathfrak{c}_3^{\mathfrak{f}}(1,0,\tau)$, and the Lie algebras are $\mathfrak{c}_1^{\mathfrak{f}}(\lambda-1,\lambda,\lambda+1)$, $\lambda \neq 1$.*

4.2. $C^{(2)}$-Family

Definition 10. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ determined by $L_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}$, where $x_2^2 \neq 0$ and $x_4^3 \neq 0$, belong to the family $C^{(2)}$.

Proposition 17. *Any $\mathfrak{c}^{\mathfrak{n}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(2)}$ is isomorphic to the algebra $\mathfrak{c}(\hat{L}_0)$ with* $\hat{L}_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 + x_4^4 & 1 \\ 0 & x_4^3 x_3^4 - x_3^3 x_4^4 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 \\ 0 & \frac{x_3^3 + x_4^4}{x_2^2} & 1 \\ 0 & \frac{x_4^3 x_3^4 - x_3^3 x_4^4}{(x_2^2)^2} & 0 \end{bmatrix}$.

Proof. A partial isomorphism $\varphi : \mathfrak{c}^{\mathfrak{n}} \to \hat{\mathfrak{c}}^{\mathfrak{n}}$ is an isomorphism if and only if the matrix equation (8) is satisfied with the given matrices L_0 and $\hat{L}_0$ in Definition 10 and Proposition 17. Firstly, we assume that $p^1 \neq 0$, hence $u^0 = 1$. The second diagonal elements imply $\hat{x}_2^2 = x_2^2$. It follows from the equation $\begin{bmatrix} p^1 & q^1 x_2^2 \\ p^2 & q^2 x_2^2 \end{bmatrix} = \begin{bmatrix} p^1 & q^1 \\ x_2^2 p^2 & x_2^2 q^2 \end{bmatrix}$ that for $x_2^2 \neq 1$ one has $p^2 = q^1 = 0$.
Secondly, we consider the case if $p^1 = 0$ and $p^2 q^1 \neq 0$. We obtain from the first two rows that $u^0 = \hat{x}_2^2 = \frac{1}{x_2^2}$ and for $x_2^2 \neq 1$ one has $q^2 = 0$.
In both cases the last column gives the expressions $r^3 = \frac{1}{x_4^3} u^0 (p^1 q^2 - p^2 q^1)$ and $r^4 = -\frac{x_4^4}{x_4^3}(p^1 q^2 - p^2 q^1)$. With these replacements we get from the third column

$$\hat{x}_3^3 = \frac{x_3^3 + x_4^4}{u^0} \quad \text{and} \quad \hat{x}_3^4 = \frac{x_4^3 x_3^4 - x_3^3 x_4^4}{(u^0)^2}. \tag{18}$$

Putting $p^3 = p^4 = q^3 = 0$ and $q^4 = \frac{r^3 x_2^3}{u^0} = \frac{x_2^3}{x_4^3}(p^1 q^2 - p^2 q^1)$ into the first two entries of the last two rows we receive the following system of equations

$$u^2 \begin{bmatrix} p^1 \\ q^1 \end{bmatrix} - u^1 \begin{bmatrix} p^2 \\ q^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \left((x_4^4 - x_2^2) \frac{x_2^3}{x_4^3} - x_2^4 \right) (p^1 q^2 - p^2 q^1) \end{bmatrix}. \tag{19}$$

We get a solution of (19) with a suitable choice of u^1 and u^2. Hence, there exist solutions of (8) and for any solution (18) is valid. This proves the claim. □

Proposition 18. *Any $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(2)}$ is isomorphic to the algebra $\mathfrak{c}(\hat{L}_0)$, where* $\hat{L}_0 = \begin{bmatrix} x_2^2 & 0 & 0 \\ 0 & x_3^3 + x_4^4 & 1 \\ \hat{x}_2^4 & 0 & 0 \end{bmatrix}$ *with* $\hat{x}_2^4 = x_2^2 x_3^3 x_4^4 + (x_4^3)^2 x_2^4 + (x_2^2 - x_4^4)(x_4^4(x_3^3 - x_2^2) + x_4^3(x_2^3 - x_3^4))$.

Proof. We take into account the matrix equation (9) describing the isomorphism condition of the partial isomorphism $\varphi : \mathfrak{c}^{\mathfrak{f}} \to \hat{\mathfrak{c}}^{\mathfrak{f}}$ using the forms of L_0 and $\hat{L}_0$ in the Definition 10 and Proposition 18. The fourth column gives that

$p^1 = x^3_4$ and the equation $(x^3_4)^2(q^3 + x^4_4 q^2) = 0$ is satisfied. This implies that $q^3 = -x^4_4 q^2$. Putting the values of p^1, q^3 into the third column we receive $\hat{x}^3_3 = x^3_3 + x^4_4$ and $u^1 = x^3_4 x^4_3 - x^3_3 x^4_4$. We get from the second column $\hat{x}^2_2 = x^2_2$, $q^4 = \left(x^4_4(2x^3_3 + x^4_4 - x^2_2) + x^3_4(x^3_2 - x^4_3)\right) q^2$ with the replacements of p^1, q^3, $\hat{x}^3_3$, u^1, and, hence, $\hat{x}^4_2 = x^2_2 x^3_3 x^4_4 + (x^3_4)^2 x^4_2 + (x^2_2 - x^4_4)\left(x^4_4(x^3_3 - x^2_2) + x^3_4(x^3_2 - x^4_3)\right)$. Finally, in the first column we assume $p^2 = 0$, and, therefore, we obtain the equation

$$(u^2 + u^3)p^1 = (\hat{x}^3_3 + u^1 - 1)p^3,$$

which can be solved by choosing a suitable value of u^2, u^3, p^3. This proves the claim. □

Definition 11. The Malcev-like algebras of normal form in the $C^{(2)}$-family are the algebras $\mathfrak{c}^{\mathfrak{n}}_5(\kappa, \lambda, \nu)$, $\mathfrak{c}^{\mathfrak{f}}_4(\kappa, \lambda, \tau)$ defined by the matrices L_0:

$$\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & \nu & 0 \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 1 \\ \tau & 0 & 0 \end{bmatrix},$$

respectively, the multiplications of which are

$\mathfrak{c}^{\mathfrak{n}}_5(\kappa, \lambda, \nu)$: $e_1e_2 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2, e_0e_3 = \lambda e_3 + \nu e_4, e_0e_4 = e_3$,
$\mathfrak{c}^{\mathfrak{f}}_4(\kappa, \lambda, \tau)$: $e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + \tau e_4, e_0e_3 = \lambda e_3, e_0e_4 = e_3$,

where $\kappa \neq 0$, with respect to a special distinguished basis.

From Propositions 17 and 18 we receive:

Corollary 19. *Any Malcev-like algebra $\mathfrak{c}^{\mathfrak{n}}$ in the $C^{(2)}$-family is isomorphic to an algebra of normal form given by $\mathfrak{c}^{\mathfrak{n}}_5(\kappa, \lambda, \nu)$, $\kappa \neq 0$. These algebras are uniquely determined in their isomorphism class up to the following isomorphism:*

$$\mathfrak{c}^{\mathfrak{n}}_5(\kappa, \lambda, \nu) \;\cong\; \mathfrak{c}^{\mathfrak{n}}_5\left(\frac{1}{\kappa}, \frac{\lambda}{\kappa}, \frac{\nu}{\kappa^2}\right), \; \kappa \neq 1.$$

Any Malcev-like algebra $\mathfrak{c}^{\mathfrak{f}}$ in the $C^{(2)}$-family is isomorphic to a unique algebra of normal form given by $\mathfrak{c}^{\mathfrak{f}}_4(\kappa, \lambda, \tau)$, $\kappa \neq 0$.

It follows from Lemma 7.9 and Corollaries 7.8, 8.2 in [10]:

Corollary 20. *Among the Malcev-like algebras $\mathfrak{c}^{\mathfrak{n}}$ in the family $C^{(2)}$ the non-Malcev binary Lie algebras are $\mathfrak{c}^{\mathfrak{n}}_5(1, \lambda, \nu)$, the Malcev algebra is the algebra $\mathfrak{c}^{\mathfrak{n}}_5(1, 1, 2)$. Among the Malcev-like algebras $\mathfrak{c}^{\mathfrak{f}}$ in the family $C^{(2)}$ there does not exist any binary Lie algebra.*

4.3. $C^{(3)}$-Family

Definition 12. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t}$=$\{\mathfrak{n},\mathfrak{f}\}$ with L_0= $\begin{bmatrix} x_2^2 & x_3^2 & 0 \\ x_2^3 & x_3^3 & 0 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}$ belong to the family $C^{(3)}$, if $x_3^2 \neq 0$, $x_4^3 = 0$ for $\mathfrak{t} = \{\mathfrak{n},\mathfrak{f}\}$, and additionally $x_2^2 x_3^3 - x_3^2 x_2^3 \neq 0$ for $\mathfrak{t} = \mathfrak{n}$.

Proposition 21. *Any $\mathfrak{c}^{\mathfrak{n}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(3)}$ is isomorphic to the algebra $\mathfrak{c}(\hat{L}_0)$, where* $\hat{L}_0 = \begin{bmatrix} x_2^2 + x_3^3 & 1 & 0 \\ x_3^2 x_2^3 - x_2^2 x_3^3 & 0 & 0 \\ 0 & 0 & x_4^4 \end{bmatrix}$.

Proof. We consider the matrix equation (8) giving the isomorphism condition for a partial isomorphism between two $\mathfrak{c}^{\mathfrak{n}}$-algebras $\mathfrak{c}(L_0)$, $\mathfrak{c}(\hat{L}_0)$ having the given forms L_0 and $\hat{L}_0$ as in Definition 12 and Proposition 21. The first row and the last two columns give

$$q^1 = 0, \quad u^0 = 1, \quad r^3 = q^2 x_3^2, \quad q^3 = -x_3^3 q^2, \quad \hat{x}_4^4 = x_4^4.$$

Substituting the values of q^3, r^3 into the second column we obtain $\hat{x}_2^2 = x_2^2 + x_3^3$ and $\hat{x}_2^3 = x_3^2 x_2^3 - x_2^2 x_3^3$. At the end, we receive the equation

$$q^4 x_2^2 + r^4 x_2^3 + p^1 q^2 x_2^4 = x_4^4 q^4 + u^1 q^2.$$

Since $q^1 = 0$ we have $p^1 q^2 \neq 0$. Therefore, the above equation can be solved by choosing a suitable value of u^1. Choosing $p^2 = p^3 = p^4 = u^2 = 0$ all equations are satisfied. This proves the assertion. □

Proposition 22. *Any $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(3)}$ is isomorphic to the algebra $\mathfrak{c}(\hat{L}_0)$ with $\hat{L}_0$ having the form* $\begin{bmatrix} x_2^2 + x_3^3 & 1 & 0 \\ 0 & 0 & 0 \\ \hat{x}_2^4 & 0 & x_4^4 \end{bmatrix}$, *where* $\hat{x}_2^4 = x_3^3 \left(x_4^4 (x_4^4 - x_3^3) - (x_2^2)^2 \right) + x_3^2 \left((x_2^2 - x_4^4)(x_2^3 - x_3^4) + x_3^2 x_2^4 \right)$.

Proof. We consider the matrix equation (9) giving the isomorphism condition for a partial isomorphism between two $\mathfrak{c}^{\mathfrak{f}}$-algebras $\mathfrak{c}(L_0)$, $\mathfrak{c}(\hat{L}_0)$ having the given forms L_0 and $\hat{L}_0$ as in Definition 12 and Proposition 22.
The second and third entries of the third column imply that $p^1 = x_3^2$, and, hence, $q^3 = -x_3^3 q^2$. Replacing these expressions into the second and third entries of the second column we get $\hat{x}_2^2 = x_2^2 + x_3^3$ and $u^1 = x_3^2 x_2^3 - x_2^2 x_3^3$. The last row yields $\hat{x}_4^4 = x_4^4$, $q^4 = \left(x_3^3 (x_3^3 - x_2^2 - x_4^4) + x_3^2 (x_2^3 - x_3^4) \right) q^2$ with the substitutions of p^1, q^3, $\hat{x}_4^4$, u^1, and $\hat{x}_2^4 = x_3^3 \left(x_4^4 (x_4^4 - x_3^3) - (x_2^2)^2 \right) + x_3^2 \left((x_2^2 - x_4^4)(x_2^3 - x_3^4) + x_3^2 x_2^4 \right)$ with the additional replacement of q^4. With the assumption $p^2 = 0$ it follows from the first column that $p^3 = u^2 = 0$. Furthermore, it remains the equation $(x_4^4 - 1) p^4 = u^3 x_3^2$, which can be solved by choosing a suitable value of u^3. This proves the assertion. □

Definition 13. The Malcev-like algebras of normal form in the $C^{(3)}$-family are the algebras $\mathfrak{c}_6^{\mathfrak{n}}(\kappa,\sigma,\mu)$, $\mathfrak{c}_5^{\mathfrak{f}}(\kappa,\tau,\mu)$ determined by the matrix L_0:

$$\begin{bmatrix} \kappa & 1 & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix},\ \sigma \neq 0,\ \begin{bmatrix} \kappa & 1 & 0 \\ 0 & 0 & 0 \\ \tau & 0 & \mu \end{bmatrix},$$

respectively, and having the multiplications

$\mathfrak{c}_6^{\mathfrak{n}}(\kappa,\sigma,\mu)$: $e_1e_2 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2, e_0e_3 = \lambda e_3 + e_4, e_0e_4 = \lambda e_4,\ \kappa, \lambda \neq 0,$

$\mathfrak{c}_5^{\mathfrak{f}}(\kappa,\tau,\mu)$: $e_1e_2 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + e_3, e_0e_3 = \kappa e_3, e_0e_4 = \mu e_4,\ \kappa \neq 0,$

with respect to a special distinguished basis.

From Propositions 21 and 22 it follows:

Corollary 23. *Any Malcev-like algebra $\mathfrak{c}^{\mathfrak{n}}$, respectively $\mathfrak{c}^{\mathfrak{f}}$ in the $C^{(3)}$-family is isomorphic to a unique algebra of normal form given by $\mathfrak{c}_6^{\mathfrak{n}}(\kappa,\sigma,\mu)$, $\sigma \neq 0$, respectively $\mathfrak{c}_5^{\mathfrak{f}}(\kappa,\tau,\mu)$.*

It follows from Lemma 7.12 and Corollaries 7.11, 8.2 in [10]:

Corollary 24. *Among the Malcev-like algebras $\mathfrak{c}^{\mathfrak{n}}$ in the $C^{(3)}$-family the non-Malcev binary Lie algebras are $\mathfrak{c}_6^{\mathfrak{n}}(\kappa, 1-\kappa, \kappa), \kappa \notin \{-1, 1\}$, the Malcev algebra is $\mathfrak{c}_6^{\mathfrak{n}}(-1, 2, -1)$. Among the Malcev-like algebras $\mathfrak{c}^{\mathfrak{f}}$ in the $C^{(3)}$-family there is no binary Lie algebra.*

4.4. $C^{(4)}$-Family

Definition 14. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ determined by $L_0 = \begin{bmatrix} x_2^2 & x_3^2 & 0 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}$, where $x_3^2 \neq 0$, $x_4^3 \neq 0$, belong to the family $C^{(4)}$.

Proposition 25. *Any $\mathfrak{c}^{\mathfrak{n}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(4)}$ is isomorphic to the algebra $\mathfrak{c}(\hat{L}_0)$ with* $\hat{L}_0 = \begin{bmatrix} x_2^2 + x_3^3 + x_4^4 & 1 & 0 \\ x_3^2x_2^3 + x_4^3x_3^4 - x_2^2x_3^3 - x_2^2x_4^4 - x_3^3x_4^4 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Proof. We consider the matrix equation (8) giving the isomorphism condition for a partial isomorphism between two $\mathfrak{c}^{\mathfrak{n}}$-algebras $\mathfrak{c}(L_0)$, $\mathfrak{c}(\hat{L}_0)$ having the given forms L_0 and $\hat{L}_0$ as in Definition 14 and Proposition 25. The first row and the last two columns give $q^1 = 0$, $u^0 = 1$, $r^3 = q^2x_3^2$, $p^1 = x_3^2x_4^3$, $r^4 = -x_3^2x_4^4q^2$, $q^3 = -(x_3^3 + x_4^4)q^2$, $q^4 = (x_3^3x_4^4 - x_4^3x_3^4)q^2$. Putting the expressions r^3, q^3 and q^4 into the second column we obtain

$$\hat{x}_2^2 = x_2^2 + x_3^3 + x_4^4, \quad \hat{x}_2^3 = x_3^2x_2^3 + x_4^3x_3^4 - x_2^2x_3^3 - x_2^2x_4^4 - x_3^3x_4^4.$$

Finally, we receive the equation $q^4x_2^2 + r^4x_2^3 + p^1q^2x_2^4 = u^1q^2$. Since $q^2 \neq 0$, this equation can be solved by choosing a suitable value of u^1. Choosing $p^2 = p^3 = p^4 = u^2 = 0$ all equations are satisfied, and the assertion follows. □

Proposition 26. *Any $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(4)}$ is isomorphic to one of the following algebras $\mathfrak{c}(\hat{L}_0)$ such that*

(i) $\hat{L}_0 = \begin{bmatrix} x_2^2 + \frac{x_3^2 x_4^4}{x_4^3} & \frac{x_3^2}{x_4^3} & 0 \\ 0 & x_3^3 + x_4^4 - \frac{x_3^2 x_4^4}{x_4^3} & 1 \\ \hat{x}_2^4 & 0 & 0 \end{bmatrix}$, $\hat{x}_2^4 = x_4^3 x_4^4 (x_3^4 - x_2^3) + (x_4^3)^2 x_2^4 + \left(x_2^2 + \frac{x_3^2 x_4^4}{x_4^3}\right)\left(x_4^4\left(2x_3^3 + x_4^4 - x_2^2 - \frac{x_3^2 x_4^4}{x_4^3}\right) + x_4^3(x_2^3 - x_3^4)\right)\left(\frac{x_4^3}{x_3^2 + x_4^3}\right) - x_3^3 (x_4^4)^2$, *for* $x_3^2 \neq -x_4^3$,

(ii) $\hat{L}_0 = \begin{bmatrix} x_2^2 - x_4^4 & -1 & 0 \\ \hat{x}_2^3 & x_3^3 + 2x_4^4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\hat{x}_2^3 = x_4^4(2(x_3^3 + x_4^4) - x_2^2) + x_4^3(x_2^3 - x_3^4)$, *for* $x_3^2 = -x_4^3$, $x_2^2 \neq x_4^4$,

(iii) $\hat{L}_0 = \begin{bmatrix} 0 & -1 & 0 \\ \hat{x}_2^3 & 2x_2^2 + x_3^3 & 1 \\ \hat{x}_2^4 & 0 & 0 \end{bmatrix}$, $\hat{x}_2^3 = x_2^2(x_2^2 + 2x_3^3) + x_4^3(x_2^3 - x_3^4)$, $\hat{x}_2^4 = x_2^2 x_4^3 (x_3^4 - x_2^3) + (x_4^3)^2 x_2^4 - (x_2^2)^2 x_3^3$, *for* $x_3^2 = -x_4^3$ *and* $x_2^2 = x_4^4$.

Proof. We consider the matrix equation (9) giving the isomorphism condition for a partial isomorphism described by the matrix A of Lemma 3 (b) between two $\mathfrak{c}^{\mathfrak{f}}$-algebras $\mathfrak{c}(L_0)$, $\mathfrak{c}(\hat{L}_0)$ having the given forms L_0 and $\hat{L}_0$ as in Definition 14 and Proposition 26. We obtain from the last column that $p^1 = x_4^3$ and $q^3 = -x_4^4 q^2$. Substituting these expressions into the third column we have $\hat{x}_3^2 = \frac{x_3^2}{x_4^3}$, $\hat{x}_3^3 = x_3^3 + x_4^4 - \frac{x_3^2 x_4^4}{x_4^3}$ and $u^1 = \frac{x_3^2 q^4}{x_4^3 q^2} - x_3^3 x_4^4 + x_4^3 x_3^4$. With the replacements of $\hat{x}_3^2$, q^3, p^1, $\hat{x}_3^3$, u^1 the second column gives

$$\begin{aligned} \hat{x}_2^2 &= x_2^2 + \frac{x_3^2 x_4^4}{x_4^3}, \\ \hat{x}_2^3 &= -\left(1 + \frac{x_3^2}{x_4^3}\right)\frac{q^4}{q^2} + x_4^4\left(2x_3^3 + x_4^4 - x_2^2 - \frac{x_3^2 x_4^4}{x_4^3}\right) + x_4^3(x_2^3 - x_3^4), \\ \hat{x}_2^4 &= \left(x_2^2 + \frac{x_3^2 x_4^4}{x_4^3}\right)\frac{q^4}{q^2} + x_4^3 x_4^4 (x_3^4 - x_2^3) + (x_4^3)^2 x_2^4 - x_3^3 (x_4^4)^2. \end{aligned}$$

If $x_3^2 \neq -x_4^3$, then $q^4 = \left(x_4^4\left(2x_3^3 + x_4^4 - x_2^2 - \frac{x_3^2 x_4^4}{x_4^3}\right) + x_4^3(x_2^3 - x_3^4)\right)\left(\frac{x_4^3 q^2}{x_3^2 + x_4^3}\right)$ yields that $\hat{x}_2^3 = 0$ and $\hat{x}_2^4$ has the form as in the assertion (i).
If $x_3^2 = -x_4^3$, then one receives $\hat{x}_3^2 = -1$ and $u^1 = -\frac{q^4}{q^2} - x_3^3 x_4^4 + x_4^3 x_3^4$. We have two cases:

- if $x_2^2 \neq x_4^4$, then $\hat{x}_2^2 = x_2^2 - x_4^4$, $\hat{x}_3^3 = x_3^3 + 2x_4^4$, $\hat{x}_2^3 = x_4^3(x_2^3 - x_3^4) + x_4^4(2(x_3^3 + x_4^4) - x_2^2)$,
  $\hat{x}_2^4 = 0$ with the choice $q^4 = \frac{x_4^3 x_4^4 (x_3^4 - x_2^3) + (x_4^3)^2 x_2^4 - x_3^3 (x_4^4)^2}{x_4^4 - x_2^2} q^2$, giving the assertion (ii),

- if $x_2^2 = x_4^4$, then $\hat{x}_2^2 = 0$, $\hat{x}_3^3 = 2x_2^2 + x_3^3$, $\hat{x}_2^3 = x_2^2(x_2^2 + 2x_3^3) + x_4^3(x_2^3 - x_3^4)$, and
  $\hat{x}_2^4 = x_2^2 x_4^3(x_3^4 - x_2^3) + (x_4^3)^2 x_2^4 - (x_2^2)^2 x_3^3$ as in the assertion (iii).

If we assume that $p^2 = p^3 = p^4 = u^2 = u^3 = 0$, then the remaining equations in the first column are satisfied. This proves the assertion. □

Definition 15. The Malcev-like algebras of normal form in the $C^{(4)}$-family are the algebras $\mathfrak{c}_7^{\mathfrak{n}}(\kappa, \sigma)$, respectively $\mathfrak{c}_6^{\mathfrak{f}}(\kappa, \eta, \lambda, \tau)$, $\mathfrak{c}_7^{\mathfrak{f}}(\kappa, \sigma, \lambda)$, $\mathfrak{c}_8^{\mathfrak{f}}(\sigma, \lambda, \tau)$ determined by the matrices L_0:

$$\begin{bmatrix} \kappa & 1 & 0 \\ \sigma & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ respectively } \begin{bmatrix} \kappa & \eta & 0 \\ 0 & \lambda & 1 \\ \tau & 0 & 0 \end{bmatrix}, \begin{bmatrix} \kappa & -1 & 0 \\ \sigma & \lambda & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ \sigma & \lambda & 1 \\ \tau & 0 & 0 \end{bmatrix},$$

and having the multiplications

$\mathfrak{c}_7^{\mathfrak{n}}(\kappa, \sigma):\ e_1e_2 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + \sigma e_3, e_0e_3 = e_2, e_0e_4 = e_3,$
$\mathfrak{c}_6^{\mathfrak{f}}(\kappa, \eta, \lambda, \tau):\ e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + \tau e_4, e_0e_3 = \eta e_2 + \lambda e_3, e_0e_4 = e_3,$
$\mathfrak{c}_7^{\mathfrak{f}}(\kappa, \sigma, \lambda):\ e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + \sigma e_3, e_0e_3 = -e_2 + \lambda e_3, e_0e_4 = e_3,$
$\mathfrak{c}_8^{\mathfrak{f}}(\sigma, \lambda, \tau):\ e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \sigma e_3 + \tau e_4, e_0e_3 = -e_2 + \lambda e_3, e_0e_4 = e_3,$

with respect to a special distinguished basis.

From Propositions 25 and 26 it follows:

Corollary 27. *Any Malcev-like algebra $\mathfrak{c}^{\mathfrak{n}}$, respectively $\mathfrak{c}^{\mathfrak{f}}$ in the $C^{(4)}$-family is isomorphic to a unique algebra of normal form given by $\mathfrak{c}_7^{\mathfrak{n}}(\kappa, \sigma)$, respectively $\mathfrak{c}_6^{\mathfrak{f}}(\kappa, \eta, \lambda, \tau)$, $\mathfrak{c}_7^{\mathfrak{f}}(\kappa, \sigma, \lambda)$ and $\mathfrak{c}_8^{\mathfrak{f}}(\sigma, \lambda, \tau)$.*

From the Propositions 7.1 and 8.1 in [10] it follows:

Corollary 28. *In the family $C^{(4)}$ there does not exist any binary Lie algebra.*

4.5. $C^{(5)}$-Family

Definition 16. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ with $L_0 = \begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \\ x_2^4 & x_3^4 & x_4^4 \end{bmatrix}$ belong to the family $C^{(5)}$ if $x_4^2 \neq 0$ for $\mathfrak{t} = \{\mathfrak{n}, \mathfrak{f}\}$, and in addition rank $\begin{bmatrix} x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{bmatrix} = 2$ for $\mathfrak{t} = \mathfrak{n}$.

Proposition 29. *Any $\mathfrak{c}^{\mathfrak{n}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(5)}$ is isomorphic to the algebra $\mathfrak{c}(\hat{L}_0)$ determined by* $\hat{L}_0 = \begin{bmatrix} \hat{x}_2^2 & 0 & 1 \\ \varepsilon & \hat{x}_3^3 & 0 \\ 0 & \hat{x}_3^4 & 0 \end{bmatrix}$, $\hat{x}_2^2 = x_2^2 + x_4^4 + \frac{x_3^2 x_4^3}{x_4^2}$, $\hat{x}_3^3 = x_3^3 - \frac{x_3^2 x_4^3}{x_4^2}$, $\varepsilon \in \{0,1\}$, $(\varepsilon, \hat{x}_3^3) \neq (0,0)$. *If* $\varepsilon = 0$, *then we have* $\hat{x}_3^4 = \delta$, $\delta \in \{0,1\}$.

Proof. We consider the matrix equation (8) giving the isomorphism condition for a partial isomorphism between two $\mathfrak{c}^{\mathfrak{n}}$-algebras $\mathfrak{c}(L_0)$, $\mathfrak{c}(\hat{L}_0)$ having the given forms L_0 and $\hat{L}_0$ as in Definition 16 and Proposition 29. The first row and the last two columns give

$$q^1 = 0, \quad u^0 = 1, \quad r^4 = q^2 x_3^2, \quad p^1 = x_4^2, \quad q^3 = -\frac{x_4^3}{x_4^2} r^3, \quad q^4 = -\left(x_4^4 + \frac{x_3^2 x_4^3}{x_4^2} \right) q^2.$$

Replacing the values of q^i, $i = 3, 4$ and r^4, into the diagonal elements we obtain

$$\hat{x}_2^2 = x_2^2 + x_4^4 + \frac{x_3^2 x_4^3}{x_4^2} \quad \text{and} \quad \hat{x}_3^3 = x_3^3 - \frac{x_3^2 x_4^3}{x_4^2}.$$

The $(3,2)$- and the $(4,3)$-entries yield $\hat{x}_2^3 = \frac{X}{(x_4^2)^2} \frac{r^3}{q^2}$ and $\hat{x}_3^4 = \frac{Y}{x_4^2} \frac{q^2}{r^3}$, where $X = x_4^2(x_4^2 x_2^3 - x_2^2 x_4^3) - x_4^3(x_3^2 x_4^3 - x_4^2 x_3^3)$, $Y = x_4^2(x_4^2 x_3^4 - x_3^2 x_4^4) - x_3^2(x_3^2 x_4^3 - x_4^2 x_3^3)$. We receive

$$\hat{x}_2^3 = \varepsilon = \begin{cases} 0, & \text{if and only if } X = 0, \\ 1, & \text{if } X \neq 0 \text{ and with the choice } q^2 = \frac{X}{(x_4^2)^2} r^3. \end{cases}$$

Suppose $\varepsilon = 0$. We obtain that

$$\hat{x}_3^4 = \delta = \begin{cases} 0, & \text{if and only if } Y = 0, \\ 1, & \text{if } Y \neq 0 \text{ and with the choice } q^2 = \frac{x_4^2}{Y} r^3. \end{cases}$$

Otherwise, if $\varepsilon = 1$ we get $\hat{x}_3^4 = \frac{XY}{(x_4^2)^3}$. At the end, we obtain the equation

$$q^4 x_2^2 + r^4 x_2^3 + p^1 q^2 x_2^4 = \hat{x}_3^4 q^3 + u^1 q^2.$$

Since $q^2 \neq 0$ this equation can be solved by choosing a suitable value of u^1. With $p^2 = p^3 = p^4 = u^2 = 0$ the remaining equations are satisfied. This proves the assertion. □

Remark 2. It follows from the matrix equation (8) that any $\mathfrak{c}^{\mathfrak{n}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(5)}$ is not isomorphic to a $\mathfrak{c}^{\mathfrak{n}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(4)}$, since an algebra $\mathfrak{c}(L_0)$ in the family $C^{(5)}$ is determined by a matrix L_0 with the entry $x_4^2 \neq 0$, whereas an algebra $\mathfrak{c}(L_0)$ in the family $C^{(4)}$ is defined by a matrix L_0 with the entry $x_4^2 = 0$.

Definition 17. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{n}}$ of normal form in the $C^{(5)}$-family are the algebras $\mathfrak{c}_8^{\mathfrak{n}}(\kappa,\lambda)$, $\mathfrak{c}_9^{\mathfrak{n}}(\kappa,\lambda)$, $\mathfrak{c}_{10}^{\mathfrak{n}}(\kappa,\lambda,\nu)$ determined by the matrices L_0:

$$\begin{bmatrix}\kappa & 0 & 1\\ 0 & \lambda & 0\\ 0 & 0 & 0\end{bmatrix},\quad \begin{bmatrix}\kappa & 0 & 1\\ 0 & \lambda & 0\\ 0 & 1 & 0\end{bmatrix},\quad \begin{bmatrix}\kappa & 0 & 1\\ 1 & \lambda & 0\\ 0 & \nu & 0\end{bmatrix},$$

and having the multiplications

$\mathfrak{c}_8^{\mathfrak{n}}(\kappa,\lambda):\ e_1e_2=e_4, e_0e_1=e_1, e_0e_2=\kappa e_2, e_0e_3=\lambda e_3, e_0e_4=e_2,$
$\mathfrak{c}_9^{\mathfrak{n}}(\kappa,\lambda):\ e_1e_2=e_4, e_0e_1=e_1, e_0e_2=\kappa e_2, e_0e_3=\lambda e_3+e_4, e_0e_4=e_2,$
$\mathfrak{c}_{10}^{\mathfrak{n}}(\kappa,\lambda,\nu):\ e_1e_2=e_4, e_0e_1=e_1, e_0e_2=\kappa e_2+e_3, e_0e_3=\lambda e_3+\nu e_4, e_0e_4=e_2,$

where $\lambda\neq 0$ in the first two cases, with respect to a special distinguished basis.

From Proposition 29 we obtain:

Proposition 30. *Any Malcev-like algebra $\mathfrak{c}^{\mathfrak{n}}$ in the $C^{(5)}$-family is isomorphic to a unique algebra of normal form given by $\mathfrak{c}_8^{\mathfrak{n}}(\kappa,\lambda)$, $\mathfrak{c}_9^{\mathfrak{n}}(\kappa,\lambda)$ and $\mathfrak{c}_{10}^{\mathfrak{n}}(\kappa,\lambda,\nu)$.*

Proposition 31. *Any $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(5)}$ defined by L_0 in Definition 16 is isomorphic to the algebra $\mathfrak{c}(\hat{L}_0)$ determined by $\hat{L}_0$ such that*

(i) *if* $x_3^2\neq -x_4^3$, *then* $\hat{L}_0=\begin{bmatrix}\hat{x}_2^2 & 1 & \frac{x_4^2}{(x_3^2+x_4^3)^2}\\ \hat{x}_2^3 & \hat{x}_3^3 & 0\\ \hat{x}_2^4 & 0 & 0\end{bmatrix}$, $\hat{x}_2^2=x_2^2+x_4^4+\frac{x_3^2x_4^3}{x_4^2}$, $\hat{x}_3^3=x_3^3-\frac{x_3^2x_4^3}{x_4^2}$,
$\hat{x}_2^3=\left(x_2^3-x_3^4+\frac{x_4^3(2x_3^3-x_2^2)+x_3^2x_4^4}{x_4^2}-\frac{2x_3^2(x_4^3)^2}{(x_4^2)^2}\right)(x_3^2+x_4^3)$,
$\hat{x}_2^4=\left(x_2^4+\frac{x_4^3(x_3^4-x_2^3)-x_2^2x_4^4}{x_4^2}+\frac{x_4^3\left(x_4^3(x_2^2-x_3^3)-x_3^2x_4^4\right)}{(x_4^2)^2}+\frac{x_3^2(x_4^3)^3}{(x_4^2)^3}\right)(x_3^2+x_4^3)^2$,

(ii) *if* $x_3^2=-x_4^3$, $x_2^3\neq x_3^4-\frac{(2x_3^3-x_2^2-x_4^4)x_4^3}{x_4^2}-\frac{2(x_4^3)^3}{(x_4^2)^2}$, *then* $\hat{L}_0=\begin{bmatrix}\hat{x}_2^2 & 0 & \frac{x_4^2}{(p^1)^2}\\ 1 & \hat{x}_3^3 & 0\\ \hat{x}_2^4 & 0 & 0\end{bmatrix}$,
$\hat{x}_2^2=x_2^2+x_4^4-\frac{(x_4^3)^2}{x_4^2}$, $\hat{x}_3^3=x_3^3+\frac{(x_4^3)^2}{x_4^2}$, $p^1=\frac{1}{x_2^3-x_3^4+\frac{(2x_3^3-x_4^4-x_2^2)x_4^3}{x_4^2}+\frac{2(x_4^3)^3}{(x_4^2)^2}}$,
$\hat{x}_2^4=\left(x_2^4+\frac{x_4^3(x_3^4-x_2^3)-x_2^2x_4^4}{x_4^2}+\frac{(x_4^3)^2(x_2^2-x_3^3+x_4^4)}{(x_4^2)^2}-\frac{(x_4^3)^4}{(x_4^2)^3}\right)(p^1)^2$,

(iii) *if* $x_3^2=-x_4^3$, *and* $x_2^3=x_3^4-\frac{(2x_3^3-x_4^4-x_2^2)x_4^3}{x_4^2}-\frac{2(x_4^3)^3}{(x_4^2)^2}$, *then*
$\hat{L}_0=\begin{bmatrix}\hat{x}_2^2 & 0 & \sigma(L_0)\\ 0 & \hat{x}_3^3 & 0\\ \frac{X_2^4}{\sigma(L_0)} & 0 & 0\end{bmatrix}$,
$\hat{x}_2^2=x_2^2+x_4^4-\frac{(x_4^3)^2}{x_4^2}$, $\hat{x}_3^3=x_3^3+\frac{(x_4^3)^2}{x_4^2}$, $\sigma(L_0)=\sigma\left(x_4^2\mathbb{K}^{\circ 2}\right)$,
$X_2^4=x_4^2x_2^4-x_2^2x_4^4+\frac{x_3^3(x_4^3)^2}{x_4^2}+\frac{(x_4^3)^3(2x_4^3-1)}{(x_4^2)^2}$.

Moreover, the $\mathfrak{c}^{\mathfrak{f}}$*-algebras determined by the matrices*

$$\begin{bmatrix} \hat{x}_2^2 & 0 & s \\ 0 & \hat{x}_3^3 & 0 \\ \frac{X_2^4}{s} & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \hat{x}_2^2 & 0 & t \\ 0 & \hat{x}_3^3 & 0 \\ \frac{X_2^4}{t} & 0 & 0 \end{bmatrix}, \; s, t \in \mathbb{K}^\circ$$

are isomorphic if and only if s *and* t *belong to the same coset of* $\mathbb{K}^{\circ 2}$ *in* $\mathbb{K}^\circ$*.*

Proof. We consider the matrix equation (9) giving the isomorphism condition for a partial isomorphism described by the matrix A of Lemma 3 (b) between two $\mathfrak{c}^{\mathfrak{f}}$-algebras $\mathfrak{c}(L_0)$, $\mathfrak{c}(\hat{L}_0)$ having the given forms L_0 and $\hat{L}_0$ as in Definition 16 and Proposition 31. Assuming $p^2 = p^3 = p^4 = u^2 = u^3 = 0$, then the equations in the first column are fulfilled. The last column yields $\hat{x}_4^2 = \frac{x_4^2}{(p^1)^2}$, $q^3 = -\frac{x_4^3}{x_4^2} p^1 q^2$, and, hence, $q^4 = \frac{1}{x_4^2}\left(\frac{(x_4^3)^2}{x_4^2} - x_4^4\right)(p^1)^2 q^2$. Substituting these expressions to the third column we obtain

$$\hat{x}_3^2 = (x_3^2 + x_4^3)\frac{1}{p^1}, \; \hat{x}_3^3 = x_3^3 - \frac{x_3^2 x_4^3}{x_4^2}, \; u^1 = \left(x_3^4 - \frac{x_3^2 x_4^4 + x_3^3 x_4^3}{x_4^2} + \frac{x_3^2 (x_4^3)^2}{(x_4^2)^2}\right) p^1.$$

Using the previous values we receive from the second column

$$\hat{x}_2^2 = x_2^2 + x_4^4 + \frac{x_3^2 x_4^3}{x_4^2}, \; \hat{x}_2^3 = \left(x_2^3 - x_3^4 + \frac{x_4^3(2x_3^3 - x_2^2) + x_3^2 x_4^4}{x_4^2} - \frac{2x_3^2 (x_4^3)^2}{(x_4^2)^2}\right) p^1,$$

$$\hat{x}_2^4 = \left(x_2^4 + \frac{x_4^3(x_3^4 - x_2^3) - x_2^2 x_4^4}{x_4^2} + \frac{x_4^3\left(x_4^3(x_2^2 - x_3^3) - x_3^2 x_4^4\right)}{(x_4^2)^2} + \frac{x_3^2 (x_4^3)^3}{(x_4^2)^3}\right)(p^1)^2.$$

If $x_3^2 \neq -x_4^3$, then with the choice $p^1 = x_3^2 + x_4^3$ we get $\hat{x}_3^2 = 1$, $\hat{x}_4^2 = \frac{x_4^2}{(x_3^2 + x_4^3)^2}$, and $\hat{x}_2^3$, $\hat{x}_2^4$ as in the assertion (i).

If $x_3^2 = -x_4^3$, then we receive $\hat{x}_3^2 = 0$, $\hat{x}_2^2 = x_2^2 + x_4^4 - \frac{(x_4^3)^2}{x_4^2}$, $\hat{x}_3^3 = x_3^3 + \frac{(x_4^3)^2}{x_4^2}$, $\hat{x}_2^3 = Y p^1$, where $Y = x_2^3 - x_3^4 + \frac{x_4^3(2x_3^3 - x_2^2 - x_4^4)}{x_4^2} + \frac{2(x_4^3)^3}{(x_4^2)^2}$. We distinguish two cases.

If $Y \neq 0$, then we get $\hat{x}_2^3 = 1$ with the choice $p^1 = \frac{1}{Y}$, and $\hat{x}_4^2$, $\hat{x}_2^4$ as in the claim (ii).

If $Y = 0$, then we obtain $\hat{x}_2^3 = 0$. The entries $\hat{x}_4^2$ and x_4^2 belong to the same coset of $\mathbb{K}^{\circ 2}$ in $\mathbb{K}^\circ$. Putting $(p^1)^2 = \frac{x_4^2}{\sigma(L_0)}$ we obtain $\hat{x}_4^2 = \sigma(L_0) = \sigma\left(x_4^2 \mathbb{K}^{\circ 2}\right) \in x_4^2 \mathbb{K}^{\circ 2}$, where $\sigma : \mathbb{K}^\circ / \mathbb{K}^{\circ 2} \to \mathbb{K}^\circ$ is an arbitrary section of the factor group $\mathbb{K}^\circ / \mathbb{K}^{\circ 2}$ in the group $\mathbb{K}^\circ$. Using the forms of $\hat{x}_2^3$ and $(p^1)^2$ we obtain the expression of X_2^4 in assertion (iii).

It follows that the second claim is true. This completes the proof. □

Remark 3. It follows from the matrix equation (9) that any $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(5)}$ is not isomorphic to a $\mathfrak{c}^{\mathfrak{f}}$-algebra $\mathfrak{c}(L_0)$ in the family $C^{(4)}$, since an algebra $\mathfrak{c}(L_0)$ in the family $C^{(5)}$ is determined by a matrix L_0 with the entry $x_4^2 \neq 0$, whereas an algebra $\mathfrak{c}(L_0)$ in the family $C^{(4)}$ is defined by a matrix L_0 with the entry $x_4^2 = 0$.

Definition 18. The Malcev-like algebras $\mathfrak{c}^{\mathfrak{f}}$ of normal form in the $C^{(5)}$-family are the algebras $\mathfrak{c}_9^{\mathfrak{f}}(\kappa, \omega, \sigma, \lambda, \tau)$, $\mathfrak{c}_{10}^{\mathfrak{f}}(\kappa, \omega, \lambda, \tau)$, $\mathfrak{c}_{11}^{\mathfrak{f}}(\kappa, \omega, \lambda, \tau)$, determined by the matrices L_0:

$$\begin{bmatrix} \kappa & 1 & \omega \\ \sigma & \lambda & 0 \\ \tau & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & \omega \\ 1 & \lambda & 0 \\ \tau & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \kappa & 0 & \omega \\ 0 & \lambda & 0 \\ \frac{\tau}{\omega} & 0 & 0 \end{bmatrix},$$

and given by the multiplications

$\mathfrak{c}_9^{\mathfrak{f}}(\kappa, \omega, \sigma, \lambda, \tau)$: $e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + \sigma e_3 + \tau e_4, e_0e_3 = e_2 + \lambda e_3, e_0e_4 = \omega e_2$,

$\mathfrak{c}_{10}^{\mathfrak{f}}(\kappa, \omega, \lambda, \tau)$: $e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + e_3 + \tau e_4, e_0e_3 = \lambda e_3, e_0e_4 = \omega e_2$,

$\mathfrak{c}_{11}^{\mathfrak{f}}(\kappa, \omega, \lambda, \tau)$: $e_1e_2 = e_3, e_1e_3 = e_4, e_0e_1 = e_1, e_0e_2 = \kappa e_2 + \frac{\tau}{\omega} e_4, e_0e_3 = \lambda e_3, e_0e_4 = \omega e_2$,

where $\omega \neq 0$, with respect to a special distinguished basis.

It follows from Proposition 31:

Proposition 32. *Any Malcev-like algebra* $\mathfrak{c}^{\mathfrak{f}}$ *in the* $C^{(5)}$*-family is isomorphic to an algebra of normal form given by* $\mathfrak{c}_9^{\mathfrak{f}}(\kappa, \omega, \sigma, \lambda, \tau)$, $\mathfrak{c}_{10}^{\mathfrak{f}}(\kappa, \omega, \lambda, \tau)$, $\mathfrak{c}_{11}^{\mathfrak{f}}(\kappa, \omega, \lambda, \tau)$. *The algebras* $\mathfrak{c}_9^{\mathfrak{f}}(\kappa, \omega, \sigma, \lambda, \tau)$, $\mathfrak{c}_{10}^{\mathfrak{f}}(\kappa, \omega, \lambda, \tau)$ *are uniquely determined in their isomorphism class. The algebras* $\mathfrak{c}_{11}^{\mathfrak{f}}(\kappa, \omega_1, \lambda, \tau)$ *and* $\mathfrak{c}_{11}^{\mathfrak{f}}(\kappa, \omega_2, \lambda, \tau)$ *are isomorphic if and only if* ω_1 *and* ω_2 *belong to the same coset of* $\mathbb{K}^{\circ 2}$ *in* $\mathbb{K}^{\circ}$.

As a consequence of Propositions 7.1 and 8.1 in [10] we obtain:

Corollary 33. *Among the Malcev-like algebras in the family* $C^{(5)}$ *there does not exist any binary Lie algebra.*

5. Summary

The normal forms of the Malcev-like algebras $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ are given in Definitions 7, 9, 11, 13, 15, 17, 18. From Sect. 4 it follows:

Theorem 34. *Any Malcev-like algebra* $\mathfrak{c}^{\mathfrak{t}}$, $\mathfrak{t} \in \{\mathfrak{n}, \mathfrak{f}\}$ *is isomorphic to an algebra of normal form given in the Table, where* $\kappa, \eta, \sigma, \lambda, \nu, \mu \in \mathbb{K}$, $\tau, \omega \in \mathbb{K}^{\circ}$. *The*

	$\mathfrak{c}^{\mathfrak{n}}$-*algebra*	$\mathfrak{c}^{\mathfrak{f}}$-*algebra*
Anti-commutative	$\mathfrak{c}^{\mathfrak{n}}_1(\kappa,\lambda,\mu), \mathfrak{c}^{\mathfrak{n}}_2(\kappa,\lambda), \mathfrak{c}^{\mathfrak{n}}_3(\kappa,\mu),$ $\mathfrak{c}^{\mathfrak{n}}_4(\kappa), \mathfrak{c}^{\mathfrak{n}}_5(\kappa,\lambda,\nu), \mathfrak{c}^{\mathfrak{n}}_6(\kappa,\sigma,\mu),$ $\mathfrak{c}^{\mathfrak{n}}_7(\kappa,\sigma), \mathfrak{c}^{\mathfrak{n}}_8(\kappa,\lambda), \mathfrak{c}^{\mathfrak{n}}_9(\kappa,\lambda),$ $\mathfrak{c}^{\mathfrak{n}}_{10}(\kappa,\lambda,\nu)$	$\mathfrak{c}^{\mathfrak{f}}_1(\kappa,\lambda,\mu), \mathfrak{c}^{\mathfrak{f}}_2(\kappa,\mu),$ $\mathfrak{c}^{\mathfrak{f}}_3(\kappa,\lambda,\tau), \mathfrak{c}^{\mathfrak{f}}_4(\kappa,\lambda,\tau),$ $\mathfrak{c}^{\mathfrak{f}}_5(\kappa,\tau,\mu), \mathfrak{c}^{\mathfrak{f}}_6(\kappa,\eta,\lambda,\tau),$ $\mathfrak{c}^{\mathfrak{f}}_7(\kappa,\sigma,\lambda), \mathfrak{c}^{\mathfrak{f}}_8(\sigma,\lambda,\tau),$ $\mathfrak{c}^{\mathfrak{f}}_9(\kappa,\omega,\sigma,\lambda,\tau),$ $\mathfrak{c}^{\mathfrak{f}}_{10}(\kappa,\omega,\lambda,\tau),$ $\mathfrak{c}^{\mathfrak{f}}_{11}(\kappa,\omega,\lambda,\tau)$
Binary Lie	$\mathfrak{c}^{\mathfrak{n}}_1(1,\lambda,\mu), \mathfrak{c}^{\mathfrak{n}}_2(1,\lambda), \mathfrak{c}^{\mathfrak{n}}_3(1,\mu),$ $\mathfrak{c}^{\mathfrak{n}}_4(1), \mathfrak{c}^{\mathfrak{n}}_1(\kappa,\lambda,\kappa+1), \mathfrak{c}^{\mathfrak{n}}_2(\kappa,\lambda),$ $\mathfrak{c}^{\mathfrak{n}}_3(\kappa,\kappa+1), \mathfrak{c}^{\mathfrak{n}}_4(\kappa), \mathfrak{c}^{\mathfrak{n}}_5(1,\lambda,\nu),$ $\mathfrak{c}^{\mathfrak{n}}_6(\kappa,1-\kappa,\kappa)$	$\mathfrak{c}^{\mathfrak{f}}_1(1,\lambda,\lambda+1),$ $\mathfrak{c}^{\mathfrak{f}}_1(\lambda-1,\lambda,\lambda+1),$ $\mathfrak{c}^{\mathfrak{f}}_2(1,3), \mathfrak{c}^{\mathfrak{f}}_3(1,0,\tau)$

Malcev-like algebras of normal form are uniquely determined in their isomorphism class up to the isomorphisms

$$\mathfrak{c}^{\mathfrak{n}}_1(\kappa,\lambda,\mu) \cong \mathfrak{c}^{\mathfrak{n}}_1\left(\frac{1}{\kappa},\frac{\lambda}{\kappa},\frac{\mu}{\kappa}\right), \mathfrak{c}^{\mathfrak{n}}_2(\kappa,\lambda) \cong \mathfrak{c}^{\mathfrak{n}}_2\left(\frac{1}{\kappa},\frac{\lambda}{\kappa}\right), \mathfrak{c}^{\mathfrak{n}}_5(\kappa,\lambda,\nu) \cong \mathfrak{c}^{\mathfrak{n}}_5\left(\frac{1}{\kappa},\frac{\lambda}{\kappa},\frac{\nu}{\kappa^2}\right),$$

$\kappa \neq 1$,

$\mathfrak{c}^{\mathfrak{f}}_3(\kappa,\lambda,\tau_1) \cong \mathfrak{c}^{\mathfrak{f}}_3(\kappa,\lambda,\tau_2)$ *if* $\tau_1\mathbb{K}^{\circ 2} = \tau_2\mathbb{K}^{\circ 2}$,

$\mathfrak{c}^{\mathfrak{f}}_{11}(\kappa,\omega_1,\lambda,\tau) \cong \mathfrak{c}^{\mathfrak{f}}_{11}(\kappa,\omega_2,\lambda,\tau)$ *if* $\omega_1\mathbb{K}^{\circ 2} = \omega_2\mathbb{K}^{\circ 2}$.

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