



Regular Articles

Strong geometric derivatives [☆]Zoltán Boros ^{*}, Péter Tóth*Institute of Mathematics, University of Debrecen, 4002 Debrecen, Pf. 400, Hungary*

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ABSTRACT

For a real valued function f , defined on an open interval I , and an arbitrary real number h , we consider the lower and upper limits of

$$2^n \left(f \left(y + \frac{h}{2^n} \right) - f(y) \right)$$

whenever n tends to infinity and y tends to a fixed element x of I . We consider two families of functions determined by the properties of these limits. The first interesting property is when these lower and upper limits are finite and equal to each other for every real number h and every x in I . The second notable family is determined by the property that both limits are finite and increasing (in a specific sense) with respect to x for every positive number h . These properties are motivated by the families of continuously differentiable functions and convex functions, respectively. However, restrictions of additive mappings belong to these classes as well. Our decomposition theorems establish that these motivating examples generate the whole classes. Namely, every function belonging to the first family can be represented as the sum of a continuously differentiable function and an additive one, while every function taken from the second family turns out to be the sum of a convex function and an additive one. We apply our results in order to give a local and approximate characterization of affine functions and Wright-convex functions, respectively.

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1. Introduction

The concept of strong differentiability was introduced by Peano [9]. A function $f : I \rightarrow \mathbb{R}$ is strongly differentiable if the finite limit of the difference quotients

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$$\frac{f(y+t) - f(y)}{t}$$

exists as t tends to 0 while y tends to any fixed $x \in I$ (where I denotes an interval of real numbers). In general, strong differentiability is closely related to continuous differentiability, as it has been already observed by Peano. Later Esser and Shisha [6] proved that a real valued function f is strongly differentiable on an open interval if, and only if, f is continuously differentiable on that interval.

Motivated by de Bruijn's decomposition theorems for functions having differences in a given class [5], the first author proved an analogous decomposition theorem for strongly $\mathbb{Q}$ -differentiable functions [1]. A real function f is strongly $\mathbb{Q}$ -differentiable if, for every real number h , the finite limit of the difference quotients

$$Q_h f(y, r) := \frac{f(y + rh) - f(y)}{r}$$

exists whenever r tends to zero through the positive rationals and y tends to a fixed element of the domain. It was proved that f is strongly $\mathbb{Q}$ -differentiable if, and only if, $f = g + A$, where g is a continuously differentiable function and A is an additive mapping. The necessary condition was reconsidered by Broszka and Grande [4], who replaced the set of positive rationals with a dense subgroup of the additive group of real numbers. Namely, they obtained the following result [4, Remark 15]:

Theorem 1.1. *Let Y denote a real Banach space. Suppose that $\mathcal{I} \subset \mathbb{R}$ such that $\mathcal{I}$ is dense in $\mathbb{R}$ and $\mathcal{I} - \mathcal{I} \subseteq \mathcal{I}$. If $f : \mathbb{R} \rightarrow Y$ is a strongly $\mathcal{I}$ -differentiable function, then there exist a continuously differentiable function $g : \mathbb{R} \rightarrow Y$ and an additive function $\phi : \mathbb{R} \rightarrow Y$ such that*

$$f(x) = g(x) + \phi(x) \quad \text{for all } x \in \mathbb{R}.$$

In the same paper [4], the authors investigate operation rules and related results for (not strongly) $\mathcal{I}$ -differentiable functions. They call $f : \mathbb{R} \rightarrow Y$ an $\mathcal{I}$ -differentiable function if the limit of $Q_h f(y, r)$ exists, for all fixed $y, h \in \mathbb{R}$, whenever $r \in \mathcal{I}$ tends to zero.

Formally, Theorem 1.1 is not a generalization of Boros' decomposition theorem [1] for two reasons. On one hand, a subgroup contains negative elements as well. So it might be reasonable to elaborate a counterpart of all of the results in [4] for functions with real variables such that the condition $\mathcal{I} \ni r \rightarrow 0$ would be replaced by $\mathcal{I} \ni r \rightarrow 0+$ in the limit defining the concept of the $\mathcal{I}$ -derivative, while the condition $\mathbb{R} \times \mathcal{I} \ni (x, r) \rightarrow (x_0, 0)$ should be replaced by $\mathbb{R} \times \mathcal{I}_+ \ni (x, r) \rightarrow (x_0, 0)$ (where $\mathcal{I}_+$ denotes the set of positive elements of $\mathcal{I}$) in the limit defining the concept of the strong $\mathcal{I}$ -derivative. On the other hand, Theorem 1.1 provides only a necessary condition for strong $\mathcal{I}$ -differentiability, which is not sufficient in general. Namely, additivity implies only (strong) $\mathbb{Q}$ -differentiability in general.

Concerning the properties of (not strongly) $\mathcal{I}$ -differentiable functions with the restriction $r > 0$ in the limit, as well as their applications for generalized convex functions, we should refer to the article by Boros and Páles [3], where $\mathcal{I}$ is supposed to be a subfield of real numbers.

In order to improve Boros' decomposition theorem for strongly $\mathbb{Q}$ -differentiable functions, keeping its "if and only if" feature, as well as admitting analogies and applications in the theories of approximately affine and approximately convex functions, we should replace the set of positive rationals by one of its proper subsets. Namely, it is reasonable to consider a simple zero sequence of positive rationals. Clearly, such an approach is not a particular case of the assumptions in Theorem 1.1. It is also reasonable to assume that f is defined on an open interval (since some convex functions defined on an interval cannot be extended to a convex function defined on the whole real line).

In this paper we consider a geometrically motivated version of these regularity concepts involving the family of convex functions into our considerations as well. For a real valued function f , defined on an open interval I , and for an arbitrary real number h , we consider the lower and upper limits of

$$2^n \left(f \left(y + \frac{h}{2^n} \right) - f(y) \right)$$

whenever n tends to infinity and y tends to a fixed element x of I . We consider two families of functions determined by the properties of these limits. The first interesting property is when these lower and upper limits are finite and equal to each other for every real number h and for every x in I . The second notable family is determined by the property that both limits are finite and increasing (in a specific sense) with respect to x , for every positive number h . According to our main theorems, these regularity concepts characterize functions of the form $f = g + \phi$, where ϕ denotes the restriction of an additive mapping, while g denotes a continuously differentiable function or a convex function, respectively. As a corollary, we obtain a characterization of continuously differentiable functions as strongly geometrically differentiable functions fulfilling, in addition, some weak regularity assumption (like boundedness from above/below on a set of positive measure). In the final two sections we apply our decomposition theorems in order to establish a local and approximate characterization of affine functions and Wright-convex functions, respectively.

2. Notions, notations and basic properties

In this chapter we introduce the concept of strong geometric derivatives. Throughout the whole paper we consider the real line $\mathbb{R}$ equipped with the standard absolute value norm. We will use the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ for the set of extended real numbers and $\mathbb{R}_+$ for the set of positive real numbers. The open ball with center $x \in \mathbb{R}$ and radius $r > 0$ is denoted by $B(x, r) := \{y \in \mathbb{R} \mid |y - x| < r\}$. From now on let $I \subseteq \mathbb{R}$ denote a nonempty open interval.

Definition 2.1. Let $f : I \rightarrow \mathbb{R}$, $x \in I$, $h \in \mathbb{R}$ and $\delta \in \mathbb{R}_+$. Let us consider the set

$$R^\diamond(f, x, h, \delta) := \left\{ 2^n \left(f \left(y + \frac{h}{2^n} \right) - f(y) \right) \mid n \in \mathbb{N} \text{ and } y, y + \frac{h}{2^n} \in B(x, \delta) \cap I \right\}.$$

Then the limits

$$\begin{aligned} \underline{D}_h^\diamond f(x) &:= \liminf_{\delta \rightarrow 0} R^\diamond(f, x, h, \delta) \\ \overline{D}_h^\diamond f(x) &:= \limsup_{\delta \rightarrow 0} R^\diamond(f, x, h, \delta) \end{aligned}$$

are called the *lower* and, respectively, *upper strong geometric derivatives of f at x in the direction h* .

Remark 2.1. Observe that $R^\diamond(f, x, h, \delta) \neq \emptyset$ always holds, since I is an open interval. Moreover, if $0 < \delta_1 < \delta_2$, then clearly $R^\diamond(f, x, h, \delta_1) \subseteq R^\diamond(f, x, h, \delta_2)$ holds, which implies

$$\begin{aligned} -\infty &\leq \inf R^\diamond(f, x, h, \delta_2) \leq \inf R^\diamond(f, x, h, \delta_1) \\ &\leq \sup R^\diamond(f, x, h, \delta_1) \leq \sup R^\diamond(f, x, h, \delta_2) \leq +\infty. \end{aligned}$$

Therefore the limits $\underline{D}_h^\diamond f(x)$ and $\overline{D}_h^\diamond f(x)$ exist in $\overline{\mathbb{R}}$, fulfilling $\underline{D}_h^\diamond f(x) \leq \overline{D}_h^\diamond f(x)$.

For a better understanding it can be helpful to interpret the lower and upper strong geometric derivatives as the following limits:

$$\underline{D}_h^\diamond f(x) = \liminf_{\substack{y \rightarrow x \\ n \rightarrow \infty}} 2^n \left(f \left(y + \frac{h}{2^n} \right) - f(y) \right)$$

and

$$\overline{D}_h^\diamond f(x) = \limsup_{\substack{y \rightarrow x \\ n \rightarrow \infty}} 2^n \left(f\left(y + \frac{h}{2^n}\right) - f(y) \right).$$

Definition 2.2. Let $f : I \rightarrow \mathbb{R}$, $x \in I$ and $h \in \mathbb{R}$. The function f is said to be *strongly geometrically differentiable at x in the direction h* if $\underline{D}_h^\diamond f(x) = \overline{D}_h^\diamond f(x) \in \mathbb{R}$. Then

$$D_h^\diamond f(x) := \underline{D}_h^\diamond f(x) = \overline{D}_h^\diamond f(x)$$

is the strong geometric derivative of f at x in the direction h . We call f *strongly geometrically differentiable* if, for all $x \in I$ and $h \in \mathbb{R}$, f is strongly geometrically differentiable at x in the direction h .

Similarly as before, we can interpret the strong geometric derivative as a limit:

$$D_h^\diamond f(x) = \lim_{\substack{y \rightarrow x \\ n \rightarrow \infty}} 2^n \left(f\left(y + \frac{h}{2^n}\right) - f(y) \right).$$

Proposition 2.1. Let $f, g : I \rightarrow \mathbb{R}$ be functions, $x \in I$ and $h \in \mathbb{R}$. Then, for any $N \in \mathbb{N}$, it holds that

$$\underline{D}_{\frac{h}{2^N}}^\diamond f(x) = \frac{1}{2^N} \underline{D}_h^\diamond f(x) \quad \text{and} \quad \overline{D}_{\frac{h}{2^N}}^\diamond f(x) = \frac{1}{2^N} \overline{D}_h^\diamond f(x).$$

Moreover, if f and g are strongly geometrically differentiable at x in the direction h while $\lambda \in \mathbb{R}$ is an arbitrary constant, then

$$D_h^\diamond (f + g)(x) = D_h^\diamond f(x) + D_h^\diamond g(x) \quad \text{and} \quad D_h^\diamond (\lambda f)(x) = \lambda D_h^\diamond f(x).$$

Proof. First let us fix $0 < \delta_0 < 2^{-(N+1)}h$. Clearly, for any radius $\delta \in]0, \delta_0[$, the inclusion $y, y + 2^{-m}h \in I \cap B(x, \delta)$ implies $m > N$. From now on, let us consider an arbitrarily fixed radius $\delta \in]0, \delta_0[$.

Let $r \in R^\diamond(f, x, 2^{-N}h, \delta)$ be an arbitrary ratio. Then there exist $n \in \mathbb{N}$ and $y \in I$ such that $y + 2^{-(N+n)}h \in I \cap B(x, \delta)$ and $r = 2^n (f(y + 2^{-(N+n)}h) - f(y))$. But this means

$$r = 2^n \left(f\left(y + \frac{h}{2^N} \frac{1}{2^n}\right) - f(y) \right) = \frac{1}{2^N} 2^{N+n} \left(f\left(y + \frac{h}{2^{N+n}}\right) - f(y) \right) =: \frac{1}{2^N} r_0$$

where $r_0 \in R^\diamond(f, x, h, \delta)$. Therefore $R^\diamond(f, x, 2^{-N}h, \delta) \subseteq 2^{-N}R^\diamond(f, x, h, \delta)$.

Similarly, if $\tilde{r} \in R^\diamond(f, x, h, \delta)$ is an arbitrary ratio, then there exist $m \in \mathbb{N}$, $z \in I$ fulfilling $z + 2^{-m}h \in I \cap B(x, \delta)$ and $\tilde{r} = 2^m (f(z + 2^{-m}h) - f(z))$. Due to the first remark of the proof, $m > N$ necessarily holds. Then we obtain that

$$\tilde{r} = 2^m \left(f\left(z + \frac{h}{2^m}\right) - f(z) \right) = 2^N 2^{m-N} \left(f\left(z + \frac{h}{2^N} \frac{1}{2^{m-N}}\right) - f(z) \right) =: 2^N \tilde{r}_0$$

where $\tilde{r}_0 \in R^\diamond(f, x, 2^{-N}h, \delta)$. Hence $2^{-N}R^\diamond(f, x, h, \delta) \subseteq R^\diamond(f, x, 2^{-N}h, \delta)$, which, together with the reverse inclusion, implies the equality of the two sets. Recalling the definition of the lower and upper strong geometric derivatives we arrive to

$$\begin{aligned} \underline{D}_{\frac{h}{2^N}}^\diamond f(x) &= \liminf_{\delta \rightarrow 0} R^\diamond(f, x, \frac{h}{2^N}, \delta) = \frac{1}{2^N} \liminf_{\delta \rightarrow 0} R^\diamond(f, x, h, \delta) = \frac{1}{2^N} \underline{D}_h^\diamond f(x) \quad \text{and} \\ \overline{D}_{\frac{h}{2^N}}^\diamond f(x) &= \limsup_{\delta \rightarrow 0} R^\diamond(f, x, \frac{h}{2^N}, \delta) = \frac{1}{2^N} \limsup_{\delta \rightarrow 0} R^\diamond(f, x, h, \delta) = \frac{1}{2^N} \overline{D}_h^\diamond f(x). \end{aligned}$$

Since the boundedness of the above sets of difference ratios is not supposed, the corresponding homogeneity equations may involve $t \cdot (+\infty) = +\infty$ or $t \cdot (-\infty) = -\infty$ with any $0 < t \in \mathbb{R}$.

In the second statement the homogeneity of the mapping $f \mapsto D_h^\diamond f(x)$ is a straightforward consequence of $R^\diamond(\lambda f, x, h, \delta) = \lambda R^\diamond(f, x, h, \delta)$. For the additivity, let us observe that $R^\diamond(f + g, x, h, \delta) \subseteq R^\diamond(f, x, h, \delta) + R^\diamond(g, x, h, \delta)$. Hence, for any ratio $r \in R^\diamond(f + g, x, h, \delta)$ we have the estimates

$$\inf R^\diamond(f, x, h, \delta) + \inf R^\diamond(g, x, h, \delta) \leq r \leq \sup R^\diamond(f, x, h, \delta) + \sup R^\diamond(g, x, h, \delta).$$

Clearly, these inequalities remain valid if r is replaced with $\inf R^\diamond(f + g, x, h, \delta)$ or $\sup R^\diamond(f + g, x, h, \delta)$. Taking the limit $\delta \rightarrow 0$ here implies $D_h^\diamond(f + g)(x) = D_h^\diamond f(x) + D_h^\diamond g(x)$. $\square$

We will often apply the abbreviation SGD for the term strongly geometrically differentiable. We give two examples for SGD functions. In fact, we will see later that the following two classes are essential when we wish to characterize the family of SGD functions.

Example 2.1. Let us suppose that $f : I \rightarrow \mathbb{R}$ is continuously differentiable. Then, for every $h \in \mathbb{R}$, f is strongly geometrically differentiable at every $x \in I$ in the direction h and $D_h^\diamond f(x) = f'(x)h$.

In order to verify this, let us choose an arbitrary ratio $2^n(f(y + 2^{-n}h) - f(y)) \in R^\diamond(f, x, h, \delta)$. Due to the differentiability of f we can use the Lagrange mean value theorem and claim that there exists $z_\delta \in B(x, \delta)$ between y and $y + 2^{-n}h$ such that

$$2^n(f(y + 2^{-n}h) - f(y)) = \frac{f(y + 2^{-n}h) - f(y)}{(y + 2^{-n}h) - y} \cdot h = f'(z_\delta)h.$$

Therefore

$$\begin{aligned} \inf\{f'(z)h \mid z \in B(x, \delta) \cap I\} &\leq \inf R^\diamond(f, x, h, \delta) \\ &\leq \sup R^\diamond(f, x, h, \delta) \leq \sup\{f'(z)h \mid z \in B(x, \delta) \cap I\}. \end{aligned}$$

Taking the limit $\delta \rightarrow 0$ and using the continuity of f' we get that $D_h^\diamond f(x) = f'(x)h$.

Example 2.2. Suppose that $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is additive, and $f(x) = \phi(x)$ for every $x \in I$. Then

$$2^n(f(y + 2^{-n}h) - f(y)) = 2^n(\phi(y) + 2^{-n}\phi(h) - \phi(y)) = \phi(h)$$

for all $y \in I$ and $n \in \mathbb{N}$ such that $y + 2^{-n}h \in I$. That is, $R^\diamond(f, x, h, \delta) = \{\phi(h)\}$, hence $D_h^\diamond f(x) = \phi(h)$ for every $x \in I$ and $h \in \mathbb{R}$.

3. Decomposition of SGD functions

In the sequel we will formulate an analogous version of the Lagrange mean value theorem, concerning SGD functions. With the help of this result we give the decomposition theorem of SGD functions.

Proposition 3.1. Let $h \in \mathbb{R}$ and suppose that $f : I \rightarrow \mathbb{R}$ is strongly geometrically differentiable in the direction h , in all points $x \in I$. Then the function

$$x \mapsto D_h^\diamond f(x) \quad (x \in I)$$

is continuous.

Proof. Let $x_0 \in I$ and $\varepsilon > 0$ be arbitrary fixed real numbers. Due to the definition of strong geometric differentiability there exists $\delta_0 > 0$ such that

$$D_h^\diamond f(x_0) - \varepsilon < \inf R^\diamond(f, x_0, h, \delta_0) \leq \sup R^\diamond(f, x_0, h, \delta_0) < D_h^\diamond f(x_0) + \varepsilon. \quad (1)$$

Suppose that $y \in B(x_0, \frac{\delta_0}{2}) \cap I$. Now for any $z \in I$, $n \in \mathbb{N}$ and $\delta \in]0, \frac{\delta_0}{2}[$ fulfilling $2^n(f(z + 2^{-n}h) - f(z)) \in R^\diamond(f, y, h, \delta)$ it holds that $z, z + 2^{-n}h \in B(y, \delta) \cap I$, because of the definition of $R^\diamond(f, y, h, \delta)$. Then obviously $z, z + 2^{-n}h \in B(y, \frac{\delta_0}{2}) \cap I$ is also true which, together with $y \in B(x_0, \frac{\delta_0}{2}) \cap I$, yields $z, z + 2^{-n}h \in B(x_0, \delta_0) \cap I$, due to the triangle inequality. Therefore $2^n(f(z + 2^{-n}h) - f(z)) \in R^\diamond(f, x_0, h, \delta_0)$, that is

$$R^\diamond(f, y, h, \delta) \subseteq R^\diamond(f, x_0, h, \delta_0) \quad \text{for all } \delta \in \left]0, \frac{\delta_0}{2}\right[.$$

This means that for all $y \in B(x_0, \frac{\delta_0}{2}) \cap I$ we have

$$\inf R^\diamond(f, x_0, h, \delta_0) \leq \inf R^\diamond(f, y, h, \delta) \leq \sup R^\diamond(f, y, h, \delta) \leq \sup R^\diamond(f, x_0, h, \delta).$$

By taking the limit $\delta \rightarrow 0$ and using the inequalities (1) we get

$$D_h^\diamond f(x_0) - \varepsilon < D_h^\diamond f(y) < D_h^\diamond f(x_0) + \varepsilon,$$

which holds for all $y \in B(x_0, \frac{\delta_0}{2}) \cap I$. As x_0 and ε were arbitrary, this means that the map $x \mapsto D_h^\diamond f(x)$ is continuous on I . $\square$

Remark 3.1. We will use the following observation in many of our later proofs. Let $f : I \rightarrow \mathbb{R}$ be an arbitrary function. Let $n \in \mathbb{N}$ and let $x = x_0 < x_1 < \dots < x_n = y$ be arbitrary elements of I . Then, by using a telescopic summation, we get

$$\frac{f(y) - f(x)}{y - x} = \sum_{k=1}^n \frac{x_k - x_{k-1}}{y - x} \cdot \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

As the right hand side is a convex combination of n difference ratios, there is at least one which is not less (and respectively, not greater) than the one on the left hand side. That is, there exist $k, l \in \{1, \dots, n\}$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \geq \frac{f(y) - f(x)}{y - x} \quad \text{and} \quad \frac{f(x_l) - f(x_{l-1})}{x_l - x_{l-1}} \leq \frac{f(y) - f(x)}{y - x}.$$

Now we formulate the Lagrange type mean value theorem mentioned earlier.

Theorem 3.1. Let $f : I \rightarrow \mathbb{R}$ be a function, moreover let $h > 0$, $a \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{0\}$ such that $a, a + 2^{-m}h \in I$. Suppose that $\underline{D}_h^\diamond f(x) \in \mathbb{R}$ and $\overline{D}_h^\diamond f(x) \in \mathbb{R}$ for all $x \in [a, a + 2^{-m}h] \subset I$. Then there exist $u, s \in [a, a + 2^{-m}h]$ such that

$$\underline{D}_h^\diamond f(u) \leq 2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right) \quad \text{and} \quad \overline{D}_h^\diamond f(s) \geq 2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right).$$

Moreover, if, for every $x \in I$, f is strongly geometrically differentiable at x in the direction h , then there exists $z \in [a, a + 2^{-m}h]$ such that

$$2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right) = D_h^\diamond f(z).$$

Proof. We define a sequence of intervals $I_n = [u_n, v_n]$ ($n \in \mathbb{N} \cup \{0\}$) with the following recursion. Let $I_0 := [a, a + 2^{-m}h]$, and if I_n is already given then let us define I_{n+1} such that

- (i) $I_{n+1} \subseteq I_n$ and they have a common endpoint;
- (ii) $v_{n+1} - u_{n+1} = \frac{v_n - u_n}{2}$;
- (iii) $\frac{f(v_{n+1}) - f(u_{n+1})}{v_{n+1} - u_{n+1}} \leq \frac{f(v_n) - f(u_n)}{v_n - u_n}$.

That is, we define a nested sequence of intervals where the subsequent interval is either half of the previous one, such that the sequence of the difference ratios of the endpoints is non-increasing. Due to Remark 3.1, such a construction is always possible.

Clearly, $u_n, v_n \in [a, a + 2^{-m}h]$ and $v_n = u_n + 2^{-(n+m)}h$ holds for all $n \in \mathbb{N} \cup \{0\}$. Therefore condition (iii) is equivalent to

$$2^{n+m+1} \left(f(u_{n+1} + 2^{-(n+m+1)}h) - f(u_{n+1}) \right) \leq 2^{n+m} \left(f(u_n + 2^{-(n+m)}h) - f(u_n) \right).$$

Furthermore (u_n) is a Cauchy-sequence, as it easily follows from $|u_{n+1} - u_n| \leq 2^{-(n+m+1)}h$. Hence it is convergent, so there exists $\lim_{n \rightarrow \infty} u_n =: u \in [a, a + 2^{-m}h]$. This implies that, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that $R^\diamond(f, u, h, \delta)$ contains the ratio

$$\begin{aligned} 2^{N+m} \left(f(u_N + \frac{h}{2^{N+m}}) - f(u_N) \right) &\leq \dots \leq 2^{1+m} \left(f(u_1 + \frac{h}{2^{m+1}}) - f(u_1) \right) \\ &\leq 2^m \left(f(u_0 + \frac{h}{2^m}) - f(u_0) \right) = 2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right). \end{aligned}$$

Therefore

$$\inf R^\diamond(f, u, h, \delta) \leq 2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right).$$

Taking the limit as δ tends to 0, we obtain

$$\underline{D}_h^\diamond f(u) \leq 2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right).$$

The existence of $s \in [a, a + 2^{-m}h]$ fulfilling the inequality $\overline{D}_h^\diamond f(s) \geq 2^m(f(a + 2^{-m}h) - f(a))$ can be proved by an analogous argument.

For the last part assume that f is SGD in the direction h . Then

$$D_h^\diamond f(u) = \underline{D}_h^\diamond f(u) \leq 2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right) \leq \overline{D}_h^\diamond f(s) = D_h^\diamond f(s).$$

However, due to Proposition 3.1, $x \mapsto D_h^\diamond f(x)$ is a continuous function, so it possesses the Intermediate Value Property. Therefore there exists z between u and s such that

$$D_h^\diamond f(z) = 2^m \left(f\left(a + \frac{h}{2^m}\right) - f(a) \right). \quad \square$$

Notation 3.1. Let $\mathbb{D}$ denote the set of dyadic rationals in $\mathbb{R}$, i.e.

$$\mathbb{D} := \left\{ \frac{m}{2^n} \mid m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}.$$

As it is well-known, $\mathbb{D}$ is a dense subring in $\mathbb{R}$.

Theorem 3.2. *Let $f : I \rightarrow \mathbb{R}$ be a function such that $D_1^\diamond f(x) = 0$ holds in all points $x \in I$. Moreover, suppose that for every $h > 0$ there exists $x_h \in I$ such that f is strongly geometrically differentiable at x_h in the direction h . Then there exists $c \in \mathbb{R}$ and an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ fulfilling*

$$f(x) = c + A(x) \quad \text{for all } x \in I.$$

Proof. Due to Proposition 2.1, $D_1^\diamond f(x) = 0$ implies $D_{\frac{1}{2^n}}^\diamond f(x) = 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $x \in I$. Now let $a \in I$ and $m, n \in \mathbb{N}$ be arbitrary numbers fulfilling $a + m2^{-n} \in I$. Applying Theorem 3.1 we get the following assertion:

$$f\left(a + \frac{m}{2^n}\right) - f(a) = \sum_{k=1}^m \left(f\left(a + \frac{k}{2^n}\right) - f\left(a + \frac{k-1}{2^n}\right) \right) = \sum_{k=1}^m D_{\frac{1}{2^n}}^\diamond f(x_k) = 0$$

where x_k is an appropriate element of the interval $[a + (k-1)2^{-n}, a + k2^{-n}]$ for every $k \in \{1, 2, \dots, m\}$. In other words, $f(a) = f(a + m2^{-n})$ holds for all $m, n \in \mathbb{N}$ and $a \in I$. Obviously, this is true for all $m \in \mathbb{Z}$ as well, so eventually we can write $f(a) = f(a + d)$ for all $a \in I$ and $d \in \mathbb{D}$ such that $a + d \in I$.

In the next step we will show that $D_h^\diamond f(x)$ exists for all $x \in I$ and $h > 0$, moreover, it is constant in x . In order to verify this, let us fix any $x \in I$ and choose $\delta_0 > 0$ such that $B(x_h, 2\delta_0) \subseteq I$ and $B(x, \delta_0) \subseteq I$. From now on let $0 < \delta < \delta_0$ be arbitrary. If $r \in R^\diamond(f, x, h, \delta)$ then there exist $y_0 \in I$ and $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$y_0, y_0 + \frac{h}{2^{n_0}} \in B(x, \delta) \quad \text{and} \quad r = 2^{n_0} \left(f\left(y_0 + \frac{h}{2^{n_0}}\right) - f(y_0) \right).$$

Let $d \in \mathbb{D} \cap B(x_h - x, \delta)$. Then, due to the triangle inequality, we have $d + y_0 + \frac{h}{2^{n_0}} \in B(x_h, 2\delta)$ and $d + y_0 \in B(x_h, 2\delta)$. However, $d \in \mathbb{D}$ ensures that

$$2^{n_0} \left(f\left(d + y_0 + \frac{h}{2^{n_0}}\right) - f(d + y_0) \right) = 2^{n_0} \left(f\left(y_0 + \frac{h}{2^{n_0}}\right) - f(y_0) \right) = r$$

also holds, so $r \in R^\diamond(f, x_h, h, 2\delta)$. That is, $R^\diamond(f, x, h, \delta) \subseteq R^\diamond(f, x_h, h, 2\delta)$, thus

$$\inf R^\diamond(f, x_h, h, 2\delta) \leq \inf R^\diamond(f, x, h, \delta) \leq \sup R^\diamond(f, x, h, \delta) \leq \sup R^\diamond(f, x_h, h, 2\delta).$$

By taking the limit $\delta \rightarrow 0$ and using the definition of strong geometric differentiability we get

$$\underline{D}_h^\diamond f(x_h) \leq \underline{D}_h^\diamond f(x) \leq \overline{D}_h^\diamond f(x) \leq \overline{D}_h^\diamond f(x_h)$$

but as the first and last terms are equal, $D_h^\diamond f(x) = D_h^\diamond f(x_h)$ follows. Thus we may define a function $m : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $m(h) = D_h^\diamond f(x_h) = D_h^\diamond f(x)$ for every $x \in I$.

Using Theorem 3.1 again, it turns out that for any $h > 0$ and $a \in I$ such that $a + h \in I$ the equation $f(a + h) - f(a) = m(h)$ holds. Hence, if $a, b \in I$ and $a < b$, then

$$f(b) - f\left(\frac{a+b}{2}\right) = m\left(\frac{b-a}{2}\right) \quad \text{and} \quad f\left(\frac{a+b}{2}\right) - f(a) = m\left(\frac{b-a}{2}\right).$$

After subtracting the second equation from the first one and rearranging, we get

$$\frac{f(a) + f(b)}{2} = f\left(\frac{a+b}{2}\right).$$

That is, f is a solution of the Jensen equation on the nonempty open interval $I \subseteq \mathbb{R}$. As a classical result in the theory of functional equations (see [7, Theorem 13.2.1.]), there exist an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $f(x) = c + A(x)$ for all $x \in I$. $\square$

After all of these preparations we can prove the decomposition theorem of strongly geometrically differentiable functions.

Theorem 3.3. *The function $f : I \rightarrow \mathbb{R}$ is strongly geometrically differentiable if, and only if, there exist a continuously differentiable function $g : I \rightarrow \mathbb{R}$ and an additive mapping $A : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$f(x) = g(x) + A(x) \quad \text{for all } x \in I.$$

Proof. Let us fix an arbitrary element $x_0 \in I$ and define the function $G : I \rightarrow \mathbb{R}$ as follows:

$$G(x) := \int_{x_0}^x D_1^\diamond f(y) dy \quad (x \in I).$$

Due to Proposition 3.1, the function $D_1^\diamond f$ is continuous, hence G is continuously differentiable. Thus $f - G$ is SGD. Moreover, due to the linearity of the strong geometric derivative (see Proposition 2.1), and because of Example 2.1, we have

$$D_1^\diamond(f - G)(x) = D_1^\diamond f(x) - D_1^\diamond G(x) = D_1^\diamond f(x) - G'(x) = D_1^\diamond f(x) - D_1^\diamond f(x) = 0$$

for all $x \in I$. Applying Theorem 3.2 for the function $f - G$, there exists $c \in \mathbb{R}$ and an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) - G(x) = c + A(x)$, for all $x \in I$. Rearranging this equation, the continuously differentiable function $g := G + c$ provides the desired decomposition $f(x) = g(x) + A(x)$ on the interval I .

The reverse implication follows immediately from Examples 2.1 and 2.2 and the fact that the sum of two SGD functions is SGD as well. $\square$

As a consequence we proceed with a characterization of continuously differentiable functions with the aid of strong geometric differentiability together with some (rather weak) regularity assumption. Similar results can be found in [1] and [4] using other types of strong derivatives.

Corollary 3.1. *The function $f : I \rightarrow \mathbb{R}$ is continuously differentiable if, and only if, f is strongly geometrically differentiable and f fulfills one of the following properties:*

- *the graph of f is not dense in $I \times \mathbb{R}$;*
- *f is bounded from above (or below) on a set of positive Lebesgue measure;*
- *f is measurable.*

Proof. The direct implication follows immediately from the continuous differentiability, in view of Example 2.1. For the reverse implication use Theorem 3.3, hence $f = g + A$ where g is continuously differentiable and A is additive. But then any of the listed additional properties ensure (see [7, Theorem 12.1.2, Theorem 9.3.1 and Theorem 9.4.3]) that A is in fact linear, so $A(x) = ax$ with some $a \in \mathbb{R}$, for all $x \in I$. Thus f is continuously differentiable. $\square$

4. Functions with increasing strong geometric derivatives

Notation 4.1. From now on we will often use a more compact notation for some specific ratios. If $f : I \rightarrow \mathbb{R}$, $h > 0$, $x \in I$ and $n \in \mathbb{N} \cup \{0\}$ such that $x, x + 2^{-n}h \in I$, let us define

$$\varrho_h^\diamond f(x, n) := 2^n \left(f \left(x + \frac{h}{2^n} \right) - f(x) \right).$$

Notation 4.2. In the next part of our work we consider functions with increasing strong geometric derivatives. More precisely, let $\mathcal{K}_+^\diamond(I)$ denote the set of functions $f : I \rightarrow \mathbb{R}$ that fulfill

$$-\infty < \overline{D}_h^\diamond f(x_1) \leq \underline{D}_h^\diamond f(x_2) < +\infty \quad (2)$$

for every $h > 0$ and for all $x_1, x_2 \in I$ such that $x_1 < x_2$.

Example 4.1. Convex functions are contained in the class $\mathcal{K}_+^\diamond(I)$. Indeed, if $f : I \rightarrow \mathbb{R}$ is convex and $y_1 < z_1 \leq y_2 < z_2$ are points in I then

$$\frac{f(z_1) - f(y_1)}{z_1 - y_1} \leq \frac{f(z_2) - f(y_2)}{z_2 - y_2}$$

(cf. [7, Theorem 7.3.1 and Theorem 7.3.2]). It is easy to check that because of this, $R^\diamond(f, x, h, \delta)$ is bounded for all $x \in I$ and $h, \delta > 0$. Moreover, if $x_1 < x_2$, $h > 0$ then each ratio in $R^\diamond(f, x_1, h, \frac{x_2 - x_1}{2})$ is not greater than those in $R^\diamond(f, x_2, h, \frac{x_2 - x_1}{2})$. Using the definition of lower and upper strong geometric derivatives we arrive to the defining condition (2) of $\mathcal{K}_+^\diamond(I)$.

Our argument in Example 2.2 shows that restrictions of (possibly discontinuous) additive mappings to I belong to the class $\mathcal{K}_+^\diamond(I)$ as well.

Proposition 4.1. Let $f \in \mathcal{K}_+^\diamond(I)$. Moreover let $x, y \in I$, $h > 0$ and $n, m \in \mathbb{N} \cup \{0\}$ such that $y + \frac{h}{2^n} \in I$ and $x < x + \frac{h}{2^m} \leq y < y + \frac{h}{2^n}$. Then

$$2^m \left(f \left(x + \frac{h}{2^m} \right) - f(x) \right) \leq 2^n \left(f \left(y + \frac{h}{2^n} \right) - f(y) \right).$$

Proof. Due to Theorem 3.1 there exist elements

$$x_1 \in \left[x, x + \frac{h}{2^{m+1}} \right], \quad x_2 \in \left[x + \frac{h}{2^{m+1}}, x + \frac{h}{2^m} \right]$$

and

$$y_1 \in \left[y, y + \frac{h}{2^{n+1}} \right], \quad y_2 \in \left[y + \frac{h}{2^{n+1}}, y + \frac{h}{2^n} \right]$$

such that $\varrho_h^\diamond f(x, m+1) \leq \overline{D}_h^\diamond f(x_1)$, $\varrho_h^\diamond f(x + \frac{h}{2^{m+1}}, m+1) \leq \overline{D}_h^\diamond f(x_2)$ and $\varrho_h^\diamond f(y, n+1) \geq \underline{D}_h^\diamond f(y_1)$, $\varrho_h^\diamond f(y + \frac{h}{2^{n+1}}, n+1) \geq \underline{D}_h^\diamond f(y_2)$. Finally, utilizing that $x_1 < y_1$, $x_2 < y_2$ and $f \in \mathcal{K}_+^\diamond(I)$, the following easy calculations show

$$\begin{aligned} \varrho_h^\diamond f(x, m) &= \frac{1}{2} \varrho_h^\diamond f(x, m+1) + \frac{1}{2} \varrho_h^\diamond f \left(x + \frac{h}{2^{m+1}}, m+1 \right) \leq \frac{1}{2} \overline{D}_h^\diamond f(x_1) + \frac{1}{2} \overline{D}_h^\diamond f(x_2) \\ &\leq \frac{1}{2} \underline{D}_h^\diamond f(y_1) + \frac{1}{2} \underline{D}_h^\diamond f(y_2) \leq \frac{1}{2} \varrho_h^\diamond f(y, n+1) + \frac{1}{2} \varrho_h^\diamond f \left(y + \frac{h}{2^{n+1}}, n+1 \right) = \varrho_h^\diamond f(y, n) \end{aligned}$$

which had to be proven. $\square$

Proposition 4.2. Let $f \in \mathcal{K}_+^\diamond(I)$, $x \in I$ and $h > 0$. Then the limits

$$\lim_{n \rightarrow \infty} 2^n \left(f(x) - f\left(x - \frac{h}{2^n}\right) \right) \quad \text{and} \quad \lim_{n \rightarrow \infty} 2^n \left(f\left(x + \frac{h}{2^n}\right) - f(x) \right)$$

exist in $\mathbb{R}$.

Remark 4.1. Using our other notation the proposition states that

$$\lim_{n \rightarrow \infty} \varrho_h^\diamond f\left(x - \frac{h}{2^n}, n\right) \in \mathbb{R} \quad \text{and} \quad \lim_{n \rightarrow \infty} \varrho_h^\diamond f(x, n) \in \mathbb{R}.$$

Of course it is necessary to remark that for finitely many indices $n \in \mathbb{N}$ the number $x - 2^{-n}h$ or $x + 2^{-n}h$ may not be in I . But I is open, so there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the points $x - 2^{-n}h$ and $x + 2^{-n}h$ are both contained in I , therefore it is reasonable to investigate the convergence of the sequences appearing in the statement.

Proof. Suppose that $n_0 \in \mathbb{N}$ is such that $x - 2^{-n_0}h, x + 2^{-n_0}h \in I$. During the whole proof let us assume that $n \geq n_0$. Then, using Proposition 4.1 we obtain

$$\begin{aligned} \varrho_h^\diamond f(x, n) &= \frac{1}{2} \varrho_h^\diamond f(x, n+1) + \frac{1}{2} \varrho_h^\diamond f\left(x + \frac{h}{2^{n+1}}, n+1\right) \\ &\geq 2 \cdot \frac{1}{2} \varrho_h^\diamond f(x, n+1) = \varrho_h^\diamond f(x, n+1). \end{aligned}$$

Hence $(\varrho_h^\diamond f(x, n))$ is a monotone decreasing sequence (from the first index n_0 for which it is defined). Similarly,

$$\begin{aligned} \varrho_h^\diamond f\left(x - \frac{h}{2^n}, n\right) &= \frac{1}{2} \varrho_h^\diamond f\left(x - \frac{h}{2^n}, n+1\right) + \frac{1}{2} \varrho_h^\diamond f\left(x - \frac{h}{2^{n+1}}, n+1\right) \\ &\leq 2 \cdot \frac{1}{2} \varrho_h^\diamond f\left(x - \frac{h}{2^{n+1}}, n+1\right) = \varrho_h^\diamond f\left(x - \frac{h}{2^{n+1}}, n+1\right). \end{aligned}$$

So $(\varrho_h^\diamond f(x - \frac{h}{2^n}, n))$ is a monotone increasing sequence (from the first index for which it is defined). Therefore the limits in the theorem surely exist in $\overline{\mathbb{R}}$. Moreover, due to Proposition 4.1, the inequalities

$$\varrho_h^\diamond f\left(x - \frac{h}{2^{n_0}}, n_0\right) \leq \varrho_h^\diamond f(x, n) \quad \text{and} \quad \varrho_h^\diamond f\left(x - \frac{h}{2^n}, n\right) \leq \varrho_h^\diamond f(x, n_0)$$

are fulfilled for all $n \geq n_0$. Hence, the monotone sequences $(\varrho_h^\diamond f(x - \frac{h}{2^n}, n))$ and $(\varrho_h^\diamond f(x, n))$ are bounded as well, so they are convergent. $\square$

Now we formulate some type of Newton–Leibniz formula for functions in $\mathcal{K}_+^\diamond(I)$.

Theorem 4.1. Let $f \in \mathcal{K}_+^\diamond(I)$ and $x \in I$, $h > 0$, $m \in \mathbb{N}$ such that $x + \frac{h}{2^m} \in I$. Then

$$\int_x^{x + \frac{h}{2^m}} \underline{D}_h^\diamond f(y) dy = h \left(f\left(x + \frac{h}{2^m}\right) - f(x) \right).$$

Proof. In the first step assume that $x, x+h \in I$ holds. From the definition of $\mathcal{K}_+^\diamond(I)$ it follows that $y \mapsto \underline{D}_h^\diamond f(y)$ is non-decreasing, so it is Riemann-integrable. Moreover, Theorem 3.1 provides that, for all $n \in \mathbb{N} \cup \{0\}$, $z \in I$ such that $z + 2^{-n}h \in I$, there exist $s_0, t_0 \in [z, z + 2^{-n}h]$ fulfilling

$$\underline{D}_h^\diamond f(s_0) \leq 2^n \left(f\left(z + \frac{h}{2^n}\right) - f(z) \right) = \varrho_h^\diamond f(z, n) \leq \overline{D}_h^\diamond f(t_0).$$

Therefore $\underline{D}_h^\diamond f(s) \leq \underline{D}_h^\diamond f(z) \leq \underline{D}_h^\diamond f(s_0) \leq \varrho_h^\diamond f(z, n)$ for all $s \leq z$. Similarly, $\varrho_h^\diamond f(z, n) \leq \overline{D}_h^\diamond f(t_0) \leq \overline{D}_h^\diamond f(z + 2^{-n}h) \leq \underline{D}_h^\diamond f(t)$ for all $t > z + 2^{-n}h$.

Furthermore, let us introduce a notation for the following real numbers and indicator functions: let $x_{n,0} := x$ and let

$$x_{n,k} := x + k \cdot \frac{h}{2^n} \quad \text{and} \quad \mathbb{I}_{n,k} := \mathbb{I}_{[x_{n,k-1}, x_{n,k}]} : \mathbb{R} \longrightarrow \{0, 1\}$$

such that

$$\mathbb{I}_{n,k}(u) = 1 \quad \text{for } u \in [x_{n,k-1}, x_{n,k}] \quad \text{and} \quad \mathbb{I}_{n,k}(u) = 0 \quad \text{otherwise} \quad (k = 1, \dots, 2^n).$$

Due to the previous observations we have

$$\underline{D}_h^\diamond f(y) \leq \sum_{k=1}^{2^n} (\mathbb{I}_{n,k}(y) \cdot \underline{D}_h^\diamond f(x_{n,k})) \leq \sum_{k=1}^{2^n-1} (\mathbb{I}_{n,k}(y) \cdot \varrho_h^\diamond f(x_{n,k}, n)) + \mathbb{I}_{n,2^n}(y) \cdot \underline{D}_h^\diamond f(x+h)$$

for all $y \in [x, x+h]$. Now let us integrate both sides from x to $x+h$.

$$\begin{aligned} \int_x^{x+h} \underline{D}_h^\diamond f(y) dy &\leq \frac{h}{2^n} \sum_{k=1}^{2^n-1} \varrho_h^\diamond f(x_{n,k}, n) + \frac{h}{2^n} \underline{D}_h^\diamond f(x+h) \\ &= h \left(f(x+h) - f\left(x + \frac{h}{2^n}\right) \right) + \frac{h}{2^n} \underline{D}_h^\diamond f(x+h) \\ &= h(f(x+h) - f(x)) + h \left(f(x) - f\left(x + \frac{h}{2^n}\right) \right) + \frac{h}{2^n} \underline{D}_h^\diamond f(x+h). \end{aligned}$$

Taking the limit $n \rightarrow \infty$, clearly $2^{-n}h \cdot \underline{D}_h^\diamond f(x+h) \rightarrow 0$, while

$$\lim_{n \rightarrow \infty} h \left(f(x) - f\left(x + \frac{h}{2^n}\right) \right) = \lim_{n \rightarrow \infty} \left(\frac{-h}{2^n} \cdot \varrho_h^\diamond f(x, n) \right) = 0 \cdot \lim_{n \rightarrow \infty} \varrho_h^\diamond f(x, n) = 0,$$

as in Proposition 4.2 we have shown that $(\varrho_h^\diamond f(x, n))$ is convergent. Eventually we have got $\int_x^{x+h} \underline{D}_h^\diamond f(y) dy \leq h(f(x+h) - f(x))$.

For the reverse inequality let us calculate as follows. For all $n \geq 2$ we have

$$\begin{aligned} \underline{D}_h^\diamond f(y) &\geq \sum_{k=1}^{2^n} (\mathbb{I}_{n,k}(y) \cdot \underline{D}_h^\diamond f(x_{n,k-1})) \geq \\ &\sum_{k=3}^{2^n} (\mathbb{I}_{n,k}(y) \cdot \varrho_h^\diamond f(x_{n,k-3}, n)) + \mathbb{I}_{n,1}(y) \cdot \underline{D}_h^\diamond f(x_{n,0}) + \mathbb{I}_{n,2}(y) \cdot \underline{D}_h^\diamond f(x_{n,1}). \end{aligned}$$

for all $y \in [x, x+h]$. Now let us integrate both sides.

$$\begin{aligned}
\int_x^{x+h} \underline{D}_h^\diamond f(y) dy &\geq \frac{h}{2^n} (\underline{D}_h^\diamond f(x_{n,1}) + \underline{D}_h^\diamond f(x_{n,0})) + \frac{h}{2^n} \sum_{k=3}^{2^n} \varrho_h^\diamond f(x_{n,k-3}, n) \\
&= \frac{h}{2^n} (\underline{D}_h^\diamond f(x_{n,1}) + \underline{D}_h^\diamond f(x_{n,0})) + h \left(f(x_{n,2^n-2}) - f(x_{n,0}) \right) \\
&\geq \frac{h}{2^n} \left(\underline{D}_h^\diamond f(x) + \underline{D}_h^\diamond f(x) \right) + h \left(f(x+h) - f(x) \right) - h \left(f(x+h) - f \left(x+h - \frac{h}{2^{n-1}} \right) \right).
\end{aligned}$$

Obviously, the first term tends to 0 as $n \rightarrow \infty$. On the other hand, for the last term we have

$$h \left(f(x+h) - f \left(x+h - \frac{h}{2^{n-1}} \right) \right) = \frac{h}{2^{n-1}} \varrho_h^\diamond f \left(x+h - \frac{h}{2^{n-1}}, n-1 \right)$$

which tends to 0, as it is a product of the zero sequence $(2^{-n+1}h)$ and the convergent sequence $(\varrho_h^\diamond f(x+h - 2^{-(n-1)}h, n-1))$, according to Proposition 4.1.

Therefore, taking the limit $n \rightarrow \infty$ on both sides of the previous inequality, we get $\int_x^{x+h} \underline{D}_h^\diamond f(y) dy \geq h(f(x+h) - f(x))$. Together with the other inequality shown earlier, we have

$$\int_x^{x+h} \underline{D}_h^\diamond f(y) dy = h(f(x+h) - f(x)).$$

for all $x \in I$, $h > 0$ such that $x+h \in I$. In order to relax the condition $x+h \in I$, for arbitrary $h > 0$ we can find $m \in \mathbb{N}$ such that $x, x+2^{-m}h \in I$, and we may apply the previous result for $2^{-m}h$ in place of h . Thus, using Proposition 2.1, we get

$$\begin{aligned}
\int_x^{x+\frac{h}{2^m}} \underline{D}_h^\diamond f(y) dy &= \int_x^{x+\frac{h}{2^m}} 2^m \underline{D}_{\frac{h}{2^m}}^\diamond f(y) dy = 2^m \int_x^{x+\frac{h}{2^m}} \underline{D}_{\frac{h}{2^m}}^\diamond f(y) dy \\
&= 2^m \frac{h}{2^m} \left(f \left(x + \frac{h}{2^m} \right) - f(x) \right) = h \left(f \left(x + \frac{h}{2^m} \right) - f(x) \right). \quad \square
\end{aligned}$$

Now we prove our main result of this section, which is the decomposition theorem of functions of $\mathcal{K}_+^\diamond(I)$.

Theorem 4.2. *The function $f : I \rightarrow \mathbb{R}$ is in $\mathcal{K}_+^\diamond(I)$ if, and only if, there exist a convex function $g : I \rightarrow \mathbb{R}$ and an additive mapping $A : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$f(x) = g(x) + A(x) \quad \text{for all } x \in I.$$

Proof. Suppose that $f \in \mathcal{K}_+^\diamond(I)$ and let us fix an arbitrary element $x_0 \in I$. Define the function $G : I \rightarrow \mathbb{R}$ as follows:

$$G(x) := \int_{x_0}^x \underline{D}_1^\diamond f(y) dy \quad (\forall x \in I).$$

Clearly, the function $\underline{D}_1^\diamond f$ is increasing (due the assumption $f \in \mathcal{K}_+^\diamond(I)$), hence we obtain that G is convex (see [12, Theorem 2.4.]), thus $G \in \mathcal{K}_+^\diamond(I)$ (see Example 4.1). Moreover, using Theorem 4.1, for any $x \in I$ and $n \in \mathbb{N} \cup \{0\}$ such that $x+2^{-n} \in I$ we have

$$\begin{aligned}
& 2^n \left((f - G) \left(x + \frac{1}{2^n} \right) - (f - G)(x) \right) \\
&= 2^n \left(f \left(x + \frac{1}{2^n} \right) - f(x) + \int_{x_0}^x \underline{D}_1^\diamond f(y) dy - \int_{x_0}^{x + \frac{1}{2^n}} \underline{D}_1^\diamond f(y) dy \right) \\
&= 2^n \left(f \left(x + \frac{1}{2^n} \right) - f(x) - \int_x^{x + \frac{1}{2^n}} \underline{D}_1^\diamond f(y) dy \right) = 2^n \cdot 0 = 0.
\end{aligned}$$

Therefore $R^\diamond(f - G, x, 1, \delta) = \{0\}$ for all $\delta > 0$, hence $D_1^\diamond(f - G)(x) = 0$ on I .

For the next step consider an arbitrary function $\varphi \in \mathcal{K}_+^\diamond(I)$ and $h > 0$. Then $\underline{D}_h^\diamond \varphi$ and $\overline{D}_h^\diamond \varphi$ are monotone increasing, therefore they are continuous on the sets $I \setminus S_1$ and $I \setminus S_2$, respectively, where S_1, S_2 are countable subsets of I . Consequently, $\underline{D}_h^\diamond \varphi$ and $\overline{D}_h^\diamond \varphi$ are both continuous on $I \setminus (S_1 \cup S_2)$ which has cardinality continuum. Let $t \in I \setminus (S_1 \cup S_2)$ be arbitrary and choose $\varepsilon > 0$ such that $[t, t + \varepsilon h] \subset I$ would hold. Then

$$\underline{D}_h^\diamond \varphi(t) \leq \overline{D}_h^\diamond \varphi(t) = \lim_{n \rightarrow \infty} \overline{D}_h^\diamond \varphi \left(t + \frac{\varepsilon h}{2n} \right) \leq \lim_{n \rightarrow \infty} \underline{D}_h^\diamond \varphi \left(t + \frac{\varepsilon h}{n} \right) = \underline{D}_h^\diamond \varphi(t),$$

which means that there is equality everywhere, so $\underline{D}_h^\diamond \varphi(t) = \overline{D}_h^\diamond \varphi(t)$. That is, $D_h^\diamond \varphi$ exists on a set with cardinality continuum.

Now apply this result for $f \in \mathcal{K}_+^\diamond(I)$ and $G \in \mathcal{K}_+^\diamond(I)$. Clearly, $f - G$ is SGD in the direction h , on a set C_h which has cardinality continuum. Specifically, C_h is nonempty, so together with the previously obtained fact that $D_1^\diamond(f - G) \equiv 0$ on I , we may apply Theorem 3.2. Hence there exists $c \in \mathbb{R}$ and an additive function $A : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) - G(x) = c + A(x)$, for all $x \in I$. Rearranging this equation, the convex function $g := G + c$ provides the desired decomposition $f(x) = g(x) + A(x)$ on the interval I .

Finally, the reverse implication follows from Examples 2.2 and 4.1. $\square$

5. A local characterization of affine functions

As an application of our decomposition theorem for SGD functions, we establish a characterization of affine functions on intervals with the aid of a local version of an approximate Pexider equation. Such investigations are motivated by the extension theorem for the Pexider equation published by Radó and Baker [10].

Throughout this section, let $1 < p \in \mathbb{R}$.

Theorem 5.1. *Let $f : I \rightarrow \mathbb{R}$ and let us assume that there exist $\varepsilon : I \rightarrow [0, +\infty[$ and $\delta : I \rightarrow]0, +\infty[$ such that, for every $x \in I$, $B(x, 2\delta(x)) \subset I$ and there exists $\phi_x : B(0, \delta(x)) \rightarrow \mathbb{R}$ fulfilling*

$$|f(y + u) - f(y) - \phi_x(u)| \leq \varepsilon(x)|u|^p \quad (3)$$

for all $y \in B(x, \delta(x))$ and $u \in B(0, \delta(x))$. Then f is affine, i.e., there exist a constant $b \in \mathbb{R}$ and an additive mapping $A : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = A(x) + b \quad \text{for every } x \in I. \quad (4)$$

Proof. For a fixed $x \in I$ and for arbitrary $u \in B(0, \delta(x))$ and $y \in B(x, \delta(x)/2)$, we have

$$\left| \phi_x(u) - 2\phi_x\left(\frac{u}{2}\right) \right|$$

$$\begin{aligned}
&\leq |\phi_x(u) + f(y) - f(y+u)| + \left| f\left(y + \frac{u}{2}\right) - f(y) - \phi_x\left(\frac{u}{2}\right) \right| + \\
&\quad \left| f(y+u) - f\left(y + \frac{u}{2}\right) - \phi_x\left(\frac{u}{2}\right) \right| \\
&\leq \varepsilon(x) (1 + 2^{1-p}) |u|^p,
\end{aligned}$$

hence, for every $m, n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&|2^m \phi_x(2^{-m}u) - 2^{m+n} \phi_x(2^{-(m+n)}u)| \\
&\leq \sum_{j=0}^{n-1} |2^{m+j} \phi_x(2^{-(m+j)}u) - 2^{m+j+1} \phi_x(2^{-(m+j+1)}u)| \\
&\leq \sum_{j=0}^{n-1} (2^{1-p})^{m+j} \varepsilon(x) (1 + 2^{1-p}) |u|^p \\
&\leq \kappa_p \varepsilon(x) 2^{(1-p)m} |u|^p,
\end{aligned}$$

where

$$\kappa_p = \frac{2^{p-1} + 1}{2^{p-1} - 1}.$$

Thus $(2^n \phi_x(2^{-n}u))$ is a Cauchy sequence. Then there exists

$$\psi_x(u) = \lim_{n \rightarrow \infty} 2^n \phi_x(2^{-n}u).$$

Moreover,

$$|2^m \phi_x(2^{-m}u) - \psi_x(u)| \leq \kappa_p \varepsilon(x) 2^{(1-p)m} |u|^p.$$

For $y \in B(x, \delta(x))$, $u \in \mathbb{R}$, and large $n \in \mathbb{N}$, we have

$$\begin{aligned}
&|2^n (f(y + 2^{-n}u) - f(y)) - \psi_x(u)| \\
&\leq 2^n |f(y + 2^{-n}u) - f(y) - \phi_x(2^{-n}u)| + |2^n \phi_x(2^{-n}u) - \psi_x(u)| \\
&\leq \frac{2^p}{2^{p-1} - 1} \varepsilon(x) 2^{(1-p)n} |u|^p.
\end{aligned}$$

Hence, f is strongly geometrically differentiable. Therefore, due to Theorem 3.3, there exist a continuously differentiable function $g : I \rightarrow \mathbb{R}$ and an additive mapping $A_0 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = g(x) + A_0(x) \quad \text{for every } x \in I. \quad (5)$$

Substituting (5) into (3) we obtain

$$|g(y+u) - g(y) + A_0(u) - \phi_x(u)| \leq \varepsilon(x) |u|^p \quad (6)$$

for every $x \in I$, $y \in B(x, \delta(x))$, and $u \in B(0, \delta(x))$. Replacing u with $2^{-n}u$ and multiplying the inequality by 2^n , we get

$$|2^n (g(y + 2^{-n}u) - g(y)) + A_0(u) - 2^n \phi_x(2^{-n}u)| \leq \varepsilon(x) 2^{(1-p)n} |u|^p.$$

Letting n tend to infinity, we obtain

$$g'(y)u + A_0(u) - \psi_x(u) = 0.$$

Taking $u \neq 0$, this yields that g' is constant on $B(x, \delta(x))$. Hence $g''(x)$ exists and $g''(x) = 0$. As x was arbitrary, we have $g'' = 0$. Therefore,

$$g(x) = ax + b \quad (x \in I)$$

for some $a, b \in \mathbb{R}$. Thus we have (4) with

$$A(x) = A_0(x) + ax. \quad \square$$

6. Approximately Wright-convex functions

Comparing Theorem 4.2 with the celebrated characterization of Wright-convex functions by Ch Tat Ng [8], it is clear that the class $\mathcal{K}_+^{\circ}(I)$ coincides with the family of Wright convex functions $f : I \rightarrow \mathbb{R}$, defined (in [13]) by Wright's inequality

$$f(\lambda x + (1 - \lambda)y) + f((1 - \lambda)x + \lambda y) \leq f(x) + f(y)$$

for every $x, y \in I$ and $\lambda \in [0, 1]$. It is therefore a natural idea that our results can be used to provide another proof for Ng's decomposition theorem in case of real variables. Moreover, motivated by analogous results for approximately convex functions with small (and thus finally vanishing) error terms ([11], [2]), we can investigate approximately Wright-convex functions with appropriately small errors.

In this section, let $0 \leq \varepsilon \in \mathbb{R}$ and $1 < p \in \mathbb{R}$.

Lemma 6.1. *Let us suppose that $f : I \rightarrow \mathbb{R}$ fulfills the inequality*

$$f(\lambda x + (1 - \lambda)y) + f((1 - \lambda)x + \lambda y) \leq f(x) + f(y) + \varepsilon (\lambda(1 - \lambda)|x - y|)^p \quad (7)$$

for every $x, y \in I$ and $\lambda \in [0, 1]$. Then

$$f(x_1 + w) - f(x_1) \leq f(x_2 + w) - f(x_2) + \varepsilon \left(\frac{w(x_2 - x_1)}{w + x_2 - x_1} \right)^p \quad (8)$$

holds for every $x_1, x_2 \in I$, $0 < w \in \mathbb{R}$ such that $x_1 \leq x_2$ and $x_2 + w \in I$. Moreover, we have

$$2^n \left(f\left(x + \frac{u}{2^n}\right) - f(x) \right) \leq f(x + u) - f(x) + \frac{1}{2^{2p}(1 - 2^{1-p})} \left(1 - 2^{(1-p)n} \right) \varepsilon |u|^p \quad (9)$$

for every $x \in I$ and $u \in \mathbb{R}$ such that $x + u \in I$.

Proof. One obtains (8) from (7) by substituting $x = x_1$, $y = x_2 + w$ and

$$\lambda = \frac{w}{w + x_2 - x_1}.$$

From (8) we can derive

$$2(f(x + v) - f(x)) \leq f(x + 2v) - f(x) + \varepsilon \left(\frac{|v|}{2} \right)^p \quad (10)$$

for every $x \in I$ and $v \in \mathbb{R}$ such that $x + 2v \in I$. Namely, if $v > 0$, we obtain (10) from (8) by substituting $x_1 = x$, $x_2 = x + v$, $w = v$, and adding $f(x + v) - f(x)$ to both sides of the inequality. For $v = 0$, both sides of the inequality (10) are equal to 0. If $v < 0$, we should apply (8) with $x_1 = x + 2v$, $x_2 = x + v$, $w = -v$, and add $f(x + 2v) + f(x + v) - 2f(x)$ to both sides of the inequality in order to obtain (10).

Now we obtain (9) from (10) by induction on n . The basic case $n = 1$ of (9) follows from (10) immediately with $v = u/2$.

Assuming that $n \in \mathbb{N}$ such that (9) is valid, we can apply (10) for $v = \frac{u}{2^{n+1}}$ as follows:

$$\begin{aligned} & 2^{n+1} \left(f \left(x + \frac{u}{2^{n+1}} \right) - f(x) \right) \\ & \leq 2^n \left(f \left(x + \frac{u}{2^n} \right) - f(x) + \varepsilon \left(\frac{1}{2} \cdot \left| \frac{u}{2^{n+1}} \right| \right)^p \right) \\ & = 2^n \left(f \left(x + \frac{u}{2^n} \right) - f(x) \right) + \frac{2^n}{2^{(n+2)p}} \varepsilon |u|^p \\ & \leq f(x + u) - f(x) + \frac{1}{2^{2p}(1 - 2^{1-p})} \left(1 - 2^{(1-p)n} \right) \varepsilon |u|^p + \frac{1}{2^{2p}} 2^{(1-p)n} \varepsilon |u|^p \\ & = f(x + u) - f(x) + \frac{1}{2^{2p}(1 - 2^{1-p})} \left(1 - 2^{(1-p)(n+1)} \right) \varepsilon |u|^p \end{aligned}$$

Thus (9) remains valid if n is replaced with $n + 1$, which completes the proof. $\square$

Lemma 6.2. *Let us suppose that $f : I \rightarrow \mathbb{R}$ fulfills (7) for all $x, y \in I$ and $\lambda \in [0, 1]$. Let $h > 0$, $\delta > 0$, $x_1, x_2 \in I$ such that $x_1 + \delta < x_2 - \delta$, and let $t_j \in R^\diamond(f, x_j, h, \delta)$ ($j = 1, 2$). Then*

$$t_1 \leq t_2 + \varepsilon \cdot c_{p,h} \cdot \delta^{p-1}, \quad (11)$$

where

$$c_{p,h} = h \cdot 2^{p-1} \left(1 + \frac{1}{2^{2p}(1 - 2^{1-p})} \right).$$

Proof. Let $m, n \in \mathbb{N}$ and

$$y_1, y_1 + \frac{h}{2^n} \in B(x_1, \delta) \cap I, \quad y_2, y_2 + \frac{h}{2^m} \in B(x_2, \delta) \cap I$$

such that

$$t_1 = 2^n \left(f \left(y_1 + \frac{h}{2^n} \right) - f(y_1) \right) \quad \text{and} \quad t_2 = 2^m \left(f \left(y_2 + \frac{h}{2^m} \right) - f(y_2) \right).$$

Our assumptions imply $\frac{h}{2^n} < 2\delta$ and $\frac{h}{2^m} < 2\delta$.

If $m = n$, we may apply (8) with $w = \frac{h}{2^n}$ in order to obtain

$$\begin{aligned} t_1 &= 2^n \left(f \left(y_1 + \frac{h}{2^n} \right) - f(y_1) \right) \\ &\leq 2^n \left(f \left(y_2 + \frac{h}{2^n} \right) - f(y_2) \right) + 2^n \varepsilon \left(\frac{\frac{h}{2^n}(y_2 - y_1)}{\frac{h}{2^n} + y_2 - y_1} \right)^p \\ &\leq t_2 + \varepsilon h \left(\frac{h}{2^n} \right)^{p-1} \leq t_2 + \varepsilon h 2^{p-1} \delta^{p-1}. \end{aligned}$$

In the remaining cases we shall combine this estimation with an appropriate application of the inequality (9).

If $n > m$, let $k = n - m$ and let us apply (9) for $x = y_1$ and $u = \frac{h}{2^m}$ as follows:

$$\begin{aligned} t_1 &= 2^n \left(f \left(y_1 + \frac{h}{2^n} \right) - f(y_1) \right) = 2^m 2^k \left(f \left(y_1 + \frac{h}{2^m 2^k} \right) - f(y_1) \right) \\ &\leq 2^m \left(f \left(y_1 + \frac{h}{2^m} \right) - f(y_1) + \frac{1 - 2^{(1-p)k}}{2^{2p}(1 - 2^{1-p})} \varepsilon \left| \frac{h}{2^m} \right|^p \right) \\ &\leq 2^m \left(f \left(y_1 + \frac{h}{2^m} \right) - f(y_1) \right) + \varepsilon \frac{1}{2^{2p}(1 - 2^{1-p})} h \left(\frac{h}{2^m} \right)^{p-1} \\ &\leq 2^m \left(f \left(y_2 + \frac{h}{2^m} \right) - f(y_2) \right) + \varepsilon h \left(\frac{h}{2^m} \right)^{p-1} + \varepsilon \frac{1}{2^{2p}(1 - 2^{1-p})} h \left(\frac{h}{2^m} \right)^{p-1} \\ &\leq t_2 + \varepsilon h 2^{p-1} \left(1 + \frac{1}{2^{2p}(1 - 2^{1-p})} \right) \delta^{p-1}. \end{aligned}$$

If $n < m$, let $k = m - n$ and let us apply (9) for $x = y_1 + \frac{h}{2^n}$ and $u = -\frac{h}{2^n}$ in the following way:

$$\begin{aligned} &2^n 2^k \left(f \left(y_1 + \frac{h}{2^n} - \frac{h}{2^n 2^k} \right) - f \left(y_1 + \frac{h}{2^n} \right) \right) \\ &\leq 2^n \left(f(y_1) - f \left(y_1 + \frac{h}{2^n} \right) + \frac{1 - 2^{(1-p)k}}{2^{2p}(1 - 2^{1-p})} \varepsilon \left| -\frac{h}{2^n} \right|^p \right) \end{aligned}$$

hence we have

$$\begin{aligned} t_1 &= 2^n \left(f \left(y_1 + \frac{h}{2^n} \right) - f(y_1) \right) \\ &\leq 2^{n+k} \left(f \left(y_1 + \frac{h}{2^n} \right) - f \left(y_1 + \frac{h}{2^n} - \frac{h}{2^n 2^k} \right) \right) + \frac{1}{2^{2p}(1 - 2^{1-p})} \varepsilon h \left(\frac{h}{2^n} \right)^{p-1} \\ &\leq 2^m \left(f \left(y_2 + \frac{h}{2^m} \right) - f(y_2) \right) + \varepsilon h \left(\frac{h}{2^n} \right)^{p-1} + \varepsilon \frac{1}{2^{2p}(1 - 2^{1-p})} h \left(\frac{h}{2^n} \right)^{p-1} \\ &\leq t_2 + \varepsilon h 2^{p-1} \left(1 + \frac{1}{2^{2p}(1 - 2^{1-p})} \right) \delta^{p-1}. \quad \square \end{aligned}$$

Theorem 6.1. If $f : I \rightarrow \mathbb{R}$ satisfies the inequality (7) for every $x, y \in I$ and $\lambda \in [0, 1]$, then $f \in \mathcal{K}_+^\diamond(I)$.

Proof. Let $h > 0$ and $x_1, x_2 \in I$ such that $x_1 < x_2$. Moreover, let $\alpha > 0$. Since $\lim_{\delta \rightarrow 0} \delta^{p-1} = 0$, there exists $\delta_0 > 0$ such that, for every $0 < \delta \leq \delta_0$, we have

$$x_1 + \delta < x_2 - \delta \quad \text{and} \quad \varepsilon \cdot c_{p,h} \cdot \delta^{p-1} < \alpha.$$

Then, due to Lemma 6.2, for every $0 < \delta \leq \delta_0$ and $t_j \in R^\diamond(f, x_j, h, \delta)$ ($j = 1, 2$), we have

$$t_1 \leq t_2 + \alpha.$$

Hence $R^\diamond(f, x_1, h, \delta)$ is bounded from above, $R^\diamond(f, x_2, h, \delta)$ is bounded from below,

$$\overline{D}_h^\diamond f(x_1) < +\infty, \quad -\infty < \underline{D}_h^\diamond f(x_2) \quad \text{and} \quad \overline{D}_h^\diamond f(x_1) \leq \underline{D}_h^\diamond f(x_2) + \alpha.$$

Since α is an arbitrary positive number, this inequality implies $\overline{D}_h^\diamond f(x_1) \leq \underline{D}_h^\diamond f(x_2)$ as well. Applying such an argument involving $x_0, x_3 \in I$ such that $x_0 < x_1$ and $x_2 < x_3$, we obtain that $\overline{D}_h^\diamond f(x_1)$ and $\underline{D}_h^\diamond f(x_2)$ are real numbers. $\square$

Corollary 6.1. *If $f : I \rightarrow \mathbb{R}$ satisfies the inequality (7) for every $x, y \in I$ and $\lambda \in [0, 1]$, then f is Wright-convex (i.e., f satisfies the inequality (7) with $\varepsilon = 0$ as well).*

A standard argument provides the validity of the localization principle for the family $\mathcal{K}_+^\diamond(I)$. We have proved that members of this family are precisely the Wright-convex functions on I . Hence Wright-convexity fulfills the localization principle on I . Namely, if $f : I \rightarrow \mathbb{R}$ such that f is locally Wright-convex, then f is globally Wright-convex.

Declaration of competing interest

None.

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