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ON THE HOLONOMY OF FINSLER MANIFOLDS

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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.

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Hereby I confirm that Balázs Attila Hubicska candidate conducted his studies with my supervision within the Differential geometry and its applications Doctoral Program of the Doctoral School of Mathematical and Computational Sciences between 2016 and 2021. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.
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I support the acceptance of the thesis.

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signature of the supervisor

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Contents

Introduction

The main topic of this thesis is the study of some specific aspects of Finsler spaces.

Finsler geometry is a relatively young area of geometric studies compared to its foundations in differential geometry, not to mention the several millenia-old urge of humanity to understand the sprawling network of shapes, forces and interactions of the world around us. Although one can not pinpoint the specific birth of differential geometry as some concepts appeared scattered over the years, it was the discovery of modern calculus in the 17th century which gave momentum and renewed interest to the study of curves and surfaces. Equipped with his "compass", René Descartes constructed some very complicated curves, such as the Folium of Descartes, whose geometric description was quite challenging at that time. Fermat however found a way to determine the minima, maxima and tangent of such curves which motivated greatly the early foundations of calculus. The field did not loose its geometrical roots, Isaac Newton and Gottfried Leibniz described the curvature of plane curves, then Leonard Euler extended the work for space curves in the 18th century.
The actual birth of differential geometry is usually attributed to the work of Carl Friedrich Gauss, who among many other topics was interested in measuring distances on curved surfaces. One of his most important ideas was that if one parametrizes the surface with two parameters, then the length of a line element can be expressed in terms of an inner product. With this construction, Gauss was able to describe the geodesics (shortest path between two points of a surface), the curvature, and proved one of the most beautiful and important result of differential geometry: Gauss's Theorema Egregium, which states that the curvature of a surface is an intrinsic quantity.

The modern form of differential geometry started with Bernhard Riemann, a student of Gauss. Riemann in his famous Habilitationsvortrag [40] introduced what we now call an n-dimensional Riemannian manifold and its curvature tensor. On a Riemannian manifold the central object is the Riemann metric, which is a positive-definite inner product on the tangent space at each point of the manifold. Interestingly, the idea of more general metrics which does not come from an inner product also appeared in his lecture, but Riemann thought they does not require essentially new principles (which is more or less holds true even now) and that these more general metrics lack nice geometric interpretations. The introduction of Finsler metrics is however a very natural step in the study of geometric spaces: here the components of the metric tensor can depend not only on the position but also on the direction. There are many interesting and natural examples in the literature. For example if one wants to cross a mountain range to get from one valley to another one can find the shortest path using classical Riemannian geometry, but it does not take the effort to climb an inclination into account. Riemannian geometry cannot model that it is easier to walk in a direction than in the opposite one, because the norm of a Riemannian metric is always symmetric. This means that an infinitesimal displacement in a direction is attributed the same length as the displacement in the opposite direction hence the distance from a point A to B is the same as from B to A. Finsler geometry can overcome this limitation, by replacing 'length' of a vector by 'effort' that is takes to travel along it. As a consequence Finsler geometry is useful in studying asymmetric problems which makes it an important tool in different areas of natural sciences. For a precise definition we refer to Section 1.2.

At the beginning of the XXth century, the intensive study of Finsler metrics was motivated by the optimal transport theory. A group of mathematicians led by C. Caratheodory aimed to adapt mathematical tools which were effective in Riemannian geometry (such as affine connections, Jacobi vector fields, sectional curvature) for a more general situation. P. Finsler was a student of Caratheodory and his dissertation [12] is one of the important steps on this way. Since then, the theory evolved in a great amount due to the works of several outstanding mathematicians (L. Berwald, H. Busemann, É. Cartan, S.S. Chern, H. Rund), but it is still far from being complete. In recent years more and more focus trends to the applications of Finsler geometry in several fields of natural sciences. From general relativity through wildfire spread and seismic ray modeling to quantum mechanics, a lot of different areas take advantage of the tools of Finsler geometry [31, 45]. The investigations of B. Russel and S. Stepney in [41, 42] opened a new way for the geometric investigations of several interesting phenomena in Quantum Information Processing (QIP) by the application of D. Bao, C. Robles and Z. Shen's theorem on the one-to-one correspondence between the solutions of the Zermelo navigational problem and Randers metrics [3].

One aspect of the geometric properties still to discover and describe is the holonomy structure, which will be the main focus of our investigations in this thesis. The holonomy group is a very natural geometric object. In a broader sense it is a geometric concept which refers to the property that if we transport an object along a closed path it may not return to its original state. This phenomena occurs in different areas of physics such as a theory what before the middle of the 20th century was not considered to be geometric in nature: quantum mechanics. The first key concept in understanding holonomy is how we transport objects: by parallelism. For an Euclidean space it is quite clear when we consider two vectors parallel since the tangent spaces at different points are naturally identified. In the Riemannian case to define parallelism we need a connection ∇ and there is a canonical choice for it: the Levi-Civita connection. This allows us to define the parallelism as follows: let $\gamma : [0,1] \to M$ be a curve joining the points $p = c(0)$ and $q = c(1)$. The vector field X along γ is called parallel, if $\nabla_{\dot{\gamma}} X = 0$. Then the parallel translation or transport

$$P_\gamma : T_pM \to T_qM \tag{1}$$

along γ can be introduced as follows: for any $v \in T_pM$ let X be the parallel vector field along γ for which $X(0) = v$. Then $P_\gamma(v) = w$ where $w = X(1)$.

The parallel translation (1) is an isomorphism between the tangent spaces T_pM and T_qM. This mapping however can have a dependence on the curve, and there is no a priori reason for parallel translations along different curves to coincide. This is where our main object of investigations, the holonomy group appears describing the parallel translation's dependence on the choice of curves. More precisely, the holonomy at a point $p \in M$ is the group generated by the automorphisms $P_c : T_pM \to T_pM$, where the curves c are closed piece-wise smooth curves starting and ending at the point p.

In the Riemannian case, the connection between the holonomy group and the geometry of the manifold was thoroughly investigated in the last century. In 1952 A. Borel and A. Lichnerowicz proved that the restricted holonomy group of a simply connected n-dimensional Riemannian manifold is a closed subgroup of the orthogonal group [7]. W. Ambrose and M. Singer described the relationship between the holonomy and the curvature [1]. Few years later, M. Berger gave the list of all possible holonomy groups of Riemannian manifolds [4]. Since then, all of these groups were shown to appear as holonomy groups of some Riemannian manifolds and therefore the complete classification of Riemannian holonomy groups is known.

In the Finslerian case, the holonomy structure can be much more complex. The fundamental differences between the holonomy properties of Riemannian and Finslerian manifolds come from the fact that in the generic case the canonical connection of a Finsler manifold is neither linear nor metrical. Indeed, in the case of Finsler manifolds the parallel translation is only 1-homogeneous, and instead of preserving the metric tensor it is preserving only the norm function. The homogeneity and the norm preserving properties allow us to consider the parallel translations as maps between the indicatrices and therefore the holonomy group as a subgroup of the diffeomorphism group of the indicatrix. Contrary to the Riemannian case, only few results are known on Finsler holonomy: for Berwald manifolds, Landsberg manifolds, and Finsler manifolds with very special curvature properties [25, 49, 37].

In this thesis we present further results on the holonomy theory of Finsler manifolds. In the preliminary Chapter 1 we will recall the necessary tools for our work: Section 1.1 introduces Spray manifolds, in Section 1.2 we recall the basic objects of Finsler geometry and in Section 1.3 we give a more detailed introduction to parallel translation and holonomy of Finsler manifolds.

Chapter 2 contains the first new results of the thesis. As a first step we introduce the tangent algebra $\mathcal{T}_o\mathcal{G}$ of a diffeomorphism group $\mathcal{G}$ of a compact manifold M and investigate its properties. We show that $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of $\mathfrak{X}(M)$, the Lie algebra of the smooth vector fields on M, and its exponential image $\exp(\mathcal{T}_o\mathcal{G})$ is contained by the topological closure of $\mathcal{G}$. Then in Section 2.2 and 2.3 we show that two very important objects, the infinitesimal holonomy algebra and the curvature algebra at a point are tangent to the holonomy group. In the later chapters we

will use these results to gain new and meaningful information about the holonomy groups of Finsler manifolds.
In Chapter 3 we show new classes of Finsler manifolds with infinite dimensional holonomy group and in some specific case we can describe the structure of the group as well. Section 3.1 investigates the holonomy group of a class of Finsler metrics naturally appearing in quantum information processing. Our results clearly demonstrate the deep difference between the Riemannian and Finslerian settings: in the Riemannian case, the holonomy group is necessarily a finite-dimensional Lie group. However, using the Finslerian model in the presence of a magnetic field, this is no longer true: the holonomy group can be a much larger, infinite dimensional group. In Section 3.2 we investigate the holonomy structure of projectively flat Randers metrics of constant flag curvature. From [38] it is already known that projectively flat non-Riemannian Finsler metrics of nonzero constant flag curvature have infinite dimensional holonomy group. Our results show more in the two dimensional Randers case: the holonomy group is maximal, and its topologial closure is the orientation preserving diffeomorphism group of the unit sphere.
Chapter 4 contains a general result about Finsler structures on a given manifold M. We show that in a sense the infinite dimensional case is the most common: In the set $\mathcal{F}$ of C^∞-smooth Finsler metrics on a manifold M of dimension $n \geq 2$, there exists a subset $\widetilde{\mathcal{F}}$ of Finsler metrics with infinite dimensional holonomy group, which is open and everywhere dense in the C^m topology where $m \geq 8$.

Chapter 1

Basic concepts and tools

In this chapter the basic notions and concepts of spray geometry and Finsler geometry are introduced which are necessary to understand the later chapters of the thesis. A basic knowledge of smooth manifolds as well as Lie groups and Lie algebras is also needed, for a comprehensive introduction to these topics we refer to [26] and [17].

1.1 Spray manifolds

Throughout the thesis, M denotes a C^∞-smooth n-dimensional manifold (unless stated otherwise), $\mathfrak{X}(M)$ is the Lie algebra of C^∞ vector fields and $\mathcal{D}iff^\infty(M)$ is the group of C^∞ diffeomorphisms of M. We will denote by TM the tangent manifold and by $\widehat{T}M = TM \setminus \{0\}$ the slit tangent manifold. Local coordinate charts (U, x^i) on M induce local coordinate charts $(\pi^{-1}(U), (x^i, y^i))$ on TM, where $\pi : TM \to M$ is the canonical projection. The second tangent bundle is denoted by (TTM, τ, TM).

A *spray* over a manifold M is a mapping $S : TM \to TTM$ satisfying the following conditions:

1. S is vector field on TM and it is a section of (TTM, π_*, TM), i.e., $\pi_* \circ S = 1_{TM}$,

2. it is smooth on the slit tangent bundle $\widehat{T}M$,

3. positive-homogeneous of degree 2.

A *spray manifold* then is a manifold M paired with a spray S. We denote spray manifolds as (M, S).

In a local coordinate system, a spray can be written in the form:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}, \tag{1.1}$$

where the functions G^i, called the *spray coefficients*, are positive-homogeneous functions of degree 2 in the y variable.

A curve $c = c(t)$ in M is called a *geodesic* of the spray S, if $\dot{c}$ is an integral curve of S, that is if it satisfies

$$\ddot{c} = S_{\dot{c}}. \tag{1.2}$$

If $\gamma(t)$ is an integral curve of S then its coordinate functions $(x^i(t), y^i(t))$ satisfy the following system of differential equations:

$$\dot{x}^i(t) = y^i(t), \tag{1.3}$$
$$\dot{y}^i(t) + 2G^i(x(t), y(t)) = 0. \tag{1.4}$$

If we consider the projection $c(t) = \pi(\gamma(t))$ then the coordinates $(c^i(t))$ of $c(t)$ satisfy

$$\ddot{c}^i(t) + 2G^i\left(c(t), \dot{c}(t)\right) = 0. \tag{1.5}$$

The canonical lift of a curve $c(t)$ is a tangent vector field, that is a curve in the tangent space defined by

$$\dot{c}(t) = \dot{c}^i(t)\frac{\partial}{\partial x^i}|_{c(t)}. \tag{1.6}$$

It is easy to show that if a curve satisfies (1.5) then its canonical lift satisfies (1.2), that is, the canonical lift of c is an integral curve of S.

Horizontal and vertical distributions

Let M be a manifold of dimension $n \in \mathbb{N}$. The *vertical distribution* on TM is defined by $\mathcal{V}TM \subset TTM$ where by $\mathcal{V}_yTM = Ker\pi_{*,y}$. In a standard local coordinate system (x^i, y^i) the vertical distribution $\mathcal{V}_yTM$ is generated by the vector fields

$$\mathcal{V}TM := span\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}.$$

Let (M, S) be a spray manifold, where the local expression of S is given by (1.1). The *horizontal distribution* denoted by $\mathcal{H}TM \subset TTM$ is the image of the horizontal lift defined in the local basis as

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - G_i^k(x,y)\frac{\partial}{\partial y^k}, \qquad (1.7)$$

where $y \in T_xM$ and $G_j^i = \frac{\partial G^i}{\partial y^j}$. The horizontal distribution is complementary to the vertical distribution, we have the decomposition

$$T_yTM = \mathcal{H}_yTM \oplus \mathcal{V}_yTM$$

with canonical projectors $h\colon TTM \to \mathcal{H}$ and $v\colon TTM \to \mathcal{V}$.

If we consider the vertical distribution over the slit tangent bundle $\widehat{T}M$, denote it by $(\mathcal{V}\widehat{T}M, \tau, \widehat{T}M)$ and the pull-back of $(\widehat{T}M, \pi, M)$ via $\pi : TM \to M$ by $(\pi^*TM, \bar{\pi}, \widehat{T}M)$ we have the canonical bundle isomorphism

$$(x, y, \xi^i\frac{\partial}{\partial y^i}) \mapsto (x, y, \xi^i\frac{\partial}{\partial x^i} :)\mathcal{V}\widehat{T}M \to \pi^*TM. \qquad (1.8)$$

This isomorphism will be used in the following for the identification of the corresponding bundles.

Covariant derivative

The *horizontal Berwald covariant derivative* of a vertical vector field ξ with respect to a vector field $X \in \mathfrak{X}(M)$ is defined by

$$\nabla_X\xi = [X^h, \xi]. \qquad (1.9)$$

In local coordinates, if $\xi = \xi^i(x,y)\frac{\partial}{\partial y_i}$ and $X(x) = X^i(x)\frac{\partial}{\partial x_i}$, then

$$\nabla_X\xi = \left(\frac{\partial \xi^i}{\partial x_j} - G_j^k\frac{\partial \xi^i}{\partial y_k} + \frac{\partial G_j^i}{\partial y_k}\xi^k\right)X^j\frac{\partial}{\partial y_i}. \qquad (1.10)$$

By defining the horizontal covariant derivative of a smooth function $\phi\colon \widehat{T}M \to \mathbb{R}$ as

$$\nabla_X \phi = \left(\frac{\partial \phi}{\partial x^j} - G_j^k(x,y) \frac{\partial \phi(x,y)}{\partial y^k} \right) X^j, \tag{1.11}$$

we can extend (1.10) to sections of the tensor bundle over $(\pi^* TM, \bar{\pi}, \widehat{T}M)$, using the canonical bundle isomorphism (1.8).

Curvature

The horizontal distribution $\mathcal{H}TM$ is, in general, non-integrable. The obstruction to its integrability is given by the *curvature tensor*

$$R_{(x,y)}(X,Y) = v[hX, hY] \tag{1.12}$$

which is the Nijenhuis torsion of the horizontal projector [14]: the horizontal distribution is integrable if and only if the curvature is identically zero. The *curvature tensor* field has the expression

$$R_{(x,y)} = R^i_{jk}(x,y) dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^i} \tag{1.13}$$

where the tensor components are

$$R^i_{jk} = \frac{\partial G^i_j}{\partial x^k} - \frac{\partial G^i_k}{\partial x^j} + G^m_j G^i_{km} - G^m_k G^i_{jm}. \tag{1.14}$$

1.1.1 Definition. A vector field $\xi \in \mathfrak{X}(TM)$ is called a *curvature vector field* if it is in the image of the curvature tensor, that is $\xi = R(X^h, Y^h)$ for some $X, Y \in \mathfrak{X}(M)$.

1.2 Finsler manifolds

Let M be a manifold and $\mathcal{F}\colon TM \to \mathbb{R}_+$ a function with the following properties:

1. $\mathcal{F}$ is C^1 on TM and C^∞ on the slit tangent bundle $\widehat{T}M$,

2. for any point $x \in M$ the restriction $\mathcal{F}_x := \mathcal{F}|_{T_xM}$ is positively homogeneous of degree one, i.e.,

$$\mathcal{F}_x(\lambda y) = \lambda \mathcal{F}_x(y), \quad \lambda > 0, y \in \widehat{T}M \tag{1.15}$$

3. for any point $x \in M$ and for any $y \in \widehat{T}M$, the fundamental form

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2 \mathcal{F}_x^2(y+su+tv)}{\partial s\,\partial t}\bigg|_{t=s=0}, \quad u,v \in T_xM,$$

is nondegenerate on T_xM where $\mathcal{F}_x := \mathcal{F}|_{T_xM}$ denotes the restriction of the Finsler function to the tangent space above the point $x \in M$.

Such a function $\mathcal{F}$ is called a *Finsler function* and a pair $(M, \mathcal{F})$ where M is a manifold is called a *Finsler manifold*. The *indicatrix* $\mathcal{I}_xM$ at $x \in M$ is a hypersurface of T_xM defined by

$$\mathcal{I}_xM = \{\, y \in T_xM : \mathcal{F}(y) = 1 \,\}. \tag{1.16}$$

Let $(M, \mathcal{F})$ be a Finsler manifold. There are several ways to define a connection on a Finsler manifold by a choice of the horizontal distribution. Although the Cartan connection, the Chern-Rund connection and the Hashiguchi connection are all arising in a natural manner, in the thesis we will use the canonical non-linear connection.

For a curve $c : [a,b] \to M$ on our manifold, define the following functional

$$\mathcal{L}(c) := \int_a^b \mathcal{E}\left(c(t), \dot{c}(t)\right) dt,$$

where $\mathcal{E} = \frac{1}{2}\mathcal{F}^2$ is the energy function. Let us assume that a curve c is an extremal of the associated variational problem. Then by standard methods we get the following system of Euler-Lagrange equations:

$$\frac{\partial \mathcal{E}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{E}}{\partial y^i} = 0, \qquad i = 1, \dots n$$

It is easy to see that these equations take the form

$$\frac{d^2 c^i}{dt^2}(t) + 2G^i(c(t), \dot{c}(t)) = 0$$

where the coefficients $G^i = G^i(x, y)$ are determined by the formula

$$G^i = \frac{1}{4} g^{il} \Big(2 \frac{\partial g_{jl}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l} \Big) \, y_j y_k,$$

where $i = 1, \dots, n$. These functions are called the geodesic coefficients, and with their help we can construct a vector field on TM as

$$S := y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

for which the coefficients G^i are positively homogeneous of degree two by the homogeneity of $\mathcal{F}$. Then this vector field S is a spray, called the *geodesic spray* of $\mathcal{F}$ or the *Finsler spray* induced by $\mathcal{F}$. In this thesis we use this spray and its associated Ehresmann connection, called the *Berwald connection*. The geodesics, covariant derivation and other geometric objects of the Finsler manifold can be obtained by considering the corresponding constructions of its associated spray manifold.

1.2.1 Definition. A vector field $\xi_x \in \mathfrak{X}(\mathcal{I}_x)$ on the indicatrix $\mathcal{I}_x \subset T_x M$ is called a curvature vector field at $x \in M$ if there exist tangent vectors $X_x, Y_x \in T_x M$ such that $\xi_x = R(X_x^h, Y_x^h)$.

The *Riemannian curvature tensor* is $R_y := R(\cdot, y)$, its components can be obtained as $R^i_j = R^i_{jk} y^k$.

The Ricci curvature $Ric(y)$ is defined to be the trace of R_y, $Ric(y) := R^m_m(x, y)$. For a tangent plane $P = Span\{\, y, u \,\} \subset T_x M$, the flag curvature is defined as

$$\mathbf{K}(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y) g_y(u, u) - g_y(y, u)^2}.$$

If a manifold has *constant flag curvature* $\mathbf{K} = \lambda \in \mathbb{R}$, then the Ricci curvature is constant in the sense that $Ric(y) = (n-1)\lambda F^2$ and the local expression of the coefficients of the curvature is

$$R^i_{jk} = \lambda \Big(\delta^i_k g_{jm}(x, y) y^m - \delta^i_j g_{km}(x, y) y^m \Big),$$

where δ^i_j is the Kronecker delta symbol. Assume that the Finsler manifold $(M, \mathcal{F})$ is locally projectively flat. Then for every point $x \in M$ there exists an *adapted* local coordinate system, that is a mapping $(x^1, \dots, x^n)$

on a neighbourhood U of x into the Euclidean space $\mathbb{R}^n$, such that the straight lines of $\mathbb{R}^n$ correspond to the geodesics of $(M, \mathcal{F})$. Then the geodesic coefficients are of the form

$$G^i = \mathcal{P}y^i, \quad G^i_k = \frac{\partial \mathcal{P}}{\partial y^k} y^i + \mathcal{P}\delta^i_k, \quad G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l} y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k \quad (1.17)$$

where $\mathcal{P}(x, y)$ is a 1-homogeneous function in y, called the *projective factor* of $(M, \mathcal{F})$. According to Lemma 8.2.1 in [11, p.155], if $(M \subset \mathbb{R}^n, \mathcal{F})$ is a projectively flat manifold, then its projective factor can be computed using the formula

$$\mathcal{P}(x, y) = \frac{1}{2\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x^i} y^i. \quad (1.18)$$

Let us consider a few relevant examples of Finsler manifolds.

Randers metrics

A Randers metric on a manifold M is a Finsler metric defined in the following form:

$$F = \alpha + \beta$$

where $\alpha = \sqrt{a_{ij}(x)y_i y_j}$ is a Riemannian norm and $\beta = b_i(x)y_i$ is a 1-form on M. In a more geometric manner, Randers metrics are the solutions of the Zermelo navigational problem on Riemannian manifolds, which means that their geodesics describe the time-optimal paths of a Riemannian manifold under the influence of an external wind or current, which is represented by a vector field on the manifold. This result is due to D. Bao, C. Robles and Z. Shen [2]. The geometry can be described in terms of a Finslerian setting with the Finsler function

$$\mathcal{F}(x, y) = \frac{\sqrt{h_x(W, y)^2 + h_x(y, y)(1 - h_x(W, W))} - h_x(W, y)}{1 - h_x(W, W)} \quad (1.19)$$

where $h_x(W, W) < 1$.

Projectively flat Randers manifolds with constant flag curvature were classified by Z. Shen in [46]. He proved that any projectively flat Randers manifold $(M, \mathcal{F})$ with non-zero constant flag curvature has negative curvature. These metrics can be normalized by a constant factor so that the curvature is $\lambda = -1/4$. In this case $(M, \mathcal{F})$ is isometric to the Finsler manifold defined by the Finsler function

$$\mathcal{F}_a(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \epsilon\left(\frac{\langle x,y\rangle}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\right) \tag{1.20}$$

on the unit ball $\mathbb{D}^n \subset \mathbb{R}^n$, where $a \in \mathbb{R}^n$ is a suitable constant vector with $|a| < 1$ and $\epsilon = \pm 1$ ([46, Theorem 1.1]). We note that the restriction of any orthogonal transformation $\phi \in \mathbb{O}(n, \mathbb{R}^n)$ on $\mathbb{D}^n$ does not change the Finsler function (1.20), therefore one can assume that $a \in \mathbb{R}^n$ has the form $a = (a_1, 0, \dots, 0)$. We can consider $(\mathbb{D}^n, \mathcal{F}_a)$ as the *standard model* of projectively flat Randers manifolds with non-zero constant flag curvature.

We remark that the computation of the coefficients of the associated connection is relatively easy: according to Lemma 8.2.1 of [11, p.155], the projective factor $\mathcal{P}(x,y)$ can be computed by the formula (1.18) which gives in the case (1.20)

$$\mathcal{P}(x,y) = \frac{1}{2}\left(\frac{\epsilon\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2} - \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\right). \tag{1.21}$$

The geodesic coefficients and the connection coefficients can be computed from (1.21) by using (1.17).

Funk metrics

A Funk metric can be described as follows. Let Ω be a bounded convex domain in $\mathbb{R}^n$ and denote its boundary by $\partial\Omega$. We can define a Finsler norm function $F_\Omega(x,y)$ in the interior of Ω for any vector $y \in T_x\Omega$ by the formulas

$$F_\Omega(x,y) > 0, \qquad x + \frac{y}{F_\Omega(x,y)} = z,$$

where $z \in \partial\Omega$. This norm function is called the Funk norm function induced by Ω. The Funk norm induced by the origo centered unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ will be called the *standard Funk norm* and will be denoted by $F_{\mathbb{B}^n}$. We denote by $o = (0, \dots, 0)$ the origin in $\mathbb{R}^n$. The standard Funk metric is a special Randers metric whose Finsler function is

$$\mathcal{F} = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}$$

1.3 Parallel translation and holonomy

In this section we will recall two important geometric constructions crucial to the main topic of this thesis. Let us first define parallel translation on a spray manifold.

Parallel translation

Let (M, S) be a spray manifold. The vector fields $X(t) = X^i(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}$ along a curve $c(t)$ are called *parallel*, if their covariant derivative is identically zero, i.e. it is a solution of the differential equation

$$D_{\dot{c}}X(t) := \left(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t))\dot{c}^j(t)\right)\frac{\partial}{\partial x^i} = 0. \tag{1.22}$$

Using the fact that the coefficients $G^i_j(x, y)$ are positive homogeneous functions of degree one, Euler's theorem for homogeneous functions gives us $D_{\dot{c}}(\lambda X(t)) = \lambda D_{\dot{c}}X(t)$. This means that if a vector field $X(t)$ is parallel along $c(t)$ then so is $\lambda X(t)$. Then the homogeneous (nonlinear) parallel translation

$$\tau_c : T_{c(0)}M \to T_{c(1)}M$$

along a curve c(t) of the spray manifold (M, S) is defined by the positive homogeneous map $\tau_c : X_0 \to X_1$ given by the value $X_1 = X(1)$ at $t = 1$ of the parallel vector field with initial value $X(0) = X_0$. Since the parallel translation of a spray manifold (M, S) is determined by its horizontal distribution $\mathcal{H}TM \subset TTM$, a spray manifold can be considered as a particular case of a fibered manifold equipped with an Ehresmann connection [48].

One can obtain parallel translation by following an elegant geometric construction: the horizontal lift of a curve $c\colon [0, 1] \to M$ with initial condition $X_0 \in T_{c(0)}M$ is a curve $c^h\colon [0, 1] \to TM$ such that $\pi \circ c^h = c$,

$\frac{dc^h}{dt} = (\frac{dc}{dt})^h$ and $c^h(0) = X_0$. Then the parallel translation of X_0 along the curve c from $c(0)$ to $c(1)$ can be expressed as

$$\mathcal{P}_c(X_0) = c^h(1). \tag{1.23}$$

Hence the parallel translation along a curve $c(t)$ joining the points p and q is the map $\tau_c : T_pM \to T_qM$ determined by the intersection points of the horizontal lift of the curve $c(t)$ with the tangent spaces T_pM and T_qM.

The holonomy group

Let (M, F) be an $n-$dimensional Finsler manifold. At any points $x \in M$ the indicatrix defined in (1.16) is an $(n-1)$-dimensional compact manifold in T_xM. Using the 1-homogeneity and norm preserving property of the parallel translation, for any closed curve c with starting and ending point $x \in M$, one can consider it as a map

$$\mathcal{P}_c : \mathcal{I}_x \to \mathcal{I}_x. \tag{1.24}$$

1.3.1 Definition. The *holonomy group* $\mathcal{H}ol_x(M, F)$ of a Finsler manifold (M, F) at a point $x \in M$ is the group generated by parallel translations along piece-wise differentiable closed curves starting and ending at the point x.

We remark, that the holonomy group can be seen as a subgroup of the diffeomorphism group of the indicatrix:

$$\mathcal{H}ol_x(M, F) \ \subset \ \mathcal{D}iff^\infty(\mathcal{I}_x),$$

where $\mathcal{D}iff^\infty(\mathcal{I}_x)$ denotes the group of smooth diffeomorphisms of $\mathcal{I}_x$ with respect to the C^∞-topology. $\mathcal{D}iff^\infty(\mathcal{I}_x)$ is an infinite dimensional Fréchet Lie group, whose Lie algebra is $\mathfrak{X}(\mathcal{I}_x)$, the Lie algebra of smooth vector fields on $\mathcal{I}_x$. We note that the holonomy group $\mathcal{H}ol_x(M, F)$ is a topological subgroup of $\mathcal{D}iff^\infty(\mathcal{I}_x)$, but its differentiable structure is not known in general.

Chapter 2

Tangent Lie algebra of a diffeomorphism group and its application to the holonomy theory

Important geometric objects, structures, or properties can often be investigated through algebraic structures. In many interesting cases (such as the topic of holonomy theory), these algebraic structures are groups, where the group operations are smooth maps. Such groups became indispensable tools for modern geometry, analysis, and theoretical physics. Lie groups and diffeomorphism groups are the most important examples for such structures.

Considering a Lie group $\mathcal{G}_L$, it is well known that most of the important information about it is captured in its tangent object, the Lie algebra $\mathfrak{g}_L$. Naturally, if $\mathcal{G}$ is a Lie sub-group of $\mathcal{G}_L$, then its Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}_L$. The Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}_L$ can be used to obtain information or eventually to determine the subgroup $\mathcal{G}$. In many relevant geometric situations, however, this framework is not general enough because of two factor: Firstly, $\mathcal{G}_L$ is not a (finite-dimensional) Lie group but the (infinite-dimensional) diffeomorphism group $\mathcal{D}iff^{\infty}(M)$ of some manifold M. Secondly, the subgroup $\mathcal{G}$ is not necessarily a Lie subgroup of $\mathcal{G}_L$. Nevertheless, natural questions arise: *can we introduce a tangential property and tangent objects to the subgroup $\mathcal{G}$ in this situation? Does the set of tangent elements possess a special algebraic structure? Can this al-*

gebraic structure be used to get information about the subgroup and thus about geometric properties? In Section 2.1 we answer these questions in the case when M is a compact manifold.

In Section 2.2 and Section 2.3 we apply the results of 2.1 by focusing on the holonomy theory of Finsler manifolds. Indeed, for Riemannian manifolds the holonomy structure has been extensively studied and now the complete classification is known [7, 4]. In particular, it is well known, that the holonomy group of a simply connected Riemannian manifold is a closed Lie subgroup of the special orthogonal group $SO(n)$. Despite the analogues in the construction, Finslerian holonomy groups can be much more complex and up to now, we do not know much about them: For special spaces the holonomy can be a finite dimensional Lie group (see [49] and [25]), but recent results show that there are Finsler manifolds with infinite dimensional holonomy group [35, 36, 37]. These latter results show the difficulties: one cannot use the well understood principal bundle machinery in the investigation because the structure group should be infinite dimensional. However, using the results of Section 2.1 we are able to introduce the notion of holonomy algebra (resp. fibered holonomy algebra) as the tangent sturcture of the holonomy group (resp. fibered holonomy group) for Finslerian manifolds. By improving the results of [33] we also prove that the curvature algebra and the infinitesimal holonomy algebra (resp. their restrictions) are Lie subalgebras of the fibered holonomy algebras (resp. the holonomy algebra). In Section 3 and 4 we use effectively these objects to get information about the Finsler holonomy strutures.

2.1 Tangent Lie algebra of a diffeomorphism group

Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^\infty(M)$ where M is a compact differentiable manifold. We do not suppose any special property on $\mathcal{G}$, in particular, we do not suppose that $\mathcal{G}$ is a Lie subgroup of $\mathcal{D}iff^\infty(M)$.

A smooth curve $c\colon I \to M$ on the manifold M has singularity of order $(k{-}1)$ at $t = 0$, if its derivatives vanish up to order $k{-}1$, $(k \geq 0)$. It is well known that if a curve c has a singularity of order $(k-1)$ at $0 \in \mathbb{R}$ then its k^{th} order derivative $c^{(k)}(0) = X_p$ is a tangent vector at $p = c(0)$. In that case, the curve c is called an *integral curve of order k* of the vector $X_p \in T_pM$. Extending this concept to vector fields, we can introduce the following

2.1.1 Definition. A C^∞−smooth curve in the diffeomorphism group $\varphi\colon I \to \mathcal{D}iff^\infty(M)$, $t \to \varphi_t$ is called an *integral curve of the vector field* $X \in \mathfrak{X}(M)$ if

(1) $\varphi_0 = id_M$,

(2) there exists $k \in \mathbb{N}$ such that for any point $p \in M$ the curve $t \to \varphi_t(p)$ is an integral curve of order k of $X(p) \in T_pM$.

This $k \in \mathbb{N}$ is called the *order* of the integral curve φ_t of the vector field X.

In particular, the flow φ_t^X of $X \in \mathfrak{X}(M)$ is an integral curve of order 1 of X. Moreover, if $k > 1$ and $t \to \varphi_t$ is an integral curve of order k of the vector field X then we have

$$\varphi_0 = id_M, \qquad \left.\frac{\partial \varphi_t}{\partial t}\right|_{t=0} = 0, \quad \dots \quad \left.\frac{\partial^{k-1} \varphi_t}{\partial t^{k-1}}\right|_{t=0} = 0, \qquad \left.\frac{\partial^{k} \varphi_t}{\partial t^{k}}\right|_{t=0} = X. \quad (2.1)$$

Let $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ be an arbitrary subgroup of the diffeomorphism group $\mathcal{D}iff^\infty(M)$. Using the terminology of Definition 2.1.1 we introduce the following

2.1.2 Definition. A vector field $X \in \mathfrak{X}(M)$ is called *tangent* to a subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ of the diffeomorphism group if there exists an integral curve of X in $\mathcal{G}$. The set of tangent vector fields of $\mathcal{G}$ is denoted by $\mathcal{T}_o\mathcal{G}$.

2.1.3 Remark. We have $X \in \mathcal{T}_o\mathcal{G}$ if and only if there exists a C^∞−smooth curve $\varphi\colon I \to Diff^\infty(M)$ such that

(*1*) $\varphi_t \in \mathcal{G}$,

(*2*) $\varphi_0 = id_M$,

(*3*) there exists $k \in \mathbb{N}$ such that equation (2.1) is satisfied.

One can observe that in Definition 2.1.2 we do not suppose that $\mathcal{G}$ is a Lie subgroup of $\mathcal{D}iff^\infty(M)$. Indeed, we use the differential structure of the later to formulate the smoothness condition on the curve in $\mathcal{G}$. Nevertheless, we have the following

2.1.4 Theorem. *If $\mathcal{G}$ is a subgroup of $\mathcal{D}iff^\infty(M)$, then $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. To prove the theorem, we have to show that

$$\begin{aligned} X, Y \in \mathcal{T}_o\mathcal{G} \quad &\Rightarrow \quad [X,Y] \in \mathcal{T}_o\mathcal{G}, && (2.2a)\\ X, Y \in \mathcal{T}_o\mathcal{G} \quad &\Rightarrow \quad X+Y \in \mathcal{T}_o\mathcal{G}, && (2.2b)\\ \lambda \in \mathbb{R},\ X \in \mathcal{T}_o\mathcal{G} \quad &\Rightarrow \quad \lambda X \in \mathcal{T}_o\mathcal{G}. && (2.2c) \end{aligned}$$

Indeed, let $X, Y \in \mathcal{T}_o\mathcal{G}$, that is $X, Y \in \mathfrak{X}(M)$ tangent to G. According to Definition 2.1.1, there exist $k, l \in \mathbb{N}$ such that $\varphi_t, \psi_t \in \mathcal{G}$ are integral curves of X and Y respectively. Let us suppose that φ_t is an integral curve of order k of X and ψ_t is an integral curve of order l of Y $(k, l \geq 1)$. Then

$$\varphi_0 = id_M, \qquad \left\{ \frac{\partial^i \varphi_t}{\partial t^i}\bigg|_{t=0} = 0 \right\}_{1\leq i<k} \qquad \frac{\partial^k \varphi_t}{\partial t^k}\bigg|_{t=0} = X, \tag{2.3}$$

and

$$\psi_0 = id_M, \qquad \left\{ \frac{\partial^j \psi_t}{\partial t^j}\bigg|_{t=0} = 0 \right\}_{1\leq j<l} \qquad \frac{\partial^l \varphi_t}{\partial t^l}\bigg|_{t=0} = Y. \tag{2.4}$$

• *Proof of* (2.2a). The computation is similar to that of [30]: Considering the group theoretical commutator

$$[\varphi_t, \psi_s] := \varphi_t^{-1} \circ \psi_s^{-1} \circ \varphi_t \circ \psi_s, \tag{2.5}$$

we get a two-parameter family of diffeomorphisms such that if one of the parameters s or t is zero then (2.5) is the identity transformation. From (2.3) and (2.4) we also know that the first, potentially nonzero derivative is the $(k+l)^{\text{th}}$ order mixed derivative:

$$\begin{aligned} \frac{\partial^{(k+l)}\left[\varphi_t, \psi_s\right]}{\partial t^k \partial s^l}\bigg|_{(0,0)}(p) &= \frac{\partial^l}{\partial s^l}\bigg|_{s=0} \left(\frac{\partial^k \left(\varphi_s^{-1} \circ \psi_t^{-1} \circ \varphi_s \circ \psi_t(p)\right)}{\partial t^k}\bigg|_{t=0} \right) \\ &= \frac{\partial^l}{\partial s^l}\bigg|_{s=0} \left(d\left(\varphi_s^{-1}\right)_{\varphi_s(p)} \circ \frac{\partial^k \psi_t^{-1}}{\partial t^k}\bigg|_{t=0} (\psi_s(p)) \right), \end{aligned} \tag{2.6}$$

where $d\left(\varphi_s^{-1}\right)_{\varphi_s(p)}$ denotes the tangent map (or Jacobi operator) of φ_s^{-1} at the point $\varphi_s(p)$. Since $d\left(\varphi_{s=0}^{-1}\right)_{\varphi_s(p)} = id$, the above formula can be written in the form

$$d\left(\frac{\partial^l \varphi_s^{-1}}{\partial s^l}\Big|_{s=0}\right)_p \frac{\partial^k \psi_t^{-1}(p)}{\partial t^k}\Big|_{t=0} + d\left(\frac{\partial^k \psi_t^{-1}}{\partial t^k}\Big|_{t=0}\right)_p \frac{\partial^l \varphi_s(p)}{\partial s^l}\Big|_{s=0}. \tag{2.7}$$

From $\varphi_t \circ \varphi_t^{-1} = id$ we get

$$0 = \frac{\partial^k}{\partial t^k}\Big|_{t=0}\left(\varphi_t \circ \varphi_t^{-1}\right) = X + \frac{\partial^k(\varphi_t^{-1})}{\partial t^k}\Big|_{t=0}$$

which yields

$$\frac{\partial^k(\varphi_t^{-1})}{\partial t^k}\Big|_{t=0} = -X. \tag{2.8}$$

Therefore we get that (2.7) can be written as

$$d\left(\frac{\partial^l \varphi_s}{\partial s^l}\Big|_{s=0}\right)_p \frac{\partial^k \psi_t(p)}{\partial t^k}\Big|_{t=0} - d\left(\frac{\partial^k \psi_t}{\partial t^k}\Big|_{t=0}\right)_p \frac{\partial^l \varphi_s(p)}{\partial s^l}\Big|_{s=0}, \tag{2.9}$$

which is the Lie bracket of the vector fields X and Y, that is

$$\frac{\partial^{k+l}\left[\varphi_t, \psi_s\right]}{\partial t^k \partial s^l}\Big|_{(0,0)} = [Y, X]\,. \tag{2.10}$$

From (2.10) we get that $t \to [\varphi_t, \psi_t]$ is an integral curve of order $(k+l)$ of $[X, Y]$ in $\mathcal{G}$. Therefore $[X, Y] \in \mathcal{T}_o\mathcal{G}$ which proves (2.2a).

- *Proof of* (2.2b).

For any $c_1, c_2, m_1, m_2 \in \mathbb{R}$, $\phi_t = \varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}$ is a smooth curve in $\mathcal{G}$ with $\phi_0 = \varphi_0 \circ \psi_0 = id_M$. Moreover, if $r =$ denotes the least common multiple of k and l and

$$m_1 = r/k, \quad m_2 = r/l, \quad c_1 = \left(m_1^k (r-k)!\right)^{-1/r}, \quad c_2 = \left(m_2^l (r-l)!\right)^{-1/r},$$

one gets

$$\frac{\partial^r \phi_t}{\partial t^r}\Big|_{t=0} = \frac{\partial^r}{\partial t^r}\Big|_{t=0}\left(\varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}\right) = X + Y, \tag{2.11}$$

showing that ψ_t is an integral curve of orderr of $X + Y$ in $\mathcal{G}$, therefore $X + Y$ is tangent to $\mathcal{G}$.

• *Proof of* (2.2c).

It is clear that in the case when $\lambda \geq 0$, one can reparametrize the integral curve of X, and using that the lower order terms are zero, we get

$$\frac{\partial^k \varphi_{\sqrt[k]{\lambda}t}}{\partial t^k}\bigg|_{t=0} = \lambda X. \tag{2.12}$$

In the case when $\lambda < 0$ one can use (2.8) and we get

$$\frac{\partial^k}{\partial t^k}\bigg|_{t=0}\left(\varphi^{-1}_{\sqrt[k]{|\lambda|}t}\right) = -|\lambda| X = \lambda X \tag{2.13}$$

From *(2.12)* and *(2.13)* we get that λX is tangent to G, that is $\lambda X \in \mathcal{T}_o\mathcal{G}$, and from *2.2b)* and *2.2c)* we get that any linear combinations of X and Y are in $\mathcal{T}_o\mathcal{G}$. □

Motivated by the results of Theorem 2.1.4 we propose the following

2.1.5 Definition. $T_0\mathcal{G}$ is called the tangent Lie algebra of the subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$.

As a direct consequence of Theorem 2.1.4 we get the following

2.1.6 Corollary. *Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^\infty(M)$ and $\mathcal{S}$ be a subset of $\mathfrak{X}(M)$ such that the elements of $\mathcal{S}$ are tangent to $\mathcal{G}$. Then the Lie subalgebra $\langle\, \mathcal{S}\,\rangle_{\mathcal{L}ie}$ of $\mathfrak{X}(M)$ generated by the elements of $\mathcal{S}$ is also tangent to $\mathcal{G}$, that is*

$$\mathcal{S} \subset \mathcal{T}_o\mathcal{G} \qquad \Rightarrow \qquad \Big\langle\, \mathcal{S}\,\Big\rangle_{Lie} \subset \mathcal{T}_o\mathcal{G}.$$

2.1.7 Remark. Slightly different tangent properties of vector fields to a subgroup $\mathcal{G}$ of the diffeomorphism group were already introduced in [33]. We will refer to the property [33, Definition 2.] as the *weak tangent property* and to [33, Definition 4.] as the *strong tangent property.* Our language is justified by the following proposition which is clarifying the relationship between the tangent property introduced in Definition 2.1.1 and the tangent properties introduced in [33]:

2.1.8 Proposition. *Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^\infty(M)$ and $X \in \mathfrak{X}(M)$. Using the terminology of Remark 2.1.7:*
(i) if X is strongly tangent to $\mathcal{G}$, then $X \in \mathcal{T}_o\mathcal{G}$.

(ii) if $X \in \mathcal{T}_o\mathcal{G}$, then it is weakly tangent to $\mathcal{G}$.

Proof. *(i)* If $X \in \mathfrak{X}(M)$ is a strongly tangent vector field to the subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$, there exists a k-parameter commutator like family of diffeomorphisms $\phi_{t_1 \dots t_k} \in \mathcal{G}$ which is C^∞-smooth in $\mathcal{D}iff^\infty(M)$, $\phi_{t_1,\dots,t_k} = id_M$ whenever one of its parameters is zero and

$$X = \frac{\partial^k \phi_{t_1 \dots t_k}}{\partial t_1 \dots \partial t_k}\bigg|_{(0\dots 0)}.$$

Consequently, if we consider the map $t \to \varphi_t$ where $\varphi_t = \phi_{t,\dots,t}$, we get a 1-parameter family of diffeomorphisms which satisfies the conditions of Definition 2.1.2. Therefore, the vector field X is tangent to $\mathcal{G}$.

To prove *(ii)*, let us suppose that φ_t is an integral curve of order k of X. Then we have (2.3) and one can write $\varphi_t(p)$ as

$$\varphi_t(p) = p + \tfrac{1}{k!}\, t^k \Big(X(p) + \omega(p,t) \Big) \tag{2.14}$$

where $\lim_{t \to 0} \omega(p,t) = 0$. The reparametrization $t \to \psi_t := \varphi_{k!\sqrt[k]{t}}$ gives a C^1-differentiable 1-parameter family of diffeomorphism in $\mathcal{D}iff^\infty(M)$ such that $\psi_0 = id_M$ and

$$\frac{\partial \psi_t}{\partial t}\bigg|_{t=0}(p) = \frac{\partial \varphi_{k!\sqrt[k]{t}}}{\partial t}\bigg|_{t=0}(p) = X(p),$$

which proves *(ii)*. □

2.1.9 Remark. One may wonder why to introduce a new tangent property when there are already two, the weak and the strong tangent properties (using the terminology of Remark 2.1.7) introduced in the literature. As an answer we point out that, the concept in [33] has a major defect: the weak tangent property is not preserved under the bracket operation, therefore it is not true in general that weakly tangent vector fields to a subgroup $\mathcal{G}$ generate a weakly tangent Lie algebra to $\mathcal{G}$. To overcome this difficulty, the authors introduced the strongly tangent property but the strongly tangent property was not preserved under the linear combination. It follows that [33] and the succeeding papers using these techniques were not able to guaranty the existence of the tangent Lie algebra $\mathcal{T}_o\mathcal{G}$ associated to $\mathcal{G}$. With our approach we are able to overcome this major deficiency.

The main feature of $\mathcal{T}_o\mathcal{G}$ is that one can obtain information about the group $\mathcal{G}$. Indeed, one has the following

2.1.10 Theorem. *Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^\infty(M)$ and $\overline{\mathcal{G}}$ its topological closure with respect to the C^∞ topology. Then the group generated by the exponential image of the tangent Lie algebra $\mathcal{T}_o\mathcal{G}$ with respect to the exponential map* $\exp\colon \mathfrak{X}(M) \to \mathcal{D}iff^\infty(M)$ *is a subgroup of $\overline{\mathcal{G}}$.*

Proof. From the proof of Proposition 2.1.8 we know that for any element $X \in \mathcal{T}_o\mathcal{G}$ there exists a C^1-differentiable 1-parameter family $\{\psi_t\} \subset \mathcal{G}$ of diffeomorphisms of M such that $\psi_0 = id_M$ and $X = \frac{\partial \psi_t}{\partial t}\Big|_{t=0}$. Then, using the argument of [43, Corollary 5.4, p. 84] on ψ_t we get that

$$\psi^n\left(\tfrac{t}{n}\right) = \psi\left(\tfrac{t}{n}\right) \circ \cdots \circ \psi\left(\tfrac{t}{n}\right)$$

in $\mathcal{G}$, as a sequence of $\mathcal{D}iff^\infty(M)$, converges uniformly in all derivatives to $\exp(tX)$. It follows that

$$\{\exp(tX) \mid t \in \mathbb{R}\} \subset \overline{\mathcal{G}},$$

for any $X \in \mathcal{T}_o\mathcal{G}$. Therefore, one has $\exp(\mathcal{T}_o\mathcal{G}) \subset \overline{\mathcal{G}}$ and if we denote by $\Big\langle \exp(\mathcal{T}_o\mathcal{G}) \Big\rangle$ the group generated by the exponential image of $\mathcal{T}_o\mathcal{G}$ we get

$$\Big\langle \exp(\mathcal{T}_o\mathcal{G}) \Big\rangle \subset \overline{\mathcal{G}},$$

which proves Theorem 2.1.10. □

We note that, assuming the manifold M is compact, we could avoid technical difficulties. Indeed, in this case, the diffeomorphism group $\mathcal{D}iff^\infty(M)$ is an (infinite dimensional) manifold and the exponential image of the flow of vector fields exists everywhere on M. For a more general and deeper discussion of the subject see [50].

The concept worked out in Definition 2.1.2 and Theorem 2.1.4 can be adapted not only for subgroups of the diffeomorphism group but for any subgroup of any (finite or infinite dimensional) Lie group:

2.1.11 Definition. Let $\mathcal{G}_L$ be a Lie group, $e \in \mathcal{G}_L$ is the identity element of $\mathcal{G}_L$ and $\mathfrak{g}_L := T_e\,\mathcal{G}_L$ the Lie algebra of $\mathcal{G}_L$. If $\mathcal{G} \subset \mathcal{G}_L$ is a subgroup of $\mathcal{G}_L$, then $X \in \mathfrak{g}_L$ is called tangent to $\mathcal{G}$ if there exist a $C^\infty-$smooth curve $\varphi\colon I \to \mathcal{G}_L$ such that

(*1*) $\varphi_t \in \mathcal{G}$,
(*2*) $\varphi_0 = e$,
(*3*) there exists $k \in \mathbb{N}$ such that $t \to \varphi_t$ is an integral curve of order k of X.

The set of tangent vector of $\mathcal{G}$ is denoted by $\mathcal{T}_o\mathcal{G}$.

Then, adapting the proof of Theorem 2.1.4 and Theorem 2.1.10 we can get the following

2.1.12 Theorem. *If $\mathcal{G}$ is a subgroup of a Lie group $\mathcal{G}_L$, then $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of $\mathfrak{g}_L$. The group $\big\langle \exp(\mathcal{T}_o\mathcal{G})\big\rangle$ generated by the exponential image of $\mathcal{T}_o\mathcal{G}$ with respect to the exponential map $\exp\colon \mathfrak{g}_L \to \mathcal{G}_L$ is a subgroup of the topological closure $\overline{\mathcal{G}}$ of $\mathcal{G}$ in $\mathcal{G}_L$.*

It is clear that in the case when $\mathcal{G}$ is a Lie subgroup of $\mathcal{G}_L$, then $\mathcal{T}_o\mathcal{G} = \mathfrak{g}$ is just the usual Lie subalgebra of $\mathfrak{g}_L$ associated to the Lie subgroup $\mathcal{G}$. Therefore Definition 2.1.11 generalizes the classical notion of the Lie subalgebra associated to a Lie subgroup.

2.1.13 Definition. We call a subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ infinite dimensional, if its tangent algebra $\mathcal{T}_o\mathcal{G}$ is infinite dimensional.

2.2 The fibered holonomy algebra and its Lie subalgebras

The notion of *fibered holonomy group* $\mathcal{H}ol_f(M)$ appeared in [33]:

2.2.1 Definition. The fibred holonomy group $\mathcal{H}ol_f(M)$ of $(M, \mathcal{F})$ consists of fibre preserving diffeomorphisms $\Phi \in \mathcal{D}iff^\infty(\mathcal{I}M)$ of the indicatrix bundle $(\mathcal{I}M, \pi, M)$ such that for any $p \in M$ the restriction $\Phi_p = \Phi|_{\mathcal{I}_pM} \in \mathcal{D}iff^\infty(\mathcal{I}_pM)$ belongs to the holonomy group $\mathcal{H}ol_p(M)$.

Let $(M, \mathcal{F})$ be a compact Finsler manifold. It is obvious that

$$\mathcal{H}ol_f(M) \subset \mathcal{D}iff^\infty(\mathcal{I}M), \tag{2.15}$$

where $\mathcal{H}ol_f(M)$ is a subgroup of the diffeomorphism group of the indicatrix bundle. Until now it is not known whether or not $\mathcal{H}ol_f(M)$ is a Lie subgroup of $\mathcal{D}iff^\infty(\mathcal{I}M)$. The set of tangent vector fields to the group $\mathcal{H}ol_f(M)$ denoted as

$$\mathfrak{hol}_f(M) := \mathcal{T}_0\Big(\mathcal{H}ol_f(M)\Big). \tag{2.16}$$

2.2.2 Definition. $\mathfrak{hol}_f(M)$ is called the *fibered holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$.

From Theorem 2.1.4 one can obtain the following

2.2.3 Corollary. *The fibered holonomy algebra $\mathfrak{hol}_f(M)$ is a Lie subalgebra of the Lie algebra of smooth vector fields $\mathfrak{X}(\mathcal{I}M)$.*

In the sequel we introduce two important Lie subalgebras of $\mathfrak{hol}_f(M)$ using the the curvature tensor (for its definition, see Paragraph 1.3): the curvature algebra and the infinitesimal holonomy algebra.

The curvature algebra

2.2.4 Definition. The *curvature algebra* $\mathfrak{R}$ is the Lie algebra generated on the indicatrix bundle by curvature vector fields:

$$\mathfrak{R} = \Big\langle R(X^h, Y^h)|_{\mathcal{I}M} \;\Big|\; X, Y \in \mathfrak{X}(M)\Big\rangle_{Lie}.$$

It is not difficult to see from the definition that a curvature vector field can be calculated as

$$\xi = R(X^h, Y^h) = \Big[X^h, Y^h\Big] - \Big[X, Y\Big]^h. \tag{2.17}$$

Moreover, we have the following

2.2.5 Proposition.

1. *The elements of the curvature algebra are tangent to the group $\mathcal{H}ol_f(M)$.*

2. *The curvature algebra $\mathfrak{R}$ is a Lie subalgebra of $\mathfrak{hol}_f(M)$.*

To prove the first part of the proposition, we have to show that the curvature vector fields are tangent to the fibered holonomy group $\mathcal{H}ol_f(M)$, that is they are elements of $\mathfrak{hol}_f(M)$. Let $\xi \in \mathfrak{X}(\mathcal{I}M)$ be a curvature vector field and $X, Y \in \mathfrak{X}(M)$ such that $\xi = R(X^h, Y^h)$. We denote by φ and ψ the integral curves of X and Y respectively. Define

$$\alpha_{t,s} := \begin{cases} \varphi_s, & 0 \leq s \leq t, \\ \psi_{s-t}\varphi_t, & t \leq s \leq 2t, \\ \varphi_{2t-s}\psi_t\varphi_t, & 2t \leq s \leq 3t, \\ \psi_{3t-s}\varphi_{-t}\psi_t\varphi_t, & 3t \leq s \leq 4t. \end{cases}$$

and

$$\beta_{t,s} := \psi_{-s}\varphi_{-s}\psi_s\varphi_s, \qquad 0 \leq s \leq t.$$

Then, for every $p \in M$ and fixed t the map $\alpha_t(p)\colon s \to \alpha_{t,s}(p)$ and $\beta_t(p)\colon s \to \beta_{t,s}(p)$ are parametrized curves: $\alpha_t(p)\colon s \to \alpha_{t,s}(p)$ is a (not necessarily closed) parallelogram and $\beta_t(p)$ joins the endpoints of $\alpha_t(p)$. Indeed, for every $p \in M$ and fixed t the endpoint of $\alpha_t(p)$ coincides with the endpoint of $\beta_t(p)$ and consequently the curve $\alpha_t(p) * \beta_t^{-1}(p)$ defined as going along the curve $\alpha_t(p)$ then continuing along $\beta_t^{-1}(p)$ (which is the curve $\beta_t(p)$ with opposed orientation) is a closed curve that starts and ends at $p \in M$. Let us consider

$$h_{t,p} := \mathcal{P}_{\alpha_t(p)*\beta_t^{-1}(p)} = \mathcal{P}_{\alpha_t(p)} \circ \mathcal{P}^{-1}_{\beta_t(p)}, \tag{2.18}$$

the parallel translation along $\alpha_t(p) * \beta_t^{-1}(p)$. We have the following

2.2.6 Lemma. *For any $p \in M$*

(1) $h_{t,p} \in \mathcal{H}ol_p(M)$,

(2) $t \to h_{t,p}$ is an integral curve of order 2 of the vector field $\xi_p := \xi\big|_{\mathcal{I}_p}$ $\Big(\xi_p \in \mathfrak{X}(\mathcal{I}_p)\Big)$.

Proof. Indeed, for every $p \in M$ and sufficiently small t the curve $\alpha_t(p) * \beta_t^{-1}(p)$ is a closed loop starting and ending at p, therefore the parallel transport $h_{t,p} : \mathcal{I}_p \to \mathcal{I}_p$ is a holonomy transformation at p and we get *(1)* of the lemma.

To show *(2)* we first remark that $\alpha_0(p)$ and $\beta_0(p)$ are the trivial curves ($s \to \alpha_{0,s}(p) = \beta_{0,s}(p) \equiv p$), therefore the parallel translation along them is the identity transformation and

$$h_{0,p} = id_{\mathcal{I}_p}. \tag{2.19}$$

On the other hand, as we have seen in Section 1.2, the parallel transport along a curve is determined by the horizontal lift of the curve. Consequently, the parallel transport along the integral curves of the vector fields X and Y can be expressed with the flows of the horizontal lifts X^h and Y^h. Let us consider first the parallel transport along the curve $\alpha_t(p)$: the parallel transport of a vector $v \in \mathcal{I}_p$ along the curve $\alpha_t(p)$ is

$$\mathcal{P}_{\alpha_t(p)}(v) = \begin{cases} \varphi_s^{X^h}(v), & 0 \leq s \leq t, \\ \varphi_{s-t}^{Y^h}\varphi_t^{X^h}(v), & t \leq s \leq 2t, \\ \varphi_{-(s-2t)}^{X^h}\varphi_t^{Y^h}\varphi_t^{X^h}(v), & 2t \leq s \leq 3t, \\ \varphi_{-(s-3t)}^{Y^h}\varphi_{-t}^{X^h}\varphi_t^{Y^h}\varphi_t^{X^h}(v), & 3t \leq s \leq 4t. \end{cases}$$

Therefore, $\mathcal{P}_{\alpha_t(p)}$ corresponds to the infinitesimal (not necessarily closed) parallelogram having as sides the integral curves of the horizontal lifts X^h and Y^h. From the well known properties of the Lie brackets (see for example [44, p.162]) we get that

$$\frac{d}{dt}\Big|_{t=0}\mathcal{P}_{\alpha_t}(v) = 0, \qquad \text{and} \qquad \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{P}_{\alpha_t}(v) = 2\left[X^h, Y^h\right]_v. \tag{2.20}$$

On the other hand, the parallel transport of a vector $w \in \mathcal{I}_{\alpha_t(p)}$ along $\beta_t^{-1}(p)$ can be calculated with the help of it's horizontal lift

$$\mathcal{P}_{\beta_t^{-1}}(w) = \mathcal{P}_{\beta_t}^{-1}(w) = ((\beta)^h(t))^{-1}(w),$$

where by the definition of the horizontal lift $\pi \circ (\beta)^h(t) = \beta(t)$ and $(\beta^{-1})^h(0) = w$ are fulfilled. Since

$$\frac{d}{dt}\Big|_{t=0}\beta_t(p) = 0, \tag{2.21}$$

$$\frac{d^2}{dt^2}\Big|_{t=0}\beta_t(p)(v) = 2\left[X, Y\right]_p, \tag{2.22}$$

we obtain

$$\frac{d}{dt}\Big|_{t=0}\mathcal{P}_{\beta_t}^{-1} = 0 \tag{2.23}$$

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{P}_{\beta_t}^{-1}(v) = -\left(2\left[X, Y\right]^h\right)_v, \tag{2.24}$$

thus, from the two equations of (2.20) and the two equations of (2.23) we get

$$\frac{d}{dt}\Big|_{t=0} h_t(v) = 0, \tag{2.25}$$

$$\frac{d^2}{dt^2}\Big|_{t=0} h_t(v) = 2\left(\left[X^h, Y^h\right] - \left[X, Y\right]^h\right)_v = 2\xi_p. \tag{2.26}$$

where we also used (2.17). To summarize, we get from (2.19) and (2.25):

$$h_{0,p} = \mathrm{id}\Big|_{\mathcal{I}_p}, \qquad \frac{d}{dt}\Big|_{t=0} h_{t,p} = 0, \qquad \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} h_{t,p} = \xi_p, \tag{2.27}$$

which means that the reparametrized map $t \to h_{t/\sqrt{2},p}$ is an integral curve of order 2 of the curvature vector field $\xi_p \in \mathfrak{X}(\mathcal{I}_p)$ and proves point *(2)* of the lemma. □

Proof of Proposition 2.2.5. Let us consider the map $h_t : \mathcal{I}M \to \mathcal{I}M$ on the indicatrix bundle, where $h_t\Big|_{\mathcal{I}_p} := h_{t,p}$. From Lemma 2.2.6 we get (by dropping the variable $p \in M$) that

(1) $h_t \in \mathcal{H}ol_f(M)$,

(2) $t \to h_t$ is an integral curve of order 2 of the vector field $\xi \in \mathfrak{X}(\mathcal{I})$.

which shows that the curvature vector field ξ is tangent to $\mathcal{H}ol_f(M)$ and proves the first part of the proposition. Applying Corollary 2.1.6, we get that the Lie algebra generated by the curvature vector field is tangent to $\mathcal{H}ol_f(M)$ which proves the second part of the proposition. □

The infinitesimal holonomy algebra

2.2.7 Definition. The *infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ of a Finsler manifold $(M, \mathcal{F})$ is the smallest Lie algebra on the indicatrix bundle which satisfies the following properties:

1) curvature vector fields are elements of $\mathfrak{hol}^*(M)$,
2) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}(M)$, then the horizontal Berwald covariant derivative $\nabla_X \xi$ is also an element of $\mathfrak{hol}^*(M)$.

We have the following

2.2.8 Proposition.

1. *The elements of the infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ *are tangent to* $\mathcal{H}ol_f(M)$.

2. *The infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ *is a Lie subalgebra of* $\mathfrak{hol}_f(M)$.

Proof. From Proposition 2.2.5 we know that the curvature vector fields are tangent to the fibered holonomy group. Moreover, from [33, Proposition 4] and from *(i)* of Remark 2.1.8 we get that the horizontal Berwald covariant derivative of tangent vector fields to $\mathcal{H}ol_f(M)$ are also tangent to $\mathcal{H}ol_f(M)$ which proves the first part of the proposition. As a consequence, the infinitesimal holonomy algebra is generated by tangent vector fields and, according to Corollary 2.1.6, it is tangent to $\mathcal{H}ol_f(M)$ proving the second part of the proposition. □

2.3 The holonomy algebra and its Lie subalgebras

Let $(M, \mathcal{F})$ be a Finsler manifold.

2.3.1 Definition. The tangent Lie algebra to the group $\mathcal{H}ol_p(M)$

$$\mathfrak{hol}_p(M) := \mathcal{T}_0\Big(\mathcal{H}ol_p(M)\Big).$$

is called the *holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$ at $p \in M$.

From Theorem 2.1.4 one obtains

2.3.2 Corollary. *The holonomy algebra* $\mathfrak{hol}_p(M)$ *of a Finsler manifold* $(M, \mathcal{F})$ *at* $p \in M$ *is a Lie subalgebra of* $\mathfrak{X}(\mathcal{I}_p)$.

In the sequel we identify two important Lie subalgebras of the holonomy algebra of Finsler manifolds. In the later chapters these objects will be used to obtain information about specific Finslerian holonomy groups.

The curvature algebra at a point of a Finsler manifold

2.3.3 Definition. The Lie algebra $\mathfrak{R}_p$ of vector fields generated by curvature vector fields at $p \in M$ is called the *curvature algebra at* p:

$$\mathfrak{R}_p = \left\langle R(X_p^h, Y_p^h)|_{\mathcal{I}_p} \;\middle|\; X_p, Y_p \in T_pM \right\rangle_{Lie}.$$

The relationship between the curvature algebra $\mathfrak{R}_p$ at $p \in M$ and the curvature algebra $\mathfrak{R}$ introduced in Definition 2.2.4 is:

$$\mathfrak{R}_p = \left\{ \xi_p = \xi|_{\mathcal{I}_p} \;\middle|\; \xi \in \mathfrak{R} \right\},$$

that is $\mathfrak{R}_p$ is the restriction of $\mathfrak{R}$ to the indicatrix $\mathcal{I}_p$. We have

2.3.4 Proposition. *The curvature algebra $\mathfrak{R}_p$ at $p \in M$ is tangent to the group $\mathcal{H}ol_p(M)$ and the curvature algebra $\mathfrak{R}_p$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.*

The proof is a direct consequence of the computation of Proposition (2.2.5).

The infinitesimal holonomy algebra at a point of a Finsler manifold

2.3.5 Definition. The *infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ of a Finsler manifold $(M, \mathcal{F})$ at a point $p \in M$ is the smallest Lie algebra on the indicatrix at p which satisfies the following properties:

1) Every curvature vector field ξ_p is an element of $\mathfrak{hol}^*_p(M)$,
2) if $\xi_p \in \mathfrak{hol}^*_p(M)$ and $X \in \mathfrak{X}(M)$, then the horizontal Berwald covariant derivative $(\nabla_X \xi)(p)$ is also an element of $\mathfrak{hol}^*_p(M)$.

2.3.6 Remark. The infinitesimal holonomy algebra of a Finsler manifold $(M, \mathcal{F})$ at a point $p \in M$ can also be considered as

$$\mathfrak{hol}^*_p(M) := \left\{ \xi|_{\mathcal{I}_p} \mid \xi \in \mathfrak{hol}^*(M) \right\}$$

of vector fields on the indicatrix $\mathcal{I}_p$ is called the infinitesimal holonomy algebra at the point $p \in M$.

From Proposition 2.2.8 we get

2.3.7 Corollary. *The infinitesimal holonomy algebra* $\mathfrak{hol}^*_p(M)$ *is tangent to the holonomy group* $\mathcal{H}ol_p(M)$ *and is a Lie subalgebra of the holonomy algebra* $\mathfrak{hol}_p(M)$*.*

We note that if the manifold M is two-dimensional, then the curvature algebra $\mathfrak{R}_p(M)$ is at most one-dimensional, and it does not give too much information about the holonomy structure of the Finsler manifold. However, even in that case, the dimension of the infinitesimal holonomy algebra $\mathfrak{hol}^*_p(M)$ can be higher, it can be even infinite dimensional. Indeed, using the notation $\xi_0 = R(\partial_{x_1}, \partial_{x_2})\Big|_{\mathcal{I}_p}$ and $\nabla_{i_1\dots i_k}\xi_0 := (\nabla_{\partial_{x_{i_1}}}\dots\nabla_{\partial_{x_{i_k}}}\xi)\Big|_{\mathcal{I}_p}$ we get

$$\mathfrak{hol}^*_p = \Big\langle \xi_0, \nabla_1\xi_0, \nabla_2\xi_0, \nabla_{11}\xi_0, \dots \Big\rangle_{\mathcal{L}ie} \tag{2.28}$$

2.3.8 Remark. The infinitesimal holonomy algebra is local in nature, that is for any open neighbourhood U of $p \in M$ we get

$$\mathfrak{hol}^*_p(M, F) = \mathfrak{hol}^*_p\Big(U, F|_{\pi^{-1}(U)}\Big).$$

For that reason, we will simplify the notation

$$\mathfrak{hol}^*_p(F) := \mathfrak{hol}^*_p(M, F)$$

by omitting the neighborhood of the point where the infinitesimal holonomy algebra is determined. Indeed, the curvature vector fields, their horizontal Berwald covariant derivatives and their Lie brackets can be computed on an arbitrarily small neighbourhood of p, therefore their value at p can be determined locally.

We note that by the construction of the infinitesimal holonomy algebra, the curvature vector fields of $\mathcal{F}$ at p are elements of $\mathfrak{hol}^*_p(M)$, therefore we have the sequence of the Lie algebras

$$\mathfrak{R}_p(M) \;\subset\; \mathfrak{hol}^*_p(M) \;\subset\; \mathfrak{hol}_p(M) \;\subset\; \mathfrak{X}(\mathcal{I}_p). \tag{2.29}$$

therefore, at the level of groups, we get

$$\exp\Big(\mathfrak{hol}^*_p(F)\Big) \subset \exp\Big(\mathfrak{hol}_p(M, F)\Big) \subset \mathcal{H}ol^c_p\Big(M, F) \subset \mathcal{D}iff^\infty(\mathcal{I}_p) \tag{2.30}$$

where $\mathcal{H}ol^c_p(M)$ denotes the topological closure of the holonomy group with respect to the C^∞–topology of $\mathcal{D}iff^\infty(\mathcal{I}_p)$. We call a Lie algebra infinite dimensional if it contains infinitely many $\mathbb{R}-$linearly independent elements.

We also remark that the first parts of the statement of Proposition 2.3.4 and 2.3.7 are improvements of the results of [33] because the tangential property of the Lie algebra is improved: we can guaranty C^∞-smoothness instead of C^1-smoothness.

Chapter 3

Some results about Finsler manifolds with infinite dimensional holonomy group

3.1 Holonomy of the quantum navigation problem

Randers model of Quantum Information Processing

In a closed finite dimensional quantum system the state Hilbert space is $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and the physical states can be identified with the rays of this space, that is the projective space $\mathbb{C}P^{n-1}$ of the underlying Hilbert space. In Quantum Information Processing (QIP) the task of the "navigator" is to find the shortest path from an initial state $|\Psi_I\rangle$ to a final state $|\Psi_F\rangle$. This means that one have to find the geodesics of the corresponding Riemannian manifold. The situation is slightly changed if we take into account an ineliminable force on the manifold, which can be represented as a vector field also referred to as the "wind". This formulation indicates that this problem is equivalent to the Zermelo navigational problem on a Riemannian manifold. In [3] the authors proved that the problem of finding the shortest path (in terms of time) in this setting is equivalent to that of finding the geodesics of a suitable Randers manifold, which is a special kind of Finsler manifold. The problem is hard to solve on the level of state space, but in [9] and in [8] the authors showed that one can lift this problem to the space acting on the states: the special unitary group $SU(n)$. The task is then to find a control Hamiltonian $\hat{H}_c(t)$ in the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$ which together with a time independent Hamiltonian $\hat{H}_0$, representing the effect of the ineliminable external field, forms $\hat{H}(t) = \hat{H}_c(t) + \hat{H}_0$ and generates the evolution of our initial state to the final state. The physical motivation and explanation can be found in [9]. Here we are focusing on the geometric background of the Finsler model. Geometric aspects and elementary solution to the time-independent quantum navigation problem can be found in [10].

Let M be an n-dimensional manifold, TM its tangent manifold, $\pi : TM \to M$ the canonical projection. Local coordinates (x^i) on M induce local coordinates (x^i, y^i) on TM where $y = y^i \frac{\partial}{\partial x^i}$.

2-dimensional quantum Zermelo problem

We consider a specific case of the two state quantum system: a single spin particle in a magnetic field. The manifold is $SU(2)$ and its tangent space can be identified with $\mathfrak{su}(2)$, which is the 3 dimensional Lie algebra of 2×2 Hermitian matrices. As in [9], we consider an invariant "wind" vector field, represented in the Lie algebra by $\hat{H}_0$ and the invariant Riemannian metric coming from the Killing form:

$$h(\hat{A}, \hat{B}) := \mathrm{tr}(\hat{A}^\dagger \hat{B}), \tag{3.1}$$

$\hat{A}, \hat{B} \in \mathfrak{su}(2)$. The Lie algebra $\mathfrak{su}(2)$ is spanned by $\hat{E}_1 = i\sigma_1, \hat{E}_2 = -i\sigma_2, \hat{E}_3 = i\sigma_3$ where the sigmas are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will work with the coordinates (x, ξ) on the tangent bundle, where $(x) = (x_1, x_2, x_3)$ are coordinates on the group $SU(2)$ and $(\xi) = (\xi_1, \xi_2, \xi_3)$ are the invariant coordinates in the Lie algebra $\mathfrak{su}(2)$ with respect the basis $\left\{\hat{E}_1, \hat{E}_2, \hat{E}_3\right\}$. The relation between the standard coordinates and the invariant coordinates on $TG = G \times \mathfrak{g}$ can be find by $\rho_{x,*}^{-1}(x, y) = (x, \xi)$, where $\rho : SU(2) \to SU(2)$ is the right translation. Modulo a rigid transformation, we can suppose that $\hat{H}_0 = c\hat{E}_1$ with $c \in \mathbb{R}$. With (3.1) and

$W = \hat{H}_0$ one can calculate (1.19) and find the corresponding invariant Randers metric:

$$\mathcal{F}(\xi) = \frac{1}{1-2c^2}\Big(\alpha_\xi + \beta_\xi\Big), \tag{3.2}$$

with a Riemannian norm α and a $1-$form β:

$$\alpha_\xi := \sqrt{2\xi_1^2 + 2\xi_2^2 + 2\xi_3^2 - 4\xi_2^2 c^2 - 4\xi_3^2 c^2}, \qquad \beta_\xi := -2c\,\xi_1. \tag{3.3}$$

For simplicity, the 3-dimensional Finler space corresponding to the 2-dimensional quantum Zermelo problem will be denoted by $\mathcal{Q} = (SU(2), \mathcal{F})$.

We note, that the indicatrix (1.16) is

$$\mathcal{I} = \Big\{\, \xi \in \mathfrak{su}(2) : \alpha_\xi = 1 + 2c\xi_1 - 2c^2 \,\Big\}\,. \tag{3.4}$$

a two-dimensional compact manifold. Let $E := \frac{1}{2}\mathcal{F}^2$ be the energy function. The invariant formulation of the Euler-Lagrange equations, called the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\partial E}{\partial \xi} = -ad^*_\xi\left(\frac{\partial E}{\partial \xi}\right), \tag{3.5}$$

can be used to determine the geodesic equation on the Lie algebra [13]. Using the basis $\Big\{\hat{E}_1, \hat{E}_2, \hat{E}_3\Big\}$, the equations (3.5) take the form

$$\frac{d}{dt}\frac{\partial E}{\partial \xi_d} = -C^b_{ad}\frac{\partial E}{\partial \xi_b}\xi_a \tag{3.6}$$

where C^b_{ad} are the structure constants with respect to the basis $\mathcal{E}$: The non zero terms are determined by $C^3_{12} = C^1_{23} = C^2_{31} = 2$. We get that

$$\dot{\xi}_1 = 0, \qquad \dot{\xi}_2 = -2c\,\xi_3\mathcal{F}(\xi), \qquad \dot{\xi}_3 = 2c\,\xi_2\mathcal{F}(\xi). \tag{3.7}$$

Curvature algebra and holonomy

The curvature tensor can be obtained from the spray coefficients. The restriction of the curvature vector fields $R(\hat{E}_i, \hat{E}_j)$ on the indicatrix (3.4) are

$$R_1 := R(\hat{E}_1, \hat{E}_2)\Big|_{\mathcal{I}} = -\frac{\xi_2}{\alpha_\xi}\,\partial_{\xi_1} + \frac{(\xi_1 - c)}{\alpha_\xi}\,\partial_{\xi_2} \qquad \in \mathfrak{X}(\mathcal{I}), \tag{3.8a}$$

$$R_2 := R(\hat{E}_1, \hat{E}_3)\Big|_{\mathcal{I}} = -\frac{\xi_3}{\alpha_\xi}\,\partial_{\xi_1} + \frac{(\xi_1 - c)}{\alpha_\xi}\,\partial_{\xi_3} \qquad \in \mathfrak{X}(\mathcal{I}), \tag{3.8b}$$

$$R_3 := R(\hat{E}_2, \hat{E}_3)\Big|_{\mathcal{I}} = -\frac{\xi_3}{\alpha_\xi}\,\partial_{\xi_2} + \frac{\xi_2}{\alpha_\xi}\,\partial_{\xi_3} \qquad \in \mathfrak{X}(\mathcal{I}), \tag{3.8c}$$

where α_ξ is defined in (3.3). The curvature algebra $\mathfrak{R}$ is generated by the commutators of the vector fields R_i, $i = 1, 2, 3$, that is:

$$\mathfrak{R} = \Big\langle R_1, R_2, R_3 \Big\rangle_{Lie}. \tag{3.9}$$

It is not difficult to calculate the first Lie backets of the curvature vector fields:

$$[R_1, R_3] = \frac{1}{\alpha_\xi} R_2 + \frac{2c\,\xi_2}{\alpha_\xi^2} R_3, \qquad [R_2, R_3] = \frac{-1}{\alpha_\xi} R_1 + \frac{2c\,\xi_3}{\alpha_\xi^2} R_3. \tag{3.10}$$

More generally, we have the following

3.1.1 Lemma. *Let $L_0 = R_2$ and denote $L_k = [L_{k-1}, R_3]$ the Lie bracket of the vector field L_{k-1} and R_2 for $k \geq 1$. Then*

$$L_k = \begin{cases} \dfrac{\epsilon_k}{\alpha_\xi^k} R_2 + 2kc\dfrac{\epsilon_k\,\xi_2}{\alpha_\xi^{k+1}} R_3, & \text{if } k \equiv 0 \mod 2, \\[2ex] \dfrac{\epsilon_k}{\alpha_\xi^k} R_1 - 2kc\dfrac{\epsilon_k\,\xi_3}{\alpha_\xi^{k+1}} R_3, & \text{if } k \equiv 1 \mod 2, \end{cases} \tag{3.11}$$

where $\epsilon_k = -1$, if $k = 4l + 1$ or $k = 4l + 2$ and $\epsilon_k = 1$, if $k = 4l + 3$ or $k = 4l$ for some $l \in \mathbb{N}$.

Proof. We prove the lemma by recurrence. Using the fact that the curvature vector fields (3.8) are tangent to the indicatrix (3.4) one can find for odd k:

$$[L_k, R_3] = \left[\frac{\epsilon_k}{\alpha_\xi^k} R_1, R_3\right] - \left[\frac{\epsilon_k 2kc\,\xi_3}{\alpha_\xi^{k+1}} R_3, R_3\right]$$

$$
\begin{aligned}
&= \frac{\epsilon_k}{\alpha_\xi^{k+1}} \left(R_2 + \frac{2c\,\xi_2}{\alpha_\xi} R_3 \right) + \frac{\epsilon_k 2kc\,\xi_2}{\alpha_\xi^{k+2}} R_3 \\
&= \frac{\epsilon_{k+1}}{\alpha_\xi^{k+1}} R_2 + 2(k+1)c \frac{\epsilon_{k+1}\,\xi_2}{\alpha_\xi^{k+2}} R_3 = L_{k+1},
\end{aligned}
$$

where we used that $\epsilon_{k+1} = \epsilon_k$ when k is odd. For even k we have analogous computation:

$$
\begin{aligned}
[L_k, R_3] &= \left[\frac{\epsilon_k}{\alpha_\xi^k} R_2, R_3 \right] + \left[\frac{\epsilon_k 2kc\,\xi_2}{\alpha_\xi^{k+1}} R_3, R_3 \right] \\
&= \frac{\epsilon_k}{\alpha_\xi^{k+1}} \left(-R_1 + \frac{2c\,\xi_3}{\alpha_\xi} R_3 \right) + \frac{\epsilon_k 2kc\,\xi_3}{\alpha_\xi^{k+2}} R_3 \\
&= \frac{\epsilon_{k+1}}{\alpha_\xi^{k+1}} R_1 - 2(k+1)c \frac{\epsilon_{k+1}\,\xi_3}{\alpha_\xi^{k+2}} R_3 = L_{k+1},
\end{aligned}
$$

where we used that $\epsilon_{i+1} = -\epsilon_i$ when k is even. □

3.1.2 Proposition. *In the presence of external wind W, the curvature algebra $\mathfrak{R}$ of the Finsler metric* (3.2) *is an infinite dimensional Lie subalgebra of $\mathfrak{X}(\mathcal{I})$ of smooth vector fields of the indicatrix* (3.4).

Proof. By definition, the curvature algebra $\mathfrak{R}$ is the Lie subalgebra of $\mathfrak{X}(\mathcal{I})$ generated by the curvature vector fields. In the case of the Finsler metric (3.2), the curvature algebra (3.9) is generated by the three curvature vector fields (3.8) and by their successive Lie brackets. If the external wind W is non-zero, then $c \neq 0$ and, using the notation of Lemma 3.1.1, we have $L_k \in \mathfrak{R}$ for any $k \in \mathbb{N}$. As the formulas of (3.11) show, every L_k can be expressed by some combination of the curvature vector fields, but the coefficients are polynomials with increasing degree in each step. Consequently,

$$
\{\, L_k \mid k \in \mathbb{N} \,\} \subset \mathfrak{R} \quad \Big(\subset \mathfrak{X}(\mathcal{I}) \Big)
$$

is an $\mathbb{R}-$linearly independent infinite set of vector fields in the curvature algebra, that is $\mathfrak{R}$ is an infinite dimensional Lie subalgebra of $\mathfrak{X}(\mathcal{I})$. □

We remark that if there is no external wind, that is $c = 0$, then (3.2) is a Riemann metric: On the indicatrix we have $\alpha_\xi = 1$ and one can easily see that $\mathfrak{R}$ is isomorphic to $\mathfrak{so}(3)$.

3.1.3 Theorem. *The holonomy group $\mathcal{H}ol(\mathcal{Q})$ of the 2-dimensional quantum Zermelo problem in the presence of an external wind W is not a finite dimensional Lie group.*

Proof. We do not know at the moment if the group $\mathcal{H}ol(\mathcal{Q})$ has a smooth Lie group structure or not, but we do know that it is a subgroup of $\mathcal{D}iff^{\infty}(\mathcal{I})$, the diffeormorphism group of the indicatrix $\mathcal{I}$. With $\mathcal{I}$ being compact, the tangent space of the group $\mathcal{H}ol(\mathcal{Q})$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{I})$, see [19, Sec. 4]. The tangent algebra of the holonomy group $\mathcal{H}ol(\mathcal{Q})$ is the holonomy algebra $\mathfrak{hol}(\mathcal{Q})$ and as (2.30) shows, it contains the curvature algebra $\mathfrak{R}$. As Proposition 3.1.2 shows, $\mathfrak{R}$ is infinite dimensional, therefore $\mathfrak{hol}(\mathcal{Q})$ is also an infinite dimensional Lie algebra. Consequently, from (2.30) we get that the holonomy group $\mathcal{H}ol(\mathcal{Q})$ cannot be a finite-dimensional Lie group.

□

We remark that if there is no external wind W, then $\mathcal{Q} = \big(SU(2), \alpha\big)$ is a 3-dimensional Riemannian manifold and its holonomy group $\mathcal{H}ol(\mathcal{Q})$ is the 3-dimensional special orthogonal group $SO(3)$.

Theorem 3.1.3 demonstrates clearly the deep difference between the Riemannian and Finslerian model: in the Riemannian setting, the holonomy group is necessarily a finite dimensional Lie group. But using the Finslerian model in the presence of a magnetic field, this is no longer true: the holonomy group can be a much larger group. Indeed, there are examples, where the topological closure of the holonomy group is the diffeomorphism group of the indicatrix [18, 37]. Until now, results on the holonomy are obtained on cases, where the parallel translations are linear (thoroughly analyzed by Z. Szabó [49]) or the flag-curvature is constant. We note, that in the 2-dimensional quantum Zermelo problem considered above the flag-curvature is not constant: with an explicit computation one can show, that $\mathcal{Q}$ has a non-constant scalar flag-curvature.

3.2 Holonomy of projectively flat Randers metrics of constant curvature

Our aim is to describe the holonomy structure of projectively flat Randers two-manifolds with non-zero constant flag curvature. As a first step, we investigate the holonomy of the standard model described in Section 1.2.

Let $(\mathbb{B}^2, \mathcal{F}_a)$ be the Finsler two-manifold where $\mathbb{B}^2$ is the unit ball in $\mathbb{R}^2$ and $\mathcal{F}_a$ is the Finsler function given by (1.20) where $a = (a_1, 0) \in \mathbb{R}^2$ is a nonzero constant vector with $|a_1| < 1$. We have the following

3.2.1 Proposition. *The holonomy group of* $(\mathbb{B}^2, \mathcal{F}_a)$ *is maximal and* $\mathcal{H}ol_x(M)^c$ *is diffeomorphic to* $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$.

Proof. We consider the case when $\epsilon = +1$ in the expression (1.20) of $\mathcal{F}_a$. The computation when $\epsilon = -1$ is analogous. The projective factor $\mathcal{P}$ and the geodesic coefficients G^i_j can be easily computed by formula (1.21) and (1.17). The expression of the curvature vector field $\xi = R\left(\partial_{x_1}, \partial_{x_2}\right)$ at the point $0 \in \mathbb{R}^2$ is

$$\begin{aligned} \xi = R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = \frac{1}{4}\frac{y_2\left(a_1 y_1 + \|y\|\right)}{\|y\|}\frac{\partial}{\partial y_1} \\ -\frac{1}{4}\left(y_1 + y_1 a_1^2 + 2\, a_1\|y\|\right)\frac{\partial}{\partial y_2}. \end{aligned} \tag{3.12}$$

Since the Minkowski norm at $0 \in \mathbb{D}^2$ is $\mathcal{F}_a(0, y) = \|y\| + \langle a, y\rangle$, the indicatrix $\mathcal{I}_0 \subset T_0M$ at 0 is defined by the equation $\sqrt{y_1^2 + y_2^2} + a_1 y_1 = 1$. Using polar coordinates (r, t) on $T_0\mathbb{R}^2$, the equation of the indicatrix $\mathcal{I}_0$ is $r(1 + a_1 \cos t) = 1$. A parametrization of $\mathcal{I}_0$ is given by

$$\phi(t) = \Big((y_1(t), y_2(t)\Big) = \left(\frac{\cos t}{1 + a_1 \cos t}, \frac{\sin t}{1 + a_1 \cos t}\right), \tag{3.13}$$

in terms of the parameter t. Using this parametrisation the coordinate expression of the restriction $\xi_0 := \xi\Big|_{\mathcal{I}_0}$ of the curvature vector field (3.12) on the indicatrix $\mathcal{I}_0$ is

$$\xi_0 := \omega(t)\frac{d}{dt} \tag{3.14}$$

where

$$\omega(t) := -\tfrac{1}{4}(1 + a_1 \cos t)^2. \tag{3.15}$$

Let us introduce the notation

$$\Sigma_n := \mathrm{Span}_{\mathbb{R}} \left\{ \xi_0^{l,m} \mid l+m \leq n \right\}, \tag{3.16}$$

where

$$\xi_0^{l,m} := (\sin^l t \cos^m t) \cdot \xi_0 \in \mathfrak{X}(\mathcal{I}_0) \tag{3.17}$$

are vector fields on the indicatrix $\mathcal{I}_0$ defined as functional multiples of (3.14). We have the following

3.2.2 Lemma. *For any $n \in \mathbb{N}$ we have $\Sigma_n \subset \mathfrak{hol}_0^*$.*

3.2.3 Remark. Using the the Pythagorean trigonometric identity $\sin^2 t + \cos^2 t = 1$, every elements of Σ_n can be expressed as a linear combination of the elements $\xi_0^{0,m} = \cos^m t \xi_0$ and $\xi_0^{1,m-1} = \sin t \cos^{m-1} t\, \xi_0$, $0 \leq m \leq n$, that is

$$\Sigma_n = \mathrm{Span}_{\mathbb{R}} \left\{ \xi_0^{0,m},\ \xi_0^{1,m-1} \mid 0 \leq m \leq n \right\}. \tag{3.18}$$

Proof of the Lemma. Taking into account Remark 3.2.3 we prove the lemma by mathematical induction by showing that the generating elements (3.18) of Σ_n are elements of $\mathfrak{hol}_0^*$.

• *First step.* From the definition of the infinitesimal holonomy algebra (see section 2.3) we know that $\xi_0^{0,0} = \xi_0$ given by (3.14) is an element of $\mathfrak{hol}_0^*$. Moreover, as (2.28) shows, the restriction of successive covariant derivatives of the curvature vector field (3.12) on $\mathcal{I}_0$ are also elements of $\mathfrak{hol}_0^*$. They can be expressed in terms of multiples of (3.14). Computing the first covariant derivatives we find that

$$(\nabla_1 \xi)\Big|_{\mathcal{I}_0} = -\tfrac{3}{2} \,(a_1 - \cos t)\, \xi_0, \tag{3.19a}$$

$$(\nabla_2 \xi)\Big|_{\mathcal{I}_0} = \tfrac{3}{2} \sin t\, \xi_0, \tag{3.19b}$$

Using a linear combination of (3.19a) and (3.19b) we get that $\xi_0^{1,0} = \sin t\, \xi_0$ and $\xi_0^{0,1} = \cos t\, \xi_0$ are element of $\mathfrak{hol}_0^*$. Therefore we have

$$\Sigma_1 = \left\{ \xi_0,\ \sin t\, \xi_0,\ \cos t\, \xi_0 \right\} \subset \mathfrak{hol}_0^*, \tag{3.20}$$

that is the statement of the Lemma is correct for $n = 1$.

• *Second step.* By definition, the second covariant derivatives of the curvature vector field are also elements of the infinitesimal holonomy algebra. Computing them we can find that

$$(\nabla_1\nabla_1\xi)\Big|_{\mathcal{I}_0} = \tfrac{3}{4}\Big(5a_1^2 - a_1\cos^3 t - 5a_1\cos t + 3\cos^2 t + 1\Big)\,\xi_0, \quad (3.21a)$$

$$(\nabla_1\nabla_2\xi)\Big|_{\mathcal{I}_0} = -\tfrac{3}{4}\Big(a_1\cos^2 t - 3a_1 - 4\cos t\Big)\sin t\,\xi_0, \quad (3.21b)$$

$$(\nabla_2\nabla_2\xi)\Big|_{\mathcal{I}_0} = \tfrac{3}{4}\Big(a_1\cos^3 t + 5 - 4\cos^2 t\Big)\,\xi_0. \quad (3.21c)$$

Using linear combinations of the elements (3.20) of Σ_1 and (3.21a)–(3.21c) we get that $\{\cos^2 t\,\xi_0,\ \sin t\cos t\,\xi_0\} \subset \mathfrak{hol}_0^*$. Completing this set with the elements of (3.20) we get that

$$\Sigma_2 \subset \mathfrak{hol}_0^*.$$

• *Third step.* Let us suppose that the statement of the lemma is true for some $n \in \mathbb{N}$, that is $\Sigma_n \subset \mathfrak{hol}_0^*$. We will show that $\Sigma_{n+1} \subset \mathfrak{hol}_0^*$ too. According to the Remark, Σ_{n+1} is generated by the elements

$$\Big\{\,\xi_0^{0,m},\ \xi_0^{1,m-1} \mid 0 \leq m \leq n\,\Big\} \cup \Big\{\,\xi_0^{0,n+1},\ \xi_0^{1,n}\,\Big\} \quad (3.22)$$

One can observe that the vector fields of the first set are elements of Σ_n, and by the inductive hypotheses, they are elements of $\mathfrak{hol}_0^*$. Hence, to prove the lemma we have to show that $\xi_0^{0,n+1}$ and $\xi_0^{1,n}$ are elements of $\mathfrak{hol}_0^*$. We have

$$\begin{aligned}\Big[\xi_0, \xi_0^{0,n-1}\Big] &= \Big[\xi_0, \cos^{n-1} t\,\xi_0\Big] = \Big(\mathcal{L}_{\xi_0}(\cos^{n-1} t)\Big)\,\xi_0\\ &= -(n-1)\omega(t)(\sin t\cos^{n-2} t)\,\xi_0 \overset{(3.15)}{=}\\ &= \tfrac{n-1}{4}(1 + 2a_1\cos t + a_1^2\cos^2 t)(\sin t\cos^{n-2} t)\,\xi_0\\ &= \tfrac{n-1}{4}\,\xi_0^{1,n-2} + \tfrac{a_1(n-1)}{2}\,\xi_0^{1,n-1} + \tfrac{a_1^2(n-1)}{4}\,\xi_0^{1,n}.\end{aligned}$$

By the inductive hypothesis, the Lie bracket on the left hand side and the first two terms in the last line are elements of $\mathfrak{hol}_0^*$. Consequently the last one must be also an element of $\mathfrak{hol}_0^*$. Moreover, the coefficients of $\xi_0^{1,n}$ is a nonzero constant, therefore we get that $\xi_0^{1,n} \in \mathfrak{hol}_0^*$.

Similarly, computing the Lie bracket of the elements ξ_0 and $\xi_0^{1,n-2}$ of the Lie algebra $\mathfrak{hol}_0^*$ we get

$$\Big[\xi_0,\xi_0^{1,n-2}\Big] = \Big[\xi_0, \sin t\cos^{n-2} t\,\xi_0\Big] = \mathcal{L}_{\xi_0}(\sin t\cos^{n-2} t)\,\xi_0$$

$$\begin{aligned}&= \omega(t)(\cos t\cos^{n-2}t - (n-2)\sin^2 t\cos^{n-3}t)\,\xi_0 \overset{(3.15)}{=}\\ &= \tfrac{n-3}{4}\xi_0^{0,n-3} + \tfrac{a_1(n-2)}{2}\xi_0^{0,n-2} + \tfrac{a_1^2(n-2)-(n-1)}{4}\xi_0^{0,n-1}\\ &\quad - \tfrac{a_1(n-1)}{2}\xi_0^{0,n-1} - \tfrac{a_1^2(n-1)}{4}\xi_0^{0,n+1}\end{aligned}$$

From the inductive hypothesis we know that the Lie bracket on the left hand side and the first four terms in the last line on the right hand side are elements of $\mathfrak{hol}_0^*$, therefore the last one must be also an element of $\mathfrak{hol}_0^*$. Since the coefficient of $\xi_0^{0,n+1}$ is nonzero we get that $\xi_0^{0,n+1} \in \mathfrak{hol}_0^*$. Consequently, the vector fields (3.22) generating Σ_{n+1} are all elements of $\mathfrak{hol}_0^*$ and $\Sigma_{n+1} \subset \mathfrak{hol}_0^*$. □

Proof of Proposition 3.2.1. From the multiple-angle formulas of the sine and cosine functions

$$\sin nt = \sum_{k=0}^{n} \binom{n}{k} \cos^k t \sin^{n-k} t \sin\left(\tfrac{n-k}{2}\pi\right),$$

$$\cos nt = \sum_{k=0}^{n} \binom{n}{k} \cos^k t \sin^{n-k} t \cos\left(\tfrac{n-k}{2}\pi\right),$$

we get that the vector fields $\sin nt\,\xi_0, \cos nt\,\xi_0 \in \mathfrak{X}(\mathcal{I}_0)$ can be expressed as a linear combination of elements of Σ_n:

$$\sin nt\,\xi_0 = \sum_{k=0}^{n} \binom{n}{k} \sin\left(\tfrac{n-k}{2}\pi\right) \xi_0^{k,n-k}, \tag{3.23a}$$

$$\cos nt\,\xi_0 = \sum_{k=0}^{n} \binom{n}{k} \cos\left(\tfrac{n-k}{2}\pi\right) \xi_0^{k,n-k}. \tag{3.23b}$$

On the other side, Lemma 3.2.2 shows that the vector fields of Σ_n are elements of the holonomy algebra $\mathfrak{hol}_0^*$. Therefore, the vector fields (3.23a) and (3.23b), $n \in \mathbb{N}$, are element of $\mathfrak{hol}_0^*$ and

$$\operatorname{Span}\{\, \sin nt\,\xi_0,\ \cos nt\,\xi_0 \mid n = 0, 1, \dots \} \subset \mathfrak{hol}_0^*. \tag{3.24}$$

Moreover, any 2π periodic smooth function can be approximated uniformly by the arithmetical means of the partial sums of its Fourier series [23, Theorem 2.12]. In particular the functions $(\sin nt)/\omega(t)$ and $(\cos nt)/\omega(t)$ can be approximated uniformly by their Fourier sums. Hence we get

$$\left\{\tfrac{d}{dt},\ \cos nt\tfrac{d}{dt},\ \sin nt\tfrac{d}{dt}\right\}_{n\in\mathbb{N}} \subset \left(\left\{\operatorname{Span}\left\{\xi_0,\ \cos nt\,\xi_0,\ \sin nt\,\xi_0\right\}_{n\in\mathbb{N}}\right\}\right)^c \subset \mathfrak{hol}_0^*. \tag{3.25}$$

The Lie algebra generated by the left hand side of (3.25) is diffeomorphic to the *Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ on $\mathbb{S}^1$. Therefore from (3.25) we get that the infinitesimal holonomy algebra contains a Lie algebra diffeomorphic to the *Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ on $\mathbb{S}^1$. Hence from Proposition 5.1 of [37] we get that the holonomy group $\mathsf{Hol}_0(\mathbb{D}^2)$ is maximal and $\left(\mathsf{Hol}_0(\mathbb{D}^2)\right)^c$ is diffeomorphic to $\mathcal{D}iff_+^\infty(\mathbb{S}^1)$. □

Using Z. Shen's classification theorem of Randers manifolds we can get the following

3.2.4 Theorem. *The holonomy group of a simply connected non-Riemannian projectively flat Randers two-manifold of constant non-zero flag curvature is maximal and $\mathcal{H}ol(M)^c$ is diffeomorphic to the orientation preserving diffeomorphism group of $\mathbb{S}^1$, that is*

$$\mathcal{H}ol(M)^c \cong \mathcal{D}iff_+^\infty(\mathbb{S}^1).$$

Proof. Let $(M, \mathcal{F})$ be a simply connected non-Riemannian projectively flat Finsler two-manifold of constant non-zero flag curvature and $x_0 \in M$. Since rescaling the metric by a constant factor does not change the connection and the parallel translation, it does not change the holonomy group either. Hence we can suppose that the metric is normalized so that the curvature is $\lambda = -\frac{1}{4}$. Using Shen's results, $\mathcal{F}$ can be locally expressed in the form $\mathcal{F}_a$ given in (1.20) where $a = (a_1, 0) \in \mathbb{R}^2$ is a nonzero constant vector with $|a_1| < 1$. From Proposition 3.2.1 we get, that the closed holonomy group of $(\mathbb{D}^2, \mathcal{F}_a)$ is maximal and diffeomorphic to $\mathcal{D}iff_+^\infty(\mathbb{S}^1)$, therefore the same is true for the closed holonomy group $\mathcal{H}ol_x(M)^c$ of $(M, \mathcal{F})$ at $x_0 \in M$. □

We can obtain the following classification:

3.2.5 Corollary. *The closure of the holonomy group $\mathcal{H}ol(M)$ of a simply connected, locally projectively flat Randers two-manifold of constant flag curvature λ is*

1. *the trivial group $\{id\}$, when $\lambda = 0$;*
2. *the rotation group $SO(2)$, when $\lambda \neq 0$ and the metric is Riemannian;*
3. *the orientation preserving diffeomorphism group of the circle $\mathcal{D}iff_+^\infty(\mathbb{S}^1)$, when $\lambda \neq 0$ and the metric is non-Riemannian.*

Proof. The holonomy structure of projectively flat Finsler manifolds was investigated in [38]: It has been proved that the holonomy group of projectively flat Finsler manifold is *a)* finite dimensional if $\lambda = 0$ or the metric is Riemannian, and *b)* infinite dimensional if $\lambda \neq 0$ and the metric is non-Riemannian. It is clear that the holonomy structures listed in *(1)* and *(2)* correspond to the (already well known) finite dimensional holonomy cases. Moreover, when $\lambda \neq 0$ and the metric is non-Riemannian we get *(3)* from Theorem 3.2.4. □

Chapter 4

Density of Finsler metrics with infinite dimensional holonomy group

Since there are examples of Finsler manifolds with finite and also with infinite dimensional holonomy group, a natural and fundamental question is whether for a generic Finsler manifold the holonomy group is infinite-dimensional. Its simplest version was explicitly asked by S.-S. Chern et al in [11, page 85]. In this section we prove that for a generic Finsler manifold the holonomy group is infinite-dimensional. More precisely, we show in Theorem 4.4.1 that in the set $\mathcal{F}$ of C^∞-smooth Finsler metrics on a manifold M of dimension $n \geq 2$, there exists a subset $\widetilde{\mathcal{F}}$ of Finsler metrics with infinite dimensional holonomy group, which is open and everywhere dense in any $\mathcal{C}^m$-topology, $m \geq 8$. This result implies that, in contrast to the Riemannian case, the closure of the holonomy group is not a compact group for most Finlser metric. Similar results for the *linear holonomy group* (defined via the linear parallel transport) were recently obtained in [22].

The proof is organised as follows. In Section 4.1 we show that if a Lie algebra of vector fields is sufficiently generic, that is satisfying the 3-jet generating property, then it is infinite-dimensional. Then in Section 4.2 we show that the standard Funk metric has 'sufficiently generic' holonomy algebra in that sense. In Section 4.3 we prove that, using the the standard Funk metric, for any Finsler metric there exists an arbitrarily small perturbation resulting a new Finsler metric with 'sufficiently large' holonomy algebra. Then the main theorem about the density of the Finsler metrics with infinite dimensional holonomy group can be found in Section 4.4.

4.1 3-jet generating Lie algebras of vector fields

4.1.1 Definition. A set $\mathcal{V} \subset \mathfrak{X}(M)$ of vector fields on a manifold M is called

- k-jet generating at $x \in M$ if the natural map $j^k_x \colon \mathcal{V} \to J^k_x(\mathfrak{X}(M))$ is surjective, and
- jet generating on M if at any $x \in M$ and for any $k \geq 0$ it is k-jet generating.

In particular, an algebra $\mathfrak{g}$ of vector fields on $U \subseteq \mathbb{R}^n$ is called 3-*jet generating* at $x \in U$, if every vector field can be approximated at x with order three by a vector field from the algebra. For example, if the algebra is *locally transitive* at x, i.e., if the elements of the algebra at x span the whole T_xU, then it is $0-$jet generating. We have the following

4.1.2 Theorem. *Let $\mathfrak{g}$ be a Lie algebra of vector fields on a manifold U. If there exists a point where it is 3-jet generating, then $\mathfrak{g}$ is infinite-dimensional.*

We remark that if dimension U is 1, the result is known and is due to Sophus Lie, see e.g. [27, Theorem 2.70]. As examples show (see e.g. the tables at the back of [27] where vector field algebras of arbitrary finite dimension are listed), the 3-jet generating property is important.

Proof of Theorem 4.1.2. We prove the theorem by contradiction: let us suppose that $\mathfrak{g} \subset \mathfrak{X}(U)$ is a *finite* dimensional Lie algebra on an n-dimensional manifold U and it is generating the third order jets at $x_0 \in U$. As before, the last property means that the 3-jet projection $\mathfrak{g} \to J^3_{x_0}(\mathfrak{X}(U))$ is onto.

We remark that a manifold with a finite dimensional Lie algebra of vector fields with locally transitive action is real analytic (in the sense that there exist a real-analytic atlas such that the vector fields of the

Lie algebra are real-analytic). Indeed, a finite dimensional Lie algebra generates a Lie group, and one can provide an analytic atlas on this Lie group so that the group multiplication is analytic. Moreover, local Lie subgroups are real analytic submanifolds, because they are images of the exponential map. For more about Lie groups and related topics, we refer to [24, 28]. It follows that locally, a manifold with transitive action of a Lie group is the factor group of the Lie group by the stabilizer of one element, which is a local Lie subgroup, then it is also analytic.

For a more detailed explanation denote the vector fields which generate the algebra by $V_1, ..., V_N$ and the structure constants of the Lie algebra by by C^i_{jk}. By definition,

$$[V_i, V_j] = \sum_{k=1}^{N} C^k_{ij} V_k.$$

Assume that the first $n = \dim M$ vector fields $V_1, ..., V_n$ are linearly independent at the point p and denote the time-t-flow of the vector field V_i by Φ^i_t. For every small t, it is a local diffeomorphism.

The key observation of the proof of this statement is the following:

4.1.3 Proposition. *For an arbitrary point q, $i = 1, ..., n$, $j = 1, ..., N$ and for any small t we consider the vectors*

$$U_j(t) := \Phi^{i*}_t \left(V_j(\Phi^i_t(q)) \right).$$

These vectors lie in T_qM since the vectors $V_j(\Phi^i_t(q))$ lie in $T_{\Phi^i_t(q)}M$. Then, $U_j(t)$ depend real-analytically on t.

Proof. Indeed, the vectors $U_j(t)$ satisfy the system of ODE

$$-\tfrac{d}{dt} U_j = \sum_{k=1}^{N} C^k_{ij} U_k.$$

This is a linear system of ODE of $N \times n$ equations in Euler form on $N \times n$ unknown functions with constant coefficients and its solution is real-analytic. □

Using 4.1.3, fix a point p in a small neighborhood and consider local coordinates $y_1, \ldots, y_n$ as follows: for a point q its coordinates are the numbers $y_1, \dots, y_n$ such that $q = \Phi^1_{y_1} \circ \cdots \circ \Phi^n_{y_n}(p)$. The inverse function theorem guarantees existence of such coordinates. Iteratively, applying 4.1.3, we see that in these coordinates the components of the vector fields are real-analytic.

We say that the order of singularity of an element $v \in \mathfrak{g}$ at x_0 is $k \in \mathbb{N}$, noted as $\mathcal{O}_{x_0}(v) = k$, if the value and all partial derivatives up to order k at x_0 are zero, and v has a non-vanishing $(k+1)$st order derivative. Each nonzero element has finite order by analyticity. Let us consider the set $\mathfrak{g}_1 \subset \mathfrak{g}$ with order of singularity at least one, that is

$$\mathfrak{g}_1 := \left\{ v \in \mathfrak{g} \quad \middle| \quad v(x_0) = 0,\ \frac{\partial v}{\partial x_i}(x_0) = 0, \quad i = 1, \dots, n \right\}.$$

It is easy to see that $\mathfrak{g}_1$ is a Lie subalgebra of $\mathfrak{g}$. Indeed, the jth component of the commutator of two vector fields $v, u \in \mathfrak{g}_1$ is given by

$$[v,u]_j = \sum_i \left(\tfrac{\partial v_j}{\partial x_i} u_i - \tfrac{\partial u_j}{\partial x_i} v_i \right)$$

and has singularity of order at least two at x_0. Actually, for any two vector fields V, U from $\mathfrak{g}_1$ such that V has order of singularity k and U has order of singularity m their commutator has order of singularity $k + m$.

Since $\mathfrak{g}$ is finite dimensional, so is $\mathfrak{g}_1$. It follows that the order of singularity is bounded on $\mathfrak{g}_1$. Indeed, if not, then there would be a sequence of vectors in $\mathfrak{g}_1$ with strictly monotone increasing order of singularity at x_0 which would produce an infinite number of linearly independent elements which is impossible.

Let $v_1 \in \mathfrak{g}_1$ be a non-zero element with maximal order $\mathcal{O}_{x_0}(v_1) = k$ of singularity at x_0. Using the 3-jet generating property, we have $k \geq 2$. Then, for any $v \in \mathfrak{g}_1$ we have $[v, v_1] \in \mathfrak{g}_1$ and $\mathcal{O}_{x_0}([v, v_1]) > k$. Since k is maximal, it follows that $[v, v_1] = 0$, and it shows that v_1 commutes with every elements of $\mathfrak{g}_1$.

On the other hand, it is possible to choose a point $\hat{x}_0 \in U$ in a neighbourhood of x_0 such that $v_1(\hat{x}_0) \neq 0$ and we have that $\mathfrak{g}_1$ has the 1-jet generating property at $\hat{x}_0$.

To see that this is possible we will use a local coordinate system centered at x_0. Since $v_1 \not\equiv 0$, therefore (using the analytical property in the neighbourhood of x_0) one can find a direction in which a higher order derivative at x_0 is nonzero. By rotating the coordinate system if it is necessary, one can suppose that this direction corresponds to the first coordinate. It follows that for sufficiently small $\varepsilon \in \mathbb{R}$ we will have points

$$\hat{\mathbf{x}} = (\varepsilon, 0, \ldots, 0) \tag{4.1}$$

with the property $v_1(\hat{\mathbf{x}}) \neq 0$.

We will show now that one can choose an arbitrarily small $\varepsilon_0 > 0$ conveniently, to have the 1-jet generating property at the corresponding point $\hat{\mathbf{x}}_0 = (\varepsilon_0, 0, \ldots 0)$. From the definition of $\mathfrak{g}_1 (\subset \mathfrak{g})$, we get that the value and the first derivatives of the coordinate functions of the elements of $\mathfrak{g}_1$ at x_0 are vanishing. The 3-jet generating property of $\mathfrak{g}$ at x_0 means that for any 3-jet $\mathcal{J}_3$ at x_0 we have an element $X \in \mathfrak{g}$ such that $\mathcal{J}_3 = j_{3,x_0}(X)$. Using this property, one can consider $\mathfrak{X} := \{X_{ij}, X_{ijk}\}_{i \leq j \leq k}$ with

$$\mathfrak{X} \subset \mathfrak{g}_1, \tag{4.2}$$

where the coordinate functions of the elements of $\mathfrak{X}$ are of the form

$$X_{ij} = x_i x_j + \sum_{|\mathbf{m}| \geq 4} P_{ij}^{\mathbf{m}} x_{\mathbf{m}} = x_i x_j + \mathcal{O}_{ij}(\mathbf{x}) \tag{4.3a}$$

$$X_{ijk} = x_i x_j x_k + \sum_{|\mathbf{m}| \geq 4} P_{ijk}^{\mathbf{m}} x_{\mathbf{m}} = x_i x_j x_k + \mathcal{O}_{ijk}(\mathbf{x}), \tag{4.3b}$$

The $P_{ij}^{\mathbf{m}}$ and $P_{ijk}^{\mathbf{m}}$ are constants in the above formulas. (In order to simplify the notation we use bold letter $\mathbf{m}$ for multi-index, its length is denoted by $|\mathbf{m}|$, $\mathcal{O}_{ij}(\mathbf{x}) := \sum_{|\mathbf{m}| \geq 4} P_{ij}^{\mathbf{m}} x_{\mathbf{m}}$, and $\mathcal{O}_{ijk}(\mathbf{x}) := \sum_{|\mathbf{m}| \geq 4} P_{ijk}^{\mathbf{m}} x_{\mathbf{m}}$ denote the higher order terms.) Since $\mathfrak{g}_1$ is a vector space, for any constant $a_{ij}, b_{ijk} \in \mathbb{R}$, we have

$$f = a_{ij} X_{ij} + b_{ijk} X_{ijk} \quad \in \mathfrak{g}_1. \tag{4.4}$$

To show the 1-jet generating property at some point $\mathbf{x}$, it is sufficient to prove that for any 1-jet $\mathcal{J}_1$, it is possible to choose a_{ij} and b_{ijk} such that the 1-jet of (4.4) satisfies $\mathcal{J}_1 = j_{1,\mathbf{x}} f$. In other words, for any given $n+1$ scalars $c_0, c_1, \ldots c_n \in \mathbb{R}$, it is possible to choose a_{ij} and b_{ijk} such that f given by (4.4) satisfies

$$c_0 = f(\mathbf{x}), \qquad c_1 = \frac{\partial f}{\partial x_1}(\mathbf{x}), \qquad c_2 = \frac{\partial f}{\partial x_2}(\mathbf{x}), \qquad \ldots \qquad c_n = \frac{\partial f}{\partial x_n}(\mathbf{x}).$$

In the sequel we investigate the 1-jet generating property at points $\hat{\mathbf{x}}$ having the special form (4.1). Using some of the elements (4.3a) and (4.3b) we consider the subset $\mathfrak{X}_1 \subset \mathfrak{X}$ where:

$$\mathfrak{X}_1 := \{X_{111}, X_{11}, X_{12}, \dots X_{1n}\}. \tag{4.5}$$

Since $\mathfrak{X}_1 \subset \mathfrak{g}_1$, every linear combination with constant coefficients

$$f = b_{111}X_{111} + a_{11}X_{11} + a_{12}X_{12} + \cdots + a_{1n}X_{1n} \tag{4.6}$$

is also an element of $\mathfrak{g}_1$. We get

$$\begin{aligned}
\partial_1 X_{111} &= \partial_1 \Big((x_1)^3 + \mathcal{O}_{111}(\hat{\mathbf{x}})\Big) \overset{(4.1)}{=} 3\varepsilon^2 + \partial_1\mathcal{O}_{111}(\varepsilon) \\
\partial_1 X_{11} &= \partial_1 \Big((x_1)^2 + \mathcal{O}_{11}(\hat{\mathbf{x}})\Big) \overset{(4.1)}{=} 2\varepsilon + \partial_1\mathcal{O}_{11}(\varepsilon) \\
\partial_1 X_{1\alpha} &= \partial_1 \left(x_1 x_\alpha + \mathcal{O}_{1\alpha}(\hat{\mathbf{x}})\right) \overset{(4.1)}{=} \partial_1\mathcal{O}_{1\alpha}(\varepsilon) && \alpha = 2, \dots, n \\
\partial_\beta X_{11} &= \partial_\beta \Big((x_1)^3 + \mathcal{O}_{111}(\hat{\mathbf{x}})\Big) \overset{(4.1)}{=} \partial_\beta\mathcal{O}_{111}(\varepsilon) && \beta = 2, \dots, n \\
\partial_\beta X_{11} &= \partial_\beta \Big((x_1)^2 + \mathcal{O}_{11}(\hat{\mathbf{x}})\Big) \overset{(4.1)}{=} \partial_\beta\mathcal{O}_{11}(\varepsilon) && \beta = 2, \dots, n \\
\partial_\beta X_{1\alpha} &= \partial_\beta \left(x_1 x_\alpha + \mathcal{O}_{1\alpha}(\hat{\mathbf{x}})\right) \overset{(4.1)}{=} \delta_{\alpha\beta}x_1 + \partial_\beta\mathcal{O}_{1\alpha}(\varepsilon) && \alpha, \beta = 2, \dots, n
\end{aligned}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, and we use a simplified notation $\mathcal{O}_{ij}(\varepsilon)$ etc. for the function obtained by evaluating $\mathcal{O}_{ij}(\hat{\mathbf{x}})$ at a point of the form (4.1). We note that for every index j, k (resp. i, j, k), $\mathcal{O}_{jk}(\varepsilon)$ is at least a fourth order, $\partial_i\mathcal{O}_{jk}(\varepsilon)$ is at least a third order – or eventually vanishing – function in the variable ε. For the function f given by (4.6) we obtain

$$f(\mathbf{x}) = a_{11}x_1^2 + b_{111}x_1^3 + \sum_{\alpha=2}^{n} a_{1\alpha}x_1x_\alpha + b_{111}\mathcal{O}_{111}(\mathbf{x}) + \sum_{i=1}^{n} a_{1i}\mathcal{O}_{1i}(\mathbf{x}) \tag{4.7a}$$

$$\partial_1 f(\mathbf{x}) = 2a_{11}x_1 + \sum_{\alpha=2}^{n} a_{1\alpha}x_\alpha + \sum_{i=1}^{n} a_{1i}\partial_1\mathcal{O}_{1i}(\mathbf{x}) \tag{4.7b}$$

$$\partial_\beta f(\mathbf{x}) = a_{1\beta}x_1 + \sum_{i=1}^{n} a_{1i}\partial_\beta\mathcal{O}_{1i}(\mathbf{x}) \quad \beta = 2, \dots n, \tag{4.7c}$$

Using (4.1) at $\hat{\mathbf{x}}$ we get

$$f(\hat{\mathbf{x}}) = a_{11}\varepsilon^2 + b_{111}\varepsilon^3 + b_{111}\mathcal{O}_{111}(\varepsilon) + \sum_{i=1}^{n} a_{1i}\mathcal{O}_{1i}(\varepsilon) \tag{4.8a}$$

$$\partial_1 f(\hat{\mathbf{x}}) \overset{(4.1)}{=} a_{11}(2\varepsilon + \partial_1\mathcal{O}_{11}(\varepsilon)) + b_{111}(3\varepsilon^2 + \partial_1\mathcal{O}_{111}(\varepsilon)) + \sum_{\alpha=2}^{n} a_{1\alpha}\partial_1\mathcal{O}_{1\alpha}(\varepsilon) \tag{4.8b}$$

(4.8c)

$$\partial_\beta f(\hat{\mathbf{x}}) \overset{(4.1)}{=} a_{1\beta}\,\varepsilon + b_{111}\partial_\beta\mathcal{O}_{111}(\varepsilon) + \sum_{i=1}^{n} a_{1i}\partial_\beta\mathcal{O}_{1i}(\varepsilon) \quad \beta = 2, \dots n, \tag{4.8d}$$

It follows that (4.1) gives a system of $n+1$ linear equations on the $n+1$ unknowns b_{111}, a_{11}, $\dots a_{1n}$:

$$\begin{pmatrix} & & \\ & A(\varepsilon) & \\ & & \end{pmatrix} \begin{pmatrix} b_{111} \\ a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{4.9}$$

where $A(\varepsilon)$ is the quadratic matrix

$$A(\varepsilon) = \begin{pmatrix} \varepsilon^3 & \varepsilon^2 & 0 & \dots & 0 \\ 3\varepsilon^2 & 2\varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \varepsilon \end{pmatrix} +$$

$$+ \begin{pmatrix} \mathcal{O}_{111}(\varepsilon) & \mathcal{O}_{11}(\varepsilon) & \mathcal{O}_{12}(\varepsilon) & \dots & \mathcal{O}_{1n}(\varepsilon) \\ \partial_1\mathcal{O}_{111}(\varepsilon) & \partial_1\mathcal{O}_{11}(\varepsilon) & \partial_1\mathcal{O}_{12}(\varepsilon) & \dots & \partial_1\mathcal{O}_{1n}(\varepsilon) \\ \partial_2\mathcal{O}_{111}(\varepsilon) & \partial_2\mathcal{O}_{11}(\varepsilon) & \partial_2\mathcal{O}_{12}(\varepsilon) & \dots & \partial_2\mathcal{O}_{1n}(\varepsilon) \\ & & & \ddots & \\ \partial_n\mathcal{O}_{111}(\varepsilon) & \partial_n\mathcal{O}_{11}(\varepsilon) & \partial_n\mathcal{O}_{12}(\varepsilon) & \dots & \partial_n\mathcal{O}_{1n}(\varepsilon) \end{pmatrix}.$$

Since $\mathcal{O}_{111}(\varepsilon)$ and $\mathcal{O}_{1i}(\varepsilon)$ are at least fourth order, $\partial_i\mathcal{O}_{jk}(\varepsilon)$ is at least third order (or eventually vanishing) functions in ε for all indices i (resp. i, j, k), we get that

$$\det(A(\varepsilon)) = -\varepsilon^{n+3} + \mathcal{O}(\varepsilon) \tag{4.10}$$

where $\mathcal{O}(\varepsilon)$ is at least an $(n+4)$th order (or eventually vanishing) function of the variable ε. It follows that there exists ε_0, arbitrarily close to 0, for which $A(\varepsilon_0)$ is regular and therefore the corresponding system (4.9) admits a (unique) solution. This ε_0 corresponds to a point $\hat{\mathbf{x}}_0$ of the form (4.1). From the above argument we obtained that at this point, for any given data $c_0, c_1, \dots c_n$, one can build an element f of $\mathfrak{g}_1$ such that the value of f and the value of its first order partial derivatives at $\hat{\mathbf{x}}_0$ coincide with the given data. That means, we have at 1-jet generating property of $\mathfrak{g}_1$ at $\hat{\mathbf{x}}_0$ is satisfied.

Then it follows that one can choose an element $v_2 \in \mathfrak{g}_1$ such that $v_2(\hat{x}_0) = 0$ but $\mathcal{D}_{v_1} v_2 \neq 0$. Therefore the commutator $[v_1, v_2]$ at $\hat{x}_0$ is non zero. This is a contradiction since v_1 is an element which commutes with every element of $\mathfrak{g}_1$. Theorem 4.1.2 is proved.

4.2 The 3-jet generating property of the Funk holonomy algebra

In this section we will investigate the holonomy structure of the Funk metric introduced in Section 1.2.

4.2.1 Remark. The holonomy of $(\mathbb{B}^2, F_{\mathbb{B}^2})$ was investigated in [37, Chapter 5]. It was proved that the infinitesimal holonomy algebra $\mathfrak{hol}^*_o(F_{\mathbb{B}^2})$ contains the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ whose elements are vector fields $f\frac{d}{dt}$ such that $f(t)$ has finite Fourier series. One has

$$\mathsf{F}(\mathbb{S}^1) \;\subset\; \mathfrak{hol}^*_o(F_{\mathbb{B}^2}) \;\subset\; \mathfrak{X}(\mathbb{S}^1). \tag{4.11}$$

Since $\mathsf{F}(\mathbb{S}^1)$ is dense in $\mathfrak{X}(\mathbb{S}^1)$, we get the same from (4.11) for $\mathfrak{hol}^*_o(F_{\mathbb{B}^2})$. Using the exponential map, one can obtain from (2.30) that the closure of the holonomy group of the Finsler surface $(\mathbb{B}^2, F_{\mathbb{B}^2})$ is $\mathcal{D}iff_+(\mathbb{S}^1)$, the group of orientation preserving diffeomorphisms of the circle [37, Theorem 5.2]. In [38] it was also proven that the infinitesimal holonomy algebra of locally projectively flat Finsler manifolds with constant flag curvature is infinite dimensional. The standard Funk metric $F_{\mathbb{B}^n}$ at $o \in \mathbb{R}^n$ has these special geometric properties, hence it also has infinite dimensional holonomy group.

Based on these results we have the following

4.2.2 Proposition. *The infinitesimal holonomy algebra* $\mathfrak{hol}_o^*(F_{\mathbb{B}^n})$ *of the standard Funk metric at the point* $o \in \mathbb{B}^n$ *has the jet generating property on the indicatrix* $\mathcal{I}_o$.

Proof. According to Definition 4.1.1, we have to show that for any $y \in \mathcal{I}_o$ and $k \in \mathbb{N}$ the jet-projection $\mathfrak{hol}_o^*(F_{\mathbb{B}^n}) \to J_y^k(\mathfrak{X}(\mathcal{I}_o))$ is onto.

In the case $n = 2$, we get from Remark 4.2.1 that $\mathfrak{hol}_o^*(F_{\mathbb{B}^2})$ is dense in $\mathfrak{X}(\mathcal{I}_o)$. It follows that the restriction of the k^{th} order jet projection on the infinitesimal holonomy algebra

$$j_y^k : \mathfrak{hol}_o^*(F_{\mathbb{B}^2}) \longrightarrow J_y^k(\mathfrak{X}(\mathcal{I}_o)), \tag{4.12}$$

is onto. In other words, any given k^{th} order jet in $J_y^k(\mathfrak{X}(\mathcal{I}_o))$ can be realized as the k-jet of an element of the infinitesimal holonomy algebra. Clearly we have the jet generating property.

Let us consider the $n > 2$ case. For each tangent 2-plane $\mathcal{K} \subset T_o\mathbb{B}^n$ the restriction of $F_{\mathbb{B}^n}$ to $\mathbb{B}^2 := \mathbb{B}^n \cap \mathcal{K}$ is the standard Funk metric on $\mathbb{B}^2$. One can suppose that $\mathcal{K}$ is the 2-plane generated by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$. Since these submanifolds are totally geodesic submanifolds of $F_{\mathbb{B}^n}$, we can apply [Theorem 3.3, [38]] and [Corollary 4.3, [38]], which gives us that $\mathfrak{hol}_o^*(F_{\mathbb{B}^n})$ contains infinitely many $\mathbb{R}$-independent vector fields which can be expressed by the curvature vector fields and their covariant derivatives.

If $y \in \mathcal{I}_o$ and $v \in T_y(\mathcal{I}_o)$, let $\mathcal{K}_{y,v}$ be the 2-plane determined by these vectors. Using the above argument we get that

$$j_y^k : \mathfrak{hol}_o^*(F_{\mathbb{B}^n}\Big|_{\mathcal{I}_o \cap \mathcal{K}_{y,v}}) \longrightarrow J_y^k(\mathfrak{X}(\mathcal{I}_o \cap \mathcal{K}_{y,v})) \tag{4.13}$$

is onto. It follows that for $y \in \mathcal{I}_o$ and $v \in T_y(\mathcal{I}_o)$, any k^{th} order v-directional derivative can be realised by elements of the holonomy algebra. Using the local coordinate system $y_1, \ldots, y_{n-1}$ on the $n-1$-dimensional indicatrix $\mathcal{I}_o$ we get that for any given $(z, z_1, \ldots, z_k) \in (\mathbb{R}^{(n-1)})^{(k+1)}$ there exist $\xi \in \mathfrak{hol}_o^*(F_{\mathbb{B}^n})$ such that

$$\xi\Big|_y = z, \quad (\mathcal{D}_v\xi)\Big|_y = z_1, \quad \ldots \quad (\mathcal{D}_v^{(k)}\xi)\Big|_y = z_k, \tag{4.14}$$

where we consider a locally constant extension of v when the higher order derivatives are computed. For the completion of the proof however we must be able to generate all k-th jet at y, that is the terms corresponding to mixed partial derivatives as well. This is possible, by using higher order derivatives corresponding to several directions. Indeed, one can use the polarization technique to show that the k^{th} order mixed partial derivatives are determined by the k^{th} order directional derivatives, a similar way as the quadratic form determines the corresponding symmetric bilinear form, or more generally, as the homogeneous from of degree k determines the corresponding symmetric multilinear k-from. Indeed, considering any $v_1, \dots v_k \in T_y(\mathcal{I}_o)$ and their constant extension in a neighbourhood of y, we get

$$\mathcal{D}_{v_1}\Big(\mathcal{D}_{v_2}\cdots(\mathcal{D}_{v_k}\xi)\Big) = \frac{1}{k!}\sum_{s=1}^{k}\sum_{1\leq j_1<\cdots<j_s\leq k}(-1)^{k-s}\mathcal{D}^{(k)}_{v_{j_1}+\cdots+v_{j_s}}\xi. \tag{4.15}$$

It follows that any mixed derivative can be realized for appropriately chosen (higher order) directional derivatives, therefore the k-jet generating property is satisfied. In the proof the choice of $y \in I_{x_0}$ is arbitrary, meaning that the argument is true for all points of I_{x_0}. Besides, given two points y_1 and y_2 from I_{x_0} one can cosider a rigid rotation of $\mathbb{R}^n$ which leaves the ball $\mathbb{B}^n$ and $o \in \mathbb{B}^n$ invariant and moves y_1 to y_2. This rigid rotation is an isometry of the standard Funk metric, which means that if one has the jet generating property in y_1, it will also be true in y_2.

□

4.2.3 Remark (The jet generating property of curvature vector fields and their derivatives)**.** One can easily show that in the 2-dimensional case, at any point $y \in \mathcal{I}_o$, the set of the curvature vector field and its derivatives up to order k contains $k+1$ linearly independent k-jet, therefore this set has the k-jet generating property. In the higher dimensional cases, from the argument of Proposition 4.2.2 using 2-dimensional planes, one can obtain that for any point $y \in \mathcal{I}_o$ and any direction $v \in T_y(\mathcal{I}_o)$, the curvature vector fields and their derivatives up to order k can be used to express the directional derivatives (4.14). From formula (4.15) one obtains that any k^{th} order derivative can be obtained by the derivatives of the curvature vector fields up to order k, that is the set

$$\{\mathcal{R}_{ij},\ \nabla_{p_1}\mathcal{R}_{ij},\ \dots,\ \nabla_{p_1\dots p_k}\mathcal{R}_{ij} \mid 1 \leq i, j, p_1\dots p_k \leq n\} \subset \mathfrak{X}(\mathcal{I}_o) \tag{4.16}$$

has the k-jet generating property.

4.3 Funk perturbation of a Finsler metric

Let (M, F) be a Finsler manifold and $x_0 \in M$ be a fixed point. We can chose an x_0-centered coordinate system (U, x) such that $x(U) \subset \mathbb{B}^n$. The associated coordinate system on TM will be denoted by $(\pi^{-1}(U), \chi = (x, y))$. We also consider a bump function $\psi \colon M \to \mathbb{R}$, such that $supp(\psi) \subset U$ and $\psi|_{\tilde{U}} = 1$ for some open neighbourhood $\tilde{U} \subset U$ of x_0. We denote by $\bar{\psi} := \psi \circ \pi$ the pull-back of ψ by the projection π.

Using the standard Funk norm function $F_{\mathbb{B}^n}$, we introduce the Finsler norm $\bar{F} : TM \to \mathbb{R}$ by the formula

$$\bar{F}^2 = \psi \cdot (F_{\mathbb{B}^n} \circ \chi)^2 + (1 - \psi) \cdot F^2. \tag{4.17}$$

We remark that $\bar{F}$ is the pull-back of the standard Funk norm function on $\pi^{-1}(\tilde{U})$.

Using (4.17) we define a smooth perturbation of the Finsler function F as a 1-parameter family of functions F_t, where

$$F_t^2 = (1 - t)F^2 + t\bar{F}^2, \quad t \in [0, 1]. \tag{4.18}$$

Then F_t is a 1-parameter family of Finsler metrics. Indeed, F and $\bar{F}$ are positively 1-homogeneous continuous function, smooth on $\widehat{T}M$, therefore F_t verifies these properties. Moreover, taking the squares in (4.18) ensures that the bilinear form

$$g^t_{ij} = (1 - t)\, g_{ij} + t\, \bar{g}_{ij}, \quad t \in [0, 1]$$

of F_t is positive definite as well.

4.3.1 Remark. From formula (1.2) we get that the geodesic coefficients $G^i(x, y)$ can be calculated in terms of the 3^{rd} order jet of the Finsler function F at (x, y). Therefore, the coefficients $R^i_{jk}(x, y)$ of the curvature tensor and the curvature vector fields

$$\mathcal{R}_{ij} := R(\delta_i, \delta_j), \qquad i, j = 1, \ldots, n, \tag{4.19}$$

can be expressed algebraically by the 5^{th} order jet of F. More generally, the value of k^{th} order covariant derivatives and k^{th} successive Lie brackets of curvature vector fields can be expressed algebraically by the $(k+5)^{\text{th}}$ order jet of F.

4.3.2 Proposition. *Any element of the infinitesimal holonomy algebra* $\mathfrak{hol}^*_{x_0}(F_t)$ *can be expressed as an algebraic fraction of polynomials in* t *whose coefficients are determined by* $j^k_{x_0}F$ *and* $j^k_{x_0}\bar{F}$ *for some* $k \in \mathbb{N}$.

Proof. The geodesic coefficients G^i_t, $i = 1, \dots, n$ of F_t can be calculated in terms of $j^3_{x_0}(F_t)$, therefore in terms of t, $j^3_{x_0}(F)$, and $j^3_{x_0}(\bar{F})$. More precisely, their expressions are algebraic fractions of polynomials in t whose coefficients are determined by the third order jets of F and $\bar{F}$. Similarly, the curvature vector fields of F_t can be expressed as algebraic fractions of polynomials in t whose coefficients are determined by $j^5_{x_0}(F)$ and $j^5_{x_0}(\bar{F})$. More generally, using Remark 4.3.1, the value of k^{th} order covariant derivatives and k^{th} successive Lie brackets of curvature vector fields of F_t can be expressed as algebraic fractions of polynomials in t whose coefficients are determined by $j^{k+5}_{x_0}(F)$ and $j^{k+5}_{x_0}(\bar{F})$.

□

4.3.3 Proposition. *For any* $y_0 \in \mathcal{I}_o$ *the set of parameters* $t \in [0,1]$, *where the 3-jet generating property of the infinitesimal holonomy algebra* $\mathfrak{hol}^*_{x_0}(F_t) \subset \mathfrak{X}(\mathcal{I}^t_{x_0})$ *of the Funk perturbation* (4.18) *is not satisfied, is finite.*

Proof. Let us suppose that $y_0 \in \mathcal{I}^t_{x_0} \subset T_{x_0}M$ for every $t \in [0,1]$. If not, then we can consider $\widetilde{F}_t(x,y) := F_t(x,y)/F_t(x_0,y_0)$ which is just a rescaling with a constant for any given t, therefore it doesn't affect the jet generating property on the indicatrix.

From Proposition 4.2.2 we know that the Funk metric has the jet generating property, therefore for $t = 1$, there are vector fields

$$\{W_1, \dots, W_l\} \in \mathfrak{hol}^*_{x_0}(F_{t=1}), \tag{4.20}$$

$l := \dim(J^3_{y_0}(\mathfrak{X}(\mathcal{I}^{t=1}_{x_0})))$ in the infinitesimal holonomy algebra, such that any 3^{rd} order jet at y_0 can be realized with their combination. Those vector fields are linear combination of curvature vector fields, their derivatives and their Lie brackets. Let us consider them for any F_t, $t \in [0,1]$. We get a set in the infinitesimal holonomy algebra of F_t at x_0:

$$\{W_1(t), \dots, W_l(t)\} \in \mathfrak{hol}^*_{x_0}(F_t). \tag{4.21}$$

Using Proposition 4.3.2, these vector fields are algebraic fractions of polynomials in t whose coefficients are determined by $j^k_{x_0}(F)$ and $j^k_{x_0}(\bar{F})$ for some $k \in \mathbb{N}$. It follows that the determinant of the $l \times l$ matrix composed by the 3rd order jet coordinates of (4.21) at y_0:

$$\mathcal{P}_t := \det \begin{pmatrix} j^3_{y_0}(W_1(t)) \\ \vdots \\ j^3_{y_0}(W_l(t)) \end{pmatrix}, \tag{4.22}$$

is an algebraic fraction of polynomials in t whose coefficients are determined by $j^3_{x_0}(F)$ and $j^3_{x_0}(\bar{F})$ for some $k \in \mathbb{N}$ with $\mathcal{P}_{t=1} \not\equiv 0$. Since every non-trivial polynomial has finitely many roots, $\mathcal{P}_t$ can only be zero at finitely many values $t \in [0, 1]$. $\square$

4.4 Almost all Finsler metrics have infinite dimensional holonomy group

4.4.1 Theorem. *In the set $\mathcal{F}$ of C^∞-smooth Finsler metrics on a manifold M of dimension $n \geq 2$, there exists a subset $\widetilde{\mathcal{F}}$ of Finsler metrics with infinite dimensional holonomy group, which is open and everywhere dense in any $\mathcal{C}^m$-topology, $m \geq 8$.*

Proof. Let $\mathcal{F}$ be the set of C^∞-smooth Finsler metrics on a given manifold M and let us consider the subset $\widetilde{\mathcal{F}} \subset \mathcal{F}$ characterized by the following property: $\tilde{F} \in \widetilde{\mathcal{F}}$ if and only if there exists a point $x_0 \in M$ such that the curvature vector fields and their derivatives

$$\{\mathcal{R}_{ij},\ \nabla_k\mathcal{R}_{ij},\ \nabla_{kl}\mathcal{R}_{ij},\ \nabla_{klh}\mathcal{R}_{ij} \mid 1 \leq i, j, k, l, h \leq n\} \subset \mathfrak{X}(\mathcal{I}_{x_0}), \tag{4.23}$$

up to order 3 has the 3−jet generating property at least at one point of the indicatrix at x_0. Then

i) the holonomy group at x_0 of any $\tilde{F} \in \widetilde{\mathcal{F}}$ is infinite dimensional,

ii) the set $\widetilde{\mathcal{F}}$ is dense in $\mathcal{F}$ with respect to the C^m topology for each $m \geq 8$ (in fact for any $m \geq 0$),

iii) the set $\widetilde{\mathcal{F}}$ is open in $\mathcal{F}$ with respect the the $C^{\widetilde{m}}$ topology for $\widetilde{m} \geq 8$.

Indeed, *i)* follows from Theorem 4.1.2: the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(\tilde{F})$ is infinite dimensional, consequently, the holonomy algebra and the holonomy group of $\tilde{F}$ at x_0 are infinite dimensional.

In order to show *ii)*, let us consider the Funk-perturbation F_t given by (4.18), a point x_0 in M and a point y_0 of the indicatrix at x_0. By Proposition 4.3.3 there exists a sufficiently small $t > 0$ such that the curvature vector fields and their derivatives up to order 3 has the $3-$jet generating property at y_0. For sufficiently small t the metric F_t is sufficiently close to F in C^m-topology.

In order to prove *iii)*, we observe that the jet-generating condition is an open condition, so if it is fulfilled at $y_0 \in \mathcal{I}_{x_0}$, it is fulfilled at any point $y_1 \in \mathcal{I}_{x_1}$ sufficiently close to y_0 in $C^{m\geq 8}$-topology on TM.

□

4.4.2 Remark. In the proof of Theorem 4.4.1, it was showed that for a convenient perturbation the infinitesimal holonomy algebra at a point is infinite-dimensional. This remains valid mircrolocally and on the level of germs.

Chapter 5

Summary

In the following we present a brief survey of the contents of each chapter.

1. Basic concepts and tools

In this chapter we give the preliminaries necessary to understand the later chapters. We collect the most indispensable concetps from spray geometry and Finsler geometry. We also standardize our notation and terminology. The main scene of our investigations is the *tangent bundle*

$$(TM, \pi, M)$$

and the *second tangent bundle*

$$(TTM, \tau, TM).$$

We introduce *spray manifolds* which are manifolds equipped with a mapping $S : TM \to TTM$ which is a vector field on TM and it is a section of (TTM, π_*, TM), i.e., $\pi_* \circ S = 1_{TM}$, smooth on the slit tangent bundle $\widehat{T}M$ an positive homogeneous of degree two. The corresponding concepts of *geodesics*, *horizontal covariant derivation*, *parallel translation* and textitcurvature are also given. The definition of a *Finsler manifold*, which is a generalization of a Riemannian manifold, and how the *geodesic spray* with all the corresponding constructions take form can be found in Section 1.2. In Section 1.3 we recall the definition of parallel translation and the main object of the thesis:

1.3.1 Definition. The *holonomy group* $\mathcal{H}ol_x(M,F)$ of a Finsler manifold (M,F) at a point $x \in M$ is the group generated by parallel translations along piece-wise differentiable closed curves starting and ending at x.

Since the parallel translation (1.23) is 1-homogeneous and preserves the norm, the holonomy group can be seen as a subgroup of the diffeomorphism group of the indicatrix:

$$\mathcal{H}ol_x(M,F) \ \subset \ \mathcal{D}iff^{\infty}(\mathcal{I}_x).$$

2. Tangent Lie algebra of a diffeomorphism group and its application to the holonomy theory

2.1 Tangent Lie algebra of a diffeomorphism group

In this section of the thesis we investigate the tangential property and tangential structure of subgroups of the diffeomorphism group.

A smooth curve $c\colon I \to M$ on the manifold M has a singularity of order $(k-1)$ at $t=0$, if its derivatives vanish up to order $k-1$, $(k \geq 0)$. It is well known that if a curve c has a singularity of order $(k-1)$ at $0 \in \mathbb{R}$ then its k^{th} order derivative $c^{(k)}(0) = X_p$ is a tangent vector at $p = c(0)$. In that case, the curve c is called an integral curve of order k of the vector $X_p \in T_pM$. Extending this concept to vector fields, we can introduce the following

2.1.1 Definition. A $C^{\infty}-$smooth curve in the diffeomorphism group $\varphi\colon I \to \mathcal{D}iff^{\infty}(M)$, $t \to \varphi_t$ is called an *integral curve of the vector field* $X \in \mathfrak{X}(M)$ if

(1) $\varphi_0 = id_M$,

(2) there exists $k \in \mathbb{N}$ such that for any point $p \in M$ the curve $t \to \varphi_t(p)$ is an integral curve of order k of $X(p) \in T_pM$.

This $k \in \mathbb{N}$ is called the *order* of the integral curve φ_t of the vector field X.

Let $\mathcal{G} \subset \mathcal{D}iff^{\infty}(M)$ be an arbitrary subgroup of the diffeomorphism group $\mathcal{D}iff^{\infty}(M)$. Using the terminology of Definition 2.1.1 we introduce the following

2.1.2 Definition. A vector field $X \in \mathfrak{X}(M)$ is called *tangent* to a subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ of the diffeomorphism group if there exists an integral curve of X in $\mathcal{G}$. The set of tangent vector fields of $\mathcal{G}$ is denoted by $\mathcal{T}_o\mathcal{G}$.

2.1.3 Remark. We have $X \in \mathcal{T}_o\mathcal{G}$ if and only if there exists a $C^\infty-$smooth curve $\varphi\colon I \to Diff^\infty(M)$ such that

(*1*) $\varphi_t \in \mathcal{G}$,

(*2*) $\varphi_0 = id_M$,

(*3*) there exists $k \in \mathbb{N}$ such that equation (2.1) is satisfied.

One can observe that in Definition 2.1.2 we do not suppose that $\mathcal{G}$ is a Lie subgroup of $\mathcal{D}iff^\infty(M)$. Indeed, we use the differential structure of the later to formulate the smoothness condition on the curve in $\mathcal{G}$. Nevertheless, we have the following

2.1.4 Theorem. If $\mathcal{G}$ is a subgroup of $\mathcal{D}iff^\infty(M)$, then $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Motivated by the results of Theorem 2.1.4 we propose the following

2.1.5 Definition. $T_0\mathcal{G}$ is called the tangent Lie algebra of the subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$.

As a direct consequence of Theorem 2.1.4 we get the following

2.1.6 Corollary. Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^\infty(M)$ and $\mathcal{S}$ be a subset of $\mathfrak{X}(M)$ such that the elements of $\mathcal{S}$ are tangent to $\mathcal{G}$. Then the Lie subalgebra $\langle\, \mathcal{S}\, \rangle_{\mathcal{L}ie}$ of $\mathfrak{X}(M)$ generated by the elements of $\mathcal{S}$ is also tangent to $\mathcal{G}$, that is

$$\mathcal{S} \subset \mathcal{T}_o\mathcal{G} \qquad \Rightarrow \qquad \Big\langle\, \mathcal{S}\, \Big\rangle_{Lie} \subset \mathcal{T}_o\mathcal{G}.$$

2.1.7 Remark. Slightly different tangent properties of vector fields to a subgroup $\mathcal{G}$ of the diffeomorphism group were already introduced in [33]. We will refer to the property [33, Definition 2.] as the *weak tangent property* and to [33, Definition 4.] as the *strong tangent property*. Our language is justified by the following proposition which is clarifying the relationship between the tangent property introduced in Definition 2.1.1 and the tangent properties introduced in [33]:

2.1.8 Proposition. Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^{\infty}(M)$ and $X \in \mathfrak{X}(M)$. Using the terminology of Remark 2.1.7:

(i) if X is strongly tangent to $\mathcal{G}$, then $X \in \mathcal{T}_o\mathcal{G}$.

(ii) if $X \in \mathcal{T}_o\mathcal{G}$, then it is weakly tangent to $\mathcal{G}$.

The main feature of $\mathcal{T}_o\mathcal{G}$ is that one can obtain information about the group $\mathcal{G}$. Indeed, one has the following

2.1.10 Theorem. Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^{\infty}(M)$ and $\mathcal{G}^c$ its topological closure with respect to the C^{∞} topology. Then the group generated by the exponential image of the tangent Lie algebra $\mathcal{T}_o\mathcal{G}$ with respect to the exponential map $\exp\colon \mathfrak{X}(M) \to \mathcal{D}iff^{\infty}(M)$ is a subgroup of $\mathcal{G}^c$.

The concept worked out in Definition 2.1.2 and Theorem 2.1.4 can be adapted not only for subgroups of the diffeomorphism group but for any subgroup of any (finite or infinite dimensional) Lie group:

2.1.11 Definition. Let $\mathcal{G}_L$ be a Lie group, $e \in \mathcal{G}_L$ is the identity element of $\mathcal{G}_L$ and $\mathfrak{g}_L := T_e\,\mathcal{G}_L$ the Lie algebra of $\mathcal{G}_L$. If $\mathcal{G} \subset \mathcal{G}_L$ is a subgroup of $\mathcal{G}_L$, then $X \in \mathfrak{g}_L$ is called tangent to $\mathcal{G}$ if there exist a $C^{\infty}-$smooth curve $\varphi\colon I \to \mathcal{G}_L$ such that

(*1*) $\varphi_t \in \mathcal{G}$,

(*2*) $\varphi_0 = e$,

(*3*) there exists $k \in \mathbb{N}$ such that $t \to \varphi_t$ is a k^{th} order integral curve of X.

The set of tangent vector of $\mathcal{G}$ is denoted by $\mathcal{T}_o\mathcal{G}$.

Then, adapting the proof of Theorem 2.1.4 and Theorem 2.1.10 we can get the following

2.1.12 Theorem. If $\mathcal{G}$ is a subgroup of a Lie group $\mathcal{G}_L$, then $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of $\mathfrak{g}_L$. The group $\left\langle \exp(\mathcal{T}_o\mathcal{G}) \right\rangle$ generated by the exponential image of $\mathcal{T}_o\mathcal{G}$ with respect to the exponential map $\exp\colon \mathfrak{g}_L \to \mathcal{G}_L$ is a subgroup of the topological closure $\mathcal{G}^c$ of $\mathcal{G}$ in $\mathcal{G}_L$.

It is clear that in the case when $\mathcal{G}$ is a Lie subgroup of $\mathcal{G}_L$, then $\mathcal{T}_o\mathcal{G} = \mathfrak{g}$ is just the usual Lie subalgebra of $\mathfrak{g}_L$ associated to the Lie subgroup $\mathcal{G}$. Therefore Definition 2.1.11 generalizes the classical notion of the Lie subalgebra associated to a Lie subgroup.

2.1.13 Definition. We call a subgroup $\mathcal{G} \subset \mathcal{D}\mathit{iff}^{\infty}(M)$ infinite dimensional, if its tangent algebra $\mathcal{T}_o\mathcal{G}$ is infinite dimensional.

2.2 The fibered holonomy algebra and its Lie subalgebras

The notion of *fibered holonomy group* $\mathcal{H}ol_f(M)$ appeared in [33]:

2.2.1 Definition. The fibred holonomy group $\mathcal{H}ol_f(M)$ of $(M, \mathcal{F})$ consists of fibre preserving diffeomorphisms $\Phi \in \mathcal{D}\mathit{iff}^{\infty}(\mathcal{I}M)$ of the indicatrix bundle $(\mathcal{I}M, \pi, M)$ such that for any $p \in M$ the restriction $\Phi_p = \Phi|_{\mathcal{I}_pM} \in \mathcal{D}\mathit{iff}^{\infty}(\mathcal{I}_pM)$ belongs to the holonomy group $\mathcal{H}ol_p(M)$.

Let $(M, \mathcal{F})$ be a compact Finsler manifold. It is obvious that

$$\mathcal{H}ol_f(M) \subset \mathcal{D}\mathit{iff}^{\infty}(\mathcal{I}M), \tag{2.15}$$

where $\mathcal{H}ol_f(M)$ is a subgroup of the diffeomorphism group of the indicatrix bundle. Until now it is not known whether or not $\mathcal{H}ol_f(M)$ is a Lie subgroup of $\mathcal{D}\mathit{iff}^{\infty}(\mathcal{I}M)$. The set of tangent vector fields to the group $\mathcal{H}ol_f(M)$ denoted as

$$\mathfrak{hol}_f(M) := \mathcal{T}_0\Big(\mathcal{H}ol_f(M)\Big). \tag{2.16}$$

2.2.2 Definition. $\mathfrak{hol}_f(M)$ is called the *fibered holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$.

From Theorem 2.1.4 one can obtain the following

2.2.3 Corollary. The fibered holonomy algebra $\mathfrak{hol}_f(M)$ is a Lie subalgebra of the Lie algebra of smooth vector fields $\mathfrak{X}(\mathcal{I}M)$.

In the sequel we introduce two important Lie subalgebras of $\mathfrak{hol}_f(M)$ using the the curvature tensor: the curvature algebra and the infinitesimal holonomy algebra.

2.2.4 Definition. The *curvature algebra* $\mathfrak{R}$ is the Lie algebra generated on the indicatrix bundle by curvature vector fields:

$$\mathfrak{R} = \Big\langle R(X^h, Y^h)|_{\mathcal{I}M} \;\Big|\; X, Y \in \mathfrak{X}(M) \Big\rangle_{Lie}.$$

It is not difficult to see from the definition of the curvature tensor that a curvature vector field can be calculated as

$$\xi = R(X^h, Y^h) = \left[X^h, Y^h\right] - \left[X, Y\right]^h. \qquad (2.17)$$

Moreover, we have the following

2.2.5 Proposition. The elements of the curvature algebra are tangent to the group $\mathcal{H}ol_f(M)$ and the curvature algebra $\mathfrak{R}$ is a Lie subalgebra of $\mathfrak{hol}_f(M)$.

2.2.7 Definition. The *infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ of a Finsler manifold $(M, \mathcal{F})$ is the smallest Lie algebra on the indicatrix bundle which satisfies the following properties:

1) curvature vector fieldsar an element of $\mathfrak{hol}^*(M)$,
2) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}(M)$, then the horizontal Berwald covariant derivative $\nabla_X \xi$ is also an element of $\mathfrak{hol}^*(M)$.

We have the following

2.2.8 Proposition. The infinitesimal holonomy algebra $\mathfrak{hol}^*(M)$ is tangent to $\mathcal{H}ol_f(M)$ and the infinitesimal holonomy algebra $\mathfrak{hol}^*(M)$ is a Lie subalgebra of $\mathfrak{hol}_f(M)$.

2.3 The holonomy algebra and its Lie subalgebras

Let $(M, \mathcal{F})$ be a Finsler manifold.

2.3.1 Definition. The tangent Lie algebra to the group $\mathcal{H}ol_p(M)$

$$\mathfrak{hol}_p(M) := \mathcal{T}_0\Big(\mathcal{H}ol_p(M)\Big).$$

is called the *holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$ at $p \in M$.

2.3.2 Corollary. The holonomy algebra $\mathfrak{hol}_p(M)$ of a Finsler manifold $(M, \mathcal{F})$ at $p \in M$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{I}_p)$.

2.3.3 Definition. The Lie algebra $\mathfrak{R}_p$ of vector fields generated by curvature vector fields at $p \in M$ is called the *curvature algebra at* p:

$$\mathfrak{R}_p = \left\langle R(X_p^h, Y_p^h)|_{\mathcal{I}_p} \;\middle|\; X_p, Y_p \in T_pM \right\rangle_{Lie}.$$

The curvature algebra $\mathfrak{R}_p$ is the restriction of $\mathfrak{R}$ to the indicatrix $\mathcal{I}_p$

2.3.4 Proposition. The curvature algebra $\mathfrak{R}_p$ at $p \in M$ is tangent to the group $\mathcal{H}ol_p(M)$ and the curvature algebra $\mathfrak{R}_p$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.

2.3.5 Definition. The *infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ of a Finsler manifold $(M, \mathcal{F})$ at a point $p \in M$ is the smallest Lie algebra on the indicatrix at p which satisfies the following properties:

1) Every curvature vector field ξ_p is an element of $\mathfrak{hol}_p^*(M)$,
2) if $\xi_p \in \mathfrak{hol}_p^*(M)$ and $X \in \mathfrak{X}(M)$, then the horizontal Berwald covariant derivative $(\nabla_X \xi)(p)$ is also an element of $\mathfrak{hol}_p^*(M)$.

2.3.6 Remark. The infinitesimal holonomy algebra of a Finsler manifold $(M, \mathcal{F})$ at a point $p \in M$ can also be considered as

$$\mathfrak{hol}_p^*(M) := \left\{ \xi|_{\mathcal{I}_p} \mid \xi \in \mathfrak{hol}^*(M) \right\}.$$

From Proposition 2.2.8 we get

2.3.7 Corollary. The infinitesimal holonomy algebra $\mathfrak{hol}_p^*(M)$ is tangent to the holonomy group $\mathcal{H}ol_p(M)$ and is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.

We remark that the first parts of the statement of Proposition 2.3.4 and 2.3.7 are improvements of the results of [33] because the tangential property of the Lie algebra is improved: we can guaranty C^∞-smoothness instead of C^1-smoothness.

Chapter 3
Some results about Finsler manifolds with infinite dimensional holonomy group

3.1 Holonomy of the quantum navigation problem

Randers model of Quantum Information Processing

In a closed **finite dimensional quantum** system the state space is $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and the physical states can be identified with the rays of this space, that is the projective space $\mathbb{C}P^{n-1}$ of the underlying Hilbert space. In Quantum Information Processing (QIP) the task of the "navigator" is to find the shortest path from an initial state $|\Psi_I\rangle$ to a final state $|\Psi_F\rangle$. The problem is hard to solve on the level of state space, but in [9] and in [8] the authors showed that one can lift this problem to the space acting on the states: the special unitary group $SU(n)$. The task is then to find a control Hamiltonian $\hat{H}_c(t)$ in the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$ which together with a time independent Hamiltonian $\hat{H}_0$, representing the effect of the ineliminable external field, forms $\hat{H}(t) = \hat{H}_c(t) + \hat{H}_0$ and generates the evolution of our initial state to the final state.

2-dimensional quantum Zermelo problem

We consider a specific case of the two state quantum system: a single spin particle in a magnetic field. As in [9], we consider an invariant "wind" vector field, represented in the Lie algebra by $\hat{H}_0$ and the invariant Riemannian metric coming from the Killing form:

$$h(\hat{A}, \hat{B}) := \mathrm{tr}(\hat{A}^\dagger \hat{B}), \tag{3.1}$$

$\hat{A}, \hat{B} \in \mathfrak{su}(2)$. The Lie algebra $\mathfrak{su}(2)$ is spanned by $\hat{E}_1 = i\sigma_1, \hat{E}_2 = -i\sigma_2, \hat{E}_3 = i\sigma_3$ where the sigmas are the Pauli matrices.

We will work with the coordinates (x, ξ) on the tangent bundle, where $(x) = (x_1, x_2, x_3)$ are coordinates on the group $SU(2)$ and $(\xi) = (\xi_1, \xi_2, \xi_3)$

are the invariant coordinates in the Lie algebra $\mathfrak{su}(2)$ with respect the basis $\left\{\hat{E}_1, \hat{E}_2, \hat{E}_3\right\}$. The relation between the standard coordinates and the invariant coordinates on $TG = G \times \mathfrak{g}$ can be find by $\rho^{-1}_{x,*}(x, y) = (x, \xi)$, where $\rho : SU(2) \to SU(2)$ is the right translation. Modulo a rigid transformation, we can suppose that $\hat{H}_0 = c\hat{E}_1$ with $c \in \mathbb{R}$. With (3.1) and $W = \hat{H}_0$ one can calculate (1.19) and find the corresponding invariant Randers metric:

$$\mathcal{F}(\xi) = \frac{1}{1 - 2c^2}\Big(\alpha_\xi + \beta_\xi\Big), \tag{3.2}$$

with a Riemannian norm α and a $1-$form β:

$$\alpha_\xi := \sqrt{2\xi_1^2 + 2\xi_2^2 + 2\xi_3^2 - 4\xi_2^2 c^2 - 4\xi_3^2 c^2}, \qquad \beta_\xi := -2c\,\xi_1. \tag{3.3}$$

For simplicity, the 3-dimensional Finler space corresponding to the 2-dimensional quantum Zermelo problem will be denoted by $\mathcal{Q} = (SU(2), \mathcal{F})$.

The invariant formulation of the Euler-Lagrange equations, called the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\partial E}{\partial \xi} = -ad^*_\xi\left(\frac{\partial E}{\partial \xi}\right), \tag{3.5}$$

can be used to determine the geodesic equation on the Lie algebra [13].

Curvature algebra and holonomy

The curvature tensor can be obtained from the spray coefficients. It is not difficult to calculate the first Lie backets of the curvature vector fields:

$$[R_1, R_3] = \frac{1}{\alpha_\xi}R_2 + \frac{2c\,\xi_2}{\alpha_\xi^2}R_3, \qquad [R_2, R_3] = \frac{-1}{\alpha_\xi}R_1 + \frac{2c\,\xi_3}{\alpha_\xi^2}R_3. \tag{3.10}$$

More generally, we have the following

3.1.1 Lemma. Let $L_0 = R_2$ and denote $L_k = [L_{k-1}, R_3]$ the Lie bracket of the vector field L_{k-1} and R_2 for $k \geq 1$. Then

$$L_k = \begin{cases} \dfrac{\epsilon_k}{\alpha_\xi^k}R_2 + 2kc\dfrac{\epsilon_k\,\xi_2}{\alpha_\xi^{k+1}}R_3, & \text{if } k \equiv 0 \mod 2, \\[2ex] \dfrac{\epsilon_k}{\alpha_\xi^k}R_1 - 2kc\dfrac{\epsilon_k\,\xi_3}{\alpha_\xi^{k+1}}R_3, & \text{if } k \equiv 1 \mod 2, \end{cases} \tag{3.11}$$

where $\epsilon_k = -1$, if $k = 4l + 1$ or $k = 4l + 2$ and $\epsilon_k = 1$, if $k = 4l + 3$ or $k = 4l$ for some $l \in \mathbb{N}$.

3.1.2 Proposition. In the presence of external wind W, the curvature algebra $\mathfrak{R}$ of the Finsler metric (3.2) is an infinite dimensional Lie subalgebra of $\mathfrak{X}(\mathcal{I})$ of smooth vector fields of the indicatrix (3.4).

We remark that if there is no external wind, that is $c = 0$, then (3.2) is a Riemann metric: On the indicatrix we have $\alpha_\xi = 1$ and one can easily see that $\mathfrak{R}$ is isomorphic to $\mathfrak{so}(3)$.

3.1.3 Theorem. The holonomy group $\mathcal{H}ol(\mathcal{Q})$ of the 2-dimensional quantum Zermelo problem in the presence of an external wind W is not a finite dimensional Lie group.

We remark that if there is no external wind W, then $\mathcal{Q} = \Big(SU(2), \alpha\Big)$ is a 3-dimensional Riemannian manifold and its holonomy group $\mathcal{H}ol(\mathcal{Q})$ is the 3-dimensional special orthogonal group $SO(3)$.

3.2 Holonomy of projectively flat Randers metrics of constant curvature

Our aim is to describe the holonomy structure of projectively flat non-Riemannian Randers two-manifolds with non-zero constant flag curvature. As a first step, we investigate the holonomy of the standard model described in Section 1.2.

Let $(\mathbb{B}^2, \mathcal{F}_a)$ be the Finsler two-manifold where $\mathbb{B}^2$ is the unit ball in $\mathbb{R}^2$ and $\mathcal{F}_a$ is the Finsler function given by (1.20) where $a = (a_1, 0) \in \mathbb{R}^2$ is a nonzero constant vector with $|a_1| < 1$. We have the following

3.2.1 Proposition. The holonomy group of $(\mathbb{B}^2, \mathcal{F}_a)$ is maximal and $\mathcal{H}ol_x(M)^c$ is diffeomorphic to $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$ which is the orientation preserving diffeomorphism group of $\mathbb{S}^1$.

3.2.2 Lemma. For any $n \in \mathbb{N}$ we have $\Sigma_n \subset \mathfrak{hol}^*_0$.

Using Z. Shen's classification theorem of Randers manifolds we can get the following

3.2.4 Theorem. The holonomy group of a simply connected non-Riemannian projectively flat Finsler two-manifold of constant non-zero flag curvature is maximal and $\overline{\mathcal{H}ol(M)}$ is diffeomorphic to the orientation preserving diffeomorphism group of $\mathbb{S}^1$, that is

$$\overline{\mathcal{H}ol(M)} \cong \mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1).$$

We can obtain the following classification:

3.2.5 Corollary. The closure of the holonomy group $\mathcal{H}ol(M)$ of a simply connected, locally projectively flat Randers two-manifold of constant flag curvature λ is

1. the trivial group $\{id\}$, when $\lambda = 0$;
2. the rotation group $SO(2)$, when $\lambda \neq 0$ and the metric is Riemannian;
3. the orientation preserving diffeomorphism group of the circle $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$, when $\lambda \neq 0$ and the metric is non-Riemannian.

4. Density of Finsler metrics with infinite dimensional holonomy group

In this section we prove that for a generic Finsler manifold the holonomy group is infinite-dimensional. More precisely, we show in Theorem 4.4.1 that in the set $\mathcal{F}$ of C^{∞}-smooth Finsler metrics on a manifold M of dimension $n \geq 2$, there exists a subset $\widetilde{\mathcal{F}}$ of Finsler metrics with infinite dimensional holonomy group, which is open and everywhere dense in any $\mathcal{C}^m$-topology, $m \geq 8$. This result implies that, in contrast to the Riemannian case, the closure of the holonomy group is not a compact group for most Finlser metric. Similar results for the *linear holonomy group* (defined via the linear parallel transport) were recently obtained in [22].

4.1 3-jet generating Lie algebras of vector fields

4.1.1 Definition. A set $\mathcal{V} \subset \mathfrak{X}(M)$ of vector fields on a manifold M is called

- k-jet generating at $x \in M$ if the natural map $j^k_x \colon \mathcal{V} \to J^k_x(\mathfrak{X}(M))$ is surjective, and

- jet generating on M if at any $x \in M$ and for any $k \geq 0$ it is k-jet generating.

In particular, an algebra $\mathfrak{g}$ of vector fields on $U \subseteq \mathbb{R}^n$ is called 3-*jet generating* at $x \in U$, if every vector field can be approximated at x with order three by a vector field from the algebra. We have the following

4.1.2 Theorem. Let $\mathfrak{g}$ be a Lie algebra of vector fields on a manifold U. If there exists a point where it is 3-jet generating, then $\mathfrak{g}$ is infinite-dimensional.

We remark that if dimension U is 1, the result is known and is due to Sophus Lie, see e.g. [27, Theorem 2.70]. As examples show (see e.g. the tables at the back of [27] where vector field algebras of arbitrary finite dimension are given), the 3-jet generating property is important.

4.2 The 3-jet generating property of the Funk holonomy algebra

In this section we investigate the holonomy structure of the Funk metric introduced in Section 1.2.

4.2.1 Remark. The holonomy of $(\mathbb{B}^2, F_{\mathbb{B}^2})$ was investigated in [37, Chapter 5]. It was proved that the infinitesimal holonomy algebra $\mathfrak{hol}^*_o(F_{\mathbb{B}^2})$ contains the Fourier algebra $\mathsf{F}(\mathbb{S}^1)$ whose elements are vector fields $f\frac{d}{dt}$ such that $f(t)$ has finite Fourier series. One has

$$\mathsf{F}(\mathbb{S}^1) \ \subset\ \mathfrak{hol}^*_o(F_{\mathbb{B}^2}) \ \subset\ \mathfrak{X}(\mathbb{S}^1). \tag{4.11}$$

Since $\mathsf{F}(\mathbb{S}^1)$ is dense in $\mathfrak{X}(\mathbb{S}^1)$, we get the same from (4.11) for $\mathfrak{hol}^*_o(F_{\mathbb{B}^2})$. Using the exponential map, one can obtain from (2.30) that the closure of the holonomy group of the Finsler surface $(\mathbb{B}^2, F_{\mathbb{B}^2})$ is $\mathcal{D}\mathit{iff}_+(\mathbb{S}^1)$, the group of orientation preserving diffeomorphisms of the circle [37, Theorem 5.2]. In [38] it was also proven that the infinitesimal holonomy algebra of locally projectively flat Finsler manifolds with constant flag curvature is infinite dimensional. The standard Funk metric $F_{\mathbb{B}^n}$ at $o \in \mathbb{R}^n$ has these special geometric properties, hence it also has infinite dimensional holonomy group.

Based on these results we have the following

4.2.2 Proposition. The infinitesimal holonomy algebra $\mathfrak{hol}^*_o(F_{\mathbb{B}^n})$ of the standard Funk metric at the point $o \in \mathbb{B}^n$ has the jet generating property on the indicatrix $\mathcal{I}_o$.

4.3 Funk perturbation of a Finsler metric

Let (M, F) be a Finsler manifold and $x_0 \in M$ be a fixed point. We can chose an x_0-centered coordinate system (U, x) such that $x(U) \subset \mathbb{B}^n$. The associated coordinate system on TM will be denoted by $(\pi^{-1}(U), \chi = (x, y))$. We also consider a bump function $\psi\colon M \to \mathbb{R}$, such that $supp(\psi) \subset U$ and $\psi|_{\tilde{U}} = 1$ for some open neighbourhood $\tilde{U} \subset U$ of x_0. We denote by $\bar{\psi} := \psi \circ \pi$ the pull-back of ψ by the projection π.

Using the standard Funk norm function $F_{\mathbb{B}^n}$, we introduce the Finsler norm $\bar{F} : TM \to \mathbb{R}$ by the formula

$$\bar{F}^2 = \psi \cdot (F_{\mathbb{B}^n} \circ \chi)^2 + (1 - \psi) \cdot F^2. \tag{4.17}$$

We remark that $\bar{F}$ is the pull-back of the standard Funk norm function on $\pi^{-1}(\tilde{U})$.

Using (4.17) we define a smooth perturbation of the Finsler function F as a 1-parameter family of functions F_t, where

$$F_t^2 = (1 - t)F^2 + t\bar{F}^2, \quad t \in [0, 1]. \tag{4.18}$$

Then F_t is a 1-parameter family of Finsler metrics.

4.3.2 Proposition. Any element of the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(F_t)$ can be expressed as an algebraic fraction of polynomials in t whose coefficients are determined by $j^k_{x_0}F$ and $j^k_{x_0}\bar{F}$ for some $k \in \mathbb{N}$.

4.3.3 Proposition. For any $y_0 \in \mathcal{I}_o$ the set of parameters $t \in [0, 1]$, where the 3-jet generating property of the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(F_t) \subset \mathfrak{X}(\mathcal{I}^t_{x_0})$ of the Funk perturbation (4.18) is not satisfied, is finite.

4.4 Almost all Finsler metrics have infinite dimensional holonomy group

4.4.1 Theorem. In the set $\mathcal{F}$ of C^∞-smooth Finsler metrics on a manifold M of dimension $n \geq 2$, there exists a subset $\widetilde{\mathcal{F}}$ of Finsler metrics with infinite dimensional holonomy group, which is open and everywhere dense in any $\mathcal{C}^m$-topology, $m \geq 8$.

4.4.2 Remark. In the proof of Theorem 4.4.1, it was showed that for a convenient perturbation the infinitesimal holonomy algebra at a point is infinite-dimensional. This remains valid mircrolocally and on the level of germs.

Chapter 6

Összefoglaló

Az alábbiakban fejezetről-fejezetre haladva tekintjük át a disszertáció tartalmát.

1. Alapfogalmak és előzmények

Ebben a fejezetben röviden összefoglaljuk a disszertáció megértéséhez szükséges legfontosabb definíciókat. Bevezetjük a differenciálgeometria szükséges alapfogalmait, valamint a Finsler geometria nélkülözhetetlen definícióit. Egységesítjük a jelölés- és elnevezésrendszert. Vizsgálataink fő színtere a

$$(TM, \pi, M)$$

érintőnyaláb és a

$$(TTM, \tau, TM)$$

második érintőnyaláb. Bevezetjük a *spray-sokaság* fogalmát, mely alatt egy sokaságot és egy hozzárendelt $S : TM \to TTM$ leképezést értünk. Ez a leképezés egy vektormező TM-en, valamint a $\pi_* : TTM \to TM$ nyaláb egy szelése, mely pozitív 2-homogén, valamint sima a $\widehat{T}M$ hasított érintőnyalábon. Bevezetjük a spray-sokaságok *geodetikusainak*, *horizontális kovariáns deriváltjának*, *párhuzamos eltolásának* és *görbületének* fogalmát. A *Finsler-sokaságok*, melyek a Riemann-sokaságok általánosításai, a 1.2 fejezetben kerülnek bevezetésre a sokasághoz rendelt *geodetikus spray*-vel és a hozzá kapcsolódó geometriai konstrukciókkal együtt. Az

1.3 alfejezetben felidézzük a párhuzamos eltolás fogalmát, valamint a disszertáció fő objektumát:

1.3.1 Definíció. Egy (M, F) Finsler-sokaság $x \in M$ pontban vett $\mathcal{H}ol_x(M, F)$ *holonómia csoportján* a sokaság x-ből induló és x-ben végződő szakaszonként sima zárt görbéi mentén vett párhuzamos eltolások által generált csoportot értjük.

Mivel a párhuzamos eltolás 1-homogén és megőrzi a Finsler normát, a holonómia csoportot tekinthetjük az indikátrix diffeomorfizmus csoportjának részcsoportjaként:

$$\mathcal{H}ol_x(M, F) \ \subset \ \mathcal{D}iff^\infty(\mathcal{I}_x).$$

2. Diffeomorfizmus csoportok érintő Lie-algebrája és alkalmazásai a holonómia elméletben

2.1 Diffeomorfizmus csoportok érintő Lie-algebrái

A disszertáció ezen fejezetében egy diffeomorfizmus csoport részcsoportjainak érintőstruktúráit és ezek tulajdonságait vizsgáljuk.

Egy $c\colon I \to M$ sima görbe az M sokaságon $(k–1)$-ad rendű szingularitással rendelkezik a $t = 0$ pontban, ha deriváltjai k–1-ad rendig eltűnnek, ahol $k \geq 0$. Jól ismert, hogy ha egy c görbének a $0 \in \mathbb{R}$ pontban $(k-1)$-rendű szingularitása van, akkor a k-ad rendű deriváltja $c^{(k)}(0) = X_p$ egy érintő vektor $p = c(0)$-ban. Ebben az esetben a c görbét a $X_p \in T_pM$ vektor *k-ad rendű integrálgörbéjének* nevezzük. Ezt a konstrukciót vektormezőkre kiterjesztve kapjuk az alábbi definíciót:

2.1.1 Definíció. Egy $\varphi\colon I \to \mathcal{D}iff^\infty(M)$, C^∞ -sima görbét a $\mathcal{D}iff^\infty(M)$ diffeomorfizmus csoportban az $X \in \mathfrak{X}(M)$ *vektormező integrálgörbéjének* nevezünk ha

(1) $\varphi_0 = id_M$,

(2) létezik $k \in \mathbb{N}$, hogy bármely $p \in M$ pont esetén a $t \to \varphi_t(p)$ görbe a $X(p) \in T_pM$ vektor egy k-ad rendű integrálgörbéje.

Ezt a $k \in \mathbb{N}$ számot az integrálgörbe rendjének nevezzük.

Legyen $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ a $\mathcal{D}iff^\infty(M)$ diffeomorfizmus csoport egy tetszőleges részcsoportja. Felhasználva a 2.1.1 Definíció elnevezéseit bevezethetjük az alábbi definíciót:

2.1.2 Definíció. Azt mondjuk, hogy egy $X \in \mathfrak{X}(M)$ vektormező *érinti* a $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ részcsoportot, ha létezik X-nek integrálgörbéje $\mathcal{G}$-ben. $\mathcal{G}$ érintő vektormezőinek halmazát $\mathcal{T}_o\mathcal{G}$-vel jelöljük.

2.1.3 Megjegyzés. $X \in \mathcal{T}_o\mathcal{G}$ akkor és csak akkor teljesül, ha létezik egy $C^\infty-$sima görbe $\varphi\colon I \to Diff^\infty(M)$ amelyre

(*1*) $\varphi_t \in \mathcal{G}$,

(*2*) $\varphi_0 = id_M$,

(*3*) létezik $k \in \mathbb{N}$, melyre (2.1) teljesül.

Felhívjuk a figyelmet a tényre, hogy a 2.1.2 definíció nem feltételezi $\mathcal{G}$-ről, hogy az a $\mathcal{D}iff^\infty(M)$ egy Lie-részcsoportja. A konstrukció során $\mathcal{D}iff^\infty(M)$ differenciálható struktúráját használjuk a $\mathcal{G}$-beli görbék simaságához.

2.1.4 Tétel. Ha $\mathcal{G}$ a $\mathcal{D}iff^\infty(M)$ diffeomorfizmus csoport részcsoportja, akkor $\mathcal{T}_o\mathcal{G}$ az $\mathfrak{X}(M)$ Lie-csoport Lie-részalgebrája.

A 2.1.4 tétel által motiválva a következő definíciót vezetjük be:

2.1.5 Definíció. $T_0\mathcal{G}$ -t a $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ részcsoport érintő algebrájának nevezzük.

A 2.1.4 tétel egy közvetlen következménye:

2.1.6 Következmény. Legyen $\mathcal{G}$ a $\mathcal{D}iff^\infty(M)$ részcsoportja, $\mathcal{S}$ pedig $\mathfrak{X}(M)$ olyan részhalmaza, melynek elemei érintik $\mathcal{G}$-t. Ekkor az $\mathcal{S}$ elemei által generált $\langle \mathcal{S} \rangle_{\mathcal{L}ie}$ rész Lie-algebra szintén érinti $\mathcal{G}$-t, azaz

$$\mathcal{S} \subset \mathcal{T}_o\mathcal{G} \qquad \Rightarrow \qquad \Big\langle \mathcal{S} \Big\rangle_{Lie} \subset \mathcal{T}_o\mathcal{G}.$$

2.1.7 Megjegyzés. Megjegyezzük, hogy [33] -ban az itt definiálttól kis mértékben eltérő érintő tulajdonság már bevezetésre került. A továbbiakban gyenge érintő tulajdonságként fogunk hivatkozni a [33, Definition 2.] -ben bevezetett, valamint erős érintő tulajdonságként a [33, Definition 4.]-ben bevezetett fogalmakra. A különböző érintőtulajdonságok közti kapcsolatot a következő állítás tisztázza:

2.1.8 Állítás. Legyen $\mathcal{G}$ a $\mathcal{D}iff^\infty(M)$ egy részcsoportja és $X \in \mathfrak{X}(M)$.

(i) ha X erősen érinti a $\mathcal{G}$ részcsoportot, akkor $X \in \mathcal{T}_o\mathcal{G}$.

(ii) ha $X \in \mathcal{T}_o\mathcal{G}$, akkor X gyengén érinti $\mathcal{G}$-t.

A $\mathcal{T}_o\mathcal{G}$ érintő algebra fontos tulajdonsága, hogy általa információt nyerhetünk a $\mathcal{G}$ részcsoportról:

2.1.10 Tétel. Legyen $\mathcal{G}$ a $\mathcal{D}iff^\infty(M)$ egy részcsoportja, és $\mathcal{G}^c$ ennek lezártja a C^∞ topológiában. Ekkor a $\mathcal{T}_o\mathcal{G}$ érintő algebra $\exp\colon \mathfrak{X}(M) \to \mathcal{D}iff^\infty(M)$ exponenciális leképezés általi képe által generált csoport a $\mathcal{G}^c$ részcsoportja.

A 2.1.2 Definícióban bevezetett fogalom és a 2.1.4 Tétel eredményét nemcsak a diffeomorfizmus csoport részcsoportjai esetén értelmezhetjük, hanem bármely véges vagy végtelen dimenziós Lie-csoport részcsoportjaira is:

2.1.11 Definíció. Legyen $\mathcal{G}_L$ egy Lie-csoport, $e \in \mathcal{G}_L$ a csoport egységeleme és $\mathfrak{g}_L := T_e\,\mathcal{G}_L$ a $\mathcal{G}_L$ csoport Lie-algebrája. Ha $\mathcal{G} \subset \mathcal{G}_L$ a $\mathcal{G}_L$ részcsoportja, akkor egy $X \in \mathfrak{g}_L$ elem érinti $\mathcal{G}$-t, ha létezik olyan C^∞-sima $\varphi\colon I \to \mathcal{G}_L$ görbe, melyre az alábbi tulajdonságok teljesülnek:

(*1*) $\varphi_t \in \mathcal{G}$,

(*2*) $\varphi_0 = e$,

(*3*) létezik olyan $k \in \mathbb{N}$, melyre a $t \to \varphi_t$ leképezés az X elem k-ad rendű integrálgörbéje.

A $\mathcal{G}$ részcsoport érinővektorainak halmazát $\mathcal{T}_o\mathcal{G}$-vel jelöljük.

A 2.1.4 Tétel valamint a 2.1.10 Tétel bizonyításában használt gondolatmenethez hasonlóan adódik az alábbi eredmény:

2.1.12 Tétel. Ha $\mathcal{G}$ a $\mathcal{G}_L$ Lie-csoport részcsoportja, akkor $\mathcal{T}_o\mathcal{G}$ a $\mathfrak{g}_L$ Lie-részalgebrája. Ennek exponenciális képe által generált $\left\langle \exp(\mathcal{T}_o\mathcal{G}) \right\rangle$ csoport a $\mathcal{G}^c$ csoport részcsoportja $\mathcal{G}_L$-ben.

Egyszerűen látható, hogy abban az esetben, ha $\mathcal{G}$ a $\mathcal{G}_L$ Lie-részcsoportja, $\mathcal{T}_o\mathcal{G} = \mathfrak{g}$ nem más, mint a $\mathcal{G}$ részcsoport Lie-algebrája. Ily módon a 2.1.11 Definíció általánosítja a Lie-részcsoportokhoz asszociált Lie-részalgebra klasszikus fogalmát.

2.1.13 Definíció. Egy $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ részcsoportot végtelen dimenziósnak nevezünk, ha $\mathcal{T}_o\mathcal{G}$-vel jelölt érintő algebrája végtelen dimenziós.

2.2 A fibrált holonómia algebra és Lie-részalgebrái

A *fibrált holonómia csoport* fogalma már korábban bevezetésre került [33] -ban:

2.2.1 Definíció. Az $(M, \mathcal{F})$ Finsler-sokaság fibrált holonómia csoportján az $(\mathcal{I}M, \pi, M)$ indikátrix nyaláb olyan fibrumtartó $\Phi \in \mathcal{D}iff^\infty(\mathcal{I}M)$ diffeomorfizmusait tartalmazó csoportot értjük, mely elemeire tetszőleges $p \in M$ esetén teljesül, hogy a $\Phi_p = \Phi|_{\mathcal{I}_pM} \in \mathcal{D}iff^\infty(\mathcal{I}_pM)$ leszűkítés benne van a $\mathcal{H}ol_p(M)$ holonómia csoportban.

Legyen $(M, \mathcal{F})$ egy kompakt Finsler-sokaság. Könnyen látható, hogy

$$\mathcal{H}ol_f(M) \subset \mathcal{D}iff^\infty(\mathcal{I}M), \tag{2.15}$$

ahol $\mathcal{H}ol_f(M)$ az indikátrix nyaláb diffeomorfizmus csoportjának részcsoportja. Még nem ismert, hogy $\mathcal{H}ol_f(M)$ Lie-részcsoportja-e a $\mathcal{D}iff^\infty(\mathcal{I}M)$ diffeomorfizmus csoportnak. A $\mathcal{H}ol_f(M)$ érintővektorainak halmazát

$$\mathfrak{hol}_f(M) := \mathcal{T}_0\Big(\mathcal{H}ol_f(M)\Big) \tag{2.16}$$

módon jelöljük.

2.2.2 Definíció. A $\mathfrak{hol}_f(M)$ halmazt az $(M, \mathcal{F})$ Finsler-sokaság *fibrált holonómia algebrájának* nevezzük.

A 2.1.4 Tétel következményeként kapjuk:

2.2.3 Következmény. A $\mathfrak{hol}_f(M)$ fibrált holonómia algebra az $\mathcal{I}M$ sima vektormezőinek $\mathfrak{X}(\mathcal{I}M)$ Lie-algebrájának Lie-részalgebrája.

A görbületi tenzor segítségével bevezetjük $\mathfrak{hol}_f(M)$ két rendkívül fontos Lie-részalgebráját: a görbületi algebrát és az infinitezimális holonómia algebrát.

2.2.4 Definíció. A $\mathfrak{R}$ *görbületi algebra* a görbületi vektormezők által generált Lie-algebra az indikátrix nyalábon:

$$\mathfrak{R} = \Big\langle R(X^h, Y^h)|_{\mathcal{I}M} \;\Big|\; X, Y \in \mathfrak{X}(M) \Big\rangle_{Lie}.$$

A görbületi tenzor definíciójából egyszerűen látszik, hogy a görbületi vektormezőket kiszámíthatjuk az alábbi formula segítségével:

$$\xi = R(X^h, Y^h) = \left[X^h, Y^h\right] - \left[X, Y\right]^h. \tag{2.17}$$

Mi több, a következő állítást is megfogalmazhatjuk:

2.2.5 Állítás. A $\mathfrak{R}$ görbületi algebra elemei érintik a $\mathcal{H}ol_f(M)$ fibrált holonómia csoportot és az általuk generált $\mathfrak{R}$ a $\mathfrak{hol}_f(M)$ Lie-részalgebrája.

2.2.6 Lemma. Bármely $p \in M$ pont esetén

(1) $h_{t,p} \in \mathcal{H}ol_p(M)$,

(2) $t \to h_{t,p}$ egy másodrendű integrálgörbéje a $\xi_p := \xi\big|_{\mathcal{I}_p}$ $\big(\xi_p \in \mathfrak{X}(\mathcal{I}_p)\big)$ vektormezőnek.

2.2.7 Definíció. Egy $(M, \mathcal{F})$ Finsler-sokaság $\mathfrak{hol}^*(M)$ *infinitezimális holonómia algebráján* az $X \in \mathfrak{X}(\mathcal{I}M)$ azon legszűkebb Lie-részalgebráját értjük, amely teljesíti az alábbi tulajdonságokat:

1) a görbületi vektormezők elemei $\mathfrak{hol}^*(M)$-nek,

2) ha $\xi \in \mathfrak{hol}^*(M)$ és $X \in \mathfrak{X}(M)$, akkor a $\nabla_X \xi$ horizontális Berwald kovariáns derivált is eleme $\mathfrak{hol}^*(M)$-nek.

2.2.8 Állítás. Az $\mathfrak{hol}^*(M)$ infinitezimális holonómia algebra érinti a $\mathcal{H}ol_f(M)$ fibrált holonómia csoportot, valamint $\mathfrak{hol}^*(M)$ a $\mathfrak{hol}_f(M)$ fibrált holonómia algebra Lie-részalgebrája.

2.3 A holonómia algebra és Lie-részalgebrái

Legyen $(M, \mathcal{F})$ egy Finsler-sokaság.

2.3.1 Definíció. A $\mathcal{H}ol_p(M)$ holonómia csoport érintő Lie-algebráját

$$\mathfrak{hol}_p(M) := \mathcal{T}_0\Big(\mathcal{H}ol_p(M)\Big)$$

módon definiáljuk és az $(M, \mathcal{F})$ Finsler-sokaság *holonómia algebrájának nevezzük* a $p \in M$ pontban.

2.3.2 Következmény. Egy $(M, \mathcal{F})$ Finsler-sokaság $p \in M$-beli $\mathfrak{hol}_p(M)$ holonómia algebrája az $\mathfrak{X}(\mathcal{I}_p)$ egy Lie-részalgebrája.

2.3.3 Definíció. A $p \in M$-beli göbületi vektormezők által generált $\mathfrak{R}_p$ Lie-algebrát a sokaság $p \in M$-beli *görbületi algebrájának* nevezzük:

$$\mathfrak{R}_p = \left\langle R(X_p^h, Y_p^h)|_{\mathcal{I}_p} \;\middle|\; X_p, Y_p \in T_pM \right\rangle_{Lie}.$$

A $\mathfrak{R}_p$ p-beli görbületi algebra a sokaság $\mathfrak{R}$ görbületi algebrájának $\mathcal{I}_p$-re való leszűkítése.

2.3.4 Állítás. A $p \in M$-beli $\mathfrak{R}_p$ görbületi algebra érinti a $\mathcal{H}ol_p(M)$ holonómia csoportot, valamint $\mathfrak{R}_p$ a $\mathfrak{hol}_p(M)$ holonómia algebra Lie-részalgebrája.

2.3.5 Definíció. Az $(M, \mathcal{F})$ Finsler-sokaság $p \in M$-beli $\mathfrak{hol}^*(M)$ *infinitezimális holonómia algebráján* azt a legkisebb Lie-algebrát értjük, melyre az alábbi tulajdonságok teljesülnek:

1) Minden ξ_p görbületi vektormező eleme $\mathfrak{hol}_p^*(M)$-nak,
2) ha $\xi_p \in \mathfrak{hol}_p^*(M)$ és $X \in \mathfrak{X}(M)$, akkor a $(\nabla_X \xi)(p)$ horizontális Berwald kovariáns derivált is eleme $\mathfrak{hol}_p^*(M)$-nak.

2.3.6 Megjegyzés. Egy $(M, \mathcal{F})$ Finsler-sokaság $p \in M$-beli infinitezimális holonómia algebráját az alábbi módon is definiálhatjuk:

$$\mathfrak{hol}_p^*(M) := \left\{ \xi|_{\mathcal{I}_p} \mid \xi \in \mathfrak{hol}^*(M) \right\}.$$

A 2.2.8 Állítás alapján kapjuk:

2.3.7 Következmény. A $\mathfrak{hol}_p^*(M)$ infinitezimális holonómia algebra érinti a $\mathcal{H}ol_p(M)$ holonómia csoportot, valamint $\mathfrak{hol}_p^*(M)$ holonómia algebra Lie-részalgebrája.

Megjegyezzük, hogy a 2.3.4 és 2.3.7 Állítások előrelépést jelentenek [33] eredményeihez képest, mivel az érintő tulajdonságban C^1-simaság helyett C^∞-simaságot kapunk.

3. Végtelen dimenziós holonómia csoporttal rendelkező Finsler-terekről

3.1 A kvantumnavigációs probléma holonómiájáról

A kvantum információ elmélet Randers modellje

Egy zárt, véges dimenziós kvantum rendszerben az állapotteret $\mathbb{C}^n$-el modellezhetjük valamely $n \in \mathbb{N}$ esetén, a fizikai állapotok pedig az alap Hilbert-tér $\mathbb{C}P^{n-1}$ projektív terének sugaraival reprezentálhatóak. A kvantumnavigációs problémában a 'navigátor' feladata, hogy megtalálja a legrövidebb útvonalat egy $|\Psi_I\rangle$ kezdeti és egy $|\Psi_F\rangle$ végállapot között. A probléma megoldása rendkívül nehéz az állapottér szintjén, azonban [9]-ben és [8]-ban a szerzők megmutatták, hogy a probléma átfogalmazható az állapotokon ható térre, mely ebben az esetben $SU(n)$. A feladat tehát ebben az értelmezésben az, hogy találjunk egy $\hat{H}_c(t)$ Hamilton-operátort az $SU(n)$ csoport $\mathfrak{su}(n)$ Lie-algebrájában, mely esetén a $\hat{H}(t) = \hat{H}_c(t) + \hat{H}_0$ operátor generálja a kezdeti állapot evolúcióját a végső állapotba egy időfüggetlen $\hat{H}_0$ Hamilton-operátorral kiegészítve. Ez a $\hat{H}_0$ operátor egy nem eliminálható külső erőhatást reprezentál, tipikusan mágneses mezőt.

2-dimenziós Kvantum Zermelo probléma

A továbbiakban egy kétállapotú kvantum rendszerrel foglalkozunk: egy mágneses mezőben vett elektron esetét vizsgáljuk. Ahogy [9]-ben, a vizsgálataink során mi is egy invariáns "szél" vektormezőt tekintünk, melyet $\hat{H}_0$ reprezentál a Lie-algebrában, a Riemann-metrika pedig a Killing formából származó invariáns Riemann-metrika:

$$h(\hat{A}, \hat{B}) := \mathrm{tr}(\hat{A}^\dagger \hat{B}), \tag{3.1}$$

$\hat{A}, \hat{B} \in \mathfrak{su}(2)$. Az $\mathfrak{su}(2)$ Lie-algebrát a $\hat{E}_1 = i\sigma_1, \hat{E}_2 = -i\sigma_2, \hat{E}_3 = i\sigma_3$ vektormezők generálják, ahol a σ_i mátrixok a Pauli mátrixokat jelölik.

A számítások során az érintőnyalábon vett (x, ξ) koordinátákkal dolgozunk, ahol $(x) = (x_1, x_2, x_3)$ az $SU(2)$ csoporton vett koordináták,

$(\xi) = (\xi_1, \xi_2, \xi_3)$ pedig a $\mathfrak{su}(2)$ Lie-algebra invariáns koordinátái az $\left\{\hat{E}_1, \hat{E}_2, \hat{E}_3\right\}$ bázisra vonatkozóan. A standard és invariáns koordináták közti kapcsolatot $TG = G \times \mathfrak{g}$-n egyszerűen meghatározhatjuk a $\rho^{-1}_{x,*}(x, y) = (x, \xi)$ formula segítégével, ahol $\rho : SU(2) \to SU(2)$ a csoport jobb eltolása. Modulo egy merev transzformáció, feltételezhetjük, hogy $\hat{H}_0 = c\hat{E}_1$, ahol $c \in \mathbb{R}$. A (3.1) egyenlet és $W = \hat{H}_0$ alapján kiszámíthatjuk a megfelelő invariáns Randers metrikát:

$$\mathcal{F}(\xi) = \frac{1}{1-2c^2}\Big(\alpha_\xi + \beta_\xi\Big), \tag{3.2}$$

ahol α egy Riemann-metrika, β pedig egy 1-forma:

$$\alpha_\xi := \sqrt{2\xi_1^2 + 2\xi_2^2 + 2\xi_3^2 - 4\xi_2^2c^2 - 4\xi_3^2c^2}, \qquad \beta_\xi := -2c\,\xi_1. \tag{3.3}$$

Az egyszerűség kedvéért a 2-dimenziós kvantum Zermelo problémának megfelelő 3-dimenziós Finsler teret $\mathcal{Q} = (SU(2), \mathcal{F})$ módon jelöljük.

Az Euler-Lagrange egyenletek invariáns megfogalmazását Euler-Poincaré egyenletek néven ismerjük:

$$\frac{d}{dt}\frac{\partial E}{\partial \xi} = -ad^*_\xi\left(\frac{\partial E}{\partial \xi}\right). \tag{3.5}$$

Ezen egyenletek segítségével meghatározhatjuk a Lie-algebra geodetikus egyenleteit [13].

A görbületi algebra és a holonómia

A görbületi tenzort a spray együtthatókból határozhatjuk meg. A görbületi vektormezők első Lie-zárójelét egyszerű számítással kapjuk:

$$[R_1, R_3] = \frac{1}{\alpha_\xi}R_2 + \frac{2c\,\xi_2}{\alpha_\xi^2}R_3, \qquad [R_2, R_3] = \frac{-1}{\alpha_\xi}R_1 + \frac{2c\,\xi_3}{\alpha_\xi^2}R_3. \tag{3.10}$$

Általánosabban:

3.1.1 Lemma. Legyen $L_0 = R_2$ és jelöljük $L_k = [L_{k-1}, R_3]$ módon a L_{k-1} és R_2 vektormezők Lie-zárójelét minden $k \geq 1$ esetén. Ekkor

$$L_k = \begin{cases} \dfrac{\epsilon_k}{\alpha_\xi^k}R_2 + 2kc\dfrac{\epsilon_k\,\xi_2}{\alpha_\xi^{k+1}}R_3, & \text{ha } k \equiv 0 \mod 2, \\[2ex] \dfrac{\epsilon_k}{\alpha_\xi^k}R_1 - 2kc\dfrac{\epsilon_k\,\xi_3}{\alpha_\xi^{k+1}}R_3, & \text{ha } k \equiv 1 \mod 2, \end{cases} \tag{3.11}$$

ahol $\epsilon_k = -1$, ha $k = 4l+1$ vagy $k = 4l+2$ és $\epsilon_k = 1$, ha $k = 4l+3$ vagy $k = 4l$ valamely $l \in \mathbb{N}$ esetén.

3.1.2 Állítás. Egy külső W erőhatás mellett a (3.2) Finsler-metrika $\mathfrak{R}$ görbületi algebrája az indikátrix sima vektormezőinek $\mathfrak{X}(\mathcal{I})$ Lie-algebrájának végtelen dimenziós Lie-részalgebrája.

3.1.3 Tétel. A 2-dimenziós kvantum Zermelo probléma $\mathcal{H}ol(\mathcal{Q})$ holonómia csoportja egy külső W erőhatás esetén nem véges dimenziós.

Megjegyezzük, hogy abban az esetben, amikor nincs külső erőhatás, azaz $W = 0$, a (3.2) metrika egy Riemann-metrika, az indikátrixon $\alpha_\xi = 1$ és könnyen látható, hogy a $\mathfrak{R}$ görbületi algebra izomorf az $\mathfrak{so}(3)$ Lie-algebrával.

3.2 Lokálisan síkprojektív, konstans zászlógörbületű Randers felületek holonómiája

Az alfejezet célja a konstans zászlógörbületű síkprojektív Randers felületek holonómia csoportjának vizsgálata. Első lépésben az 1.2 alfejezetben bevezetett standard modell holonómiáját vizsgáljuk.

Legyen $(\mathbb{B}^2, \mathcal{F}_a)$ egy Finsler felület, ahol $\mathbb{B}^2$ az $\mathbb{R}^2$ tér nyílt körlemeze, $\mathcal{F}_a$ pedig a (1.20) egyenlet alapján meghatározott Finsler függvény $a = (a_1, 0) \in \mathbb{R}^2$ választással, ahol $|a_1| < 1$ nemnulla konstans vektor.

3.2.1 Állítás. A $(\mathbb{B}^2, \mathcal{F}_a)$ Finsler-sokaság holonómia csoportja maximális és $\mathcal{H}ol^c_x(M)$ diffeomorf $\mathcal{D}iff^\infty_+(\mathbb{S}^1)$-vel, az $\mathbb{S}^1$ egységkör irányítástartó diffeomorfizmus csoportjával.

Z. Shen Randers-sokaságokra vonatkozó klasszifikációs tételét alkalmazva kapjuk

3.2.4 Tétel. Egy egyszeresen összefüggő nem-Riemann és nemzérus konstans zászlógörbületű lokálisan síkprojektív Randers felület holonómia csoportja maximális, valamint $\mathcal{H}ol^c(M)$ diffeomorf $\mathcal{D}iff^\infty_+(\mathbb{S}^1)$-vel , azaz

$$\mathcal{H}ol^c(M) \cong \mathcal{D}iff^\infty_+(\mathbb{S}^1).$$

A tétel alapján a következő klasszifikációt kapjuk:

3.2.5 Következmény. Egy egyszeresen összefüggő konstans zászlógörbületű lokálisan síkprojektív Randers felület holonómia csoportjának lezártja

1. az $\{id\}$ triviális csoport, ha $\lambda = 0$;
2. az $SO(2)$ forgatás csoport, ha $\lambda \neq 0$ és a metrika Riemann;
3. az egységkör irányítástartó diffeomorfizmus csoportja $\mathcal{D}iff^{\infty}_{+}(\mathbb{S}^1)$, ha $\lambda \neq 0$ és a metrika nem-Riemann.

4. A végtelen dimenziós holonómia csoporttal rendelkező Finsler-terek halmazának sűrűségéről

Ebben a fejezetben bebizonyítjuk, hogy egy generikus Finsler-sokaság holonómia csoportja végtelen dimenziós. Pontosan szólva a 4.4.1 Tételben megmutatjuk, hogy egy $n \geq 2$ dimenziós M sokaságon értelmezett C^{∞}-sima Finsler metrikák halmazának létezik végtelen dimenziós holonómia csoporttal rendelkező olyan $\widetilde{\mathcal{F}}$ részhalmaza, mely nyílt és mindenhol sűrű bármely $\mathcal{C}^m$-topológiában, ahol $m \geq 8$. Ez az eredmény rávilágít, hogy a Riemann esettel szemben a legtöbb Finsler-sokaság holonómia csoportjának lezártja nem egy kompakt csoport. Hasonló eredményre jutottak a szerzők [22]-ban a *lineáris holonómia csoportra vonatkozóan,* melyet lineáris párhuzamos eltolással definiálunk.

4.1 Vektormezők 3-jet generáló Lie-alegbrái

4.1.1 Definíció. Egy M sokaságon vett vektormezők $\mathcal{V} \subset \mathfrak{X}(M)$ halmazát

- k-jet generálónak nevezzük az $x \in M$ pontban, ha a természetes $j^k_x \colon \mathcal{V} \to J^k_x(\mathfrak{X}(M))$ leképezés szürjektív,
- jet generálónak nevezzük M-en, ha bármely $x \in M$ és bármely $k \geq 0$ esetén k-jet generáló.

A definíció értelmében az $U \subseteq \mathbb{R}^n$-en értelmezett vektormezők $\mathfrak{g}$ algebráját 3-*jet generálónak* nevezzük $x \in U$-ban, ha minden vektormezőt közelíthetünk harmadrendben az algebra elemeivel az x pontban. A következő tételhez jutunk:

4.1.2 Tétel. Legyen $\mathfrak{g}$ egy U sokaságon értelmezett vektormezők Lie-algebrája. Ha létezik olyan pont, melyben $\mathfrak{g}$ rendelkezik a 3-jet generáló tulajdonsága, akkor $\mathfrak{g}$ végtelen dimenziós.

Megjegyezzük, hogy ha U dimenziója 1, az eredmény ekvivalens Sophus Lie nevezetes eredményével, lásd pl. [27, Theorem 2.70]. Ahogy a [27] végén található táblázatban szereplő példák mutatják, melyben véges dimenziós vektormező algebrák szerepelnek, a 3-jet generáló tulajdonság fontos szereppel bír.

4.2 A Funk-metrika holonómia algebrájának 3-jet generáló tulajdonsága

Ebben a fejezetben a 1.2 Alfejezetben bemutatott Funk-metrika holonómia struktúráját vizsgáljuk.

4.2.1 Megjegyzés. A $(\mathbb{B}^2, F_{\mathbb{B}^2})$ Finsler-sokaság holonómiáját korábban [37, Chapter 5] vizsgálta. A szerzők bebizonyították, hogy a $(\mathbb{B}^2, F_{\mathbb{B}^2})$ Finsler felület holonómia csoportjának lezártja $\mathcal{D}iff_+(\mathbb{S}^1)$, az egységkör irányítástartó diffeomorfizmus csoportjának lezártja [37, Theorem 5.2].

A fenti eredményt felhasználva kapjuk a következő állítást:

4.2.2 Állítás. A standard Funk-metrika $\mathfrak{hol}^*_o(F_{\mathbb{B}^n})$-módon jelölt $o \in \mathbb{B}^n$ pontbeli infinitezimális holonómia algebrája rendelkezik a jet generáló tulajdonsággal az $\mathcal{I}_o$ indikátrixon.

4.3 Finsler metrikák Funk perturbációi

Legyen (M, F) egy Finsler-sokaság, $x_0 \in M$ pedig egy rögzített pontja. Ekkor választhatunk egy olyan x_0-ba centrált (U, x) koordinátarendszert, melyre $x(U) \subset \mathbb{B}^n$. Az ehhez TM-en asszociált koordinátarendszert $(\pi^{-1}(U), \chi = (x, y))$ módon jelöljük. Tekintsünk továbbá egy $\psi \colon M \to \mathbb{R}$ dudorfüggvényt, melyre $supp(\psi) \subset U$ és $\psi|_{\tilde{U}} = 1$ teljesül x_0 valamely $\tilde{U} \subset U$ nyílt környezetében. Jelöljük $\bar{\psi} := \psi \circ \pi$ módon a ψ leképezés π általi visszahúzottját.

A standard Funk-norma függvényt használva bevezetünk egy új $\bar{F}$: $TM \to \mathbb{R}$ Finsler-normát az alábbi módon:

$$\bar{F}^2 = \psi \cdot (F_{\mathbb{B}^n} \circ \chi)^2 + (1-\psi) \cdot F^2. \tag{4.17}$$

Megjegyezzük, hogy $\bar{F}$ a standard Funk-norma visszahúzottja $\pi^{-1}(\tilde{U})$-n.

Felhasználva a (4.17) egyenletet, definiáljuk az F sima perturbációját F_t függvények 1-paraméteres családjaként, ahol

$$F_t^2 = (1-t)F^2 + t\bar{F}^2, \quad t \in [0,1]. \tag{4.18}$$

Könnyen elenőrizhető, hogy ekkor minden F_t egy Finsler-norma lesz M-en.

4.3.2 Állítás. A $\mathfrak{hol}^*_{x_0}(F_t)$ infinitezimális holonómia algebra minden eleme kifejezhető t polinomjainak törtkifejezéseként, ahol a polinomok együtthatói meghatározhatóak a $j^k_{x_0}F$ és $j^k_{x_0}\bar{F}$ jetekkel valamely $k \in \mathbb{N}$ esetén.

4.3.3 Állítás. Bármely $y_0 \in \mathcal{I}_o$ esetén azon $t \in [0,1]$ paraméterek halmaza, melyekre a Funk perturbáció $\mathfrak{hol}^*_{x_0}(F_t) \subset \mathfrak{X}(\mathcal{I}^t_{x_0})$ infinitezimális holonómia algebrája nem rendelkezik a 3-jet generáló tulajdonsággal, véges.

4.4 Majdnem minden Finsler-metrika holonómia csoportja végtelen dimenziós

4.4.1 Tétel. Egy $n \geq 2$ dimenziós M sokaságon értelmezett C^∞-sima Finsler metrikák $\mathcal{F}$ halmazának létezik $m \geq 8$ esetén a $\mathcal{C}^m$-topológiában véve olyan nyílt és mindenütt sűrű részhalmaza, mely elemeinek holonómia csoportja végtelen dimenziós.

4.4.2 Megjegyzés. A 4.4.1 Tétel bizonyításában megmutattuk, hogy egy megfelelő perturbáció esetén az infinitezimális holonómia algebra egy pontban végtelen dimenziós. Ez az állítás igaz marad a pont kis környezetében.

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