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COMPARISON AND SUBHOMOGENEITY OF INTEGRAL MEANS

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Abstract. The differential and integral means were defined and studied by Elezović and Pečarić [3]. They gave sufficient conditions for the comparison of two integral (differential) means and applied this to obtain some other inequalities.

Our aim is to give necessary and sufficient conditions for the comparison of two integral means and in this way improve theorems 2, 3 of [3]. We also deal with the subhomogeneity and homogeneity of integral means. As they are special Cauchy means we can use the method of the author [7] to prove our results.

1. Differential and integral means

If f is a continuous real function on an (open or closed) interval I and f is differentiable on I° (being the interior of I) then for every $x, y \in I, x < y$ there is a point $t \in]x, y[$ such that

$$f'(t) = \frac{f(y) - f(x)}{y - x}.$$

This is Lagrange's mean value theorem.

If f' is invertible then t is unique and

$$t = (f')^{-1}\left(\frac{f(y) - f(x)}{y - x}\right).$$

This number t is called the *differential* f*-mean of* x and y and denoted by $D_f(x, y)$.

Similarly if $g : I \to \mathbb{R}$ is continuous and strictly monotonic on I then for every $x, y \in I, x < y$ there is a unique point $s \in]x, y[$ such that

$$g(s) = \frac{1}{y - x}\int_x^y g(u)\, du.$$

thus

$$s = g^{-1}\left(\frac{1}{y - x}\int_x^y g(u)\, du\right).$$

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This number s is called the *integral* g*-mean of* x and y and denoted by $I_g(x,y)$.

Clearly, (requiring D_f, I_g to have the mean property or requiring them to be continuous) we have for equal arguments

$$D_f(x,x) = I_g(x,x) = x \quad (x \in I).$$

These means were defined and studied by Elezović and Pečarić [3] (we slightly changed their definition). They gave sufficient conditions for the comparison of differential and integral means and applied these to obtain some other inequalities.

Our aim is to give necessary and sufficient conditions for the comparison of differential and integral means and in this way improve theorems 2, 3 of [3]. For this purpose we shall use the method and results of the author [7] *as the means* D_f, I_g *are special Cauchy means.* Due to the importance of the above means we give a proof for the comparison theorem which is independent from the one in [7]. We also discuss the subhomogeneity and homogeneity of these means.

As

$$D_f = I_{f'}$$

it is enough to study the means I_f only.

Parallel to our means we consider the discrete quasi–arithmetic means

$$\mathscr{M}_\phi(\mathbf{u}) := \phi^{-1}\left(\frac{\sum\limits_{k=1}^{n} \phi(u_k)}{n}\right) \quad (\mathbf{u} = (u_1,\dots,u_n) \in I^n, n \geqslant 2)$$

and quasi–arithmetic integral means

$$\mathscr{I}_\phi(u) := \phi^{-1}\left(\frac{\int\limits_a^b \phi(u(t))\,dt)}{b-a}\right) \quad (u \in R[a,b])$$

where $\phi : I \to \mathbb{R}$ is a strictly monotonic and continuous function on the interval I and the latter case $I = [a,b]$ and $R[a,b]$ denotes the set of all functions $u : [a,b] \to \mathbb{R}$ which are Riemann integrable on $[a,b]$.

If ψ is also a continuous and strictly monotonic function on I then the comparison of discrete quasi–arithmetic means

$$\mathscr{M}_\phi(\mathbf{u}) \leqslant \mathscr{M}_\psi(\mathbf{u}) \quad (\mathbf{u} \in I^n, n \geqslant 2)$$

or the comparison of quasi–arithmetic integral means

$$\mathscr{I}_\phi(u) \leqslant \mathscr{I}_\psi(u) \quad (u \in R[a,b])$$

holds if and only if

$$\begin{cases} \text{either} \quad \phi \text{ is strictly increasing on } I \text{ and } \quad \phi\circ\psi^{-1} \text{ is concave on } \psi(I), \\ \text{or} \quad \phi \text{ is strictly decreasing on } I \text{ and } \quad \phi\circ\psi^{-1} \text{ is convex on } \psi(I) \end{cases} \tag{1}$$

where $\psi(I)$ is the range of ψ over I (see e.g. Hardy, Littlewood and Pólya [4], Losonczi [6]).

As

$$I_g(x,y) = \mathscr{I}_g(u) \quad \text{if} \quad u(t) = t \in I = I_{x,y} = [x,y]$$

and the validity of (1) on I implies its validity on any subinterval, (1) (with $\phi = f,\ \psi = g$) is clearly sufficient for the comparison

$$I_f(x,y) \leqslant I_g(x,y) \quad (x,y \in I).$$

In Section 2 we show that assuming some differentiability conditions (1) (with $\phi = f,\ \psi = g$) *is also necessary* for the above comparison.

2. Comparison and equality of integral means

The strict monotonicity and continuity of f guarantees the existence of I_f. We need however stronger regularity conditions to find criteria for the comparison of integral means. Denote by $\mathscr{E}(I)$ the set of functions $f : I \to \mathbb{R}$ which are differentiable on I and their derivatives do not vanish on I. Clearly $f \in \mathscr{E}(I)$ ensures the existence of I_f. As usual $\mathscr{C}_n(I)$ denotes the space of all functions $f : I \to \mathbb{R}$ which have continuous nth derivative on I.

THEOREM 1. *Suppose that I is a real interval, $f,g \in \mathscr{E}(I)$; $f,g \in \mathscr{C}_2(I)$. The inequality*

$$I_f(x,y) \leqslant I_g(x,y) \quad (x,y \in I) \tag{2}$$

holds if and only if one of the eight conditions

$$\frac{f''(x)}{f'(x)} \leqslant \frac{g''(x)}{g'(x)} \quad (x \in I) \tag{3}$$

$$\ln\left|\frac{f'}{g'}\right| \text{ is decreasing on } I \tag{4}$$

$$\ln\left|\frac{g'}{f'}\right| \text{ is increasing on } I \tag{5}$$

$$\frac{f(u)-f(v)}{f'(v)} \leqslant \frac{g(u)-g(v)}{g'(v)} \quad (u,v \in I) \tag{6}$$

$$f'\left(g^{-1}(t)\right)\frac{d^2}{dt^2}f\left(g^{-1}(t)\right) \leqslant 0 \quad (t \in f(I)) \tag{7}$$

$$g'\left(f^{-1}(t)\right)\frac{d^2}{dt^2}g\left(f^{-1}(t)\right) \geqslant 0 \quad (t \in g(I)) \tag{8}$$

$$\begin{cases} \text{either} & f \text{ is strictly increasing on } I \text{ and } f\circ g^{-1} \text{ is concave on } g(I), \\ \text{or} & f \text{ is strictly decreasing on } I \text{ and } f\circ g^{-1} \text{ is convex on } g(I) \end{cases} \tag{9}$$

$$\begin{cases} \text{either} & g \text{ is strictly decreasing on } I \text{ and } g\circ f^{-1} \text{ is concave on } f(I), \\ \text{or} & g \text{ is strictly increasing on } I \text{ and } g\circ f^{-1} \text{ is convex on } f(I) \end{cases} \tag{10}$$

is satisfied.

Proof. Necessity. We show that (2) implies (3) and that (3) is equivalent to (4), (5), (6), (7), (8), (9), (10).

It is easy to check that I_f can be written in the form

$$I_f(x,y) = f^{-1}\left(\int_0^1 f((1-t)x+ty)\,dt\right).$$

The advantage of this form is that it is valid for all $x,y \in I$.

Differentiating behind the integral sign we obtain that

$$\partial_1 I_f(x,y) = \frac{\int_0^1 (1-t)f'((1-t)x+ty)\,dt}{f'[I_f(x,y)]},$$

$$\partial_1^2 I_f(x,y) = \frac{\int_0^1 (1-t)^2 f''((1-t)x+ty)\,dt}{f'[I_f(x,y)]} - \frac{\left(\int_0^1 (1-t)f'((1-t)x+ty)\,dt\right)^2 f''[I_f(x,y)]}{f'[I_f(x,y)]^3}$$

where ∂_1 denotes partial differentiation operator with respect to the first variable. From this, by $I_f(x,x)=x$, $\int_0^1 (1-t)\,dt = 1/2$, $\int_0^1 (1-t)^2\,dt = 1/3$ we obtain

$$\partial_1 I_f(x,y)|_{y=x} = \frac{1}{2},$$

$$\partial_1^2 I_f(x,y)|_{y=x} = \frac{1}{12}\frac{f''(x)}{f'(x)}.$$

Denoting the difference of the right and left hand sides of (2) by $\Phi(x,y)$ we get for any $x,y \in I$ by Taylor's formula that

$$\Phi(x,y) = \Phi(y,y) + \partial_1\Phi(y,y)(x-y) + \frac{\partial_1^2\Phi(\xi,y)}{2}(x-y)^2 = \frac{\partial_1^2\Phi(\xi,y)}{2}(x-y)^2 \geqslant 0$$

where ξ is a value between x and y. Thus $\partial_1^2\Phi(\xi,y) \geqslant 0$ and taking the limit $y \to x$ we obtain, by the continuity of f'', g'', that

$$\partial_1^2 \Phi(x,x) = \frac{1}{12}\left(\frac{f''(x)}{f'(x)} - \frac{g''(x)}{g'(x)}\right) \geqslant 0$$

proving that (3) is necessary for (2).

Next we show that (3 is equivalent to (4), (5), (6), (7), (8), (9), (10) provided that $f,g \in \mathscr{E}(I)$; $f,g \in \mathscr{C}_2(I)$. Using the identity $(\ln|f'|)' = f''/f'$ we get

$$\frac{g''(x)}{g'(x)} - \frac{f''(x)}{f'(x)} = -\left(\ln\left|\frac{f'}{g'}\right|\right)' = \left(\ln\left|\frac{g'}{f'}\right|\right)'$$

which shows that (3) is equivalent to (4) and (5).

Let

$$H(t) := f\left(g^{-1}(t)\right) \leqslant 0 \quad (t \in g(I)).$$

then

$$f'\left(g^{-1}(t)\right) H''(t) = f'\left(g^{-1}(t)\right)\left(\frac{f''\left(g^{-1}(t)\right)}{\left(g'\left(g^{-1}(t)\right)\right)^2} - \frac{f'\left(g^{-1}(t)\right) g''\left(g^{-1}(t)\right)}{\left(g'\left(g^{-1}(t)\right)\right)^3}\right)$$

$$= \left(\frac{f'\left(g^{-1}(t)\right)}{g'\left(g^{-1}(t)\right)}\right)^2 \left(\frac{f''\left(g^{-1}(t)\right)}{f'\left(g^{-1}(t)\right)} - \frac{g''\left(g^{-1}(t)\right)}{g'\left(g^{-1}(t)\right)}\right)$$

which shows that (3) and (7) are equivalent. We can prove the equivalence of (3) and (8) in a similar way rewriting the second derivative of $K(t) := g\left(f^{-1}(t)\right)$ $(t \in g(I))$ in a suitable form. Assume now that $f'(x) > 0$ $(x \in I)$. Then (7) means exactly that H is concave on $g(I)$ A well-known characterization of the concavity of H is

$$H(s) - H(t) \leqslant (s-t)H'(t) \quad (t, s \in g(I))$$

(see e.g. [8]) which can be written as

$$f\left(g^{-1}(s)\right) - f\left(g^{-1}(t)\right) \leqslant (s-t)\frac{f'\left(g^{-1}(t)\right)}{g'\left(g^{-1}(t)\right)} \quad (t, s \in J).$$

Substituting here $s = g(u), t = g(v)$ $u, v \in I$ one can see that (6) is equivalent to the concavity of H on $g(I)$ i.e. to (7) and thus also to (3). The case when $f'(x) < 0$ $(x \in I)$ can be treated similarly.

Using characterization of $\mathscr{C}_2$ convex (concave), strictly monotonic functions it is obvious that (9) is equivalent to (7) and (10) is equivalent to (8).

We remark that the equivalence of (3) and (7) has also been discussed in [2].

Sufficiency. We show that (6) implies (2). Assume again that $f'(x) > 0$ $(x \in I)$. Let $x, y \in I$ and substitute

$$u = (1-t)x + ty, \quad v = I_g(x, y)$$

in (6) and integrate from 0 to 1 with respect to t. The right hand side of the inequality so obtained will be zero by the definition of I_g thus we have

$$\frac{\int_0^1 f((1-t)x+ty)\,dt - f\left(I_g(x,y)\right)}{f'\left(I_g(x,y)\right)} \leqslant 0$$

hence

$$(I_f(x,y) =) \;\; f^{-1}\left(\int_0^1 f((1-t)x+ty)\,dt\right) \leqslant I_g(x,y)$$

as we stated. The case $f' < 0$ is similar. $\square$

REMARK 1. According to Theorem 1 (2) $\Leftrightarrow$ (9) (or (10)) provided that $f,g \in \mathscr{E}(I)$; $f,g \in \mathscr{C}_2(I)$. As we have seen the implication (9) (or (10)) $\Rightarrow$ (2) holds if we assume only the minimal conditions on f and g: their strict monotonicity and continuity. It is quite natural to ask if the reverse implication holds if only the strict monotonicity and continuity of f,g is assumed. This is very likely to be true but our method is definitely not suitable to prove it.

$H''(t) \geqslant 0$ $(t \in g(I))$ if and only if $H'(t) = f'\left(g^{-1}(t)\right)/g'\left(g^{-1}(t)\right)$ is increasing on $g(I)$. This holds if and only if either f'/g' is increasing and g is increasing on I or f'/g' is decreasing and g is decreasing on I. The inequality $H''(t) \leqslant 0$ $(t \in g(I))$ can be characterized similarly. By this we get still another form of the comparison (cf. [3] Theorem 2).

THEOREM 2. *Suppose that I is a real interval, $f,g \in \mathscr{E}(I); f,g \in \mathscr{C}_2(I)$.*
The inequality

$$I_f(x,y) \leqslant I_g(x,y) \quad (x,y \in I) \tag{2}$$

holds if and only if one of the following conditions hold:

(i) *f is increasing, g is increasing and f'/g' decreasing on I,*
(ii) *f is increasing, g is decreasing and f'/g' increasing on I,*
(iii) *f is decreasing, g is increasing and f'/g' increasing on I,*
(iv) *f is decreasing, g is decreasing and f'/g' decreasing on I.*

Applying this result and Theorem 1 and using the identity $D_g = I_{g'}$ we obtain

THEOREM 3. *Suppose that I is a real interval, $f \in \mathscr{E}(I); f \in \mathscr{C}_3(I)$ and $f''(x) \neq 0$ $(x \in I)$.*
The inequality

$$D_g(x,y) \leqslant I_g(x,y) \quad (x,y \in I)$$

holds if and only if one of the following conditions hold:

(i) *g is convex, g is increasing and g''/g' decreasing on I,*
(ii) *g is convex, g is decreasing and g''/g' increasing on I,*
(iii) *g is concave, g is increasing and g''/g' increasing on I,*
(iv) *g is concave, g is decreasing and g''/g' decreasing on I.*

THEOREM 4. *Suppose that I is a real interval, $f \in \mathscr{E}(I)$; $f \in \mathscr{C}_3(I)$ and $f''(x) \neq 0$ $(x \in I)$.*
The inequality

$$D_g(x,y) \leqslant I_g(x,y) \quad (x,y \in I)$$

holds if and only if

$$\text{the function } \ln\left|\frac{g''}{g'}\right| \text{ is decreasing on } I.$$

The inequality

$$D_g(x,y) \geqslant I_g(x,y) \quad (x,y \in I)$$

holds if and only if

$$\text{the function } \ln\left|\frac{g''}{g'}\right| \text{ is increasing on } I.$$

Concerning the equality of integral means we have

THEOREM 5. *Suppose that I is a real interval, $f,g \in \mathscr{E}(I)$; $f,g \in \mathscr{C}_2(I)$. The equality*

$$I_f(x,y) = I_g(x,y) \quad (x,y \in I) \tag{11}$$

holds if and only if there is are constants $c \neq 0, d$ such that

$$f(x) = cg(x) + d \quad (x \in I). \tag{12}$$

Proof. (11) holds if and only if both (2) and its reverse inequality

$$I_f(x,y) \geqslant I_g(x,y) \quad (x,y \in I)$$

are satisfied. By Theorem 1 these hold if and only if $\ln\left|\frac{f'}{g'}\right|$ is both decreasing and increasing on I i.e. if and only if it is a constant function, which is equivalent to (12). □

Similarly we get

THEOREM 6. *Suppose that I is a real interval, $f,g \in \mathscr{E}(I)$; $f \in \mathscr{C}_3(I), g \in \mathscr{C}_2(I)$. The equality*

$$D_f(x,y) = I_g(x,y) \quad (x,y \in I)$$

holds if and only if there are constants $c \neq 0, d$ such that

$$f'(x) = cg(x) + d \quad (x \in I).$$

3. Subhomogeneity of integral means

The concept of subhomogeneous functions was introduced in [5].

THEOREM 7. *Suppose that $f \in \mathscr{E}(\mathbb{R}_+)$; $f \in \mathscr{C}_3(\mathbb{R}_+)$ where $\mathbb{R}_+ =]0,\infty[$. Let $F(x) = xf'(x)$ $(x \in \mathbb{R}_+)$ and assume that $F'(x) = f'(x) + xf''(x) \neq 0$ for $x \in \mathbb{R}_+$. The mean I_f is positive subhomogeneous with respect to the function $(t,x) \to tx$, i.e.*

$$I_f(tx,ty) \leqslant tI_f(x,y) \quad (x,y \in \mathbb{R}_+, t \in]1,\infty[) \tag{13}$$

holds if and only if

$$x \to \operatorname{sign}\left[1 + x\frac{f''(x)}{f'(x)}\right] \ln\left|1 + x\frac{f''(x)}{f'(x)}\right| \quad (x \in \mathbb{R}_+) \text{ is decreasing on } \mathbb{R}_+. \tag{14}$$

Proof. By Theorem 2 of [5] (13) holds if and only if

$$x\partial_1 I_f(x,y) + y\partial_2 I_f(x,y) \leqslant I_f(x,y) \quad (x,y \in \mathbb{R}_+).$$

This can be written as

$$\frac{\int_0^1 ((1-t)x+ty)\, f'((1-t)x+ty)\; dt}{f'(I_f(x,y))} \leqslant I_f(x,y)$$

or, by the help of the function $F(x) := xf'(x)$ $(x \in \mathbb{R}_+)$, if $f' > 0$

$$F(I_F(x,y)) \leqslant F(I_f(x,y)) \quad (x,y \in \mathbb{R}_+).$$

If $F' > 0$ this is equivalent to

$$I_F(x,y) \leqslant I_f(x,y) \quad (x,y \in \mathbb{R}_+). \tag{15}$$

Thus, if $\operatorname{sign}\left[\frac{F'(x)}{f'(x)}\right] = 1$ then (13) holds if and only if (15) is valid while in the case $\operatorname{sign}\left[\frac{F'(x)}{f'(x)}\right] = -1$ (13) holds if and only if the reverse of (15) is valid. Applying Theorem 1 (using condition (4)) we get (14). □

REMARK 2. (14) is also necessary and sufficient for the inequality

$$I_f(tx,ty) \geqslant tI_f(x,y) \quad (x,y \in \mathbb{R}_+,\, t \in]0,1[).$$

If in (13) we require $t \in]0,1[$ or reverse the inequality sign in (13) (but keep $t \in]1,\infty[$) then Theorem 7 remains valid if we replace the word decreasing by increasing in (14).

THEOREM 8. *Suppose that* $f \in \mathscr{E}(\mathbb{R}_+)$; $f \in \mathscr{C}_2(\mathbb{R}_+)$. *The mean* I_f *is homogeneous i.e.*

$$I_f(tx,ty) = tI_f(x,y) \quad (x,y,t \in \mathbb{R}_+) \tag{16}$$

holds if and only if

$$\begin{aligned} \text{either } I_f(x,y) &= \left(\frac{y^{\beta+1}-x^{\beta+1}}{(\beta+1)(y-x)}\right)^{1/\beta} && (x,y \in \mathbb{R}_+, x \neq y) \\ \text{or } I_f(x,y) &= \exp\left(\frac{y\ln(y/e) - x\ln(x/e)}{y-x}\right) && (x,y \in \mathbb{R}_+, x \neq y) \end{aligned} \tag{17}$$

where $\beta \neq 0$ *is an arbitrary constant.*

Proof. Assuming $f \in \mathscr{C}_3(\mathbb{R}_+)$ and $(xf'(x))' \neq 0$ we could easily find f using Theorem 7. To avoid the excess regularity conditions we apply another method.

For any fixed $t \in \mathbb{R}_+$ we have

$$\frac{1}{t} I_f(tx, ty) = \frac{1}{t} f^{-1}\left(\frac{\int_{tx}^{ty} f(u)\,du}{ty - tx}\right) = \frac{1}{t} f^{-1}\left(\frac{\int_{x}^{y} f(tv)\,dv}{y - x}\right) = I_{f_t}(x, y)$$

where $f_t(x) := f(tx)$ $(x \in \mathbb{R}_+)$. Thus (16) is equivalent to

$$I_{f_t}(x, y) = I_f(x, y) \quad (x, y, t \in \mathbb{R}_+)\,.$$

By Theorem 5 this is valid if and only if there is a function $c : \mathbb{R}_+ \to \mathbb{R} - \{0\}$ such that

$$f_t'(x) = tf'(tx) = c(t)f'(x) \quad (x, y, t \in \mathbb{R}_+)$$

is satisfied. With $x = 1$ we obtain that $c(t) = tf'(t)/f'(1)$. Substituting this and introducing the function $h(x) := f'(x)/f'(1)$ our equation goes over into the Cauchy functional equation

$$h(xt) = h(t)h(x) \quad (t, x \in \mathbb{R}_+).$$

The solutions continuous at one point of this equation are (see e.g. Aczél [1]) of the form $h(x) = x^\alpha$, $(x \in \mathbb{R}_+)$ where $\alpha \in \mathbb{R}$ is an arbitrary constant. From $f'(x)/f'(1) = h(x) = x^\alpha$ we obtain (by separating the cases $\alpha = -1, \alpha \neq -1$ and introducing new constants) that

$$\begin{aligned} \text{either } f(x) &= ax^\beta + b \quad (x \in \mathbb{R}_+) \\ \text{or } f(x) &= a\ln x + b \quad (x \in \mathbb{R}_+) \end{aligned}$$

where $\beta \neq 0, a \neq 0, b \in \mathbb{R}$ arbitrary constants. The corresponding means are clearly the ones given by (17). □

REMARK 3. From (17) one can see that the homogeneous integral means are exactly the Stolarsky [9], [10] means with parameters $(\beta + 1, 1)$ with $\beta \neq 0$ and $(1, 1)$ corresponding to $\beta = 0$.

THEOREM 9. *Suppose that* $f \in \mathscr{E}(\mathbb{R}), f \in \mathscr{C}_3(\mathbb{R})$ *and* $f''(x) \neq 0$ $(x \in \mathbb{R})$. *The mean* I_f *is positive subhomogeneous with respect to the function* $(t, x) \to x + t$, *i.e.*

$$I_f(x + t, y + t) \leqslant I_f(x, y) + t \quad (x, y \in \mathbb{R}, t \in \mathbb{R}_+) \tag{18}$$

holds if and only if

$$x \to \operatorname{sign}\left[\frac{f''(x)}{f'(x)}\right] \ln\left|\frac{f''(x)}{f'(x)}\right| \quad (x \in \mathbb{R}) \text{ is decreasing on } \mathbb{R}. \tag{19}$$

Proof. By Theorem 1 of [5] (18) holds if and only if

$$\partial_1 I_f(x, y) + \partial_2 I_f(x, y) \leqslant 1 \quad (x, y \in \mathbb{R}).$$

This can be written as

$$\frac{\int_0^1 f'\left((1 - t)x + ty\right)\,dt}{f'\left(I_f(x, y)\right)} \leqslant 1$$

or in the case $f'(x) > 0$

$$f'\left(D_f(x,y)\right) \leqslant f'\left(I_f(x,y)\right) \quad (x,y \in \mathbb{R})$$

which is equivalent to

$$D_f(x,y) \leqslant I_f(x,y) \quad (x,y \in \mathbb{R} \tag{20}$$

provided that $f''(x) > 0$. Thus, if $\operatorname{sign}\left[\frac{f''(x)}{f'(x)}\right] = 1$ the inequality (18) is equivalent to (20) while if $\operatorname{sign}\left[\frac{f''(x)}{f'(x)}\right] = -1$ the inequality (18) is equivalent to the reverse of (20). Applying Theorem 4 completes the proof. □

REMARK 4. (19) is necessary and sufficient for the inequality

$$I_f(x+t, y+t) \geqslant I_f(x,y) + t \quad (x,y \in \mathbb{R}, t \in]-\infty, 0[)\,.$$

If in (18) we require $t \in]-\infty, 0[$ or reverse the inequality sign in (18) (but keep $t \in]0, \infty[$) then our Theorem 9 remains valid if we replace the word decreasing by increasing in (19).

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