



Coulomb gas and sine-Gordon model in arbitrary dimension

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Abstract

The sine-Gordon (SG), i.e. periodic scalar field theory is known to play an important role in $d = 2$ dimensions. A paradigmatic example is the topological phase transition of the vortex dynamics in superfluid films and layered superconductors which are described by SG type models. Periodic scalar potentials find applications in $d = 4$ dimensions, too. Higgs, inflaton and axion physics are examples where scalar fields naturally appear, thus, the SG model can be used instead of the usual polynomial one. The SG quantum field theory can be mapped onto the neutral Coulomb-gas (CG) in arbitrary dimension and the renormalization group (RG) study of the d -dimensional CG model was obtained in the dilute gas approximation. It signals a single phase for $d > 2$, however, it was shown recently, that a suitable generalization of the SG model can possess a topological phase transitions in $d = 4$ dimensions. Our goals in this work are (i) to map out the phase structure of the (original) SG and the equivalent neutral CG models by the functional RG method in arbitrary dimension, (ii) to compare the 3-dimensional SG and isotropic XY spin models and show that they belong to different universality classes, (iii) to study the consequences of the findings on higgs, inflaton, axion models and on the topological phase transition in higher dimensions.

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1. Introduction

Universality expresses the idea that different microscopic physics can give rise to the same scaling behaviour at a phase transition. If the scaling behaviour of different models agrees or disagrees it is possible to decide whether they belong to the same universality class or not. If two different models can be mapped onto each other they belong to the same universality class hence by the study of one model one can obtain the scaling behaviour of the other one. Furthermore, models of the same class of universality can be used to test and compare various types of methods. A canonical example for model equivalences is the sine-Gordon (SG) scalar field theory defined by the Euclidean action [1–5]

$$S = \int d^d x \left[\frac{1}{2} (\partial_\mu \varphi_x)^2 + u \cos(\beta \varphi_x) \right], \quad (1)$$

which belongs to the universality class of the neutral Coulomb gas (CG) [6] and the isotropic XY classical spin model [1] for $d = 2$. They undergo a topological, i.e., Kosterlitz-Thouless-Berezinski [7] (KTB) phase transition. Moreover, the KTB universality regulates the scaling behaviour of the $O(N)$ symmetric field theory with $d = N = 2$. It is worth noting that signatures of the KTB scaling in this model have been also observed by the Functional Renormalization Group (FRG) method [8]. Furthermore, the SG scalar theory is the bose form of the 2-dimensional fermionic Thirring model [9] and the bosonic counterpart of multi-flavor Quantum Electrodynamics and multi-colour Quantum Chromodynamics are also SG type models [10]. Bosonization is well established only in $d = 2$ dimensions but the mapping between the SG theory (1) and the neutral CG holds in arbitrary dimension [1,3].

In $d = 2$ dimensions, the SG model has been used to describe the vortex dynamics in superfluid films and magnetically coupled layered superconductors [11]. In addition, the holographic RG flow [12] and the c -function [13] of the SG model have also been studied.

In $d > 2$ dimensions one might expect no room for any physical application for the SG model, however, there are various cases where periodic scalar fields can play a role in 4-dimensional physics. The most natural three situations among these are the following (i) the periodic inflationary potential, (ii) the mass generation by a periodic Higgs potential, (iii) the axion potential which naturally appears as a periodic function. The main concepts of these three cases are summarized in the appendices. For example, in inflationary cosmology the so called natural inflation i.e., a periodic potential has already been used as a competing inflationary model, see Appendix A. Another possible application is a periodic self-interaction which is proposed here as a possible extension of the standard model Higgs potential. Using the usual parametrization of the field around the first minimum, one recovers the Lagrangian of the Higgs sector but with a periodic interaction term for a single-component real scalar field, see Appendix B. Finally, one has to mention the periodic axion potential which was proposed to retain the CP conserving nature of QCD where gauge symmetry and renormalizability allow the inclusion of CP violating terms but experimental data do not favour such an extension, see Appendix C.

Recently [14], an extension of the SG scalar theory has been studied in $d = 4$ dimensions. Eq. (2) of [14] has an unusual kinetic term $(\Delta \varphi_x)^2$ where Δ is the 4-dimensional Laplacian and φ is a real scalar field. On the one hand, it was shown in [14] by the exact FRG study performed in the momentum space that such extension of the SG model undergoes a topological phase transitions. On the other hand, it is also known that the neutral Coulomb-gas which is identical to the SG model (1) has a single phase in $d > 2$ dimensions, at least this is the result of the real space renormalization group (RG) study obtained in the so called dilute gas approximation [15].

In the real space RG approach the RG transformations are performed in the coordinate space by using a sharp cutoff, i.e., the charges of the Coulomb gas are represented by solid discs with finite diameter and in the dilute gas approximation, only two-body interactions are taken into account. This approximation is valid if the fugacity of the Coulomb gas remains small. Since the fugacity of the Coulomb gas is directly related to the Fourier amplitude of the SG model, the dilute gas approximation corresponds to the case where the Fourier amplitude is assumed to be small, so the RG equations can be linearized with respect to the amplitude which gives the linearized RG flow. The single phase picture is based on these dilute gas or linearized RG equations. Thus, it is a relevant question whether the exact phase structure of the SG model (1) possesses any topological phase transitions in higher dimensions.

Indeed, the RG study of the phase structure of the d -dimensional SG model (1) has already been performed in [2–5] using various approximations. For example, in Refs. [3–5] the RG equations were obtained in the leading order of the derivative expansion of the action, i.e., in the local potential approximation (LPA) where the wave function renormalization is kept to be constant. In Eq. (1), the scalar field can be rescaled ($\varphi \rightarrow \beta\varphi$) and as a result, the frequency parameter β appears in the kinetic term and related to the wave function renormalization ($z \equiv 1/\beta^2$). Thus, LPA means constant dimensionful frequency. Another type of approximation has been used in [2], where RG equations were obtained beyond LPA but they are expanded in terms of the Fourier amplitude. Let us note, that the leading order in this approach is equivalent to the dilute gas approximation of the real space RG approach of the corresponding neutral Coulomb-gas [3].

The goals of this work are (i) to go beyond the previously used approximations and to map out the exact phase structure of the d -dimensional SG model in the framework of the functional RG method, (ii) to consider possible equivalences between the SG scalar field theory, the neutral CG and the XY spin models in $d \neq 2$ dimensions, (iii) to study the appearance of the topological phase transition in higher dimensions, (iv) to draw some conclusions on phase structure of a periodic Higgs, inflaton and axion potentials.

First the functional RG study of the single and double frequency SG model is presented and the phase structures of the SG, CG and XY models are compared. The findings of the present work are compared to recent results, then applied for Higgs, inflaton and axion physics. Finally results are summarized in the conclusion.

2. Functional RG study of the SG model

The main purpose here is to perform the functional RG study of the SG model in arbitrary dimension (in particular in $d = 4$) going beyond approximations used in previous works [2–5]. This is achieved by extending the RG analysis of the single-frequency two-dimensional SG model [16] for higher dimensions and for higher harmonics. There is an interest in the literature to improve the RG study of the d -dimensional periodic model (or the equivalent Coulomb gas), e.g., the PhD thesis [17] was devoted to the RG study of the Coulomb gas (in the dilute gas approximation which is equivalent to the linearized RG flow) for arbitrary dimensions. Findings of this work recover these “dilute gas results” and go beyond that by studying the full RG flow equations including wave function renormalization and including higher harmonics in arbitrary dimension.

We use the effective average action FRG method [18,19] where the evolution equation reads as

$$k\partial_k\Gamma_k[\varphi] = \frac{1}{2}\text{Tr}\left[\frac{k\partial_k R_k}{R_k + \delta_\varphi^2\Gamma_k[\varphi]}\right], \quad (2)$$

with the running momentum cutoff k and the Tr stands for the integration over all momenta and R_k is the regulator function specified later. In order to solve the FRG equation, we consider the following ansatz for the SG model

$$\Gamma_k = \int d^d x \left[\frac{1}{2} z_k (\partial_\mu \varphi_x)^2 + V_k(\varphi_x) \right], \quad V_k(\varphi_x) = u_k \cos(\beta \varphi_x) \quad (3)$$

where the local potential contains a single Fourier mode. The Fourier amplitude u_k and the field independent wavefunction renormalization z_k depend on the RG scale k . The dimensionful frequency β , is scale independent, i.e., it remains constant over the RG flow because the RG transformation retains the periodicity of the dimensionful model with an unchanged period length. Thus, β is a free parameter of the model which can be chosen arbitrarily. Although RG transformations generate higher harmonics, we use the ansatz (3) which contains a single Fourier mode since in case of the 2-dimensional SG model [20] it was found to be an appropriate approximation [16,21]. (Later the role of higher harmonics is investigated.) Eq. (11) leads to the RG evolution equations for the blocked potential $V_k(\varphi)$ and the wavefunction renormalization z_k ,

$$k \partial_k V_k = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \mathcal{D}_k k \partial_k R_k, \quad (4)$$

$$k \partial_k z_k = \left(\frac{\beta}{2\pi} \int_0^{2\pi/\beta} d\varphi \right) (V_k''')^2 \int \frac{d^d p}{(2\pi)^d} \mathcal{D}_k^2 k \partial_k R_k \left(\frac{2}{d} \frac{\partial^2 \mathcal{D}_k}{\partial p^2 \partial p^2} p^2 + \frac{\partial \mathcal{D}_k}{\partial p^2} \right), \quad (5)$$

with $\mathcal{D}_k = (z_k p^2 + R_k + V_k'')^{-1}$ where $V_k'' \equiv \partial_\varphi^2 V_k$ and $V_k''' \equiv \partial_\varphi^3 V_k$. Since the l.h.s. of (5) is independent of the field, a projection onto the field-independent subspace has been introduced on the r.h.s. of (5). The scale k covers the momentum interval from the high-energy/ultraviolet (UV) cutoff Λ to zero. Inserting the ansatz (3) into Eqs. (4), (5), flow equations for the dimensionful couplings can be derived

$$k \partial_k u_k = \int_p \frac{k \partial_k R_k}{\beta^2 u_k} \left(\frac{P - \sqrt{P^2 - (\beta^2 u_k)^2}}{\sqrt{P^2 - (\beta^2 u_k)^2}} \right), \quad (6)$$

$$k \partial_k z_k = \int_p \frac{\beta^2 k \partial_k R_k}{2} \left[\frac{-(\beta^2 u_k)^2 P (\partial_{p^2} P + \frac{2}{d} p^2 \partial_{p^2}^2 P)}{[P^2 - (\beta^2 u_k)^2]^{5/2}} + \frac{(\beta^2 u_k)^2 p^2 (\partial_{p^2} P)^2 (4P^2 + (\beta^2 u_k)^2)}{d [P^2 - (\beta^2 u_k)^2]^{7/2}} \right], \quad (7)$$

where $P = z_k p^2 + R_k$ and $\int_p = \int d p p^{d-1} \Omega_d / (2\pi)^d$ with the d-dimensional solid angle Ω_d . Since the dimensionful frequency is scale-independent, it is convenient to merge it with the scale-dependent wave function renormalization z_k . Thus, we introduce $\hat{z}_k = z_k / \beta^2$, $\hat{R}_k = R_k / \beta^2$ and $\hat{P} = P / \beta^2 = \hat{z}_k p^2 + \hat{R}_k$ and the RG flow equations (6) and (7) can be written as,

$$k \partial_k u_k = \int_p \frac{k \partial_k \hat{R}_k}{u_k} \left(\frac{\hat{P} - \sqrt{\hat{P}^2 - u_k^2}}{\sqrt{\hat{P}^2 - u_k^2}} \right), \quad (8)$$

$$k\partial_k\hat{z}_k = \int_p \frac{k\partial_k\hat{R}_k}{2} \left[\frac{-u_k^2\hat{P}(\partial_{p^2}\hat{P} + \frac{2}{d}p^2\partial_{p^2}^2\hat{P})}{[\hat{P}^2 - u_k^2]^{5/2}} + \frac{u_k^2p^2(\partial_{p^2}\hat{P})^2(4\hat{P}^2 + u_k^2)}{d[\hat{P}^2 - u_k^2]^{7/2}} \right]. \quad (9)$$

Before we further study the RG flow equations, we show that they can be derived by using the rescaled version of the original action for the SG model,

$$\Gamma_k[\theta] = \int d^d x \left[\frac{1}{2}\hat{z}_k(\partial_\mu\theta_x)^2 + u_k \cos(\theta_x) \right], \quad (10)$$

where the rescaled (dimensionless) field $\theta = \beta\varphi$ is introduced. Let us note, that the field carries a dimension for $d \neq 2$ thus the frequency of the SG model (1) becomes a dimensionful parameter for $d \neq 2$, i.e., $\beta^2 = k^{2-d}\tilde{\beta}_k^2$ where $\tilde{\beta}_k$ is dimensionless, so the rescaled wavefunction renormalization has a dimension of k^{d-2} , i.e., $\hat{z}_k = \tilde{z}_k k^{d-2}$. The corresponding FRG equation reads as

$$k\partial_k\Gamma_k[\theta] = \frac{1}{2}\text{Tr} \left[\frac{k\partial_k R_k}{R_k + \beta^2\delta_\theta^2\Gamma_k[\theta]} \right] = \frac{1}{2}\text{Tr} \left[\frac{k\partial_k\hat{R}_k}{\hat{R}_k + \delta_\theta^2\Gamma_k[\theta]} \right] \quad (11)$$

where the rescaled regulator function $\hat{R}_k = R_k/\beta^2$ has been used. Inserting the ansatz (10) into Eqs. (11), the RG flow equations (8) and (9) can be obtained. Thus, equations (8) and (9) are derived in two different ways.

Momentum integrals have to be performed numerically, except the linearized form of Eqs. (8), (9) around the Gaussian fixed point where analytical results are available. This requires a special choice for the regulator function R_k such as the power-law [19] or the Litim-type [22] ones,

$$R_{\text{pow}}(k) = p^2 \left(\frac{k^2}{p^2} \right)^b, \quad R_{\text{Litim}}(k) = p^2 \left(\frac{k^2}{p^2} - 1 \right) \Theta(k^2 - p^2) \quad (12)$$

where $\Theta(x)$ stands for the Heaviside step function and b is a free parameter of the power-law regulator. In general, the regulator function beyond LPA should be given by the inclusion (multiplicative approach) or the exclusion (additive approach) of the field independent wavefunction renormalization z_k . In the multiplicative approach, the rescaled regulator \hat{R}_k contains the rescaled wavefunction renormalization \hat{z}_k . In the additive approach, the frequency can be absorbed by the overall multiplicative constant of the rescaled regulator or can be chosen arbitrarily since it is a scale-independent free parameter of the model. Important to note, that the additive approach requires the use of the power-law regulator function.

The phase structure should be independent whether we use the multiplicative or additive approaches for the definition of the regulator and of its parameters such as b . For example, one can choose $b = 2$, see Ref. [23]. Regarding the regulator-dependence we note that the linearized RG flow equations can always be obtained analytically for the power-law type regulator, thus, there is no real need for the use of the Litim-type one. The full RG flow (with higher harmonics) requires a numerical treatment anyway (even for the Litim-type regulator) and produces us a complete picture of the phase diagram.

Let us first discuss the linearized RG flow equations obtained by the additive approach of the regulator function where dimensionless couplings $\tilde{u}_k = k^{-d}u_k$, and $\tilde{z}_k = k^{2-d}\hat{z}_k$ are introduced. In this case, the RG flow equations have the following forms for $d = 1$ dimension

$$(1 + k\partial_k)\tilde{u}_k = \frac{1}{4b \sin(\frac{\pi}{2b})} \tilde{z}_k^{\frac{1-2b}{2b}} \tilde{u}_k + \mathcal{O}(\tilde{u}_k^2) \quad (13)$$

$$(-1 + k\partial_k)\tilde{z}_k = -\frac{c_1(b)}{4\pi}\tilde{z}_k^{\frac{5-4b}{2b}}\tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3) \quad (14)$$

with $c_1(b) = \frac{5\pi(55+4b(5b-18))}{48b\sin[5\pi/(2b)]}$. Eqs. (13), (14) have a non-trivial fixed point at $\tilde{z}_* = [4b\sin(\pi/(2b))]^{2b/(1-2b)}$, $\tilde{u}_*^2 = (4\pi/c_1)\tilde{z}_*^{3-5/(2b)}$. For $d = 2$, linearized RG equations are

$$(2 + k\partial_k)\tilde{u}_k = \frac{1}{4\pi\tilde{z}_k}\tilde{u}_k + \mathcal{O}(\tilde{u}_k^2) \quad (15)$$

$$k\partial_k\tilde{z}_k = -\frac{c_2(b)}{8\pi}\tilde{z}_k^{\frac{2-2b}{b}}\tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3) \quad (16)$$

with $c_2(b) = \frac{2\pi(b-2)(b-1)}{3b^2\sin[2\pi/b]}$ and result in a KTB type (i.e. infinite order) phase transition with $\tilde{z}_* = 1/(8\pi)$ [16,20]. Important to note that $c_2(b) > 0$ for $b > 1$. Finally, for $d = 3$, linearized RG equations are

$$(3 + k\partial_k)\tilde{u}_k = \frac{1}{8\pi b\sin(\frac{\pi}{2b})}\tilde{z}_k^{\frac{-1-2b}{2b}}\tilde{u}_k + \mathcal{O}(\tilde{u}_k^2) \quad (17)$$

$$(1 + k\partial_k)\tilde{z}_k = -\frac{c_3(b)}{8\pi^2}\tilde{z}_k^{\frac{3-4b}{2b}}\tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3) \quad (18)$$

with $c_3(b) = \frac{\pi(3+4b(b-2))}{16b^2\sin[3\pi/(2b)]}$. Due to the tree-level scaling of \tilde{z}_k , the non-trivial fixed point appears for $d < 2$ and disappears for $d > 2$ in the RG flow. However, a “turning point” can be identified for $d > 2$ where the irrelevant coupling \tilde{u}_k turns to a relevant one. For $d = 3$ the turning point is at $\tilde{z}_* = [24\pi b\sin(\pi/(2b))]^{-2b/(1+2b)}$.

In the multiplicative approach where the definition of the regulator R_k includes z_k beyond LPA, the linearized RG flow equations obtained for the dimensionless couplings are almost identical to those obtained by the additive case. The prefactors of the r.h.s. of the flow equations and the power of the Fourier amplitude \tilde{u} are identical in the multiplicative and additive cases. The difference is due to the power of the dimensionless wavefunction renormalization \tilde{z}_k . In the multiplicative case, the r.h.s. of the flow equations of the Fourier amplitude contain \tilde{z}_k^{-1} and the flow equations of the wavefunction renormalization have \tilde{z}_k^{-2} independently of choice of the dimension d and the parameter b . Since the flow equations of the multiplicative and additive approaches should give the same phase structure, in this work we focus on the more complex additive case.

In general, RG equations (6) and (7) (exact for a single Fourier mode) have to be solved numerically. Important feature of the exact RG flow the emergence of a new low-energy/infrared (IR) fixed point related to the degeneracy of the blocked action. Namely, Eqs. (6), (7) become singular at the momentum scale where $\bar{k} - k^{2-d}u_k = 0$ with $\bar{k} = \min_{p^2} P = bk^2[\tilde{z}_k/(b-1)]^{1-1/b}$. Therefore, it is convenient to redefine the dimensionless coupling constant as $\bar{u}_k \equiv k^{2-d}u_k/\bar{k} = k^2\tilde{u}/\bar{k}$ which tends to one in case of degeneracy. Exact RG equations (6), (7) were solved for $b = 2$ and flow diagrams are plotted in Fig. 1 for $d = 1$, in Fig. 2 for $d = 2$, and in Fig. 3 for $d = 3$ dimensions. The IR fixed point ($\bar{u}_* = 1$, $1/\tilde{z}_* = 0$) which corresponds to degeneracy was found in any dimensions. The $d = 1$ case needs further improvement because in quantum mechanics spontaneous symmetry breaking is not allowed. The appearance of the broken phase, was found for the anharmonic oscillator, see e.g. [24] where the higher order terms in the gradient expansion handle the problem.

Let us emphasize that only the exact RG flow with the inclusion of the wave function renormalization is suitable for the determination of the IR fixed point since the perturbative (truncated)

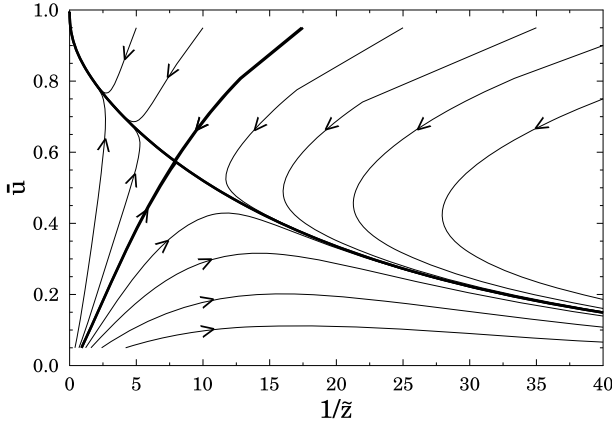


Fig. 1. The exact phase diagram of the SG and the equivalent CG models for $d = 1$ dimensions (similar RG flow can be drawn for $d < 2$). Arrows indicate the direction of the flow.

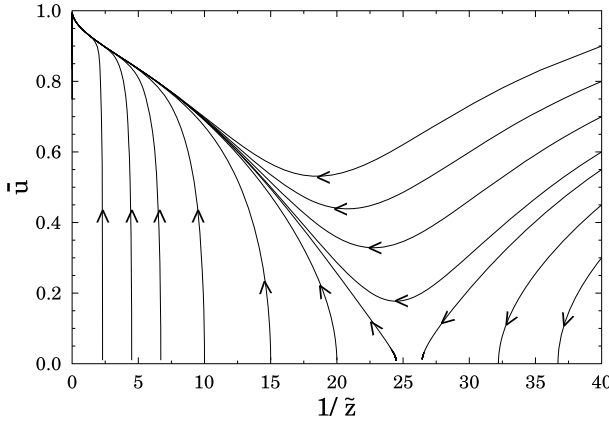


Fig. 2. The exact phase diagram of the SG and the equivalent CG models for $d = 2$ dimensions which shows the Kosterlitz-Thouless-Berezinski type phase transition.

RG equations are non-singular [2]. Thus any previous attempts based on perturbation theory [2] or by the usage of the local potential approximation [3–5] were unable to determine the IR behaviour in a reliable manner. Furthermore, the exact RG flow changes the position of the non-trivial fixed point obtained for $d = 1$. Indeed, for $b = 2$, linearized RG equations (13), (14) result in $1/\tilde{z}_* = 2^{10/3} \approx 10.08$ and $\tilde{u}_* = \tilde{u}_*/(2\sqrt{\tilde{z}_*}) \approx 0.23$ which have been modified by the exact RG, see Fig. 1. Similarly, the turning point of the $d = 3$ case determined by the exact RG coincides with that of the linearized RG which is $1/\tilde{z}_* = 4(6\pi)^{4/5} \approx 41.9$ only for small \tilde{u} , see Fig. 3. For $d = 2$ dimensions the critical value $1/\tilde{z}_c = \beta_c^2 = 8\pi$ which separates the phases of the model (see Fig. 2) is found to be scheme-independent. The comparison of exact and perturbative (linearized) RG flow can be used for optimizing the renormalization schemes.

Finally, let us discuss the possibility of having a topological phase transitions in higher dimensions. In [14] it was argued that the extended version of the SG model possesses a topological phase transition in $d = 4$ dimensions. The kinetic terms of the extended model and the usual one

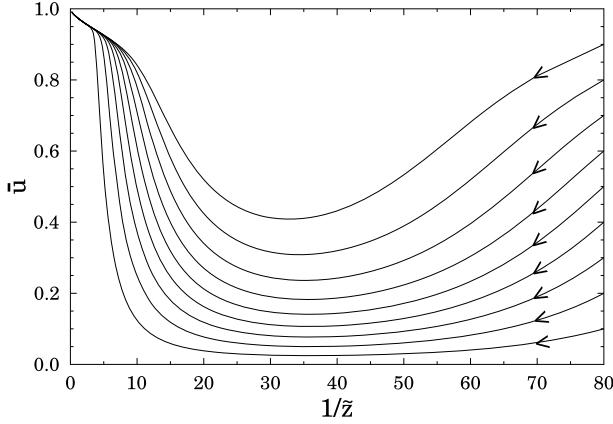


Fig. 3. The exact phase diagram of the SG and the equivalent CG models for $d = 3$ dimensions which indicates a single phase for $d > 2$.

(1) are different and this is the reason why one can observe a phase transition for the extended SG theory. Indeed, the Δ^2 kinetic term is sub-leading with respect to the standard ∇^2 term and, so, the topological (or KTB) phase transition in $d = 4$ dimension is expected to be of “higher order” with respect to the standard one in $d = 2$, and so, it needs a specific tuning of the coupling in the ∇^2 in order to be observed. If one rescales the field by the frequency ($\varphi \rightarrow \beta\varphi$) in both versions of the SG model, the dimension of the wavefunction renormalization can be determined. In the extended SG model it is *dimensionless* which requires no tree-level scaling and it makes possible (under some other conditions) to observe a topological phase transition. However, the kinetic term of the usual SG model (1) has a tree-level scaling which has serious consequence, the RG flow has a turning instead of a critical point. The exact RG flow, see Fig. 5 in the following sections, confirms this expectation. So, if the usual kinetic term is generated by the RG flow, the absence of the topological phase transition is unavoidable in higher dimensions $d > 2$.

3. Double-frequency SG model

Let us study the influence of higher harmonics on the phase structure of the single-frequency SG model. For $d = 2$ the higher frequency modes of the SG model correspond to vortices with higher vorticity of the related XY model and it is known to play no role in the vortex dynamics (the excitation of vortices with higher vorticity has lower probability). The equivalence between the SG and the XY models has been proven in $d = 2$ dimensions, so it is important to clarify the results of the previous section which were obtained for the single-frequency SG model for $d \neq 2$. Let us consider the double-frequency SG model

$$\Gamma_k = \int_x \left[\frac{\bar{z}_k}{2} (\partial_\mu \theta_x)^2 + u_{1k} \cos(\theta_x) + u_{2k} \cos(2\theta_x) \right] \quad (19)$$

where the first Fourier amplitude is equivalent to that of the single-frequency SG model, $u_{1k} \equiv u_k$. In case of the double-frequency model the parameter space is three-dimensional thus its projection to the \bar{u} , $1/\bar{z}$ plane is used to compare RG trajectories of the 1-dimensional SG model with single and double frequency, see Fig. 4. Let us note, the same normalized Fourier amplitude \bar{u} is used both for the single- and the double-frequency cases. The definition of \bar{u} is given by the

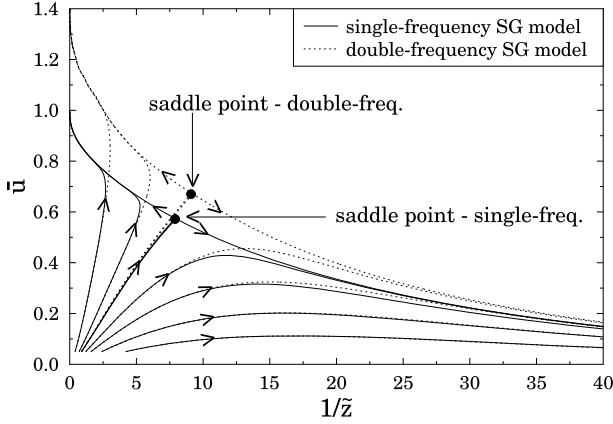


Fig. 4. Lower half of the phase portrait of the SG model with single (10) and double frequency (19) are compared (in the \bar{u} , $1/\bar{z}$ plane) for $d = 1$. Results are in qualitative agreement.

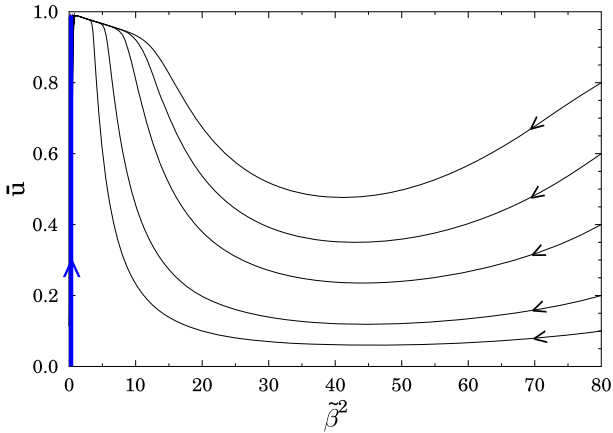


Fig. 5. Phase diagram of the SG model in $d = 4$ dimensions.

low-energy behaviour of the inverse propagator (see the previous section) and it is normalized in such a way that for the single-frequency model it tends to 1.0 in the low-energy limit of the broken phase. The low-energy limit of \bar{u} in the broken phase of the double-frequency SG model is similar to that of the single-frequency model. Similarly, a non-trivial saddle point appears in $d = 1$ dimension in the RG flow of the SG model both for the single- and double-frequency cases but its position differs from each other, see Fig. 4. Let us also mention that the numerical results shown in Fig. 4 are obtained by solving the RG equations (4) and (5) both for the single- and double-frequency SG models in $d = 1$ using the same initial condition for $u_{1\Lambda} \equiv u_\Lambda$ and for the double-frequency case the other Fourier amplitude is chosen as $u_{2\Lambda} = 0.1u_{1\Lambda}$.

The main conclusion is that results of the RG analysis of the single- and double-frequency SG models are in qualitative agreement which demonstrates that similarly to the $d = 2$ case, the higher harmonics do not modify the phase structure given by the single-frequency model.

4. Phase structure of the neutral CG

The CG model has been the subject of an intense study in last decades [2,3,6,15,25,26] and there is a continuous interest in the use of the SG representation of CG systems [2,3,6,26] since it has an important relevance in various models of critical phenomena (e.g. superfluid transition in He films, melting of crystals in $d = 2$, arrays of Josephson junctions, roughening transition). However, the complete understanding of the critical behaviour of the d -dimensional neutral CG which is a plasma of equal number of positive and negative charges is still lacking. For example, previous investigations (even for the well studied 2-dimensional CG model) were restricted to the case of low densities (limit of low fugacity) and there are different competing views of what happens to the neutral plasma at high densities [25,26]. Thus, an RG study of the d -dimensional CG beyond the dilute gas approximation is required which is one of the goals of the present work.

Since the mapping between the CG and SG models holds in arbitrary dimension [3,6] (and it is exact in case of point-like charges) the RG study of the d -dimensional SG model can be directly used to map out the phase structure of the CG model. In the framework of perturbation theory (limit of low fugacity) RG equations of the CG model in arbitrary dimension is given in [15] and reads as

$$\frac{dx}{dl} = -x(xy^2 + d - 2), \quad \frac{dy}{dl} = -y(x - d) \quad (20)$$

with $\partial_l = -k\partial_k$, $y \sim \tilde{u}_k$ and $x \sim 1/\tilde{T} \sim 1/\tilde{z}_k$ where \tilde{T} is the temperature and y is the fugacity. Using these identities Eq. (20) can be rewritten as

$$((d - 2) + k\partial_k)\tilde{z}_k = -c_z\tilde{u}_k^2, \quad (d + k\partial_k)\tilde{u}_k = c_u\frac{1}{\tilde{z}_k}\tilde{u}_k, \quad (21)$$

with constants c_u , c_z and they are found to be similar to the linearized RG equations obtained for the SG model, see Eqs. (13), (14), Eqs. (15), (16) and Eqs. (17), (16). For example, the non-trivial fixed point of the 1-dimensional SG model can be identified in the flow generated by (21), too. In the so called sharp cutoff limit ($b \rightarrow \infty$) the linearized RG equation derived for the Fourier amplitude of the SG model reduces to $(d + k\partial_k)\tilde{u}_k = \Omega_d\tilde{u}_k/(\tilde{z}_k 2(2\pi)^d)$ which is identical to the second equation of (21). However, the sharp cutoff does not support to the derivative expansion at higher order, i.e. for $b \rightarrow \infty$ the multiplicative constant $c_d(b)$ of the RG equation obtained for \tilde{z}_k becomes infinity.

Most important new results of the exact RG flow are the existence of the high temperature ($\tilde{u}_* = 1$, $1/\tilde{z}_* = 0$) and the absence of new further non-trivial fixed points. Let us note that, the perturbative (truncated) RG has a non-singular structure hence it is not able to recover the degenerate potential therefore it is not suitable to observe the high temperature fixed point [15]. Moreover, the exact RG changes the position of the non-trivial fixed point for $d < 2$ and the turning point for $d > 2$. Furthermore, since the mapping between the SG and CG models is exact for point-like charges the exact RG flow indicates a single phase for the CG for $d > 2$.

5. Checking the results

So far RG equations beyond the local potential approximation (LPA) have been derived for the SG model (with single and double frequency) in d -dimensions. By using the mapping of the SG theory onto the CG model, its RG flow and phase structure has been discussed beyond the dilute

gas approximation. In this section let us check the findings of the present work by comparing them to known recent results [17,27].

Let us first rewrite the linearized RG flow equations (13), (14), (15), (16) and (17), (18) in a single form valid in arbitrary dimensions

$$(d + k\partial_k)\tilde{u}_k = A_{b,d} \tilde{z}_k^{\frac{(2-d)-2b}{2b}} \tilde{u}_k + \mathcal{O}(\tilde{u}_k^2) \quad (22)$$

$$((d-2) + k\partial_k)\tilde{z}_k = -B_{b,d} \tilde{z}_k^{\frac{(6-d)-4b}{2b}} \tilde{u}_k^2 + \mathcal{O}(\tilde{u}_k^3) \quad (23)$$

where $A_{b,d}$ and $B_{b,d}$ are constants depend on the dimension d and the regulator parameter b . For the comparison it is convenient to use the frequency instead of the wave function renormalization ($\tilde{\beta}^2 = 1/\tilde{z}$),

$$k\partial_k\tilde{u}_k = \left[A_{b,d} (\tilde{\beta}_k^2)^{\frac{2b-(2-d)}{2b}} - d \right] \tilde{u} \quad (24)$$

$$k\partial_k\tilde{\beta}_k^2 = \tilde{\beta}_k^2 \left[(d-2) + B_{b,d} \tilde{u}_k^2 (\tilde{\beta}_k^2)^{\frac{6b-(6-d)}{2b}} \right], \quad (25)$$

and take the limit $b \rightarrow \infty$ which is identical to the so called sharp cutoff regulator (as mentioned in the previous section),

$$k\partial_k\tilde{u}_k = \left[A_{\infty,b} \tilde{\beta}_k^2 - d \right] \tilde{u} \quad (26)$$

$$k\partial_k\tilde{\beta}_k^2 = \tilde{\beta}_k^2 \left[(d-2) + B_{\infty,b} \tilde{u}_k^2 \tilde{\beta}_k^6 \right]. \quad (27)$$

It is important to note that $B_{\infty,b}$ cannot be defined unambiguously in the functional RG approach since the sharp cutoff confront to the derivative expansion, however, it is possible in the real space RG even beyond LPA.

Let us first compare Eqs. (26) and (27) to the RG equations (3.2.8) and (3.2.9) of [17] which is obtained in the dilute gas approximation and reads

$$k\partial_k\tilde{u}_k = \left[\frac{K_d}{2} \tilde{\beta}_k^2 - d \right] \tilde{u} \quad (28)$$

$$k\partial_k\tilde{\beta}_k^2 = \tilde{\beta}_k^2 \left[(d-2) + \left(\frac{I_1 K_d}{2} + B \right) \frac{\tilde{u}_k^2}{4} \tilde{\beta}_k^6 \right], \quad (29)$$

by using the following identifications $2z \equiv \tilde{u}$, $\alpha \equiv \beta$ and $\partial_l \equiv -k\partial_k$ where K_d , I_1 and B are constants. It is clearly demonstrated that Eqs. (26) and (27) are identical to (28) and (29) up to some constant.

Finally, let us consider the flow equations above (45) in [27] which are obtained for the SG model in $d = 2$ using the Wilson-Kadanoff blocking relation up to leading order terms and reads as

$$k\partial_k\tilde{u}_k = \left(\frac{\tilde{\beta}_k^2}{4\pi} - 2 \right) \tilde{u} \quad (30)$$

$$k\partial_k\tilde{\beta}_k^2 = \frac{3\tilde{\beta}_k^6 \tilde{u}_k^2}{4\pi \Lambda^3} \quad (31)$$

where the identifications $g \equiv \tilde{u}$ and $\partial_l \equiv -k\partial_k$ are used. One finds agreement between Eqs. (24) (25) and Eqs. (30) (31) since $A_{b,2} = 1/(4\pi)$ and $b = 2$ is chosen. Thus, it was shown that the findings of the present work recover known (approximate) results which validates our conclusions.

6. Isotropic classical XY spin model

The partition function of the d -dimensional isotropic classical XY spin model [1] can be expressed in terms of topological excitations of the original degrees of freedom. For $d = 2$ dimensions, the dual theory is the vortex gas which is known to belong to the class of universality of the neutral CG [28,29,32]. Two-dimensional generalized models are well known where both the CG and the vortex gas are included as particular limiting cases [28,29,32] and are self-dual under the duality transformation. For $d = 3$, the dual theory is the gas of interacting vortex loops [30,31] (i.e. the lattice CG [32]). Corresponding flow equations have been derived for the parameters K (i.e. the coupling between the spins) and y (i.e. the fugacity of the vortex loops) by real-space RG method ($a \sim 1/k$ where a is the running cutoff in the coordinate space while k is the running momentum cutoff) in the limit of low fugacity [3,30,31],

$$a\partial_a K = K - \frac{4\pi^3}{3} K^2 y, \quad a\partial_a y = \left(6 - \pi^2 K L\right) y, \quad (32)$$

where L approaches a constant in the IR limit ($L \rightarrow 1$) [30], or it is weakly divergent ($L \rightarrow \ln(a/a_c) + 1$) [31]. Since $a\partial_a = -k\partial_k$ and by using the identities $y \equiv \tilde{u}$, $K \equiv 1/\tilde{z}$, RG equations of (32) are rewritten as

$$k\partial_k \tilde{z}_k = \tilde{z}_k - \frac{4\pi^3}{3} \tilde{u}_k^2, \quad k\partial_k \tilde{u}_k = \left(\frac{1}{\tilde{z}_k} \frac{\pi^2 L}{2} - 3\right) \tilde{u}_k, \quad (33)$$

which can be compared to the linearized RG obtained for the SG (17), (18) and the CG (21) models in $d = 3$ dimensions. The only qualitative disagreement is the sign of the tree-level scaling term of \tilde{z}_k but it has important consequences since the RG flow of the vortex-loop gas has a non-trivial fixed point which is absent in case of the 3-dimensional SG and CG models. Therefore, it demonstrates that the vortex-loop gas has a different scaling behaviour, thus it belongs to a class of universality different from that of the SG and CG models for $d \neq 2$. For $d = 2$, the couplings K , \tilde{z} have no tree-level scaling thus SG, CG, XY models are in the same universality class. Let us emphasize that RG equations are compared at the linearized level, however the exact RG study of the SG and CG models is required since it shows the absence of new non-trivial fixed points, thus there is no room to find a mapping between the parameters of the vortex-loop gas and the 3-dimensional CG which could produce the same phase structure.

7. Application to Higgs, inflaton and axion physics

In this section we apply the results of the functional RG study of the SG model obtained in $d = 4$ dimensions for Higgs, inflaton and axion physics. The phase diagram of the four-dimensional SG model is shown in Fig. 5.

Let us first discuss its consequences on the stability of the periodic Higgs potential in the framework of the Higgs-Yukawa model [51]

$$\Gamma_k = \int_k \left[\frac{1}{2} Z_{\phi,k} (\partial_\mu \phi)^2 + U_k(\rho) + Z_{\psi,k} \bar{\psi} i \not{\partial} \psi + i h_k \phi \bar{\psi} \psi \right] \quad (34)$$

where the Yukawa coupling (h_k) has been introduced in order to couple the Higgs field (ϕ) to a fermion field (ψ) (top quark). The scalar potential is assumed to be a periodic one, i.e., $U_k(\rho) \sim \cos(\beta\rho)$ with $\rho = \frac{1}{2}\phi^2$. The functional RG equation suitable for the Higgs-Yukawa model reads as [51]

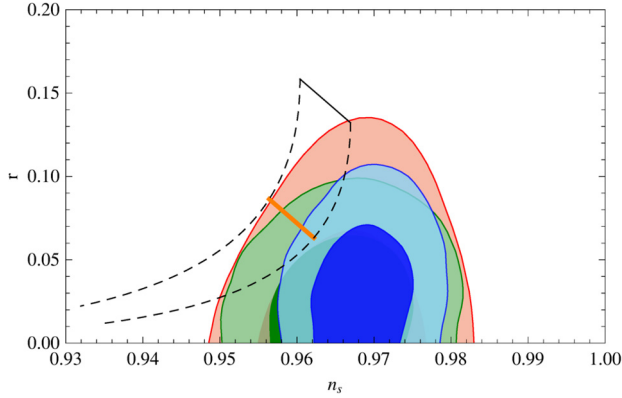


Fig. 6. Orange line represents the best fit of the periodic inflationary potential (for $N \in [50, 60]$) with respect to its frequency to Planck results [35]. This gives the optimal value which is $\beta = 0.15$. The black line stands for the quadratic inflationary potential (for $N \in [50, 60]$) which is almost excluded. Dashed lines indicate allowed regions for the periodic model (for various frequencies where $\beta = 0$ recovers the quadratic potential).

$$k \partial_k \Gamma_k = \frac{1}{2} \text{STr} \frac{k \partial_k R_k}{R_k + \Gamma_k^{(2)}}. \tag{35}$$

If the Yukawa coupling contains a linear term in the Higgs field (which is the usual case) and the scalar potential periodic in the field components, then the Hessian of the running action $\Gamma_k^{(2)}$ with respect to the fields $(\phi, \psi, \bar{\psi})$ has a fully periodic part. In other words, the free boson contribution for the functional RG equation remains periodic and the application of the RG study of the pure periodic scalar model on the stability problem [49,50] represents a reliable approximation. Although, the scalar potential is periodic in the magnitude, i.e., in ρ which requires a more careful analysis where one has to treat the RG running of every couplings in a complete study but this is not the goal here. Coming back to the RG results of the pure SG model the following statements can be done. Since the dimensionless Fourier amplitude (\tilde{u}_k) tends to a constant value (see, Fig. 5) the dimensionless periodic potential remains bounded from below and above which guarantee the stability at this level of approximation. A polynomial potential can lose stability if it becomes unbounded from below, but a periodic potential with finite Fourier amplitude cannot be unstable. This is why we argue that stability of the periodic scalar potential is guaranteed because the RG study of the SG model results in a finite value for the amplitude. Furthermore, it also means that the dimensionful Fourier amplitude (and consequently the dimensionful mass term for the Higgs) tends to a small number in the deep IR limit. This is required since the Higgs mass at the scale of inflation (UV scale) should be large fixed by Cosmic Microwave Background Radiation (CMBR) data and it should tend to the measured Higgs mass in the low-energy, IR limit which orders of magnitude smaller than at the UV scale. Thus, the RG result of the pure periodic model supports to consider the periodic model as a possible UV completion of the SM Higgs potential.

Let us turn to the discussion of the RG running of the (dimensionless) frequency $\tilde{\beta}$ of the periodic inflationary potential, for the details see Appendix A. Conditions for the slow-roll-down inflation require a small frequency ($\tilde{\beta} = 0.15$) which is shown by Fig. 6. The blue line of Fig. 5 stands for the RG trajectory for such a small frequency. It is an almost vertical straight line, i.e. $\tilde{\beta}$ does not change over inflation and in the post-inflationary period. Moreover, it tends to zero in the deep IR limit which allows the Taylor expansion of the periodic potential and results in a

simple quadratic potential. The RG study presented here, can be applied for a modified version of the periodic potential in which, in addition to the periodic self-interaction, a linear term appears

$$V = g\phi + u \cos(\beta\phi) \quad (36)$$

which was considered in [39] as a new type of inflationary potential. The idea is similar to the Higgs case, the Tr-term of the functional RG equation remains periodic since not the potential but its Hessian appears there which is periodic even for the above potential.

Finally, let us consider the consequences of the RG study of the SG model on axion physics. In Ref. [55] it was shown that RG transformations result in a flat (dimensionful) axion potential at LPA. Findings of this work are suitable to go beyond LPA. Indeed, the flow diagram Fig. 5 clearly takes into account the (non-trivial) RG running of the frequency which can be seen only beyond LPA. Since the dimensionless Fourier amplitude tends to a constant value the corresponding dimensionful parameter vanishes in the IR limit, the dimensionful potential flattens out. This confirms the flattening of the axion potential beyond LPA.

8. Summary

In this work the functional RG study of the periodic scalar field theory, i.e. the sine-Gordon (SG) model has been performed in arbitrary dimension and the wave function renormalization is included. This allows us to go beyond approximations used in previous works and to consider the consequences of the RG study of the periodic scalar model on Higgs, inflaton and axion physics and to consider the possibility of a topological phase transition in higher dimensions.

For example a periodic Higgs potential was proposed as a possible extension of the standard quartic one. Using the usual parametrization of the field around the first minimum, one recovers the Lagrangian of the Higgs sector but with a periodic type self-interaction term for a single-component real scalar field. The exact phase structure of the pure SG scalar field theory (with no gauge and fermionic fields) was mapped out by the functional RG method in arbitrary dimension. It was found that the SG model has a single phase in $d > 2$ and the potential remains bounded from below. This result opens a new platform to perform stability studies for models with periodic type Higgs potentials, as an example one can mention the so called massive sine-Gordon model which contains quadratic and periodic terms [48].

The RG running of the frequency parameter is used to answer to the question whether the conditions for slow-roll inflation remain valid under RG transformations. It was shown that starting from a small value for the frequency (which is required), it does not evolve along the corresponding RG trajectory. Thus, our analysis supports the viability of a periodic inflationary potential.

Finally, it was also shown that the dimensionful periodic potential flattens out in the IR limit which confirms previous results on the flattening of the axion potential.

The findings were used to determine the phase structure of the equivalent neutral Coulomb-gas (CG) too. The high temperature fixed point of the d -dimensional SG and CG was determined and shown that the perturbative RG is not suitable to recover it. The position of the non-trivial fixed point of the RG flow for $d < 2$ and the turning point for $d > 2$ dimensions given by the exact flow is compared to the linearized one. Note, that for $d = 1$ dimension, the broken phase should be vanished if the approximations used are improved [33].

The absence of new further non-trivial fixed points were also demonstrated which results in a single phase for the SG scalar theory and the neutral CG with point-like charges for $d > 2$, thus, (i) no further phase transition can be identified at high densities for the neutral CG as opposed to e.g. [25,26], (ii) no topological phase transition can be observed for the SG model with the

usual kinetic term (1) in higher dimension $d > 2$, (iii) there is no way to find a mapping between the phase structure of the neutral 3-dimensional CG and the vortex-loop gas which is the gas of topological defects of the isotropic XY spin model for $d = 3$, i.e., they belong to different universality classes.

CRedit authorship contribution statement

1. Conception or design of the work: István Nándori
2. Data collection (numerical results, codes): István Nándori
3. Data analysis and interpretation: István Nándori
4. Drafting the article: István Nándori
5. Critical revision of the article: István Nándori
6. Final approval of the version to be published: István Nándori

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Periodic inflationary potential

A possible application of the higher dimensional periodic, i.e. sine-Gordon type scalar field theory is inflationary cosmology. Let us start with the Friedman-Robertson-Walker (FRW) metric $g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$ using natural units $c = \hbar = 1$ where $a = a(t)$ is the scale factor of which time-dependence is determined by the Friedman equation (derived from the Einstein equation for the FRW metric). The solution of the Friedman equation with the assumption for the equation of state $p = -\rho$ results in exponential expansion, i.e., $a(t) \sim e^{Ht}$ where H is the Hubble constant. The key observation is that scalar fields can mimic the required equation of state (under some conditions) thus represent excellent models for inflation [34]. The Lagrangian of the Einstein-Hilbert action involving a real scalar field minimally coupled to gravity is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{m_p^2}{2} R + \mathcal{L}_\phi \right],$$

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \frac{1}{a^2} (\nabla\phi)^2 - V(\phi)$$

with $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})} = a^3$. The first observation is that over inflation the field can be considered to be homogeneous ($\nabla\phi/a = 0$). The simplest example for inflationary potential is the so-called chaotic monomial or large field inflation where the potential reads

$$V = \lambda m_p^{4-\alpha} \phi^\alpha, \quad m_p^2 = \frac{1}{8\pi G}, \quad m_p = 2.4 \times 10^{18} \text{ GeV}$$

where m_p is the (reduced) Planck mass, G is the gravitational constant and $\alpha = 2, 4$ is a typical choice. Conditions for the potential can be derived in order to fulfil the requirements of prolonged inflation (with slow roll down). These conditions are written in terms of parameters (ϵ , η and N) which are the following for chaotic or large field inflation (LFI)

$$\epsilon = \frac{1}{2} m_p^2 \left(\frac{V'}{V} \right)^2 = \frac{1}{2} m_p^2 \left(\frac{\lambda \alpha m_p^{4-\alpha} \phi^{\alpha-1}}{\lambda m_p^{4-\alpha} \phi^\alpha} \right)^2 = \frac{m_p^2}{2} \frac{\alpha^2}{\phi^2} \ll 1$$

$$\eta = m_p^2 \frac{V''}{V} = m_p^2 \left(\frac{\lambda \alpha (\alpha - 1) m_p^{4-\alpha} \phi^{\alpha-2}}{\lambda m_p^{4-\alpha} \phi^\alpha} \right) = m_p^2 \frac{\alpha (\alpha - 1)}{\phi^2} \ll 1$$

$$N = -\frac{1}{m_p^2} \int_{\phi_i}^{\phi_f} d\phi \frac{V}{V'} = -\frac{1}{m_p^2} \int_{\phi_i}^{\phi_f} d\phi \frac{\phi}{\alpha} = \frac{1}{2\alpha m_p^2} (\phi_i^2 - \phi_f^2) = [50 - 60]$$

Since inflation ends when $\epsilon \approx 1$ or $\eta \approx 1$ therefore one has to choose

$$\phi_f = \alpha m_p \implies \epsilon = \frac{1}{2} \quad \eta = \frac{\alpha - 1}{\alpha}, \quad \phi_i = \sqrt{2\alpha m_p^2 N + \phi_f^2} \approx \sqrt{2\alpha N} m_p$$

where at the last approximation we used that $\phi_i \gg \phi_f$. From the above parameters one can build up measurable quantities such as the spectral index ($n_s \approx 2\eta - 6\epsilon$) and the tensor-to-scalar ratio ($r \approx 16\epsilon$) which relates to each other. For the quadratic case ($\alpha = 2$) the relation is independent of N and valid up to order ($1/N$)

$$(n_s - 1) + \frac{r}{4} = 0,$$

and can be compared to observations directly. For example, result of the Planck mission [35] put strong constraints on the applicability of the quadratic LFI model, i.e., it is almost excluded by measured data on the thermal fluctuations of the cosmic microwave background radiation (CMBR). Indeed, testing of competing scenarios for scalar inflationary models against observation is a general issue, see for example [36,37].

Let us now turn to our original plan, i.e., to use a periodic scalar field as an inflationary model. Consider the sine-Gordon model in the following representation which is the so called Pseudo-Nambu-Goldstone boson potential (PNGB) [38],

$$V = u [1 - \cos(\beta\phi)]$$

where u, β are dimensionful parameters. Generalization of that can be found also in Ref. [39]. In $d = 4$ dimensions the scalar field carries a dimension, $\phi = k^{(d-2)/2} \tilde{\phi}$ where $\tilde{\phi}$ is dimensionless and k is an arbitrary chosen momentum scale convenient to take at the planck mass $k = m_p$. Thus, the corresponding dimensionless parameters are $\beta = m_p^{-1} \tilde{\beta}$ and $u = m_p^4 \tilde{u}$. Calculating ϵ , η and N parameters one finds

$$\epsilon = \frac{1}{2} m_p^2 \left(\frac{V'}{V} \right)^2 = \frac{1}{2} \tilde{\beta}^2 \frac{\sin^2(\tilde{\beta}\tilde{\phi})}{[1 - \cos(\tilde{\beta}\tilde{\phi})]^2} \ll 1$$

$$\eta = m_p^2 \frac{V''}{V} = \tilde{\beta}^2 \frac{\cos(\tilde{\beta}\tilde{\phi})}{1 - \cos(\tilde{\beta}\tilde{\phi})} \ll 1$$

$$N = -\frac{1}{m_p^2} \int_{\phi_i}^{\phi_f} d\phi \frac{V}{V'} = -\frac{2}{\tilde{\beta}^2} \log \cos \left(\frac{\tilde{\beta}\tilde{\phi}}{2} \right) \Big|_{\tilde{\phi}_i}^{\tilde{\phi}_f} = [50 - 60]$$

and assuming a small value for the frequency $\tilde{\beta}$ the conditions can be fulfilled [40] which validates the use of the periodic scalar field theory in inflationary cosmology for small frequencies $\tilde{\beta} \ll 1$. In this limit one can expand the periodic potential and finds [40]

$$\boxed{(n_s - 1) + \frac{r}{4} = -\tilde{\beta}^2}$$

where the last equation can be compared to the that of the quadratic potential. Precise measurements on n_s and r can be used to put constraints on the dimensionless frequency $\tilde{\beta}$ [40]. The important difference between the quadratic and the periodic cases is the presence of the frequency in the above formula which is a running parameter of the model thus renormalization effects should be taken into account at least in the post-inflation period.

Indeed, one of our goal is to determine the RG running of the pure sine-Gordon theory in $d = 4$ dimensions and to discuss the consequences on the use of the periodic model to inflationary physics.

Appendix B. Periodic Higgs potential

In the standard model (SM) of particle physics the Brout-Englert-Higgs (BEH) mechanism [41,42] has been used to generate mass for the weak gauge bosons. Above a critical temperature the electroweak symmetry is unbroken, all elementary particles are massless. Below the critical temperature the symmetry is spontaneously broken and the W and Z bosons acquire masses. Furthermore, fundamental fermions can also acquire mass as a result of their interaction with the Higgs field. The SM Higgs field is an $SU(2)$ complex scalar doublet with four real components

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}.$$

The underlying symmetry of the electroweak sector is $SU(2)_L \times U(1)_Y$, thus, the Higgs Lagrangian reads as

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - V(\phi) - \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}),$$

$$V = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2, \quad (D_\mu = \partial_\mu + ig \mathbf{T} \cdot \mathbf{W}_\mu + ig' y_j B_\mu)$$

and the vacuum expectation of the Higgs field is either at zero field for $\mu^2 > 0$ or at $\sqrt{\phi^* \phi} = \sqrt{-\mu^2/(2\lambda)} = v/\sqrt{2}$ for $\mu^2 < 0$. After the spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y$ into $U(1)_Q$ the field around the groundstate can be parametrized as

$$\phi(x) = \frac{1}{\sqrt{2}} \exp \left(i \frac{\mathbf{T} \cdot \boldsymbol{\xi}(x)}{v} \right) \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

with $v = 246$ GeV known from low-energy experiments and the unitary phase can be dropped by choosing an appropriate gauge. As a consequence of the BEH mechanism the photon remains massless but the weak gauge bosons acquire masses. Three degrees of freedom of the Higgs field (out of the four) mix with weak gauge bosons while the remaining degree of freedom becomes the Higgs boson which is discovered at CERN's Large Hadron Collider. The complete Lagrangian for the Higgs sector reads

$$\mathcal{L} = \frac{\partial_\mu h \partial^\mu h}{2} - \frac{M_h^2 h^2}{2} - \frac{M_h^2 h^3}{2v} - \frac{M_h^2 h^4}{8v^2} + \left(M_W^2 W_\mu^+ W^{-\mu} + \frac{M_Z^2 Z_\mu Z^\mu}{2} \right) \left(1 + \frac{2h}{v} + \frac{h^2}{v^2} \right)$$

with $M_h = \sqrt{-2\mu^2} = \sqrt{2\lambda v^2}$. The measured value for the Higgs mass $M_h = 125.6$ GeV [43,44] implies $\lambda = 0.13$. This is close to the predicted value based on the assumption of the absence of new physics between the Fermi and Planck scales and the asymptotic safety of gravity [45]. This enables us to extrapolate the SM up to very high energies and to interpret the Higgs boson as the inflaton by using two competing scenarios, (i) non-minimal coupling to gravity, (ii) higgs-inflation from false vacuum.

Let us first discuss the large non-minimal coupling to gravity [46], which results in the following action

$$S = \int d^4x \sqrt{-\bar{g}} \frac{m_p^2}{2} \left[(1 + \xi \tilde{h}^2) \bar{R} - \bar{g}^{\mu\nu} \partial_\mu \tilde{h} \partial_\nu \tilde{h} - 2m_p^2 \frac{\lambda}{4} \left(\tilde{h}^2 + \frac{2v^2}{m_p^2} \right)^2 \right]$$

where $\bar{g}^{\mu\nu}$ is the metric in the Jordan frame, ξ represents the coupling between gravity and the dimensionless Higgs scalar \tilde{h} . To perform the slow-roll study, the action is usually rewritten in the Einstein frame where it takes the form

$$S = \int d^4x \sqrt{-g} \left[m_p^2 \frac{R}{2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m_p^4 \lambda}{4\xi^2} \left(1 - e^{-\sqrt{2/3}\phi/m_p} \right)^2 \right]$$

in which the metric tensor being denoted by $g^{\mu\nu}$ and the new field variable is introduced via $d\phi/d\tilde{h} = m_p \sqrt{1 + \xi(1 + 6\xi)\tilde{h}^2/(1 + \xi\tilde{h}^2)}$ but as a possible drawback of the method, perturbative unitarity can be violated for $\xi \neq 0$.

In case of a Higgs inflation from false vacuum a minimal coupling is used, i.e., $\xi = 0$ which implies $\phi = m_p \tilde{h}$ with the same shape of potentials in both frames but the SM Higgs potential is extended and assumed to develop a second (or more) minimum [47,48]. This scenario can have difficulties to achieve an exit from the inflationary phase. In addition, the measured Higgs mass is close to the lower limit, 126 GeV, ensuring absolute vacuum stability within the SM [49] although it was also shown [50] that traditional Higgs inflation can be possible within a minimalistic framework even if the SM vacuum is not completely stable.

Therefore, the UV completion of the Higgs potential which try to make connection between Higgs and inflationary physics, is required to have no just a global but local minima which can be realized by adding new interaction terms to the Lagrangian, such as $\lambda_2(\phi^*\phi)^3$. Indeed, the stability study of various types of polynomial Higgs potentials in the framework of simplified models has been performed by using functional renormalization group (RG) technique [51]. However, in the RG point of view the phase structure has not been modified significantly if the Higgs potential remains polynomial. Instead, a periodic self-interaction,

$$V = u[\cos(\beta\sqrt{\phi^*\phi}) - 1] = -\frac{u\beta^2}{2}|\phi^*\phi| + \frac{u\beta^4}{4}(\phi^*\phi)^2 + \dots$$

can influence more drastically the phase structure and the RG running of the couplings due to the periodicity which should be protected by the RG. The frequency β can be chosen to keep the first minimum of the potential to be equal to that of the standard quartic Higgs potential, i.e., $\beta = \sqrt{2\pi}/v$. The inclusion of higher harmonics can shift the second minimum to take place at large fields. There are various scenarios to identify the Higgs mass by the appropriate choice of

u (i) using the Taylor expansion of the (tree-level, bare) potential around the zero field where one finds $\mu^2 = -\frac{1}{2}u\beta^2$, (ii) including the RG running of the parameters in the co-called massive phase of the periodic model where the Taylor expansion is more justified and the mass parameter is given by the infra-red (IR) values $\mu^2 = -\frac{1}{2}u_{\text{IR}}\beta_{\text{IR}}^2$, (iii) by means of the soliton mass.

With the same parametrization of the field around the (first) minimum, the Lagrangian of the Higgs sector looks similar to the usual one but with a periodic self-interaction term for $h(x)$.

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h - u \left[\cos\left(\frac{1}{\sqrt{2}}\beta|v + h(x)|\right) - 1 \right] + \left(M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \right) \left(1 + 2\frac{h}{v} + \frac{h^2}{v^2} \right).$$

A similar structure is found for the $O(N)$ extended version of the sine-Gordon model [52] which was studied in $d = 2$ dimensions and the dependence of its phase structure on N was determined but it has not been investigated in higher dimensions. A rigorous functional RG study provides us a very powerful tool to attack such problems as it has been done for the pure $O(N)$ model in higher dimensions where the triviality of the model is re-examined [53]. The assumption of a periodic Higgs potential means infinitely many degenerate minima and requires a careful functional RG treatment.

Another goal of this work is to discuss some of the consequences of the renormalization of the pure sine-Gordon model to Higgs inflation and to the stability problem in the framework of the Higgs-Yukawa theory.

Appendix C. Periodic axion potential

Our last example for applications of sine-Gordon type scalar field theory in higher dimensions is related to the axion. Constraints from symmetry and renormalizability on the standard model QCD action allow to extend it by a CP violating term. However, experimental data do not favour such an extension although the standard model Lagrangian is not CP symmetric, so, QCD could be CP violating as well. Peccei and Quinn proposed a mechanism and introduced a new hypothetical scalar field with $U(1)$ symmetry in order to build up a CP conserving theory from a model with massive fermions coupled to a non-Abelian gauge field [54]. The axion appears as a phase of a Goldstone mode for a complex scalar Φ with a vacuum expectation value $\langle \Phi \rangle = f e^{i\theta}$ corresponds to the spontaneous break down of the $U(1)$ symmetry at the scale f . Integrating over the QCD degrees of freedom one arrives at the following effective action

$$S = \int d^4x \left(\frac{f^2}{2} \partial_\mu \theta \partial^\mu \theta + u[1 - \cos(\theta)] \right) = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + u[1 - \cos(\beta\phi)] \right),$$

with $\phi = f\theta$ and $\beta = 1/f$ where the periodic potential appears naturally and the rescaling of the field has been done by using the assumption that f is independent of the spacetime. In Ref. [55] it was shown that the axion potential flattens out under RG transformations which were taken in the so called local potential approximation. Our goal here is to perform the functional RG study of the periodic scalar model, i.e., the sine-Gordon theory in $d = 4$ beyond the local potential approximation in order to clarify the flattening of the axion potential.

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