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Linear recursive sequences and factorials

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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.

The results published in the thesis are not reported in any other PhD theses.

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Hereby I confirm that Tamás Szakács candidate conducted his studies with my supervision within the Diophantine and constructive number theory Doctoral Program of the Doctoral School of Mathematical and Computational Sciences between 2012 and 2024. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.

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signature of the supervisor

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Contents

1 | Introduction

The history of linear recursive sequences goes back several centuries. One of the first mentions was made by Leonardo Pisano (more commonly known as Fibonacci). The Fibonacci numbers $1, 2, 3, 5, 8, 13, \ldots$ were employed in 1202 in a problem on the number of offspring of a pair of rabbits. We shall write F_n for the nth term of $0, 1, 1, 2, 3, 5, \ldots$ derived by prefixing 0,1 to the former sequence. Albert Girard in 1634 writed down the law $F_{n+2} = F_{n+1} + F_n$ for the Fibonacci numbers (see sequence A000045 in OEIS [35]). These types of sequences are called recursive. Robert Simson observed in 1753 that this sequence is given by the successive convergents to the continued fraction for $\frac{\sqrt{5}+1}{2}$. J. P. M. Binet published in 1843 the formula named after him:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

E. Lucas in 1876 stated theorems on the sequence of Fibonacci. He was the first to publish about the laws found among the elements of the Fibonacci sequence. The sum of the first n terms equals $F_{n+2} - 2$, the sum of those terms taken with alternate signs equals $(-1)^n \cdot F_{n-1}$. It is here that we first encounter the concept of a characteristic equation and the proof of the relationship between the roots and the members of the sequence. The interesting facts about the sequence are still researched by mathematicians with unbroken enthusiasm.

In the second chapter of this work, when discussing linear recursive sequences, many results can be read about Fibonacci numbers. We discuss theorems about recursive sequences, where the quotient of adjacent terms of the sequences approaches the golden ratio. Later, integer sequences are constructed as linear combinations of Fibonacci numbers and rational polynomials. In the last part of the chapter, we deal with the convolution of second order linear recursive sequences, where Fibonacci numbers and other famous sequences also appear.

By the previously mentioned Binet's formula for $n \geq 0$, we can use the following explicit form for the nth term of the Fibonacci numbers

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

where φ and ψ are the roots of the characteristic polynomial of $\{F_n\}_{n=0}^{\infty}$, that is, $p(x) = x^2 - x - 1$, $p(\varphi) = p(\psi) = 0$, $\varphi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$. φ is also known as the golden ratio, and

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

We investigate linear recursive sequences $\{G_n\}_{n=1}^{\infty}$ of real numbers, where the sequences $\left\{\frac{G_{n+1}}{G_n}\right\}$ converge quicker to the golden ratio than $\left\{\frac{F_{n+1}}{F_n}\right\}$.

Several problems of combinatorics have solution in the form

$$W_n = u(n)F_n + v(n)F_{n-1} + c(n), \tag{1.1}$$

where $u(x)$, $v(x)$ and $c(x)$ are rational polynomials of the variable x. It is not obvious, at least ab ovo, that the terms of $\{W_n\}_{n=0}^{\infty}$ in (1.1) are integer since the coefficient polynomials are rational. For example, if $\{A_n\}_{n=0}^{\infty}$ gives the number of parts in all compositions of $n + 1$ with no 1's, then

$$A_n = \frac{2n+3}{5}F_n - \frac{n}{5}F_{n-1}, \tag{1.2}$$

see sequence A010049 in OEIS [35]. Here $u(x) = (2x + 3)/5$ and $v(x) = -x/5$ are linear polynomials with non-integer rational coefficients (and $c(x)$ vanishes), but $\{A_n\}_{n=0}^{\infty}$ is an integer sequence. There are polynomials with higher degree appearing in (1.1). For instance, look at sequence $\{B_n\}_{n=0}^{\infty}$ (see A129707 of [35]) which describes the number of inversions in all Fibonacci binary words of length n. The formula

$$B_{n-3} = \frac{5n^2 - 37n + 50}{50}F_n + \frac{4n-4}{50}F_{n-1} \tag{1.3}$$

given in the encyclopedia leads to

$$B_n = \frac{5n^2 - n - 4}{25} F_n + \frac{5n^2 + n}{50} F_{n-1}$$

via the identity $z_3 F_{n+3} + z_2 F_{n+2} = (3z_3 + 2z_2)F_n + (2z_3 + z_2)F_{n-1}$. The last identity comes immediately as one applies the Fibonacci recurrence thrice. We study the general problem, how to give conditions for the rational functions $p_i(x)$ to guarantee sequence $\{W_n\}_{n=0}^{\infty}$

$$W_n = p_1(n)F_{n-j_1} + p_2(n)F_{n-j_2} + \cdots + p_s(n)F_{n-j_s}$$

to be integer.

The convolution of two linear recursive sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$ is the sequence

$$C_n = \sum_{k=0}^{n} G_k H_{n-k}.$$

W. Zhang in [47] gave formula for the convolution of the second order linear recursive sequences $\{U_n\}_{n=0}^{\infty}$ with themself

$$\sum_{a_1+a_2+\cdots+a_k=n} U_{a_1} U_{a_2} \cdots U_{a_k}$$

and wrote down some new convolution properties. We work with Lucas sequences of the first and second kind $G_n(G_0, G_1, A_1, B_1)$ and $H_n(H_0, H_1, A_2, B_2)$, where the initial terms are $G_0 = 0, G_1 = 1$ and $H_0 = 2, H_1 = A_1$, respectively. At first, we consider the convolution of those sequences where the characteristic polynomials have no common root, then exactly one common root, after that, we deal that case when the characteristic polynomials have two common roots, that is, the characteristic polynomials are the same ones. There are some results in the last case from W. Zhang ([47]), and from Z. Zhang and P. He ([48]) for which we get different formulas, but using some well known identities they could be convertible to each other.

In the third chapter, we discuss the question of finding all products of factorials yielding a factorial, which is a long standing problem, studied by many authors. Here we only mention a few related results, for a survey of the topic see e.g. R. K. Guy: Unsolved Problems in Number Theory [19], section B23. Consider the equation

$$n! = \prod_{i=1}^{r} a_i!,$$

which has trivial solutions, for example $6! = 5!3!$ or $12! = 11!3!2!$. On the other hand, according to a conjecture of Surányi, the only non-trivial solution with $r = 2$ is $10! = 7!6!$, while a conjecture of Hickerson predicts that the only non-trivial solutions for arbitrary r are given by $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$, $16! = 14!5!2!$ (see e.g. Erdős [10], pp. 27-28). These conjectures have been checked for $n \leq 10^6$ by Caldwell [7]. We consider the equation as

$$A!B! = C!$$

with positive integers A, B, C satisfying $C \geq B \geq A > 1$. Our purpose is to show the finiteness of the solutions to this equation with $k := B - A$ bounded. Our main result provides an explicit upper bound for C in terms of k. Certainly, this immediately implies that for any fixed k, it has only finitely many solutions. Further, we show that the only non-trivial solution with $k \leq 10^6$ is the well-known $10! = 7!6!$.

Later, we study the so-called multiplying balancing numbers n which satisfy the equation

$$1 \cdot 2 \cdots (n-1) = (n+1)(n+2) \cdots (n+r)$$

for some positive integer r. If we multiply the equation by $n!$, we get $(n-1)!n! = (n+r)!$ which is an equation of the type $A!B! = C!$, with $k = B - A = 1$. This means that $6! = 8 \cdot 9 \cdot 10$ is the only solution of the equation. We will show an alternative proof of this statement.

2 | Linear recursive sequences

The second order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ is defined by the recurrence relation

$$G_n = AG_{n-1} + BG_{n-2} \quad (n \geq 2),$$

where the initial terms G_0, G_1 and the weights A, B are fixed real numbers with $|G_0| + |G_1| \neq 0$ and $AB \neq 0$. Sometimes the following notation $G_n(G_0, G_1, A, B)$ is used, too. The polynomial

$$p(x) = x^2 - Ax - B$$

is known as the characteristic polynomial of the sequence $\{G_n\}_{n=0}^{\infty}$. If its discriminant $D = A^2 + 4B \neq 0$ then the Binet's formula of $\{G_n\}_{n=0}^{\infty}$ is

$$G_n = \frac{G_1 - \beta G_0}{\alpha - \beta}\alpha^n - \frac{G_1 - \alpha G_0}{\alpha - \beta}\beta^n,$$

where α, β are distinct roots of $p(x)$.

If $G_0 = 0$ and $G_1 = 1$ then $\{G_n\}_{n=0}^{\infty}$ is known as Lucas sequence of the first kind $\{R_n\}_{n=0}^{\infty}$ with its Binet's formula

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.1}$$

while if $G_0 = 2$ and $G_1 = A$ then the sequence is known as Lucas sequence of the second kind $\{V_n\}_{n=0}^{\infty}$ with its Binet's formula

$$V_n = \alpha^n + \beta^n. \tag{2.2}$$

It's known that the generating function of $\{G_n\}_{n=0}^{\infty}$ is

$$g(x) = \frac{G_0 + (G_1 - AG_0)x}{1 - Ax - Bx^2}. \tag{2.3}$$

There are some well-known Lucas sequences, such as Fibonacci, Pell, Jacobsthal, Mersenne, and their associate sequences. The following table contains the initial terms, characteristic polynomials and generating functions of these sequences.

Name	$G_n(G_0, G_1, A, B)$	Charact. polynom.	Gen. function
Fibonacci	$F_n(0, 1, 1, 1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{x}{1-x-x^2}$
Pell	$P_n(0, 1, 2, 1$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{x}{1-2x-x^2}$
Jacobsthal	$J_n(0, 1, 1, 2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{x}{1-x-2x^2}$
Mersenne	$M_n(0, 1, 3, -2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{x}{1-3x+2x^2}$
Lucas	$L_n(2, 1, 1, 1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{2-x}{1-x-x^2}$
P-Lucas	$p_n(2, 2, 2, 1)$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{2-2x}{1-2x-x^2}$
J-Lucas	$j_n(2, 1, 1, 2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{2-x}{1-x-2x^2}$
M-Lucas	$m_n(2, 3, 3, -2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{2-3x}{1-3x+2x^2}$

Table 2.1: Named sequences

A positive integer n is called by Finkelstein [16] a numerical center, and independently from him, Behera and Panda [3] called a balancing number if

$$1 + \cdots + (n-1) = (n+1) + \cdots + (n+r)$$

holds for some positive integer r. The sequence of balancing numbers is denoted by $\{B_m\}_{m=1}^{\infty}$. As one can easily check, we have $B_1 = 6$ and $B_2 = 35$. Note that by a result of Behera and Panda [3], we have

$$B_{m+1} = 6B_m - B_{m-1} \qquad (m > 1).$$

In that paper they proved that, there are infinitely many balancing numbers. Liptai [23] searched for those balancing numbers which are Fibonacci numbers, too. Using the results of A. Baker and G. Wüstholz [2] he proved that there are no Fibonacci balancing numbers. Similarly in [24] he proved that there are no Lucas balancing numbers. Using an other method L. Szalay [45] got the same result. Liptai, Luca, Pintér and Szalay [25] generalized the concept of balancing numbers in the following way. Let y, k, l be fixed positive integers with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

They proved (several effective and ineffective finiteness results in their paper) that for any fixed positive integer $k > 1$, there are only finitely many positive pairs of integers (y, l) such that y possesses a (k, l)-power numerical center. Later G.K. Panda and P.K. Ray [37] slightly modified the definition of balancing number and introduced the notion of cobalancing number. A positive integer n is called a cobalancing number if

$$1 + 2 + \cdots + (n-1) + n = (n+1) + (n+2) + \cdots + (n+K)$$

for some $K \in \mathbb{N}$. In this case K is called the cobalancer of n. They also proved that the cobalancing numbers fulfill the following recurrence relation

$$B^c_{m+1} = 6B^c_m - B^c_{m-1} + 2 \quad (n > 1),$$

where $B^c_1 = 2$ and $B^c_2 = 14$. Moreover they found that every balancer is a cobalancing number and every cobalancer is a balancing number. As a generalization of the notion of a balancing number A. Bérczes, K. Liptai and I. Pink [4] call a binary recurrence $R = R(A, B, R_0, R_1)$ a balancing sequence if

$$R_1 + R_2 + \ldots + R_{n-1} = R_{n+1} + R_{n+2} + \ldots + R_{n+k}$$

holds for some $k \geq 1$ and $n \geq 2$. They proved that any sequence $R = R(A, B, 0, R_1)$ with the condition $D = A^2 + 4B > 0$, $(A, B) \neq (0, 1)$ is not a balancing sequence. T. Kovács, K. Liptai and P. Olajos [22] extended the concept of balancing numbers to arithmetic progressions. Let $a > 0$ and $b \geq 0$ be coprime integers. If for some positive integers n and r we have

$$(a + b) + \cdots + (a(n-1) + b) = (a(n+1) + b) + \cdots + (a(n+r) + b)$$

then we say that $an + b$ is an (a, b)-balancing number. They proved several effective finiteness and explicit results about them. In the proofs they combined the Baker's method, the modular method developed by Wiles and others, the Chabauty method and the theory of elliptic curves.

Let us consider now the definition of recursive sequences with arbitrary order k. Let $A_0, A_1, \ldots, A_{k-1}$ be given real numbers with $A_{k-1} \neq 0$, where

$k \geq 2$ fixed integer. A linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of order k is defined by the recurrence

$$G_n = A_0 G_{n-1} + A_1 G_{n-2} + \cdots + A_{k-1} G_{n-k} \qquad (n \geq k),$$

where the initial terms $G_0, G_1, \ldots, G_{k-1}$ are fixed real numbers with $|G_0| + |G_1| + \cdots + |G_{k-1}| \neq 0$. The polynomial

$$p(x) = x^k - A_0 x^{k-1} - A_1 x^{k-2} - \cdots - A_{k-2} x - A_{k-1}$$

is said to be the characteristic polynomial of the sequence $\{G_n\}_{n=0}^{\infty}$, the roots of the equation $p(x) = 0$ are denoted by α_i's $(1 \leq i \leq k)$. In the sequel, we suppose that the root α_1 is of the largest absolute value, that is, $|\alpha_1| > |\alpha_2| \geq \cdots \geq |\alpha_k| > 0$ and the multiplicity of α_1 is 1. According to the literature [31] and [32], α_1 is called as the dominant root, and if we denote by m_i the multiplicity of the distinct α_i's $(1 \leq i \leq l, \sum\limits_{i=1}^{l} m_i = k)$ then the Binet's formula for the term G_n is as follows

$$G_n = a\alpha_1^n + p_2(n)\alpha_2^n + p_3(n)\alpha_3^n + \cdots + p_l(n)\alpha_l^n,$$

where the degree of the polynomial p_i $(2 \leq i \leq l)$ is less than m_i. The constant a and the polynomials p_i belong to the ring $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_l)[x]$, and we suppose that the initial terms are chosen such that $a \neq 0$.

2.1 | The golden ratio

In mathematics, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. Expressed algebraically, for quantities a and b with $a > b > 0$, a is in a golden ratio to b if

$$\frac{a+b}{a} = \frac{a}{b} = \varphi,$$

where the Greek letter phi (φ) denotes the golden ratio. The constant φ satisfies the quadratic equation $\varphi^2 = \varphi + 1$ and is an irrational number with a value of $\varphi = \frac{1+\sqrt{5}}{2} = 1.618033988749....$ The golden ratio was called the extreme and mean ratio by Euclid and the divine proportion by Luca Pacioli and also goes by several other names. Mathematicians have studied the golden ratio's properties since antiquity. It is the ratio of a regular pentagon's diagonal to its side and thus appears in the construction of the dodecahedron and icosahedron. A golden rectangle, that is, a rectangle with an aspect ratio of φ may be cut into a square and a smaller rectangle with the same aspect ratio. The golden ratio has been used to analyze the proportions of natural objects and artificial systems such as financial markets, in some cases based on dubious fits to data. The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves and other parts of vegetation.

The Fibonacci numbers were obtained when solving a 13th-century problem, which can be given as follows. Let $n \geq 2$, $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2},$$

which is the well known Fibonacci sequence. The dominant root of the characteristic polynomial is φ, the golden ratio. We also know that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

It can be read in [17] that the authors, Gatta and D'Amico gave infinitely many sequences, where the quotient of adjacent terms approaches the golden ratio. If

$$H_0 = a - b,\ H_1 = b \quad \text{and} \quad H_n = H_{n-1} + H_{n-2},$$

then for $a, b \in \mathbb{R},\ b \neq 0$

$$H_n = aF_{n-1} + bF_{n-2} \quad \text{and} \quad \lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \varphi.$$

The sequence $\{H_n\}_{n=0}^{\infty}$ differs from the sequence $\{F_n\}_{n=0}^{\infty}$ only in the initial terms, the characteristic polynomial is the same. In [21] T. Komatsu obtained similar results not only for the golden ratio. The question then arises, does the limit give a quicker convergence to φ? According to the literature [31] and [32]: Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be convergent sequences of real numbers with $\lim\limits_{n\to\infty} x_n = y_n = z$, we say that $\{y_n\}_{n=0}^{\infty}$ converges quicker than $\{x_n\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{y_n - z}{x_n - z} = 0. \tag{2.4}$$

We investigate linear recursive sequences $\{G_n\}_{n=1}^{\infty}$ of real numbers, where the sequences $\left\{\frac{G_{n+1}}{G_n}\right\}$ converge quicker to the golden ratio than $\left\{\frac{F_{n+1}}{F_n}\right\}$. Naturally, we suppose that division by zero never occurs.

Binary linear recursive sequences and the golden ratio

By the known Binet's formula for $\{H_n\}_{n=0}^{\infty}$ $(n \geq 0)$

$$H_n = \frac{c\varphi^n - d\psi^n}{\sqrt{5}},$$

where $c = (b - \psi(a-b)) \neq 0$ and $d = (b - \varphi(a-b))$, so one can easily verify that

$$\lim_{n\to\infty} \frac{\frac{H_{n+1}}{H_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \frac{d}{c},$$

which implies the following statement. If $d \neq 0$, then the sequence $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$ does not converge quicker to φ than the sequence $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$. If $d = 0$, then $\{H_n\}_{n=0}^{\infty}$ is a simple geometric sequence, where $\frac{H_{n+1}}{H_n} = \varphi$.

Let us consider now the second order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of real numbers,

$$G_n = AG_{n-1} + BG_{n-2}$$

with its characteristic polynomial

$$p(x) = x^2 - Ax - B = (x - \alpha_1)(x - \alpha_2),$$

where $\alpha_1 = \varphi$ is the dominant root, that is, $|\alpha_2| < \varphi$ and $\alpha_2 \in \mathbb{R}$. Using $A = \alpha_1 + \alpha_2,\ B = -\alpha_1\alpha_2$, we get that

$$G_n = (\alpha_1 + \alpha_2)G_{n-1} - \alpha_1\alpha_2 G_{n-2},$$

where $n \geq 2$ and $G_0, G_1 \in \mathbb{R}$. By the Binet's formula this sequence has an explicit form for $n \geq 0$,

$$G_n = a\varphi^n + b\alpha_2^n,$$

where a, b are computable constants depending only on the initial terms and the roots, and we suppose that $ab \neq 0$. In this case, it can be easily verified that

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \varphi \quad \text{and} \quad \lim_{n\to\infty} \frac{\frac{G_{n+1}}{G_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \frac{b(\alpha_2 - \varphi)}{a(\varphi - \psi)} \cdot \left(\frac{\alpha_2}{\psi}\right)^n,$$

which implies the following theorem.

Theorem 1 (T. Szakács [44], 2017)**.** *The sequence* $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ *converges quicker to* φ *than* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$ *if and only if* $|\alpha_2| < |\psi|$.

Ternary linear recursive sequences and the golden ratio

In [17] F. Gatta and A. D'Amico investigated the following ternary sequence

$$H_{n+1} = 2H_n - H_{n-2}, \quad (n \geq 3) \tag{2.5}$$

with the real initial terms H_1, H_2, H_3, $(H_3\varphi^2 - H_1\varphi - H_2 \neq 0)$ and they proved that

$$\lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \varphi,$$

that is, they proved that there exist infinitely many third order linear recursive sequences, where the ratio of the consecutive terms tends to the golden ratio. Let us use the explicit form

$$H_n = a\varphi^n + b\psi^n + c, \tag{2.6}$$

for $n \geq 1$, where $a (= H_3\varphi^2 - H_1\varphi - H_2 \neq 0), b, c$ are computable real constants depending only on the initial terms and the roots of its characteristic polynomial

$$p(x) = x^3 - 2x^2 + 1 = (x^2 - x - 1)(x - 1) = (x - \varphi)(x - \psi)(x - 1).$$

Investigating the following limit we can obtain:

$$\lim_{n\to\infty} \frac{\frac{H_{n+1}}{H_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \left(-b + \frac{c(1-\varphi)}{\sqrt{5}\cdot\psi^n} \right) \cdot \frac{1}{a},$$

which implies the following result for the sequence $\{H_n\}_{n=0}^{\infty}$.

Theorem 2 (T. Szakács [44], 2017)**.** *Let us use the notation of* (2.6)*.*

- *If $c \neq 0$, then the sequence $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$ does not converge quicker to φ than the sequence $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$. More precisely the sequence $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$ converges quicker than the sequence $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$.*
- *If $c = 0$ and $b \neq 0$, then the sequence $\left\{\frac{H_{n+1}}{H_n}\right\}_{n=1}^{\infty}$ does not converge quicker to φ than the sequence $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$.*
- *If $c = 0$ and $b = 0$, then $H_n = a\varphi^n$, which is a simple geometric sequence.*

The previous theorem dealt with the sequence (2.5) investigated by F. Gatta and A. D'Amico in [17], but – omitting the details – similar results can be obtained in the following case, too:

$$H_{n+1} = 2H_{n-1} + H_{n-2},$$

where $n \geq 3$ and H_1, H_2, H_3 are the initial terms. The characteristic polynomial is $p(x) = x^3 - 2x - 1 = (x^2 - x - 1)(x + 1) = (x - \varphi)(x - \psi)(x + 1)$.

Let us consider now the third order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of real numbers

$$G_n = A_0G_{n-1} + A_1G_{n-2} + A_2G_{n-3},$$

with its characteristic polynomial

$$p(x) = x^3 - A_0x^2 - A_1x - A_2 = (x - \varphi)(x - \alpha_1)(x - \alpha_2),$$

where $A_0, A_1, A_2 \in \mathbb{R}, (A_2 \neq 0)$, α_1, α_2 are non zero complex numbers, and $|\alpha_1| < \varphi, |\alpha_2| < \varphi$.

Theorem 3 (T. Szakács [44], 2017)**.** *The sequence* $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ *converges quicker to* φ *than* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$ *if and only if* $|\alpha_1| < |\psi|, |\alpha_2| < |\psi|$.

Proof. We use the limit in (2.4) and examine three different cases.

(i) α_1, α_2 are distinct real numbers. By the Binet's formula

$$G_n = a\varphi^n + b\alpha_1^n + c\alpha_2^n,$$

we can obtain the following:

$$\lim_{n\to\infty} \frac{\frac{G_{n+1}}{G_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \frac{\frac{G_{n+1}-\varphi G_n}{G_n}}{\frac{F_{n+1}-\varphi F_n}{F_n}} = \lim_{n\to\infty} \frac{G_{n+1} - \varphi G_n}{F_{n+1} - \varphi F_n} \cdot \frac{F_n}{G_n} =$$
$$\lim_{n\to\infty} \frac{a\varphi^{n+1} + b\alpha_1^{n+1} + c\alpha_2^{n+1} - \varphi(a\varphi^n + b\alpha_1^n + c\alpha_2^n)}{\varphi^{n+1} - \psi^{n+1} - \varphi(\varphi^n - \psi^n)} \cdot$$
$$\frac{\varphi^n - \psi^n}{a\varphi^n + b\alpha_1^n + c\alpha_2^n} =$$
$$\lim_{n\to\infty} \frac{b\alpha_1^n(\alpha_1 - \varphi) + c\alpha_2^n(\alpha_2 - \varphi)}{\psi^n(\varphi - \psi)} \cdot \frac{\varphi^n - \psi^n}{a\varphi^n + b\alpha_1^n + c\alpha_2^n} =$$
$$\lim_{n\to\infty} \frac{b\left(\frac{\alpha_1}{\psi}\right)^n(\alpha_1 - \varphi) + c\left(\frac{\alpha_2}{\psi}\right)^n(\alpha_2 - \varphi)}{a\sqrt{5}}.$$

(ii) α_1, α_2 are real numbers and $\alpha_1 = \alpha_2$. By the Binet's formula

$$G_n = a\varphi^n + (bn + c)\alpha_1^n,$$

we can obtain the following:

$$\lim_{n\to\infty}\frac{\frac{G_{n+1}}{G_n}-\varphi}{\frac{F_{n+1}}{F_n}-\varphi}=\lim_{n\to\infty}\frac{\frac{G_{n+1}-\varphi G_n}{G_n}}{\frac{F_{n+1}-\varphi F_n}{F_n}}=\lim_{n\to\infty}\frac{G_{n+1}-\varphi G_n}{F_{n+1}-\varphi F_n}\cdot\frac{F_n}{G_n}=$$

$$\lim_{n\to\infty}\frac{a\varphi^{n+1}+(b(n+1)+c)\alpha_1^{n+1}-\varphi(a\varphi^n+(bn+c)\alpha_1^n)}{\varphi^{n+1}-\psi^{n+1}-\varphi(\varphi^n-\psi^n)}\cdot$$

$$\frac{\varphi^n-\psi^n}{a\varphi^n+(bn+c)\alpha_1^n}=$$

$$\lim_{n\to\infty}\frac{(bn+c)\alpha_1^n(\alpha_1-\varphi)+b\alpha_1^{n+1}}{\psi^n(\varphi-\psi)}\cdot\frac{\varphi^n-\psi^n}{a\varphi^n+(bn+c)\alpha_1^n}=$$

$$\lim_{n\to\infty}\frac{(bn+c)\left(\frac{\alpha_1}{\psi}\right)^n(\alpha_1-\varphi)+b\alpha_1\left(\frac{\alpha_1}{\psi}\right)^n}{a\sqrt{5}}.$$

(iii) $\alpha_1=z$, $\alpha_2=\overline{z}$ are non real, complex numbers and by the Binet's formula

$$G_n=a\varphi^n+bz^n+c\overline{z}^n,$$

we can obtain the following:

$$\lim_{n\to\infty}\frac{\frac{G_{n+1}}{G_n}-\varphi}{\frac{F_{n+1}}{F_n}-\varphi}=\lim_{n\to\infty}\frac{\frac{G_{n+1}-\varphi G_n}{G_n}}{\frac{F_{n+1}-\varphi F_n}{F_n}}=\lim_{n\to\infty}\frac{G_{n+1}-\varphi G_n}{F_{n+1}-\varphi F_n}\cdot\frac{F_n}{G_n}=$$

$$\lim_{n\to\infty}\frac{a\varphi^{n+1}+bz^{n+1}+c\overline{z}^{n+1}-\varphi(a\varphi^n+bz^n+c\overline{z}^n)}{\varphi^{n+1}-\psi^{n+1}-\varphi(\varphi^n-\psi^n)}\cdot$$

$$\frac{\varphi^n-\psi^n}{a\varphi^n+bz^n+c\overline{z}^n}=$$

$$\lim_{n\to\infty}\frac{bz^n(z-\varphi)+c\overline{z}^n(\overline{z}-\varphi)}{\psi^n(\varphi-\psi)}\cdot\frac{\varphi^n-\psi^n}{a\varphi^n+bz^n+c\overline{z}^n}=$$

$$\lim_{n\to\infty}\frac{b\left(\frac{z}{\psi}\right)^n(z-\varphi)+c\left(\frac{\overline{z}}{\psi}\right)^n(\overline{z}-\varphi)}{a\sqrt{5}}.$$

These limits imply that in all the above cases the limits are equal to zero if and only if in (i) and in (ii) $|\alpha_1|<|\psi|, |\alpha_2|<|\psi|$, while in (iii) $|z|<|\psi|$, that is, our theorem has been proved. □

k-order linear recursive sequences and the golden ratio

Let us consider now the k-order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of real numbers,

$$G_n = A_0G_{n-1} + A_1G_{n-2} + \cdots + A_{k-1}G_{n-k}$$

with its characteristic polynomial

$$p(x) = x^k - A_0x^{k-1} - A_1x^{k-2} - \cdots - A_{k-2}x - A_{k-1}.$$

The Binet's formula for the term G_n is the following:

$$G_n = a\alpha_1^n + p_2(n)\alpha_2^n + p_3(n)\alpha_3^n + \cdots + p_l(n)\alpha_l^n,$$

where $\alpha_1 = \varphi$ is the dominant root, and is equal to the golden ratio.

Theorem 4 (T. Szakács [44], 2017). *The sequence* $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ *converges quicker to the golden ratio than* $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$, *if* $|\alpha_i| < |\psi|, i = 2, 3 \ldots, l$.

Proof.

$$\lim_{n\to\infty} \frac{\frac{G_{n+1}}{G_n} - \varphi}{\frac{F_{n+1}}{F_n} - \varphi} = \lim_{n\to\infty} \frac{\frac{G_{n+1}-\varphi G_n}{G_n}}{\frac{F_{n+1}-\varphi F_n}{F_n}} = \lim_{n\to\infty} \frac{G_{n+1} - \varphi G_n}{F_{n+1} - \varphi F_n} \cdot \frac{F_n}{G_n} =$$

$$\lim_{n\to\infty} \frac{a\varphi^{n+1} + \sum_{i=2}^{l} p_i(n+1)\alpha_i^{n+1} - \varphi(a\varphi^n + p_2(n)\alpha_2^n + \cdots + p_l(n)\alpha_l^n)}{\varphi^{n+1} - \psi^{n+1} - \varphi(\varphi^n - \psi^n)}$$

$$\cdot \frac{\varphi^n - \psi^n}{a\varphi^n + \cdots + p_l(n)\alpha_l^n} =$$

$$\lim_{n\to\infty} \frac{p_2(n+1)\alpha_2^{n+1} + \cdots + p_l(n+1)\alpha_l^{n+1} - \varphi p_2(n)\alpha_2^n - \cdots - \varphi p_l(n)\alpha_l^n}{\psi^n\sqrt{5}} \cdot$$

$$\frac{\varphi^n - \psi^n}{a\varphi^n + \cdots + p_l(n)\alpha_l^n} =$$

$$\lim_{n\to\infty} \frac{\alpha_2^n \overbrace{(p_2(n+1)\alpha_2 - p_2(n)\varphi)}^{r_2(n)} + \cdots + \alpha_l^n \overbrace{(p_l(n+1)\alpha_l - p_l(n)\varphi)}^{r_l(n)}}{\psi^n a\sqrt{5}} =$$

$$\lim_{n\to\infty} \frac{\left(\frac{\alpha_2}{\psi}\right)^n r_2(n) + \left(\frac{\alpha_3}{\psi}\right)^n r_3(n) + \cdots + \left(\frac{\alpha_l}{\psi}\right)^n r_l(n)}{a\sqrt{5}}.$$

Investigating different cases of the limit above, one can see that the limit is equal to zero if $\left(\frac{\alpha_i}{\psi}\right)^n$ tends to zero for all $i = 2, 3\dots, l$. □

2.2 | On certain Fibonacci representations

Terms with negative subscripts $-n$ ($n \in \mathbb{N}$) can be introduced via the equality

$$F_{-n} = -F_{-n+1} + F_{-n+2}, \tag{2.7}$$

and it turns out that $F_{-n} = (-1)^{n+1}F_n$. An extension having similar flavour is based on the well-known Fibonacci identity $F_{n-j} = F_nF_{-j} + F_{n-1}F_{-j+1}$ with $n \in \mathbb{Z}, j \in \mathbb{N}$. Combining it with (2.7), we have

$$F_{n-j} = \left((-1)^j F_{j-1}\right) F_n + \left((-1)^{j+1} F_j\right) F_{n-1}. \tag{2.8}$$

Let $0 \le j_1 < j_2 < \cdots < j_s$ be nonnegative integers, and $p_1(x), p_2(x), \dots,$ $p_s(x) \in \mathbb{Q}[x]$ such that $\deg(p_i(x)) = d_i$. Put $d^\star = \max_i\{d_i\}$, which is a nonnegative integer. Define the sequence $\{W_n\}_{n=0}^{\infty}$ by

$$W_n = p_1(n)F_{n-j_1} + p_2(n)F_{n-j_2} + \cdots + p_s(n)F_{n-j_s}. \tag{2.9}$$

We give conditions for the rational functions $p_i(x)$ to guarantee sequence $\{W_n\}_{n=0}^{\infty}$ to be integer. Multiple application of (2.8) transforms (2.9) into the form

$$W_n = P_0(n)F_n + P_1(n)F_{n-1}, \tag{2.10}$$

where $P_0(x)$ and $P_1(x)$ are suitable rational polynomials depend on the subscripts $j_1, j_2, \dots, j_s$ and on the polynomials $p_1(x), p_2(x), \dots, p_s(x)$. Hence, essentially, it is sufficient to consider only (2.10). But, sometimes the form (2.9) promises a more advantageous starting point in the investigations.

2.2.1 | General approach

In the foregoing, we have seen the idea how to simplify (2.9) to get (2.10), here we choose a slightly different way. At the beginning, we assume that $p_j(x) \in \mathbb{C}[x]$ for $j = 1, 2, \ldots, s$. The Binet's formula implies that

$$\begin{aligned} W_n &= p_1(n)F_{n-j_1} + p_2(n)F_{n-j_2} + \cdots + p_s(n)F_{n-j_s} \qquad (2.11)\\ &= \sum_{t=1}^{s} p_t(n)\frac{\alpha^{n-j_t} - \beta^{n-j_t}}{\sqrt{5}}\\ &= \sum_{t=1}^{s} \left(\frac{p_t(n)}{\alpha^{j_t}}\frac{\alpha^n}{\sqrt{5}} - \frac{p_t(n)}{\beta^{j_t}}\frac{\beta^n}{\sqrt{5}}\right). \end{aligned}$$

Then there exist polynomials $q_\alpha(x), q_\beta(x) \in \mathbb{C}[x]$ (if $p_j(x) \in \mathbb{Q}[x]$, then $q_\alpha(x)$ and $q_\beta(x)$ are from $\mathbb{Q}(\alpha)[x]$) such that

$$W_n = q_\alpha(n)\alpha^n - q_\beta(n)\beta^n.$$

Clearly,

$$q_\alpha(n) = \sum_{t=1}^{s} \frac{p_t(n)}{\sqrt{5}\alpha^{j_t}}, \quad q_\beta(n) = \sum_{t=1}^{s} \frac{p_t(n)}{\sqrt{5}\beta^{j_t}}.$$

Let $d_\alpha = \deg(q_\alpha(x))$ and $d_\beta = \deg(q_\beta(x))$. Put $\tilde{d} = d_\alpha + d_\beta + 2$, which gives the order of the recursive sequence $\{W_n\}_{n=0}^\infty$. The characteristic polynomial of $\{W_n\}_{n=0}^\infty$ is

$$w(x) = (x-\alpha)^{d_\alpha+1}(x-\beta)^{d_\beta+1} = (x^2-x-1)^{d_{\alpha\beta}+1}(x-\alpha)^{d_\alpha-d_{\alpha\beta}}(x-\beta)^{d_\beta-d_{\alpha\beta}},$$

where $d_{\alpha\beta} = \min\{d_\alpha, d_\beta\}$. Note that at least one of $d_\alpha - d_{\alpha\beta}$ and $d_\beta - d_{\alpha\beta}$ is zero. Before investigating the principal problem we analyse the question of equality of degrees d_α and d_β. In the case $s = 2$, $j_1 = 0$, $j_2 = 1$, the example

$$W_n = (n+1)F_n + (-\alpha n + 2)F_{n-1},$$

where $p_1(n) = n + 1$, $p_2(n) = -\alpha n + 2$ admits $q_\alpha(n) = 1$ and $q_\beta(n) = \alpha n - 1$, so it might happen that d_α differs from d_β. In the example above, the coefficients are not from $\mathbb{Q}$ but from $\mathbb{Q}(\sqrt{5})$, and this is the reason why

$d_\alpha \neq d_\beta$ may happen. The situation differs when we assume $p_j(x) \in \mathbb{Q}[x]$ for all possible j. In this case, one can show easily that $d_\alpha = d_\beta$. Here we skip the proof because it is rather technical, but we point on the crucial point. The leading coefficient of $q_\alpha(x)$ and $q_\beta(x)$ are conjugates in $\mathbb{Q}(\sqrt{5})$, so they can vanish only together.

2.2.2 | Specific cases with equal degrees

In the sequel, suppose that the coefficient polynomials are from $\mathbb{Q}$, i.e. $d_\alpha = d_\beta$. Consequently $d_{\alpha\beta} = d_\alpha = d_\beta$, and then $\tilde{d} = 2(d_{\alpha\beta} + 1)$ holds, moreover

$$w(x) = (x^2 - x - 1)^{d_{\alpha\beta}+1}.$$

We note in advance that the method we use in the following cases can be applied for other given coefficient polynomials $p_j(x)$. We always obtain a system of parametric linear equations, where the unknowns are the coefficients of the polynomials $p_j(x)$ and their multipliers come from the initial values of $\{W_n\}$. The evaluation of the solution leads to the desired conditions.

Case $s = 2$, $j_1 = 0$, $j_2 = 1$, $d_1 = d_2 = 1$

Assume that $a \neq 0$, b, $c \neq 0$, d are rational numbers and

$$W_n = (an + b)F_n + (cn + d)F_{n-1}. \tag{2.12}$$

Following the list of equivalent transformations in (2.11) it leads to

$$W_n = \frac{(a\alpha + c)n + (b\alpha + d)}{\alpha\sqrt{5}}\alpha^n - \frac{(a\beta + c)n + (b\beta + d)}{\beta\sqrt{5}}\beta^n,$$

the initial values are

$$\begin{aligned} W_0 &= d, \\ W_1 &= a + b, \\ W_2 &= 2a + b + 2c + d, \\ W_3 &= 6a + 2b + 3c + d. \end{aligned}$$

The characteristic polynomial of $\{W_n\}$ is

$$w(x) = (x-\alpha)^2(x-\beta)^2 = (x^2-x-1)^2 = x^4-2x^3-x^2+2x+1,$$

hence the recurrence relation

$$W_n = 2W_{n-1} + W_{n-2} - 2W_{n-3} - W_{n-4} \tag{2.13}$$

holds for $n \geq 4$.

Now we investigate what rational coefficients a, b, c and d guarantee the integrity of $\{W_n\}$. Clearly, the initial values W_0, W_1, W_2 and W_3 must be integer. Consequently, $d = W_0$ must be integer, further solving the system

$$\begin{aligned} z_1 &= a+b, \\ z_2 &= 2a+b+2c, \\ z_3 &= 6a+2b+3c \end{aligned}$$

in a, b, c with arbitrary integer parameters $z_1 = W_1$, $z_2 = W_2 - d$, $z_3 = W_3 - d$ we obtain

$$a = \frac{-z_1 - 3z_2 + 2z_3}{5}, \quad b = \frac{6z_1 + 3z_2 - 2z_3}{5}, \quad c = \frac{-2z_1 + 4z_2 - z_3}{5}.$$

This result, together with (2.13) guarantees that $\{W_n\}$ is an integer sequence. Hence we proved

Theorem 5 (K. Liptai, L. Németh, T. Szakács and L. Szalay [26], 2024)**.** *The terms*

$$W_n = (an+b)F_n + (cn+d)F_{n-1}$$

form an integer sequence $\{W_n\}$ *if and only if* d *is integer and*

$$a = \frac{-z_1 - 3z_2 + 2z_3}{5}, \quad b = \frac{6z_1 + 3z_2 - 2z_3}{5}, \quad c = \frac{-2z_1 + 4z_2 - z_3}{5},$$

where z_1, z_2, z_3 *are integers, too.*

Note that once we have $d \in \mathbb{Z}$ and $a, b, c \in \mathbb{Q}$ are so as given in the theorem above, then the initial values of the recursive sequence $\{W_n\}$ of order four are $W_0 = d$, $W_1 = z_1$, $W_2 = z_2 + d$, and $W_3 = z_3 + d$. Thus the integer sequence (2.13) with suitable initial values has also an other interpretation given by (2.12). For example, let $d = 0$, moreover $z_1 = z_2 = 1$, $z_3 = 3$. In this case, we get the integer sequence

$$W_n = \frac{2n+3}{5}F_n - \frac{n}{5}F_{n-1},$$

which is the sequence (1.2) in the introduction.

Case $s = 2$, $j_1 = 0$, $j_2 = 1$, $d_1 = d_2 = 2$

This part is devoted to study the case when the coefficient polynomials are quadratic. The treatment is analogous to the previous subsection, hence we notify only the results of computations.
Assume that $a \neq 0$, b, c, $d \neq 0$, e, f are rational numbers, and

$$W_n = (an^2 + bn + c)F_n + (dn^2 + en + f)F_{n-1}. \tag{2.14}$$

Now sequence $\{W_n\}$ satisfies

$$\begin{aligned} W_n = & \frac{(a\alpha + d)n^2 + (b\alpha + e)n + (c\alpha + f)}{\alpha\sqrt{5}}\alpha^n - \\ & - \frac{(a\beta + d)n^2 + (b\beta + e)n + (c\beta + f)}{\beta\sqrt{5}}\beta^n, \end{aligned}$$

with initial values

$$\begin{aligned} W_0 &= f, \\ W_1 &= a + b + c, \\ W_2 &= 4a + 2b + c + 4d + 2e + f, \\ W_3 &= 18a + 6b + 2c + 9d + 3e + f, \\ W_4 &= 48a + 12b + 3c + 32d + 8e + 2f, \\ W_5 &= 125a + 25b + 5c + 75d + 15e + 3f. \end{aligned} \tag{2.15}$$

The characteristic polynomial of $\{W_n\}$ is

$$w(x) = (x-\alpha)^3(x-\beta)^3 = (x^2-x-1)^3 = x^6-3x^5+5x^3-3x-1,$$

hence

$$W_n = 3W_{n-1} - 5W_{n-3} + 3W_{n-5} + W_{n-6}. \tag{2.16}$$

Clearly, f must be integer, further eliminating f from system (2.15) and solving it in a, b, c, d, e we obtain

$$\begin{aligned} a &= \frac{-z_1+3z_2+z_3-3z_4+z_5}{10},\\ b &= \frac{-5z_1-75z_2+15z_3+45z_4-17z_5}{50},\\ c &= \frac{30z_1+30z_2-10z_3-15z_4+6z_5}{25},\\ d &= \frac{3z_1-4z_2-3z_3+4z_4-z_5}{10},\\ e &= \frac{-45z_1+80z_2+15z_3-40z_4+11z_5}{50}, \end{aligned} \tag{2.17}$$

where $z_1 = W_1$, $z_2 = W_2 - f$, $z_3 = W_3 - f$, $z_4 = W_4 - 2f$, $z_5 = W_5 - 3f$ arbitrary integer parameters. A summary of the result of this subsection is

Theorem 6 (K. Liptai, L. Németh, T. Szakács and L. Szalay [26], 2024)**.** *The rational coefficients a, b, c, d, e and f determine integer sequences in the form*

$$W_n = (an^2+bn+c)F_n + (dn^2+en+f)F_{n-1}$$

if and only if $f \in \mathbb{Z}$ and a, b, c, d, e are given in (2.17).

Thus the integer sequence (2.16) with suitable initial values has also an other interpretation given by (2.14). For example, let $f = 0$, $z_1 = 0$, $z_2 = 1$, $z_3 = 4$, $z_4 = 12$, and $z_5 = 31$. In this particular case, we get the integer sequence

$$W_n = \frac{5n^2-n-4}{25}F_n + \frac{5n^2+n}{50}F_{n-1},$$

which is equivalent to the result (1.3) given in OEIS [35].

Case: $s=2,\ j_1=0,\ j_2=1,\ d_1=2,\ d_2=1$

Now $a\neq 0$, b, c, $d\neq 0$, e all are in $\mathbb{Q}$, and

$$W_n=(an^2+bn+c)F_n+(dn+e)F_{n-1}.$$

Using the usual technique we obtain that sequence $\{W_n\}$ satisfies

$$\begin{aligned} W_n = &\frac{a\alpha n^2+(b\alpha+d)n+(c\alpha+e)}{\alpha\sqrt{5}}\alpha^n \\ &-\frac{a\beta n^2+(b\beta+d)n+(c\beta+e)}{\beta\sqrt{5}}\beta^n \end{aligned}$$

with initial values

$$\begin{aligned} W_0&=e,\\ W_1&=a+b+c,\\ W_2&=4a+2b+c+2d+e, \\ W_3&=18a+6b+2c+3d+e,\\ W_4&=48a+12b+3c+8d+2e. \end{aligned} \tag{2.18}$$

The characteristic polynomial of $\{W_n\}$ is

$$w(x)=(x-\alpha)^3(x-\beta)^3=(x^2-x-1)^3=x^6-3x^5+5x^3-3x-1,$$

hence

$$W_n=3W_{n-1}-5W_{n-3}+3W_{n-5}+W_{n-6}.$$

Clearly, e must be integer, further eliminating e from (2.18) and solving it in a,b,c,d we obtain

$$\begin{aligned} a&=\frac{2z_1-z_2-2z_3+z_4}{10},\\ b&=\frac{-56z_1-7z_2+66z_3-23z_4}{50},\\ c&=\frac{48z_1+6z_2-28z_3+9z_4}{25},\\ d&=\frac{-6z_1+18z_2-9z_3+2z_4}{25}, \end{aligned} \tag{2.19}$$

where $z_1=W_1$, $z_2=W_2-e$, $z_3=W_3-e$, $z_4=W_4-2e$ arbitrary integer parameters. A summary of the result of this subsection is

Theorem 7 (K. Liptai, L. Németh, T. Szakács and L. Szalay [26], 2024)**.** *The rational coefficients a, b, c, d and e determine integer sequence in the form*

$$W_n = (an^2 + bn + c)F_n + (dn + e)F_{n-1}$$

if and only if e and z_i $(i = 1, \ldots, 4)$ are integers and a, b, c and d given in (2.19).

For example, let $e = z_1 = z_2 = z_3 = 1$ and $z_4 = 2$. In this particular case, we get the integer sequence

$$W_n = \frac{5n^2 - 43n + 88}{50}F_n + \frac{14n + 50}{50}F_{n-1}.$$

This sequence $\{W_n\}_{n=0}^{\infty} = (1, 1, 2, 2, 4, 7, 15, 32, 69, 146, 303, \ldots)$ does not appear in OEIS.

2.2.3 | A modified problem

In the introduction, (1.1) offers a further polynomial $c(x)$. Németh [34] investigated a related question, namely the problem of walks on tiled square boards, and proved, among others, that the tiling-walking sequence $\{r_n\}$ of the $(2 \times n)$-board with only dominoes is recursively given by a sixth order recurrence having explicit form

$$r_n = \frac{4n}{5}F_{n+1} + \frac{3n + 3}{5}F_n + \frac{1}{2} + \frac{1}{2}(-1)^n. \tag{2.20}$$

This is sequence A054454 in OEIS [35].

Our purpose now is to examine the sequence

$$W_n = (an + b)F_n + (cn + d)F_{n-1} + e + f(-1)^n, \tag{2.21}$$

where the coefficients $a, b, \ldots, f$ are rational numbers again in order to have integrity condition for $\{W_n\}$. Since the method is detailed in the previous parts, here we record the statement, and compare it to Németh's equality (2.20).

Theorem 8 (K. Liptai, L. Németh, T. Szakács and L. Szalay [26], 2024)**.** *Let the initial values* $W_0, W_1, \ldots, W_5$ *be integers. If*

$$\begin{aligned}
a &= \frac{3W_0 + 2W_1 - 7W_2 - W_3 + 4W_4 - W_5}{5} \\
b &= \frac{-3W_0 - 2W_1 - 3W_2 + 6W_3 + 6W_4 - 4W_5}{5} \\
c &= \frac{-4W_0 - W_1 + 11W_2 - 2W_3 - 7W_4 - 3W_5}{5} \\
d &= 2W_1 + W_2 + 2W_3 - W_4 \\
e &= \frac{W_0 + 3W_1 + W_2 - 3W_3 - W_4 + W_5}{2} \\
f &= \frac{W_0 + W_1 - 3W_2 - W_3 + 3W_4 - W_5}{2},
\end{aligned}$$

then $W_n = (an + b)F_n + (cn + d)F_{n-1} + e + f(-1)^n$ *is an integer sequence. The reversal of the statement is also true.*

As an example, let $W_0 = 0$, $W_1 = 1$, $W_2 = 2$, $W_3 = 6$, $W_4 = 12$, $W_5 = 26$. Now $a = 4/5$, $b = -4/5$, $c = 3/5$, $d = 0$, $e = 1/2$, $f = -1/2$. Then

$$W_n = \frac{4n - 4}{5}F_n + \frac{3n}{5}F_{n-1} + \frac{1}{2} - \frac{1}{2}(-1)^n.$$

This coincides with (2.20) via $r_n = W_{n+1}$.

Finally, we give a well-known sequence for $f = 0$ in (2.21). The sequence of Leonardo numbers is defined by $L_n = L_{n-1} + L_{n-2} + 1$, with initial terms $L_0 = 1$, $L_1 = 1$ (cited as A001595 in OEIS [35]). It is easy to see that

$$L_n = 2F_n + 2F_{n-1} - 1.$$

2.3 | Convolution

We consider the sequence $\{C_n\}_{n=0}^{\infty}$ given by the convolution of two second order linear recursive sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$:

$$C_n = \sum_{k=0}^{n} G_k H_{n-k},$$

where $\{G_n\}$ and $\{H_n\}$ are Lucas sequences of the first or second kind, see the Binet's formulas (2.1) and (2.2). The applied methods for proofs require the separation of the cases when the characteristic polynomials have or dont't have common root. We give convolution formulas in different cases, where the formulas depend only on the initial terms and on the roots of the characteristic polynomial. After each theorem, we show some special cases in corollaries using the named sequences, listed in Table 2.1 (Fibonacci, Pell, Jacobsthal, Mersenne, Lucas, P-Lucas, J-Lucas, M-Lucas). In the following, we will use the notations:

$$\begin{aligned} a &= (A_1 - A_2)\alpha + B_1 - B_2, \\ b &= (A_1 - A_2)\beta + B_1 - B_2, \\ c &= (A_2 - A_1)\gamma + B_2 - B_1, \\ d &= (A_2 - A_1)\delta + B_2 - B_1, \end{aligned} \tag{2.22}$$

where $bd \neq 0$, α, β and γ, δ are the roots of the characteristic polynomials $p(x) = x^2 - A_1 x - B_1$ and $q(x) = x^2 - A_2 x - B_2$ of $G_n(G_0, G_1, A_1, B_1)$ and $H_n(H_0, H_1, A_2, B_2)$, respectively.

In the proofs, we use the method of partial-fraction decomposition, the generating functions of second order linear recursive sequences, the idea that C_n is the coefficient of x^n in

$$g(x)h(x) = \sum_{n=0}^{\infty} G_n x^n \cdot \sum_{n=0}^{\infty} H_n x^n = \sum_{n=0}^{\infty} C_n x^n,$$

where $g(x)$, $h(x)$ are the generating functions of sequences $\{G_n\}_{n=0}^{\infty}$, $\{H_n\}_{n=0}^{\infty}$ respectively and the following well-known identities.

$$\frac{1}{1-\alpha x} = \sum_{n=0}^{\infty} (\alpha x)^n, \quad (0 < |\alpha x| < 1).$$

$$\begin{aligned}\frac{1}{(1-\alpha x)^2} &= \left(\frac{1}{\alpha(1-\alpha x)}\right)' = \left(\frac{1}{\alpha}\sum_{n=0}^{\infty}(\alpha x)^n\right)' \\ &= \frac{1}{\alpha}\sum_{n=1}^{\infty} n\alpha^n x^{n-1} = \frac{1}{\alpha}\sum_{n=0}^{\infty}(n+1)\alpha^{n+1}x^n \\ &= \sum_{n=0}^{\infty}(n+1)(\alpha x)^n, \quad (0 < |\alpha x| < 1).\end{aligned}$$

The characteristic polynomials have no common root

At first, we suppose that all the roots are real numbers and the characteristic polynomials have no common roots. The following theorem deals with the convolution of two different Lucas sequences of the first kind.

Theorem 9 (T. Szakács [42], 2016). *The convolution of* $G_n(0, 1, A_1, B_1)$ *and* $H_n(0, 1, A_2, B_2)$ *can be written as*

$$C_n = \sum_{k=0}^{n} G_k H_{n-k} = \frac{\frac{\alpha^{n+1}}{a} - \frac{\beta^{n+1}}{b}}{\alpha - \beta} + \frac{\frac{\gamma^{n+1}}{c} - \frac{\delta^{n+1}}{d}}{\gamma - \delta}.$$

Proof of Theorem 9. Using (2.3), the generating functions of the sequences
$G_n(0, 1, A_1, B_1)$ and $H_n(0, 1, A_2, B_2)$ are

$$g(x) = \frac{x}{1 - A_1 x - B_1 x^2} = \frac{x}{(1-\alpha x)(1-\beta x)}$$

and

$$h(x) = \frac{x}{1 - A_2 x - B_2 x^2} = \frac{x}{(1-\gamma x)(1-\delta x)},$$

where α, β and γ, δ are the roots of the characteristic polynomials of $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$, respectively. The generating functions can be written as (by the method of partial-fraction decomposition)

$$g(x) = \frac{1}{\alpha - \beta}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)$$

and

$$h(x) = \frac{1}{\gamma - \delta}\left(\frac{1}{1-\gamma x} - \frac{1}{1-\delta x}\right).$$

From this it follows that

$$\begin{aligned}
&g(x)h(x)(\alpha-\beta)(\gamma-\delta) = \\
&\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right)\left(\frac{1}{1-\gamma x} - \frac{1}{1-\delta x}\right) = \\
&\frac{1}{(1-\alpha x)(1-\gamma x)} - \frac{1}{(1-\alpha x)(1-\delta x)} - \frac{1}{(1-\beta x)(1-\gamma x)} + \\
&\frac{1}{(1-\beta x)(1-\delta x)} = \\
&\frac{\frac{\alpha}{\alpha-\gamma}}{1-\alpha x} - \frac{\frac{\gamma}{\alpha-\gamma}}{1-\gamma x} - \frac{\frac{\alpha}{\alpha-\delta}}{1-\alpha x} + \frac{\frac{\delta}{\alpha-\delta}}{1-\delta x} - \frac{\frac{\beta}{\beta-\gamma}}{1-\beta x} + \frac{\frac{\gamma}{\beta-\gamma}}{1-\gamma x} + \frac{\frac{\beta}{\beta-\delta}}{1-\beta x} - \\
&\frac{\frac{\delta}{\beta-\delta}}{1-\delta x} = \\
&\frac{\frac{\alpha(\gamma-\delta)}{(A_1-A_2)\alpha+B_1-B_2}}{1-\alpha x} - \frac{\frac{\beta(\gamma-\delta)}{(A_1-A_2)\beta+B_1-B_2}}{1-\beta x} + \frac{\frac{\gamma(\alpha-\beta)}{(A_2-A_1)\gamma+B_2-B_1}}{1-\gamma x} - \frac{\frac{\delta(\alpha-\beta)}{(A_2-A_1)\delta+B_2-B_1}}{1-\delta x}
\end{aligned}$$

Now using that C_n is the coefficient of x^n in $g(x)h(x)$ and e.g.

$$\frac{1}{1-\alpha x} = \sum_{n=0}^{\infty} (\alpha x)^n,$$

we get

$$C_n = \frac{1}{\alpha-\beta}\left(\frac{\alpha^{n+1}}{(A_1-A_2)\alpha+B_1-B_2} - \frac{\beta^{n+1}}{(A_1-A_2)\beta+B_1-B_2}\right) + \\ + \frac{1}{\gamma-\delta}\left(\frac{\gamma^{n+1}}{(A_2-A_1)\gamma+B_2-B_1} - \frac{\delta^{n+1}}{(A_2-A_1)\delta+B_2-B_1}\right).$$

□

Let us see now the convolution formulas of some famous, named Lucas sequences. In the remarks, the reader can check which one is already known and which is a new formula in the OEIS [35].

Corollary 1. *The convolution of Fibonacci and Pell numbers is:*

$$C_n = \sum_{k=0}^{n} F_k P_{n-k} = P_n - F_n.$$

Remark 1. *In [35], (A106515) it can be found that*

$$C_n = \sum_{k=0}^{n} F_{n-k-1} P_{k+1} = P_n - F_n + P_{n+1},$$

where because of the different indices the term P_{n+1} occures, as well.

Corollary 2. *The convolution of Fibonacci and Jacobsthal numbers is:*

$$C_n = \sum_{k=0}^{n} F_k J_{n-k} = J_{n+1} - F_{n+1}.$$

Remark 2. *In [35], (A094687) the formula*

$$C_n = \sum_{k=0}^{n} F_k J_{n-k} = C_{n-1} + 2C_{n-2} + F_{n-1}$$

can be found. After a short calculation one can easily verify that the two formulas for C_n are the same ones.

Corollary 3. *The convolution of Fibonacci and Mersenne numbers is:*

$$C_n = \sum_{k=0}^{n} F_k M_{n-k} = m_{n+1} - F_{n+4}.$$

Corollary 4. *The convolution of Pell and Jacobsthal numbers is:*

$$C_n = \sum_{k=0}^{n} P_k J_{n-k} = \frac{P_{n+1} + P_n - J_{n+2}}{2}.$$

Corollary 5. *The convolution of Pell and Mersenne numbers is:*

$$C_n = \sum_{k=0}^{n} P_k M_{n-k} = \frac{P_{n+2} + P_{n+1} - M_{n+2}}{2}.$$

In the following theorem, we deal with the convolution of a Lucas sequence of the first and second kind.

Theorem 10 (T. Szakács [42], 2016)**.** *The convolution of* $G_n(0,1,A_1,B_1)$ *and* $H_n(2,A_2,A_2,B_2)$ *can be written as*

$$C_n = \sum_{k=0}^{n} G_k H_{n-k} =$$
$$= \frac{\frac{\alpha^{n+1}(2\alpha-A_2)}{a} - \frac{\beta^{n+1}(2\beta-A_2)}{b}}{\alpha-\beta} + \frac{\frac{\gamma^{n+1}(2\gamma-A_2)}{c} - \frac{\delta^{n+1}(2\delta-A_2)}{d}}{\gamma-\delta}.$$

Corollary 6. *The convolution of Fibonacci and P-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} F_k p_{n-k} = p_n - F_{n-1}.$$

Corollary 7. *The convolution of Fibonacci and J-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} F_k j_{n-k} = j_{n+1} - L_{n+1}.$$

Remark 3. *This our convolution has the same form as of Griffiths and Bramham in [18].*

Corollary 8. *The convolution of Fibonacci and M-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} F_k m_{n-k} = M_{n+1} - F_{n+1}.$$

Remark 4. *The sequence A228078 in [35] defined by* $a(n) = 2^n - F_n - 1 = M_n - F_n$ *and our formula have the same terms (apart from the shifting indices).*

Corollary 9. *The convolution of Pell and Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} P_k L_{n-k} = P_n + p_n - L_n.$$

Corollary 10. *The convolution of Pell and J-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} P_k j_{n-k} = \frac{8P_{n+1} + p_{n+1} - 2j_{n+2}}{4}.$$

Corollary 11. *The convolution of Pell and M-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} P_k m_{n-k} = \frac{4P_{n+2} + p_{n+1} - 2m_{n+2}}{4}.$$

Corollary 12. *The convolution of Jacobsthal and Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} J_k L_{n-k} = j_{n+1} - L_{n+1}.$$

Remark 5. *The convolution of Lucas and Jacobsthal numbers was also investigated by Griffiths and Bramham in [18], the two formulas are the same ones.*

Corollary 13. *The convolution of Jacobsthal and P-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} J_k p_{n-k} = 2(P_{n+1} - J_{n+1}).$$

Corollary 14. *The convolution of Mersenne and Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} M_k L_{n-k} = 3m_{n+1} - L_{n+4} - 2.$$

In the following theorem, we deal with the convolution of two Lucas sequences of the second kind.

Theorem 11 (T. Szakács [42], 2016)**.** *The convolution of the sequences $G_n(2, A_1, A_1, B_1)$ and $H_n(2, A_2, A_2, B_2)$ can be written as*

$$\begin{aligned} C_n &= \sum_{k=0}^{n} G_k H_{n-k} = \\ &= \frac{\frac{\alpha^{n+1}(2\alpha - A_1)(2\alpha - A_2)}{a} - \frac{\beta^{n+1}(2\beta - A_1)(2\beta - A_2)}{b}}{\alpha - \beta} + \\ &+ \frac{\frac{\gamma^{n+1}(2\gamma - A_1)(2\gamma - A_2)}{c} - \frac{\delta^{n+1}(2\delta - A_1)(2\delta - A_2)}{d}}{\gamma - \delta}. \end{aligned}$$

Corollary 15. *The convolution of Lucas and P-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} L_k p_{n-k} = 2F_{n+1} - 6F_n + 2P_{n+1} + 6P_n.$$

Corollary 16. *The convolution of Lucas and J-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} L_k j_{n-k} = 9J_{n+1} - 5F_{n+1}.$$

Corollary 17. *The convolution of Lucas and M-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} L_k m_{n-k} = 3M_{n+1} - L_{n+1} + 2.$$

Corollary 18. *The convolution of P-Lucas and J-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} p_k j_{n-k} = 2P_{n+2} + p_{n+1} - 2j_{n+1}.$$

Corollary 19. *The convolution of P-Lucas and M-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} p_k m_{n-k} = 2P_{n+2} + 4P_{n+1} - M_{n+2} - 1.$$

The characteristic polynomials have one common root

We suppose that $p(\alpha) = q(\alpha) = 0$, $p(\beta) = 0$, $q(\beta) \neq 0$, while $q(\delta) = 0$, $p(\delta) \neq 0$, that is, β and δ are distinct roots, while α is the common root. In the following theorem, we deal with the convolution of two different Lucas sequences of the first kind, that is, when the initial terms are $0, 1$.

Theorem 12 (T. Szakács [43], 2017)**.** *The convolution of* $G_n(0,1,A_1,B_1)$ *and* $H_n(0,1,A_2,B_2)$ *is*

$$C_n = \sum_{k=0}^{n} G_k H_{n-k} = \frac{\alpha^n(n+1) + \alpha^n \frac{B_1+B_2-2\alpha^2}{(\alpha-\beta)(\alpha-\delta)} - \beta^{n+1}\frac{\alpha-\delta}{b} - \delta^{n+1}\frac{\alpha-\beta}{d}}{(\alpha-\beta)(\alpha-\delta)}.$$

Corollary 20. *The convolution of Jacobsthal and Mersenne numbers is:*

$$C_n = \sum_{k=0}^{n} J_k M_{n-k} = \frac{2n + (2n-3)M_n - J_n}{6}.$$

If we use the values of A_1, B_1, A_2, B_2 and the Binet's formula (2.1), then the result of the corollary can be reached from Table 2.1, e.g. in this special case the sequences are $G_n = J_n(0,1,1,2)$ and $H_n = M_n(0,1,3,-2)$.

$$\alpha = 2, \beta = -1, \quad \alpha = 2, \delta = 1.$$

By (2.22), we get that

$$\begin{aligned} b &= 6, \\ d &= -2. \end{aligned}$$

Applying Theorem 12 and (2.1), we get the result

$$\begin{aligned} C_n &= \frac{2^n(n+1) - 2^n \cdot \frac{8}{3} - \frac{(-1)^{n+1}}{6} + \frac{3}{2}}{3} \\ &= \frac{2n \cdot 2^n - \left(2^n \cdot \frac{1}{3} - (-1)^n \cdot \frac{1}{3}\right) - (2^n \cdot 3 - 3)}{6} \\ &= \frac{2n(M_n + 1) - J_n - 3M_n}{6} = \frac{2n + (2n-3)M_n - J_n}{6}. \end{aligned}$$

In the following theorem, we deal with the convolution of a Lucas sequence of the first and second kind, that is, when the initial terms are $0, 1$ and $2, A_2$.

Theorem 13 (T. Szakács [43], 2017)**.** *The convolution of* $G_n(0, 1, A_1, B_1)$ *and* $H_n(2, A_2, A_2, B_2)$ *is*

$$C_n = \sum_{k=0}^{n} G_k H_{n-k} =$$

$$\frac{\alpha^n(n+1)(2\alpha - A_2) + \alpha^n \frac{B_1 - B_2}{\alpha - \beta} - \beta^{n+1} \frac{(\alpha-\delta)(2\beta - A_2)}{b} - \delta^{n+1} \frac{(\alpha-\beta)(2\delta - A_2)}{d}}{(\alpha - \beta)(\alpha - \delta)}.$$

Corollary 21. *The convolution of Jacobsthal and M-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} J_k m_{n-k} = \frac{2n + (2n+3)M_n + 5J_n}{6}.$$

Corollary 22. *The convolution of Mersenne and J-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} M_k j_{n-k} = \frac{2n + (2n-1)M_n + J_n}{2}.$$

In the following theorem, we deal with the convolution of two different Lucas sequences of the second kind, that is, when the initial terms are $2, A_1$ and $2, A_2$.

Theorem 14 (T. Szakács [43], 2017)**.** *The convolution of the sequences* $G_n(2, A_1, A_1, B_1)$ *and* $H_n(2, A_2, A_2, B_2)$ *is*

$$
\begin{aligned}
C_n = \sum_{k=0}^{n} G_k H_{n-k} &= \alpha^n(n+1) + \alpha^n \frac{B_1 + B_2 + 2\alpha^2}{(\alpha-\beta)(\alpha-\delta)} \\
&- \beta^{n+1} \frac{2\beta - A_2}{(\alpha-\beta)(A_1 - A_2)} + \delta^{n+1} \frac{2\delta - A_1}{(\alpha-\delta)(A_1-A_2)}.
\end{aligned}
$$

Corollary 23. *The convolution of J-Lucas and M-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} j_k m_{n-k} = \frac{n + 1 + (n+2)M_{n+1} + 5J_{n+1}}{2}.$$

The characteristic polynomials have two common roots

That is, $p(x) = q(x)$ and so $p(\alpha) = q(\alpha) = 0$, $p(\beta) = q(\beta) = 0$. In the following theorem, we deal with the convolution of a Lucas sequence of the first kind with itself, that is, the initial terms are $0, 1$. Zhang W. in [47] has generalized this type of problem, now we give different formulas.

Theorem 15 (T. Szakács [43], 2017)**.** *The convolution of* $R_n(0, 1, A_1, B_1)$ *with itself is*

$$C_n = \sum_{k=0}^{n} R_k R_{n-k} = \frac{1}{(\alpha-\beta)^2}\Big((n+1)V_n - 2R_{n+1}\Big),$$

where V_n *is the associate sequence of* R_n.

Corollary 24. *The convolution of Fibonacci numbers with themself:*

$$C_n = \sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}\Big((n+1)L_n - 2F_{n+1}\Big).$$

Remark 6. *The formula given by Zhang W. in [47] was the following.*

$$\sum_{a+b=n} F_a F_b = \frac{1}{5}\Big((n-1)F_n + 2nF_{n-1}\Big), \quad n \geq 1.$$

It can be easily verifyed that the two formulas are the same ones using some well known identities between the Fibonacci and Lucas numbers: $2F_{n+1} = F_n + L_n$ *and* $L_n = F_{n-1} + F_{n+1}$*.*

$$\begin{aligned}
&\frac{1}{5}\Big((n+1)L_n - 2F_{n+1}\Big) = \frac{1}{5}\Big((n+1)L_n - L_n - F_n\Big) = \frac{1}{5}\Big(nL_n - F_n\Big) \\
=&\frac{1}{5}\Big(n(F_{n-1} + F_{n+1}) - F_n\Big) = \frac{1}{5}\Big(nF_{n-1} + n(F_n + F_{n-1}) - F_n\Big) \\
=&\frac{1}{5}\Big((n-1)F_n + 2nF_{n-1}\Big).
\end{aligned}$$

S. Vajda in [46] on page 183 gave the same formula for the convolution of Fibonacci numbers like us in Corollary 24.

In the following theorem, we deal with the convolution of a Lucas sequence of the first and second kind, that is, the initial terms are $0, 1$ and $2, A_1$.

Theorem 16 (T. Szakács [43], 2017)**.** *The convolution of* $R_n(0, 1, A_1, B_1)$ *and* $V_n(2, A_1, A_1, B_1)$ *is*

$$C_n = \sum_{k=0}^{n} R_k V_{n-k} = (n+1)R_n.$$

Corollary 25. *The convolution of Fibonacci and Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} F_k L_{n-k} = (n+1)F_n.$$

Remark 7. *The OEIS [35] contains the sequence with id: A099920.*

Corollary 26. *The convolution of Pell and P-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} P_k p_{n-k} = (n+1)P_n.$$

Corollary 27. *The convolution of Jacobsthal and J-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} J_k j_{n-k} = (n+1)J_n.$$

Corollary 28. *The convolution of Mersenne and M-Lucas numbers is:*

$$C_n = \sum_{k=0}^{n} M_k m_{n-k} = (n+1)M_n.$$

Remark 8. *The OEIS [35] contains this sequence as number of labeled acyclic digraphs with n nodes containing exactly $n-1$ points of in-degree zero. (id: A058877)*

In the following theorem, we deal with the convolution of a Lucas sequence of the second kind with itself, that is, the initial terms are $2, A_1$.

Theorem 17 (T. Szakács [43], 2017)**.** *The convolution of the sequence $V_n(2, A_1, A_1, B_1)$ with itself is*

$$C_n = \sum_{k=0}^{n} V_k V_{n-k} = (n+1)V_n + 2R_{n+1},$$

where V_n is the associate sequence of R_n.

Corollary 29. *The convolution of Lucas numbers with themself is:*

$$C_n = \sum_{k=0}^{n} L_k L_{n-k} = (n+1)L_n + 2F_{n+1}.$$

Remark 9. *In the paper of Zhang Z. and He P. [48] Corollary 1 contains another formula for the convolution of Lucas numbers with themself:*

$$\sum_{a+b=n} L_a L_b = \frac{1}{5}\Big(2L_{n+1} + (5n+9)L_n\Big).$$

It can be easily verifyed that the two formulas are the same ones using some well known identities between the Fibonacci and Lucas numbers: $L_{n+2} + L_n = 5F_{n+1}$ *and* $L_{n+2} = L_{n+1} + L_n$.

$$\begin{aligned}
&\frac{1}{5}\Big(2L_{n+1} + (5n+9)L_n\Big) = \frac{1}{5}\Big(2L_{n+1} + 2L_n + 5nL_n + 7L_n\Big)\\
=&\frac{1}{5}\Big(2L_{n+2} + 2L_n + 5nL_n + 5L_n\Big) = \frac{1}{5}\Big(10F_{n+1} + 5nL_n + 5L_n\Big)\\
=&(n+1)L_n + 2F_{n+1}.
\end{aligned}$$

S. Vajda in [46] on page 183 gave the same formula for the convolution of Lucas numbers like us in Corollary 29. The OEIS [35] contains the sequence with the following id: A099924.

3 | Factorials

Consider the equation

$$n! = \prod_{i=1}^{r} a_i! \tag{3.1}$$

with $r \geq 2$ in positive integers $n, a_1, \dots, a_r$, with $a_1 \geq \dots \geq a_r > 1$. Observe that this equation has infinitely many solutions given by

$$n = a_2! \dots a_r!, \ a_1 = n - 1, \text{ with } a_2, \dots, a_r \text{ arbitrary.}$$

For example, we have $6! = 5!3!$ or $12! = 11!3!2!$. Such solutions are called trivial. Obviously, equation (3.1) has infinitely many trivial solutions. On the other hand, according to a conjecture of Surányi, the only non-trivial solution to (3.1) with $r = 2$ is $10! = 7!6!$, while a conjecture of Hickerson predicts that the only non-trivial solutions to (3.1) are given by $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$, $16! = 14!5!2!$ (see e.g. Erdős [10], pp. 27-28). These conjectures have been checked for $n \leq 10^6$ by Caldwell [7]. Erdős [10] (see Theorem 2) proved that writing $P(m)$ for the largest prime factor of the positive integer m (with the convention $P(1) = 1$), the assertion

$$P(n(n+1)) > 4 \log n \tag{3.2}$$

would imply that equation (3.1) has only finitely many non-trivial solutions - however, (3.2) is far from being established. (See also [13], p. 70.) Luca [28] proved that assuming the *abc*-conjecture, (3.1) has only finitely many solutions. This result (beside obtaining other related theorems) has been made more explicit by Luca, Saradha and Shorey [29].
We also mention that after multiplying both sides of (3.1) by $n!$, we get an equation of the form

$$n! \prod_{i=1}^{r} a_i! = y^2. \tag{3.3}$$

This equation also attracted a lot of attention. For related results, here we only mention a classical paper of Erdős and Graham [12] together with the recent paper of Luca, Saradha and Shorey [29], and the references there.

3.1 | On the equation A!B!=C!

We consider the case $r = 2$, and rewrite equation (3.1) as

$$A!B! = C! \tag{3.4}$$

with positive integers A, B, C satisfying $C \geq B \geq A > 1$. As we noted already, the problem of finding all solutions to equation (3.4) is still open. Beside the results mentioned so far, we recall a theorem of Erdős [11] saying that in all solutions of (3.4) with C large enough, we have $C - B \leq 5 \log\log C$. This result has been recently sharpened by Bath and Ramachandra [5] to $C - B \leq ((1+\varepsilon)/\log 2) \log\log C$ for $C > C_\varepsilon$, with arbitrary $\varepsilon > 0$. Recalling the result of Luca, under the *abc*-conjecture we have $C - B = 1$ for C large enough. Note that, however, if we would assume that say $C - B = 2$, equation (3.4) would still remain very hard to solve. In this direction, we only refer to a paper of Luca [27] and the references there.

Our purpose is to show the finiteness of the solutions to (3.4) with $k = B - A$ bounded. Our main result provides an explicit upper bound for C in terms of k. Certainly, this immediately implies that for any fixed k, (3.4) has only finitely many solutions. Further, we show that the only non-trivial solution to (3.4) with $k \leq 10^6$ is the well-known $10! = 7!6!$, mentioned earlier.

Theorem 18 (L. Hajdu, Á. Papp and T. Szakács [20], 2018). *Writing* $k = B - A$ *for all non-trivial solutions of equation* (3.4) *different from* $(A, B, C) = (6, 7, 10)$ *we have* $C < 5k$. *Further, if* $k \leq 10^6$, *then the only non-trivial solution to* (3.4) *is given by* $(A, B, C) = (6, 7, 10)$.

To prove the theorem, we need some lemmas. The first one provides explicit lower- and upper bounds for the prime counting function $\pi(x)$.

Lemma 1. *We have*

(i) $\frac{x}{\log x}\left(1+\frac{1}{2\log x}\right) < \pi(x)$ *for* $x \geq 59$,

(ii) $\pi(x) < \frac{x}{\log x}\left(1+\frac{3}{2\log x}\right)$ *for* $x \geq 1$.

Proof. Parts (i) and (ii) are formulas (3.1) and (3.2), respectively in *Approximate formulas for some functions of prime numbers* from Rosser and Schoenfeld [39]. □

We shall also need an explicit estimate for the number of primes in an interval.

Lemma 2. *Let* $M > 0$ *and* $N > 1$. *Then we have*

$$\pi(M+N) - \pi(M) \leq 2\pi(N).$$

Proof. The statement is formula (1.12) in *The large sieve* from Montgomery and Vaughan [33]. □

We shall also need bounds for the n-th prime p_n.

Lemma 3. *We have*

(i) $n(\log n + \log\log n - 3/2) < p_n$ *for* $n \geq 2$,

(ii) $p_n < n(\log n + \log\log n - 1/2)$ *for* $n \geq 20$.

Proof. Parts (i) and (ii) are formulas (3.10) and (3.11) in [39], respectively. □

We need a simple variant of the Stirling-formula, too.

Lemma 4. *For all* $n \geq 1$ *we have*

$$\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n} \leq n! \leq e \cdot n^{n+1/2} \cdot e^{-n}.$$

Proof. For $n = 1$ the assertion can be readily checked. For $n \geq 2$ the statement immediately follows from the more refined bounds

$$\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n+1/(12n+1)} < n! < \sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n+1/12n}$$

given by Robbins [38]. □

Now, we have all the tools to prove our theorem.

Proof of Theorem 18. In view of $C > B$, the equation $A!B! = C!$ can be rewritten as

$$A! = (B+1)\cdots C. \tag{3.5}$$

Observe that no prime p with $C/2 < p \leq C$ can appear on either side of the above equation. That is, all such primes must belong to the interval $(A, B]$. So for all solutions A, B, C we have

$$\pi(C) - \pi(C/2) \leq \pi(B) - \pi(A).$$

Using parts (i) and (ii) of Lemma 1 to bound the left hand side and Lemma 2 to bound the right hand side of the above inequality, recalling the notation $k = B - A$ we obtain

$$\frac{C}{\log C}\left(1 + \frac{1}{2\log C}\right) - \frac{C/2}{\log C/2}\left(1 + \frac{3}{2\log C/2}\right) < \frac{2k}{\log k}. \tag{3.6}$$

(Note that here we tacitly assumed that $C \geq 59$, whence $k \geq 2$. However, $C < 59$ would be a much better bound for C than the one we get by the general argument.) If contrary to what we want to prove, $C \geq 5k$ would hold, then (3.6) would imply

$$\frac{C}{\log C}\left(1 + \frac{1}{2\log C}\right) - \frac{C/2}{\log C/2}\left(1 + \frac{3}{2\log C/2}\right) < \frac{2C/5}{\log 2C/5}.$$

It is obvious that for large C, the above inequality cannot hold. A simple calculation with Magma [6] shows that this is the case whenever $C > 10^6$. However, by Caldwell's result [7] mentioned earlier, we know that the only solution to (3.4) with $C \leq 10^6$ is given by $(A, B, C) = (6, 7, 10)$. Hence we get that apart from this solution we always have $C < 5k$, and the first part of the theorem follows.

Now we consider the second statement. Assume first that $k \leq 850000$. Observe that in (3.5), none of $B+1, B+2, \ldots, C$ can be a prime. Let p_{n+1} be the first prime greater than C. By Bertrand's postulate we have that $p_n > C/2$. This shows that $A < p_n \leq B$ must be valid. Thus by (3.4), we obtain that

$$(p_n + 1) \cdots (p_{n+1} - 1) \geq (B + 1)(B + 2) \cdots C = A! \geq (p_n - k)! \ . \quad (3.7)$$

Now using Lemmas 3, 4 and $k \leq 850000$, a simple Magma calculation gives $n+1 \leq 78200$, whence $C \leq 10^6$. So in this case the theorem follows from the result of Caldwell [7].
Assume now that $850000 < k \leq 10^6$. Then by what we have proved already, we get $C < 5k \leq 5 \cdot 10^6$. By Caldwell's result we may also assume that $10^6 < C$. By a simple Magma program we get that the length of the longest prime-free interval inside $(10^6, 5 \cdot 10^6)$ is 153. If $C > 1000507$ (which is a prime), then $A \geq 507$. However, then

$$10^{1153} < 507! \leq A! = (B + 1)(B + 2) \cdots C < (5 \cdot 10^6)^{153} < 10^{1071}$$

yields a contradiction. So we are left with the cases $1000000 < C < 1000507$. Writing $p_n < C < p_{n+1}$, based upon (3.7) we must have

$$(p_n + 1) \cdots (p_{n+1} - 1) \geq (p_n - 10^6)! \ .$$

Checking the few possibilities corresponding to the remaining values of C by Magma, we obtain that C must belong to one of the intervals

$$(999983, 1000003), \quad (1000003, 1000033), \quad (1000039, 1000081).$$

Note that the endpoints of each of the above intervals are certainly consecutive primes. To exclude these cases, observe that if a prime p divides the product $(B+1)(B+2)\ldots C$, then (3.4) implies that all primes q with $q < p$ must also divide this product. However, a simple check by Magma assures that it is impossible to find such products with terms in any of the above intervals. Hence the theorem follows. □

3.2 | Multiplying balancing numbers

In this section, we study a further approach of balancing numbers. A positive integer n is called a multiplying balancing number if

$$1 \cdot 2 \cdots (n - 1) = (n + 1)(n + 2) \cdots (n + r) \quad (3.8)$$

for some positive integer r. The number r is called the balancer corresponding to the multiplying balancing number n.
We will show that the only multiplying balancing number is $n = 7$ with the balancer $r = 3$. This means that $6! = 8 \cdot 9 \cdot 10$ is the only solution of the equation (3.8). Notice that this is the same solution as before we have in Theorem 18. If we multiply the equation (3.8) by $n!$, we get $(n-1)!n! = (n+r)!$ which is an equation of the type $A!B! = C!$, with $k = B - A = 1$.

Let us use the function $\alpha_2 \colon \mathbb{N} \to \mathbb{N}$, $\alpha_2(x) := \sum_{k=1}^{[\log_2 x]} \left[\frac{x}{2^k}\right]$, where $x \geq 2$ and $\alpha_2(x)$ calculates the exponent of 2 in the canonical form of $x!$.

Lemma 5 (T. Szakács [41], 2011)**.** $x - \log_2 x - 2 < \alpha_2(x) < x$

Proof.

$$\alpha_2(x) = \left[\frac{x}{2^1}\right] + \left[\frac{x}{2^2}\right] + \left[\frac{x}{2^3}\right] + \cdots + \left[\frac{x}{2^k}\right] \leq \frac{x}{2^1} + \frac{x}{2^2} + \frac{x}{2^3} + \cdots + \frac{x}{2^k} =$$
$$= x\left(\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right) = x\left(1 - \frac{1}{2^k}\right) \leq x - 1 < x$$
$$\alpha_2(x) > \underbrace{\left(\frac{x}{2^1} - 1\right) + \left(\frac{x}{2^2} - 1\right) + \cdots + \left(\frac{x}{2^k} - 1\right)}_{[\log_2 x]} =$$
$$= x\left(1 - \frac{1}{2^k}\right) - [\log_2 x] = x - \frac{x}{2^k} - [\log_2 x] > x - \log_2 x - 2$$

☐

Lemma 6 (T. Szakács [41], 2011)**.** *Let n be a multiplying balancing number and r be the balancer. If $n > 64$, then*

$$\frac{3(n+1)}{2} < n + r.$$

Proof. The equation (3.8) could be rewritten as $(n-1)! \cdot (n)! = (n+r)!$. Now using the function $\alpha_2(n)$ and Lemma 5 for both sides of the equation, we get

$$\alpha_2(n-1) + \alpha_2(n) = \alpha_2(n+r)$$

$$2n - 2\log_2 n - 5 < n + r$$

We can replace $\log_2 n$ with $\frac{n}{8}$ if $n > 64$, that is

$$\frac{3(n+1)}{2} < \frac{3(n+1)}{2} + \frac{2n}{8} - 6.5 = 2n - \frac{2n}{8} - 5 < n + r$$

□

Using a results of M. El Bachraoui [9] we get that, if $n \geq 2$ then there exists a p prime satisfying the inequality

$$n < p < \frac{3(n+1)}{2} < n + r.$$

Hence the right side of the equation (3.8) contains a prime factor if $n > 64$. This is impossible, so the equation cannot be true. It can be checked easily by any computer algebra system (CAS), for example Magma, that for $n = 2, \ldots, 64$ there is only one number satisfying the equation (3.8). So we get that $n = 7$, is the only multiplying balancing number.

Multiplying cobalancing number could be defined similarly, but let's first see if they exist at all. Using the concept of multiplying balancing numbers, we get the following equation.

$$1 \cdot 2 \cdots (n-1)n = (n+1)(n+2) \cdots (n+r) \tag{3.9}$$

Multiplying both sides of the equation by $n!$, we get a familiar one (3.3), mentioned earlier. This is the case when a factorial equals a perfect square. This problem has been solved in 1975 by P. Erdős and J.L. Selfridge (see paper [14]). So the multiplying cobalancing numbers don't exist, but let us show an alternative proof for this statement.

Let p be the greatest odd prime, which is less than n, where $n \geq 4$. Using our notation, we show that the following inequalities are true

$$p < n < 2p \leq n + r < 3p.$$

Suppose that $n \geq 2p$. The interval $[p, 2p]$ always contains a prime, so there is a prime greater than p and lower than n which is impossible because of the definition of p. Hence $n < 2p$. On both sides of the equation (3.9) the exponent of p is 1 in the prime decomposition. So we can write the following inequalities $2p \leq n + r < 3p$.

Using a result of Chebyshev we get that there is a prime z between p and $2p$. Three cases have to be analyzed now: $z = n$, $z > n$, and $z < n$. If $z > n$, then the prime decomposition of the left and right sides is not the same. Now let $z < n$. This situation contradicts the fact that p is the greatest odd prime which is less than n. The last case is $z = n$. Hence $n + r \geq 2z$ because of the prime factor z. Thus the left side of the equation (3.9) has at most as many factors as the right side has which is impossible. First and last there are no multiplying cobalancing numbers.

Using the concept of K. Liptai, F. Luca, Á. Pintér and L. Szalay [25] we can get another approach to balancing numbers. Let m, k, l be fixed positive integers with $m \geq 4$. A positive integer n with $n \leq m - 2$ is called a (k, l)-power multiplying balancing number for m if

$$1^k \cdots (n-1)^k = (n+1)^l \cdots (m-1)^l.$$

Using the previous ideas, it could be easily seen that if $n \geq 4$, then there is only one (k, l)-power multiplying balancing number, which is $n = 7$ with $m = 11$ and $k = l$.

Remark 10. *If $p = 2$ and $n = 3$ we get the equation*

$$1^k \cdot 2^k = 4^l.$$

In this case $n = 3$ is (k, l)-power numerical center for $m = 5$ and there are infinitely many (k, l) pairs with $k = 2l$.

4 | Summary

The thesis presents own results in the field of linear recursive sequences and factorials, embedded among the already known results and open questions of the given topics.

In the second chapter, we discuss the results of linear recursive sequences, so let's first review the important definitions related to the topic.
Let $A_0, A_1, \ldots, A_{k-1}$ be given real numbers with $A_{k-1} \neq 0$, where $k \geq 2$ fixed integer. A linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ of order k is defined by the recurrence

$$G_n = A_0 G_{n-1} + A_1 G_{n-2} + \cdots + A_{k-1} G_{n-k} \qquad (n \geq k),$$

where the initial terms $G_0, G_1, \ldots, G_{k-1}$ are fixed real numbers with $|G_0| + |G_1| + \cdots + |G_{k-1}| \neq 0$. The polynomial

$$p(x) = x^k - A_0 x^{k-1} - A_1 x^{k-2} - \cdots - A_{k-2} x - A_{k-1}$$

is said to be the characteristic polynomial of the sequence $\{G_n\}_{n=0}^{\infty}$, the roots of the equation $p(x) = 0$ are denoted by α_i's $(1 \leq i \leq k)$. The root α_1 is of the largest absolute value, that is, $|\alpha_1| > |\alpha_2| \geq \cdots \geq |\alpha_k| > 0$ and the multiplicity of α_1 is 1. According to the literature, α_1 is called as the dominant root, and if we denote by m_i the multiplicity of the distinct α_i's $(1 \leq i \leq l, \sum_{i=1}^{l} m_i = k)$ then the Binet's formula for the term G_n is as follows

$$G_n = a\alpha_1^n + p_2(n)\alpha_2^n + p_3(n)\alpha_3^n + \cdots + p_l(n)\alpha_l^n,$$

where the degree of the polynomial p_i $(2 \leq i \leq l)$ is less than m_i. The constant $a \neq 0$ and the polynomials p_i belong to the ring $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_l)[x]$. In the special case $k = 2$, we get the second order linear recursive sequence $\{G_n\}_{n=0}^{\infty}$ defined by the recurrence relation

$$G_n = AG_{n-1} + BG_{n-2} \quad (n \geq 2),$$

where the initial terms G_0, G_1 and the weights A, B are fixed real numbers with $|G_0| + |G_1| \neq 0$ and $AB \neq 0$. Sometimes the following notation $G_n(G_0, G_1, A, B)$ is used, too. Using the roots of the characteristic polynomial $p(x) = x^2 - Ax - B$, the Binet's formula could be given as

$$G_n = \frac{G_1 - \beta G_0}{\alpha - \beta}\alpha^n - \frac{G_1 - \alpha G_0}{\alpha - \beta}\beta^n.$$

If $R_0 = 0$ and $R_1 = 1$ then $\{R_n\}_{n=0}^{\infty}$ is known as Lucas sequence of the first kind with its Binet's formula

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

while if $V_0 = 2$ and $V_1 = A$ then the sequence is known as Lucas sequence of the second kind $\{V_n\}_{n=0}^{\infty}$ with its Binet's formula

$$V_n = \alpha^n + \beta^n.$$

The generating function of $\{G_n\}$ is

$$g(x) = \frac{G_0 + (G_1 - AG_0)x}{1 - Ax - Bx^2}.$$

There are some well-known Lucas sequences of the first kind, such as Fibonacci, Pell, Jacobsthal, Mersenne, and their associate sequences. The following table contains the initial terms, characteristic polynomials and generating functions of these sequences.

Name	$G_n(G_0, G_1, A, B)$	Charact. polynom.	Gen. function
Fibonacci	$F_n(0,1,1,1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{x}{1-x-x^2}$
Pell	$P_n(0,1,2,1$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{x}{1-2x-x^2}$
Jacobsthal	$J_n(0,1,1,2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{x}{1-x-2x^2}$
Mersenne	$M_n(0,1,3,-2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{x}{1-3x+2x^2}$
Lucas	$L_n(2,1,1,1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{2-x}{1-x-x^2}$
P-Lucas	$p_n(2,2,2,1)$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{2-2x}{1-2x-x^2}$
J-Lucas	$j_n(2,1,1,2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{2-x}{1-x-2x^2}$
M-Lucas	$m_n(2,3,3,-2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{2-3x}{1-3x+2x^2}$

Table 4.1: Named sequences

One of the most famous of these second order linear recursive sequences is the Fibonacci sequence which was obtained when solving a 13th-century problem. Let $F_0 = 0$, $F_1 = 1$, and the recurrence relation is

$$F_n = F_{n-1} + F_{n-2},$$

furthermore the dominant root of the characteristic polynomial is $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio and the other root is $\psi = \frac{1-\sqrt{5}}{2}$. We also know that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

We investigate linear recursive sequences $\{G_n\}_{n=1}^{\infty}$ of real numbers, where the sequences $\left\{\frac{G_{n+1}}{G_n}\right\}$ converge quicker to the golden ratio than $\left\{\frac{F_{n+1}}{F_n}\right\}$. According to the literature, let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be convergent sequences of real numbers with $\lim\limits_{n\to\infty} x_n = y_n = z$, we say that $\{y_n\}$ converges quicker to z than $\{x_n\}$ if

$$\lim_{n\to\infty} \frac{y_n - z}{x_n - z} = 0.$$

In general, we obtain that a sequence $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ will converge faster than the quotient sequence of adjacent terms of the Fibonacci numbers if the absolute value of all roots of the characteristic polynomial of $\{G_n\}$ are less than $\psi = \frac{1-\sqrt{5}}{2}$.

Theorem 1 (T. Szakács [44], 2017). *Let $\{G_n\}_{n=1}^{\infty}$ be a linear recursive sequence of order k, the dominant root of the characteristic polynomial be φ, and the other roots be α_i. The sequence $\left\{\frac{G_{n+1}}{G_n}\right\}$ converges quicker to the golden ratio than $\left\{\frac{F_{n+1}}{F_n}\right\}$, if $|\alpha_i| < |\psi|, i = 2, 3 \dots, l$.*

Next, sequences are studied whose terms are product of rational polynomials and Fibonacci numbers. Let $0 \le j_1 < j_2 < \cdots < j_s$ be nonnegative integers, and $p_1(x), p_2(x), \dots, p_s(x) \in \mathbb{Q}[x]$ such that $\deg(p_i(x)) = d_i$.

Put $d^\star = \max_i\{d_i\}$, which is a nonnegative integer. Define the sequence $\{W_n\}_{n=0}^\infty$ by

$$W_n = p_1(n)F_{n-j_1} + p_2(n)F_{n-j_2} + \cdots + p_s(n)F_{n-j_s}.$$

We give conditions for the rational functions $p_i(x)$ to guarantee sequence $\{W_n\}$ to be integer. Only the first case is discussed here, the interested reader will find the rest in the section 2.2.2.

Case 1: $s = 2$**,** $j_1 = 0$**,** $j_2 = 1$**,** $d_1 = d_2 = 1$
Assume that $a \neq 0$, b, $c \neq 0$, d are rational numbers and

$$W_n = (an + b)F_n + (cn + d)F_{n-1}.$$

Equivalent transformations lead to

$$W_n = \frac{(a\alpha + c)n + (b\alpha + d)}{\alpha\sqrt{5}}\alpha^n - \frac{(a\beta + c)n + (b\beta + d)}{\beta\sqrt{5}}\beta^n,$$

the initial values are

$$\begin{aligned} W_0 &= d, \\ W_1 &= a + b, \\ W_2 &= 2a + b + 2c + d, \\ W_3 &= 6a + 2b + 3c + d. \end{aligned}$$

The characteristic polynomial of $\{W_n\}$ is

$$w(x) = (x - \alpha)^2(x - \beta)^2 = (x^2 - x - 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1,$$

hence the recurrence relation

$$W_n = 2W_{n-1} + W_{n-2} - 2W_{n-3} - W_{n-4}$$

holds for $n \geq 4$.

Theorem 2 (K. Liptai, L. Németh, T. Szakács and L. Szalay [26], 2024)**.** *The terms*

$$W_n = (an + b)F_n + (cn + d)F_{n-1}$$

form an integer sequence $\{W_n\}$ *if and only if* d *is integer and*

$$a = \frac{-z_1 - 3z_2 + 2z_3}{5}, \quad b = \frac{6z_1 + 3z_2 - 2z_3}{5}, \quad c = \frac{-2z_1 + 4z_2 - z_3}{5},$$

where z_1, z_2, z_3 *are integers, too.*

Remark 1. *For example, let* $d = 0$*, moreover* $z_1 = z_2 = 1$*,* $z_3 = 3$*. In this case, we get the integer sequence*

$$W_n = \frac{2n+3}{5} F_n - \frac{n}{5} F_{n-1},$$

which is the sequence A010049 in the OEIS [35].

Later, we discuss the convolution of two second order linear recursive sequences $\{G_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$ which could be denoted by the sequence

$$C_n = \sum_{k=0}^{n} G_k H_{n-k}.$$

We work with Lucas sequences of the first and second kind $G_n(0, 1, A_1, B_1)$ and $H_n(2, A_1, A_2, B_2)$, respectively. (See some named sequences in Table 4.1.) There are several different cases depending on how many common roots the characteristic polynomials have and which types of sequences is being used for the convolution. The only case mentioned here is when the polynomials do not have a common root, and both sequences are Lucas sequences of the first kind. For other cases, see section 2.3.

Theorem 3 (T. Szakács [42], 2016)**.** *Let* $p(x) = x^2 - A_1 x - B_1$ *and* $q(x) = x^2 - A_2 x - B_2$ *be the characteristic polynomials of* $G_n(G_0, G_1, A_1, B_1)$ *and* $H_n(H_0, H_1, A_2, B_2)$*, and let* α, β *and* γ, δ *be the roots of these polynomials, respectively. The convolution of* $G_n(0, 1, A_1, B_1)$ *and* $H_n(0, 1, A_2, B_2)$ *can be written as*

$$C_n = \sum_{k=0}^{n} G_k H_{n-k} = \frac{\frac{\alpha^{n+1}}{a} - \frac{\beta^{n+1}}{b}}{\alpha - \beta} + \frac{\frac{\gamma^{n+1}}{c} - \frac{\delta^{n+1}}{d}}{\gamma - \delta},$$

where $bd \neq 0$, *and*

$$\begin{aligned} a &= (A_1 - A_2)\alpha + B_1 - B_2, \\ b &= (A_1 - A_2)\beta + B_1 - B_2, \\ c &= (A_2 - A_1)\gamma + B_2 - B_1, \\ d &= (A_2 - A_1)\delta + B_2 - B_1. \end{aligned}$$

As a consequence of the above theorem, let's look at two convolutions, one of which was already included in the OEIS before the publication of the result, and the other was not.

Corollary 1. *The convolution of Fibonacci and Pell numbers is:*

$$C_n = \sum_{k=0}^{n} F_k P_{n-k} = P_n - F_n.$$

Remark 2. *In the OEIS, (A106515) it can be found that*

$$C_n = \sum_{k=0}^{n} F_{n-k-1} P_{k+1} = P_n - F_n + P_{n+1},$$

where because of the different indices the term P_{n+1} *occures, as well.*

Corollary 2. *The convolution of Pell and Mersenne numbers is:*

$$C_n = \sum_{k=0}^{n} P_k M_{n-k} = \frac{P_{n+2} + P_{n+1} - M_{n+2}}{2}.$$

Remark 3. *In the OEIS, (A307572), the sequence has been included since* 2019*, but it is defined not as a convolution.*

In the third chapter, factorials were investigated. Consider the equation

$$n! = \prod_{i=1}^{r} a_i!$$

with $r \geq 2$ in positive integers $n, a_1, \ldots, a_r$, with $a_1 \geq \cdots \geq a_r > 1$. According to a conjecture of Surányi, the only non-trivial solution with $r = 2$ is $10! = 7!6!$, while a conjecture of Hickerson predicts that the only non-trivial solutions are given by $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$, $16! = 14!5!2!$ (see e.g. Erdős [10], pp. 27-28). These conjectures have been checked for $n \leq 10^6$ by Caldwell [7]. Rewrite the original equation into the following form, where A, B, C are positive integers:

$$A!B! = C!$$

Here we deal with the case $r = 2$, especially when $k = B - A$ is fixed. Our main result provides an explicit upper bound for C in terms of k. Certainly, this immediately implies that for any fixed k, the equation has only finitely many solutions. Further, we show that the only non-trivial solution with $k \leq 10^6$ is the well-known $10! = 7!6!$.

Theorem 4 (L. Hajdu, Á. Papp and T. Szakács [20], 2018)**.** *Let A, B, C be integers and $k = B - A$. For all non-trivial solutions of the equation $A!B! = C!$ different from $(A, B, C) = (6, 7, 10)$, it is true that $C < 5k$. Further, if $k \leq 10^6$, then the only non-trivial solution is given by $(A, B, C) = (6, 7, 10)$.*

A positive integer n is called a multiplying balancing number if

$$1 \cdot 2 \cdots (n-1) = (n+1)(n+2) \cdots (n+r)$$

for some positive integer r. The number r is called the balancer corresponding to the multiplying balancing number n. We will show in section 3.2, that the only multiplying balancing number is $n = 7$ with the balancer $r = 3$. This means that $6! = 8 \cdot 9 \cdot 10$ is the only solution of the equation. If we multiply the equation by $n!$, we get $(n-1)!n! = (n+r)!$ which is an equation of the type $A!B! = C!$, with $k = B - A = 1$.

5 | Összefoglaló

A dolgozatban a lineáris rekurzív sorozatok és faktoriálisok témakörében született saját eredmények szerepelnek, beágyazva az adott terület már ismert eredményei, illetve nyitott kérdései közé.

A második fejezetben lineáris rekurzív sorozatokkal kapcsolatos eredményeket tárgyalunk, így először tekintsük át az ezzel kapcsolatos fontosabb definíciókat.
Legyenek $A_0, A_1, \ldots, A_{k-1}$ adott valós számok, $A_{k-1} \neq 0$, ahol $k \geq 2$ előre rögzített egész szám. Az alábbi rekurzióval definiált $\{G_n\}_{n=0}^{\infty}$ sorozatot k-ad rendű lineáris rekurzív sorozatnak nevezzük

$$G_n = A_0 G_{n-1} + A_1 G_{n-2} + \cdots + A_{k-1} G_{n-k} \qquad (n \geq k),$$

ahol $G_0, G_1, \ldots, G_{k-1}$ rögzített valós számok lesznek a kezdőelemek, és $|G_0| + |G_1| + \cdots + |G_{k-1}| \neq 0$. A

$$p(x) = x^k - A_0 x^{k-1} - A_1 x^{k-2} - \cdots - A_{k-2} x - A_{k-1}$$

polinomot a sorozat karakterisztikus polinomjának nevezzük, a gyökeit pedig α_i-vel jelöljük $(1 \leq i \leq k)$. Legyen a legnagyobb abszolút értékű gyök α_1, azaz $|\alpha_1| > |\alpha_2| \geq \cdots \geq |\alpha_k| > 0$, és legyen a multiplicitása 1. Az ilyen gyököt a szakirodalom domináns gyöknek nevezi. Ha a különböző α_i gyökök multiplicitását m_i-vel jelöljük $(1 \leq i \leq l, \sum_{i=1}^{l} m_i = k)$, akkor a sorozathoz tartozó Binet-formulát a következő egyenlőség adja meg

$$G_n = a\alpha_1^n + p_2(n)\alpha_2^n + p_3(n)\alpha_3^n + \cdots + p_l(n)\alpha_l^n,$$

ahol a p_i polinomok foka $(2 \leq i \leq l)$ kisebb, mint m_i. Az a $(a \neq 0)$ konstans és a p_i polinomok a $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_l)[x]$ gyűrű elemei.
A $k = 2$ esetben másodrendű lineáris rekurzív sorozatot kapunk

$$G_n = AG_{n-1} + BG_{n-2} \quad (n \geq 2),$$

G_0, G_1 kezdőelemekkel, és A, B súlyokkal, ahol $|G_0| + |G_1| \neq 0$ és $AB \neq 0$. Használatos a $G_n(G_0, G_1, A, B)$ jelölés is. A $p(x) = x^2 - Ax - B$ karakterisztikus polinom gyökeinek segítségével felírható a sorozat Binet-formulája

$$G_n = \frac{G_1 - \beta G_0}{\alpha - \beta}\alpha^n - \frac{G_1 - \alpha G_0}{\alpha - \beta}\beta^n.$$

Ha a kezdőelemek $R_0 = 0$ és $R_1 = 1$, akkor az $\{R_n\}_{n=0}^{\infty}$ sorozatot elsőfajú Lucas-sorozatnak nevezzük, és Binet-formulája

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Ezen sorozat asszociáltja a másodfajú Lucas-sorozat $\{V_n\}_{n=0}^{\infty}$, melynek kezdőtagjai $V_0 = 2$ és $V_1 = A$, ekkor a Binet-formula a következő

$$V_n = \alpha^n + \beta^n.$$

A sorozat generátor függvényének szokás nevezni a következő függvényt

$$g(x) = \frac{G_0 + (G_1 - AG_0)x}{1 - Ax - Bx^2}.$$

A következő táblázatban áttekintjük néhány nevezetes elsőfajú Lucas-sorozat (Fibonacci, Pell, Jacobsthal, Mersenne, és asszociált sorozataik) alapadatait, mint a kezdőtagok, súlyok, karakterisztikus polinom és generátor függvény.

Név	$G_n(G_0, G_1, A, B)$	Karakterisztikus p.	Generátor fv.
Fibonacci	$F_n(0, 1, 1, 1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{x}{1-x-x^2}$
Pell	$P_n(0, 1, 2, 1$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{x}{1-2x-x^2}$
Jacobsthal	$J_n(0, 1, 1, 2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{x}{1-x-2x^2}$
Mersenne	$M_n(0, 1, 3, -2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{x}{1-3x+2x^2}$
Lucas	$L_n(2, 1, 1, 1)$	$p(x) = x^2 - x - 1$	$g(x) = \frac{2-x}{1-x-x^2}$
P-Lucas	$p_n(2, 2, 2, 1)$	$p(x) = x^2 - 2x - 1$	$g(x) = \frac{2-2x}{1-2x-x^2}$
J-Lucas	$j_n(2, 1, 1, 2)$	$p(x) = x^2 - x - 2$	$g(x) = \frac{2-x}{1-x-2x^2}$
M-Lucas	$m_n(2, 3, 3, -2)$	$p(x) = x^2 - 3x + 2$	$g(x) = \frac{2-3x}{1-3x+2x^2}$

Table 5.1: Nevezetes sorozatok

Az egyik leghíresebb ilyen másodrendű sorozat a Fibonacci-sorozat, amely Leonardo Pisano nevéhez fűződik és már több, mint 800 éve izgalommal tölti el a kutatóit. A sorozat kezdőtagjai $F_0 = 0$, $F_1 = 1$, és

$$F_n = F_{n-1} + F_{n-2}.$$

A karakterisztikus polinom domináns gyöke $\varphi = \frac{1+\sqrt{5}}{2}$, ami a jól ismert aranyarány, a másik gyök pedig $\psi = \frac{1-\sqrt{5}}{2}$. Továbbá, ha megvizsgáljuk a szomszédos tagok hányadosának sorozatát, akkor azt tapasztaljuk, hogy

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

Legelőször azt a kérdést járjuk körbe, hogy milyen feltételek mellett konvergál gyorsabban egy ilyen hányados-sorozat az aranyarányhoz, mint a Fibonacci-számokból álló $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$. A konvergenciagyorsasághoz a következő definíciót vesszük alapul: Legyenek $\{x_n\}_{n=0}^{\infty}$ és $\{y_n\}_{n=0}^{\infty}$ konvergens valós számsorozatok $\lim\limits_{n\to\infty} x_n = y_n = z$, azt mondjuk, hogy $\{y_n\}_{n=0}^{\infty}$ gyorsabban konvergál z-hez, mint $\{x_n\}_{n=0}^{\infty}$, ha

$$\lim_{n\to\infty} \frac{y_n - z}{x_n - z} = 0.$$

Általános esetben azt az eredményt kapjuk, hogy egy $\left\{\frac{G_{n+1}}{G_n}\right\}_{n=1}^{\infty}$ sorozat gyorsabban fog konvergálni, mint a Fibonacci-sorozat szomszédos tagjainak hányados-sorozata, ha $\{G_n\}$ karakterisztikus polinomjának minden gyöke abszolút értékben kisebb, mint a $\psi = \frac{1-\sqrt{5}}{2}$.

Tétel 1 (T. Szakács [44], 2017)**.** *Legyen $\{G_n\}_{n=1}^{\infty}$ k-ad rendű lineáris rekurzív sorozat, karakterisztikus polinomjának domináns gyöke φ, többi gyöke pedig α_i. A $\left\{\frac{G_{n+1}}{G_n}\right\}$ sorozat gyorsabban konvergál az aranyarányhoz, mint $\left\{\frac{F_{n+1}}{F_n}\right\}$, ha $|\alpha_i| < |\psi|, i = 2, 3, \ldots, l$.*

Ezt követően olyan sorozatokat vizsgálunk, amelyek tagjai racionális polinomok és a Fibonacci-számok szorzataként állíthatók elő. Legyenek $0 \leq$

$j_1 < j_2 < \cdots < j_s$ nemnegatív egészek, és $p_1(x), p_2(x), \ldots, p_s(x) \in \mathbb{Q}[x]$ racionális polinomok. Legyen továbbá $\deg(p_i(x)) = d_i$, és $d^\star = \max_i\{d_i\}$, amely szintén nemnegatív egész. Definiáljuk a $\{W_n\}_{n=0}^{\infty}$ sorozatot a következőképpen

$$W_n = p_1(n)F_{n-j_1} + p_2(n)F_{n-j_2} + \cdots + p_s(n)F_{n-j_s}.$$

Megadunk néhány speciális esetben feltételeket a $p_i(x)$ polinomokra vonatkozóan úgy, hogy $\{W_n\}$ sorozat egész tagú legyen. Itt most csak az első esetet taglaljuk, az érdeklődő olvasó megtalálja a többit a 2.2.2 fejezetben.

1. eset: $s = 2$**,** $j_1 = 0$**,** $j_2 = 1$**,** $d_1 = d_2 = 1$
Az első esetben tegyük fel, hogy $a \neq 0$, b, $c \neq 0$, d racionális számok, és

$$W_n = (an + b)F_n + (cn + d)F_{n-1}.$$

Ekvivalens átalakításokkal a sorozatunk a következő alakra hozható

$$W_n = \frac{(a\alpha + c)n + (b\alpha + d)}{\alpha\sqrt{5}}\alpha^n - \frac{(a\beta + c)n + (b\beta + d)}{\beta\sqrt{5}}\beta^n,$$

ahol a kezdőtagok az alábbiak

$$\begin{aligned} W_0 &= d, \\ W_1 &= a + b, \\ W_2 &= 2a + b + 2c + d, \\ W_3 &= 6a + 2b + 3c + d. \end{aligned}$$

A karakterisztikus polinom ebből

$$w(x) = (x - \alpha)^2(x - \beta)^2 = (x^2 - x - 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1,$$

valamint a rekurzív összefüggés $n \geq 4$ esetén

$$W_n = 2W_{n-1} + W_{n-2} - 2W_{n-3} - W_{n-4}.$$

Tétel 2 (K. Liptai, L. Németh, T. Szakács és L. Szalay [26], 2024)**.** *A*

$$W_n = (an + b)F_n + (cn + d)F_{n-1}$$

sorozat tagjai akkor és csakis akkor egész számok, ha d egész és

$$a = \frac{-z_1 - 3z_2 + 2z_3}{5}, \quad b = \frac{6z_1 + 3z_2 - 2z_3}{5}, \quad c = \frac{-2z_1 + 4z_2 - z_3}{5},$$

ahol z_1, z_2, z_3 szintén egészek.

Megjegyzés 1. *Ha az előző sorozathoz úgy választjuk meg a kezdőértékeket, hogy $d = 0$, $z_1 = z_2 = 1$, $z_3 = 3$, akkor a következő egész tagú sorozatot kapjuk*

$$W_n = \frac{2n+3}{5} F_n - \frac{n}{5} F_{n-1},$$

amely megtalálható az OEIS-ben [35] a következő kóddal: A010049.

Ezt követően két másodrendű lineáris rekurzív sorozat konvolúciójával foglalkozunk

$$C_n = \sum_{k=0}^{n} G_k H_{n-k},$$

ahol $\{G_n\}$ és $\{H_n\}$ elsőfajú-, és másodfajú Lucas-sorozatok (lásd a néhány felsorolt nevezetes sorozatot a fenti táblázatban, Fibonacci, Pell, Jacobsthal, Mersenne, Lucas, P-Lucas, J-Lucas, M-Lucas). Számos különböző eset adódik attól függően, hogy a karakterisztikus polinomnak hány közös gyöke van, illetve mely típusú sorozatok konvolúcióját írjuk fel. Itt csak azt az esetet említjük meg, amikor a polinomoknak nincs közös gyöke, illetve mindkét sorozat elsőfajú Lucas-sorozat. A többi esetet lásd a 2.3 fejezetben.

Tétel 3 (T. Szakács [42], 2016). *Ha $G_n(G_0, G_1, A_1, B_1)$ és $H_n(H_0, H_1, A_2, B_2)$ karakterisztikus polinomjai rendre $p(x) = x^2 - A_1 x - B_1$ és $q(x) = x^2 - A_2 x - B_2$, akkor legyenek α, β és γ, δ ezen polinomok gyökei. A $G_n(0, 1, A_1, B_1)$ és $H_n(0, 1, A_2, B_2)$ sorozatok konvolúciója a következő alakban írható.*

$$C_n = \sum_{k=0}^{n} G_k H_{n-k} = \frac{\frac{\alpha^{n+1}}{a} - \frac{\beta^{n+1}}{b}}{\alpha - \beta} + \frac{\frac{\gamma^{n+1}}{c} - \frac{\delta^{n+1}}{d}}{\gamma - \delta},$$

ahol $bd \neq 0$, és

$$\begin{aligned} a &= (A_1 - A_2)\alpha + B_1 - B_2, \\ b &= (A_1 - A_2)\beta + B_1 - B_2, \\ c &= (A_2 - A_1)\gamma + B_2 - B_1, \\ d &= (A_2 - A_1)\delta + B_2 - B_1. \end{aligned}$$

A fenti tétel következményeként nézzünk meg két olyan konvolúciót, amelyek közül az egyik már szerepelt az OEIS-ben az eredmény publikálása előtt, a másik pedig nem.

Következmény 1. *A Fibonacci- és Pell-számok konvolúciója:*

$$C_n = \sum_{k=0}^{n} F_k P_{n-k} = P_n - F_n.$$

Megjegyzés 2. *Az OEIS-ben, A106515 kóddal megtalálható a sorozat, ahol csak a különböző indexelés miatt van némi eltérés a miénkhez képest:*

$$C_n = \sum_{k=0}^{n} F_{n-k-1} P_{k+1} = P_n - F_n + P_{n+1}$$

Következmény 2. *A Pell- és Mersenne-számok konvolúciója:*

$$C_n = \sum_{k=0}^{n} P_k M_{n-k} = \frac{P_{n+2} + P_{n+1} - M_{n+2}}{2}.$$

Megjegyzés 3. *Az OEIS-ben, A307572 kóddal* 2019-*óta megtalálható a sorozat, de nem mint konvolúció van definiálva. A konvolúcióval kapcsolatos eredményt [42] jelen dolgozat szerzője* 2016-*ban publikálta.*

A dolgozat harmadik fejezetében faktoriálisokkal foglalkozunk. Tekintsük a következő egyenletet $r \geq 2$ esetén.

$$n! = \prod_{i=1}^{r} a_i!,$$

ahol $n, a_1, \ldots, a_r$ pozitív egészek a következő feltétellel $a_1 \geq \cdots \geq a_r > 1$. Hickerson sejtése szerint az egyenlet nem-triviális megoldásai $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$, $16! = 14!5!2!$, Caldwell [7] igazolta a sejtést $n \leq 10^6$ esetben. A dolgozatban $r = 2$ esettel foglalkozunk, azon belül is azzal, amikor $k = B - A$ rögzített. Írjuk át az eredeti egyenletet a következő alakba, ahol A, B, C pozitív egészek:

$$A!B! = C!$$

Adott k érték mellett explicit felső korlátot adunk C-re, ami azt is jelenti, hogy véges sok megoldás lesz csak.

Tétel 4 (L. Hajdu, Á. Papp és T. Szakács [20], 2018)**.** *Tekintsük az* $A!B! = C!$ *egyenletet, és legyen* $k = B - A$*. Minden olyan nem-triviális megoldásra, amely különbözik* $(A, B, C) = (6, 7, 10)$*-től igaz, hogy* $C < 5k$*. Továbbá, ha* $k \leq 10^6$*, akkor az egyenlet egyetlen nem-triviális megoldása az* $(A, B, C) = (6, 7, 10)$.

Az előző egyenlet egy speciális eseteként tekintsük a következő összefüggést:

$$1 \cdot 2 \cdots (n-1) = (n+1)(n+2) \cdots (n+r)$$

Ezen egyenlettel definiált számokat multiplikatív balansz számoknak nevezzük. Ez az egyenlet az előző probléma $k = 1$ esete, és egyetlen megoldása az $n = 7$.

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