ON THE HOLONOMY OF FINSLER MANIFOLDS

Thesis for the Degree of Doctor of Philosophy (PhD)

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Debrecen, 2026.

Hereby I declare that I prepared this thesis within the Doctoral Council for Natural Sciences and Engineering, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.
The results published in the thesis are not reported in any other PhD theses.

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Hereby I confirm that Asma Mezrag candidate conducted her studies with my supervision within the Differential Geometry and its Applications Doctoral Program of the Doctoral School of Mathematical and Computational Sciences between 2022 and 2026. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.
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signature of the supervisor

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Contents

1 Introduction

The end of the nineteenth century and the beginning of the twentieth century witnessed a profound development in mathematical thought, as new directions emerged aiming to extend the classical framework of Riemannian geometry. Although Bernhard Riemann had already indicated in his celebrated 1854 habilitation lecture the possibility of generalizing the notion of a metric beyond one arising from an inner product, this idea did not crystallize into an independent field until the work of Paul Finsler. His dissertation, written under the influence of the calculus of variations, laid the methodological foundations of what is now known as Finsler geometry by broadening the concept of length so that it depends not only on position but also on direction, thereby providing a more flexible and comprehensive geometric framework than the traditional Riemannian model. At the beginning of the twentieth century, the intensive study of Finsler metrics was further stimulated by problems arising in the calculus of variations and optimal transport theory.

Following Finsler's foundational work [23], the theory developed through the contributions of several outstanding mathematicians, including C. Carathéodory, H. Busemann, É. Cartan, L. Berwald, and S. S. Chern. In particular, ideas from the calculus of variations and from Riemannian geometry—such as affine connections, Jacobi fields, and curvature notions—were gradually adapted to this broader setting. Although the theory has evolved significantly, many structural questions remain open. In recent decades, increasing attention has also been directed toward applications of Finsler geometry in the natural sciences, ranging from general relativity and seismic ray modeling to wildfire spread and quantum mechanics [38, 2]. In particular, the investigations of B. Russell and S. Stepney [53, 54] have opened new perspectives in Quantum Information Processing by exploiting the theorem of D. Bao, C. Robles, and Z. Shen on the one-to-one correspondence between solutions of Zermelo's navigation problem and Randers metrics [6].

One important geometric feature that still requires a deeper understanding is the holonomy structure, which forms one of the central themes of this thesis. The holonomy group is a natural geometric invariant associated with a manifold endowed with a connection, and it reflects how the local geometric data encoded by the connection influences the global structure of the manifold. The subject of this thesis is the investigation of the holonomy structure of Finsler manifolds and some related aspects of Finsler geometry.

The notion of holonomy has its historical roots in classical mechanics, where it appeared at the end of the nineteenth century. It was Heinrich Hertz who introduced the terms *holonomic* and *non-holonomic* This sentence needs revision. constraints in his posthumously published work *Die Prinzipien der Mechanik, in neuen Zusammenhängen dargestellt* (*The Principles of Mechanics Presented in a New Form*), which appeared in 1895. In its modern mathematical meaning, however, holonomy emerged only later as a genuinely geometric concept.

In the mathematical setting, the notion of holonomy of an affine connection and the corresponding parallel transport seems to have first appeared in the work of É. Cartan [13, 14, 15]. Working with the Levi–Civita connection of a Riemannian manifold M, Cartan showed that the corresponding holonomy group is a subgroup of the orthogonal group. He also proved that, if M is simply connected, then the holonomy group is connected. Furthermore, he observed that for any two points $p, q \in M$ on a connected manifold, the holonomy groups $\mathcal{H}ol_p(M)$ and $\mathcal{H}ol_q(M)$ are conjugate via parallel translation along any curve joining p to q. Therefore, the holonomy group $\mathcal{H}ol(M) \subset Gl(n, \mathbb{R})$ is well defined up to conjugation.

The geometric meaning of holonomy is that it captures important information about the global structure of a manifold. Informally, the holonomy group describes the fact that parallel transport may depend on the chosen path: when a geometric object is transported along a closed curve, it may fail to return to its original position. This is one of the basic effects of curvature, and it appears not only in differential geometry but also in physics and classical mechanics.

To explain this more precisely, we first recall the notion of parallel transport. In Euclidean space, vectors at different points can be compared directly, since the tangent spaces are naturally identified. On a curved manifold, however, such a comparison requires additional structure, namely a connection. In the Riemannian case, the natural choice is the Levi–Civita connection ∇. Let $\gamma : [0, 1] \to M$ be a curve joining p and q. A vector field V along γ is said to be parallel if it satisfies

$$\nabla_{\dot{\gamma}} V = 0. \tag{1.1}$$

For such a curve, the parallel transport is the map

$$\mathcal{P}_\gamma \colon T_pM \to T_qM \tag{1.2}$$

defined as follows: for a given vector $v \in T_pM$, let V be the parallel vector field along γ with initial value $V(0) = v$. Then the parallel transport of v along γ is given by

$$\mathcal{P}_\gamma(v) = w, \tag{1.3}$$

where $w = V(1)$. The map $\mathcal{P}_\gamma$ is an isomorphism from T_pM onto T_qM. In general, however, this isomorphism depends on the chosen curve γ, and there is no reason why parallel transports along different curves should coincide. This dependence on the path is described by the holonomy group, which is one of the main objects of our investigation. More precisely, the holonomy group at a point $p \in M$ is the group of all automorphisms to (1.2) In the Riemannian case, the holonomy groups have been extensively studied and now their complete classification is known, due to the work of excellent mathematicians. In 1952, A. Borel and A. Lichnerowicz showed that the holonomy group of a simply connected n-dimensional Riemannian manifold is a closed Lie subgroup of the orthogonal group [9]. In that same year, de Rham [21] established the result now known as the de Rham decomposition theorem. It states that when the holonomy group of a Riemannian manifold is reducible, the manifold locally splits into a product of Riemannian manifolds. Shortly afterwards, W. Ambrose and M. Singer clarified the close relationship between holonomy and curvature [1]. A few years later, M. Berger in his doctoral thesis established his celebrated list of all possible holonomy groups of Riemannian manifolds [8]. Since then, each group on Berger's list has been realized as the holonomy group of a suitable Riemannian manifold. In 1998, S. Merkulov and and L. J. Schwachhöfer classified all irreducible holonomy groups of torsion-free connections [37, 57]. This classification revealed new symplectic holonomies, namely holonomy groups associated with symplectic manifolds carrying a parallel symplectic form. After Bryant's first example [10], further work produced an infinite family of such connections [19, 20]. In the Finslerian setting, some early contributions to holonomy in Finsler geometry already appeared in the classical literature. In particular, Barthel's paper [7] on nonlinear connections and their holonomy groups is a fundamental work in this direction; see also Kozma's survey [34] for a clear summary. Barthel also refers to an earlier paper of V.V. Wagner [62] on two-dimensional Finsler spaces with finite and continuous holonomy groups.

In Finsler geometry, the holonomy structure may be very different from that of the Riemannian case. The main reason is that the canonical connection of a Finsler manifold is, in general, neither linear nor compatible with a metric in the usual sense. As a result, parallel transport in Finsler geometry can behave differently: it is only positively homogeneous of degree one and preserves the norm instead of the metric tensor. Parallel transport acts naturally on the indicatrices, and the holonomy group can therefore be viewed as a subgroup of the diffeomorphism group of the indicatrix. Nevertheless, our knowledge of Finslerian holonomy groups remains rather limited, and this topic continues to be an active area of modern geometric research. The first significant contributions to the study of holonomy in Finsler geometry were made by Z. Szabó [58] and later by L. Kozma [35]. Their work established fundamental results for certain special classes of Finsler manifolds and showed that, in these cases, holonomy theory still exhibits strong similarities with the classical Riemannian setting.

However, the situation changes fundamentally in more general Finslerian setting. In [42], it was established that the holonomy groups of Finsler manifolds need not be compact or finite-dimensional. In [44], explicit two-dimensional examples were exhibited whose holonomy groups are infinite-dimensional and isomorphic to $\mathcal{D}iff_+(\mathbb{S}^1)$, the orientation-preserving diffeomorphism group of the circle. Further systematic investigations have since deepened our understanding. In [31], it was shown that the holonomy group of a simply connected, non-Riemannian, projectively flat Finsler two-manifold of constant nonzero flag curvature is maximal and isomorphic to $\mathcal{D}iff_+(\mathbb{S}^1)$. These results suggest that infinite-dimensional holonomy is not an anomaly but rather a common phenomenon among Finsler metrics. The results of [29] showed that infinite-dimensional holonomy is the typical situation in Finsler geometry: on every manifold, there exists an open and dense subset of Finsler metrics whose holonomy groups are infinite-dimensional. Consequently, finite-dimensional holonomy appears only in exceptional cases. These include flat metrics, as well as Riemannian, Berwald, and Landsberg-type Finsler metrics [9, 58, 35]. In addition, for locally projectively flat

Randers manifolds, [43] proved that finite-dimensional holonomy occurs exactly in the flat or Riemannian case.

This thesis is devoted to further results on the holonomy theory of Finsler manifolds. In the preliminary Chapter 2, we introduce the basic notions and concepts of spray geometry and Finsler geometry that will be used throughout the thesis. The chapter is organized as follows. In Section 2.1, we recall the notion of connections and the geometric structures associated with them. In Section 2.2, we introduce sprays and the geometric objects naturally related to them. Finally, in Section 2.3, we present the basic definitions and fundamental concepts of Finsler manifolds.

In Chapter 3, we investigate the holonomy structure of Finsler manifolds with maximal holonomy. Section 3.1 recalls the basic notions of parallel transport and holonomy, and introduces the main tools used later, namely the holonomy algebra and the infinitesimal holonomy algebra. In Section 3.2, we establish a general result showing that if the holonomy algebra at a point is dense in the Lie algebra of smooth vector fields on the indicatrix, then the holonomy group is maximal, meaning that its closure is isomorphic to the identity component of the diffeomorphism group of the indicatrix. In Section 3.3, we apply this theorem to spherically symmetric projective Finsler metrics of nonzero constant flag curvature and derive explicit descriptions of their holonomy groups. In particular, this yields the holonomy groups of the standard Funk metric and the Bryant–Shen metrics in arbitrary dimension. These results are generalizations of the results of [44]. This chapter is based on the results of [39].

In Chapter 4, we investigate the holonomy of two-dimensional Randers manifolds with constant flag curvature and provide a complete classification of their holonomy groups. Our starting point is the classification of Randers metrics of constant flag curvature [6] given in Theorem 4.2, which yields a finite family of models. In Section 4.1, we first analyze the infinitesimal holonomy structure by determining the corresponding infinitesimal holonomy algebras, and show that they are finite-dimensional in some cases and infinite-dimensional in others. In Section 4.2, we then turn our attention to the holonomy groups. We prove that when the infinitesimal holonomy algebra is infinite-dimensional, then the holonomy is maximal, that is, its closure is isomorphic to the group of orientation-preserving diffeomorphisms of the circle. On the other hand, when the infinitesimal holonomy algebra is finite-dimensional, we determine explicitly the holonomy groups. In particular, we show that a proper (i.e., one that is neither Riemannian, Berwald, nor Landsberg type) non-flat Finsler metric can have a finite-dimensional holonomy group. This chapter is based on the results of [40].

In Chapter 5, we introduce a natural parallelism associated with navigation data (h, W). D. Bao, C. Robles, and Z. Shen proved in [6] that the Zermelo's navigation problem is equivalent to considering geodesics of Randers-type Finsler metrics. The construction of the metric structure associated to the navigation data is easy to understand (the sets of unit vectors, called indicatrices, are blown away by the wind W), however, the affine structure (the parallel translation) is not so easy or natural to understand [51]. Moreover, the holonomy group can be very large even in cases when the metric structure is relatively simple [30]. For this reason, we consider the geometric setting of navigation data and introduce a natural parallel translation using the Riemannian parallelism. The geometry obtained in this way has some nice and natural features: a natural parallel translation is homogeneous (but in general nonlinear), preserves the Randers type Finslerian norm constituted by the navigation data, and the holonomy group is finite-dimensional. This chapter is based on the results of [41].

2 Preliminaries

In this chapter, we present the basic notions and concepts of spray geometry and Finsler geometry that are essential for the developments in the subsequent chapters. Throughout this thesis, M denotes a smooth n-dimensional manifold, unless stated otherwise. We write $\mathfrak{X}(M)$ for the Lie algebra of smooth vector fields on M, and $\mathcal{D}iff(M)$ for the group of smooth diffeomorphisms of M. The tangent bundle of M is denoted by TM, and $TM \setminus \{0\}$ the slit tangent bundle. If (U, x^i) is a local coordinate chart on M, then it induces coordinates (x^i, y^i) on TM, where $\pi : TM \to M$ is the canonical projection. The second tangent bundle is denoted by (TTM, τ, TM). The vector 1-form J on TM, defined locally as $J = \frac{\partial}{\partial y^i} \otimes dx^i$, is known as the natural almost-tangent structure of TM, the vertical vector field $\mathcal{C} = y^i \frac{\partial}{\partial y^i}$ on TM is called the Liouville vector field.

2.1 Connections and associated geometric structures

In this section, we present the differential–algebraic formulation of the connection theory introduced in [28], which will play a central role in the developments that follow.

Connections and the horizontal–vertical decomposition

2.1 Definition. A *connection* on M is a tensor field of type $(1, 1)$, denoted by Γ, on TM (i.e. $\Gamma \in \Psi^1(TM)$) satisfying

$$J\Gamma = J, \qquad \Gamma J = -J. \tag{2.1}$$

The connection is called *homogeneous* if $[\mathcal{C}, \Gamma] = 0$, and if it is $\mathcal{C}^\infty$ on $TM \setminus \{0\}$ and $\mathcal{C}^0$ on TM. In particular, if Γ is $\mathcal{C}^1$ on the entire tangent manifold TM (including the zero section), then it is called *linear*.

Let Γ be a connection. Then $\Gamma^2 = I$, and the eigenspace corresponding to the eigenvalue -1 coincides with the vertical space. Consequently, for any nonzero point $z \in TM$, the tangent space T_zTM admits the decomposition

$$T_zTM = \mathcal{H}_z \oplus \mathcal{V}_z, \tag{2.2}$$

where the subspace $\mathcal{H}_z$, called the *horizontal space*, is the eigenspace corresponding to the eigenvalue $+1$.

In the natural local basis $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}$, the matrix representation of Γ takes the form

$$\Gamma = \begin{pmatrix} \delta_i^j & 0 \\ -2\Gamma_i^j & -\delta_i^j \end{pmatrix}, \tag{2.3}$$

where the functions $\Gamma_j^i = \Gamma_j^i(x,y)$ are called the *coefficients of the connection.* If the connection is homogeneous (resp. linear), then the coefficients $\Gamma_i^j(x,y)$ are homogeneous of degree one (resp. linear) with respect to y. The associated *horizontal* and *vertical* projectors are defined by

$$\mathfrak{h} := \tfrac{1}{2}(Id + \Gamma), \qquad \nu := \tfrac{1}{2}(Id - \Gamma), \tag{2.4}$$

respectively. In local coordinates, the horizontal projector satisfies

$$\mathfrak{h}\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}, \qquad \mathfrak{h}\Big(\frac{\partial}{\partial y^i}\Big) = 0.$$

The *vertical distribution* on TM is the subbundle $\mathcal{V}TM \subset TTM$ defined by $\mathcal{V}_uTM := \ker(\pi_{*,u})$. In local coordinates (x^i, y^i) on TM, the vertical distribution is given by

$$\mathcal{V}TM = \operatorname{span}\left\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\}. \tag{2.5}$$

And the *horizontal distribution* $\mathcal{H}TM \subset TTM$ defined as the image of the horizontal projector. In local coordinates, it is given by

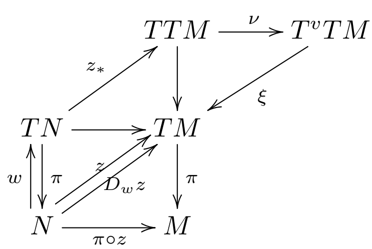
$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}, \qquad y \in T_xM. \tag{2.6}$$

Covariant derivative

2.2 Definition. Given two manifolds N and M, let $w \in \mathfrak{X}(N)$ and $z \in \mathfrak{X}_N(M)$, that is $z\colon N \to TM$ be is a smooth mapping. Using the natural isomorphism $\xi_z : T_z^vTM \to T_{\pi(z)}M$, the *covariant derivative* of z with respect to w is defined by:

$$D_w z = \xi_z(\nu \circ z_* \circ w). \tag{2.7}$$

We have the following diagram:



(2.8)

Particular cases:

1. If $N = M$ and $w, z \in \mathfrak{X}(M)$, then

$$D_w z = w^i \left(\frac{\partial z^j}{\partial x^i} + \Gamma^j_i \circ z \right) \frac{\partial}{\partial x^j}. \tag{2.9}$$

2. If $N = I$ is an interval of $\mathbb{R}$, $w = \frac{d}{dt}$ and $z : I \to TM$, $z(t) = \big(x(t), y(t)\big)$ is a vector field along a curve $\gamma\colon I \to M$ (that is $\gamma = \pi \circ z$), we arrive at:

$$D_{\frac{d}{dt}} z = \left(\frac{dy^j}{dt} + \Gamma^j_i(x(t), y(t)) \frac{dx^i}{dt} \right) \frac{\partial}{\partial x^j}. \tag{2.10}$$

2.3 Definition. A vector field z along a curve γ is called *parallel* if $D_{\frac{d}{dt}} z = 0$. An *autoparallel curve* is a curve $\gamma\colon [a, b] \to M$ such that $D_{\frac{d}{dt}} \gamma' = 0$.

2.4 Remark. From Definitions 2.2 and 3.1 we get that a vector field z is parallel along a curve γ if and only if $\nu \circ z' = 0$, that is z' is horizontal, and a curve γ is autoparallel if and only if $\nu \circ \gamma'' = 0$, that is γ'' is horizontal.

Let ξ be a vertical vector field and let $X \in \mathfrak{X}(M)$. The *horizontal Berwald covariant derivative* of ξ in the direction of X is defined by

$$\nabla_X \xi = [X^h, \xi], \tag{2.11}$$

where X^h denotes the horizontal lift of X. In local coordinates, if

$$\xi = \xi^i(x, y) \frac{\partial}{\partial y^i}, \qquad X = X^j(x) \frac{\partial}{\partial x^j},$$

then the covariant derivative takes the form

$$\nabla_X \xi = \left(\frac{\partial \xi^i}{\partial x^j} - G^k_j \frac{\partial \xi^i}{\partial y^k} + \frac{\partial G^i_j}{\partial y^k} \xi^k \right) X^j \frac{\partial}{\partial y^i}. \tag{2.12}$$

In the sequel we will use the simplified notation

$$\nabla_k \xi = \nabla_{\frac{\partial}{\partial x^k}} \xi \tag{2.13}$$

Curvature

In general, the horizontal distribution is not integrable. The failure of integrability is measured by the curvature tensor, which encodes the nontrivial interaction between horizontal directions. It is defined by

$$R(X, Y) = v[X^h, Y^h]. \tag{2.14}$$

which is related to the Nijenhuis torsion of the horizontal projector [28]. Hence, the horizontal distribution is integrable if and only if the curvature vanishes identically. The curvature tensor field has the expression

$$R = R^i_{jk}(x, y)\, dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^i}, \tag{2.15}$$

with components

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y). \quad (2.16)$$

2.5 Definition. A vector field $\xi \in \mathfrak{X}(TM)$ is called a *curvature vector field* if it lies in the image of the curvature tensor, that is, $\xi = R(X,Y)$ for some vector fields $X, Y \in \mathfrak{X}(M)$.

2.2 Sprays and associated geometric structures

2.6 Definition. A *spray* on a manifold M is a vector field $S : TM \to TTM$ which is a section of the second tangent bundle, that is $\tau \circ S = \mathrm{Id}_{TM}$, is smooth on the slit tangent bundle $TM \setminus \{0\}$, and is positively homogeneous of degree two in the fibre variable. A pair (M,S) consisting of a manifold and a spray is called a *spray manifold*.

In local coordinates, any spray can be written in the form

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \quad (2.17)$$

where the functions $G^i(x,y)$, called the *spray coefficients*, are positively homogeneous of degree two in y.

2.7 Definition (Geodesics of a spray)**.** Let S be a spray on a manifold M. A smooth curve $\gamma(t)$ on M is called a *geodesic* of the spray S if its velocity vector field $\dot\gamma(t)$ is an integral curve of S, i.e.,

$$\ddot\gamma(t) = S(\dot\gamma(t)). \quad (2.18)$$

In local coordinates this condition leads to the system

$$\dot x^i(t) = y^i(t), \qquad \dot y^i(t) + 2G^i(x(t), y(t)) = 0. \quad (2.19)$$

Projecting this system to the base manifold, we obtain the second-order differential equations

$$\ddot\gamma^i(t) + 2G^i(\gamma(t), \dot\gamma(t)) = 0. \quad (2.20)$$

The *canonical lift* of a curve $\gamma(t)$ is the curve in the tangent bundle defined by

$$\dot\gamma(t) = \dot\gamma^i(t)\frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}. \quad (2.21)$$

If $\gamma(t)$ satisfies (2.20), then its canonical lift is an integral curve of the spray S.

2.8 Definition (spray associated with a connection)**.** Let Γ be a homogeneous connection, and consider its horizontal projector $\mathfrak{h}$. The spray S associated to the connection is defined by $S = \mathfrak{h}\tilde S$, where $\tilde S$ is an arbitrary spray.

Locally, the coefficients of the spray associated to the connection (2.3) are

$$G^i = \tfrac{1}{2} y^j \Gamma^i_j, \tag{2.22}$$

where the $\Gamma^i_j(x, y)$ are the coefficients of the connection.

2.9 Definition (Connection associated with a spray)**.** Let S be a spray on the slit tangent bundle $TM \setminus \{0\}$. The *homogeneous connection associated with the spray S* is the $(1,1)$–tensor field $\Gamma := [J, S]$. This tensor defines a nonlinear connection on M, called the canonical (or spray) connection associated with S. If the spray has a local expression (2.17), then the coefficients of the connection are:

$$\Gamma^i_j = \frac{\partial G^i}{\partial y^j}. \tag{2.23}$$

If S is a quadratic spray, that is, then the associated nonlinear connection $\Gamma = [J, S]$ is linear.

2.10 Remark. Although every spray S canonically determines a homogeneous nonlinear connection Γ_S (see Definition 2.9), and a nonlinear connection Γ is uniquely determining a spray S_Γ (see Definition 2.8), the correspondence $S \longmapsto \Gamma_S$, and $\Gamma \longmapsto S_\Gamma$ are not inverse to each other in general in the sense that the spray associated with a given connection associated with a spray need not coincide with the original spray.

2.3 Finsler manifolds

2.11 Definition. A *Finsler manifold* is a pair $(M, \mathcal{F})$, where M is a smooth manifold and $\mathcal{F} : TM \to \mathbb{R}_+$ is a function, called the *Finsler function*, satisfying the following conditions:

1. $\mathcal{F}$ is smooth on the slit tangent bundle $TM \setminus \{0\}$;
2. for each $x \in M$, the restriction $\mathcal{F}_x := \mathcal{F}|_{T_xM}$ is positively homogeneous of degree one, that is,
$$\mathcal{F}_x(\lambda y) = \lambda \mathcal{F}_x(y), \qquad \lambda > 0, \; y \in T_xM; \tag{2.24}$$
3. for each $x \in M$ and $y \in T_xM \setminus \{0\}$, The symmetric bilinear form
$$g_{x,y} \colon (u, v) \;\mapsto\; g_{ij}(x, y) u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s \, \partial t}\Big|_{t=s=0} \tag{2.25}$$
is positive definite at every $y \in T_xM \setminus \{0\}$.

The hypersurface of T_xM defined by

$$\mathcal{I}_x = \{\, y \in T_xM \mid \mathcal{F}(x, y) = 1 \,\}, \tag{2.26}$$

is called the *indicatrix* at $x \in M$. We note that at any point $x \in M$ the indicatrix is diffeomorphic to $\mathbb{S}^{n-1}$, the $(n-1)$-dimensional sphere. Let $(M, \mathcal{F})$ be a Finsler manifold. Let $\gamma : [a, b] \to M$ be a smooth curve on the manifold. Consider the variational problem

$$L(\gamma) = \int_a^b F\big(\gamma(t), \dot{\gamma}(t)\big)\, dt. \tag{2.27}$$

Suppose that the curve $\gamma(t)$ is an extremal of this functional. Then, by the Euler–Lagrange equations, we obtain

$$\frac{\partial \mathcal{F}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{F}}{\partial y^i}\right) = 0, \qquad i = 1, \dots, n. \tag{2.28}$$

These equations can be written in the second–order form

$$\frac{d^2\gamma^i}{dt^2}(t) + 2G^i\big(\gamma(t), \dot{\gamma}(t)\big) = 0, \tag{2.29}$$

where the functions $G^i = G^i(x, y)$ are given by

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\Big) y^j y^k. \tag{2.30}$$

These functions are called the *geodesic coefficients*. Using them, we define a vector field on TM by

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}. \tag{2.31}$$

Since the coefficients $G^i(x, y)$ are positively homogeneous of degree two in y, the vector field S is a spray. It is called the *geodesic spray* or the *Finsler spray* induced by the metric F. In this thesis we use this spray and its associated Ehresmann connection, called the *Berwald connection*. The geodesics, covariant derivative, and other geometric objects of the Finsler manifold are obtained from the corresponding constructions on the associated spray manifold.

2.12 Definition. A vector field $\xi_x \in \mathfrak{X}(I_x)$ on the indicatrix $I_x \subset T_xM$ is called a *curvature vector field* at $x \in M$ if there exist tangent vectors $X_x, Y_x \in T_xM$ such that $\xi_x = R(X_x, Y_x)$.

The Riemann curvature tensor is defined by $R_y := R(\,\cdot\,, y)$, with components $R^i{}_j = R^i{}_{jk} y^k$. The Ricci curvature is the trace of R_y, $\mathrm{Ric}(y) := R^m{}_m(x, y)$. For a tangent plane $P = \mathrm{Span}\{y, u\} \subset T_xM$, the *flag curvature* is defined by

$$K(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y) g_y(u, u) - g_y(y, u)^2}. \tag{2.32}$$

If the manifold has constant flag curvature $K = \lambda \in \mathbb{R}$, then the Ricci curvature satisfies

$$\mathrm{Ric}(y) = (n-1)\lambda F^2, \tag{2.33}$$

and the curvature coefficients take the form

$$R^i{}_{jk} = \lambda\left(\delta^i_k\, g_{jm}(x, y) y^m - \delta^i_j\, g_{km}(x, y) y^m\right), \tag{2.34}$$

where δ^i_j denotes the Kronecker delta.

2.13 Lemma ([45]). *The horizontal covariant derivative $\nabla_W R$ of the tensor field $R = R^i_{jk}(x,y)dx^j \wedge dx^k \frac{\partial}{\partial x^i}$ vanishes.*

Special classes of Finsler manifolds

In this subsection, we present different types of Finsler manifolds (M, F) where the metrics have special geometric properties.

Berwald and Landsberg manifolds

The *Landsberg tensor* $\mathcal{L} = L^i_{jk}\, dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ is defined by

$$L^i_{jk} = -\frac{1}{2} F \frac{\partial \mathcal{F}}{\partial y^s} \frac{\partial^3 G^s}{\partial y^i\, \partial y^j\, \partial y^k}. \tag{2.35}$$

A Finsler manifold $(M, \mathcal{F})$ is called a *Berwald manifold* if, in any standard local coordinate system (x^i, y^i) on TM, the spray coefficients $G^i(x,y)$ are quadratic in $y \in T_xM$ for all $x \in M$. The manifold $(M, \mathcal{F})$ is called a *Landsberg manifold* if the Landsberg tensor vanishes identically, that is, $L^i_{jk} = 0$. We note that any Berwald manifold is also a Landsberg manifold; however, it is still an open question whether there exists a non-Berwaldian Landsberg manifold.

Projective Finsler manifold

A Finsler function F on an open subset $D \subset \mathbb{R}^n$ is said to be *projective* or projectively flat, if all geodesic curves are straight lines in D. A Finsler manifold is said to be *locally projective* or locally projectively flat, if at any point there is a local coordinate system (x^i) in which F is projective. Let $(x^1, \ldots, x^n)$ be a local coordinate system on M corresponding to the canonical coordinates of the Euclidean space which is projectively related to (M, F). Then the geodesic coefficients (7.1) and their derivatives have the form

$$G^i = \mathcal{P}(x,y) y^i, \qquad G^i_k = \frac{\partial \mathcal{P}}{\partial y^k} y^i + \mathcal{P}\delta^i_k, \qquad G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l} y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k. \tag{2.36}$$

where $\mathcal{P}$ is a 1-homogeneous function in y, called the projective factor of $(M, \mathcal{F})$. According to [17, Lemma 8.2.1, p. 155] the projective factor can be computed using the formula

$$\mathcal{P}(x,y) = \frac{1}{2\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x^i} y^i. \tag{2.37}$$

Randers manifolds and Zermelo navigation

In 1941, G. Randers [50] introduced a class of Finsler metrics by modifying a Riemannian norm $a = \sqrt{a_{ij} y^i y^j}$ by a 1-form $b = b_i dx^i$. The Randers-type Finsler norm function is defined as

$$\mathcal{F}(x,y) = a(x,y) + b(x,y) = \sqrt{a_{ij} y^i y^j} + b_i y^i, \tag{2.38}$$

where a_{ij} are the components of a Riemannian metric and the b_i are those of a 1-form. To ensure the positivity of $\mathcal{F}$ and the regularity of the energy function $E = \frac{1}{2}\mathcal{F}^2$ one has to impose the condition $\|b\| = \sqrt{b_i b^i} < 1$ on the norm of b with respect to the Riemannian metric. For a comprehensive analysis of Randers metrics, see [4, 5, 6]. In [6] the authors proved the connection between Randers metrics and the navigation problem: the shortest paths of the navigation problem are precisely the geodesics of a Randers metric. Conversely, every Randers metric can be expressed in terms of suitable navigation data (h, W) where h is a Riemannian metric and W is a vector field on M (the "wind") whose Riemannian norm is smaller than 1, establishing a one-to-one correspondence between Randers metrics and solutions to Zermelo's problem. The Randers metric arising from navigation data takes the form

$$\mathcal{F}(x, y) = a(x, y) + b(x, y) = \frac{\sqrt{\lambda\,|y|^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \tag{2.39}$$

where

$$\|y\|^2 = h(y, y), \qquad \lambda = 1 - |W|^2, \qquad W_0 = h(y, W). \tag{2.40}$$

Comparing (2.38) and (2.39) we can get that the coefficients a_{ij} of the Riemannian metric a and the 1-form b of the Randers norm function corresponding to the navigation data (h, W) are given by

$$a_{ij} = \frac{\lambda h_{ij} + W_i W_j}{\lambda^2}, \qquad b_i = -\frac{W_i}{\lambda}. \tag{2.41}$$

The condition $|W| < 1$ guarantees that a_{ij} defines a positive definite Riemannian metric. Moreover, since $h(W, W) = a(b, b)$, this condition also ensures the positivity of F. As a result, F is well-defined on the open submanifold

$$\{\, x \in M \mid \|W_x\| < 1 \,\} \subset M. \tag{2.42}$$

It is easy to see that the indicatrix (4.91) at $x \in M$ of the Randers type norm function associated to the navigation data (h, W) can be obtained from the unit sphere of the Riemannian metric h by a translation along $W(x)$, that is

$$\mathcal{I}_x^F = \mathcal{I}_x^h + W_x \tag{2.43}$$

2.14 Remark. Let $\mathcal{F}$ be the Randers norm associated to the navigation data (h, W). At any point $p \in M$ a vector V_p° has a unit Finsler norm if and only if $V_p^\circ - W_p$ has a unit Riemannian norm, that is

$$\mathcal{F}(V_p^\circ) = 1 \qquad \Leftrightarrow \qquad \left\|V_p^\circ - W_p\right\| = 1. \tag{2.44}$$

The geodesic spray coefficients G_h^i of the Riemannian metric h are given by

$$G_h^i(x, y) = \frac{1}{2}\mathcal{A}_{kj}^i(x) y^j y^k, \tag{2.45}$$

where $\mathcal{A}_{kj}^i$ are the Christoffel symbols of h, defined as

$$\mathcal{A}_{kj}^i = \frac{1}{2} h^{is}\left(\frac{\partial h_{sj}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^s} + \frac{\partial h_{ks}}{\partial x^j}\right). \tag{2.46}$$

The geodesic coefficients of a Randers metric F are related to those of the underlying Riemannian metric a by a modification that incorporates the additional navigation data [4]. Likewise, the relation between the geodesic spray coefficients of a and h can be found in [5, page 235]. C. Robles established in [51] that the geodesic coefficients of F are related to those of h by:

$$G^i(x,y) := G^i_h(x,y) + \zeta^i(x,y), \tag{2.47}$$

where

$$\zeta^i = \frac{1}{4}\left(\frac{1}{\mathcal{F}}y^i - W^i\right)\left(2\mathcal{F}\mathcal{S}_0 - \mathcal{L}_{00} - \mathcal{F}^2\mathcal{L}_{WW}\right) - \frac{1}{4}\mathcal{F}^2\left(\mathcal{S}^i + \mathcal{T}^i\right) - \frac{1}{2}\mathcal{F}\mathcal{C}^i_0 \tag{2.48}$$

with

$$\mathcal{L}_{ij} = W_{i:j} + W_{j:i}, \quad \mathcal{C}_{ij} = W_{i:j} - W_{j:i}, \quad \mathcal{S}_i = W^s\mathcal{L}_{si}, \quad \mathcal{T}_i = W^s\mathcal{C}_{si}. \tag{2.49}$$

where the colon denotes covariant differentiation with respect to the Levi-Civita connection of h:

$$W_{i:j} = \frac{\partial W_i}{\partial x^j} - W_s\mathcal{A}^s_{ij}, \tag{2.50}$$

with, $W_i = h_{ij}W^j$. Indices on these tensors are raised using the inverse (h^{ij}) of (h_{ij}). For instance,

$$\mathcal{S}^i = h^{ij}\mathcal{S}_j, \qquad \mathcal{T}^i = h^{ij}\mathcal{T}_j. \tag{2.51}$$

Furthermore, as before, the subscript 0 denotes contraction with y, so that

$$\mathcal{S}_0 = W^j\mathcal{L}_{js}y^s, \qquad \mathcal{L}_{00} = \mathcal{L}_{js}y^jy^s, \qquad \mathcal{C}^i_0 = h^{ij}\mathcal{C}_{jk}y^k. \tag{2.52}$$

Finally, we define

$$\mathcal{L}_{WW} = W^iW^j\mathcal{L}_{ij}. \tag{2.53}$$

2.15 Remark. If the vector field W is an infinitesimal homothety of h, that is if the Lie derivative of h with respect of the wind W satisfies $\mathcal{L}_W h = \sigma h$, then we get

$$\mathcal{S}^i = \sigma W^i. \tag{2.54}$$

3 Finsler manifold with maximal holonomy

The holonomy group of a Riemannian or Finsler manifold is a fundamental algebraic object naturally associated with its geometric structure. It is defined as the group generated by parallel translations along loops with respect to the canonical connection. In Riemannian geometry, holonomy theory has been intensively developed, culminating in a complete classification of possible holonomy groups. In contrast, the holonomy theory of Finsler manifolds is much less understood and reveals phenomena that do not occur in the Riemannian setting. Z. I. Szabó proved in [58] that when the Finslerian parallel translation is linear, one can associate a Riemannian metric sharing the same holonomy group. For Landsberg metrics, L. Kozma showed in [35] that the holonomy group is a compact Lie group acting by isometries on the indicatrix with respect to a naturally induced Riemannian metric. However, the situation changes drastically in general. In [42], it was demonstrated that the holonomy group of a Finsler manifold need not be a compact Lie group. Moreover, in [44], examples of two-dimensional Finsler manifolds were constructed whose holonomy groups are infinite-dimensional, being isomorphic to $\mathcal{D}\mathit{iff}_+(\mathbb{S}^1)$ in the orientable case and to $\mathcal{D}\mathit{iff}(\mathbb{S}^1)$ in the non-orientable case. It is worth emphasizing that explicit descriptions of holonomy groups for nonlinear Finsler connections were previously known only in dimension two.

In this chapter we are focusing on the holonomy structure of n-dimensional Finsler manifolds. We use the method developed in [31], which studies holonomy via the tangent Lie algebra of the holonomy group, called the holonomy algebra, together with its infinitesimal counterpart. The chapter is organized as follows:

In Section 3.1, we present the fundamental tools and basic notions related to parallelism and holonomy. We begin by recalling the concept of parallel transport associated with a connection. We then introduce the holonomy group, together with the corresponding holonomy algebra and infinitesimal holonomy algebra.

In Section 3.2, we formulate and prove the main theorem of this chapter, which states that if the holonomy algebra at a point is dense in the Lie algebra of smooth vector fields on the indicatrix, then the holonomy group is maximal, and its closure is isomorphic to the connected component of the identity in the diffeomorphism group of the $(n-1)$-dimensional sphere.

In Section 3.3, we apply this theorem to spherically symmetric projective Finsler metrics of nonzero constant flag curvature. In the simply connected case, the holonomy group is shown to be isomorphic to $\mathcal{D}\mathit{iff}_o(\mathbb{S}^{n-1})$, the identity component of the diffeomorphism group of the $(n-1)$-dimensional sphere. Consequently, the closure of the holonomy groups of the n-dimensional standard Funk metric and the Bryant–Shen metrics are also isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$. These results provide an explicit description of the holonomy group of n-dimensional non-Berwaldian Finsler manifolds, that is, in the case where the canonical connection is genuinely nonlinear. This chapter is based on the results of [39].

3.1 Parallelism and holonomy

Parallel translation

3.1 Definition (Parallel vector field and autoparallel curve)**.** Let (M, S) be a spray manifold. A vector field

$$V(t) = V^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \tag{3.1}$$

along a curve $\gamma(t)$ is called *parallel* if its covariant derivative vanishes identically along the curve, that is, if it satisfies the differential equation

$$D_{\dot\gamma}V := \Big(\frac{dV^i}{dt} + G^i_j(\gamma, V)\cdot\dot\gamma^j\Big)\frac{\partial}{\partial x^i} = 0, \tag{3.2}$$

Since the coefficients $G^i_j(x, y)$ are positively homogeneous of degree one in the variable y, yields

$$D_{\dot\gamma}(\lambda V(t)) = \lambda D_{\dot\gamma}V(t), \tag{3.3}$$

for every scalar $\lambda > 0$. Therefore, if a vector field $V(t)$ is parallel along the curve $\gamma(t)$, then any positive scalar multiple $\lambda V(t)$ is also parallel. This property leads naturally to the notion of homogeneous (nonlinear) parallel translation along a curve in a spray manifold (M, S). For a curve $\gamma(t)$, the parallel translation

$$\mathcal{P}_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M, \tag{3.4}$$

is defined as the positively homogeneous map that assigns to an initial vector $V_0 \in T_{\gamma(0)}M$ the endpoint $V(1)$ of the parallel vector field $V(t)$ along $\gamma(t)$ satisfying $V(0) = V_0$. Since parallel translation on a spray manifold is completely determined by the horizontal distribution $\mathcal{H}TM \subset TTM$, such a manifold may be viewed as a special case of a fibered manifold endowed with an Ehresmann connection [59]. Parallel translation also admits a geometric description in terms of horizontal lifts. Given a curve $\gamma : [0, 1] \to M$ and an initial vector $V_0 \in T_{\gamma(0)}M$, the horizontal lift of γ is a curve $\gamma^h : [0, 1] \to TM$ satisfying

$$\pi \circ \gamma^h = \gamma, \tag{3.5}$$

whose tangent vectors lie in the horizontal distribution. More precisely, the horizontal lift $\gamma^h(t)$ is characterized by

$$\frac{d\gamma^h}{dt} = \Big(\frac{d\gamma}{dt}\Big)^h, \qquad \gamma^h(0) = V_0. \tag{3.6}$$

The parallel translation of the vector V_0 along the curve γ from $\gamma(0)$ to $\gamma(1)$ is then given by

$$\mathcal{P}_\gamma(V_0) = \gamma^h(1). \tag{3.7}$$

Thus, if $v(t)$ joins the points $p = \gamma(0)$ and $q = \gamma(1)$, the parallel translation along γ is the map

$$\mathcal{P}_\gamma : T_pM \to T_qM, \tag{3.8}$$

which assigns to each vector $X_0 \in T_pM$ the endpoint of the horizontal lift of $\gamma(t)$ starting from V_0.

Holonomy group and holonomy algebra

Let $(M, \mathcal{F})$ be a Finsler manifold. For a curve $\gamma\colon [0,1] \to M$, denote by

$$\mathcal{P}_\gamma\colon T_{\gamma(0)}M \to T_{\gamma(1)}M \tag{3.9}$$

the *parallel translation* along γ. Since $\mathcal{P}_\gamma$ preserves the Finsler norm (see [17, Lemma 4.1.2]) and acts differentiably on the slit tangent bundle, it induces a well-defined mapping

$$\mathcal{P}_\gamma\colon \mathcal{I}_{\gamma(0)} \to \mathcal{I}_{\gamma(1)} \tag{3.10}$$

between the indicatrices.

3.2 Definition. The *holonomy group* $\mathcal{H}ol_x(\mathcal{F})$ at a point $x \in M$ is the subgroup of transformations generated by parallel transport along all piecewise smooth closed loops based at $x \in M$.

By the norm-preservation of parallel transport, for any closed curve γ based at x, the associated parallel transport map induces a diffeomorphism

$$\mathcal{P}_\gamma\colon \mathcal{I}_x \to \mathcal{I}_x. \tag{3.11}$$

Hence, the holonomy group can be naturally identified with a subgroup of the diffeomorphism group of the indicatrix:

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}iff(\mathcal{I}_x). \tag{3.12}$$

The tangent Lie algebra (see [31]) of the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is called the *holonomy algebra* at x and is denoted by $\mathfrak{hol}_x(\mathcal{F})$. The holonomy algebra can give information about the holonomy property of the Finsler manifold.. Considering the tangent spaces of both sides in (3.12) we obtain

$$\mathfrak{hol}_x(\mathcal{F}) \ \subset \ \mathfrak{X}(\mathcal{I}_x). \tag{3.13}$$

3.3 Proposition ([31])**.** *Let $(M, \mathcal{F})$ be a Finsler manifold. Then the holonomy algebra*

$$\mathfrak{hol}_x(\mathcal{F}) \ \subset \ \mathfrak{X}(\mathcal{I}_x). \tag{3.14}$$

is a Lie subalgebra of $\mathfrak{X}(\mathcal{I}_x)$, and its exponential image is in the topological closure of the holonomy group, that is

$$\exp\big(\mathfrak{hol}_x(\mathcal{F})\big) \subset \overline{\mathcal{H}ol_x(\mathcal{F})}, \tag{3.15}$$

where the overline denotes the topological closure of the holonomy group with respect to the C^∞–topology of $\mathcal{D}iff(\mathcal{I}_x)$.

3.4 Definition. The holonomy group is called *finite (resp. infinite) dimensional*, if its tangent Lie algebra, the holonomy algebra , is finite (resp. infinite) dimensional. In the finite dimensional case, the dimension of the holonomy algebra is called the dimension of the holonomy group.

From Proposition 3.3 one can obtain, that a Lie subalgebra of $\mathfrak{X}(\mathcal{I}_x)$ generated by any subset of $\mathfrak{hol}_x(\mathcal{F})$ is also a Lie subalgebra of $\mathfrak{hol}_x(\mathcal{F})$, and in particular, its elements have the tangent property to the holonomy group $\mathcal{H}ol_x(\mathcal{F})$. One can show that for any point $x \in M$ and any tangent vectors $X_1, X_2 \in T_xM$, the curvature vector fields $R(X_1, X_2) \in \mathfrak{X}(\mathcal{I}_x)$ considered as

$$y \to R_{(x,y)}(X_1, X_2), \tag{3.16}$$

and their successive convariant derivatives are tangent to the holonomy group [31]. It follows that the Lie subalgebra of vector fields on the indicatrix $\mathcal{I}_x$ generated by the curvature vector fields (3.16) and their successive covariant derivatives

$$\mathfrak{hol}^*_x(\mathcal{F}) := \big\langle \nabla_{X_k} \dots \nabla_{X_3} R(X_1, X_2) \;\big|\; X_1, \dots, X_k \in \mathfrak{X}(M) \big\rangle_{Lie}, \tag{3.17}$$

is Lie subalgebra of the holonomy algebra:

$$\mathfrak{hol}^*_x(\mathcal{F}) \subset \mathfrak{hol}_x(\mathcal{F}), \tag{3.18}$$

3.5 Definition. The Lie algebra $\mathfrak{hol}^*_x(\mathcal{F})$ defined in (3.17) is called the *infinitesimal holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$ at the point $x \in M$.

From (3.18) and (3.14) we obtain the inclusions of Lie algebras

$$\mathfrak{hol}^*_x(\mathcal{F}) \subset \mathfrak{hol}_x(\mathcal{F}) \subset \mathfrak{X}(\mathcal{I}_x). \tag{3.19}$$

Moreover, by (3.15),

$$\exp\big(\mathfrak{hol}^*_x(\mathcal{F})\big) \subset \exp\big(\mathfrak{hol}_x(\mathcal{F})\big) \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}iff(\mathcal{I}_x). \tag{3.20}$$

Since $\mathfrak{hol}^*_x(\mathcal{F})$ is a Lie subalgebra of $\mathfrak{hol}_x(\mathcal{F})$, we have

$$\dim\big(\mathfrak{hol}^*_x(\mathcal{F})\big) \leq \dim\big(\mathfrak{hol}_x(\mathcal{F})\big). \tag{3.21}$$

therefore the dimension of the infinitesimal holonomy algebra gives the lower estimate for the dimension of the holonomy group:

$$\dim(\mathfrak{hol}^*_x(\mathcal{F})) \leq \dim(\mathcal{H}ol_x(\mathcal{F})). \tag{3.22}$$

3.2 Maximal holonomy

Before proving the main theorem, we establish a fundamental topological property of the holonomy group in the simply connected case.

3.6 Lemma. *Let $(M, \mathcal{F})$ be an n-dimensional simply connected Finsler manifold, and let $x \in M$. Then the holonomy group at x is contained in the connected component of the identity in the diffeomorphism group of the indicatrix:*

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}\!\mathit{iff}_o(\mathcal{I}_x). \tag{3.23}$$

Proof. Assume that M is simply connected. Then every loop based at x is homotopic to the trivial loop. Since parallel transport depends continuously on the curve, the corresponding holonomy transformations depend continuously on the homotopy parameter. In particular, each holonomy transformation can be connected to the identity by a continuous path in $\mathcal{H}ol_x(\mathcal{F})$. Consequently, $\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}\!\mathit{iff}_o(\mathcal{I}_x)$.

□

3.7 Lemma. *Let $\mathcal{I}$ be a n-dimensional manifold and set*

$$G := \big\langle \exp\big(\mathfrak{X}(\mathcal{I})\big) \big\rangle_{\text{group}} \subset \mathcal{D}\!\mathit{iff}_o(\mathcal{I}). \tag{3.24}$$

Then G is a non-trivial normal subgroup of $\mathcal{D}\!\mathit{iff}_o(\mathcal{I})$. In particular, if $\mathcal{D}\!\mathit{iff}_o(\mathcal{I})$ is simple, then $G = \mathcal{D}\!\mathit{iff}_o(\mathcal{I})$.

Proof. We first show that G is invariant under conjugation by elements of $\mathcal{D}\!\mathit{iff}_o(\mathcal{I})$. Let $h \in \mathcal{D}\!\mathit{iff}_o(\mathcal{I})$ and $\xi \in \mathfrak{X}(\mathcal{I})$. Recall that the exponential map associates to the vector field ξ its flow $\exp(s\xi)$, $s \in \mathbb{R}$. The conjugation of $\exp(s\xi)$ by h is given by

$$h \circ \exp(s\xi) \circ h^{-1}. \tag{3.25}$$

A standard property of flows implies that this map is again the flow of a vector field, namely

$$h \circ \exp(s\xi) \circ h^{-1} = \exp\big(s\, Ad_h \xi\big), \tag{3.26}$$

where the vector field $Ad_h\xi$ is defined by

$$Ad_h\xi := h_*\xi \circ h^{-1} \in \mathfrak{X}(\mathcal{I}). \tag{3.27}$$

Since $Ad_h\xi$ is again a smooth vector field on N, it follows that $\exp(s\, Ad_h\xi) \in G$ whenever $\exp(s\xi) \in G$. Therefore, $hGh^{-1} \subset G$, which shows that G is a normal subgroup of $\mathcal{D}\!\mathit{iff}_o(\mathcal{I})$. Next, we prove that G is non-trivial. Since $\mathfrak{X}(\mathcal{I})$ contains non-zero vector fields, choose $\xi \neq 0$. For sufficiently small $s \neq 0$, the flow $\exp(s\xi)$ is not equal to the identity diffeomorphism. Hence G contains non-identity elements, and therefore it is a non-trivial subgroup of $\mathcal{D}\!\mathit{iff}_o(\mathcal{I})$. Finally, assume that $\mathcal{D}\!\mathit{iff}_o(\mathcal{I})$ is a simple group. By definition, a simple group has no non-trivial proper normal subgroups. Since G is a non-trivial normal subgroup of $\mathcal{D}\!\mathit{iff}_o(\mathcal{I})$, it follows that $G = \mathcal{D}\!\mathit{iff}_o(\mathcal{I})$. This completes the proof. □

We are now in a position to state the main theorem of this section, which provides a sufficient condition for the holonomy group to be maximal in terms of the holonomy algebra in the n-dimensional case.

3.8 Theorem. *Let $(M, \mathcal{F})$ be an n-dimensional simply connected Finsler manifold and let $x \in M$. If the holonomy algebra $\mathfrak{hol}_x(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_x)$ of smooth vector fields on the indicatrix $\mathcal{I}_x$, then the holonomy group at x is maximal. More precisely,*

$$\overline{\mathcal{H}ol_x(\mathcal{F})} \cong \mathcal{D}iff_o(\mathbb{S}^{n-1}), \tag{3.28}$$

where $\mathcal{D}iff_o(\mathbb{S}^{n-1})$ denotes the connected component of the identity in the diffeomorphism group of the $(n-1)$-dimensional sphere.

Proof. Let $(M, \mathcal{F})$ be an n-dimensional simply connected Finsler manifold and fix a point $x \in M$. By Lemma 3.6, we have

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}iff_o(\mathcal{I}_x). \tag{3.29}$$

On the other hand, by hypothesis, the infinitesimal holonomy algebra $\mathfrak{hol}_x(F)$ is dense in $\mathfrak{X}(\mathcal{I}_x)$, therefore its closure satisfies

$$\overline{\mathfrak{hol}_x(\mathcal{F})} = \mathfrak{X}(\mathcal{I}_x). \tag{3.30}$$

Since the exponential mapping

$$\exp : \mathfrak{X}(\mathcal{I}_x) \longrightarrow \mathcal{D}iff_o(\mathcal{I}_x)$$

is continuous (cf. [49, Lemma 4.1]), we obtain

$$\exp\big(\overline{\mathfrak{hol}_x(\mathcal{F})}\big) \subset \overline{\exp\big(\mathfrak{hol}_x(\mathcal{F})\big)}. \tag{3.31}$$

Moreover, by the definition of the infinitesimal holonomy algebra (3.20),

$$\exp\big(\mathfrak{hol}_x(\mathcal{F})\big) \subset \overline{\mathcal{H}ol_x(\mathcal{F})}. \tag{3.32}$$

Hence, taking into account (3.30), (3.31), (3.32), and (3.29), respectively, we obtain

$$\exp\big(\mathfrak{X}(\mathcal{I}_x)\big) = \exp\big(\overline{\mathfrak{hol}_x(\mathcal{F})}\big) \subset \overline{\exp\big(\mathfrak{hol}_x(\mathcal{F})\big)} \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}iff_o(\mathcal{I}_x). \tag{3.33}$$

For the generated groups this yields

$$\big\langle \exp\big(\mathfrak{X}(\mathcal{I}_x)\big)\big\rangle_{\text{group}} \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}iff_o(\mathcal{I}_x). \tag{3.34}$$

Applying Lemma 3.7 with $\mathcal{I} = \mathcal{I}_x$, we obtain that the group $\big\langle \exp\big(\mathfrak{X}(\mathcal{I}_x)\big)\big\rangle_{\text{group}}$ is a non-trivial normal subgroup of $\mathcal{D}iff_o(\mathcal{I}_x)$. On the other hand, since the indicatrix $\mathcal{I}_x$ is a closed manifold, it follows from Thurston's theorem [60, Theorem 1] that the group $\mathcal{D}iff_o(\mathcal{I}_x)$ is simple, hence its only non-trivial normal subgroup is itself. Therefore,

$$\big\langle \exp\big(\mathfrak{X}(\mathcal{I}_x)\big)\big\rangle_{\text{group}} = \mathcal{D}iff_o(\mathcal{I}_x). \tag{3.35}$$

Substituting this into (3.34), we obtain

$$\mathcal{D}\mathit{iff}_o(\mathcal{I}_x) \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}\mathit{iff}_o(\mathcal{I}_x), \tag{3.36}$$

that is,

$$\overline{\mathcal{H}ol_x(\mathcal{F})} = \mathcal{D}\mathit{iff}_o(\mathcal{I}_x). \tag{3.37}$$

Finally, since $\mathcal{I}_x$ is diffeomorphic to $\mathbb{S}^{n-1}$, their diffeomorphism groups and their connected components are isomorphic. Hence,

$$\overline{\mathcal{H}ol_x(\mathcal{F})} \cong \mathcal{D}\mathit{iff}_o(\mathbb{S}^{n-1}), \tag{3.38}$$

which completes the proof. □

Theorem 3.8 provides a general criterion for maximal holonomy in simply connected n-dimensional Finsler manifolds in terms of the density of the holonomy algebra. In the following section, we apply this criterion to spherically symmetric projective Finsler metrics.

3.3 Spherically symmetric projective Finsler metrics

In the previous section, we established a general criterion for maximal holonomy in simply connected n-dimensional Finsler manifolds in terms of the density of the infinitesimal holonomy algebra. In order to illustrate the strength of this criterion and to obtain concrete geometric results, we now apply it to a distinguished class of Finsler metrics. Projective Finsler metrics with constant flag curvature form one of the most important and well-studied families in Finsler geometry. In particular, Z. Shen [56] obtained a complete classification of such metrics in the x-analytic case. He showed that a projective Finsler metric $\mathcal{F}$ with constant flag curvature is completely determined, in an x_0-centered coordinate system, by the functions $\psi(y) = \mathcal{F}(0, y)$ and $\phi(y) = \mathcal{P}(0, y)$, where $\mathcal{P}$ is the projective factor.

In this section, we focus on the geometrically natural situation in which these data are spherically symmetric. More precisely, we assume that at a fixed point $x_0 \in M$ the Finsler function and the projective factor are both proportional to the Euclidean norm, that is,

$$\mathcal{F}(x_0, y) = c_1\|y\|, \qquad \mathcal{P}(x_0, y) = c_2\|y\|, \tag{3.39}$$

where $c_1, c_2 \neq 0$ are constants. Since multiplying the Finsler function by a positive constant does not affect the geodesic equation, the parallel translation, nor the holonomy structure, we may normalize the metric by choosing $c_1 = 1$. Thus, without loss of generality, we may assume that

$$\mathcal{F}(x_0, y) = \|y\|, \qquad \mathcal{P}(x_0, y) = c\,\|y\|, \tag{3.40}$$

where $c \neq 0$ is a constant. Under this normalization, the indicatrix at x_0 is given by

$$\mathcal{I}_{x_0} = \{\, y \in T_{x_0}M \mid \mathcal{F}(x_0, y) = 1 \,\} = \{\, y \in T_{x_0}M \mid \|y\| = 1 \,\} \cong \mathbb{S}^{n-1} \subset \mathbb{R}^n, \tag{3.41}$$

that is, the indicatrix coincides with the $(n-1)$-dimensional Euclidean sphere.

3.9 Proposition. *Under the assumptions* (3.40)*, the geodesic coefficients at x_0 are given by*

$$\begin{aligned} G^i(x_0,y) &= c\|y\|\, y^i, \\ G^i_j(x_0,y) &= c\left(\frac{y^iy^j}{\|y\|} + \|y\|\delta^i_j\right), \\ G^i_{jk}(x_0,y) &= c\left(\frac{y^i}{\|y\|}\delta^j_k + \frac{y^j}{\|y\|}\delta^i_k + \frac{y^k}{\|y\|}\delta^i_j - \frac{y^iy^jy^k}{\|y\|^3}\right). \end{aligned} \tag{3.42}$$

Moreover, the projective factor satisfies

$$\frac{\partial \mathcal{P}}{\partial x^m}(x_0,y) = (c^2-\lambda)y^m, \tag{3.43}$$

and the derivatives of the geodesic coefficients with respect to the base coordinates satisfy

$$\frac{\partial G^i_k}{\partial x^m}(x_0,y) = (c^2-\lambda)\big(y^i\delta^m_k + y^m\delta^i_k\big). \tag{3.44}$$

$$\frac{\partial G^i_{kl}}{\partial x^m} = (c^2-\lambda)(\delta^m_k\delta^i_l + \delta^m_l\delta^i_k) \tag{3.45}$$

Proof. Throughout the proof we work in an x_0–centered coordinate system and use the normalization (3.40), i.e.

$$\mathcal{F}(x_0,y) = \|y\|, \qquad \mathcal{P}(x_0,y) = c\,\|y\| \quad (c \neq 0).$$

Since F is projective, its spray coefficients have the form

$$G^i(x,y) = \mathcal{P}(x,y)\,y^i. \tag{3.46}$$

Evaluating (3.46) at $x = x_0$ and using $\mathcal{P}(x_0,y) = c\|y\|$ yields

$$G^i(x_0,y) = c\|y\|\,y^i. \tag{3.47}$$

Differentiating with respect to y^j gives

$$G^i{}_j(x_0,y) = \frac{\partial}{\partial y^j}\big(c\|y\|\,y^i\big) = c\left(\frac{y^j}{\|y\|}y^i + \|y\|\delta^i_j\right) = c\left(\frac{y^iy^j}{\|y\|} + \|y\|\delta^i_j\right), \tag{3.48}$$

and a further differentiation with respect to y^k yields

$$\begin{aligned} G^i{}_{jk}(x_0,y) &= \frac{\partial}{\partial y^k}\left[c\left(\frac{y^iy^j}{\|y\|} + \|y\|\delta^i_j\right)\right] \\ &= c\left(\frac{\delta^i_k y^j + \delta^j_k y^i}{\|y\|} - \frac{y^iy^jy^k}{\|y\|^3} + \frac{y^k}{\|y\|}\delta^i_j\right) \\ &= c\left(\frac{y^i}{\|y\|}\delta^j_k + \frac{y^j}{\|y\|}\delta^i_k + \frac{y^k}{\|y\|}\delta^i_j - \frac{y^iy^jy^k}{\|y\|^3}\right). \end{aligned} \tag{3.49}$$

which proves (3.42). Next we compute $\partial\mathcal{P}/\partial x^m$ at x_0. For projective Finsler metrics with constant flag curvature λ, Chern–Shen [17, Lemma 8.2.1] gives

$$\frac{\partial\mathcal{P}}{\partial x^m} = \mathcal{P}\,\frac{\partial\mathcal{P}}{\partial y^m} - \lambda\,\mathcal{F}\,\frac{\partial\mathcal{F}}{\partial y^m} = \tfrac{1}{2}\,\frac{\partial(\mathcal{P}^2-\lambda\mathcal{F}^2)}{\partial y^m}. \tag{3.50}$$

At $x = x_0$ we have $\mathcal{P}(x_0,y) = c\|y\|$ and $\mathcal{F}(x_0,y) = \|y\|$, hence $\mathcal{P}^2 - \lambda\mathcal{F}^2 = (c^2 - \lambda)\|y\|^2$. Since $\partial_{y^m}\|y\|^2 = 2y^m$, the last expression in (3.50) yields

$$\frac{\partial\mathcal{P}}{\partial x^m}(x_0,y) = \tfrac{1}{2}\,(c^2-\lambda)\,\partial_{y^m}\|y\|^2 = (c^2-\lambda)\,y^m, \tag{3.51}$$

which is (3.43). Finally, differentiating (3.46) with respect to y^k gives

$$G^i{}_k = \frac{\partial\mathcal{P}}{\partial y^k}\,y^i + \mathcal{P}\,\delta^i_k. \tag{3.52}$$

Differentiating (3.52) with respect to x^m and evaluating at $x = x_0$ we obtain

$$\frac{\partial G^i{}_k}{\partial x^m}(x_0,y) = \frac{\partial^2\mathcal{P}}{\partial x^m\partial y^k}(x_0,y)\,y^i + \frac{\partial\mathcal{P}}{\partial x^m}(x_0,y)\,\delta^i_k. \tag{3.53}$$

From (3.43) we have $\partial\mathcal{P}/\partial x^m(x_0,y) = (c^2-\lambda)y^m$, hence

$$\frac{\partial^2\mathcal{P}}{\partial x^m\partial y^k}(x_0,y) = \frac{\partial}{\partial y^k}\big((c^2-\lambda)y^m\big) = (c^2-\lambda)\delta^m_k. \tag{3.54}$$

Substituting these into the previous identity gives

$$\frac{\partial G^i{}_k}{\partial x^m}(x_0,y) = (c^2-\lambda)\delta^m_k\,y^i + (c^2-\lambda)y^m\,\delta^i_k = (c^2-\lambda)\big(y^i\delta^m_k + y^m\delta^i_k\big), \tag{3.55}$$

which is (3.44). Differentiating (3.53) with respect to y^l, we find

$$\frac{\partial G^i_{kl}}{\partial x^m} = \frac{\partial^3\mathcal{P}}{\partial x^m\partial y^k\partial y^l}y^i + \frac{\partial^2\mathcal{P}}{\partial x^m\partial y^k}\delta^i_l + \frac{\partial^2\mathcal{P}}{\partial x^m\partial y^l}\delta^i_k = (c^2-\lambda)(\delta^m_k\delta^i_l + \delta^m_l\delta^i_k). \tag{3.56}$$

This completes the proof. □

3.10 Corollary. *Under the assumptions of Proposition 3.9, the curvature tensor at x_0 is given by*

$$R^l_{ij}(x_0,y) = \lambda\big(\delta^l_j y^i - \delta^l_i y^j\big), \tag{3.57}$$

and the corresponding curvature vector fields on the indicatrix are

$$\xi_{ij} = \lambda\left(y^i\frac{\partial}{\partial y^j} - y^j\frac{\partial}{\partial y^i}\right). \tag{3.58}$$

Proof. By (2.16) the curvature tensor of the canonical connection can be written at $x = x_0$ in local coordinates as

$$R^l_{ij}(x_0,y) = \frac{\partial G^l_i}{\partial x^j}(x_0,y) - \frac{\partial G^l_j}{\partial x^i}(x_0,y) + G^m_i(x_0,y)\,G^l_{jm}(x_0,y) - G^m_j(x_0,y)\,G^l_{im}(x_0,y). \tag{3.59}$$

Using (3.44), we obtain

$$\frac{\partial G^l_i}{\partial x^j}(x_0,y)-\frac{\partial G^l_j}{\partial x^i}(x_0,y)=(c^2-\lambda)\big(y^l\delta^j_i+y^j\delta^l_i-y^l\delta^i_j-y^i\delta^l_j\big)=(c^2-\lambda)\big(y^j\delta^l_i-y^i\delta^l_j\big) \tag{3.60}$$

On the other hand, using (3.42), a direct contraction over the index m yields

$$G^m{}_i(x_0,y)\,G^l{}_{jm}(x_0,y)-G^m{}_j(x_0,y)\,G^l{}_{im}(x_0,y)=c^2\big(y^i\delta^l_j-y^j\delta^l_i\big). \tag{3.61}$$

Inserting the above identities into (3.59) we obtain

$$R^l{}_{ij}(x_0,y)=(c^2-\lambda)\big(y^j\delta^l_i-y^i\delta^l_j\big)+c^2\big(y^i\delta^l_j-y^j\delta^l_i\big)=\lambda\big(\delta^l_jy^i-\delta^l_iy^j\big), \tag{3.62}$$

which proves (3.57). Finally, the curvature vector field corresponding to the pair (i,j) is $\xi_{ij}=R^s{}_{ij}(x_0,y)\,\frac{\partial}{\partial y^s}$. Substituting (3.57) gives

$$\xi_{ij}=\lambda\left(\delta^s_jy^i-\delta^s_iy^j\right)\frac{\partial}{\partial y^s}=\lambda\left(y^i\frac{\partial}{\partial y^j}-y^j\frac{\partial}{\partial y^i}\right), \tag{3.63}$$

which is (3.58). This completes the proof. □

We introduce a multi-index notation. For a multi-index $\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n$ we define its length by $\ell(\mathbf{m})=m_1+\cdots+m_n$, and set

$$\mathbf{y}^{\mathbf{m}}:=\prod_{k=1}^{n}(y^k)^{m_k}=(y^1)^{m_1}\cdots(y^n)^{m_n}.$$

For each integer $p\geq 0$, we define the real vector space

$$\mathcal{A}_p:=\operatorname{span}_{\mathbb{R}}\left\{\frac{\mathbf{y}^{\mathbf{m}}}{\|y\|^{\ell(\mathbf{m})}}\,\xi_{ij}\Big|_{\widehat{T}_{x_0}M}\;\Big|\;1\leq i<j\leq n,\;|\ell(\mathbf{m})=p\right\}, \tag{3.64}$$

and we introduce the Lie algebra

$$\mathcal{A}:=\oplus_{p=0}^{\infty}\mathcal{A}_p. \tag{3.65}$$

3.11 Remark. The elements of $\mathcal{A}$ can be interpreted as 1-homogeneous vector fields on the slit tangent space $\widehat{T}_{x_0}M$. Equivalently, after restriction to the indicatrix $\mathcal{I}_{x_0}\simeq\mathbb{S}^{n-1}$, they can be regarded as vector fields on $\mathbb{S}^{n-1}$ with polynomial coefficients. Indeed, in (3.64) the factor $\|y\|^{-|\mathbf{m}|}$ is identically equal to 1 on the indicatrix, since $\|y\|=1$ on $\mathcal{I}_{x_0}$.

The Lie algebra $\mathcal{A}$ constructed above plays a fundamental role in the description of the infinitesimal holonomy algebra. We now show that it is contained in $\mathfrak{hol}^*_{x_0}(\mathcal{F})$.

3.12 Lemma. *The Lie algebra $\mathcal{A}$ satisfies $\mathcal{A}\subset\mathfrak{hol}^*_{x_0}(\mathcal{F})$.*

Proof. Recall that the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(F)$ is generated by the curvature vector fields, their successive horizontal covariant derivatives, and all Lie brackets of these vector fields. We prove the statement by mathematical induction on

$p \geq 0$, showing that every generator of $\mathcal{A}_p$ can be expressed as a linear combination of elements of $\mathfrak{hol}^*_{x_0}(\mathcal{F})$.

Base step ($p = 0$). By definition, the elements of $\mathcal{A}_0$ are linear combinations of the curvature vector fields ξ_{ij} with constant coefficients. Since each curvature vector field ξ_{ij} belongs to $\mathfrak{hol}^*_{x_0}(\mathcal{F})$, it follows immediately that $\mathcal{A}_0 \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$.

Induction step for $p = 1$. By Lemma 2.13, the horizontal covariant derivative of the curvature tensor vanishes. Consequently, for the curvature vector fields $\xi_{ij} = R\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big)$, their horizontal covariant derivatives satisfy

$$\nabla_k \xi_{ij} = G^s_{ki}\, \xi_{sj} + G^s_{kj}\, \xi_{is}, \tag{3.66}$$

At the particular point $x_o \in M$, using the antisymmetry of the curvature tensor, we may rewrite (3.66) in the form

$$\nabla_k \xi_{ij} = G^s_{ki}\, \xi_{sj} - G^s_{kj}\, \xi_{si}. \tag{3.67}$$

Expanding the summation over the index s, we obtain

$$\nabla_k \xi_{ij} = \sum_{s=1}^{n} \left(G^s_{ki}\, \xi_{sj} - G^s_{kj}\, \xi_{si} \right). \tag{3.68}$$

Separating in (3.68) the terms corresponding to $s = i$ and $s = j$, we obtain

$$\nabla_k \xi_{ij} = \left(G^i_{ki}\, \xi_{ij} - G^i_{kj}\, \xi_{ii} \right) + \left(G^j_{ki}\, \xi_{jj} - G^j_{kj}\, \xi_{ji} \right) + \sum_{\substack{s=1 \\ s \neq i,j}}^{n} \left(G^s_{ki}\, \xi_{sj} - G^s_{kj}\, \xi_{si} \right). \tag{3.69}$$

Since $\xi_{ii} = 0$ and $\xi_{jj} = 0$, and $\xi_{ji} = -\xi_{ij}$, this reduces to

$$\nabla_k \xi_{ij} = \left(G^i_{ki} + G^j_{kj} \right) \xi_{ij} + \sum_{\substack{s=1 \\ s \neq i,j}}^{n} \left(G^s_{ki}\, \xi_{sj} - G^s_{kj}\, \xi_{si} \right). \tag{3.70}$$

We now determine the coefficient of the curvature vector field ξ_{ij} in (3.70). This coefficient is given by

$$G^i_{ki} + G^j_{kj}. \tag{3.71}$$

Using the explicit expression of the coefficients G^s_{km} at x_0 (Proposition 3.9), we compute

$$\begin{aligned} G^i_{ki} + G^j_{kj} &= c \left(\frac{y^k}{\|y\|} + \frac{y^i}{\|y\|} \delta^i_k + \frac{y^i}{\|y\|} \delta^k_i - \frac{y^k (y^i)^2}{\|y\|^3} \right) \\ &\quad + c \left(\frac{y^k}{\|y\|} + \frac{y^j}{\|y\|} \delta^j_k + \frac{y^j}{\|y\|} \delta^k_j - \frac{y^k (y^j)^2}{\|y\|^3} \right) \\ &= c \left(2 \frac{y^k}{\|y\|} + 2 \frac{y^i}{\|y\|} \delta^i_k + 2 \frac{y^j}{\|y\|} \delta^j_k - \frac{y^k \left((y^i)^2 + (y^j)^2 \right)}{\|y\|^3} \right). \end{aligned} \tag{3.72}$$

We now examine the terms in (3.70) corresponding to indices $s \neq i, j$. These terms are of the form $G^s_{ki}\,\xi_{sj} - G^s_{kj}\,\xi_{si}$. Using the explicit expression of the coefficients G^s_{km} at x_0 (Proposition 3.9), we obtain

$$\begin{aligned} G^s_{ki}\,\xi_{sj} &= c\left(\frac{y^k}{\|y\|}\delta^s_i + \frac{y^i}{\|y\|}\delta^s_k + \frac{y^s}{\|y\|}\delta^k_i - \frac{y^k y^i y^s}{\|y\|^3}\right)\xi_{sj}, \\ G^s_{kj}\,\xi_{si} &= c\left(\frac{y^k}{\|y\|}\delta^s_j + \frac{y^j}{\|y\|}\delta^s_k + \frac{y^s}{\|y\|}\delta^k_j - \frac{y^k y^j y^s}{\|y\|^3}\right)\xi_{si}. \end{aligned} \tag{3.73}$$

Since $s \neq i, j$, we have $\delta^s_i = \delta^s_j = 0$. Therefore, subtracting the two expressions in (3.73) yields

$$\begin{aligned} G^s_{ki}\,\xi_{sj} - G^s_{kj}\,\xi_{si} = {} & c\left(\frac{y^i}{\|y\|}\delta^s_k + \frac{y^s}{\|y\|}\delta^k_i - \frac{y^k y^i y^s}{\|y\|^3}\right)\xi_{sj} \\ & - c\left(\frac{y^j}{\|y\|}\delta^s_k + \frac{y^s}{\|y\|}\delta^k_j - \frac{y^k y^j y^s}{\|y\|^3}\right)\xi_{si}. \end{aligned} \tag{3.74}$$

A direct arrangement shows that the right-hand side of (3.74) can be written as a linear combination of the curvature vector fields ξ_{ij}, ξ_{is} and ξ_{js}. More precisely, we obtain

$$G^s_{ki}\,\xi_{sj} - G^s_{kj}\,\xi_{si} = c\left(\frac{y^s}{\|y\|}\delta^s_k - \frac{y^k (y^s)^2}{\|y\|^3}\right)\xi_{ij} + c\frac{y^s}{\|y\|}\delta^k_j\,\xi_{is} - c\frac{y^s}{\|y\|}\delta^k_i\,\xi_{js}. \tag{3.75}$$

Coming back to the decomposition (3.70), we have

$$\nabla_k \xi_{ij} = \left(G^i_{ki} + G^j_{kj}\right)\xi_{ij} + \sum_{\substack{s=1 \\ s\neq i,j}}^{n} \left(G^s_{ki}\,\xi_{sj} - G^s_{kj}\,\xi_{si}\right). \tag{3.76}$$

Using (3.72)

$$G^s_{ki}\,\xi_{sj} - G^s_{kj}\,\xi_{si} = c\left(\frac{y^s}{\|y\|}\delta^s_k - \frac{y^k (y^s)^2}{\|y\|^3}\right)\xi_{ij} + c\frac{y^s}{\|y\|}\delta^k_j\,\xi_{is} - c\frac{y^s}{\|y\|}\delta^k_i\,\xi_{js}, \tag{3.77}$$

valid for every $s \neq i, j$, we obtain

$$\begin{aligned} \nabla_k \xi_{ij} = {} & c\left(2\frac{y^k}{\|y\|} + 2\frac{y^i}{\|y\|}\delta^i_k + 2\frac{y^j}{\|y\|}\delta^j_k - \frac{y^k\left((y^i)^2 + (y^j)^2\right)}{\|y\|^3}\right)\xi_{ij} \\ & + c\sum_{\substack{s=1 \\ s\neq i,j}}^{n}\left(\frac{y^s}{\|y\|}\delta^s_k - \frac{y^k (y^s)^2}{\|y\|^3}\right)\xi_{ij} \\ & + c\sum_{\substack{s=1 \\ s\neq i,j}}^{n}\frac{y^s}{\|y\|}\,\delta^k_j\,\xi_{is} - c^2\sum_{\substack{s=1 \\ s\neq i,j}}^{n}\frac{y^s}{\|y\|}\,\delta^k_i\,\xi_{js}. \end{aligned} \tag{3.78}$$

Finally, since

$$\frac{(y^i)^2 + (y^j)^2}{\|y\|^2} + \sum_{\substack{s=1 \\ s\neq i,j}}^{n} \frac{(y^s)^2}{\|y\|^2} = 1, \tag{3.79}$$

the ξ_{ij}–coefficients in (3.78) simplify to

$$2\frac{y^k}{\|y\|} + 2\frac{y^i}{\|y\|}\delta^i_k + 2\frac{y^j}{\|y\|}\delta^j_k - \frac{y^k}{\|y\|} + \frac{y^k}{\|y\|}\delta^k_k, \tag{3.80}$$

and therefore we arrive at the compact identity

$$\nabla_k \xi_{ij} = 2c\frac{y^k}{\|y\|}\,\xi_{ij} + c\,\delta^k_j \sum_{s=1}^{n} \frac{y^s}{\|y\|}\,\xi_{is} - c\,\delta^k_i \sum_{s=1}^{n} \frac{y^s}{\|y\|}\,\xi_{js}. \tag{3.81}$$

In particular, from (3.81) we obtain the following special cases. Let $i \neq j$.
For $i \neq j$ and $k \neq i, j$, we have

$$\nabla_k \xi_{ij} = 2c\,\frac{y^k}{\|y\|}\,\xi_{ij}. \tag{3.82}$$

For $i \neq j$, the covariant derivative in the direction x^i is given by

$$\nabla_i \xi_{ij} = 3c\,\frac{y^i}{\|y\|}\,\xi_{ij} - c\sum_{\substack{s=1 \\ s\neq i}}^{n} \frac{y^s}{\|y\|}\,\xi_{js}. \tag{3.83}$$

Similarly, in the direction x^j we obtain

$$\nabla_j \xi_{ij} = 3c\,\frac{y^j}{\|y\|}\,\xi_{ij} + c\sum_{\substack{s=1 \\ s\neq j}}^{n} \frac{y^s}{\|y\|}\,\xi_{is}. \tag{3.84}$$

As a consequence, for any pairwise distinct indices i, j, k we obtain

$$\frac{y^k}{\|y\|}\,\xi_{ij} = \frac{1}{2c}\,\nabla_k \xi_{ij} \in \mathfrak{hol}^*_{x_o}(F), \qquad i \neq j,\ i \neq k,\ j \neq k, \tag{3.85}$$

showing that the vector fields of the form $\frac{y^k}{\|y\|}\xi_{ij}$ belong to $\mathfrak{hol}^*_{x_o}(F)$. Moreover, for $k = i$, using (3.83), we obtain the linear system,

$$\begin{pmatrix} \nabla_1 \xi_{1j} \\ \vdots \\ [\nabla_j \xi_{jj}] \\ \vdots \\ \nabla_n \xi_{nj} \end{pmatrix} = c \begin{pmatrix} 3 & 1 & \dots & 1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 3 \end{pmatrix} \begin{pmatrix} \frac{y^1}{\|y\|}\xi_{1j} \\ \vdots \\ \left[\frac{y^j}{\|y\|}\xi_{jj}\right] \\ \vdots \\ \frac{y^n}{\|y\|}\xi_{nj} \end{pmatrix}, \qquad j = 1, \dots, n, \tag{3.86}$$

where in the column vectors the trivially vanishing terms indicated in square brackets are omitted. Since the $(n-1)\times(n-1)$ matrix in (3.86) is invertible, it follows that each vector field $\frac{y^i}{\|y\|}\xi_{ij}$ can be expressed as a linear combination of covariant derivatives of curvature vector fields. Consequently, $\frac{y^i}{\|y\|}\xi_{ij} \in \mathfrak{hol}^*_{x_o}(F)$, and therefore the generating elements of $\mathcal{A}_1$ belong to $\mathfrak{hol}^*_{x_o}(F)$. Hence, $\mathcal{A}_1 \subset \mathfrak{hol}^*_{x_o}(F)$.

Induction step for $p = 2$. By definition, the second horizontal covariant derivatives of the curvature vector fields belong to the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(F)$. More precisely, for any indices i, j, k, m we have$\nabla_m \nabla_k \xi_{ij} \in \mathfrak{hol}^*_{x_o}(F)$. Recall that the first horizontal covariant derivatives of the curvature vector fields are given by

$$\nabla_{x^k}\xi_{ij} = G^s_{ki}\,\xi_{sj} - G^s_{kj}\,\xi_{si}. \tag{3.87}$$

Applying ∇_{x^m} to (3.87), we obtain

$$\nabla_{x^m}\nabla_{x^k}\xi_{ij} = \nabla_{x^m}\left(G^s_{ki}\xi_{sj} - G^s_{kj}\xi_{si}\right), \tag{3.88}$$

that is,

$$\nabla_{x^m}\nabla_{x^k}\xi_{ij} = (\nabla_{x^m}G^s_{ki})\,\xi_{sj} - (\nabla_{x^m}G^s_{kj})\,\xi_{si} + G^s_{ki}\,(\nabla_{x^m}\xi_{sj}) - G^s_{kj}\,(\nabla_{x^m}\xi_{si}). \tag{3.89}$$

From the definition of the horizontal covariant derivative, we have

$$\nabla_{x^m}G^i_{jk} = \partial_{x^m}G^i_{jk} - G^l_m\,\partial_{y^l}G^i_{jk}. \tag{3.90}$$

Using the explicit expressions of the coefficients G^i_{jk} at x_0, we obtain

$$\begin{aligned}\nabla_m G^i_{jk}(x_0, y) = &-\lambda\big(\delta^m_k\delta^i_j + \delta^m_j\delta^i_k\big) - c^2\,\delta^i_m\delta^j_k - 3c^2\,\frac{y^iy^jy^ky^m}{\|y\|^4}\\ &+ \frac{c^2}{\|y\|^2}\Big(y^iy^j\delta^m_k + y^iy^k\delta^m_j + y^iy^m\delta^j_k + y^jy^k\delta^m_i + y^jy^m\delta^i_k + y^ky^m\delta^i_j\Big).\end{aligned} \tag{3.91}$$

Substituting (3.91) and (3.81) into (3.89), we obtain the explicit expression for the second horizontal covariant derivatives of the curvature vector fields – after simplification – gives

$$\begin{aligned}\nabla_m\nabla_k\xi_{ij} = &\ (\lambda + c^2)(\delta^m_j\xi_{ki} - \delta^m_i\xi_{kj}) + c^2(\delta^k_j\xi_{mi} - \delta^k_i\xi_{mj}) + 2(c^2 - \lambda)\delta^m_k\xi_{ij}\\ &+ 4\frac{c^2}{\|y\|^2}\Big(\delta^i_k y^my^s\xi_{sj} - \delta^j_k y^my^s\xi_{si} - \delta^j_m y^ky^s\xi_{si} - \delta^i_m y^ky^s\xi_{sj} - y^my^k\xi_{ij}\Big).\end{aligned} \tag{3.92}$$

In particular, we obtain the following special cases of the second horizontal covariant derivatives:

$$\nabla_m\nabla_k\xi_{ij} = -4c^2\frac{y^my^k}{\|y\|^2}\,\xi_{ij}, \tag{3.93a}$$

$$\nabla_k\nabla_k\xi_{ij} = 2(c^2 - \lambda)\,\xi_{ij} - 4c^2\frac{(y^k)^2}{\|y\|^2}\,\xi_{ij}, \tag{3.93b}$$

$$\nabla_i\nabla_i\xi_{ij} = -3\lambda\xi_{ij} - 4c^2\frac{(y^i)^2}{\|y\|^2}\,\xi_{ij}\xi_{sj}, \tag{3.93c}$$

$$\nabla_i \nabla_k \xi_{ij} = -(\lambda + c^2)\, \xi_{kj} - 4c^2 \frac{y^i y^k}{\|y\|^2}\, \xi_{ij} - 4c^2 \sum_{s=1}^{n} \frac{y^k y^s}{\|y\|^2}\, \xi_{sj}. \tag{3.93d}$$

Equation (3.93a) shows that $\frac{y^k y^m}{\|y\|^2}\, \xi_{ij}$, can be expressed as a constant multiple of the second covariant derivatives of the curvature vector fields. Hence, it belongs to the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Similarly, equations (3.93b) and (3.93c) imply that $\frac{(y^k)^2}{\|y\|^2}\, \xi_{ij}$, and $\frac{(y^i)^2}{\|y\|^2}\, \xi_{ij}$ can be expressed as linear combinations of curvature vector fields and their second horizontal covariant derivatives. Therefore, these vector fields also belong to the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Finally, equation (3.93d) can be interpreted as a linear system for the quantities $\frac{y^k y^i}{\|y\|^2}\, \xi_{ij}$. More precisely, we obtain the matrix equation

$$\begin{pmatrix} \nabla_1 \nabla_k \xi_{1j} + (c^2+\lambda)\xi_{kj} \\ \vdots \\ [\nabla_j \nabla_k \xi_{jj} + (c^2+\lambda)\xi_{kj}] \\ \vdots \\ \nabla_n \nabla_k \xi_{nj} + (c^2+\lambda)\xi_{kj} \end{pmatrix} = -4c^2 \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{y^k y^1}{\|y\|^2} \xi_{1j} \\ \vdots \\ \left[\frac{y^k y^j}{\|y\|^2} \xi_{jj}\right] \\ \vdots \\ \frac{y^k y^n}{\|y\|^2} \xi_{nj} \end{pmatrix}. \tag{3.94}$$

where the elements on the left-hand side belong to the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_0}(\mathcal{F})$. The $(n-1) \times (n-1)$ matrix appearing in (3.94) is regular; hence the terms $\frac{y^k y^i}{\|y\|^2}\, \xi_{ij}$ can be expressed as linear combinations of the elements on the left-hand side, which themselves belong to $\mathfrak{hol}^*_{x_0}(\mathcal{F})$. Consequently, $\frac{y^k y^i}{\|y\|^2}\, \xi_{ij} \in \mathfrak{hol}^*_{x_0}(\mathcal{F})$. Therefore, all elements generating $\mathcal{A}_2$ can be expressed as linear combinations of elements of the infinitesimal holonomy algebra. Hence, $\mathcal{A}_2 \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$.

Induction step for $p > 2$. we will assume that $\mathcal{A}_\ell \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$, for all $1 \leq \ell \leq p$, and we will show that $\mathcal{A}_{p+1} \subset \mathfrak{hol}^*_{x_0}(M, \mathcal{F})$. First, observe that

$$\sum_{s=1}^{n} \frac{y^s}{\|y\|} \xi_{ks} = \frac{1}{\lambda} \left(\sum_{s=1}^{n} \frac{y^s y^k}{\|y\|} \frac{\partial}{\partial y^s} - \|y\| \frac{\partial}{\partial y^k} \right) = \frac{1}{\lambda} \left(\frac{y^k}{\|y\|}\, C - \|y\| \frac{\partial}{\partial y^k} \right) \in \mathcal{A}_1, \tag{3.95}$$

where $\mathcal{C}$ denotes the canonical Liouville (radial) vector field. Let $\mathbf{m} = (m_1, \dots, m_n)$ be a multiindex of length $\ell(\mathbf{m}) = m_1 + \dots + m_n = p$, and consider the vector field

$$\frac{\mathbf{y}^{\mathbf{m}}}{\|\mathbf{y}\|^{\ell(\mathbf{m})}}\, \xi_{ij} = \frac{y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}}{\|y\|^p}\, \xi_{ij} \in \mathcal{A}_p. \tag{3.96}$$

By the induction hypothesis, $\mathcal{A}_1 \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$, and $\mathcal{A}_p \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$. Since $\mathfrak{hol}^*_{x_0}(\mathcal{F})$ is a Lie algebra, it follows that

$$[\mathcal{A}_1, \mathcal{A}_p] \subset \mathfrak{hol}^*_{x_0}(\mathcal{F}). \tag{3.97}$$

In particular, the Lie bracket of the elements (4.28) and (3.96) belongs to $\mathfrak{hol}^*_{x_0}(\mathcal{F})$. If the indices i, j, k are pairwise distinct, then

$$\left[\frac{\mathbf{y}^{\mathbf{m}}}{\|\mathbf{y}\|^{\ell(\mathbf{m})}}\, \xi_{ij}, \sum_{s=1}^{n} \frac{y^s}{\|y\|} \xi_{ks} \right] = m_k \frac{\mathbf{y}^{\mathbf{m}-\mathbf{1}_k}}{\|\mathbf{y}\|^{p-1}}\, \xi_{ij} - p \frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_k}}{\|\mathbf{y}\|^{p+1}}\, \xi_{ij}, \tag{3.98}$$

where $\mathbf{1_k} = (0, \ldots, 1, \ldots 0)$ denotes the multiindex with 1 in the kth position. By (3.97), the left-hand side of (3.100) belongs to $\mathfrak{hol}^*_{x_0}(\mathcal{F})$, and by the induction hypothesis, $\frac{\mathbf{y}^{\mathbf{m}-\mathbf{1}_k}}{\|\mathbf{y}\|^{p-1}}\,\xi_{ij} \in \mathcal{A}_{p-1} \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$. Therefore,

$$\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_k}}{\|\mathbf{y}\|^{p+1}}\,\xi_{ij} \in \mathfrak{hol}^*_{x_0}(\mathcal{F}), \qquad i \neq j,\ k \neq i,\ k \neq j. \tag{3.99}$$

Similarly, one computes

$$\left[\frac{\mathbf{y}^{\mathbf{m}}}{\|\mathbf{y}\|^{\ell(\mathbf{m})}}\,\xi_{ij}, \sum_{s=1}^{n}\frac{y^s}{\|y\|}\xi_{is}\right] = m_i\frac{\mathbf{y}^{\mathbf{m}-\mathbf{1}_i}}{\|\mathbf{y}\|^{p-1}}\,\xi_{ij} + (1-p)\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_i}}{\|\mathbf{y}\|^{p+1}}\,\xi_{ij} + \sum_{s=1, s\neq i}^{n}\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_s}}{\|\mathbf{y}\|^{p+1}}\,\xi_{sj}. \tag{3.100}$$

Using (3.99), we conclude that

$$\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_i}}{\|\mathbf{y}\|^{p+1}}\,\xi_{ij} \in \mathfrak{hol}^*_{x_0}(\mathcal{F}), \qquad i \neq j. \tag{3.101}$$

Combining (3.99) and (3.101), we obtain $\mathcal{A}_{p+1} \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$, which completes the proof. □

The previous lemma shows that, for every $p \in \mathbb{N}$, the space $\mathcal{A}_p$ is contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. that is by using (3.19) they are in the holonomy algebra $\mathfrak{hol}_{x_o}(\mathcal{F})$. In particular, this implies that $\mathfrak{hol}_{x_o}(\mathcal{F})$ contains all polynomial-type vector fields on the indicatrix $\mathcal{I}_{x_o}$ obtained from curvature vector fields by multiplication with with polynomial coefficients. Under the assumption (3.40), these vector fields form a dense subalgebra of the Lie algebra $\mathfrak{X}(\mathcal{I}_{x_o})$ of smooth vector fields on the indicatrix. As a consequence, the holonomy algebra is dense in $\mathfrak{X}(\mathcal{I}_{x_o})$, which we formulate precisely in the following proposition.

3.13 Proposition. *Let $(M, \mathcal{F})$ be a projectively flat, spherically symmetric Finsler manifold of constant flag curvature $\lambda \neq 0$, and let $x_o \in M$ be a point at which condition (3.40) is satisfied. Then the holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_{x_o})$ of smooth vector fields on the indicatrix $\mathcal{I}_{x_o}$.*

Proof. Let $(M, \mathcal{F})$ be a projectively flat, spherically symmetric Finsler manifold of constant curvature $\lambda \neq 0$, and let $x_o \in M$ be a point at which condition (3.40) is satisfied. According to Remark 3.11, the elements of $\mathcal{A}$ can be interpreted as vector fields on the indicatrix $\mathcal{I}_{x_o} \simeq \mathbb{S}^{n-1}$ with polynomial coefficients. More precisely,

$$\mathcal{A} = \left\{\, Q^{ij}\,\xi_{ij} \mid Q^{ij} \in \mathcal{P}ol(\mathbb{S}^{n-1}) \,\right\}, \tag{3.102}$$

where

$$\mathcal{P}ol(\mathbb{S}^{n-1}) = \mathbb{R}[y^1, \ldots, y^n]\big|_{\mathbb{S}^{n-1}} = \left\{ p\big|_{\mathbb{S}^{n-1}} \;\middle|\; p \in \mathbb{R}[y^1, \ldots, y^n] \right\} \tag{3.103}$$

denotes the algebra of polynomial functions on the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Nachbin's theorem [46, 47] provides an analogue of the Stone–Weierstrass theorem for algebras of real-valued C^k functions on a C^k-manifold, $k = 1, \ldots, \infty$. It states that if $\mathcal{S}$ is a subalgebra of $C^k(M)$ on a finite-dimensional C^k manifold M, and if $\mathcal{S}$ separates points of M and also separates tangent vectors in the sense that for every $x \in M$ and every $v \in T_xM$ there exists an $f \in \mathcal{S}$ such that $df_x(v) \neq 0$, then $\mathcal{S}$ is dense in $C^k(M)$. Clearly, the algebra $\mathcal{P}ol(\mathbb{S}^{n-1})$ of polynomial functions on $\mathbb{S}^{n-1}$ satisfies the assumptions of Nachbin's theorem. Consequently, $\mathcal{P}ol(\mathbb{S}^{n-1})$ is dense in $C^\infty(\mathbb{S}^{n-1})$ with respect to the C^∞-topology:

$$\overline{\mathcal{P}ol(\mathbb{S}^{n-1})} = C^\infty(\mathbb{S}^{n-1}). \tag{3.104}$$

On the other hand, any vector field $X \in \mathfrak{X}(\mathbb{S}^{n-1})$ can be written in the form

$$X = X^{ij}\,\xi_{ij}, \tag{3.105}$$

where X^{ij} are smooth functions on $\mathbb{S}^{n-1}$ and ξ_{ij} are the infinitesimal generators of rotations, which coincide with the curvature vector fields defined in (3.58). Hence,

$$\mathfrak{X}(\mathbb{S}^{n-1}) = \left\{\, X^{ij}\,\xi_{ij} \mid X^{ij} \in C^\infty(\mathbb{S}^{n-1}) \,\right\}. \tag{3.106}$$

Together with (3.104), this implies that (3.102) is dense in (3.106) with respect to the C^k-topology, that is,

$$\overline{\mathcal{A}} = \mathfrak{X}(\mathbb{S}^{n-1}). \tag{3.107}$$

By Lemma 4.10, we have

$$\mathcal{A} \subset \mathfrak{hol}^*_{x_o}(\mathcal{F}) \subset \mathfrak{hol}_{x_o}(\mathcal{F}) \subset \mathfrak{X}(\mathbb{S}^{n-1}), \tag{3.108}$$

and therefore

$$\overline{\mathcal{A}} \subset \overline{\mathfrak{hol}_{x_o}(\mathcal{F})} \subset \mathfrak{X}(\mathbb{S}^{n-1}). \tag{3.109}$$

Combining this inclusion with (3.107), we obtain

$$\overline{\mathfrak{hol}_{x_o}(\mathcal{F})} = \mathfrak{X}(\mathbb{S}^{n-1}), \tag{3.110}$$

that is, the infinitesimal holonomy algebra $\mathfrak{hol}_{x_o}(\mathcal{F})$ is dense in $\mathfrak{X}(\mathbb{S}^{n-1})$ with respect to the C^∞ topology. □

We now pass from the holonomy algebra to the holonomy group. In the simply connected case, the structure of the holonomy group is determined by its holonomy algebra. The following maximality result is a direct application of Theorem 3.8 from the previous section.

3.14 Theorem. *Let $(M, \mathcal{F})$ be a simply connected, projectively flat, spherically symmetric Finsler manifold of constant curvature $\lambda \neq 0$, and let $x_o \in M$ be a point at which condition (3.40) is satisfied. Then the holonomy group $\mathcal{H}ol_{x_o}(\mathcal{F})$ is maximal; that is, its closure is isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$, the connected component of the identity in the group of smooth diffeomorphisms of the $(n-1)$-dimensional sphere.*

Proof. The statement follows directly from Theorem 3.8 and Proposition 3.13. Let $(M, \mathcal{F})$ be a projectively flat, spherically symmetric Finsler manifold of constant curvature $\lambda \neq 0$, and let $x_o \in M$ satisfy condition (3.40). By Proposition 3.13, the holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_{x_o})$ of smooth vector fields on the indicatrix and since. On the other hand, Theorem 3.8 implies that, in this situation, the holonomy group is maximal in the sense that its closure coincides with the connected component of the identity of the full diffeomorphism group of the indicatrix. More precisely,

$$\overline{\mathcal{H}ol_{x_o}(\mathcal{F})} \cong \mathcal{D}iff_o(\mathbb{S}^{n-1}). \tag{3.111}$$

This completes the proof. □

We conclude this section by presenting two classical examples which illustrate the applicability of Theorem 3.14. In both cases, the metrics under consideration are projectively flat and have constant flag curvature, and the assumptions of the theorem can be verified explicitly at a fixed point. These examples show how the general theory applies to well-known Finsler metrics.

3.15 Example. (P. Funk [26, 27]) The *standard Funk manifold* $(\mathbb{D}^n, \mathcal{F})$ is defined on the unit open disk $\mathbb{D}^n \subset \mathbb{R}^n$ by the Finsler metric

$$\mathcal{F}(x,y) = \frac{\sqrt{|y|^2 - \big(|x|^2|y|^2 - \langle x,y\rangle^2\big)}}{1-|x|^2} \pm \frac{\langle x,y\rangle}{1-|x|^2}. \tag{3.112}$$

This metric is projectively flat and has constant flag curvature $\lambda = -\frac{1}{4}$. Its projective factor can be computed from (2.37) and is given by

$$\mathcal{P}(x,y) = \frac{1}{2}\,\frac{\pm\sqrt{|y|^2 - \big(|x|^2|y|^2 - \langle x,y\rangle^2\big)} + \langle x,y\rangle}{1-|x|^2}. \tag{3.113}$$

At the origin $x_o = (0, \ldots, 0) \in \mathbb{D}^n$, we obtain

$$\mathcal{F}(x_o, y) = |y|, \qquad \mathcal{P}(x_o, y) = \pm\frac{1}{2}\,|y|. \tag{3.114}$$

Hence the standard Funk metric satisfies the assumptions of Theorem 3.14 at x_o. Consequently, its holonomy group $\mathcal{H}ol_{x_o}(\mathcal{F})$ is maximal and isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$.

3.16 Example. The *Bryant–Shen metrics* $\mathcal{F}_\alpha$, with $|\alpha| < \frac{\pi}{2}$, form a one-parameter family of projectively flat Finsler metrics with constant flag curvature $\lambda = 1$. They are given by

$$\mathcal{F}_\alpha = \sqrt{\frac{\sqrt{A}+B}{2D} + \left(\frac{C}{D}\right)^2} + \frac{C}{D}, \qquad \mathcal{P}_\alpha = -\sqrt{\frac{\sqrt{A}-B}{2D} - \left(\frac{C}{D}\right)^2} - \frac{\widetilde{C}}{D}, \tag{3.115}$$

where

$$\begin{aligned}
A &= \big(\cos(2\alpha)|y|^2 + |x|^2|y|^2 - \langle x,y\rangle^2\big)^2 + \big(\sin(2\alpha)|y|^2\big)^2,\\
B &= \cos(2\alpha)|y|^2 + |x|^2|y|^2 - \langle x,y\rangle^2,\\
C &= \sin(2\alpha)\langle x,y\rangle,\\
\widetilde{C} &= \big(\cos(2\alpha) + |x|^2\big)\langle x,y\rangle,\\
D &= |x|^4 + 2\cos(2\alpha)|x|^2 + 1.
\end{aligned} \tag{3.116}$$

In Euclidean coordinates centered at the origin $0 \in \mathbb{R}^n$, the norm function and the projective factor satisfy

$$\mathcal{F}_\alpha(0,y) = |y|\cos\alpha, \qquad \mathcal{P}_\alpha(0,y) = |y|\sin\alpha, \qquad |\alpha| < \frac{\pi}{2}. \tag{3.117}$$

R. Bryant introduced and studied this class of Finsler metrics on $\mathbb{S}^2$ in [11, 12], where great circles are geodesics. Z. Shen later generalized this construction and obtained the expression (3.115) in [56, Example 7.1]. Since the assumptions of Theorem 3.14 are satisfied, the holonomy group $\mathcal{H}ol_{x_o}(\mathcal{F}_\alpha)$ of the Bryant–Shen metric is maximal and isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$.

4 Holonomy of Randers surfaces with constant flag curvature

Recent developments in Finsler geometry have shown that holonomy groups may be infinite-dimensional, and that this is in fact typical in the Finslerian setting. In contrast to the Riemannian case, where holonomy groups are always finite-dimensional Lie groups, Finsler manifolds exhibit a much richer behavior. As we could see in the previous chapter, the holonomy group of a Finsler manifold can be even infinite-dimensional [43, 31, 39]. This picture was further strengthened in [29], where it was shown that, on every manifold M, in the set of all Finsler metrics $\mathcal{F}(M)$, there exists an open and dense subset $\mathcal{F}_o(M) \subset \mathcal{F}(M)$, such that for every Finsler function $\mathcal{F} \in \mathcal{F}_o(M)$, the holonomy group $\mathcal{H}ol(\mathcal{F})$ is infinite-dimensional.

Thus, finite-dimensional holonomy appears to be exceptional in Finsler geometry. It occurs, for instance, for flat metrics, as well as for Riemannian, Berwald, and Landsberg-type Finsler metrics [9, 58, 35]. Moreover, in the class of locally projectively flat Randers manifolds, finite-dimensional holonomy occurs only in the flat or Riemannian case [43]. These observations lead to the following question:

1 Question. *Can a proper (i.e., non-Riemannian, non-Berwald, and non-Landsberg-type) non-flat Finsler metric have a finite-dimensional holonomy group?*

In this chapter, we address this question by studying the holonomy of Randers two-manifolds of constant flag curvature. Randers metrics are of particular geometric interest, since they provide some of the most natural and illustrative examples of non-Riemannian Finsler manifolds [4, 50], and they also have important applications. A Randers metric may be regarded as a Riemannian metric perturbed by a one-form, and, as shown in [6], such metrics arise naturally from Zermelo navigation problems on Riemannian manifolds. To analyze the holonomy of two-dimensional Randers manifolds with constant flag curvature, we rely on the classification results of D. Bao, C. Robles, and Z. Shen [6, Theorem 3]. We first recall the following characterization theorem.

4.1 Theorem ([6, Theorem 3])**.** *Let $(M, \mathcal{F})$ be a Randers manifold defined by the navigation data (h, W). Then $\mathcal{F}$ has constant flag curvature K if and only if the following conditions hold:*

1. *the Riemannian manifold (M, h) has constant sectional curvature $K + \frac{1}{16}\sigma^2$,*
2. *the vector field W is an infinitesimal homothety of h, satisfying $\mathcal{L}_W h = -\sigma h$.*

Moreover, the constant σ must be zero unless h is flat, in which case W must be a Killing vector field of h.

In [6], the authors provide a description of such metrics, which we state in the following theorem.

4.2 Theorem ([6, Proposition 5. and 6.]). *Let $\mathcal{F}(x, y)$ be a Randers metric with constant flag curvature K defined by the navigation data (h, W) on the n-dimensional manifold M. Then, up to local isometry, (h, W) must belong to one of the following families:*

- *$K = 0$, then h is the Euclidean metric, $W = Qx+C$, with $(Qx+C)\cdot(Qx+C) < 1$;*
- *$K > 0$, then h is $1/K$ times the standard metric on the unit n-sphere, and $W = Qx + C + (x \cdot C)x$, with $\frac{1}{1+(x\cdot x)}((Qx + C) \cdot (Qx + C) + (x \cdot C)^2) < K$,*
- *$K < 0$, there are two possibilities:*
  - · *Case $(\mathcal{K}_1^-)$: h is the Klein metric of sectional curvature K on the unit ball $\mathbb{B}^n$, and $W = Qx + C - (x \cdot C)x$, where $\frac{1}{1-(x\cdot x)}((Qx + C) \cdot (Qx + C) - (x \cdot C)^2) < |K|$,*
  - · *Case $(\mathcal{K}_2^-)$: h is the Euclidean metric, and $W = Qx + C - \frac{\sigma}{2}x$ satisfies $(Qx + C) \cdot (Qx + C) + \sigma x \cdot (\frac{\sigma}{4}x - C) < 1$, where $\sigma = \pm\sqrt{|K|}$,*

where $Q = (Q^i_j)$ is a constant skew-symmetric matrix and $C = (C_i)$ is a constant vector, $Qx = (Q^i_j x^j)$, $x = (x^i)$, and "$\cdot$" denotes the standard Euclidean dot product.

The classification given in Theorem 4.2 provides a finite collection of local models for Randers manifolds with constant flag curvature. Our goal is to give a complete classification of the holonomy groups of two-dimensional Randers manifolds of constant flag curvature, and to determine precisely when they are finite-dimensional and when they are infinite-dimensional.

In Section 4.1, we analyse the infinitesimal holonomy structure by determining the infinitesimal holonomy algebras. The results show that in some cases this local structure is finite-dimensional, while in others it is infinite-dimensional.

In Section 4.2, we study the corresponding holonomy groups. In the infinite-dimensional case, we prove that the holonomy is maximal; that is, its closure is isomorphic to the group of orientation-preserving diffeomorphisms of the circle. In the finite-dimensional cases, we obtain a complete classification of the holonomy groups. In particular, we establish the existence of Randers surfaces of constant curvature with non-trivial finite-dimensional holonomy. This chapter is based on the results of [40].

4.1 Infinitesimal holonomy algebra of Randers surfaces with constant flag curvature

In this section, we study the infinitesimal holonomy algebra of two-dimensional simply connected Randers manifolds of constant flag curvature arising from navigation data.

Working within the model geometries given by Theorem 4.2, we distinguish three cases according to the sign of the curvature: vanishing curvature ($K = 0$), positive constant curvature ($K > 0$), and negative constant curvature ($K < 0$), which are analyzed separately below.

4.3 Remark. By the classification of Randers manifolds of constant curvature (see [6, Section 6.]), we may assume, without loss of generality, that the matrices Q and C take the canonical form

$$Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \qquad C = (c, 0), \tag{4.1}$$

where $q, c \in \mathbb{R}$ are real constants.

4.1.1 Randers surfaces of vanishing curvature ($K = 0$)

In this case, the situation is particularly simple. Indeed, when the curvature tensor (2.15) vanishes identically, all curvature vector fields (3.16), together with their successive covariant derivatives, vanish as well. Consequently, every element of the infinitesimal holonomy algebra defined in (3.17) is identically zero, and hence the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is trivial at every point $x \in M$.

4.1.2 Randers surface of positive constant curvature

We begin with a lemma determining the dimension of the infinitesimal holonomy algebra in dimension two.

4.4 Lemma. *If the curvature tensor of a 2-dimensional Finsler manifold (M, F) is recurrent with respect to the horizontal Berwald derivative (2.12), then $\dim\big(\mathfrak{hol}^*_x\big) = 1$ at any point $x \in M$.*

Proof of Lemma 4.4. The curvature tensor is recurrent with respect to the horizontal Berwald derivative if and only if the curvature vector field $\xi = R(\partial_{x_1}, \partial_{x_2})$ satisfies

$$\nabla_i \xi = \lambda_i \cdot \xi, \tag{4.2}$$

with some functions $\lambda_i \in \mathcal{C}^\infty(M)$, $i = 1, 2$. We now show that this property is preserved under iterated covariant derivatives. More precisely, for any $1 \leq i_1 \dots i_k \leq k$, the higher order covariant derivatives of the curvature vector field have the form

$$\nabla_{i_k} \dots \nabla_{i_1} \xi = \lambda_{i_k \dots i_1} \cdot \xi, \tag{4.3}$$

with some function $\lambda_{i_k \dots i_1} \in C^\infty(M)$. Indeed, using mathematical induction. For $k = 1$, the statement follows directly from the assumption (4.2). Assume now that (4.3) holds for some $k \geq 1$. Using the definition of the covariant derivative (2.12), we obtain

$$\begin{aligned}\nabla_{i_{k+1}}\nabla_{i_k}\dots\nabla_{i_1}\xi &= \nabla_{i_{k+1}}(\nabla_{i_k}\dots\nabla_{i_1}\xi) = \nabla_{i_{k+1}}(\lambda_{i_k\dots i_1}\xi)\\ &= (\nabla_{i_{k+1}}\lambda_{i_k\dots i_1})\xi + \lambda_{i_k\dots i_1}(\nabla_{i_{k+1}}\xi)\\ &= \left(\frac{\partial\lambda_{i_k\dots i_1}}{\partial x^{k+1}} + \lambda_{i_k\dots i_1}\lambda_{i_{k+1}}\right)\cdot\xi,\end{aligned} \tag{4.4}$$

Thus (4.3) holds for the $k+1$st covariant derivatives with

$$\lambda_{i_{k+1}\dots i_1} = \frac{\partial\lambda_{i_k\dots i_1}}{\partial x^{k+1}} + \lambda_{i_k\dots i_1}\lambda_{i_{k+1}} \quad \in C^\infty(M). \tag{4.5}$$

which completes the induction. On the other hand, since the curvature vector field $\xi = R(\partial_{x_1}, \partial_{x_2})$ is vertical, the Lie bracket of any two of its scalar multiples vanishes:

$$\left[\lambda\,\xi, \mu\,\xi\right] = 0, \tag{4.6}$$

for any functions $\lambda, \mu \in C^\infty(M)$. Consequently, the Lie algebra generated by ξ and its covariant derivatives satisfies

$$\Big\langle \nabla_{i_k}\cdots\nabla_{i_1}\xi \Big\rangle_{\mathrm{Lie}} \subset \big\{\,\lambda\,\xi \mid \lambda \in C^\infty(M)\big\}. \tag{4.7}$$

Restricting to the indicatrix $\mathcal{I}_x$ at a point $x \in M$, we conclude that

$$\mathfrak{hol}^*_x(\mathcal{F}) = \Big\langle \nabla_{i_k}\cdots\nabla_{i_1}\xi\big|_{\mathcal{I}_x} \Big\rangle_{\mathrm{Lie}} = \mathrm{span}_{\mathbb{R}}\big\{\xi\big|_{\mathcal{I}_x}\big\}. \tag{4.8}$$

Hence, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is one-dimensional at every point $x \in M$, generated by the restriction of the curvature vector field ξ to the indicatrix. □

4.5 Proposition. *Let $(M, \mathcal{F})$ be a two-dimensional Randers manifold of constant positive curvature. Then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is one-dimensional at every point $x \in M$.*

Proof. Using Theorem 4.2, and up to a local isometry and a rescaling – these operations do not change the holonomy group – we can consider as model the Randers metric corresponding to the navigation data (h, W), where $h = h_{ij}dx^i \otimes dx^j$ is the standard round metric on $\mathbb{S}^2$ given by

$$h_{ij} = \frac{\big(1+(x^1)^2+(x^2)^2\big)\delta^i_j - x^i x^j}{\big(1+(x^1)^2+(x^2)^2\big)^2}, \tag{4.9}$$

and the wind vector field W is defined by

$$W(x^1,x^2) = \big(c(x^1)^2 + qx^2 + c\big)\frac{\partial}{\partial x^1} + \big(cx^1x^2 - qx^1\big)\frac{\partial}{\partial x^2}. \tag{4.10}$$

Since

$$\mathcal{L}_{ij} = W_{i:j} + W_{j:i} = 0, \tag{4.11}$$

it follows that $\mathcal{L}_W h = 0$, and hence W is an infinitesimal isometry of the Riemannian metric h. Consequently, by (2.52), (2.53), and (2.54), we obtain

$$\mathcal{S}_0 = \mathcal{L}_{00} = \mathcal{L}_{WW} = \mathcal{S}^i = 0. \tag{4.12}$$

Therefore, the geodesic coefficients of the Randers spray (2.47) reduce to

$$G^i(x,y) = G^i_h(x,y) - \tfrac{\mathcal{F}^2}{4}\,\mathcal{T}^i(x) - \tfrac{\mathcal{F}}{2}\,\mathcal{C}^i_0(x,y), \tag{4.13}$$

where G^i_h denote the geodesic coefficients of the Riemann metric h, and $\mathcal{T}^i$ and $\mathcal{C}^i_0$ are given by (2.51) and (2.52), respectively. A direct computation shows that the first covariant derivatives $\nabla_1\xi$ and $\nabla_2\xi$ of the curvature vector field $\xi = R(\partial_{x_1}, \partial_{x_2})$ fulfill the relation (4.2) with the functions

$$\lambda_i = \frac{-3x^i}{1 + (x^1)^2 + (x^2)^2}, \qquad i = 1, 2. \tag{4.14}$$

By Lemma 4.4, it follows that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is one-dimensional at every point $x \in M$. $\square$

4.1.3 Randers surface of negative constant curvature

Up to local isometry, two distinct model families describe two-dimensional Randers manifolds of constant negative curvature. These models correspond precisely to cases $(\mathcal{K}_1^-)$ and $(\mathcal{K}_2^-)$ in Theorem 4.2.

Case $(\mathcal{K}_1^-)$: Klein metric with wind

Using Theorem 4.2, up to a local isometry and a rescaling, the corresponding navigation data (h, W) is given by the Klein metric $h = h_{ij}dx^i \otimes dx^j$ on the unit disk $\mathbb{B}^2$:

$$h_{ij} = \frac{\big(1 - (x^1)^2 - (x^2)^2\big)\,\delta^i_j + x^i x^j}{\big(1 - (x^1)^2 - (x^2)^2\big)^2}, \tag{4.15}$$

and a wind vector field W of the form

$$W(x^1, x^2) = \big(-c(x^1)^2 + qx^2 + c\big)\frac{\partial}{\partial x^1} - \big(cx^1x^2 + qx^1\big)\frac{\partial}{\partial x^2}, \tag{4.16}$$

where $c, q \in \mathbb{R}$ are constants.

4.6 Proposition. *Let $(M, \mathcal{F})$ be a two-dimensional Randers manifold of constant negative curvature whose navigation data (h, W) is given by the Klein metric (4.15) and the wind field (4.16). Then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is one-dimensional at every point $x \in M$.*

Proof. The proof follows the same strategy as that of Proposition 4.5. A direct computation shows that the wind field W is a Killing vector field of the Klein metric h, that is, $\mathcal{L}_W h = 0$. Consequently, condition (4.12) holds, and the geodesic coefficients of the spray associated with the Randers metric F are given by (4.13). Furthermore, the first covariant derivatives $\nabla_1\xi$ and $\nabla_2\xi$ of the curvature vector field ξ satisfy condition (4.2) with coefficients

$$\lambda_i = \frac{-3x^i}{1-(x^1)^2-(x^2)^2}, \qquad i = 1, 2. \tag{4.17}$$

Hence, by Lemma 4.4, it follows that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is one-dimensional. □

Case $(\mathcal{K}_2^-)$: Euclidean metric with wind

Using 4.2, up to a local isometry, the navigation data (h, W) is given by the Euclidean metric defined by the coefficients $h_{ij} = \delta^i_j$, by the wind W given by:

$$W(x^1, x^2) = \big(qx^2 + c - \tfrac{\sigma}{2}x^1\big)\frac{\partial}{\partial x^1} - \big(\tfrac{\sigma}{2}x^2 + qx^1\big)\frac{\partial}{\partial x^2}. \tag{4.18}$$

4.7 Proposition. *Let $(M, \mathcal{F})$ be a two-dimensional Randers manifold of constant negative curvature arising from the navigation data (h, W), where h is the Euclidean metric and W is the wind vector field given by (4.18). Then the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_{x_o})$ of vector fields on the indicatrix at the origin x_o.*

In order to prove this proposition, we will calculate the curvature vector field $\xi = R(\partial_{x^1}, \partial_{x^2})$ at the origin $x_o = (0,0) \in \mathbb{R}^2$. The indicatrix at x_o is given by

$$\mathcal{I}_{x_o} = \Big\{(y^1, y^2) \;\Big|\; \sqrt{(y^1)^2 + (1-c^2)(y^2)^2} - c\,y^1 = 1 - c^2\Big\}. \tag{4.19}$$

Geometrically, $\mathcal{I}_{x_o}$ is a circle translated by the vector $(c, 0)$. A convenient parametrization of $\mathcal{I}_{x_o}$ is given by the map $\gamma\colon [0, 2\pi] \to \mathcal{I}_{x_o}$, defined as

$$\gamma(t) = \left(\frac{(1-c^2)\cos t}{1 - c\cos t}, \frac{\sqrt{1-c^2}\sin t}{1-c\cos t}\right). \tag{4.20}$$

This representation will be used in the sequel to simplify computations. In particular, along the parametrization $\gamma(t)$ one has

$$\sqrt{(y^1)^2 + (1-c^2)(y^2)^2} = \frac{1-c^2}{1-c\cos t}. \tag{4.21}$$

Using the curvature formula (2.16), we compute the curvature vector field ξ and consider its restriction to the indicatrix,

$$\xi_0 := \xi\big|_{\mathcal{I}_{x_o}} \in \mathfrak{hol}^*_{x_o}(\mathcal{F}), \tag{4.22}$$

which is an element of the infinitesimal holonomy algebra defined in (3.17). With respect to the parameter t, the vector field ξ_0 takes the form

$$\xi_0 = \omega(t)\,\frac{d}{dt}, \tag{4.23}$$

where

$$\omega(t) = \frac{\sigma^2(1-c\cos t)^2}{16\,(c^2-1)\sqrt{1-c^2}}. \tag{4.24}$$

For later use, we introduce the family of subspaces

$$\Sigma_n := \mathrm{span}_{\mathbb{R}}\big\{\xi_0^{l,m}\;\big|\;l+m\le n\big\}, \tag{4.25}$$

where

$$\xi_0^{l,m} := (\sin^l t\,\cos^m t)\cdot\xi_0 \in \mathfrak{X}(\mathcal{I}_{x_o}),$$

are vector fields on the indicatrix $\mathcal{I}_{x_o}$ obtained as functional multiples of ξ_0.

4.8 Lemma. *For every $n\in\mathbb{N}$, one has $\Sigma_n \subset \mathfrak{hol}^*_{x_o}(\mathcal{F})$.*

4.9 Remark. Owing to the trigonometric identity $\sin^2 t+\cos^2 t=1$, any element of Σ_n can be written as a linear combination of the vector fields $\xi_0^{0,m}=\cos^m t\,\xi_0$ and $\xi_0^{1,m-1}=\sin t\,\cos^{m-1}t\,\xi_0$, with $0\le m\le n$. In particular,

$$\Sigma_n = \mathrm{span}_{\mathbb{R}}\big\{\,\xi_0^{0,m},\;\xi_0^{1,m-1}\;\big|\;0\le m\le n\big\}. \tag{4.26}$$

Proof. We will prove the statement by induction on n, showing that the generating vector fields of Σ_n belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ for every $n\in\mathbb{N}$.

For $n=1$. By formula (3.17), the infinitesimal holonomy algebra at x_o contains the restriction of the curvature vector field (4.22) to the indicatrix, as well as the restrictions of its covariant derivatives. A direct computation of the first covariant derivatives yields

$$\xi_1 := \nabla_1\xi\big|_{\mathcal{I}_{x_o}} = \frac{3\sigma}{4(c^2-1)}\big(c-\cos t\big)\,\xi_0,\quad \in\mathfrak{hol}^*_{x_o}(\mathcal{F}), \tag{4.27a}$$

$$\xi_2 := \nabla_2\xi\big|_{\mathcal{I}_{x_o}} = \frac{3\sigma}{4\sqrt{1-c^2}}\sin t\,\xi_0,\qquad \in\mathfrak{hol}^*_{x_o}(\mathcal{F}), \tag{4.27b}$$

where we use the simplified notation (2.13). Since ξ_0, ξ_1, and ξ_2 all lie in $\mathfrak{hol}^*_{x_o}(\mathcal{F})$, taking appropriate linear combinations with constant coefficients shows that $\cos t\,\xi_0$ and $\sin t\,\xi_0$ also belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Therefore,

$$\Sigma_1 = \mathrm{span}_{\mathbb{R}}\big\{\xi_0,\;\cos t\,\xi_0,\;\sin t\,\xi_0\big\}\subset\mathfrak{hol}^*_{x_o}(\mathcal{F}). \tag{4.28}$$

For $n=2$. Since Σ_2 is generated by $\Sigma_1\cup\big\{\cos^2 t\,\xi_0,\;\sin t\cos t\,\xi_0\big\}$, and $\Sigma_1\subset\mathfrak{hol}^*_{x_o}(\mathcal{F})$ by (4.28), it suffices to show that $\xi_0^{0,2}=\cos^2 t\,\xi_0$ and $\xi_0^{1,1}=\sin t\cos t\,\xi_0$, belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$, that is, $\big\{\cos^2 t\,\xi_0,\;\sin t\cos t\,\xi_0\big\}\subset\mathfrak{hol}^*_{x_o}(\mathcal{F})$. By definition, all second covariant derivatives of the curvature vector field belong to the infinitesimal holonomy algebra. A direct computation shows that

$$\xi_{11} = \nabla_1\nabla_1\xi\,|_{\mathcal{I}_{x_0}} = \frac{3\sigma(c^2-1)}{c^6-2c^2+1}\Big(c\sigma\cos^3 t + 4c\sigma\cos^2 t + \frac{4q\cos^2 t\sin t}{\sqrt{1-c^2}} - \frac{8q\cos t\sin t}{\sqrt{1-c^2}} - 11c\sigma\cos t + \frac{+4qc(2-c^2)\sin t}{\sqrt{1-c^2}} + \frac{(5c^4-4c^2-1)\sigma}{\sqrt{1-c^2}}\Big)\xi_0, \tag{4.29a}$$

$$\xi_{12} = \nabla_1\nabla_2\xi\,|_{\mathcal{I}_{x_0}} = \frac{3\sigma}{c^4-2c^2+1}\Big(4cq\cos^3 t + c\sigma\sqrt{1-c^2}\cos^2 t\sin t + 4\sigma\sqrt{1-c^2}\cos t\sin t - 8q\cos^2 t + 5c\sigma\sqrt{1-c^2}\sin t + 4cq\cos t - 4c^2 q + 4q\Big)\xi_0, \tag{4.29b}$$

$$\xi_{22} = \nabla_2\nabla_2\xi\,|_{\mathcal{I}_{x_0}} = \frac{3\sigma}{c^4-2c^2+1}\Big(c\sigma(c^2-1)\cos^3 t + 4q\sqrt{1-c^2}\cos^2 t\sin t - 5c^2\sigma + 5\sigma + 4c\sigma(c^2-1)\cos^2 t - 8q\sqrt{1-c^2}\cos t\sin t + 4qc(2-c^2)\sqrt{1-c^2}\sin t\Big)\xi_0. \tag{4.29c}$$

Considering the linear combination of the vector fields (4.29), and by using (4.28), we can deduce that

$$\Big(c\sigma\cos^3 t + 4c\sigma\cos^2 t + \frac{4q\cos^2 t\sin t}{\sqrt{1-c^2}} - 11c\sigma\cos t - \frac{8q\cos t\sin t}{\sqrt{1-c^2}}\Big)\xi_0, \tag{4.30a}$$

$$\Big(4cq\cos^3 t + c\sigma\sqrt{1-c^2}\cos^2 t\sin t + 4\sigma\sqrt{1-c^2}\cos t\sin t - 8q\cos^2 t\Big)\xi_0, \tag{4.30b}$$

are in $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. On the other hand, since $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is a Lie algebra, the Lie brackets of the vector fields ξ_0, $\xi_0^{0,1}$, $\xi_0^{1,0}$ also belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Their explicit expressions are

$$[\xi_0, \xi_0^{0,1}] = [\xi_0, \cos t\,\xi_0] = \big(c^2\sin t\cos^2 t - 2c\cos t\sin t + \sin t\big)\xi_0 \in \mathfrak{hol}^*_{x_o}(\mathcal{F}), \tag{4.31a}$$

$$[\xi_0, \xi_0^{1,0}] = [\xi_0, \sin t\,\xi_0] = \big(c^2\cos^3 t - 2c\cos^2 t + \cos t\big)\xi_0 \in \mathfrak{hol}^*_{x_o}(\mathcal{F}). \tag{4.31b}$$

Combining (4.30) and (4.31), the vector fields $\cos^2 t\,\xi_0$ and $\sin t\cos t\,\xi_0$ can be expressed as linear combinations of elements of $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Hence, (4.25) holds for $n = 2$.
For $n > 2$. Assume that $\Sigma_n \subset \mathfrak{hol}^*_{x_o}(\mathcal{F})$. We will show that $\Sigma_{n+1} \subset \mathfrak{hol}^*_{x_o}(\mathcal{F})$, also holds. Since Σ_{n+1} is generated by

$$\Sigma_n \cup \big\{\,\xi_0^{1,n},\ \xi_0^{0,n+1}\big\}, \tag{4.32}$$

and $\Sigma_n \subset \mathfrak{hol}^*_{x_o}(\mathcal{F})$ by the induction hypothesis, it suffices to prove that

$$\xi_0^{0,n+1} = \cos^{n+1} t\,\xi_0 \quad \text{and} \quad \xi_0^{1,n} = \sin t\,\cos^n t\,\xi_0$$

belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Since $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is a Lie algebra and $\Sigma_n \subset \mathfrak{hol}^*_{x_o}(\mathcal{F})$, the following Lie brackets also belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$:

$$[\xi_0, \xi_0^{0,n-1}] = \frac{\sigma^2(n-1)}{16(1-c^2)^{\frac{3}{2}}}\big(c^2\cos^n t\,\sin t - 2c\cos^{n-1} t\,\sin t + \cos^{n-2} t\,\sin t\big)\xi_0, \tag{4.33a}$$

$$[\xi_0, \xi_0^{1,n-2}] = -\frac{\sigma^2(n-1)}{16(1-c^2)^{\frac{3}{2}}}\Big(c^2\cos^{n+1}t - 2c\cos^n t + (1-c^2)\cos^{n-1}t + 2(c-2)\cos^{n-2}t + \frac{2-n}{n-1}\cos^{n-3}t\Big)\xi_0. \tag{4.33b}$$

By the induction hypothesis, the last two terms in (4.33a) belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Consequently, the first term must also lie in $\mathfrak{hol}^*_{x_o}(\mathcal{F})$, that is $\cos^n t\,\sin t\,\xi_0 \in \mathfrak{hol}^*_{x_o}(\mathcal{F})$. Similarly, in (4.33b) the last four terms belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$, which implies that the leading term $\cos^{n+1}t\,\xi_0$ is also an element of $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. As a result, we have $\Sigma_{n+1} \subset \mathfrak{hol}^*_{x_o}$. and the induction is complete.

□

In order to describe a sufficiently large subalgebra of the infinitesimal holonomy algebra at x_o and to prepare the density argument on the indicatrix, we introduce the following family of vector fields. Let $\mathcal{A}$ denote the Lie algebra of vector fields on the indicatrix $\mathcal{I}_{x_o}$ generated by trigonometric multiples of the curvature vector field ξ_0, namely

$$\mathcal{A} := \mathrm{span}_{\mathbb{R}}\big\{\xi_0,\ \cos(nt)\,\xi_0,\ \sin(nt)\,\xi_0 \;\big|\; n\in\mathbb{N}\big\} \subset \mathfrak{X}(\mathcal{I}_{x_o}). \tag{4.34}$$

4.10 Lemma. *The Lie algebra $\mathcal{A}$ is contained in the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$.*

Proof. The classical multiple-angle formulas for the sine and cosine functions imply that, for each $n\in\mathbb{N}$, the functions $\sin(nt)$ and $\cos(nt)$ can be written as finite linear combinations of monomials of the form $\sin^\ell t\,\cos^m t$ with $\ell+m=n$. Consequently, the vector fields $\sin(nt)\,\xi_0$ and $\cos(nt)\,\xi_0$ admit representations in terms of the generators $\xi_0^{\ell,m}$:

$$\sin(nt)\,\xi_0 = \sum_{k=0}^{n}\binom{n}{k}\sin\!\big(\tfrac{n-k}{2}\pi\big)\ \xi_0^{k,n-k}, \qquad \cos(nt)\,\xi_0 = \sum_{k=0}^{n}\binom{n}{k}\cos\!\big(\tfrac{n-k}{2}\pi\big)\ \xi_0^{k,n-k}. \tag{4.35}$$

By Lemma 4.8, all vector fields $\xi_0^{k,n-k}$ belong to $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. It follows that $\sin(nt)\,\xi_0$ and $\cos(nt)\,\xi_0$ are elements of $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ for every $n\in\mathbb{N}$. Since $\mathcal{A}$ is generated by these vector fields, we conclude that,

$$A \subset \mathfrak{hol}^*_{x_o}(\mathcal{F}) \tag{4.36}$$

□

Proof of Proposition 4.7. Let $(M,\mathcal{F})$ be a Randers manifold determined by the navigation data (h,W), where h is the Euclidean metric and the wind vector field W is given by (4.18). As shown above, the Lie algebra $\mathcal{A}$ consists of vector fields on the indicatrix $\mathcal{I}_{x_o}$, which is diffeomorphic to the circle $\mathbb{S}^1$, and whose coefficients are trigonometric functions. Identifying $\mathcal{I}_{x_o}$ with $\mathbb{S}^1$ via the parameter t, the elements of $\mathcal{A}$ may be viewed as vector fields with Fourier-type coefficients. It is well known that any 2π-periodic smooth function can be approximated uniformly by the arithmetic means of the partial sums of its Fourier series (see [32, Theorem 2.12]). In particular, the functions $(\sin nt)/\omega(t)$ and $(\cos nt)/\omega(t)$ admit uniform approximation by finite

Fourier sums. Consequently, the closure of the Lie algebra generated by $\mathcal{A}$ contains the Lie algebra

$$\mathbb{F} := \left\{ \tfrac{d}{dt},\ \cos(nt)\tfrac{d}{dt},\ \sin(nt)\tfrac{d}{dt} \ \middle|\ n \in \mathbb{N} \right\}, \tag{4.37}$$

which is naturally identified with the classical Fourier algebra on $\mathbb{S}^1$. Since the Fourier algebra is dense in the Lie algebra $\mathfrak{X}(\mathbb{S}^1)$ of smooth vector fields on the circle, we obtain

$$\overline{\mathcal{F}} = \mathfrak{X}(\mathcal{I}_{x_o}).$$

On the other hand, by (4.36) we have the chain of inclusions

$$\mathfrak{X}(\mathcal{I}_{x_o}) = \overline{\mathbb{F}} \subset \overline{\mathcal{A}} \subset \overline{\mathfrak{hol}^*_{x_o}(\mathcal{F})} \subset \mathfrak{X}(\mathcal{I}_{x_o}),$$

which implies

$$\overline{\mathfrak{hol}^*_{x_o}(\mathcal{F})} = \mathfrak{X}(\mathcal{I}_{x_o}).$$

Therefore, the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is dense in $\mathfrak{X}(\mathcal{I}_{x_o})$ with respect to the C^∞ topology. □

Notes. We summarize the results of this section as follows. Let $(M, \mathcal{F})$ be a 2-dimensional Randers manifold of constant curvature K, associated with the navigation data (h, W). Then the following possibilities occur:

- $\dim(\mathfrak{hol}^*_x) = 0$ when $K = 0$,
- $\dim(\mathfrak{hol}^*_x) = 1$ when either $K > 0$, or $K < 0$ and $\mathcal{K}^-_1$ is satisfied,
- $\dim(\mathfrak{hol}^*_x) = \infty$ when $K < 0$ and $\mathcal{K}^-_2$ is satisfied.

The above results on the infinitesimal holonomy algebra naturally lead us to the study of the holonomy group. Indeed, the occurrence of a one-dimensional infinitesimal holonomy algebra suggests that the corresponding holonomy group may likewise be one-dimensional.

4.2 Holonomy groups of Randers surfaces with constant flag curvature

In this section we investigate the global holonomy group of two-dimensional, simply connected Randers manifolds of constant curvature. Motivated by the explicit computations of the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ obtained in the previous section, our goal is to provide a complete classification of the holonomy groups of Randers surfaces of constant curvature. In particular, we distinguish between the cases in which the infinitesimal holonomy algebra is one-dimensional and those in which it is infinite-dimensional. These two situations lead to qualitatively different holonomy structures, which we analyze in detail in the two-dimensional setting.

4.2.1 Trivial holonomy ($K = 0$)

In the case of a simply connected two-dimensional Randers manifold of vanishing curvature. In this case the holonomy group is trivial. In fact, this statement holds in a much more general setting.

4.11 Theorem. *Let $(M, \mathcal{F})$ be a a simply connected two-dimensional Randers manifold of vanishing curvature. Then its holonomy group is trivial, that is,*

$$\mathcal{H}ol_x(\mathcal{F}) = \{Id\}. \tag{4.38}$$

Proof. Parallel translation on a Finsler manifold can be described using the associated Ehresmann connection; see [59]. The horizontal distribution is defined via the horizontal lift

$$T_xM \longrightarrow T_{(x,y)}TM,$$

which, in local coordinates, is given by

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - G_i^k(x,y)\,\frac{\partial}{\partial y^k}, \tag{4.39}$$

where $y \in T_xM$. Since the horizontal distribution is complementary to the vertical distribution, the tangent bundle of TM decomposes as

$$T_yTM = \mathcal{H}_y \oplus \mathcal{V}_y,$$

with canonical projections $h\colon TTM \to \mathcal{H}$ and $v\colon TTM \to \mathcal{V}$. The subbundle $\mathcal{H} \subset TTM$ is called the *horizontal distribution.* Let $\gamma\colon [0,1] \to M$ be a smooth curve and $X_0 \in T_{\gamma(0)}M$. The horizontal lift of γ with initial condition X_0 is the unique curve $\gamma^h\colon [0,1] \to TM$ satisfying

$$\pi \circ \gamma^h = \gamma, \qquad \frac{d\gamma^h}{dt} = \left(\frac{d\gamma}{dt}\right)^h, \qquad \gamma^h(0) = X_0. \tag{4.40}$$

The parallel translation along γ is then given by $\mathcal{P}_\gamma(X_0) = \gamma^h(1)$. If the curvature tensor (2.15) vanishes, the horizontal distribution $\mathcal{H}$ is integrable. Consequently, the horizontal lift of any closed curve γ with $\gamma(0) = \gamma(1)$ is also closed, that is,$\gamma^h(0) = \gamma^h(1)$. Hence parallel translation along any closed curve is trivial, and the holonomy group reduces to the identity. □

4.2.2 Finite-dimensional nontrivial holonomy

By Definition 3.4, in order to show that the holonomy group is finite-dimensional, it suffices to verify that $\mathfrak{hol}_x(\mathcal{F})$ is finite-dimensional. The infinitesimal structure of the holonomy of 2-dimensional Randers manifolds with constant non-vanishing curvature was analyzed in Section 4.1.2. These results show that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is one-dimensional at every point whenever the curvature is positive, or when $K < 0$ with $\mathcal{K}_2^-$ are satisfied. From a local point of view this strong restriction of the infinitesimal holonomy suggests that the global holonomy group itself might be finite-dimensional, and possibly even one-dimensional.

However, the relation (3.18) between the infinitesimal holonomy algebra and the holonomy algebra provides only a lower bound (3.22) on the dimension of the holonomy group. Indeed, since the holonomy group is a global geometric object, one can easily envision situations where a nontrivial holonomy element in some point $\hat{x}$ induces, via conjugation by parallel translation along a curve connecting $\hat{x}$ and another point x, a nontrivial element in the holonomy group at x. Such effects may increase the dimension of the holonomy group beyond what is suggested by local considerations. The following lemma provides a general criterion ensuring that the holonomy group is one-dimensional.

4.12 Lemma. *Let $(M, \mathcal{F})$ be a Finsler surface and assume that the indicatrix bundle $\mathcal{I}M$ admits a coordinate system with global α as a fiber coordinate. Suppose that there exist smooth functions Φ on $\mathcal{I}M$, and Ψ on TM respectively, such that for every curve $\gamma(t)$, the parallelism of unit vector fields along γ is described by a differential equation*

$$\frac{d}{dt}\big(\Phi(\gamma(t),\alpha(t)\big) = \Psi(\gamma(t),\dot{\gamma}(t)). \tag{4.41}$$

Then the holonomy group at any point $x \in M$ is one-dimensional.

The coordinate α will be called the *angular parameter* on the indicatrix.

Proof. In order to calculate the dimension of the holonomy group $\mathcal{H}ol_x$, we should investigate the dimension of its tangent space, the holonomy algebra. For this, we divide the proof into the following steps:

Step 1: Holonomy transformation associated to a closed curve.

Let $\gamma\colon [0,1] \to M$ be a closed curve based at x, and let $V(t)$ be a parallel unit vector field along the curve $\gamma(t)$. On the indicatrix bundle, using the angular coordinate α we can write

$$V(t) = (\gamma(t), \alpha(t)), \qquad 0 \le t \le 1, \tag{4.42}$$

The value $\alpha(0)$ corresponds to the initial vector $V_0 = V(0)$ at the starting point of γ, while $\alpha(1)$ corresponds to the vector $V_1 = V(1)$ obtained after parallel transport along the curve. Consequently, the holonomy transformation along γ is the map

$$\mathcal{H}_\gamma : \mathcal{I}_x \to \mathcal{I}_x, \qquad \mathcal{H}_\gamma(V_0) = V_1. \tag{4.43}$$

In terms of the angular parameter α on the indicatrix $\mathcal{I}_x$, this transformation can be written as

$$\mathcal{H}_\gamma : [0, 2\pi] \to [0, 2\pi], \qquad \mathcal{H}_\gamma\big(\alpha(0)\big) = \alpha(1). \tag{4.44}$$

By the assumption, $\alpha(t)$ satisfies a scalar differential equation of the form (4.41), and by integrating we obtain

$$\Phi(\gamma(1),\alpha(1)) - \Phi(\gamma(0),\alpha(0)) = \int_0^1 \frac{d}{dt}\Phi\big(\gamma(t),\alpha(t)\big)\,dt = \int_0^1 \Psi\big(\gamma(t),\dot{\gamma}(t)\big)dt. \tag{4.45}$$

Since γ is a closed curve based on x we have $\gamma_0 = \gamma_1 = x$, we get the equation

$$\Phi\big(x, \mathcal{H}_\gamma(\alpha)\big) = \Phi(x,\alpha) + \oint_\gamma \Psi\big(\gamma(t), \dot\gamma(t)\big)dt. \tag{4.46}$$

implicitly determining the holonomy transformation (4.44) associated with γ.

Step 2: Holonomy algebra

The holonomy algebra $\mathfrak{hol}_x(\mathcal{F})$ is the tangent space of the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ at its unit element of the group, that is at the identity. In order to compute its elements, let us consider a smooth curve in $\mathcal{H}ol_x$ passing through the unit element. The correspondent 1-parameter family of closed curves $\{\gamma_s\}$ can be considered as $\gamma\colon\,]-\varepsilon,\varepsilon[\times[0,1]\to M$ where for each parameter $s \in]-\varepsilon,\varepsilon[$, the curve

$$t \longrightarrow \gamma_s(t), \qquad [0,1] \longrightarrow M, \tag{4.47}$$

is a closed curve emanating from the fixed base point $x \in M$, and in particular for $s = 0$ we have $\gamma_0(t) = x$ the trivial curve. Then we have a 1-parameter family of holonomy transformation

$$s \longrightarrow \mathcal{H}_{\gamma_s}, \quad]-\varepsilon,\varepsilon[\longrightarrow \mathcal{H}ol_x \tag{4.48}$$

is a curve in the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ with the property that for $s = 0$ we have $\mathcal{H}_{\gamma_s} = id$. The derivative of the map (4.48) at $s = 0$ is a vector field on the indicatrix $\mathcal{I}_x$:

$$\mathfrak{h}_\gamma := \frac{d}{ds}\Big|_{s=0}\mathcal{H}_{\gamma_s} \qquad \in \mathfrak{X}(\mathcal{I}_x), \tag{4.49}$$

and its value at a vector $V_0 \in \mathcal{I}_x$ can be calculated as

$$\mathfrak{h}_\gamma(V_0) = \frac{d}{ds}\Big|_{s=0}\mathcal{H}_{\gamma_s}(V_0). \tag{4.50}$$

Using the representation (4.44) of the holonomy transformations as a map on $[0,2\pi]$, the derivative (4.49) can be interpreted as a vector field $\mathfrak{h}_\gamma$ on $[0,2\pi]$, where for each value of angular parameter $\alpha_0 \in [0,2\pi]$ we have

$$\mathfrak{h}_\gamma(\alpha_0) = \frac{d}{ds}\Big|_{s=0}\mathcal{H}_{\gamma_s}(\alpha_0). \tag{4.51}$$

For the calculation we use the results of step 1: for each $s \in (-\varepsilon,\varepsilon)$ we consider the parallel translation on the closed curve γ_s the unit vector corresponding to the angular parameter $\alpha_s(0) = \alpha_0$. Using the notation (4.44), the holonomy transformation along γ_s is

$$\mathcal{H}_{\gamma_s}(\alpha_0) = \alpha_s(1), \qquad -\varepsilon \le s \le \varepsilon, \tag{4.52}$$

and α_s satisfies the equation (4.41). Introducing $k_s := \int_0^1 \Psi\big(\gamma_s(t), \dot\gamma_s(t)\big)dt$, we get

$$\Phi\big(x, \mathcal{H}_{\gamma_s}(\alpha)\big) = \Phi(x,\alpha) + k_s. \tag{4.53}$$

Differentiating with respect to the parameter s we get

$$\partial_\alpha \Phi\big(x, \mathcal{H}_{\gamma_0}(\alpha)\big) \cdot \tfrac{d}{ds}\Big|_{s=0} \mathcal{H}_{\gamma_s}(\alpha) = \frac{dk_s}{ds}\Big|_{s=0}. \tag{4.54}$$

Note, that by assumption, $\gamma_{s=0} \equiv \{x\}$ is the trivial curve, therefore its holonomy $\mathcal{H}_{\gamma_0}$ is trivial, that is $\mathcal{H}_{\gamma_0}(\alpha) = \alpha$, therefore we get

$$\partial_\alpha \Phi\big(x, \alpha\big) \cdot \tfrac{d}{ds}\Big|_{s=0} \mathcal{H}_{\gamma_s}(\alpha) = \kappa_\gamma, \tag{4.55}$$

where

$$\kappa_\gamma := \frac{dk_s}{ds}\Big|_{s=0}, \tag{4.56}$$

is a real number, depending on the 1-parameter family $\{\gamma_s\}$ of closed curve. Using the holonomy notation (4.51) we have

$$\mathfrak{h}_\gamma(\alpha) = \frac{d}{ds}\Big|_{s=0} \mathcal{H}_s(\alpha) = \frac{\kappa_\gamma}{\partial_\alpha \Phi\big(x, \alpha\big)} \frac{\partial}{\partial \alpha}, \tag{4.57}$$

As a consequence, the vector field (4.49) on the indicatrix obtaind as the tangent vector of the 1-parameter family of holonomy tranformation, then

$$\mathfrak{hol}_x = \{\mathfrak{h}_\gamma \mid \gamma \in \Gamma_x\}. \tag{4.58}$$

Introducing the notation

$$\mathfrak{h}_x := \frac{1}{\partial_\alpha \Phi\big(x, \alpha\big)} \frac{\partial}{\partial \alpha} \in \mathfrak{X}(\mathcal{I}_x), \tag{4.59}$$

we get from (4.57)

$$\mathfrak{hol}_x(\mathcal{F}) = \{\kappa\, \mathfrak{h}_x \mid \kappa \in \mathbb{R}\}, \tag{4.60}$$

It follows that the tangent space of the holonomy group is 1-dimensional, and by Definition (3.4) we have $\dim \mathcal{H}ol_x = 1$. □

4.13 Remark. In order to obtain a convenient local form of the Randers metric for the study of parallel translation and holonomy, we first introduce adapted coordinates in which the Randers metric associated with the navigation data admits a simpler representation. In these coordinates, the underlying Riemannian metric h becomes orthogonal and the wind field W is aligned with one of the coordinate directions, which considerably simplifies the subsequent calculations. The construction of these adapted coordinates relies on the special structure of the navigation data. Starting from the wind vector field W, we choose a second vector field X such that X is h–orthogonal to W and commutes with W on an open subset of M. Once such a pair (X, W) is obtained, we introduce local coordinates by integrating these vector fields and using their flows as coordinate directions. That is there exists an "orthogonal" coordinate system $(\hat{x}^1, \hat{x}^2)$ on M such that

$$W = \partial_{\hat{x}^1}, \qquad X = \partial_{\hat{x}^2}, \qquad h(W, X) = 0. \tag{4.61}$$

In the corresponding adapted frame, any tangent vector $y \in T_x M$ can be written as

$$y = \hat{y}^2 X + \hat{y}^2 W. \tag{4.62}$$

Since X and W are h–orthogonal, it follows that

$$h_x(y, y) = A(x)u^2 + B(x)v^2, \qquad h_x(y, W) = B(x)v, \tag{4.63}$$

substituting these expressions into the Randers metric (2.39), we obtain the local form

$$\mathcal{F}(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2) = \sqrt{\frac{A}{\lambda}\,(\hat{y}^1)^2 + \frac{B}{\lambda^2}\,(\hat{y}^2)^2} - \frac{B}{\lambda}\,\hat{y}^2, \tag{4.64}$$

where

$$A := h(X, X), \qquad B = h(W, W), \qquad \lambda = 1 - B. \tag{4.65}$$

Finite-dimensional holonomy with positive curvature ($K > 0$)

We consider the navigation data (h, W) given by Theorem 4.2, where the metric h and the vector field W are described by (4.9) and (4.10). The h-norm of the wind field is given by

$$h(W, W)(x) = \frac{\omega(x^1)^2 + (qx^2 + c)^2}{1 + (x^1)^2 + (x^2)^2} < 1. \tag{4.66}$$

We note that if $c = q = 0$, then the Finsler norm reduces to the standard Riemannian norm on the sphere $\mathbb{S}^2$. Since this case is already Riemannian, we assume in the following that at least one of the parameters c and q is nonzero. The adapted coordinate system introduced in Remark 4.13 is given explicitly in the following lemma.

4.14 Lemma. *The adapted coordinates $(\hat{x}^1, \hat{x}^2)$, is given by*

$$\hat{x}^1 = \frac{1}{\sqrt{\omega}} \arctan\left(\frac{x^1\sqrt{\omega}}{c + qx^2}\right), \quad \hat{x}^2 = \frac{1}{2\omega} \ln\left(\frac{\delta(x^1, x^2)}{(cx^2 - q)^2}\right), \tag{4.67}$$

where

$$\omega := q^2 + c^2, \qquad \delta(x^1, x^2) := \omega \cdot (x^1)^2 + (c + qx^2)^2. \tag{4.68}$$

In these coordinates, we obtain

$$A = \frac{\omega e^{-\omega \hat{x}^2}}{1 + e^{-\omega \hat{x}^2}}, \qquad B = \frac{\omega^2\, e^{-\omega \hat{x}^2}}{(1 + e^{-\omega \hat{x}^2})^2}, \qquad \lambda = \frac{1 + (1 - \omega)e^{-\omega \hat{x}^2}}{1 + e^{-\omega \hat{x}^2}}. \tag{4.69}$$

Proof. Let $Y = Y^1 \frac{\partial}{\partial x^1} + Y^2 \frac{\partial}{\partial x^2}$, be a vector field on $\mathbb{D}^2$. Requiring Y to be h–orthogonal to the wind field W, that is, $h(W, Y) = 0$, leads, after a straightforward computation using the round metric and the explicit form of W, to a linear relation between the components of Y. More precisely, the orthogonality condition is equivalent to

$$(qx^2)\, Y^1 - qx^1\, Y^2 = 0. \tag{4.70}$$

A convenient nontrivial solution of (4.70) is given by the vector field

$$Y_0 = qx^1 \frac{\partial}{\partial x^1} + (c + qx^2) \frac{\partial}{\partial x^2}, \tag{4.71}$$

which therefore spans the h–orthogonal distribution $W^\perp$ on $\mathbb{D}^2$. A direct computation using the explicit expressions of W and Y_0 shows that

$$[W, Y_0] = -c\, x^1\, Y_0. \tag{4.72}$$

Thus Y_0 is not invariant under the flow of W, but its Lie bracket with W is proportional to Y_0 itself. We therefore look for a vector field of the form $X := \mu\, Y_0$, where μ is a nonvanishing scalar function, such that X commutes with W. Since X is colinear to Y_0, the orthogonality condition $h(W, X) = 0$ is automatically satisfied. Using (4.72), the commutation condition $[W, X] = 0$ reduces to the first-order equation

$$W(\mu) = c\, x^1\, \mu. \tag{4.73}$$

Assuming $\mu = \mu(x^2)$, this equation can be integrated explicitly, yielding

$$\mu(x^2) = cx^2 - q. \tag{4.74}$$

Consequently, we get

$$X = (cx^2 - q)\Big(qx^1 \frac{\partial}{\partial x^1} + (c + qx^2) \frac{\partial}{\partial x^2}\Big) \tag{4.75}$$

A direct computation of the h–norm of X yields

$$h(X, X)(x) = \frac{\big((q^2 + c^2)(x^1)^2 + (qx^2 + c)^2\big)(qx^2 - c)^2}{(1 + (x^1)^2 + (x^2)^2)^2}. \tag{4.76}$$

Since the vector fields X and W are linearly independent and commute, they define local coordinates $(\hat{x}^1, \hat{x}^2)$ such that

$$W = \partial_{\hat{x}^1}, \qquad X = \partial_{\hat{x}^2}. \tag{4.77}$$

Equivalently, the coordinate functions $\hat{x}^1 = \hat{x}^1(x^1, x^2)$ and $\hat{x}^2 = \hat{x}^2(x^1, x^2)$ are determined by the first-order systems

$$\begin{cases} W^1 \dfrac{\partial \hat{x}^1}{\partial x^1} + W^2 \dfrac{\partial \hat{x}^1}{\partial x^2} = 1, \\ X^1 \dfrac{\partial \hat{x}^1}{\partial x^1} + X^2 \dfrac{\partial \hat{x}^1}{\partial x^2} = 0, \end{cases} \qquad \begin{cases} W^1 \dfrac{\partial \hat{x}^2}{\partial x^1} + W^2 \dfrac{\partial \hat{x}^2}{\partial x^2} = 0, \\ X^1 \dfrac{\partial \hat{x}^2}{\partial x^1} + X^2 \dfrac{\partial \hat{x}^2}{\partial x^2} = 1. \end{cases} \tag{4.78}$$

W^1, W^2 are the components of the wind field W, and X^1, X^2 are the components of the commuting h–orthogonal vector field X. Solving the system (4.78) for the partial derivatives yields

$$\hat{x}^1_{x^1} = \frac{qx^2 + c}{\delta(x^1, x^2)}, \qquad \hat{x}^1_{x^2} = -\frac{qx^1}{\delta(x^1, x^2)}, \tag{4.79}$$

$$\hat{x}^2_{x^1} = -\frac{x^1}{\delta(x^1,x^2)}, \qquad \hat{x}^2_{x^2} = \frac{c(x^1)^2 + qx^2 + c}{(cx^2-q)\,\delta(x^1,x^2)}, \tag{4.80}$$

where

$$\delta(x^1,x^2) := \omega(x^1)^2 + (c+qx^2)^2, \qquad \omega := q^2 + c^2. \tag{4.81}$$

Note that $\delta(x^1,x^2)$ and (cx^2-q) do not vanish on the unit disk $\mathbb{B}^2$, since their product coincides with the numerator of the h–norm of the vector field X defined in (4.76). Integrating (4.79) and (4.80) gives

$$t(x^1,x^2) = \frac{1}{\sqrt{\omega}}\arctan\left(\frac{x^1\sqrt{\omega}}{c+qx^2}\right), \tag{4.82}$$

and

$$s(x^1,x^2) = \frac{1}{2\omega}\ln\left(\frac{\delta(x^1,x^2)}{(cx^2-q)^2}\right). \tag{4.83}$$

Solving the defining relations for $\hat{x}^1$ and $\hat{x}^2$ yields the inverse transformation in the form

$$\begin{aligned} x^1(\hat{x}^1,\hat{x}^2) &= \frac{\mp\sqrt{\omega}\,e^{-\omega\hat{x}^2}\,\sin\!\left(\sqrt{\omega}\hat{x}^1\right)}{q\mp c\,e^{-\omega\hat{x}^2}\,\cos\!\left(\sqrt{\omega}\hat{x}^1\right)}, \\ x^2(\hat{x}^1,\hat{x}^2) &= \frac{\mp q\,e^{-\omega\hat{x}^2}\,\cos\!\left(\sqrt{\omega}\hat{x}^1\right) - c}{q\mp c\,e^{-\omega\hat{x}^1}\,\cos\!\left(\sqrt{\omega}\hat{x}^1\right)}. \end{aligned} \tag{4.84}$$

Substituting the expressions of x^1 and x^2 from into (4.66) and (4.76), we obtain

$$\|W\|^2 = h(W,W) = \frac{\omega e^{-\omega\hat{x}^2}}{1+e^{-\omega\hat{x}^2}}, \qquad h(X,X) = \frac{\omega^2\,e^{-\omega\hat{x}^2}}{(1+e^{-\omega\hat{x}^2})^2}. \tag{4.85}$$

Consequently,

$$\|W\|^2 = A = \frac{\omega e^{-\omega\hat{x}^2}}{1+e^{-\omega\hat{x}^2}}, \qquad B = \frac{\omega^2\,e^{-\omega\hat{x}^2}}{(1+e^{-\omega\hat{x}^2})^2}, \qquad \lambda = \frac{1+(1-\omega)e^{-\omega\hat{x}^2}}{1+e^{-\omega\hat{x}^2}}. \tag{4.86}$$

□

4.15 Remark. Further simplification is possible by considering the coordinates (r,θ) where

$$r := \exp\!\left(-\omega\,\hat{x}^2\right), \qquad \theta := \sqrt{\omega}\,\hat{x}^1. \tag{4.87}$$

With this coordinates, the functions A, B, and λ take the form

$$\|W\|^2 = A = \frac{\omega r^2}{1+r^2}, \qquad B = \frac{\omega^2\,r^2}{(1+r^2)^2}, \qquad \lambda = \frac{1+(1-\omega)r^2}{1+r^2}. \tag{4.88}$$

4.16 Theorem. *Let $(M,\mathcal{F})$ be a simply connected two-dimensional Randers manifold of constant positive curvature. Then, for any point $x\in M$, the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is one-dimensional.*

Proof. Let $(M, \mathcal{F})$ be a simply connected two-dimensional Randers manifold of constant positive curvature, isometric to the Randers manifold associated with the navigation data (h, W), where h is the round metric (4.9) on $\mathbb{S}^2$ and W is given by (4.10). Using the notation (r, θ, u, v) for the adapted coordinate system on TM we get

$$\mathcal{F}(r,\theta,u,v) = \sqrt{\frac{u^2}{(1+r^2)(1+(1-\omega)r^2)} + \frac{(1+r^2)r^2v^2}{(1+(1-\omega)r^2)^2}} - \frac{\sqrt{\omega}\,r^2}{1+(1-\omega)r^2}\,v. \tag{4.89}$$

The corresponding navigation data in the polar coordinates (r, θ) are given by

$$h_{r\theta} = \begin{pmatrix} \dfrac{1}{(1+r^2)^2} & 0 \\ 0 & \dfrac{r^2}{1+r^2} \end{pmatrix}, \qquad W_{r\theta} = \sqrt{\omega}\,\frac{\partial}{\partial\theta}. \tag{4.90}$$

The indicatrix at the point (r, θ) is given by

$$\mathcal{I}_{(r,\theta)} = \left\{ (u,v) \;\middle|\; \left(\frac{u}{1+r^2}\right)^2 + \left(\frac{(v-\sqrt{\omega})\,r}{\sqrt{1+r^2}}\right)^2 = 1 \right\}, \tag{4.91}$$

and (r, θ, α) gives a parametrization of the indicatrix bundle $\mathcal{I}M$ where

$$u = (1+r^2)\cos\alpha, \qquad v = \frac{\sqrt{1+r^2}}{r}\sin\alpha + \sqrt{\omega}. \tag{4.92}$$

In order to determine the differential equation of the parallel transport, let $\gamma : [0,1] \to M$ be a curve given by

$$\gamma(t) = (r(t), \theta(t)), \qquad 0 \le t \le 1, \tag{4.93}$$

and let $V(t)$ be a unit vector field along γ. Using the angular coordinate α introduced in (4.92), we see that $V(t)$ has the form

$$V(t) = \Big((1+r^2(t))\,\cos(\alpha(t))\Big)\frac{\partial}{\partial r} + \Big(\tfrac{\sqrt{1+r^2(t)}}{r(t)}\,\sin(\alpha(t)) + \sqrt{\omega}\Big)\frac{\partial}{\partial\theta}. \tag{4.94}$$

If the vector field (4.94) is parallel, then it satisfies the differential equation (3.2) where we have

$$G_1^1(\gamma, V) = -2r\cos\alpha + \frac{1}{\sqrt{1+r^2}}\,\frac{\sqrt{\omega}\cos\alpha\sin\alpha}{\Big(1+\frac{\sqrt{\omega}r}{\sqrt{1+r^2}}\sin\alpha\Big)}, \tag{4.95a}$$

$$G_2^1(\gamma, V) = -\frac{\sqrt{1+r^2}\sin\alpha}{1+\frac{\sqrt{\omega}r}{\sqrt{1+r^2}}\sin\alpha}, \tag{4.95b}$$

$$G_1^2(\gamma, V) = \frac{\sin\alpha}{r^2\sqrt{1+r^2}} - \frac{\sqrt{\omega}\cos^2\alpha}{(1+\frac{\sqrt{\omega}r}{\sqrt{1+r^2}}\sin\alpha)(1+r^2)r^2}, \tag{4.95c}$$

$$G_2^2(\gamma, V) = \frac{\cos\alpha}{r(1+\frac{\sqrt{\omega}r}{\sqrt{1+r^2}}\sin\alpha)}. \tag{4.95d}$$

The equation (3.2) for $i = 1$ is

$$\frac{dV^1}{dt} + \dot{r}\, G^1_1(\gamma, V) + \dot{\theta}\, G^1_2(\gamma, V) = 0, \tag{4.96}$$

and using (4.94) gives

$$-\dot{\alpha}\sin(\alpha)(1+r^2) + 2r\dot{r}\cos(\alpha) + \dot{r}\ G^1_1(\gamma, V) + \dot{\theta}\ G^1_2(\gamma, V) = 0, \tag{4.97}$$

Equivalently,

$$-\dot{\alpha}\sin\alpha(1+r^2) + \dot{r}\big(2r\cos\alpha + G^1_1(\gamma, V)\big) + \dot{\theta}\, G^1_2(\gamma, V) = 0, \tag{4.98}$$

that is, with (4.122a) and (4.122b) after some simplification we get

$$-\dot{\alpha}\sin\alpha(1+r^2) + \dot{r}\left(\frac{1}{\sqrt{1+r^2}}\frac{\sqrt{\omega}\cos\alpha\sin\alpha}{1+\frac{\sqrt{\omega}r}{\sqrt{1+r^2}}\sin\alpha}\right) - \dot{\theta}\left(\frac{\sqrt{1+r^2}\sin\alpha}{1+\frac{\sqrt{\omega}r}{\sqrt{1+r^2}}\sin\alpha}\right) = 0. \tag{4.99}$$

Factoring out the common denominator, further simplifying, and rearranging the terms, we can rewrite the equation (4.99) as

$$-\dot{\alpha}\frac{r\sin\alpha\ \sqrt{\omega}}{\sqrt{1+r^2}} - \dot{\alpha} + \dot{r}\frac{\sqrt{\omega}\cos\alpha}{(1+r^2)\sqrt{1+r^2}} - \dot{\theta}\frac{1}{\sqrt{1+r^2}} = 0. \tag{4.100}$$

Observe that the first term can be expressed as

$$-\dot{\alpha}\frac{r\sqrt{\omega}\sin\alpha}{\sqrt{1+r^2}} = \frac{d}{dt}\left(\frac{r\cos\alpha\ \sqrt{\omega}}{\sqrt{1+r^2}}\right) - \dot{r}\frac{\sqrt{\omega}\cos\alpha}{(1+r^2)\sqrt{1+r^2}}. \tag{4.101}$$

Substituting this into the previous equation and simplifying gives:

$$\frac{d}{dt}\left(\frac{r\sqrt{\omega}}{\sqrt{1+r^2}}\cos\alpha - \alpha\right) = \frac{\dot{\theta}}{\sqrt{1+r^2}}. \tag{4.102}$$

Equation (4.102) can be written in the form (4.41). Indeed, it corresponds to the choice

$$\Phi(\gamma, \alpha) = \frac{r\sqrt{\omega}}{\sqrt{1+r^2}}\cos\alpha - \alpha, \qquad \Psi(\gamma, \dot{\gamma}) = \frac{\dot{\theta}}{\sqrt{1+r^2}}. \tag{4.103}$$

Therefore, by Lemma 4.12, the holonomy group at x is one-dimensional.

□

Finite-dimensional holonomy with negative curvature ($K < 0$ with $\mathcal{K}_1^-$)

In this case the navigation data (h, W) is given, up to an isometry and a rescaling, by the Klein metric (4.15) on the unit disk $\mathbb{B}^2 \subset \mathbb{R}^2$, with the vector field (4.16).

4.17 Lemma. *The adapted coordinates $(\hat{x}^1, \hat{x}^2)$, is given by different cases depending on the value of*

$$\omega := q^2 - c^2. \tag{4.104}$$

1. *In the **case** $\omega \neq 0$. The adapted coordinates are*

$$\hat{x}^1(x^1, x^2) = \begin{cases} \dfrac{1}{\sqrt{\omega}} \operatorname{arctan}\left(\dfrac{x^1\sqrt{\omega}}{c + qx^2}\right), & \omega > 0, \\ \dfrac{1}{\sqrt{-\omega}} \operatorname{artanh}\left(\dfrac{x^1\sqrt{-\omega}}{c + qx^2}\right), & \omega < 0, \end{cases} \tag{4.105}$$

$$\hat{x}^2(x^1, x^2) = \frac{1}{2\omega} \ln\left(\frac{|\delta(x^1, x^2)|}{(cx^2 + q)^2}\right).$$

$$A = \frac{\omega^2 e^{2\omega\hat{x}^2}}{(1 - e^{2\omega\hat{x}^2})^2}, \quad B = \frac{\omega e^{2\omega\hat{x}^2}}{1 - e^{2\omega\hat{x}^2}}, \quad \lambda = \frac{1 - (1 + \omega)e^{2\omega\hat{x}^2}}{1 - e^{2\omega\hat{x}^2}}. \tag{4.106}$$

2. *In the **case** $\omega = 0$. The adapted coordinates are*

$$\hat{x}^1 = \frac{x^1}{c(1 + x^2)}, \qquad \hat{x}^2 = \frac{(x^1)^2}{2c^2(1 + x^2)^2} - \frac{1}{c^2(1 + x^2)}. \tag{4.107}$$

In his case,

$$A = \frac{c^2}{(1 + 2c^2\hat{x}^2)^2}, \quad B = -\frac{c^2}{1 + 2c^2\hat{x}^2}, \quad \lambda = \frac{1 + c^2(2\hat{x}^2 + 1)}{1 + 2c^2\hat{x}^2}. \tag{4.108}$$

Proof. The proof follows the same argument as in the previous lemma. The field field X which is h–orthogonal to W and commutes with it is given by

$$X = (cx^2 + q)\Big(qx^1 \frac{\partial}{\partial x^1} + (c + qx^2) \frac{\partial}{\partial x^2}\Big). \tag{4.109}$$

A direct computation shows that the squared norms of the vector fields W and X are

$$h(W, W) = \frac{(c + qx^2)^2 + \omega(x^1)^2}{1 - (x^1)^2 - (x^2)^2}, \qquad h(X, X) = \frac{(cx^2 + q)^2\big((c + qx^2)^2 + \omega(x^1)^2\big)}{\big(1 - (x^1)^2 - (x^2)^2\big)^2}. \tag{4.110}$$

The coordinate functions $\hat{x}^1 = \hat{x}^1(x^1, x^2)$ and $\hat{x}^2 = \hat{x}^2(x^1, x^2)$

$$\hat{x}^1_{x^1} = \frac{c + qx^2}{\delta(x^1, x^2)}, \qquad \hat{x}^1_{x^2} = -\frac{qx^1}{\delta(x^1, x^2)}, \tag{4.111}$$

$$\hat{x}^2_{x^1} = \frac{x^1}{\delta(x^1,x^2)}, \qquad \hat{x}^2_{x^2} = \frac{-c(x^1)^2 + qx^2 + c}{(cx^2+q)\,\delta(x^1,x^2)}, \tag{4.112}$$

where $\delta(x^1,x^2) := \omega(x^1)^2 + (c+qx^2)^2$. It is clear that $\delta(x^1,x^2)$ and (cx^2+q) do not vanish, since their product appears in the numerator of the h–norm of the vector field X.

Case $\omega \neq 0$. The system (4.111)–(4.112) admits the solutions

$$\hat{x}^1(x^1,x^2) = \begin{cases} \dfrac{1}{\sqrt{\omega}} \arctan\left(\dfrac{x^1\sqrt{\omega}}{c+qx^2}\right), & \omega > 0, \\ \dfrac{1}{\sqrt{-\omega}} \operatorname{artanh}\left(\dfrac{x^1\sqrt{-\omega}}{c+qx^2}\right), & \omega < 0, \end{cases} \qquad \hat{x}^2(x^1,x^2) = \frac{1}{2\omega}\ln\left(\frac{|\delta(x^1,x^2)|}{(cx^2+q)^2}\right).$$

Solving the defining relations for $\hat{x}^1$ and $\hat{x}^2$ yields the inverse transformation $(x^1,x^2) \mapsto (\hat{x}^1,\hat{x}^2)$ in the form

$$\begin{aligned} x^1(\hat{x}^1,\hat{x}^2) &= \frac{\pm\sqrt{|\omega|}\, e^{\omega s}\, S_\omega}{q \mp c\, e^{\omega s}\, C_\omega}, \\ x^2(\hat{x}^1,\hat{x}^2) &= \frac{\pm q\, e^{\omega s}\, C_\omega - c}{q \mp c\, e^{\omega s}\, C_\omega}, \end{aligned} \tag{4.113}$$

where

$$S_\omega = \begin{cases} \sin\left(\sqrt{\omega}\,\hat{x}^1\right), & \omega > 0, \\ \sinh\left(\sqrt{-\omega}\,\hat{x}^1\right), & \omega < 0, \end{cases} \qquad C_\omega = \begin{cases} \cos\left(\sqrt{\omega}\,\hat{x}^1\right), & \omega > 0, \\ \cosh\left(\sqrt{-\omega}\,\hat{x}^1\right), & \omega < 0. \end{cases}$$

Substituting x^1 and x^2 from (4.113) into (4.110), we obtain

$$A = \frac{\omega^2 e^{2\omega\hat{x}^2}}{(1-e^{2\omega\hat{x}^2})^2}, \qquad B = \frac{\omega e^{2\omega\hat{x}^2}}{1-e^{2\omega\hat{x}^2}}, \qquad \lambda = \frac{1-(1+\omega)e^{2\omega\hat{x}^2}}{1-e^{2\omega\hat{x}^2}}.$$

Case $\omega = 0$. In this case the system integrates to

$$\hat{x}^1(x^1,x^2) = \frac{x^1}{c(1+x^2)}, \qquad \hat{x}^2(x^1,x^2) = \frac{(x^1)^2}{2c^2(1+x^2)^2} - \frac{1}{c^2(1+x^2)}. \tag{4.114}$$

The inverse transformation is

$$x^1(\hat{x}^1,\hat{x}^2) = \frac{2\hat{x}^1}{c((\hat{x}^1)^2 - 2\hat{x}^2)}, \qquad x^2(\hat{x}^1,\hat{x}^2) = \frac{2}{c^2((\hat{x}^1)^2 - 2\hat{x}^2)} - 1. \tag{4.115}$$

By substituting these expressions into (4.110), we get

$$A = \frac{c^2}{(1+2c^2\hat{x}^2)^2}, \quad B = -\frac{c^2}{1+2c^2\hat{x}^2}, \quad \lambda = \frac{1+c^2(2\hat{x}^2+1)}{1+2c^2\hat{x}^2}. \tag{4.116}$$

□

4.18 Theorem. *Let $(M, \mathcal{F})$ be a simply connected two-dimensional Randers manifold of constant negative curvature with $\mathcal{K}_1^-$. Then, for any point $x \in M$, the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is one-dimensional.*

Proof. Let $(M, \mathcal{F})$ be a simply connected two–dimensional Randers manifold isometric to the Randers manifold determined by the navigation data (h, W), where h is the Klein metric (4.15) on the unit disk $\mathbb{B}^2$ and W is the wind field given by (4.16). In each of the following cases, we derive the parallel transport equation on the indicatrix.

Case $\omega \neq 0$. More convenient coordinate system is given by (r, θ) where

$$r := \exp\big(\omega\, \hat{x}^2\big), \qquad \theta := \sqrt{|\omega|}\, \hat{x}^1. \tag{4.117}$$

The norm of the wind in this coordinate:

$$\|W\| = \frac{\sqrt{|\omega|}r}{\sqrt{r^2-1}}. \tag{4.118}$$

Using the notation $\varepsilon := \operatorname{sgn}(\omega)$, the Finsler norm function is

$$\mathcal{F}(r,\theta,u,v) = \sqrt{\frac{u^2}{(1-r^2)\big(1-(1+\omega)r^2\big)} + \frac{\varepsilon r^2(1-r^2)v^2}{\big(1-(1+\omega)r^2\big)^2}} - \frac{\varepsilon\sqrt{|\omega|}\, r^2\, v}{1-(1+\omega)r^2}. \tag{4.119}$$

The indicatrix at (r, θ) is determined by

$$\left(\frac{u}{1-r^2}\right)^2 + \left(\frac{r\big(v-\sqrt{|\omega|}\big)}{\sqrt{|1-r^2|}}\right)^2 = 1, \tag{4.120}$$

It admits a parametrization

$$u = \varepsilon(1-r^2)\cos\alpha, \qquad v = \frac{\sqrt{|1-r^2|}}{r}\sin\alpha + \sqrt{|\omega|}. \tag{4.121}$$

Using this parametrization, the connection coefficients are

$$G^1_1(\gamma, V) = 2\varepsilon r\cos\alpha + \varepsilon\frac{\sqrt{1-r^2}}{1-r^2}\frac{\sqrt{|\omega|}\cos\alpha\sin\alpha}{1+\frac{\sqrt{|\omega|}r}{\sqrt{1-r^2}}\sin\alpha}, , \tag{4.122a}$$

$$G^1_2(\gamma, V) = -\varepsilon\frac{\sqrt{1-r^2}\sin\alpha}{1+\frac{\sqrt{|\omega|}r}{\sqrt{1-r^2}}\sin\alpha}, \tag{4.122b}$$

$$G^2_1(\gamma, V) = \varepsilon\frac{\sin\alpha}{r^2\sqrt{1-r^2}} - \varepsilon\frac{\sqrt{|\omega|}\cos^2\alpha}{(1+\frac{\sqrt{|\omega|}r}{\sqrt{1-r^2}}\sin\alpha)(1-r^2)r^2}, \tag{4.122c}$$

$$G^2_2(\gamma, V) = \varepsilon\frac{\cos\alpha}{r(1+\frac{\sqrt{|\omega|}r}{\sqrt{1-r^2}}\sin\alpha)}. \tag{4.122d}$$

Substituting into the first parallel transport equation gives

$$\frac{d}{dt}\left(\alpha - \frac{\sqrt{|\omega|}\, r}{\sqrt{|1-r^2|}}\cos\alpha\right) + \frac{\varepsilon\dot\theta}{\sqrt{|1-r^2|}} = 0. \tag{4.123}$$

This is of the form (4.41) with

$$\Phi(r,\theta,\alpha) = \alpha - \frac{\sqrt{|\omega|}\, r}{\sqrt{|1-r^2|}}, \qquad \Psi(r,\theta,\dot r,\dot\theta) = \frac{-\varepsilon\dot\theta}{\sqrt{|1-r^2|}}. \tag{4.124}$$

From Lemma 4.12 we get that the holonomy group is 1-dimensional.

Case $\omega = 0$. In this the adapted coordinate system the Finsler norm function is

$$\begin{aligned}\mathcal{F}(\hat x^1,\hat x^2;\hat y^1,\hat y^2) = \sqrt{\frac{c^2\,(\hat y^1)^2}{(1+2c^2\hat x^2)\big(1+c^2(2\hat x^2+1)\big)} + \frac{c^2(1+2c^2 s)\,(\hat y^2)^2}{\big(1+c^2(2\hat x^2+1)\big)^2}} \\ -\frac{c^2\,\hat y^2}{1+c^2(2\hat x^2+1)}.\end{aligned} \tag{4.125}$$

The indicatrix is

$$\left(\frac{c\,\hat y^1}{1+2c^2\hat x^2}\right)^2 + \left(\frac{c(\hat y^2-1)}{\sqrt{-1-2c^2\,\hat x^2}}\right)^2 = 1, \tag{4.126}$$

with parametrization

$$\hat y^1 = \frac{-(1+2c^2\hat x^2)}{c}\cos\alpha, \qquad \hat y^2 = \frac{\sqrt{-1-2c^2\hat x^2}}{c}\sin\alpha + 1. \tag{4.127}$$

A direct computation yields the following connection coefficients for $c>0$:

$$G^1_1(\gamma,V) = 2c\cos\alpha + \frac{\sqrt{-1-2c^2\hat x^2}}{-1-2c^2\hat x^2}\,\frac{c^2\cos\alpha\sin\alpha}{1+\dfrac{c}{\sqrt{-1-2c^2\hat x^2}}\sin\alpha}, \tag{4.128}$$

$$G^1_2(\gamma,V) = -\frac{c\sqrt{-1-2c^2\hat x^2}\,\sin\alpha}{1+\dfrac{c}{\sqrt{-1-2c^2\hat x^2}}\sin\alpha}, \tag{4.129}$$

$$G^2_1(\gamma,V) = \frac{c\sin\alpha}{\sqrt{-1-2c^2\hat x^2}} - \frac{c^2\cos^2\alpha}{1+\dfrac{c}{\sqrt{-1-2c^2\hat x^2}}\sin\alpha}, \tag{4.130}$$

$$G^2_2(\gamma,V) = \frac{c^2\cos\alpha}{1+\dfrac{c}{\sqrt{-1-2c^2\hat x^2}}\sin\alpha}. \tag{4.131}$$

Substituting these coefficients into the first parallel transport equation yields

$$\frac{d}{dt}\left(\alpha - \frac{c}{\sqrt{-1-2c^2\hat x^2}}\cos\alpha\right) + \dot{\hat x}^2\,\frac{c^2}{\sqrt{-1-2c^2\hat x^2}} = 0. \tag{4.132}$$

which is of the form (4.41) with

$$\Phi(\hat{x}^1,\hat{x}^2,\alpha)=\alpha-\frac{c\cos\alpha}{\sqrt{-(1+2c^2\hat{x}^2)}},\qquad \Psi(\hat{x}^1,\hat{x}^2,\dot{\hat{x}}^1,\dot{\hat{x}}^2)=\frac{-c^2\dot{\hat{x}}^2}{\sqrt{-(1+2c^2\hat{x}^2)}}. \tag{4.133}$$

For $c<0$, the same computation leads to the same parallel transport equation, with the corresponding choice

$$\Phi(\hat{x}^1,\hat{x}^2,\alpha)=\alpha+\frac{c\cos\alpha}{\sqrt{-1-2c^2\hat{x}^2}},\qquad \Psi(\hat{x}^1,\hat{x}^2,\dot{\hat{x}}^1,\dot{\hat{x}}^2)=\frac{-c^2\dot{\hat{x}}^2}{\sqrt{-(1+2c^2\hat{x}^2)}}. \tag{4.134}$$

From Lemma 4.12 we get that the holonomy group is 1-dimensional. □

Isomorphism between Finsler and Riemannian holonomy groups

Let us consider a Randers metric $\mathcal{F}$ of constant curvature determined by the navigation data (h,W) considered in Section 4.2.2, that is when either $K>0$ or $K<0$ with $\mathcal{K}_1^-$. As we could see in section4.2.2, the parallelism of a unit vector field $V(t)$ along a curve $\gamma(t)$ is determined by the differential equation (4.41) where the functions Φ and Ψ are determined by (4.103), (4.124), and (4.133),(4.134) respectively.

4.19 Remark (Case $W=0$)**.** In the case when the wind W is identically zero, then the Finsler norm function F reduces to the Riemannian norm function, the parallelism is with respect to its Levi-Civita connection associated with h. Moreover, we have $\Psi(x,\alpha)=\alpha$, and (4.41) gives

$$\frac{d\alpha}{dt}=\Psi(\gamma(t),\dot{\gamma}(t)) \tag{4.135}$$

and (4.46) gives the Riemannian holonomy transformation $\mathcal{H}^R_\gamma$:

$$\mathcal{H}^R_\gamma\colon \mathcal{I}_x\to\mathcal{I}_x,\qquad \mathcal{H}^R_\gamma(\alpha)=\alpha+\oint_\gamma\Psi\big(\gamma(t),\dot{\gamma}(t)\big)dt. \tag{4.136}$$

4.20 Remark (Case $W\neq 0$)**.** Let us consider

$$\beta:=\Phi(x,\alpha)=\alpha-\|W_x\|\cos\alpha. \tag{4.137}$$

Since $\|W\|<1$, we have $\partial_\alpha\Phi\neq 0$, therefore β can be regarded as a new angular coordinate on the indicatrix bundle. The parallel transport of a unit vector field along a curve $\gamma:[0,1]\to M$ is given by

$$\frac{d\beta}{dt}=\Psi(\gamma(t),\dot{\gamma}(t)), \tag{4.138}$$

so that

$$\beta(1)=\beta(0)+\int_0^1\Psi(\gamma(t),\dot{\gamma}(t)). \tag{4.139}$$

Consequently, for a closed curve γ, the Finslerian holonomy transformation is

$$\mathcal{H}_\gamma\colon \mathcal{I}_x\to\mathcal{I}_x,\qquad \mathcal{H}_\gamma(\beta)=\beta+\oint_\gamma\Psi\big(\gamma(t),\dot{\gamma}(t)\big)dt. \tag{4.140}$$

From the Remarks 4.19 and 4.20 we get the following Theorem:

4.21 Theorem. *Let $(M,\mathcal{F})$ be a simply connected two-dimensional Randers manifold of constant curvature K associated wight the navigation data (h,W) where either $K>0$ or $K<0$ with $\mathcal{K}_1^-$. Then at any $x\in M$ the Finsler holonomy group $\mathcal{H}ol(\mathcal{F})$ and the Riemannian holonomy group $\mathcal{H}ol(h)$ are isomorphic:*

$$\mathcal{H}ol_x(\mathcal{F})\cong\mathcal{H}ol_x^R(h). \tag{4.141}$$

Proof. Let γ be a closed curve based on x. Comparing the Riemannian holonomy transformation (4.136) and the Finslerian holonomy transformation (4.140) one can see, that both acts on the indicatrix as a translation with the same constant $C_\gamma := \oint_\gamma \Psi\big(\gamma(t),\dot\gamma(t)\big)dt$. More precisely, rewriting (4.137) as $\beta=\Psi_x(\alpha):=\Psi(x,\alpha)$ we get that $\mathcal{H}_\gamma(\beta)=\mathcal{H}_\gamma^R(\alpha)$, that is

$$\mathcal{H}_\gamma=\Psi_x\circ\mathcal{H}_\gamma^R\circ\psi^{-1}, \tag{4.142}$$

for any closed curve γ, which proves (4.141). □

4.22 Corollary. *Let $(M,\mathcal{F})$ be a simply connected two-dimensional Randers manifold of constant curvature K associated with the navigation data (h,W) where either $K>0$ or $K<0$ with $\mathcal{K}_1^-$. Then its holonomy group is isomorphic to the rotation group:*

$$\mathcal{H}ol_x(\mathcal{F})\cong SO(2). \tag{4.143}$$

4.23 Remark. One can verify that, when the wind W does not vanish, in both of the previous two cases, that is when $K>0$ or $K<0$ with $\mathcal{K}_1^-$, W is not parallel with respect to the Riemannian metric h. Consequently, the associated Finsler metric $\mathcal{F}$ is not Berwaldian [5]. One can also compute the Landsberg tensor (2.35) and check that it does not vanish; hence F is not a Landsberg metric either.

It follows that these Randers metrics do not belong to any of the special classes of Finsler metrics (Riemannian, Berwald, or Landsberg) for which the holonomy is known to be finite-dimensional. Nevertheless, Theorems 4.16 and 4.18 demonstrate that, although most Finsler metrics have infinite-dimensional holonomy, there exist genuine Finsler metrics whose holonomy is finite-dimensional. This provides a positive answer to Question 1.

4.2.3 Infinite-dimensional holonomy ($K<0$ with $\mathcal{K}_2^-$)

As shown in Proposition 4.7, the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(\mathcal{F})$ of the class of Randers surfaces of constant curvature considered in Section (4.1.3) is infinite dimensional. It follows from (3.22) that the corresponding holonomy group is infinite dimensional. However, a stronger result holds.

4.24 Theorem. *Let $(M,\mathcal{F})$ be a 2-dimensional Randers manifold of constant negative curvature with $\mathcal{K}_2^-$. Then the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is maximal, its closure is isomorphic to $\mathcal{D}iff_+(\mathbb{S}^1)$.*

Proof. Let $(M, \mathcal{F})$ be a 2-dimensional Randers manifold of constant negative curvature, isometric to the Randers surface associated to navigation data (h, W), where h is the Euclidean metric and W is the wind field determined by equation (4.18). By Proposition 4.7, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is dense in $\mathfrak{X}(\mathcal{I}_x)$, in the Lie algebra of smooth vector fields on the indicatrix $\mathcal{I}_x$. According to [39, Theorem 3.3], if the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ of an n-dimensional Finsler manifold is dense in $\mathfrak{X}(\mathcal{I}_x)$, then the holonomy group is maximal and isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$, the identity component of the diffeomorphism group of the $(n-1)$-sphere. In the two-dimensional case, $\mathcal{D}iff_o(\mathbb{S}^1) = \mathcal{D}iff_+(\mathbb{S}^1)$, the group of orientation preserving diffeomorphisms of the circle $\mathbb{S}^1$. This completes the proof.

□

Notes. The classification of the holonomy groups of Randers surfaces of constant curvature can be summarized as follows: For a Randers metric of constant flag curvature on a simply connected 2-dimensional manifold, then the following possibilities can occur:

1. if $K = 0$, then the holonomy group is trivial, that is $\mathcal{H}ol = \{id\}$,
2. if $K > 0$, then the holonomy group is 1-dimensional and $\mathcal{H}ol \cong SO(2)$,
3. if $K < 0$ with $\mathcal{K}_1^-$, then the holonomy group is 1-dimensional and $\mathcal{H}ol \cong SO(2)$,
4. if $K < 0$ with $\mathcal{K}_2^-$, then the holonomy group is maximal, that is $\overline{\mathcal{H}ol} \cong \mathcal{D}iff_+(\mathbb{S}^1)$,

5 Natural parallelism associated with navigation data

Navigation data consist of a pair (h, W), where h is a Riemannian metric and W is a smooth vector field on a manifold M. Zermelo's navigation problem asks for the paths of shortest travel time on a Riemannian manifold (M, h) under the influence of a wind or current represented by the vector field W. As proved by D. Bao, C. Robles, and Z. Shen in [6], Zermelo's navigation problem is equivalent to the study of geodesics of Randers-type Finsler metrics. The construction of the corresponding metric structure from the navigation data is geometrically intuitive: the sets of unit vectors, that is, the indicatrices, are shifted by the wind field W. By contrast, the associated affine structure, in particular the parallel translation, is much less immediate and natural to describe [51]. Moreover, the holonomy group may be very large even in situations where the metric structure itself is relatively simple [30].

In this chapter we introduce a natural way a parallelism associated with the navigation data. In Section 5.1, we give a formal definition of this parallel translation, and we investigate its basic properties: the natural parallel translation is homogeneous (but in general nonlinear), preserves the Randers type Finslerian norm constituted by the navigation data, and the holonomy group is finite-dimensional. We show that it reduces to the Riemannian parallel translation along curves on which W is h-parallel. We prove that the holonomy group of the natural parallel translation is isomorphic to the Riemannian holonomy group of h.

In Section 5.2, we develop the geometric structures induced by this parallelism. We derive the horizontal distribution and the coefficients of the associated (generally nonlinear) connection, define the corresponding covariant derivative, and characterize its parallel vector fields. We then study the autoparallel curves of the construction and derive the associated differential equation, spray, and *natural symmetric connection*. Finally, we clarify the relationship between these objects and the geometry of the Riemannian metric h, including the projective relation to the Riemannian spray. Illustrative examples are given. This chapter is based on the results of [41].

5.1 Parallel translation associated to navigation data

We briefly recall the navigation framework. Let M be a smooth manifold equipped with a Riemannian metric h. A *navigation datum* is a pair (h, W), where $W \in \mathfrak{X}(M)$ is a smooth vector field satisfying

$$\|W(x)\|_h < 1 \qquad \text{for all } x \in M.$$

This subunit condition ensures that, for each $x \in M$, translating the h-unit sphere in T_xM by the vector $W(x)$ produces a smooth, strongly convex hypersurface; equivalently, it defines a Finsler indicatrix.

In these terms, the classical Zermelo navigation problem asks for time-minimizing curves on (M, h) under the drift imposed by W. The fundamental theorem of Bao–Robles–Shen identifies this variational problem with the geodesic problem of a Randers-type Finsler metric naturally associated with (h, W) [6]. Accordingly, the datum (h, W) determines a Randers norm $\mathcal{F} : TM \setminus \{0\} \to (0, \infty)$ (see (2.38)), whose unit spheres are precisely the translated h-unit spheres. We will repeatedly use the equivalence between the Finslerian unit condition $\mathcal{F}(V^\circ) = 1$ and the h-unit condition for the shifted vector $V^\circ - W$ (cf. (2.44)).

Although the metric component of the construction is dictated by the indicatrix geometry, the choice of a compatible notion of parallel transport is not intrinsic from the Finsler viewpoint. In the navigation setting, however, the presence of the Riemannian metric h suggests a natural choice: one may translate vectors by first removing the wind component, performing Riemannian parallel transport, and then adding back the wind at the endpoint. We therefore introduce the following notion of natural parallel translation.

5.1 Definition (Natural parallel translation)**.** Let c be a smooth curve joining p to q. Let $V_p^\circ \in T_pM$ be a unit vector with respect to the Randers norm $\mathcal{F}$ associated with the navigation data (h, W). The *natural parallel translation* of V_p° along c is defined by

$$\mathcal{P}(V_p^\circ) := \mathcal{P}_R(V_p^\circ - W_p) + W_q, \tag{5.1}$$

where $\mathcal{P}_R$ denotes the Riemannian parallel translation along c with respect to h. For an arbitrary nonzero vector $V_p \in T_pM \setminus \{0\}$ we extend the definition by homogeneity:

$$\mathcal{P}(V_p) := \mathcal{F}(V_p) \cdot \mathcal{P}\left(\frac{1}{\mathcal{F}(V_p)} V_p\right). \tag{5.2}$$

The natural parallel translation defined above has some immediate properties. From its construction, it is homogeneous but in general not linear. At the same time, since it is built from the Riemannian parallel translation, it preserves the Randers norm. Moreover, if the wind vector field is parallel along a curve, then the natural parallel translation reduces to the usual Riemannian one. These facts are formulated in the following properties.

5.2 Proposition (Homogeneity, nonlinearity, and norm preservation)**.** *The natural parallel translation $\mathcal{P}$ is homogeneous, but in general it is not additive. Moreover, it preserves the Randers norm function $\mathcal{F}$:*

$$\mathcal{F}(\mathcal{P}(V_p)) = \mathcal{F}(V_p), \qquad V_p \in T_pM \setminus \{0\}. \tag{5.3}$$

Proof. Homogeneity follows directly from the definition (5.2). Additivity fails in general, since even though $\mathcal{P}_R$ is linear, the presence of the normalization by $\mathcal{F}(\,\cdot\,)$ in (5.2) prevents $\mathcal{P}$ from satisfying $\mathcal{P}(U_p + V_p) = \mathcal{P}(U_p) + \mathcal{P}(V_p)$.

For norm preservation, let V_p° be $\mathcal{F}$-unit. Using the relation between $\mathcal{F}$ and the h-norm (cf. (2.44)), and the fact that $\mathcal{P}_R$ preserves the h-norm, we obtain

$$\begin{aligned} 1 = \mathcal{F}(V_p^\circ) &= \|V_p^\circ - W_p\|_h = \|\mathcal{P}_R(V_p^\circ - W_p)\|_h \\ &= \mathcal{F}\big(\mathcal{P}_R(V_p^\circ - W_p) + W_q\big) \\ &= \mathcal{F}(\mathcal{P}(V_p^\circ)). \end{aligned}$$

Finally, the homogeneity of both $\mathcal{P}$ and $\mathcal{F}$ extends the conclusion to any nonzero V_p. □

5.3 Theorem. *Let (h, W) be a navigation data on the manifold M. If W is parallel along the curve c with respect to the Riemannian metric, then the natural and the Riemannian parallel transports on c coincide.*

Proof. Indeed, let c be a curve from p to q and suppose that W to be parallel with respect to the Riemannian metric h along the curve c. Then $\mathcal{P}_R(W_p) = W_q$. Let $V_p^\circ \in T_pM$ be a Finslerian unit vector at p. Using the linearity of the Riemannian parallel translation $\mathcal{P}_R$:

$$\mathcal{P}(V_p^\circ) = \mathcal{P}_R(V_p^\circ - W_p) + W_q = \mathcal{P}_R(V_p^\circ) - \mathcal{P}_R(W_p) + W_q = \mathcal{P}_R(V_p^\circ), \tag{5.4}$$

showing that $\mathcal{P}$ and $\mathcal{P}_R$ coincide on Finslerian unit vectors. The statement follows from the homogeneity property of the parallel translations $\mathcal{P}$ and $\mathcal{P}_R$. □

We now consider the holonomy of the natural parallel translation. Since this translation is constructed directly from the Riemannian parallel transport, a close relation between the two holonomy groups is to be expected. The following theorem shows that the holonomy group of the natural parallel translation is in fact isomorphic to the Riemannian one, and therefore it is finite-dimensional.

5.4 Theorem (Holonomy of the natural parallel translation)**.** *Let (h, W) be navigation data on the manifold M. The holonomy group $\mathcal{H}ol(\mathcal{P})$ associated to the natural parallel translation is isomorphic to the Riemannian holonomy group $\mathcal{H}ol(\mathcal{P}_R)$. In particular, the holonomy group of $\mathcal{H}ol(\mathcal{P})$ is finite dimensional.*

Proof. Fix a point $p \in M$ and consider parallel translations along closed curves based at p. Let $\varphi \in \mathcal{H}ol(\mathcal{P})$ and $\varphi_R \in \mathcal{H}ol(\mathcal{P}_R)$ denote the corresponding natural and Riemannian holonomy transformations. From the definition of the natural parallel translation, for any $V_p \in T_pM$ we obtain

$$\varphi(V_p) = \varphi_R(V_p) - \mathcal{F}(V_p)\big(\varphi_R(W_p) - W_p\big). \tag{5.5}$$

This defines a correspondence $\varphi \leftrightarrow \varphi_R$ between the elements of the two holonomy groups. To recover φ_R from φ, substitute $V_p = W_p$ into (5.5). This yields

$$\varphi_R(W_p) = \frac{\varphi(W_p) - \mathcal{F}(W_p)W_p}{1 - \mathcal{F}(W_p)}.$$

Using this expression, we obtain the inverse relation

$$\begin{aligned}\varphi_R(V_p) &= \varphi(V_p) + \mathcal{F}(V_p)\left(\frac{\varphi(W_p) - \mathcal{F}(W_p)W_p}{1 - \mathcal{F}(W_p)} - W_p\right) \\ &= \varphi(V_p) + \mathcal{F}(V_p)\frac{\varphi(W_p) - W_p}{1 - \mathcal{F}(W_p)}.\end{aligned}$$

Thus the correspondence $\varphi \leftrightarrow \varphi_R$ is one-to-one.

It remains to verify that this correspondence preserves the group operation. Let $\varphi, \psi \in \mathcal{H}ol(\mathcal{P})$ be induced by $\varphi_R, \psi_R \in \mathcal{H}ol(\mathcal{P}_R)$, respectively. Using (5.5), we compute

$$\begin{aligned}\psi \circ \varphi(V_p) &= \psi_R(\varphi(V_p)) - \mathcal{F}(\varphi(V_p))\big(\psi_R(W_p) - W_p\big) \\ &= \psi_R \circ \varphi_R(V_p) - \mathcal{F}(V_p)\big(\psi_R \circ \varphi_R(W_p) - W_p\big),\end{aligned}$$

where we used the fact that $\mathcal{F}(\varphi(V_p)) = \mathcal{F}(V_p)$, since the natural parallel translation preserves the Randers norm. This shows that the composition $\psi \circ \varphi$ corresponds to $\psi_R \circ \varphi_R$, and hence the correspondence between the two holonomy groups is a group isomorphism. Since the Riemannian holonomy group is finite-dimensional, the same holds for $\mathcal{H}ol(\mathcal{P})$. □

5.5 Remark. It is known from [29] that, in general, homogeneous parallel translations associated with Finsler metrics have infinite-dimensional holonomy groups. Finite-dimensional holonomy appears only in exceptional situations. From this perspective, it is natural to look for examples of homogeneous (but nonlinear) parallel translations whose holonomy groups are still finite-dimensional. The preceding proposition shows that the natural parallel translation belongs to this distinguished class.

We now investigate the geometric structures determined by this parallelism. In particular, we derive the horizontal distribution, the associated connection, the covariant derivative, and the torsion. These objects provide a differential-geometric description of the natural parallel translation in terms of the navigation data (h, W).
Remark 2.4 implies that parallel translation on M can be encoded as horizontality in TM: if $V(t)$ is parallel along $c(t)$, then the lifted curve $(c(t), V(t)) \subset TM$ is horizontal. Consequently, the horizontal distribution of a given parallelism can be obtained by differentiating parallel vector fields. We follow this approach for the natural parallel translation.

Let $\mathcal{A}_{ij}^k = \mathcal{A}_{ij}^k(x)$ be the Christoffel symbols of the Levi–Civita connection ∇^R of the Riemannian metric h:

$$\nabla^R_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \mathcal{A}_{ij}^k \frac{\partial}{\partial x^k}. \tag{5.6}$$

Fix a smooth curve $c(t)$ in M. Denote by $\mathcal{P}_t$ the natural parallel translation along c, and by $(\mathcal{P}_R)_t$ the Riemannian parallel translation with respect to h. By (7.6), for any $\mathcal{F}$-unit vector $V_0 \in T_{c(0)}M$ the natural parallel field is given by

$$\mathcal{P}_t(V_0) = (\mathcal{P}_R)_t(V_0 - W_0) + W(c(t)). \tag{5.7}$$

Since $(c(t), \mathcal{P}_t(V_0))$ is horizontal, its velocity must lie in the horizontal space at $(c(t), \mathcal{P}_t(V_0))$:

$$\frac{d}{dt}\mathcal{P}_t(V_0) \in \mathcal{H}_{(c(t),\mathcal{P}_t(V_0))}. \tag{5.8}$$

In local coordinates (x^k, y^k) on TM, the curve $(c(t), \mathcal{P}_t(V_0))$ has the form $(c^k(t), \mathcal{P}_t(V_0)^k)$, hence

$$\big(c^k(t), \mathcal{P}_t(V_0)^k, \dot{c}^k(t), \tfrac{d}{dt}[\mathcal{P}_t(V_0)]^k\big) \in TTM.$$

Differentiating (5.7) and using that $(\mathcal{P}_R)_t$ is Riemannian parallel transport, we obtain

$$\begin{aligned}
\tfrac{d}{dt}[\mathcal{P}_t(V_0)]^k &= \tfrac{d}{dt}[\mathcal{P}_R(V_0 - W_0) + W(c(t))]^k \\
&= \tfrac{d}{dt}[\mathcal{P}_R(V_0 - W_0)]^k + \tfrac{d}{dt}[W(c(t))]^k \\
&= -\mathcal{A}^k_{ij}(c(t)) \cdot \dot{c}^i(t) \cdot \mathcal{P}_R(V_0 - W_0) + \frac{\partial W^k}{\partial x^i}\Big|_{c(t)} \dot{c}^i(t).
\end{aligned}$$

That is ,

$$\frac{d}{dt}[\mathcal{P}_t(V_0)]^k = -\mathcal{A}^k_{ij}(c(t))\, \dot{c}^i(t)\, (\mathcal{P}_R)^j_t(V_0 - W_0) + \frac{\partial W^k}{\partial x^i}(c(t))\, \dot{c}^i(t). \tag{5.9}$$

Let $\Gamma^k_i = \Gamma^k_i(x, y)$ be the coefficients of the connection determined by the natural parallelism, and let

$$\mathfrak{h} : TTM \to \mathcal{H} \subset TTM \tag{5.10}$$

be the corresponding horizontal projector. Locally, $\mathcal{H}$ is spanned by

$$\mathfrak{h}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial y^j}. \tag{5.11}$$

Comparing (5.8) with the coordinate expression (5.9) yields

$$\Gamma^k_i\big(c(t), \mathcal{P}_t(V_0)\big)\, \dot{c}^i(t) = \mathcal{A}^k_{ij}(c(t))\, \dot{c}^i(t)\, \big[(\mathcal{P}_R)_t(V_0 - W_0)\big]^j - \dot{c}^i(t)\, \frac{\partial W^k}{\partial x^i}\Big|_{c(t)}. \tag{5.12}$$

Setting $t = 0$ (so that $\mathcal{P}_0 = (\mathcal{P}_R)_0 = \mathrm{id}$) we obtain at $x = c(0)$ the formula

$$\Gamma^k_i(x, V_0) = \mathcal{A}^k_{ij}(x)\big(V^j_0 - W^j(x)\big) - \frac{\partial W^k}{\partial x^i}\Big|_x, \tag{5.13}$$

valid for every $\mathcal{F}$-unit vector $V_0 \in T_xM$. Using homogeneity, we extend it to arbitrary $y \in T_xM \setminus \{0\}$ by taking $V_0 = \frac{1}{\mathcal{F}(x,y)}y$, which gives

$$\Gamma^k_i(x, y) = \mathcal{A}^k_{ij}(x)\big(y^j - \mathcal{F}(x, y)W^j(x)\big) - \mathcal{F}(x, y)\, \frac{\partial W^k}{\partial x^i}\Big|_x. \tag{5.14}$$

With the abbreviations $\mathcal{A}_{ij}^k = \mathcal{A}_{ij}^k(x)$ and $\mathcal{F}(y) = \mathcal{F}(x,y)$, this can be written as

$$\Gamma_i^k(x,y) = \mathcal{A}_{is}^k\, y^s - \mathcal{F}(y)\, \mathcal{A}_{is}^k\, W^s - \mathcal{F}(y)\, \frac{\partial W^k}{\partial x^i}. \tag{5.15}$$

It is also convenient to describe $\mathcal{H}$ via horizontal lifts. For $X \in T_xM$, the horizontal lift $l_{(x,y)}(X) \in \mathcal{H}_{(x,y)}$ satisfies

$$l_{(x,y)}(X) = l_{(x,y)}^R(X) + \mathcal{F}(x,y)\left(\nabla_X^R W\right)^v, \tag{5.16}$$

where $l_{(x,y)}^R$ is the Riemannian horizontal lift and $(\cdot)^v$ denotes vertical lift.

5.6 Definition. Let (h, W) be navigation data on M, and let $\mathfrak{h}$ be the horizontal projector (5.10) induced by the natural parallelism. The connection

$$\Gamma := 2\mathfrak{h} - \mathrm{Id} \tag{5.17}$$

is called the *natural connection*. Its coefficients are given by (5.15).

The horizontal distribution determines the notion of parallel vector fields along curves. This, in turn, gives rise to a covariant derivative associated with the natural parallelism. We now describe this covariant derivative and its basic properties. A vector field $V(t)$ along a curve $c(t)$ is parallel if the lifted curve $(c(t), V(t)) \subset TM$ is horizontal. In terms of the horizontal lift, this condition can be written as

$$\dot{V}_t = l_{(c,V_t)}(\dot{c}(t)). \tag{5.18}$$

Equivalently, the vector field is parallel if its covariant derivative along the curve vanishes. Using the expression of the horizontal lift, one obtains the following formula for the covariant derivative of a vector field along $c(t)$:

$$\frac{\nabla V_t}{dt} = \left(\frac{dV_t^k}{dt} + \Gamma_i^k(c(t), V_t)\, \dot{c}^i(t)\right) \frac{\partial}{\partial x^k}. \tag{5.19}$$

This formula naturally leads to the definition of the (generally nonlinear) covariant derivative of a vector field V in the direction of a vector field X:

$$\nabla_X V = X^i \left(\frac{\partial V^k}{\partial x^i} + \Gamma_i^k(x, V)\right) \frac{\partial}{\partial x^k}. \tag{5.20}$$

Thus we obtain a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad (X, Y) \mapsto \nabla_X Y. \tag{5.21}$$

This operator is $C^\infty(M)$-linear in its first argument, but only positively homogeneous over $\mathbb{R}$ in the second one; in particular, it is not additive in the second variable. The notion of parallelism can therefore be expressed in terms of this nonlinear covariant derivative.

5.7 Property. *A vector field $V(t)$ along a curve $c(t)$ is parallel if and only if $\nabla_{\dot{c}} V = 0$.*

5.8 Property. *The coordinate-free expression of the nonlinear covariant derivative* (5.21) *associated with the natural parallelism is*

$$\nabla_X Y = \nabla^R_X Y - \mathcal{F}(Y)\, \nabla^R_X W, \tag{5.22}$$

where ∇^R *denotes the Levi–Civita connection of the Riemannian metric* h.

5.9 Remark. Formula (5.22) immediately shows that if the wind vector field W is parallel along a curve, then the natural and the Riemannian parallel translations along that curve coincide. In particular, W is parallel with respect to h if and only if the nonlinear covariant derivative ∇ reduces to the Riemannian connection ∇^R.

5.10 Proposition. *The covariant differentiations along the integral curves of the vector field* W *coincide if and only if the integral curves of* W *are Riemannian geodesics.*

Proof. Substituting $X = W$ in formula (5.22) we have the same covariant derivatives along the integral curves of the vector field W if and only if $\nabla^R_{\dot c} W = 0$, that is the integral curves of W are Riemannian geodesics. □

5.11 Proposition. *The integral curves of* W *are pre-geodesics (resp. geodesics) of* ∇ *if and only if they are pre-geodesics (resp. geodesics) of* ∇^R.

Proof. Substituting $X = Y = W$ in formula (5.22) we have that

$$\nabla_W W = (1 - \mathcal{F}(W))\, \nabla^R_W W. \tag{5.23}$$

Since W is one of the navigation data, it follows that

$$\mathcal{F}(W) = \frac{|W|}{1 + |W|} < 1, \tag{5.24}$$

and the acceleration vector fields $\nabla_W W$ and $\nabla^R_{\dot c} W$ are proportional to $\dot c = W \circ c$ at the same time as follows:

$$\nabla_{\dot c} W(t) = \varphi(t)\dot c(t) \qquad \Leftrightarrow \qquad \nabla^R_{\dot c} W(t) = \rho(t)\dot c(t), \tag{5.25}$$

with

$$\rho(t) = \frac{\varphi(t)}{1 - \mathcal{F}(\dot c(t))}. \tag{5.26}$$

As (5.25) shows, the integral curves of W are pre-geodesics of the natural connection ∇ if and only if they are pre-geodesics of the connection ∇^R and, in case of $\varphi(t) = \rho(t) = 0$, the integral curves of W are geodesics of ∇, if and only if they are geodesics of ∇^R.

□

5.2 Connection associated to the natural parallelism

In this section we describe the connection corresponding to the natural parallel translation introduced above. In particular, we characterize the autoparallel curves and derive the associated differential equation and spray.

Let (h, W) be navigation data on M. The curves associated with the natural parallelism are those for which the velocity vector is transported parallel to itself along the curve. Using the covariant derivative ∇ introduced in previous section , this condition is written as

$$\nabla_{\dot{c}}\dot{c} = 0. \tag{5.27}$$

Substituting the expression (5.22) of the covariant derivative in terms of the Levi–Civita connection ∇^R of the Riemannian metric h, we obtain

$$\nabla^R_{\dot{c}}\dot{c} - \mathcal{F}(\dot{c})\,\nabla^R_{\dot{c}}W = 0. \tag{5.28}$$

Thus the acceleration with respect to the Riemannian connection is modified by a term depending on the wind field. In local coordinates this leads to the second-order system

$$\ddot{c}^k + \dot{c}^i\left(\mathcal{A}^k_{ij}\dot{c}^j - \mathcal{F}(\dot{c})\,\mathcal{A}^k_{ij}W^j - \mathcal{F}(\dot{c})\,\frac{\partial W^k}{\partial x^i}\right) = 0, \tag{5.29}$$

where $\mathcal{A}^k_{ij}$ are the Christoffel symbols of ∇^R. The system (5.29) is generated by the spray

$$S = y^k\frac{\partial}{\partial x^k} - 2G^k(x,y)\frac{\partial}{\partial y^k}, \tag{5.30}$$

whose coefficients are

$$G^k(x,y) = \tfrac{1}{2}\Big(\mathcal{A}^k_{ij}y^iy^j - \mathcal{F}(x,y)\,y^i\mathcal{A}^k_{ij}W^j - \mathcal{F}(x,y)\,y^i\frac{\partial W^k}{\partial x^i}\Big). \tag{5.31}$$

The spray obtained above determines a natural connection on the tangent bundle. For later reference, we introduce the following terminology.

5.12 Definition. The spray (5.30) with coefficients (5.31) corresponding to the natural parallelism will be called the *natural spray.* The connection

$$\overline{\Gamma} := [J, S] \tag{5.32}$$

generated by this spray will be called the *natural symmetric connection.*

The natural symmetric connection should be compared with the natural connection introduced earlier. In general, these two connections do not coincide, as the following remark shows.

5.13 Remark. The natural connection (5.17) and the natural symmetric connection (5.32) are different in general. Indeed, the torsion of the natural symmetric connection is identically zero, while the torsion of the natural connection is typically nonzero.

The next lemma will be used to describe the relation between these connections.

5.14 Lemma. *Let ρ be a one-form on the base manifold and let φ be a zero-homogeneous function on the tangent bundle. Then the relation*

$$y^i \rho_i^k \circ \pi = \varphi(x, y)\, y^k \tag{5.33}$$

holds if and only if φ depends only on the base point, that is $\varphi(x, y) = \varphi(x)$, and

$$\rho_j^k(x) = \varphi(x)\, \delta_j^k. \tag{5.34}$$

Proof. Starting from (5.33), we differentiate both sides with respect to y^l. This gives

$$\rho_l^k \circ \pi = \varphi_{y^l}\, y^k + \delta_l^k\, \varphi. \tag{5.35}$$

Contracting this relation by setting $k = l$, we obtain

$$\rho_k^k \circ \pi = \varphi_{y^k}\, y^k + n\, \varphi. \tag{5.36}$$

Since φ is zero homogeneous, the Euler relation implies $y^k \varphi_{y^k} = 0$. Hence $\varphi = \frac{1}{n}\, \rho_k^k \circ \pi$, which shows that φ depends only on the base point. Substituting this back into (5.33), we get

$$y^i \rho_i^k \circ \pi = \varphi(x)\, y^k. \tag{5.37}$$

Differentiating this identity with respect to y^j yields $\rho_j^k(x) = \varphi(x)\, \delta_j^k$, which proves the statement. The converse direction is immediate. □

5.15 Definition. Let (M, h) be a Riemannian manifold. A vector field $W \in \mathfrak{X}(M)$ is called *concircular* with respect to the metric h if there exists a smooth function $\varphi \in C^\infty(M)$ such that

$$\nabla_X^R W = \varphi\, X \tag{5.38}$$

for every vector field $X \in \mathfrak{X}(M)$, where ∇^R denotes the Levi–Civita connection of h. The function φ is referred to as the *potential function* of W.

5.16 Proposition. *Let (h, W) be navigation data on a manifold M. The natural spray S is projectively related to the quadratic spray S^R of the Levi–Civita connection ∇^R if and only if the vector field W is concircular with respect to the Riemannian metric h.*

Proof. Two sprays are projectively related if and only if they have the same geodesics as point sets, that is, if they differ only by a reparametrization. It is well known that this is equivalent to the existence of a positively one-homogeneous function $P(x, y)$ such that

$$S = S^R + PC, \tag{5.39}$$

where C denotes the canonical (Liouville) vector field. The quadratic spray corresponding to the Levi–Civita connection ∇^R has the local form

$$S^R = y^k \frac{\partial}{\partial x^k} - \mathcal{A}_{ij}^k(x) y^i y^j \frac{\partial}{\partial y^k}. \tag{5.40}$$

Comparing this expression with the natural spray (5.30)–(5.31), we see that the condition $S = S^R + PC$ is equivalent to the relation

$$\mathcal{F}(x, y)\, y^i \rho_i^k(x) = P(x, y)\, y^k, \tag{5.41}$$

where the vector-valued one-form ρ is given by

$$\rho(X) = \nabla_X^R W. \tag{5.42}$$

Applying Lemma 5.14 to (5.41) with $\varphi = P/\mathcal{F}$, we conclude that the above identity holds if and only if

$$P(x, y) = \varphi(x)\mathcal{F}(x, y) \quad \text{and} \quad \rho_j^k(x) = \varphi(x)\delta_j^k. \tag{5.43}$$

The latter condition is precisely

$$\nabla_X^R W = \varphi(x) X \tag{5.44}$$

for all vector fields X, which means that W is a concircular vector field with respect to the metric h. This proves the statement. □

We conclude this section with two examples illustrating the behavior of the natural spray. The first example shows a case in which the natural spray is metrizable, while the second one demonstrates that this property does not hold in general.

5.17 Example. Let M be the interior of the Euclidean unit ball in $\mathbb{R}^n$, equipped with the standard Euclidean metric $h_{ij} = \delta_{ij}$. Consider the radial wind vector field

$$W = -x^i \frac{\partial}{\partial x^i}. \tag{5.45}$$

A direct computation shows that

$$\nabla_X^R W = -X \tag{5.46}$$

for every vector field X. Hence W is a concircular vector field with potential function $\varphi = -1$. Substituting this into the expression of the natural spray, we obtain

$$S = y^i \frac{\partial}{\partial x^i} - \mathcal{F}(x, y)\, C, \tag{5.47}$$

where C denotes the Liouville vector field. In this case, the Randers metric $\mathcal{F}$ coincides with the Funk metric [48]. Consequently, the spray (5.47) is precisely the canonical spray of the Funk metric, and therefore the natural spray is Finsler metrizable.

5.18 Example. Consider the Euclidean plane $\mathbb{R}^2$ with the standard metric $h_{ij} = \delta_{ij}$, and let the wind vector field be the infinitesimal rotation

$$W = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}. \tag{5.48}$$

For this choice of wind, the horizontal distribution of the natural parallelism exhibits a completely different behavior than in the previous example. Computing successive Lie brackets of the horizontal vector fields δ_1, δ_2 shows that they generate new, independent vertical directions. As a result, the distribution obtained from all iterated brackets,

$$\mathcal{D}_{\mathcal{H}} := \langle \mathcal{H} \rangle_{\text{Lie}}, \tag{5.49}$$

known as the holonomy distribution, becomes four-dimensional. In fact, it coincides with the whole tangent space of the tangent bundle:

$$\mathcal{D}_{\mathcal{H}} = TTM. \tag{5.50}$$

Suppose now that the natural spray were Finsler metrizable. Then the corresponding Finsler norm function F would have to be invariant under the holonomy distribution [24]. This would imply that $\mathcal{L}_X \mathcal{F} = 0$ for every vector field $X \in \mathcal{D}_{\mathcal{H}}$. Since $\mathcal{D}_{\mathcal{H}} = TTM$, this would force F to be constant on TM, which is impossible for a Finsler norm. Therefore the natural spray in this example is not Finsler metrizable.

6 Summary

In the summary, we follow the same section structure as in the dissertation, with simplified numbering to aid the reader.

1. Introduction

This thesis investigates the holonomy structure of Finsler manifolds, a natural generalization of Riemannian geometry in which the notion of length depends on both position and direction. While holonomy theory is well understood in the Riemannian setting, its Finslerian counterpart exhibits fundamentally different and often more complex behavior, frequently leading to infinite-dimensional structures. The aim of this work is to deepen the understanding of these phenomena by analyzing maximal holonomy, classifying holonomy groups in specific cases, and introducing new geometric frameworks related to navigation problems.

2. Preliminaries

The second chapter of the dissertation introduces the fundamental notions of spray and Finsler geometry used throughout the thesis. It covers connections and their associated horizontal–vertical decomposition, covariant derivatives, curvature, as well as sprays and their geodesics, given by

$$\ddot{\gamma}^i + 2G^i(x, \dot{\gamma}) = 0.$$

Finsler manifolds are defined via a norm $F(x, y)$, with metric

$$g_{ij}(x, y) = \tfrac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

and its associated geodesic spray $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ where

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y) \Big(2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \Big) y^j y^k. \tag{6.1}$$

Key curvature quantities, such as the flag curvature

$$K(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y) g_y(u, u) - g_y(y, u)^2},$$

are also introduced, together with important classes like Berwald, Landsberg, and Randers manifolds.

3. Finsler manifold with maximal holonomy

This chapter of the dissertation introduces the parallel translation and the holonomy group. In this chapter we study the holonomy of n-dimensional Finsler manifolds via holonomy algebras, proves that the density of the holonomy algebra implies maximal holonomy, and applies this result to spherically symmetric projective Finsler metrics of nonzero constant flag curvature.

Let $(M, \mathcal{F})$ be a Finsler manifold. The notion of *parallel translation* along a curve $\gamma\colon [0,1] \to M$ is described by a smooth mapping

$$\mathcal{P}_\gamma\colon T_{\gamma(0)}M \to T_{\gamma(1)}M, \tag{6.2}$$

determined the usual way by parallel vector fields along γ. The *holonomy group* $\mathcal{H}ol_x(\mathcal{F})$ at a point $x \in M$ is the subgroup of transformations generated by parallel translation along all piece-wise smooth closed loops based at $x \in M$. Due to the norm-preserving property of parallel translation, each holonomy transformation can be regarded as a diffeomorphism of the indicatrix

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}iff(\mathcal{I}_x). \tag{6.3}$$

The *holonomy algebra* $\mathfrak{hol}_x(\mathcal{F})$, introduced in [31], is the tangent space of the holonomy group at its unit element, and is a Lie subalgebra (3.14) of the Lie algebra of vector fields on the indicatrix $\mathcal{I}_x$.

A key class of vector fields in $\mathfrak{hol}_x(\mathcal{F})$ arises from the curvature tensor (2.15) along with their iterated covariant derivatives, generate the *infinitesimal holonomy algebra* (3.17) at a point $x \in M$.

3.1 Maximal holonomy

The main theorem of this section, which provides a sufficient condition for the holonomy group to be maximal in terms of the holonomy algebra in the n-dimensional case.

3.1 Theorem. Let $(M, \mathcal{F})$ be an n-dimensional simply connected Finsler manifold and let $x \in M$. If the holonomy algebra $\mathfrak{hol}_x(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_x)$ of smooth vector fields on the indicatrix $\mathcal{I}_x$, then the holonomy group at x is maximal. More precisely,

$$\overline{\mathcal{H}ol_x(\mathcal{F})} \cong \mathcal{D}iff_o(\mathbb{S}^{n-1}), \tag{6.4}$$

where $\mathcal{D}iff_o(\mathbb{S}^{n-1})$ denotes the connected component of the identity in the diffeomorphism group of the $(n-1)$-dimensional sphere.

3.2 Spherically symmetric projective Finsler metrics

In this section, we apply the Theorem 3.1 to spherically symmetric projective Finsler metrics with constant flag curvature. Here we assume that there is a point $x_0 \in M$ where the Finsler function and the projective factor are both proportional to the

Euclidean norm, that is (3.39) is satisfied. We introduce a multi-index notation and for each integer $p \geq 0$, we define the real vector space

$$\mathcal{A}_p := \operatorname{span}_{\mathbb{R}} \left\{ \frac{\mathbf{y}^{\mathbf{m}}}{\|y\|^{\ell(\mathbf{m})}} \, \xi_{ij} \Big|_{\widehat{T}_{x_0} M} \;\Big|\; 1 \leq i < j \leq n, \; |\ell(\mathbf{m}) = p \right\}, \tag{6.5}$$

and we introduce the Lie algebra $\mathcal{A} := \oplus_{p=0}^{\infty} \mathcal{A}_p$. We have $\mathcal{A} \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$ and as a consequence, we get the following

3.2.1 Proposition. Let $(M, \mathcal{F})$ be a projectively flat, spherically symmetric Finsler manifold of constant flag curvature $\lambda \neq 0$, and let $x_o \in M$ be a point at which condition (3.40) is satisfied. Then the holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_{x_o})$ of smooth vector fields on the indicatrix $\mathcal{I}_{x_o}$.

We now pass from the holonomy algebra to the holonomy group.

3.2.2 Theorem. Let $(M, \mathcal{F})$ be a simply connected, projectively flat, spherically symmetric Finsler manifold of constant curvature $\lambda \neq 0$, and let $x_o \in M$ be a point at which condition (3.40) is satisfied. Then the holonomy group $\mathcal{H}ol_{x_o}(\mathcal{F})$ is maximal; , its closure is isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$.

3.2.3 Example. The holonomy groups of the Funk metric, and the Bryant–Shen metric are maximal, and isomorphic to $\mathcal{D}iff_o(\mathcal{I}_x)$.

4. Holonomy of Randers surfaces with constant flag curvature

This chapter gives a classification of the holonomy groups of Randers surfaces of constant curvature, which are among the most illustrative two-dimensional Finsler structures arising from navigation data. Both finite- and infinite-dimensional holonomy groups occur. The classification shows that the two negatively curved classes have essentially different holonomy behavior, and proves that, although generic Finsler holonomy groups are infinite-dimensional, there exist genuine Finsler metrics with finite-dimensional holonomy groups. To study these holonomy groups, we use the classification of Randers metrics of constant flag curvature obtained by Bao, Robles, and Shen [6].

4.1 Infinitesimal holonomy algebras

In this section, we study the infinitesimal holonomy algebra of simply connected Randers surfaces of constant flag curvature arising from navigation data.

4.1.1 Proposition. Let (M, F) be a 2-dimensional Randers manifold of constant curvature K. Then one has the following possibilities:

- $\dim(\mathfrak{hol}^*_x) = 0$ when $K = 0$,
- $\dim(\mathfrak{hol}^*_x) = 1$ when either $K > 0$, or $K < 0$ with $\mathcal{K}^-_1$ are satisfied,
- $\dim(\mathfrak{hol}^*_x) = \infty$ when $K < 0$ with $\mathcal{K}^-_2$ are satisfied.

4.2 Holonomy groups

In this section, we study the holonomy groups of simply connected Randers surfaces of constant curvature. We obtained the following theorem:

4.2.1 Theorem (Classification of the holonomy groups of Randers surfaces of constant curvature)**.** Let F be a Randers metric of constant flag curvature K on the simply connected 2-dimensional manifold M. Then the following possibilities can occur:

1. if $K = 0$, then the holonomy group is trivial, that is $\mathcal{H}ol = \{id\}$,
2. if $K > 0$, then the holonomy group is 1-dimensional and $\mathcal{H}ol \cong SO(2)$,
3. if $K < 0$ with $\mathcal{K}_1^-$, then the holonomy group is 1-dimensional and $\mathcal{H}ol \cong SO(2)$,
4. if $K < 0$ with $\mathcal{K}_2^-$, then the holonomy group is maximal, that is $\overline{\mathcal{H}ol} \cong \mathcal{D}iff_+(\mathbb{S}^1)$,

5. Natural parallelism associated with navigation data

Navigation data (h, W) define Randers metrics via Zermelo navigation, where geodesics describe time-minimizing paths under a wind field [6]. We introduce a natural parallel translation associated with (h, W), preserving the Randers norm and yielding finite-dimensional holonomy isomorphic to that of h, and study the induced geometric structures.

5.1 Parallel translation associated to navigation data

A navigation datum (h, W) consists of a Riemannian metric h and a vector field W with $\|W\|_h < 1$, which defines a Randers metric whose indicatrices are obtained by translating the h-unit spheres.

5.1.1 Definition. Let c be a smooth curve joining p to q. The *natural parallel translation* along c is defined by

$$\mathcal{P}(V_p^\circ) := \mathcal{P}_R(V_p^\circ - W_p) + W_q, \tag{6.6}$$

where $\mathcal{P}_R$ denotes the Riemannian parallel translation along c with respect to h. For an arbitrary nonzero vector we extend the definition by homogeneity.

5.1.2 Proposition. The natural parallel translation $\mathcal{P}$ is homogeneous, preserves the Randers norm function $\mathcal{F}$, but in general it is not additive.

5.1.3 Theorem. Let (h, W) be navigation data on the manifold M. The holonomy group $\mathcal{H}ol(\mathcal{P})$ associated to the natural parallel translation is isomorphic to the Riemannian holonomy group $\mathcal{H}ol(\mathcal{P}_R)$. In particular, the holonomy group of $\mathcal{H}ol(\mathcal{P})$ is finite dimensional.

5.2 Connection associated to the natural parallelism

In this section, we study the connection associated with the natural parallel translation, focusing on its autoparallel curves. For navigation data (h, W), these curves are characterized by the condition $\nabla_{\dot c}\dot c = 0$, where ∇ is the covariant derivative associated to the parallel translation $\mathcal{P}$. Using the Levi–Civita connection ∇^R of the h, we obtain

$$\nabla^R_{\dot c}\dot c - \mathcal{F}(\dot c)\,\nabla^R_{\dot c}W = 0. \tag{6.7}$$

Thus the acceleration with respect to the Riemannian connection is modified by a term depending on the wind field. In local coordinates this leads to the second-order system

$$\ddot c^k + \dot c^i\left(\mathcal{A}^k_{ij}\dot c^j - \mathcal{F}(\dot c)\,\mathcal{A}^k_{ij}W^j - \mathcal{F}(\dot c)\,\frac{\partial W^k}{\partial x^i}\right) = 0, \tag{6.8}$$

where $\mathcal{A}^k_{ij}$ are the Christoffel symbols of ∇^R. The spray coefficients corresponding to the system (6.8) are

$$G^k(x,y) = \tfrac{1}{2}\Big(\mathcal{A}^k_{ij}y^iy^j - \mathcal{F}(x,y)\,y^i\mathcal{A}^k_{ij}W^j - \mathcal{F}(x,y)\,y^i\frac{\partial W^k}{\partial x^i}\Big). \tag{6.9}$$

5.2.1 Definition. The spray S with coefficients (6.9) corresponding to the natural parallelism will be called the *natural spray*. The connection

$$\overline{\Gamma} := [J, S] \tag{6.10}$$

generated by this spray will be called the *natural symmetric connection*.

5.2.2 Remark. The natural connection (5.17) and the natural symmetric connection (6.10) are different in general. Indeed, the torsion of the natural symmetric connection is identically zero, while the torsion of the natural connection is typically nonzero.

5.2.3 Proposition. Let (h, W) be navigation data on a manifold M. The natural spray S is projectively related to the quadratic spray S^R of the Levi–Civita connection ∇^R if and only if the vector field W is concircular with respect to the Riemannian metric h.

7 Összefoglaló

Az összefoglalóban követjük a disszertáció fejezeteit, valamint – az olvasó tájékozódásának megkönnyítése érdekében – az összefoglalóra vonatkozó egyszerűsített számozást használunk.

1. Bevezetés

Jelen értekezés a Finsler-sokaságok holonómiaszerkezetének vizsgálatával foglalkozik. A Finsler-geometria a Riemann-geometria természetes általánosítása, amelyben a vektorok hosszának fogalma nemcsak a helytől, hanem az iránytól is függhet. Míg a holonómiaelmélet a Riemann-esetben jól ismert és részletesen kidolgozott, addig Finsler-geometriai megfelelője alapvetően eltérő, és gyakran lényegesen bonyolultabb viselkedést mutat, amely sok esetben végtelen dimenziós struktúrák megjelenéséhez vezet. A dolgozat célja ezen jelenségek mélyebb megértése: egyrészt a maximális holonómia feltételeinek vizsgálatával, másrészt speciális esetekben a holonómiacsoportok osztályozásával, valamint új, navigációs problémákhoz kapcsolódó geometriai mennyiségek bevezetésével.

2. Előzmények

A disszertáció második fejezetében áttekintjük azokat a Finsler-geometriai alapfogalmakat, amelyek a dolgozatban fontos szerepet játszanak. Tárgyaljuk a konnexiókat és azok által származtatott horizontális, illetve vertikális felbontást, a kovariáns deriválást és a görbület fogalmát, továbbá a spray-ket és azok geodetikusait, amelyek általánosan az alábbi differenciálegyenlettel adhatók meg:

$$\ddot{\gamma}^i + 2G^i(x, \dot{\gamma}) = 0.$$

A Finsler-sokaságokat egy $F(x, y)$ norma segítségével definiáljuk, a hozzájuk tartozó metrika

$$g_{ij}(x, y) = \tfrac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

valamint az ehhez tartozó geodetikus spray $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$, ahol

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\Big) y^j y^k. \tag{7.1}$$

Bevezetjük továbbá a legfontosabb görbületi mennyiségeket, például a zászlógörbületet:

$$K(P,y) = \frac{g_y(R_y(u),u)}{g_y(y,y)g_y(u,u) - g_y(y,u)^2},$$

és ismertetjük a Finsler-sokaságok néhány jelentős osztályát, mint például a Berwald-, Landsberg- és Randers-típusú sokaságokat.

3. Maximális holonómiájú Finsler-sokaságok

A disszertáció ezen fejezetében bevezetjük a párhuzamos eltolást, valamint a holonómia csoport fogalmát. Vizsgáljuk az n-dimenziós Finsler-sokaságok holonómiáját a holonómiaalgebrák segítségével. Megmutatjuk, hogy amennyiben a holonómiaalgebra sűrű az indikatrix sima vektormezőinek Lie-algebrájában, akkor a holonómia csoport maximális. Ezt az eredményt alkalmazzuk nemzérus állandó zászlógörbületű, gömbszimmetrikus projektív Finsler-metrikákra.

Legyen $(M,\mathcal{F})$ egy Finsler-sokaság. A *párhuzamos eltolás* fogalmát egy $\gamma\colon [0,1] \to M$ görbe mentén egy

$$\mathcal{P}_\gamma\colon T_{\gamma(0)}M \to T_{\gamma(1)}M \tag{7.2}$$

sima leképezés írja le, amelyet a γ mentén vett párhuzamos vektormezők segítségével a Riemann-féle párhuzamos eltoláshoz hasonlóan vezethetünk be. Az $x \in M$ pontban értelmezett *holonómiacsoport* $\mathcal{H}ol_x(\mathcal{F})$ azon transzformációk csoportja, amelyeket az x-ből induló és oda visszatérő, szakaszonként sima zárt hurkok mentén vett párhuzamos eltolás generál. A párhuzamos eltolás normamegőrző tulajdonsága miatt minden ilyen transzformáció az indikátrix diffeomorfizmusaként értelmezhető:

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}i\!f\!f(\mathcal{I}_x). \tag{7.3}$$

A $\mathfrak{hol}_x(\mathcal{F})$ holonómiaalgebra a holonómiacsoport egységelemében vett érintőtér, ami az indikátrixon értelmezett vektormezők Lie-algebrájának egy részalgebrája [31]. A $\mathfrak{hol}_x(\mathcal{F})$ egy fontos részalgebrája a görbületi vektormezők (2.15) és valamint ezek iterált kovariáns deriváltjai által származtatott $x \in M$ pont beli infinitézimális holonómiaalgebra (3.17).

3.1 Maximális holonómia

Ennek a fejezetnek a fő eredménye a holonómiaalgebra segítségével ad elegendő feltételt a holonómiacsoport maximalitására.

3.1 Tétel. Legyen $(M,\mathcal{F})$ egy n-dimenziós, egyszeresen összefüggő Finsler-sokaság, és legyen $x \in M$. Ha a holonómiaalgebra $\mathfrak{hol}_x(\mathcal{F})$ sűrű az indikátrixon értelmezett sima vektormezők $\mathfrak{X}(\mathcal{I}_x)$ Lie-algebrájában, akkor az x-beli holonómiacsoport maximális. Pontosabban:

$$\overline{\mathcal{H}ol_x(\mathcal{F})} \cong \mathcal{D}i\!f\!f_o(\mathbb{S}^{n-1}), \tag{7.4}$$

ahol $\mathcal{D}i\!f\!f_o(\mathbb{S}^{n-1})$ az $(n-1)$-dimenziós gömb diffeomorfizmuscsoportjának egységkomponensét jelöli.

3.2 Gömbszimmetrikus projektív Finsler-metrikák

Ebben a részben a 3.1 tételt alkalmazzuk állandó zászlógörbületű gömbszimmetrikus projektív Finsler-metrikák esetére. Feltesszük, hogy létezik egy $x_0 \in M$ pont, ahol a Finsler-függvény és a projektív faktor egyaránt arányos az euklideszi normával, azaz teljesül az (3.39) feltétel. Bevezetünk egy multiindex jelölést, és minden $p \geq 0$ egész számra definiáljuk az

$$\mathcal{A}_p := \operatorname{span}_{\mathbb{R}} \left\{ \frac{\mathbf{y}^{\mathbf{m}}}{\|y\|^{\ell(\mathbf{m})}} \, \xi_{ij}\Big|_{\widehat{T}_{x_0}M} \;\Big|\; 1 \leq i < j \leq n, \; |\ell(\mathbf{m}) = p \right\}, \tag{7.5}$$

valós vektorteret, valamint az $\mathcal{A} := \oplus_{p=0}^{\infty}\mathcal{A}_p$ Lie-algebrát. Ekkor $\mathcal{A} \subset \mathfrak{hol}^*_{x_0}(\mathcal{F})$, amiből következik az alábbi állítás:

3.2.1 Állítás. Legyen $(M, \mathcal{F})$ projektíven, gömbszimmetrikus Finsler-sokaság állandó $\lambda \neq 0$ zászlógörbülettel, és legyen $x_o \in M$ olyan pont, ahol a (3.40) feltétel teljesül. Ekkor a $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ holonómiaalgebra sűrű az $\mathfrak{X}(\mathcal{I}_{x_o})$ Lie-algebrában.

Most áttérünk a holonómiaalgebráról a holonómiacsoportra.

3.2.2 Tétel. Legyen $(M, \mathcal{F})$ egyszeresen összefüggő, projektív, gömbszimmetrikus Finsler-sokaság $\lambda \neq 0$ állandó görbülettel, és legyen $x_o \in M$ olyan pont, ahol a (3.40) feltétel teljesül. Ekkor a $\mathcal{H}ol_{x_o}(\mathcal{F})$ holonómiacsoport maximális, és lezártja izomorf a $\mathcal{D}iff_o(\mathbb{S}^{n-1})$ csoporttal.

3.2.3 Példa. A Funk-metrika és a Bryant–Shen-metrika holonómiacsoportja maximális.

4. Állandó zászlógörbületű Randers-felületek holonómiája

Ebben a fejezetben megadjuk az állandó görbületű Randers-felületek holonómiacsoportjainak osztályozását. Ezen geometriai terek a navigációs problémából származó kétdimenziós Finsler-struktúrák legszemléletesebb példái. Mind véges, mind végtelen dimenziós holonómiacsoportok előfordulnak. Az osztályozás megmutatja, hogy a két negatív görbületű eset lényegesen különböző holonómia tulajdonságot mutat. Érdekesség, hogy bár a generikus Finsler-holonómiacsoportok végtelen dimenziósak, léteznek olyan valódi Finsler-metrikák, amelyek holonómiacsoportja véges dimenziós. E holonómiacsoportok vizsgálatához felhasználjuk az állandó zászlógörbületű Randers-metrikáknak a Bao, Robles és Shen által adott osztályozását [6].

4.1 Infinitézimális holonómiaalgebrák

Ebben a részben az egyszeresen összefüggő, állandó zászlógörbületű, navigációs adatokból származó Randers-felületek infinitézimális holonómiaalgebráját vizsgáljuk.

4.1.1 Állítás. Legyen (M, F) egy kétdimenziós, állandó K görbületű Randers-sokaság. Ekkor az alábbi esetek lehetségesek:

- $\dim(\mathfrak{hol}^*_x) = 0$, ha $K = 0$,
- $\dim(\mathfrak{hol}^*_x) = 1$, ha $K > 0$, vagy ha $K < 0$ és teljesül a $\mathcal{K}_1^-$ feltétel,
- $\dim(\mathfrak{hol}^*_x) = \infty$, ha $K < 0$ és teljesül a $\mathcal{K}_2^-$ feltétel.

4.2 Holonómiacsoportok

Ebben a részben az egyszeresen összefüggő, állandó görbületű Randers-felületek holonómiacsoportjait vizsgáljuk.

4.2.1 Tétel. Legyen F egy állandó K zászlógörbületű Randers-metrika az egyszeresen összefüggő kétdimenziós M sokaságon. Ekkor az alábbi esetek lehetségesek:

1. ha $K = 0$, akkor a holonómiacsoport triviális, azaz $\mathcal{H}ol = \{id\}$,
2. ha $K > 0$, akkor a holonómiacsoport egydimenziós és $\mathcal{H}ol \cong SO(2)$,
3. ha $K < 0$ és $\mathcal{K}_1^-$, akkor a holonómiacsoport egydimenziós és $\mathcal{H}ol \cong SO(2)$,
4. ha $K < 0$ és $\mathcal{K}_2^-$, akkor a holonómiacsoport maximális, azaz $\overline{\mathcal{H}ol} \cong \mathcal{D}iff_+(\mathbb{S}^1)$,

5. Navigációs adatokhoz kapcsolódó természetes párhuzamosság

A (h, W) navigációs "adatok" a Zermelo-féle navigációs problémán keresztül Randers-metrikákat határoznak meg [6]. A disszertációban bevezetünk egy, a (h, W)-hez kapcsolódó természetes párhuzamos eltolást, amely megőrzi a Randers-normát, és olyan véges dimenziós holonómiát eredményez, amely izomorf a h Riemann-metrika holonómiájával.

5.1 Navigációs adatokhoz tartozó párhuzamos eltolás

Egy (h, W) navigációs adat egy h Riemann-metrikából és egy W vektormezőből áll, amelyre $\|W\|_h < 1$. Ez meghatároz egy olyan Randers-metrikát, amelynek indikátrixai a h-egységgömbjeinek eltolásával adódnak.

5.1.1 Definíció. Legyen c egy sima görbe, amely p-t q-val köti össze. Egy V_p° egységvektor *természetes párhuzamos eltoltja* a c görbe mentén a következőképpen definiált:

$$\mathcal{P}(V_p^\circ) := \mathcal{P}_R(V_p^\circ - W_p) + W_q, \tag{7.6}$$

ahol $\mathcal{P}_R$ a c menti h metrikához tartozó Riemann-féle párhuzamos eltolás. Tetszőleges nemzérus vektorra a definíciót homogenitási feltétel segítségével terjesztjük ki.

5.1.2 Állítás. A $\mathcal{P}$ természetes párhuzamos eltolás homogén, megőrzi a Randers-normát, azonban általában nem additív.

5.1.3 Tétel. Legyen (h, W) egy navigációs adat az M sokaságon. A természetes párhuzamos eltoláshoz tartozó $\mathcal{H}ol(\mathcal{P})$ holonómiacsoport izomorf a $\mathcal{H}ol(\mathcal{P}_R)$ Riemann-féle holonómiacsoporttal. Következésképpen a $\mathcal{H}ol(\mathcal{P})$ véges dimenziós.

5.2 A természetes párhuzamossághoz tartozó konnexió

Ebben a részben a természetes párhuzamos eltoláshoz tartozó konnexiót vizsgáljuk, különös tekintettel annak autoparallel görbéire. A (h, W) navigációs adatok esetén ezek a görbék a $\nabla_{\dot{c}}\dot{c} = 0$ feltétellel jellemezhetők, ahol ∇ a $\mathcal{P}$ párhuzamos eltoláshoz tartozó kovariáns derivált. Az h metrika Levi–Civita-féle konnexióját ∇^R-rel jelölve kapjuk:

$$\nabla^R_{\dot c}\dot c - \mathcal{F}(\dot c)\,\nabla^R_{\dot c}W = 0, \tag{7.7}$$

így a gyorsulás a Riemann-féle konnexióhoz képest egy, a szélmezőtől függő taggal módosul. Lokális koordinátarendszerben ez a következő másodrendű differenciálegyenlet-rendszerhez vezet:

$$\ddot c^k + \dot c^i\left(\mathcal{A}^k_{ij}\dot c^j - \mathcal{F}(\dot c)\,\mathcal{A}^k_{ij}W^j - \mathcal{F}(\dot c)\,\frac{\partial W^k}{\partial x^i}\right) = 0, \tag{7.8}$$

ahol $\mathcal{A}^k_{ij}$ a ∇^R konnexió Christoffel-féle szimbólumokat jelöli. Az (7.8) rendszerhez tartozó spray-együtthatók:

$$G^k(x,y) = \tfrac{1}{2}\Big(\mathcal{A}^k_{ij}y^iy^j - \mathcal{F}(x,y)\,y^i\mathcal{A}^k_{ij}W^j - \mathcal{F}(x,y)\,y^i\frac{\partial W^k}{\partial x^i}\Big). \tag{7.9}$$

5.2.1 Definíció. Az (7.9) együtthatókkal rendelkező, a természetes párhuzamossághoz tartozó S spray-t *természetes spray*-nek nevezzük. Az általa generált

$$\overline{\Gamma} := [J, S] \tag{7.10}$$

konnexiót *természetes szimmetrikus konnexiónak* nevezzük.

5.2.2 Megjegyzés. A természetes konnexió (5.17) és a természetes szimmetrikus konnexió (7.10) általában különböznek: a természetes szimmetrikus konnexió torziója azonosan zérus, míg a természetes konnexió torziója általában nem zérus.

5.2.3 Állítás. Legyen (h, W) navigációs adat az M sokaságon. A természetes spray S akkor és csak akkor projektíven ekvivalens a Levi–Civita-féle ∇^R konnexió kvadratikus S^R spray-jével, ha a W vektormező koncirkuláris a h Riemann-metrikára nézve.

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