Aequat. Math.

DOI 10.1007/s00010-014-0258-6

Aequationes Mathematicae

An alternative equation for polynomial functions

Zoltán Boros and Włodzimierz Fechner

Dedicated to Prof. János Aczél on the occassion of his 90^{th} birthday

Abstract. In this paper we prove that if a generalized polynomial function f satisfies the condition $f(x)f(y) = 0$ for all solutions of the equation $x^2 + y^2 = 1$, then f is identically equal to 0.

Mathematics Subject Classification (2000). 39B52.

Keywords. Polynomial functions, Alternative equation.

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the set of all real numbers, rationals, and positive integers, respectively. We call a function $f : \mathbb{R} \to \mathbb{R}$ *additive* if $f(x + y) = f(x) + f(y)$ holds for all $x, y \in \mathbb{R}$. The function f is called $\mathbb{Q}$-*homogeneous* if the equation $f(qx) = qf(x)$ is fulfilled by every $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. As it is also well-known [10, Theorem 5.2.1], if $f : \mathbb{R} \to \mathbb{R}$ is additive, then f is $\mathbb{Q}$-homogeneous as well.

Among several problems in the theory of functional equations, J. Aczél listed the following problem of Halperin [1]: is every additive mapping $f : \mathbb{R} \to \mathbb{R}$, which satisfies

$$f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x)$$

for all $x \neq 0$, of the form $f(x) = f(1)x$ for all $x \in \mathbb{R}$ (i.e., linear)? Two independent affirmative answers to Halperin's question are due to Kurepa [11]

Research of Z. Boros has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK 81402 (contributing to the technical and logistic background) and by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4.A/2-11/1-2012-0001 'National Excellence Program' (providing personal grant). Research of W. Fechner was supported by the Polish Ministry of Science and Higher Education in the years 2013–2014, under project No IP2012 011072.

Published online: 23 March 2014

Birkhäuser

and Jurkat [6]. These results were extended in various directions by several authors ([5,7,8], [10, Theorem 14.3.3], [12,13]).

The following problem was formulated by Benz in 1989 [3]. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive function satisfying $yf(x) = xf(y)$ for every $x, y \in \mathbb{R}$ such that $x^2 + y^2 = 1$. Does it imply that f is linear? This question, together with a similar one for derivations, was answered in the affirmative by Boros and Erdei [4].

Motivated by the question of Benz, Szabó [14] posed the following problem: suppose that $f : \mathbb{R} \to \mathbb{R}$ is additive and $f(x)f(y) = 0$ for all solutions of the equation $x^2 + y^2 = 1$. Does it imply that f is identically equal to zero? The solution was published in a joint paper by Kominek et al. [9], where they proved that the implication is true.

The purpose of the present paper is to extend the last result by providing an analogue of this statement for polynomial mappings. In order to formulate such a generalization, we have to introduce the related concepts.

Let $n \in \mathbb{N}$. A function $F : \mathbb{R}^n \to \mathbb{R}$ is called *n-additive* if, for every $i \in \{1, 2, \dots, n\}$ and for every $x_1, \dots, x_n, y_i \in \mathbb{R}$,

$$
\begin{aligned}
&F(x_1, \dots, x_{i-1}, x_i + y_i, x_{i+1}, \dots, x_n) \\
&\quad = F(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) + F(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n),
\end{aligned}
$$

i.e., F is additive in each of its variables $x_i \in \mathbb{R}$, $i = 1, \dots, n$. Clearly, an n–additive function is also $\mathbb{Q}$-homogeneous in each variable.

Given a function $F : \mathbb{R}^n \to \mathbb{R}$, by the *diagonalization (or trace)* of F we understand the function $f : \mathbb{R} \to \mathbb{R}$ arising from F by putting all the variables (from $\mathbb{R}$) equal:

$$
f(x) = F(x, \dots, x) \qquad (x \in \mathbb{R}).
$$

If, in particular, f is a diagonalization of an n–additive function $F : \mathbb{R}^n \to \mathbb{R}$, we say that f is a *generalized monomial of degree* n. It is convenient to assume that generalized monomials of degree zero are precisely constant mappings.

If f is a sum of generalized monomials of degrees $n_1, n_2, \dots n_k$, respectively, and $n = \max\{n_1, n_2, \dots, n_k\}$, then f is called a *generalized polynomial of degree* n. We note that the degree of a generalized polynomial is not uniquely determined in our context as we do not exclude identically zero terms. This approach is convenient when we formulate our statements and arguments.

For more information concerning these notions the reader is referred to the monograph by Kuczma [10, Chapter 15.9].

Now we can establish our main theorem.

Theorem 1. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is a generalized polynomial of degree* $n \in \mathbb{N}$ *and* $f(x)f(y) = 0$ *for all solutions of the equation* $x^2 + y^2 = 1$. *Then* f *is identically equal to zero.*

Proof. Given a generalized polynomial f of degree n, we can associate $k-$additive and symmetric functionals $A_k\colon \mathbb{R}^k \to \mathbb{R}$ for $k = 0, 1, \dots n$ with f in such a way that

$$f(x) = \sum_{k=0}^{n} A_k(x, \dots, x) \tag{1}$$

for all $x \in \mathbb{R}$.

Now, let $x, y \in \mathbb{R}$ be arbitrary solutions of $x^2 + y^2 = 1$. If α, β are such that $\alpha^2 + \beta^2 = 1$, then it is straightforward to check that the following identity holds true:

$$(\alpha x - \beta y)^2 + (\beta x + \alpha y)^2 = 1.$$

Next, assume, in addition, that α and β are rationals, u is a real number such that $|u| < 1$, and $x = -u$.

Take $y = \sqrt{1 - x^2}$; we have

$$f(\alpha x - \beta y) f(\beta x + \alpha y) = 0\,.$$

Denote

$$a_{k,l} = A_k(\underbrace{x, \dots, x}_{l}, \underbrace{y, \dots, y}_{k-l})$$

for $k = 0, 1, \dots n$ and $l = 0, 1, \dots k$.

With this notation we can calculate that

$$f(\alpha x - \beta y) = \sum_{k=0}^{n} A_k(\alpha x - \beta y, \dots, \alpha x - \beta y) = \sum_{k=0}^{n}\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \alpha^l \beta^{k-l} a_{k,l}$$

and

$$f(\beta x + \alpha y) = \sum_{k=0}^{n} A_k(\beta x + \alpha y, \dots, \beta x + \alpha y) = \sum_{k=0}^{n}\sum_{l=0}^{k} \binom{k}{l} \alpha^{k-l} \beta^l a_{k,l}\,.$$

Clearly, for every pair (α, β) chosen as above at least one of the foregoing expressions is equal to zero.

What is more, we can find infinitely many distinct pairs (α_i, β_i) such that $\alpha_i^2 + \beta_i^2 = 1$ and both α_i and β_i are rationals, so let us take

$$\alpha_i = \frac{i^2 - 1}{i^2 + 1} \quad \text{and} \quad \beta_i = \frac{2i}{i^2 + 1} \tag{2}$$

for $i = 1, 2, \dots$.

Thus, for every $i \in \mathbb{N}$, we have either

$$0 = \sum_{k=0}^{n}\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(\frac{i^2 - 1}{i^2 + 1}\right)^{l} \left(\frac{2i}{i^2 + 1}\right)^{k-l} a_{k,l}$$

or

$$0=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\left(\frac{i^2-1}{i^2+1}\right)^{k-l}\left(\frac{2i}{i^2+1}\right)^{l}a_{k,l}\,.$$

Multiplying both equations by $(i^2+1)^n$ and introducing the functions

$$P_n(i)=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(i^2-1)^{l}(2i)^{k-l}(i^2+1)^{n-k}a_{k,l}\,,$$
$$\tilde{P}_n(i)=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}(i^2-1)^{k-l}(2i)^{l}(i^2+1)^{n-k}a_{k,l}\,,$$

we have $P_n(i)=0$ or $\tilde{P}_n(i)=0$ for each positive integer i. Hence either P_n or $\tilde{P}_n$ has infinitely many zeros. On the other hand, both P_n and $\tilde{P}_n$ are polynomials of degree not greater than $2n$. Therefore, one of them has to be identically equal to 0. So either

$$0=P_n(0)=\sum_{k=0}^{n}(-1)^k a_{k,k}=\sum_{k=0}^{n}(-1)^k A_k(x\,,\,x\,,\,\dots\,,\,x)$$
$$=\sum_{k=0}^{n}A_k(-x\,,\,-x\,,\,\dots\,,\,-x)=f(-x)=f(u)$$

or

$$0=\tilde{P}_n(-1)=\sum_{k=0}^{n}(-2)^k 2^{n-k}a_{k,k}=2^n\sum_{k=0}^{n}(-1)^k a_{k,k}$$
$$=2^n f(-x)=2^n f(u)\,,$$

i.e., $f(u)=0$.

We have thus proved that f vanishes on the open interval $(-1,1)$.

Now let us consider an arbitrary non-zero real number x and any rational number r fulfilling $|r|<1/|x|$. Then, according to the representation (1), we have

$$0=f(rx)=\sum_{k=0}^{n}A_k(rx,\dots,rx)=\sum_{k=0}^{n}r^k A_k(x,\dots,x)\,.$$

The last expression in this equation is an "ordinary" polynomial with respect to r. Since it has infinitely many zeros (as the inequality condition for r is satisfied by infinitely many rational numbers), it must be identically zero, hence it vanishes at $r=1$ as well, thus $f(x)=0$.

We note that our last argument can be replaced by a reference to Székelyhidi's regularity theorem [15]. Namely, since f vanishes on the open interval $(-1,1)$, it is, in particular, locally bounded around zero. However, every generalized polynomial which satisfies this condition is continuous. Therefore, f is an "ordinary" polynomial, and thus $f = 0$ on $\mathbb{R}$. □

Remark 1. Generalized polynomials are obtained from polynomials of the form $f(x) = \sum_{k=0}^{n} c_k x^k$ by replacing the monomial terms $c_k x^k$ with generalized monomials of degree k. Another way to generalize such a representation of f is obtained by replacing the finite sum with an infinite one. This idea leads to the well-known concept of analytic functions. We can, actually, establish an analogy of Theorem 1 for analytic functions: If $f\colon \mathbb{R} \to \mathbb{R}$ is analytic and $f(x)f(y) = 0$ for all solutions of the equation $x^2 + y^2 = 1$, then f is identically equal to zero. In fact, according to our assumption, f has infinitely many zeros in the interval $[-1, 1]$, hence it is identically equal to zero.

On the other hand, the regularity assumption that f is analytic cannot be replaced with a weaker one. Clearly, every mapping $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies $f(t) = 0$ for every $t \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ fulfils the condition $f(x)f(y) = 0$ for all solutions of the equation $x^2 + y^2 = 1$. Therefore, there exist infinitely many times differentiable functions which satisfy this condition and are not identically equal to zero.

References

[1] Aczél, J.: Some unsolved problems in the theory of functional equations. Arch. Math. **15**, 435–444 (1964)
[2] Aczél, J., Dhombres, J.: Functional equations in several variables. Encyclopaedia of mathematics and its applications, vol. 31. Cambridge Univ. Press, Cambridge (1989)
[3] Benz, W.: 5. Problem in report of meeting: the twenty-seventh international symposium on functional equations. Aequationes Math. **39**, 302 (1990)
[4] Boros, Z., Erdei, P.: A conditional equation for additive functions. Aequationes Math. **70**, 309–313 (2005)
[5] Grząślewicz, A.: Some remarks to additive functions. Math. Japon. **23**, 573–578 (1978/79)
[6] Jurkat, W.B.: On Cauchy's functional equation. Proc. Am. Math. Soc. **16**, 683–686 (1965)
[7] Kannappan, Pl., Kurepa, S.: Some relations between additive functions I. Aequationes Math. **4**, 163–175 (1970)
[8] Kannappan, Pl., Kurepa, S.: Some relations between additive functions II. Aequationes Math. **6**, 46–58 (1971)
[9] Kominek, Z., Reich, L., Schwaiger, J.: On additive functions fulfilling some additional condition. Sitzungsber. Abt. II **207**, 35–42 (1998)
[10] Kuczma, M.: An introduction to the theory of functional equations and inequalities (2nd edn). Birkhäuser, Basel (2009)
[11] Kurepa, S.: The Cauchy functional equation and scalar product in vector spaces. Glasnik Mat.-Fiz. Astronom. Ser. II Društvo Mat. Fiz. Hrvatske **19**, 23–36 (1964)

[12] Kurepa, S.: Remarks on the Cauchy functional equation. Publ. Inst. Math. (Beograd) (N.S.) **5**(19), 85–88 (1965)
[13] Nishiyama, A., Horinouchi, S.: On a system of functional equations. Aequationes Math. **1**, 1–5 (1968)
[14] Szabó, Gy.: 20. Problem, report of meeting, the thirtieth international symposium on functional equations. Aequationes Math. **46**, 294 (1993)
[15] Székelyhidi, L.: Regularity properties of polynomials on groups. Acta Math. Hung. **45**(1–2), 15–19 (1985)

Zoltán Boros
Institute of Mathematics
University of Debrecen
P. O. Box: 12
4010 Debrecen Hungary
e-mail: zboros@science.unideb.hu

Włodzimierz Fechner
Institute of Mathematics
University of Silesia
Bankowa 14
40–007 Katowice Poland
e-mail: fechner@math.us.edu.pl;
wlodzimierz.fechner@us.edu.pl

Received: July 15, 2013
Revised: February 5, 2014