# Dilative stability

## Ph.D. Thesis

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# Contents

# Chapter 1

# Introduction

In this dissertation we define and study a scaling property of stochastic processes, which we call dilative stability. An infinitely divisible process $\{X(t),\, t \geq 0\}$ (satisfying certain natural conditions) is said to be $(\alpha, \delta)$-dilatively stable, if

$$\forall T > 0 : X(Tt) \overset{\text{fd}}{\sim} T^{\alpha - \frac{\delta}{2}} X^{\circledast T^{\delta}}(t),$$

where $\alpha > 0$ and $\delta \leq 2\alpha$ are constants (we say that such an $(\alpha, \delta)$ is admissible), $\overset{\text{fd}}{\sim}$ denotes that the finite-dimensional distributions are the same, and $X^{\circledast c}$ means the $0 < c$-th convolution power. This property is comparable with the well-known self-similarity: a stochastic process $\{X(t),\, t \geq 0\}$ is called $\alpha$-self-similar, if

$$\forall T > 0 : X(Tt) \overset{\text{fd}}{\sim} T^{\alpha} X(t),$$

where $\alpha > 0$. It can be seen that $(\alpha, 0)$-dilative stability is just $\alpha$-self-similarity, i.e., dilative stability is a generalization of self-similarity, at least when both are defined. The main difference between them is that, heuristically, the branching or confluent structure of the distribution of the processes is inessential in or even incongruos with self-similarity, but it plays a significant role in dilative stability. Accordingly, dilative stability is defined only for infinitely divisible processes, while self-similarity has nothing to do with infinite divisibility. Another important difference lies in the tools used: while the cumulants are unnecessary in the case of self-similarity, they are indispensable even for dilative stability to be uniquely defined, and they provide a key tool in the proofs.

The dissertation consists of five parts. After Chapter 1, which is the introduction, Chapter 2 compares self-similarity and dilative stabity. It was Lamperti's paper [22], which had attracted attention to the significance of the former property. We present the dilatively stable analogues of Lamperti's theorems, and show that a lot of results can be transferred from the self-similar to the dilatively stable case. Moreover, as we will see, dilative stability provides a new possibility for long memory modelling. Some examples suggest that dilatively stable processes are simply certain transforms, namely, the so-called power function-th convolution powers of some self-similar processes. It will turn out that this is not so in general, so dilative stability is not a redundant concept.

Then we study the Lamperti transform. It is a known fact that this transform gives the correspondence between self-similar and stationary processes. The analogous statement we prove is that the Lamperti transform connects dilatively stable processes and the so-called translatively stable processes. A particular attention will be paid to the correspondence between dilatively stable processes with independent increments and translatively stable wide sense Ornstein–Uhlenbeck (OU) type processes, because this is the analogue of the known correspondence between self-similar processes with independent increments and stationary OU type processes.

There is a section about self-similar and dilatively stable processes with stationary increments. The importance of that processes lies in the fact that the autocovariance function of a non-degenerate dilatively stable process with stationary increments is the same as that of the corresponding self-similar process with $L^2$-stationary increments, i.e. as the autocovariance function of a fractional Brownian motion. Having stated this we arrived to the long memory property.

Then we treat the dilatively stable renormalization operators. We show that the analogy with the self-similar case holds true in this regard as well. That is, the set of dilatively stable renormalization operators (for any fixed admissible parameter) constitute a semigroup, so the renormalization operators can be iterated. Furthermore, a process is dilatively stable if and only if it is a fixed point of the corresponding dilatively stable renormalization operator. This explains why exactly the renormalized processes happen to converge in dilatively stable (functional) limit theorems.

In Chapter 3 we deal with the superposition of stationary OU type processes, and we state a dilatively stable renormalization functional limit theorem for the integrated superposition processes. OU type processes are homogeneous Markov processes with a particular property called regular affinity, defined in Duffie et al. [9]. Regular affine processes are exactly those regular, homogeneous Markov processes, which are infinitely decomposable (roughly, the transition probability distribution is infinitely divisible). On the other hand, infinitely decomposable processes are just those processes which are suitable for being superposed. The two types of regular affine processes: OU type processes and continuous state branching processes with immigration (CBI processes). This is the reason why we happen to superpose OU type processes in Chapter 3, and CBI processes in Chapter 4. In the latter part the particular case when the CBI process is the so-called diffusion process with linear generator (DLG process), is treated separately. We state dilatively stable renormalization functional limit theorems for the integrated superposition processes. The development of Chapter 4 is parallel with that of Chapter 3, but it contains also some new results with respect to CBI processes.

The last part, Chapter 5 deals with renormalization functional limit theorems for Cox processes. According to Grandell [13, Thm. 4.2.2], if a self-similar renormalization functional limit theorem holds for the intensity process of a Cox process, then a similar self-similar renormalization functional limit theorem holds also for the Cox process itself. In Chapter 5 we present the analogous statement for dilatively stable renormalization functional limit theorems. As a corollary we obtain a dilatively stable renormalization functional limit theorem for the superposition of death counting (SDC) processes of stationary birth and death processes with immigration.

Existing results, as far as possible, are given with references to their origin. Hence every result presented without citation is the author's own result, or folklore.

Special notations not defined in the text:

| | |
|---|---|
| $\sim, \stackrel{\mathrm{1d}}{\sim}, \stackrel{\mathrm{fd}}{\sim}$ : | equivalence of the distributions, 1-dimensional distributions, finite-dimensional distributions |
| $\circledast, \,^{\circledast c}$ : | convolution, $c$-th convolution power |
| $\xrightarrow{\mathrm{w}}$ : | weak convergence of distributions (probability measures) |
| $\xrightarrow{\mathrm{fd}}$ : | weak convergence of the finite-dimensional distributions |
| $\mathsf{cum}, \mathsf{cum}_n$ : | (joint) cumulant, cumulant of order $n$ |
| $\lfloor\ \rfloor$ : | integer part |
| $(t_1^*, \ldots, t_m^*)$ : | the components of $(t_1, \ldots, t_m)$ listed in nondecreasing order |

| | |
|---|---|
| $\Gamma$ : | the Gamma function |
| $C[0,\infty)$ : | the space of continuous functions on $[0,\infty)$ with the local uniform topology and Borel (equiv. the cylinder) $\sigma$-algebra |
| $D[0,\infty)$ : | the set of càdlàg functions on $[0,\infty)$ with the extended Skorokhod topology and Borel (equiv. the cylinder) $\sigma$-algebra (see Lindvall [23] or Jacod–Shiryaev [18]) |

# Chapter 2

# Self-similarity and dilative stability

Self-similarity is a scaling property of stochastic processes. It was Lamperti's paper [22] which called attention to the significance of this property. In this chapter we compare self-similarity and a more general scaling property, which we call dilative stability. We present the dilative stable analogues of Lamperti's theorems, and show that a lot of results can be transferred from the self-similar to the dilatively stable case. Moreover, as we will see, dilative stability provides a new opportunity to model long memory processes.

## 2.1 Definitions and examples

### 2.1.1 Notation.[1]

$$\mathcal{S} \doteq \Big\{\text{stochastic process } \{X(t), t \geq 0\} : X(0) = 0, \\ \{X(t), t \geq 0\} \not\equiv 0 \text{ and right-continuous in distribution}\Big\}.$$

$$\mathcal{I} \doteq \Big\{\{X(t),\, t \geq 0\} \in \mathcal{S} : X(1) \text{ is non-Gaussian, the} \\ \text{finite-dimensional distributions are infinitely divisible, and functions} \\ c_n(t) \doteq \mathsf{cum}_n(X(t)),\ t \geq 0,\ n \geq 2, \text{ exist and are right-continuous}\Big\}.$$

[1] $\mathcal{S}$ is for **S**tochastic process, $\mathcal{I}$ is for **I**nfinitely divisible.

**2.1.2 Remark.** The cumulant of order $m \geq 3$ of an infinitely divisible distribution is the moment of order $m$ of the Lévy measure in the Lévy–Khintchine representation. Thus, the following implication holds:

$$\{X(t),\, t \geq 0\} \in \mathcal{I} \implies \forall\, m \in \mathbb{N} : c_{2m}(1) > 0. \tag{2.1.1}$$

**2.1.3 Definition. (SS)** (Lamperti [22]) Let $f : (0,\infty) \to (0,\infty)$. A process $\{X(t),\, t \geq 0\} \in \mathcal{S}$ is called *$f$-self-similar* (Lamperti [22] uses the term "$f$-semi-stable") if

$$\forall\, T > 0 : X(Tt) \overset{\text{fd}}{\sim} f(T)X(t). \tag{2.1.2}$$

**(DS)** Let $f, g : (0,\infty) \to (0,\infty)$. Process $\{X(t),\, t \geq 0\} \in \mathcal{I}$ is called *$(f,g)$-dilatively stable* if

$$\forall\, T > 0 : X(Tt) \overset{\text{fd}}{\sim} \frac{f(T)}{\sqrt{g(T)}}\, X^{\circledast g(T)}(t). \tag{2.1.3}$$

**2.1.4 Proposition.** ***(SS)*** *The function $f$ in Definition 2.1.3 (SS) is uniquely determined.*

***(DS)*** *The pair $(f,g)$ in Definition 2.1.3 (DS) is unique.*

**Proof.** The self-similar case is obvious. So, let $\{X(t),\, t \geq 0\}$ be a dilatively stable process. Then by (2.1.1) $c_2(1) \neq 0$ and $c_4(1) \neq 0$. By (2.1.3), for all $T > 0$ we have the system of equations

$$\begin{aligned} c_2(T) &= f^2(T)c_2(1) \\ c_4(T) &= \frac{f^4(T)}{g(T)}\, c_4(1), \end{aligned}$$

which can be solved uniquely with respect to $(f(T), g(T))$. □

In Definition 2.1.3 (DS) stable processes are excluded because of the moment condition and the non-Gaussianity condition (see Notation 2.1.1). This is reasonable because for that processes the convolution power and the multiplication by scalar can suitably replace each other, therefore the pair $(f,g)$ in Definition 2.1.3 (DS) would not be unique. In the strict 2-stable case, i.e. for a Gaussian process with zero mean, only $g$ would not be unique in Definition 2.1.3 (DS). Besides, in such a case the process would be $f$-self-similar. Eventually, instead of the assumption on the cumulants and non-Gaussianity (see Notation

2.1.1) a weaker condition would be enough in the definition of dilative stability. Namely, Proposition 2.1.4 (DS) and most of the further statements about dilative stability would hold true under the assumption that not all the finite-dimensional distributions of the process are strictly pseudo-stable (i.e. there exists an $n \in \mathbb{N}$ and a finite-dimensional distribution $(X(t_1), \ldots, X(t_n))$ such that for any $c, d > 0$ $c\,(X(t_1), \ldots, X(t_n)) \overset{\text{fd}}{\nsim} (X(t_1), \ldots, X(t_n))^{\circledast d}$). In the authors' opinion, the assumption in $\mathcal{I}$ is more comfortable than the one mentioned here.

Here are some examples for self-similar and for dilatively stable processes.

**2.1.5 Example. (SS)** *Product of a white noise and a power function*: Let $\alpha > 0$, $\{W(t),\, t \geq 0\} \not\equiv 0$ a white noise consisting of i.i.d. random variables and

$$X(t) \doteq t^\alpha W(t), \quad t \geq 0.$$

Then $\{X(t),\, t \geq 0\} \in \mathcal{S}$ and

$$\forall T > 0 : X(Tt) \overset{\text{fd}}{\sim} T^\alpha X(t),$$

i.e. the process $\{X(t),\, t \geq 0\}$ is $f(t) = t^\alpha$-self-similar.

**(DS)** Let $\alpha > 0$, $\delta \leq 2\alpha$, $\{W(t),\, t > 0\}$ a white noise consisting of i.i.d. random variables such that $W(1)$ has finite moments of all orders, it is infinitely divisible and non-Gaussian. Let $\{X(t),\, t \geq 0\}$ be a process which has the following finite-dimensional distributions:

$$(X(t_1), \ldots, X(t_n)) \overset{\text{fd}}{\sim} \left(t_1^{\alpha-\frac{\delta}{2}} W^{\circledast t_1^\delta}(t_1), \ldots, t_n^{\alpha-\frac{\delta}{2}} W^{\circledast t_n^\delta}(t_n)\right), \qquad 0 < t_1, \ldots, t_n,\ n = 1, 2, \ldots,$$

where the random variables $W^{\circledast t_1^\delta}(t_1)$, ..., $W^{\circledast t_n^\delta}(t_n)$ are independent, and let $X(0) \doteq 0$. Then $\{X(t),\, t \geq 0\} \in \mathcal{I}$ and

$$\begin{aligned}\forall T > 0 : \{X(Tt),\, t > 0\} &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} \left\{t^{\alpha-\frac{\delta}{2}} W^{\circledast T^\delta t^\delta}(Tt),\ t > 0\right\} \\ &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} \left\{\left(t^{\alpha-\frac{\delta}{2}} W^{\circledast t^\delta}\right)^{\circledast T^\delta}(t),\ t > 0\right\} \qquad (2.1.4) \\ &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} \left\{X^{\circledast T^\delta}(t),\ t > 0\right\}\end{aligned}$$

(this can be obtained using characteristic functions), and the identity corresponding to (2.1.4) also holds if the first component of the finite-dimensional distribution is taken at $t=0$. Therefore

$$\forall\, T>0 : X(Tt) \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} X^{\circledast T^\delta}(t),$$

i.e. $\{X(t),\, t\geq 0\}$ is $(f,\, g)$-dilatively stable, where $f(t)=t^\alpha$, $g(t)=t^\delta$.

**2.1.6 Example.** **(SS)** *Product of a power function and a random variable*: Let $\alpha>0$, $\xi\not\equiv 0$ a random variable and let

$$X(t) \doteq t^\alpha \xi, \quad t\geq 0.$$

Then $\{X(t),\, t\geq 0\}\in\mathcal{S}$ and

$$\forall\, T>0 : X(Tt) \overset{\text{fd}}{\sim} T^\alpha X(t),$$

i.e. process $\{X(t),\, t\geq 0\}$ is $f(t)=t^\alpha$-self-similar.

**(DS)** *Product of a power function and a Lévy process with a power function time-change*: Let $\alpha>0$, $\delta\leq 2\alpha$ and let $\{L(t),\, t\geq 0\}$ be a non-Gaussian Lévy process such that all the cumulants of $L(1)$ exist. Let

$$X(t) \doteq \begin{cases} t^{\alpha-\frac{\delta}{2}} L(t^\delta) & \text{if } t>0 \\ 0 & \text{if } t=0 \end{cases}.$$

Then $\{X(t),\, t\geq 0\}\in\mathcal{I}$ and

$$\begin{aligned} X(Tt) &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} t^{\alpha-\frac{\delta}{2}} L(T^\delta t^\delta) \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} t^{\alpha-\frac{\delta}{2}} L^{\circledast T^\delta}(t^\delta) \\ &\qquad \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} \left(t^{\alpha-\frac{\delta}{2}} L(t^\delta)\right)^{\circledast T^\delta} \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} X^{\circledast T^\delta}(t) \end{aligned}$$

for all $T, t>0$, and the finite-dimensional distributions are the same even if the first component of them is the value at $t=0$. Therefore

$$\forall\, T>0 : X(Tt) \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} X^{\circledast T^\delta}(t),$$

i.e. the process $\{X(t),\, t\geq 0\}$ is $(f,\, g)$-dilatively stable, where $f(t)=t^\alpha$, $g(t)=t^\delta$.

The next example shows non-trivial self-similar and dilatively stable processes having stationary increments.

**2.1.7 Example. (SS)** *Fractional Brownian motion (FBM, Mandelbrot–Van Ness [26]):*

$$B^{(H)}(t) \doteq \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \int\limits_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) B(ds), \quad t \geq 0, \tag{2.1.5}$$

where $\{B(t),\, t \in \mathbb{R}\}$ is the Brownian motion over $\mathbb{R}$ and $H \in (0,1)$ (the Hurst parameter or memory parameter). The integral in (2.1.5) is an $L^2$-integral. The FBM has stationary increments, and since $\{B^{(H)}(t),\, t \geq 0\} \in \mathcal{S}$ and

$$\forall T > 0 : B^{(H)}(Tt) \overset{\text{fd}}{\sim} T^H B^{(H)}(t), \tag{2.1.6}$$

process $\{B^{(H)}(t),\, t \geq 0\}$ is $f(t) = t^H$-self-similar.

**(DS)** *Non-Gaussian fractional Lévy process (FLP, Marquardt [27]):*

$$M^{(H)}(t) \doteq \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \int\limits_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) L(ds), \quad t \geq 0, \tag{2.1.7}$$

where $\{L(t),\, t \in \mathbb{R}\}$ is a non-Gaussian two-sided Lévy process over $\mathbb{R}$ with zero mean and finite moments of all orders and $H \in (1/2, 1)$ (the Hurst parameter or long memory parameter). (Marquardt [27] studies the case when the Lévy process has no Brownian component, however, we allow a Brownian component). The integral in (2.1.7) is simply an $L^2$-integral. The FLP has stationary increments. Its first and second order moment structures are the same as those of the FBM, particularly we have

$$\mathsf{E}\big(M^{(H)}(t)\big)^2 = \mathsf{E}\,(L(1))^2\, t^{2H}, \quad t \geq 0.$$

Thus Kolmogorov's condition for a.s. sample path continuity is fulfilled, implying also continuity in distribution. Therefore $\{M^{(H)}(t),\, t \geq 0\} \in \mathcal{S}$. In fact, $\{M^{(H)}(t),\, t \geq 0\} \in \mathcal{I}$, since functions

$$\begin{aligned} c_n(t) = \mathsf{cum}_n\left(M^{(H)}(t)\right) &= \int\limits_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right)^n ds\ \mathsf{cum}_n(L(1)) \\ &= t^{n\left(H-\frac{1}{2}\right)+1} \int\limits_{0}^{\infty} \left(s^{H-\frac{1}{2}} - (s-1)_+^{H-\frac{1}{2}}\right)^n ds\ \mathsf{cum}_n(L(1)), \quad t \geq 0, \end{aligned}$$

$n = 2, 3, \ldots,$ are right-continuous. Moreover, since for all $T > 0$ we have

$$\begin{aligned} M^{(H)}(Tt) &= \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \int_{-\infty}^{Tt} \left((Tt-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) L(ds) \\ &\overset{\text{fd}}{\sim} \frac{T^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} \int_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) L^{\circledast T}(ds) \\ &\overset{\text{fd}}{\sim} T^{H-\frac{1}{2}} \left(M^{(H)}\right)^{\circledast T}(t), \end{aligned}$$

$\{M^{(H)}(t),\, t \geq 0\}$ is $(f,g)$-dilatively stable, where $f(t) = t^H$, $g(t) = t$.

## 2.2 Theorems on self-similarity and dilative stability

**2.2.1 Theorem.** ***(SS)*** *(Lamperti [22]) If a process $\{X(t),\, t \geq 0\}$ is $f$-self-similar, then $f$ is necessarily a power function with a positive exponent, i.e. there exists an $\alpha > 0$ such that $f(t) = t^\alpha$. Therefore instead of the expression "$t^\alpha$-self-similar" we simply use "$\alpha$-self-similar".*

***(DS)*** *If a process $\{X(t),\, t \geq 0\}$ is $(f,g)$-dilatively stable, then $f$ and $g$ are necessarily power functions, i.e. there exist*

$$\alpha > 0, \quad \delta \leq 2\alpha, \tag{2.2.1}$$

*such that $f(t) = t^\alpha$ and $g(t) = t^\delta$. Therefore instead of the expression "$(t^\alpha, t^\delta)$-dilatively stable" we simply use "$(\alpha,\delta)$-dilatively stable".*

**Proof.** The proof of the self-similar case can be found in Lamperti [22], so, only the dilatively stable case will be treated here. The substitutions $T = s$, $t = t$, then $T = st$, $t = 1$ in (2.1.3) yield

$$X(st) \overset{\text{1d}}{\sim} \frac{f(s)}{\sqrt{g(s)}} X^{\circledast g(s)}(t), \tag{2.2.2}$$

$$X(st) \overset{\text{1d}}{\sim} \frac{f(st)}{\sqrt{g(st)}} X^{\circledast g(st)}(1). \tag{2.2.3}$$

By (2.1.1) we have

$$c_2(1) \neq 0 \text{ and } \exists n \geq 3 : c_n(1) \neq 0. \tag{2.2.4}$$

Taking the second order cumulants in (2.2.2) and in (2.2.3) we obtain

$$c_2(st) = f^2(s)c_2(t) = f^2(s)f^2(t)c_2(1),$$

$$c_2(st) = f^2(st)c_2(1).$$

Because of the equality of the right hand sides and $c_2(1) \neq 0$, for the function $A_2(t) \doteq f^2(t)$ we have the functional equation

$$A_2(s)A_2(t) = A_2(st). \tag{2.2.5}$$

Since the function $c_2$ is right-continuous, so is the function $A_2$ over the interval $(0, \infty)$, therefore the solution of (2.2.5) is

$$A_2(t) = t^{\alpha_2}, \tag{2.2.6}$$

with some $\alpha_2 \in \mathbb{R}$. Similarly, with an $n$ of (2.2.4) instead of 2, if $A_n(t) \doteq f^n(t)g^{1-\frac{n}{2}}(t)$, then $A_n(t) = t^{\alpha_n}$, with some $\alpha_n \in \mathbb{R}$. Hence $f(t) = t^{\alpha}$, $g(t) = t^{\delta}$ with some $\alpha, \delta \in \mathbb{R}$. On the other hand putting $s = 1$ in (2.2.3), and taking the cumulants, it implies

$$c_2(t) = f^2(t)c_2(1) = t^{\alpha}c_2(1), \tag{2.2.7}$$

$$c_n(t) = f^n(t)g^{1-\frac{n}{2}}(t)c_n(1) = t^{n\alpha+(1-\frac{n}{2})\delta}c_n(1) \tag{2.2.8}$$

for all $t > 0$. Because of the right-continuity of functions $c_2$ and $c_n$ at $t = 0$ and the equation $c_2(0) = c_n(0) = 0$ (since $X(0) = 0$), from (2.2.7) and (2.2.8) we obtain

$$\alpha > 0, \quad n\alpha + \left(1 - \frac{n}{2}\right)\delta > 0. \tag{2.2.9}$$

Since we have $c_n(1) \neq 0$ for infinitely many $n$s, the inequalities (2.2.9) are possible only if (2.2.1) holds. □

**2.2.2 Remark.** If (2.1.2) or (2.1.3) is assumed to be fulfilled only for the one-dimensional distributions instead of all the finite-dimensional ones, then the corresponding property can be called marginal self-similarity and marginal dilative stability, respectively. Proposition 2.1.4 holds true also when only marginal self-similarity or marginal dilative stability is assumed. The same is true for Theorem 2.2.1.

**2.2.3 Proposition.** ***(SS)*** *For all* $\alpha > 0$ *there exists an* $\alpha$*-self-similar process.*
***(DS)*** *For all* $(\alpha, \delta)$ *in the parameter domain (2.2.1) there exists an* $(\alpha, \delta)$*-dilatively stable process.*

**Proof. (SS)** Processes $\{X(t),\, t \geq 0\}$ in Example 2.1.5 (SS) are of this kind.
**(DS)** Processes $\{X(t),\, t \geq 0\}$ in Example 2.1.5 (DS) are of this kind. □

**2.2.4 Remark.** In the set $\mathcal{I}$ dilative stability is a generalization of self-similarity: if $\alpha > 0$, then a process in $\mathcal{I}$ is $(\alpha, 0)$-dilatively stable if and only if it is $\alpha$-self-similar.

**2.2.5 Notation. (SS)** Let $\alpha > 0$. $\mathcal{S}_{\alpha\mathsf{ss}} \doteq \{\alpha\text{-self-similar processes}\}$.
**(DS)** Let $\alpha > 0$, $\delta \leq 2\alpha$. $\mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \doteq \{(\alpha, \delta)\text{-dilatively stable processes}\}$.

**2.2.6 Remark. (SS)** Self-similar processes with finite moments of all orders can also be called *unifractals*. In case of an $\{X(t),\, t \geq 0\} \in \mathcal{S}_{\alpha\mathsf{ss}}$ this means that

$$\log\left|\mathsf{cum}_m(X(t))\right| = \alpha\, m \log(t) + \log\left|\mathsf{cum}_m(X(1))\right|, \quad t > 0,\ m = 1, 2, \ldots .$$

The points of interest are that the coefficient of $m\log(t)$ does not depend on $t$ (this is why “fractal”), and does not depend on the order $m$ either (this is why “uni”).

**(DS)** Dilatively stable processes can also be called *multifractals*. In case of an $\{X(t),\, t \geq 0\} \in \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$ this means that

$$\log\left|\mathsf{cum}_m(X(t))\right| = \left(\alpha - \frac{\delta}{2} + \frac{\delta}{m}\right) m \log(t) + \log\left|\mathsf{cum}_m(X(1))\right|, \qquad t > 0,\ m = 1, 2, \ldots .$$

The points of interest are that the coefficient of $m\log(t)$ does not depend on $t$ (this is why “fractal”), and does depend on the order $m$ (this is why “multi”).

In the literature the notions of uni- and multifractal are usually defined by the help of the moments instead of the cumulants. On the other hand, there exist asymptotic versions of these notions: asymptotical unifractal and asymptotical multifractal. Then the scaling equations hold only asymptotically, as $t \to \infty$. Now, in this asymptotical sense it’s just the same whether they are defined through the moments or the cumulants.

By the following theorem the processes in $\mathcal{S}$ or in $\mathcal{I}$ which are the limit processes in renormalization limit theorems, are exactly the self-similar or the dilatively stable processes, respectively.

**2.2.7 Theorem.** ***(SS)*** *(Lamperti [22]) If $\{X(t),\, t \geq 0\} \in \mathcal{S}$ is a process such that there exist a process $\{Y(t),\, t \geq 0\}$ and a function $f:(0,\infty)\to(0,\infty)$ for which*

$$\frac{1}{f(T)}\, Y(Tt) \xrightarrow[T\to\infty]{fd} X(t), \tag{2.2.10}$$

*then there exists an $\alpha > 0$ such that $\{X(t),\, t \geq 0\}$ is $\alpha$-self-similar and $f$ is regularly varying of order $\alpha$, i.e. $f(t) = t^{\alpha}\ell_f(t)$, where $\ell_f$ is a slowly varying function. Conversely, every self-similar process $\{X(t),\, t \geq 0\}$ is a limit process in the previous sense.*

***(DS)*** *If $\{X(t),\, t \geq 0\} \in \mathcal{I}$ is a process such that there exist an infinitely divisible process $\{Y(t),\, t \geq 0\}$ with finite moments of all orders and functions $f, g:(0,\infty)\to(0,\infty)$ for which*

$$\frac{\sqrt{g(T)}}{f(T)}\, Y^{\circledast \frac{1}{g(T)}}(Tt) \xrightarrow[T\to\infty]{fd} X(t), \tag{2.2.11}$$

*and the convergence corresponding to (2.2.11) holds also for every cumulants of the one-dimensional distributions, then there exist*

$$\alpha > 0, \quad \delta \leq 2\alpha, \tag{2.2.12}$$

*such that process $\{X(t),\, t \geq 0\}$ is $(\alpha,\delta)$-dilatively stable and $f$ and $g$ are regularly varying functions of order $\alpha$ and $\delta$, resp., i.e.*

$$f(t) = t^{\alpha}\ell_f(t), \quad g(t) = t^{\delta}\ell_g(t), \tag{2.2.13}$$

*where $\ell_f$ and $\ell_g$ are slowly varying functions. Conversely, every dilatively stable process $\{X(t),\, t \geq 0\}$ is a limit process in the previous sense.*

**Proof.** The proof of the self-similar case can be found in Lamperti [22].
Proof of the dilatively stable case: Since $\{X(t),\, t \geq 0\} \in \mathcal{I}$, by (2.1.1) we have

$$c_{2,X}(1) \neq 0 \text{ and } \exists\, n \geq 3 : c_{n,X}(1) \neq 0. \tag{2.2.14}$$

The assumption with respect to the convergence of the cumulants, for 2 and for an $n$ in (2.2.14):

$$\frac{1}{f^2(T)}\, c_{2,Y}(Tt) \underset{T\to\infty}{\longrightarrow} c_{2,X}(t),$$

$$\frac{g^{\frac{n}{2}-1}(T)}{f^n(T)}\, c_{n,Y}(Tt) \underset{T\to\infty}{\longrightarrow} c_{n,X}(t).$$

Writing these first for $T := T,\ t := 1,$ then for $T := Tt,\ t := 1/t,$ we have

$$\frac{1}{f^2(T)}\, c_{2,Y}(T) \underset{T\to\infty}{\longrightarrow} c_{2,X}(1), \tag{2.2.15}$$

$$\frac{g^{\frac{n}{2}-1}(T)}{f^n(T)}\, c_{n,Y}(T) \underset{T\to\infty}{\longrightarrow} c_{n,X}(1), \tag{2.2.16}$$

$$\frac{1}{f^2(Tt)}\, c_{2,Y}(T) \underset{T\to\infty}{\longrightarrow} c_{2,X}\left(\frac{1}{t}\right), \tag{2.2.17}$$

$$\frac{g^{\frac{n}{2}-1}(Tt)}{f^n(Tt)}\, c_{n,Y}(T) \underset{T\to\infty}{\longrightarrow} c_{n,X}\left(\frac{1}{t}\right). \tag{2.2.18}$$

Let us consider (2.2.17–2.2.18) for values of $t$ in a sufficiently small left neighbourhood $(1-\varepsilon,\, 1)$ of $t = 1$, for which $c_{2,X}(1/t) \neq 0$ and $c_{n,X}(1/t) \neq 0$ (because of (2.2.14) and the right-continuity of $c_{2,X}$ and $c_{n,X}$ there really exists such a left neighbourhood of $t = 1$). From (2.2.15–2.2.18) we obtain

$$\frac{f(Tt)}{f(T)} \underset{T\to\infty}{\longrightarrow} \sqrt{\frac{c_{2,X}(1)}{c_{2,X}\left(\frac{1}{t}\right)}} > 0, \tag{2.2.19}$$

$$\frac{g(Tt)}{g(T)} \underset{T\to\infty}{\longrightarrow} \left(\frac{c_{n,X}\left(\frac{1}{t}\right)}{c_{n,X}(1)}\right)^{\frac{2}{n-2}} \left(\frac{c_{2,X}(1)}{c_{2,X}\left(\frac{1}{t}\right)}\right)^{\frac{n}{n-2}} > 0. \tag{2.2.20}$$

(2.2.19–2.2.20) hold for all $t \in (1-\varepsilon,\, 1)$, so according to Bingham et al. [3, Thm. 1.4.1] (which states that if a convergence of the type (2.2.19) holds on a set of $t$s of positive measure, where the limit function is positive, then function $f$ is regularly varying) it follows that $f$ and $g$ are regularly varying, i.e. there exist $\alpha, \delta \in \mathbb{R}$ and slowly varying functions $\ell_f$ and $\ell_f$ such that (2.2.13) holds.

The proof of $(\alpha, \delta)$-dilative stability of process $\{X(t),\, t \geq 0\}$ is the following. Let $k \in \mathbb{N}$, $\underline{t} \doteq (t_1, \ldots, t_k) \in (0,\infty)^k$, $T \in (0,\infty)$ and let us denote the characteristic function of a random vector of the form $X(\underline{t}) \doteq (X(t_1), \ldots, X(t_k))$ by $\varphi_{X(\underline{t})}(\underline{u})$. Then (2.2.11) yields

$$\varphi^{\frac{1}{g(S)}}_{\frac{\sqrt{g(S)}}{f(S)} Y(S\underline{t})}(\underline{u}) \underset{S\to\infty}{\longrightarrow} \varphi_{X(\underline{t})}(\underline{u})$$

and rewriting the same for $T\underline{t}$ instead of $\underline{t}$ gives

$$\varphi^{\frac{1}{g(S)}}_{\frac{\sqrt{g(S)}}{f(S)} Y(TS\underline{t})}(\underline{u}) \underset{S\to\infty}{\longrightarrow} \varphi_{X(T\underline{t})}(\underline{u}). \tag{2.2.21}$$

But we have $f(S) = S^{\alpha}\ell_f(S)$ and $g(S) = S^{\delta}\ell_g(S)$, hence

$$\begin{aligned} \frac{f(TS)}{f(S)} &= T^{\alpha}\frac{\ell_f(TS)}{\ell_f(S)} = T^{\alpha} I_f(S), \\ \frac{g(TS)}{g(S)} &= T^{\delta}\frac{\ell_g(TS)}{\ell_g(S)} = T^{\delta} I_g(S), \end{aligned}$$

where $\lim_{S\to\infty} I_f(S) = 1$ and $\lim_{S\to\infty} I_g(S) = 1$. From these and (2.2.21) it follows that

$$\varphi^{T^{\delta} I_g(S)\frac{1}{g(TS)}}_{T^{\alpha-\frac{\delta}{2}} I_f(S)(I_g(S))^{-\frac{1}{2}}\frac{\sqrt{g(TS)}}{f(TS)} Y(TS\underline{t})}(\underline{u}) \underset{S\to\infty}{\longrightarrow} \varphi_{X(T\underline{t})}(\underline{u}). \tag{2.2.22}$$

On the other hand we have

$$\begin{aligned} &\varphi^{T^{\delta} I_g(S)\frac{1}{g(TS)}}_{T^{\alpha-\frac{\delta}{2}} I_f(S)(I_g(S))^{-\frac{1}{2}}\frac{\sqrt{g(TS)}}{f(TS)} Y(TS\underline{t})}(\underline{u}) \\ &= \left(\varphi^{\frac{1}{g(TS)}}_{\frac{\sqrt{g(TS)}}{f(TS)} Y(TS\underline{t})}\left(T^{\alpha-\frac{\delta}{2}} I_f(S)(I_g(S))^{-\frac{1}{2}}\underline{u}\right)\right)^{T^{\delta} I_g(S)} \underset{S\to\infty}{\longrightarrow} \left(\varphi_{X(\underline{t})}\left(T^{\alpha-\frac{\delta}{2}}\underline{u}\right)\right)^{T^{\delta}} \\ &= \varphi_{T^{\alpha-\frac{\delta}{2}} X^{\circledast T^{\delta}}(\underline{t})}(\underline{u}) \end{aligned} \tag{2.2.23}$$

(since the family of characteristic functions converges uniformly over bounded intervals). But $k$, $\underline{t}$ and $\underline{u}$ were arbitrary, thus from (2.2.22–2.2.23) we conclude

$$X(Tt) \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} X^{\circledast T^{\delta}}(t), \tag{2.2.24}$$

and since $T > 0$ was also arbitrary, the process $\{X(t),\, t \geq 0\}$ is $(f, g)$-dilatively stable, where $f(t) = t^{\alpha}$ and $g(t) = t^{\delta}$. The inequalities (2.2.12) follow from Theorem 2.2.1 (DS).

The converse statement of the theorem is obvious. □

**2.2.8 Remark.** The statement of the (DS) part of the previous theorem holds true if we assume instead of (2.2.11)

- the corresponding convergence of the joint cumulants, i.e.

$$\mathsf{cum}\left(\frac{\sqrt{g(T)}}{f(T)}\, Y^{\circledast \frac{1}{g(T)}}(Tt_1), \ldots, \frac{\sqrt{g(T)}}{f(T)}\, Y^{\circledast \frac{1}{g(T)}}(Tt_m)\right)$$
$$\underset{T\to\infty}{\longrightarrow} \mathsf{cum}\big(X(t_1), \ldots, X(t_m)\big), \qquad 0 \leq t_1, \ldots, t_m, \quad m = 1, 2, \ldots,$$

- and that the finite-dimensional distributions of process $\{X(t),\, t \geq 0\}$ are uniquely determined by their joint cumulants.

The reason is that (2.2.11) follows from these assumptions.

**2.2.9 Definition. (SS)** Let $\alpha > 0$. A process $\{Y(t),\, t \geq 0\}$ is called *asymptotically $\alpha$-self-similar* if there exist an $\alpha$-self-similar process $\{X(t),\, t \geq 0\}$ and a slowly varying function $\ell$ such that

$$\frac{1}{T^{\alpha}\ell(T)}\, Y(Tt) \xrightarrow[T\to\infty]{\text{fd}} X(t). \tag{2.2.25}$$

**(DS)** Let $\alpha > 0$, $\delta \leq 2\alpha$. An infinitely divisible process $\{Y(t),\, t \geq 0\}$ is called *asymptotically $(\alpha, \delta)$-dilatively stable* if there exist an $(\alpha, \delta)$-dilatively stable process $\{X(t),\, t \geq 0\}$ and slowly varying functions $\ell_1$, $\ell_2$ such that

$$\frac{1}{T^{\alpha-\frac{\delta}{2}}\ell_1(T)}\, Y^{\circledast \frac{1}{T^{\delta}\ell_2(T)}}(Tt) \xrightarrow[T\to\infty]{\text{fd}} X(t). \tag{2.2.26}$$

In terms of the above definition, Theorem 2.2.7 essentially says that under its conditions, that is, in renormalization limit theorems the limit processes must be self-similar or dilatively stable and the starting processes must be asymptotically self-similar or asymptotically dilatively stable, respectively. An asymptotically self-similar and an asymptotically dilatively stable process, which is not (exactly) self-similar and not (exactly) dilatively stable, is given in the following example.

**2.2.10 Example. (SS)** Let $\alpha > 0$, $\ell$ be a strictly positive, slowly varying function, $\{W(t),\, t \geq 0\} \not\equiv 0$ be a white noise consisting of i.i.d. random variables,

$$Y(t) \doteq t^{\alpha}\ell(t)W(t), \quad t > 0, \qquad Y(0) \doteq 0,$$

and let $\{X(t),\, t \geq 0\}$ be a process given in Example 2.1.5 (SS) (product of a white noise and a power function). Then

$$\forall T > 0 : \frac{1}{T^{\alpha}\ell(T)}\, Y(Tt) = \frac{\ell(Tt)}{\ell(T)}\, t^{\alpha}W(Tt) \overset{\text{fd}}{\sim} \frac{\ell(Tt)}{\ell(T)} X(t) \xrightarrow[T\to\infty]{\text{fd}} X(t),$$

hence process $\{Y(t),\, t \geq 0\}$ is asymptotically $\alpha$-self-similar. It is clearly not (exactly) $\alpha$-self-similar, unless the function $\ell$ is a constant.

**(DS)** Let $\alpha > 0$, $\delta \leq 2\alpha$, $\ell_1$ and $\ell_2$ be strictly positive, slowly varying functions, $\{W(t),\, t > 0\}$ be a white noise consisting of i.i.d. random variables such that $W(1)$ has finite moments of all orders, it is infinitely divisible and non-Gaussian. Let $\{Y(t),\, t \geq 0\}$ be a process which has the following finite-dimensional distributions:

$$\big(Y(t_1), \dots, Y(t_n)\big) \overset{\text{fd}}{\sim} \Big(t_1^{\alpha-\frac{\delta}{2}}\ell_1(t_1)W^{\circledast t_1^{\delta}\ell_2(t_1)}(t_1), \dots, t_n^{\alpha-\frac{\delta}{2}}\ell_1(t_n)W^{\circledast t_n^{\delta}\ell_2(t_n)}(t_n)\Big),$$
$$0 < t_1, \dots, t_n,\ n \in \mathbb{N},$$

where $\Big(W^{\circledast t_1^{\delta}\ell_2(t_1)}(t_1), \dots, W^{\circledast t_n^{\delta}\ell_2(t_n)}(t_n)\Big)$ are independent and let $Y(0) \doteq 0$. Moreover, let $\{X(t),\, t \geq 0\}$ be a process given in Example 2.1.5 (DS). Then

$$\begin{aligned}\forall T > 0 : \quad Y(Tt) &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}}\ell_1(T)t^{\alpha-\frac{\delta}{2}}\, \frac{\ell_1(Tt)}{\ell_1(T)}\, W^{\circledast t^{\delta}T^{\delta}\ell_2(Tt)}(Tt) \\ &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}}\ell_1(T)\left(t^{\alpha-\frac{\delta}{2}}\, \frac{\ell_1(Tt)}{\ell_1(T)}\, W^{\circledast t^{\delta}\frac{\ell_2(Tt)}{\ell_2(T)}}\right)^{\circledast T^{\delta}\ell_2(T)}(t)\end{aligned}$$

(this can be checked using characteristic functions). Hence

$$\frac{1}{T^{\alpha-\frac{\delta}{2}}\ell_1(T)}\, Y^{\circledast \frac{1}{T^{\delta}\ell_2(T)}}(Tt) \xrightarrow[T\to\infty]{\text{fd}} t^{\alpha-\frac{\delta}{2}}W^{\circledast t^{\delta}}(t) \overset{\text{fd}}{\sim} X(t).$$

This means that the process $\{Y(t),\, t \geq 0\}$ is asymptotically $(\alpha, \delta)$-dilatively stable. It is clearly not (exactly) $(\alpha, \delta)$-dilatively stable, unless functions $\ell_1$ and $\ell_2$ are constants.

By Theorem 2.2.7 (DS), the dilatively stable renormalization limit theorems are of the form (2.2.26). As the following proposition shows, using independent sums instead of the convolution powers, (2.2.26) can be rewritten into a more simple form.

**2.2.11 Proposition.** *Let $\alpha > 0$, $\delta \leq 2\alpha$ and assume that $\{Y_j(t),\, t \geq 0\}$, $j = 1, 2, \ldots$, are i.i.d. asymptotically $(\alpha, \delta)$-dilatively stable processes, namely, let (2.2.26) hold for them, where either $\delta < 0$ or $\delta = 0$ and $\lim_{T\to\infty} \ell_2(T) = 0$. Then*

$$\frac{1}{T^{\alpha-\frac{\delta}{2}}\ell_1(T)} \sum_{j=1}^{\left\lfloor \frac{1}{T^{\delta}\ell_2(T)} \right\rfloor} Y_j(Tt) \xrightarrow[T\to\infty]{fd} X(t). \tag{2.2.27}$$

**Proof.** What we have to prove is that

$$\frac{1}{T^{\alpha-\frac{\delta}{2}}\ell_1(T)} Y_1^{\circledast\left(\left\lfloor \frac{1}{T^{\delta}\ell_2(T)} \right\rfloor - \frac{1}{T^{\delta}\ell_2(T)}\right)}(Tt) \xrightarrow[T\to\infty]{\text{fd}} 0.$$

But this is true indeed, since

$$\begin{aligned} &\frac{1}{T^{\alpha-\frac{\delta}{2}}\ell_1(T)} Y_1^{\circledast\left(\left\lfloor \frac{1}{T^{\delta}\ell_2(T)} \right\rfloor - \frac{1}{T^{\delta}\ell_2(T)}\right)}(Tt) \\ &\overset{\text{fd}}{\sim} \left( \frac{1}{T^{\alpha-\frac{\delta}{2}}\ell_1(T)} Y_1^{\circledast \frac{1}{T^{\delta}\ell_2(T)}} \right)^{\circledast \frac{\left\lfloor \frac{1}{T^{\delta}\ell_2(T)} \right\rfloor - \frac{1}{T^{\delta}\ell_2(T)}}{\frac{1}{T^{\delta}\ell_2(T)}}} (Tt) \xrightarrow[T\to\infty]{\text{fd}} X^{\circledast 0}(t) = 0. \end{aligned}$$

$\square$

**2.2.12 Remark.** Under the conditions of Proposition 2.2.11 the left-hand side of (2.2.27) becomes even more concrete if we assume that $\delta < 0$ and $\lim\limits_{T\to\infty} \ell_2(T) = \gamma > 0$:

$$\frac{1}{(\gamma n)^{\frac{\delta-2\alpha}{2\delta}} \ell_1\left((\gamma n)^{-\frac{1}{\delta}}\right)} \sum_{j=1}^{n} Y_j\left((\gamma n)^{-\frac{1}{\delta}}\, t\right) \xrightarrow[n\to\infty]{\text{fd}} X(t).$$

The following theorem states, roughly, that the convolution operation preserves asymptotical self-similarity and asymptotical dilative stability. Its practical consequence will be treated in Remark 2.7.6.

**2.2.13 Proposition.** *(maximum renormalization principle) Let* $n \geq 2$ *be a fixed natural number,* $0 < \alpha_1, \dots, \alpha_{n-1} < \alpha_n$*, let* $\{Y_i(t), t \geq 0\}$*,* $i = 1, \dots, n$*, be independent processes and let* $Y(t) \doteq \sum_{i=1}^n Y_i(t)$*,* $t \geq 0$*.*

***(SS)*** *If for each* $i$ *the process* $\{Y_i(t), t \geq 0\}$ *is asymptotically* $\alpha_i$*-self-similar with the slowly varying function* $\ell_i$ *and the limit process* $\{X_i(t), t \geq 0\}$ *in (2.2.25), then*

$$\frac{1}{T^{\alpha_n} \ell_n(T)} Y(Tt) \xrightarrow[T \to \infty]{fd} X_n(t).$$

*That is, the (finite) sum of independent, asymptotically self-similar processes is asymptotically self-similar with the largest parameter* $\alpha$*.*

***(DS)*** *If for each* $i$ *the process* $\{Y_i(t), t \geq 0\}$ *is an* $L^2$*-process with zero mean, and it is asymptotically* $(\alpha_i, \delta_i)$*-dilatively stable with the slowly varying functions* $\ell_{1,i}$*,* $\ell_{2,i}$ *and the limit process* $\{X_i(t), t \geq 0\}$ *in (2.2.26), and the corresponding convergence holds also for the second order cumulant (i.e. for the variance), i.e., if*

$$\lim_{T \to \infty} \mathsf{D}^2 \left( \frac{1}{T^{\alpha_i - \frac{\delta_i}{2}} \ell_{1,i}(T)} Y_i^{\circledast \frac{1}{T^{\delta_i} \ell_{2,i}(T)}}(Tt) \right) = \mathsf{D}^2 X_i(t), \quad t \geq 0, \tag{2.2.28}$$

*then*

$$\frac{1}{T^{\alpha_n - \frac{\delta_n}{2}} \ell_{1,n}(T)} Y^{\circledast \frac{1}{T^{\delta_n} \ell_{2,n}(T)}}(Tt) \xrightarrow[T \to \infty]{fd} X_n(t).$$

*That is, the (finite) sum of independent, asymptotically dilatively stable* $L^2$*-processes with zero mean is asymptotically dilatively stable with a parameter corresponding to the largest parameter* $\alpha$*, provided that the convergences stating the asymptotic dilative stability hold also for the variances.*

**Proof.** The proof of the (SS) part is very simple, so we omit it. The (DS) part follows from the fact that for each $i < n$ we have

$$\begin{aligned}
&\mathsf{D}^2 \left( \frac{1}{T^{\alpha_n - \frac{\delta_n}{2}} \ell_{1,n}(T)} Y_i^{\circledast \frac{1}{T^{\delta_n} \ell_{2,n}(T)}}(Tt) \right) \\
&= \mathsf{D}^2 \left( \frac{1}{T^{\alpha_i - \frac{\delta_i}{2}} \ell_{1,i}(T)} Y_i^{\circledast \frac{1}{T^{\delta_i} \ell_{2,i}(T)}}(Tt) \right) T^{2(\alpha_i - \alpha_n)} \frac{\ell_{1,i}^2(T) \ell_{2,i}(T)}{\ell_{1,n}^2(T) \ell_{2,n}(T)} \xrightarrow[T \to \infty]{} 0, \\
&\qquad t \geq 0.
\end{aligned}$$

## 2.3 The relation of self-similarity and dilative stability

To describe the relation of self-similarity and dilative stability, a new operation called the "function-th" convolution power will be needed. The following lemma establishes it.

**2.3.1 Lemma.** *Let $T \subset \mathbb{R}$ be an interval, $f : T \to [0, \infty]$ be an increasing or decreasing function and let $\mathcal{D}_f$ be the set of infinitely divisible processes $\{X(t),\, t \in T\}$, which are zero at points where $f$ is zero or infinity, i.e. $X(t) = 0$ if $f(t) = 0$ or $f(t) = \infty$. Let us consider the mapping $\{\,\cdot\,\}^{\circledast f} : \mathcal{D}_f \to \mathcal{D}_f$, which carries a process $\{X(t),\, t \in T\} \in \mathcal{D}_f$ into $\{X(t),\, t \in T\}^{\circledast f}$ with the finite-dimensional distributions*

$$\begin{aligned}\big(X(t_1), \ldots, X(t_n)\big)^{\circledast(f(t_1),\ldots,f(t_n))} \doteq\; & \big(X(t_1), \ldots, X(t_n)\big)^{\circledast f(t_1)} \\ & \circledast \big(0, X(t_2), \ldots, X(t_n)\big)^{\circledast(f(t_2)-f(t_1))} \\ & \vdots \\ & \circledast \big(0, \ldots, 0, X(t_n)\big)^{\circledast(f(t_n)-f(t_{n-1}))}, \\ & t_1 \le \ldots \le t_n \in T,\ n \in \mathbb{N},\end{aligned} \tag{2.3.1}$$

*if $f$ is increasing, and*

$$\begin{aligned}\big(X(t_1), \ldots, X(t_n)\big)^{\circledast(f(t_1),\ldots,f(t_n))} \doteq\; & \big(X(t_1), \ldots, X(t_n)\big)^{\circledast f(t_n)} \\ & \circledast \big(X(t_1), \ldots, X(t_{n-1}), 0\big)^{\circledast(f(t_{n-1})-f(t_n))} \\ & \vdots \\ & \circledast \big(X(t_1), 0 \ldots, 0\big)^{\circledast(f(t_1)-f(t_2))}, \\ & t_1 \le \ldots \le t_n \in T,\ n \in \mathbb{N},\end{aligned} \tag{2.3.2}$$

*if $f$ is decreasing. (For simplicity, the distributions are denoted by the random vectors themselves, and we consider $(0, \ldots, 0)^{\circledast\infty}$ to be $(0, \ldots, 0)$.) Then the process $\{X(t),\, t \in T\}^{\circledast f}$ is well-defined in distribution. When the exponent is a constant function, then $\{X(t),\, t \in T\}^{\circledast f}$ coincides with the usual convolution power with constant exponent. In any case, the mapping $\{\,\cdot\,\}^{\circledast f}$ is an injection and its values are infinitely divisible processes.*

**Proof.** Since the process $\{X(t),\, t \in T\}$ is infinitely divisible, the right hand side of (2.3.1) or (2.3.2) determines a distribution on $\mathbb{R}^n$. Checking Kolmogorov's compatibility condition is straightforward but very space-consuming, therefore we omit it. It is also obvious from (2.3.1–2.3.2) that the "function-th" convolution power is a generalization of the ordinary "constant-th" convolution power.

To prove the injectivity of the "function-th" convolution power let us assume that $f$ is increasing, say, and nowhere zero or infinity. If

$$\varphi_{(X(t_1),\ldots,X(t_n))^{\circledast(f(t_1),\ldots,f(t_n))}} = \varphi_{(Y(t_1),\ldots,Y(t_n))^{\circledast(f(t_1),\ldots,f(t_n))}}$$

($\varphi$ denotes the characteristic function of its index), then writing (2.3.1) both for $\big(X(t_1),\ldots,X(t_n)\big)^{\circledast(f(t_1),\ldots,f(t_n))}$ and $\big(Y(t_1),\ldots,Y(t_n)\big)^{\circledast(f(t_1),\ldots,f(t_n))}$ with characteristic functions, the equality of the right hand sides yields

$$\begin{aligned} &\varphi^{f(t_1)}_{X(t_1),\ldots,X(t_n)}\big(u_1,\ldots,u_n\big)\ \varphi^{f(t_2)-f(t_1)}_{0,X(t_2),\ldots,X(t_n)}\big(u_1,u_2,\ldots,u_n\big)\cdots \\ &\qquad\qquad\times \varphi^{f(t_n)-f(t_{n-1})}_{0,\ldots,0,X(t_n)}\big(u_1,\ldots,u_{n-1},u_n\big) \\ &= \varphi^{f(t_1)}_{Y(t_1),\ldots,Y(t_n)}\big(u_1,\ldots,u_n\big)\ \varphi^{f(t_2)-f(t_1)}_{0,Y(t_2),\ldots,Y(t_n)}\big(u_1,u_2,\ldots,u_n\big)\cdots \\ &\qquad\qquad\times \varphi^{f(t_n)-f(t_{n-1})}_{0,\ldots,0,Y(t_n)}\big(u_1,\ldots,u_{n-1},u_n\big). \end{aligned} \tag{2.3.3}$$

From (2.3.3) with the substitution $u_1 = \ldots = u_{n-1} = 0$ we obtain

$$\varphi^{f(t_n)}_{X(t_n)}\big(u_n\big) = \varphi^{f(t_n)}_{Y(t_n)}\big(u_n\big),$$

hence

$$\varphi^{f(t_n)-f(t_{n-1})}_{0,\ldots,0,X(t_n)}\big(u_1,\ldots,u_{n-1},u_n\big) = \varphi^{f(t_n)-f(t_{n-1})}_{0,\ldots,0,Y(t_n)}\big(u_1,\ldots,u_{n-1},u_n\big). \tag{2.3.4}$$

In the next step dividing (2.3.3) by (2.3.4) (which can be done since an infinitely divisible characteristic function is nowhere zero), with the substitution $u_1 = \ldots = u_{n-2} = 0$ we obtain

$$\varphi^{f(t_{n-1})}_{X(t_{n-1}),X(t_n)}\big(u_{n-1},u_n\big) = \varphi^{f(t_{n-1})}_{Y(t_{n-1}),Y(t_n)}\big(u_{n-1},u_n\big),$$

hence

$$\begin{aligned} &\varphi^{f(t_{n-1})-f(t_{n-2})}_{0,\ldots,0,X(t_{n-1}),X(t_n)}\big(u_1,\ldots,u_{n-2},u_{n-1},u_n\big) \\ &\qquad\qquad = \varphi^{f(t_{n-1})-f(t_{n-2})}_{0,\ldots,0,Y(t_{n-1}),Y(t_n)}\big(u_1,\ldots,u_{n-2},u_{n-1},u_n\big). \end{aligned} \tag{2.3.5}$$

In the next step dividing (2.3.3) by (2.3.4) and by (2.3.5), with the replacement $u_1 = \cdots = u_{n-3} = 0$ we obtain

$$\varphi^{f(t_{n-2})}_{X(t_{n-2}),X(t_{n-1}),X(t_n)}\big(u_{n-2}, u_{n-1}, u_n\big) = \varphi^{f(t_{n-2})}_{Y(t_{n-2}),Y(t_{n-1}),Y(t_n)}\big(u_{n-2}, u_{n-1}, u_n\big).$$

Continuing this procedure, in the last step we have

$$\varphi^{f(t_1)}_{X(t_1),\dots,X(t_n)}\big(u_1, \dots, u_n\big) = \varphi^{f(t_1)}_{Y(t_1),\dots,Y(t_n)}\big(u_1, \dots, u_n\big),$$

i.e.

$$\big(X(t_1), \dots, X(t_n)\big) \sim \big(Y(t_1), \dots, Y(t_n)\big),$$

proving that the mapping $\{\,\cdot\,\}^{\circledast f}$ is injective.

One can check that the injectivity also follows when $f$ can take the values zero and infinity. By (2.3.1–2.3.2) it is also obvious that the finite-dimensional distributions of $\{X(t),\, t \in T\}^{\circledast f}$ inherit the infinite divisibility. $\square$

**2.3.2 Definition.** We call the process $\{X(t),\, t\in T\}^{\circledast f}$ of Lemma 2.3.1 (or, more precisely, its distribution) the *$f$-th convolution power* of process $\{X(t),\, t\in T\} \in \mathcal{D}_f$. When necessary, we use also the notation $\{X(t),\, t\in T\}^{\circledast f(t)} \doteq \{X(t),\, t\in T\}^{\circledast f}$ (e.g. $\{X(t),\, t \in T\}^{\circledast t^\delta}$), so, when the argument of the function in the convolution exponent is $t$ (the same as the parameter of the process), then the convolution power is meant as the $f$-th convolution power and not the constant $f(t)$-th convolution power.

**2.3.3 Remark.** If $f$ is not a constant function, then operators $\{\,\cdot\,\}^{\circledast f}$ and $\{\,\cdot\,\}^{\circledast \frac{1}{f}}$ defined in Lemma 2.3.1 are not the inverse of each other. For example, in case of $f(t_1) \neq f(t_2)$, we have

$$\Big(\big(X(t_1), X(t_2)\big)^{\circledast(f(t_1), f(t_2))}\Big)^{\circledast\left(\frac{1}{f(t_1)}, \frac{1}{f(t_2)}\right)} \sim \big(X(t_1), X(t_2)\big)$$

if and only if $X(t_1)$ and $X(t_2)$ are independent. The reason is that although the operators $\{\,\cdot\,\}^{\circledast f}$ corresponding to increasing functions $f$ constitute a semigroup, as well the operators corresponding to decreasing functions, all the operators together no longer constitute a group. This is evident, because the product of an increasing $f$ and a decreasing $g$ is, in general, not monotone, thus the composition operator $\big(\{\,\cdot\,\}^{\circledast f}\big)^{\circledast g}$ is not defined at all. And, naturally, functions $f$ and $1/f$ are neither simultaneously increasing, nor simultaneously decreasing.

**2.3.4 Lemma.** *Let* $T \subset \mathbb{R}$ *be an interval,* $f : T \to [0, \infty]$ *and* $g : T \to [0, \infty]$ *be an increasing and a decreasing function, respectively, and let* $\{X(t),\, t \in T\}$ *be a process in* $\mathcal{D}_f$ *or* $\mathcal{D}_g$*. If* $\{X(t),\, t \in T\}$ *has independent increments, then so does* $\{X(t),\, t \in T\}^{\circledast f}$*, and* $\{X(t),\, t \in T\}^{\circledast g}$ *has independent increments if and only if* $\{X(t),\, t \in T\}$ *is a trivial process, i.e. a deterministic function, or the function* $g$ *is a constant.*

**Proof.** Let us say that a random vector $(\xi_1, \ldots, \xi_m)$ is of independent increments if the random variables $\xi_1, \xi_2 - \xi_1, \ldots, \xi_m - \xi_{m-1}$ are independent. Now, the first statement is a consequence of (2.3.1), because each row on the right-hand side of (2.3.1) is a vector of independent increments and the rows are independent (remember that, for simplicity, we do not distinguish between random vectors and their distributions), therefore, the result is also of independent increments.

The second statement is a consequence of (2.3.2) taken for $n = 2$ with $g$ instead of $f$ and with $t_1 < t_2$. Indeed, in this case the right-hand side is

$$\big(X(t_1), X(t_2)\big)^{\circledast g(t_2)} \circledast \big(X(t_1), 0\big)^{\circledast g(t_1) - g(t_2)},$$

where the first random vector is of independent increments but the second is not (unless $X(t_1)$ is a constant, or $g(t_1) = g(t_2)$), therefore their independent sum is, in general, not of independent increments. (Here we used the fact that if the independent sum of two infinitely divisible random vectors and either of the two are of independent increments, then the other is also of independent increments. This easily follows, since the characteristic function of an infinitely divisible distribution is nowhere zero.) $\square$

**2.3.5 Example.** A Lévy process (in law) $\{L(t),\, t \geq 0\}$ is the $t$-th convolution power of the time-independent process $\{L(1),\, t \geq 0\}$, i.e.

$$\big\{L(t),\, t \geq 0\big\} \overset{\text{fd}}{\sim} \big\{L(1),\, t \geq 0\big\}^{\circledast t}.$$

More generally, if $f : [0, \infty) \to [0, \infty)$ is a time-change function (i.e. strictly increasing, continuous, $f(0) = 0$ and $\lim_{t \to \infty} f(t) = \infty$), then the time-changed Lévy process (in law) $\{L(f(t)),\, t \geq 0\}$ is the $f$-th convolution power of the time-independent process $\{L(1),\, t \geq 0\}$, i.e.

$$\big\{L(f(t)),\, t \geq 0\big\} \overset{\text{fd}}{\sim} \big\{L(1),\, t \geq 0\big\}^{\circledast f}.$$

This follows, using (2.3.1) for $n = 1$, Lemma 2.3.4 and the fact that the finite-dimensional distributions of a process with independent increments are uniquely determined by the one-dimensional distributions.

The next example suggests the connection between self-similar and dilatively stable processes.

**2.3.6 Example.** Let $\alpha > 0$, $\delta < 2\alpha$ and consider the process of Example 2.1.5 (SS) with the modification that its parameter is $\alpha - \frac{\delta}{2}$ instead of $\alpha$ and that it is infinitely divisible, non-Gaussian and all of its moments are finite. So, let

$$Y(t) \doteq t^{\alpha - \frac{\delta}{2}} W(t), \quad t \geq 0,$$

where $\{W(t),\, t \geq 0\} \not\equiv 0$ is a non-Gaussian, infinitely divisible, i.i.d. white noise with finite moments of all orders. Consider also the process of Example 2.1.5 (DS):

$$X(t) \doteq t^{\alpha - \frac{\delta}{2}} (W(t))^{\circledast t^{\delta}}, \quad t > 0,$$

$$X(0) \doteq 0.$$

Then $\{Y(t),\, t \geq 0\} \in \mathcal{S}_{(\alpha - \frac{\delta}{2})\mathsf{ss}}$, $\{X(t),\, t \geq 0\} \in \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$, and

$$\{X(t),\, t \geq 0\} = \{Y(t),\, t \geq 0\}^{\circledast t^{\delta}}. \tag{2.3.6}$$

The following theorem specifies what the connection between self-similar and dilatively stable processes can be.

**2.3.7 Theorem.** *Let $\alpha > 0$, $\delta \leq 2\alpha$.*
*1) If for some $\{X(t),\, t \geq 0\} \in \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$ there exists an infinitely divisible process $\{Y(t),\, t \geq 0\} \in \mathcal{S}$ such that (2.3.6) holds, then $\delta < 2\alpha$ and $\{Y(t),\, t \geq 0\} \in \mathcal{S}_{(\alpha - \frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I}$. Conversely, if $\delta < 2\alpha$ and $\{Y(t),\, t \geq 0\} \in \mathcal{S}_{(\alpha - \frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I}$, then the process in (2.3.6) is $(\alpha, \delta)$-dilatively stable.*
*2) Let $\delta < 2\alpha$. If for some $\{Y(t),\, t \geq 0\} \in \mathcal{S}_{(\alpha - \frac{\delta}{2})\mathsf{ss}}$ there exists an $\{X(t),\, t \geq 0\} \in \mathcal{I}$ such that*

$$\{Y(t),\, t \geq 0\} \doteq \{X(t),\, t \geq 0\}^{\circledast t^{-\delta}}, \tag{2.3.7}$$

*then $\{X(t),\, t \geq 0\} \in \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$. Conversely, if $\{X(t),\, t \geq 0\} \in \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$, then process (2.3.7) is $(\alpha - \delta/2)$-self-similar and it is in $\mathcal{I}$.*

**Proof.** 1) By the $(\alpha, \delta)$-dilative stability of $\{X(t),\, t \geq 0\}$ and (2.3.6) we have for any $T > 0$:

$$\left(Y(Tt)\right)^{\circledast (Tt)^{\delta}} \overset{\text{fd}}{\sim} T^{\alpha - \frac{\delta}{2}} \left( \left(Y(t)\right)^{\circledast t^{\delta}} \right)^{\circledast T^{\delta}},$$

or, equivalently,

$$\left(\left(Y(Tt)\right)^{\circledast t^\delta}\right)^{\circledast T^\delta} \overset{\text{fd}}{\sim} \left(T^{\alpha-\frac{\delta}{2}}\left(Y(t)\right)^{\circledast t^\delta}\right)^{\circledast T^\delta},$$

hence

$$\left(Y(Tt)\right)^{\circledast t^\delta} \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}}\left(Y(t)\right)^{\circledast t^\delta} \overset{\text{fd}}{\sim} \left(T^{\alpha-\frac{\delta}{2}}Y(t)\right)^{\circledast t^\delta},$$

from which by the injectivity of the $t^\delta$-th convolution power (see Lemma 2.3.1) we obtain

$$Y(Tt) \overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}}Y(t). \tag{2.3.8}$$

Since $\{Y(t),\, t\geq 0\}\in\mathcal{S}$, it is right-continuous in distribution also at $t=0$, therefore $\delta<2\alpha$ and $\{Y(t),\, t\geq 0\}$ is $(\alpha-\delta/2)$-self-similar. Moreover, by (2.3.1) or (2.3.2) (depending on whether $\delta\geq 0$ or $\delta\leq 0$) the joint cumulants exist and $\{Y(t),\, t\geq 0\}$ is a non-Gaussian process. Finally, the right-continuity of the cumulant functions follows from (2.3.8) and from that $\delta<2\alpha$:

$$\mathsf{cum}_m(Y(t)) = t^{m\left(\alpha-\frac{\delta}{2}\right)}\mathsf{cum}_m(Y(1)), \quad t>0,$$

completing the proof of that $\{Y(t),\, t\geq 0\}\in\mathcal{I}$.

Conversely, if $\{Y(t),\, t\geq 0\}\in\mathcal{I}$ is $(\alpha-\delta/2)$-self-similar, then for any $T>0$ we have

$$\begin{aligned} X(Tt) \overset{\text{fd}}{\sim} \left(Y(Tt)\right)^{\circledast (Tt)^\delta} &\overset{\text{fd}}{\sim} \left(\left(T^{\alpha-\frac{\delta}{2}}Y(t)\right)^{\circledast t^\delta}\right)^{\circledast T^\delta} \\ &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}}\left(\left(Y(t)\right)^{\circledast t^\delta}\right)^{\circledast T^\delta} = T^{\alpha-\frac{\delta}{2}}\left(X(t)\right)^{\circledast T^\delta}, \end{aligned}$$

which means that $\{X(t),\, t\geq 0\}\in\mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$.

2) The proof is similar and therefore is omitted. □

Example 2.3.6 raises the question of whether any dilatively stable process can be obtained as the convolution power of some self-similar process. The first statement of Theorem 2.3.7 answers the question straightaway: an $(\alpha,2\alpha)$-dilatively stable process is surely not the convolution power of any self-similar process.

Considering only the $\delta < 2\alpha$ case, the answer is not so simple. Now, the precise question is the following one. Given parameters $\alpha > 0,\ \delta < 2\alpha,$ is the mapping

$$\begin{gathered} T_S : \mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I} \longrightarrow \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}\,, \\ \{Y(t),\, t \geq 0\} \longmapsto T_S\Big(\{Y(t),\, t \geq 0\}\Big) \doteq \{Y(t),\, t \geq 0\}^{\circledast\, t^{\delta}} \end{gathered} \tag{2.3.9}$$

a surjective mapping? The converse question is also meaningful: given parameters $\alpha > 0,\ \delta < 2\alpha,$ is the mapping

$$\begin{gathered} T_D : \mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \longrightarrow \mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I}\,, \\ \{X(t),\, t \geq 0\} \longmapsto T_D\Big(\{X(t),\, t \geq 0\}\Big) \doteq \{X(t),\, t \geq 0\}^{\circledast\, t^{-\delta}} \end{gathered} \tag{2.3.10}$$

a surjective mapping? Clearly, $T_S$ and $T_D$ are not the inverse of each other, unless $\delta = 0,$ when both mappings are identities. What is more, neither mapping $T_S \circ T_D$ nor $T_D \circ T_S$ has any fixed point, unless $\delta = 0$ (see Remark 2.3.3). In spite of this, $T_S$ could even be a bijective mapping between the set of $(\alpha-\delta/2)$-self-similar processes in $\mathcal{I}$ and the set of $(\alpha,\delta)$-dilatively stable processes. The same is true for $T_D$. However, the following theorem states that neither mapping is surjective. This implies that dilative stability is not simply the combination of self-similarity and "a power function-th convolution power", so dilative stability is not a redundant concept.

**2.3.8 Theorem.** *Let $\alpha > 0$ and $\delta < 2\alpha$. Mapping (2.3.9) is an injection. It is a bijection if and only if $\delta = 0$. The same is true for mapping (2.3.10).*

**Proof.** The injectivity of both mappings follows from the last statement of Lemma 2.3.1. If $\delta = 0,$ then both mappings are the identity. So, let $\delta \neq 0.$

First of all we note that using (2.3.1–2.3.2) and its notation and assuming that the cumulants exist, we have

$$\mathsf{cum}\left(\big(X(t_1),\ldots,X(t_n)\big)^{\circledast(f(t_1),\ldots,f(t_n))}\right) = f(t_1)\mathsf{cum}\left(X(t_1),\ldots,X(t_n)\right), \qquad t_1 \leq \ldots \leq t_n,$$

if function $f$ is increasing, and

$$\mathsf{cum}\left(\big(X(t_1),\ldots,X(t_n)\big)^{\circledast(f(t_1),\ldots,f(t_n))}\right) = f(t_n)\mathsf{cum}\left(X(t_1),\ldots,X(t_n)\right), \qquad t_1 \leq \ldots \leq t_n,$$

if $f$ is decreasing. This fact will be used for $n = 2,$ to compute the covariance.

Let $0 < t_1 < t_2$. For any $\{Y(t),\, t \geq 0\} \in \mathcal{S}_{(\alpha - \frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I}$ we have

$$\mathsf{Cov}\big(T_S(Y)(t_1), T_S(Y)(t_2)\big) = \begin{cases} (t_1)^\delta \mathsf{Cov}\big(Y(t_1), Y(t_2)\big) & \text{if } \delta > 0, \\ (t_2)^\delta \mathsf{Cov}\big(Y(t_1), Y(t_2)\big) & \text{if } \delta < 0, \end{cases}$$

hence

$$\mathsf{Corr}\big(T_S(Y)(t_1), T_S(Y)(t_2)\big) = \frac{\mathsf{Cov}\big(T_S(Y)(t_1), T_S(Y)(t_2)\big)}{\mathsf{D}\big(T_S(Y)(t_1)\big)\mathsf{D}\big(T_S(Y)(t_2)\big)}$$

$$= \left\{ \begin{array}{ll} \dfrac{(t_1)^\delta \mathsf{Cov}\big(Y(t_1), Y(t_2)\big)}{t_1^{\frac{\delta}{2}} \mathsf{D}\big(Y(t_1)\big) t_2^{\frac{\delta}{2}} \mathsf{D}\big(Y(t_2)\big)} & \text{if } \delta > 0 \\ \\ \dfrac{(t_2)^\delta \mathsf{Cov}\big(Y(t_1), Y(t_2)\big)}{t_1^{\frac{\delta}{2}} \mathsf{D}\big(Y(t_1)\big) t_2^{\frac{\delta}{2}} \mathsf{D}\big(Y(t_2)\big)} & \text{if } \delta < 0 \end{array} \right\} = \left(\frac{t_1}{t_2}\right)^{\frac{|\delta|}{2}} \mathsf{Corr}\big(Y(t_1), Y(t_2)\big).$$

Similarly, if $0 < t_1 < t_2,$ then for any $\{X(t),\, t \geq 0\} \in \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$ we have

$$\mathsf{Cov}\big(T_D(X)(t_1), T_D(X)(t_2)\big) = \begin{cases} (t_2)^{-\delta} \mathsf{Cov}\big(X(t_1), X(t_2)\big) & \text{if } \delta > 0, \\ (t_1)^{-\delta} \mathsf{Cov}\big(X(t_1), X(t_2)\big) & \text{if } \delta < 0, \end{cases}$$

hence

$$\mathsf{Corr}\big(T_D(X)(t_1), T_D(X)(t_2)\big) = \frac{\mathsf{Cov}\big(T_D(X)(t_1), T_D(X)(t_2)\big)}{\mathsf{D}\big(T_D(X)(t_1)\big)\mathsf{D}\big(T_D(X)(t_2)\big)}$$

$$= \left\{ \begin{array}{ll} \dfrac{(t_2)^{-\delta} \mathsf{Cov}\big(X(t_1), X(t_2)\big)}{t_1^{-\frac{\delta}{2}} \mathsf{D}\big(X(t_1)\big) t_2^{-\frac{\delta}{2}} \mathsf{D}\big(X(t_2)\big)} & \text{if } \delta > 0 \\ \\ \dfrac{(t_1)^{-\delta} \mathsf{Cov}\big(X(t_1), X(t_2)\big)}{t_1^{-\frac{\delta}{2}} \mathsf{D}\big(X(t_1)\big) t_2^{-\frac{\delta}{2}} \mathsf{D}\big(X(t_2)\big)} & \text{if } \delta < 0 \end{array} \right\} = \left(\frac{t_1}{t_2}\right)^{\frac{|\delta|}{2}} \mathsf{Corr}\big(X(t_1), X(t_2)\big).$$

Thus, $T_S$ and $T_D$ decrease $|\mathsf{Corr}(X(t_1), X(t_2))|$ and $|\mathsf{Corr}(Y(t_1), Y(t_2))|$, resp., by a factor $(t_1/t_2)^{|\delta|/2},$ where $t_1 < t_2$.

Next we prove that $T_D$ is not surjective. For this, let process $\{Y(t),\, t \geq 0\}$ be as in Example 2.1.6 (SS), but with $(\alpha - \delta/2)$ instead of $\alpha$ and, of course, in $\mathcal{I}$. That is

$$Y(t) \doteq t^{\alpha - \frac{\delta}{2}}\xi, \quad t \geq 0,$$

where $\xi$ is some non-Gaussian, infinitely divisible random variable with finite moments of all orders. Then $\{Y(t),\, t \geq 0\}$ is an $(\alpha - \delta/2)$-self-similar process in $\mathcal{I}$. But $\mathsf{Corr}(Y(t_1), Y(t_2)) = 1$, for any $0 < t_1 < t_2$, therefore $\{Y(t),\, t \geq 0\}$ can not be the image of any process in the domain of $T_D$, since $T_D$ decrease the modulus of the correlation coefficient by a factor $(t_1/t_2)^{|\delta|/2} < 1$.

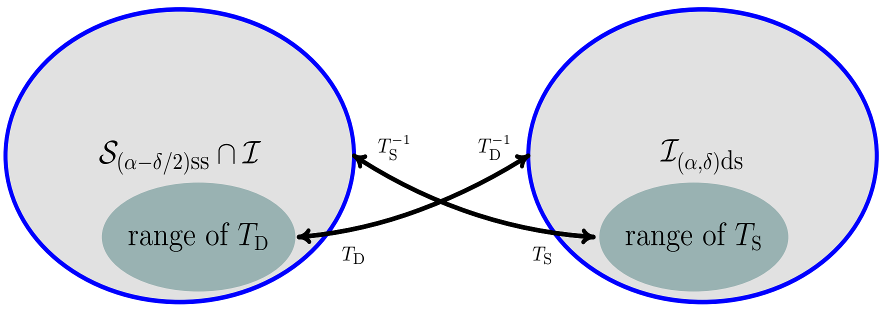
Finally, we prove that $T_S$ is not surjective either. For this, let $\{X_1(t),\, t \geq 0\}$ be an $(\alpha, \delta)$-dilatively stable process, let

$$V(t) \doteq C\, t^{\alpha}\xi, \quad t \geq 0,$$

where $C > 0$ is a constant, $\xi$ is a standard Gaussian random variable, independent of process $\{X_1(t),\, t \geq 0\}$ and let

$$X(t) \doteq X_1(t) + V(t), \quad t \geq 0. \tag{2.3.11}$$

Then, it is easy to check that $\{X(t),\, t \geq 0\} \in \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$. It is also obvious, that $\lim_{C\to\infty} \mathsf{Corr}\big(X(t_1), X(t_2)\big) = 1$, for any fixed pair $(t_1, t_2)$. Therefore, for a sufficiently large $C$, the $(\alpha, \delta)$-dilatively stable process $\{X(t),\, t \geq 0\}$ can not be the image of any process in the domain of $T_S$, since $T_S$ decrease the modulus of the correlation coefficient by a factor $(t_1/t_2)^{|\delta|/2} < 1$. □




**2.3.1 Figure.** The connection between self-similarity and dilative stability. Mappings $T_S$ and $T_D$ are defined by (2.3.9–2.3.10).

## 2.4 The Lamperti transforms as the connection between dilative and translative stability

The Lamperti transforms introduced by Lamperti [22], give a one-to-one correspondence between self-similar processes and stationary processes. In this section we define the notion of translative stability, a generalization of stationarity, and we show that the Lamperti transforms also give a one-to-one correspondence between dilatively stable processes and translatively stable processes.

**2.4.1 Notation.**

$$\begin{aligned}\widetilde{\mathcal{I}} \doteq \big\{&\text{stochastic processes } \{S(t),\, t\in\mathbb{R}\} : S(0) \text{ is non-Gaussian, the}\\ &\text{finite-dimensional distributions are infinitely divisible, and functions}\\ &c_n(t) \doteq \mathsf{cum}_n(S(t)),\ t\in\mathbb{R},\ n\geq 2,\ \text{exist and are right-continuous}\big\}.\end{aligned}$$

**2.4.2 Remark.** There holds the equivalent of the implication in Remark 2.1.2:

$$\{S(t),\, t\in\mathbb{R}\}\in\widetilde{\mathcal{I}} \implies \forall\, m\in\mathbb{N} : \mathsf{cum}_{2m}(S(0)) > 0.$$

**2.4.3 Definition.** Let $g:\mathbb{R}\to(0,\infty)$. Process $\{S(t),\, t\in\mathbb{R}\}\in\widetilde{\mathcal{I}}$ is called *$g$-translatively stable* if

$$\forall\, T\in\mathbb{R} : S(t+T) \overset{\text{fd}}{\sim} S^{\circledast g(T)}(t). \tag{2.4.1}$$

**2.4.4 Proposition.** *If process $\{S(t),\, t\in\mathbb{R}\}$ is $g$-translatively stable, then $g$ is necessarily an exponential function, i.e. there exists a $\delta\in\mathbb{R}$ such that $g(t)=e^{\delta t}$. Therefore instead of the expression "$e^{\delta t}$-translatively stable" we simply use "$\delta$-translatively stable".*

**Proof.** We have

$$\begin{aligned}\mathsf{D}^2\left(S\left(T_1+T_2\right)\right) = \mathsf{D}^2\left(S^{\circledast g(T_2)}\left(T_1\right)\right) &= \mathsf{D}^2\left(S^{\circledast g(T_1)g(T_2)}(0)\right)\\ &= g\left(T_1\right) g\left(T_2\right) \mathsf{D}^2(S(0)), \quad T_1,T_2\in\mathbb{R},\end{aligned}$$

and at the same time

$$\mathsf{D}^2\left(S\left(T_1+T_2\right)\right) = \mathsf{D}^2\left(S^{\circledast g(T_1+T_2)}(0)\right) = g\left(T_1+T_2\right)\mathsf{D}^2(S(0)), \quad T_1,T_2\in\mathbb{R},$$

hence

$$g\left(T_1+T_2\right)=g\left(T_1\right)g\left(T_2\right),\quad T_1,T_2\in\mathbb{R}.$$

Since function $\mathsf{D}^2(S(T))=g(T)\mathsf{D}^2(S(0))$, $T\in\mathbb{R}$, is right-continuous (see Notation 2.4.1), so is $g$. Thus, $g(T)=e^{\delta T}$, with some constant $\delta\in\mathbb{R}$. □

**2.4.5 Notation.** $\mathcal{S}_{\mathsf{st}}\doteq\big\{$strictly stationary processes over $(-\infty,\infty)\big\}$.

**2.4.6 Notation.** $\mathcal{I}_{\delta\mathsf{ts}}\doteq\big\{\delta$-translatively stable processes$\big\}$, $\delta\in\mathbb{R}$.

**2.4.7 Remark.** $\mathcal{I}_{0\mathsf{ts}}=\mathcal{S}_{\mathsf{st}}\bigcap\widetilde{\mathcal{I}}$.

**2.4.8 Example.** *Lévy process with an exponential function time-change:* Let $\delta\in\mathbb{R}$ and let $\{L(t),\,t\geq 0\}$ be a non-Gaussian Lévy process (in law) such that all the cumulants of $L(1)$ exist. Let

$$S(t)\doteq L\left(e^{\delta t}\right),\quad t\in\mathbb{R}.$$

Then it is easy to check that $\{S(t),\,t\in\mathbb{R}\}\in\widetilde{\mathcal{I}}$ and for any $T\in\mathbb{R}$ we have

$$S(t+T)=L\left(e^{\delta T}e^{\delta t}\right)\overset{\text{fd}}{\sim}L^{\circledast e^{\delta T}}\left(e^{\delta t}\right)\overset{\text{fd}}{\sim}S^{\circledast e^{\delta T}}(t),\quad t\in\mathbb{R},$$

i.e. $\{S(t),\,t\in\mathbb{R}\}\in\mathcal{I}_{\delta\mathsf{ts}}$.

**2.4.9 Theorem.** *(Lamperti transforms)* ***(SS)*** *([22]) Let* $\alpha>0$*. The mapping*

$$\begin{aligned}&L_{ST}:\mathcal{S}_{\mathsf{st}}\longrightarrow\mathcal{S}_{\alpha\mathsf{ss}},\\&X(t)\doteq\big(L_{ST}(S)\big)(t)\doteq t^{\alpha}S(\log t),\quad t>0,\\&X(0)\doteq 0,\end{aligned}\tag{2.4.2}$$

*is well-defined, i.e. it maps* $\mathcal{S}_{\mathsf{st}}$ *into* $\mathcal{S}_{\alpha\mathsf{ss}}$*. Conversely, the mapping*

$$\begin{aligned}&L_{SS}:\mathcal{S}_{\alpha\mathsf{ss}}\longrightarrow\mathcal{S}_{\mathsf{st}},\\&S(t)\doteq\big(L_{SS}(X)\big)(t)\doteq e^{-\alpha t}X(e^{t}),\quad t\in\mathbb{R},\end{aligned}\tag{2.4.3}$$

*is well-defined, i.e. it maps* $\mathcal{S}_{\alpha\mathsf{ss}}$ *into* $\mathcal{S}_{\mathsf{st}}$*. The two mappings are the inverse of each other.*

***(DS)*** *Let* $\alpha > 0,\ \delta \le 2\alpha$. *The mapping*

$$\begin{aligned} &L_{TS} : \mathcal{I}_{\delta\mathsf{ts}} \longrightarrow \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}, \\ &X(t) \doteq \big(L_{TS}(S)\big)(t) \doteq t^{\alpha-\frac{\delta}{2}} S(\log t), \quad t > 0, \\ &X(0) \doteq \big(L_{TS}(S)\big)(0) \doteq 0, \end{aligned} \tag{2.4.4}$$

*is well-defined, i.e. it maps* $\mathcal{I}_{\delta\mathsf{ts}}$ *into* $\mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$. *Conversely, the mapping*

$$\begin{aligned} &L_{DS} : \mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \longrightarrow \mathcal{I}_{\delta\mathsf{ts}}, \\ &S(t) \doteq \big(L_{DS}(X)\big)(t) \doteq e^{-\left(\alpha-\frac{\delta}{2}\right)t} X(e^t), \quad t \in \mathbb{R}, \end{aligned} \tag{2.4.5}$$

*is well-defined, i.e. it maps* $\mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$ *into* $\mathcal{I}_{\delta\mathsf{ts}}$. *The two mappings are the inverse of each other.*

**Proof.** The (SS) part can be easily checked. (DS) part: $X(1) = S(0)$ is non-Gaussian and $L_{TS}$ preserves the infinite divisibility of the finite-dimensional distributions. It also preserves the right-continuity of the cumulant functions, because

$$\begin{aligned} \mathsf{cum}_m(X(t)) = t^{m\left(\alpha-\frac{\delta}{2}\right)} \mathsf{cum}_m(S(\log t)) &= t^{m\left(\alpha-\frac{\delta}{2}\right)} e^{\delta \log t} \mathsf{cum}_m(S(0)) \\ &= t^{m\left(\alpha-\frac{\delta}{2}\right)+\delta} \mathsf{cum}_m(S(0)), \quad t > 0 \end{aligned}$$

and $\mathsf{cum}_m(X(t))$ is right-continuous even in $t = 0$. Therefore $L_{TS}$ maps into $\mathcal{I}$. The image processes of $L_{TS}$ are $(\alpha, \delta)$-dilatively stable processes, since

$$\begin{aligned} X(Tt) = T^{\alpha-\frac{\delta}{2}} t^{\alpha-\frac{\delta}{2}} S(\log t + \log T) &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} t^{\alpha-\frac{\delta}{2}} S^{\circledast e^{\delta \log T}}(\log t) \\ &\overset{\text{fd}}{\sim} T^{\alpha-\frac{\delta}{2}} \left(t^{\alpha-\frac{\delta}{2}} S(\log t)\right)^{\circledast T^\delta} = T^{\alpha-\frac{\delta}{2}} X^{\circledast T^\delta}(t), \quad T > 0. \end{aligned}$$

The proof of the converse statement is analogous, so we omit it.

$L_{TS}$ is a one-to-one mapping between the paths. Its inverse is $L_{DS}$, so the two mappings are the inverse of each other. □

**2.4.10 Example.** Let $\alpha > 0,\ \delta \le 2\alpha$ and let $\{L(t),\, t \ge 0\}$ be a non-Gaussian Lévy process such that all the cumulants of $L(1)$ exist. Then the $(\alpha, \delta)$-dilatively stable process of Example 2.1.6(DS):

$$X(t) \doteq \begin{cases} t^{\alpha-\frac{\delta}{2}} L(t^\delta) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and the $\delta$-translatively stable process of Example 2.4.8:

$$S(t) \doteq L\left(e^{\delta t}\right), \quad t \in \mathbb{R},$$

correspond to each other by the Lamperti transforms:

$$S(t) = \big(L_{DS}(X)\big)(t) = e^{-\left(\alpha-\frac{\delta}{2}\right)t} X\left(e^t\right), \quad t \in \mathbb{R}$$

$$X(t) = \big(L_{TS}(S)\big)(t) = t^{\alpha-\frac{\delta}{2}} S(\log t), \quad t \geq 0.$$

**2.4.11 Remark.** Let us observe that while in the (SS) case the Lamperti transforms are not defined for $\alpha = 0$ (since 0-self-similarity does not exist), in the (DS) case they are defined even if $\alpha - \delta/2 = 0$ (since $(\alpha, \delta)$-dilative stability does exist even for such $(\alpha, \delta)$).

## 2.5 Translatively stable wide sense Ornstein–Uhlenbeck type processes

This section is a digression to the field of processes in the title. These processes are the translatively stable analogues of the stationary Ornstein–Uhlenbeck type processes and they will be used in the next section.

**2.5.1 Definition.** (see Sato [29]) Let $T = (-\infty, \infty)$ or $T = [0, \infty)$. A stochastic process $\{K(t),\, t \in T\}$ is called an *additive process* on $T$ if
- it has independent increments,
- it is stochastically continuous,
- it has a.s. càdlàg paths and
- $K(0) = 0$ a.s..

The set of additive processes on $[0, \infty)$ is denoted by $\mathcal{I}_{\mathsf{add}}$.

**2.5.2 Remark.** Additive processes are infinitely divisible. This follows from the facts that the one-dimensional distributions (and so the distributions of the increments) are infinitely divisible (see Sato [29, Thm. 9.1]) and that the increments are independent.

**2.5.3 Definition.** (see Sato [31, p. 214] or Maejima–Sato [25, Def. 3.2]) Let $\{K(t),\, t \in \mathbb{R}\}$ be an additive process on $(-\infty, \infty)$ with the generating triplet $(\sigma^2(t), \gamma(t), \nu(t,\cdot))$, $t \in \mathbb{R}$, i.e. with Lévy–Khintchine representation

$$\log\Big(\mathsf{E}\, e^{iuK(t)}\Big) = -\frac{\sigma^2(t)}{2}\, u^2 \,+\, i\gamma(t)u \,+ \int_{-\infty}^{\infty}\Big(e^{iux} - 1 - iuxc(x)\Big)\nu(t, dx), \quad u \in \mathbb{R}$$

were $c(x)$ is any fixed truncation function. Then $\{K(t),\, t \in \mathbb{R}\}$ is called a *natural additive process* if the location parameter $\gamma(t)$, $t \in \mathbb{R}$, is of locally bounded variation (equivalently, if $\{K(t),\, t \in \mathbb{R}\}$ is a semimartingale).

**2.5.4 Notation.** Let $b \in \mathbb{R}$, $f : (-\infty, b] \to \mathbb{R}$ be a locally bounded measurable function and $\{K(t),\, t \in \mathbb{R}\}$ be a natural additive process. Then

$$\int_{-\infty}^{b} f(t)\, dK(t) \doteq \lim_{a \to -\infty} \int_{a}^{b} f(t)\, dK(t), \quad \text{a.s.},$$

if the limit a.s. exists.

**2.5.5 Definition.** (see Maejima–Sato [25]) Let $\lambda \in \mathbb{R}$ and $\{K(t),\, t \in \mathbb{R}\}$ be a natural additive process. We call a process $\{S(t),\, t \in \mathbb{R}\}$ a *wide sense Ornstein–Uhlenbeck (OU) type process* with parameter $\lambda$ and driving process $\{K(t),\, t \in \mathbb{R}\}$ if

- $$S(t) = S(0) + \lambda \int_0^t S(s)\, ds + K(t), \quad t \in \mathbb{R}, \tag{2.5.1}$$

shortly
$$dS(t) = \lambda S(t)\, dt + dK(t),$$

- where $S(0)$ and $\{K(t),\, t \geq 0\}$ are independent
- and $\{S(t),\, t \in \mathbb{R}\}$ has a.s. càdlàg paths.

We call a wide sense OU type process $\{S(t),\, t \in \mathbb{R}\}$ *non-trivial* if it really

depends on $t$, i.e. $S \not\equiv S(0)$ (equivalently, if its parameter $\lambda$ and driving process $\{K(t),\, t \geq 0\}$ are not simultaneously zero).
A wide sense OU type process is called simply an *OU type process* if its driving process is a two-sided Lévy process.

**2.5.6 Remark.** Wide sense OU type processes are infinitely divisible, since natural additive processes are so.

**2.5.7 Notation.** $\mathcal{I}_{\mathsf{wsOU}\lambda} \doteq \{\text{wide sense OU type processes with parameter } \lambda\}$, $\mathcal{I}_{\mathsf{OU}\lambda} \doteq \{\text{OU type processes with parameter } \lambda\}$, $\lambda \in \mathbb{R}$.

**2.5.8 Remark.** In Definition 2.5.5 the càdlàg property is assumed to ensure the a.s. local integrability of $\{S(t),\, t \geq 0\}$ in (2.5.1). Since by (2.5.1) process $\{K(t),\, t \geq 0\}$ inherits the càdlàg property from $\{S(t),\, t \geq 0\}$, this property had to be assumed in Definition 2.5.1, too.

**2.5.9 Remark.** For any $\lambda \in \mathbb{R}$, natural additive process $\{K(t),\, t \in \mathbb{R}\}$ and random variable $\xi$ independent of $\{K(t),\, t \geq 0\}$, process

$$S(t) = e^{\lambda t} \left( \xi + \int_0^t e^{-\lambda s} dK(s) \right), \quad t \in \mathbb{R},$$

is the unique wide sense OU type process with parameter $\lambda$, driving process $\{K(t),\, t \in \mathbb{R}\}$ and initial value $S(0) = \xi$. This can be proved using integration by parts, see Maejima–Sato [25, Thm. 4.1].

**2.5.10 Notation.** For a random variable $\xi$ with finite mean let $\xi_c \doteq \xi - \mathsf{E}\xi$ (the *centered* random variable).

The following theorem characterizes translatively stable non-trivial wide sense OU type processes.

**2.5.11 Theorem.** *1) i) There exists a $\delta$-translatively stable non-trivial wide sense OU type process with parameter $\lambda$ and driving process $\{K(t),\, t \in \mathbb{R}\}$ if and only if the following conditions (ii–iv) are fulfilled:*

*ii)* $$\lambda \leq 0;$$

*iii)* $$\delta > 2\lambda;$$

*iv)* $\{K(t),\, t\in\mathbb{R}\}$ *is the time-changed two-sided Lévy process*

$$K(t) = L\left(\frac{e^{\delta t}-1}{e^{\delta}-1}\right), \quad t\in\mathbb{R}, \tag{2.5.2}$$

*(for* $\delta = 0$ *the time-change function is* $\lim_{\delta\to 0}\big((e^{\delta t}-1)/(e^{\delta}-1)\big) = t,\ t\geq 0$*),*

$$L(t) = \begin{cases} L_1(t) & \text{if } t\geq 0, \\ -L_2(-t) & \text{if } t<0, \end{cases} \tag{2.5.3}$$

*where* $\{L_i(t),\, t\geq 0\},\ i=1,2,$ *are independent Lévy processes,* $L_1(1)\sim L_2(1)$ *is non-Gaussian with finite moments of all orders and* $\mathsf{E}L(1) = 0,$ *if* $\delta = \lambda \neq 0.$

*2) Under the condidions (i) or (ii–iv) a stochastic process* $\{S(t),\, t\in\mathbb{R}\}$

*v) is a* $\delta$*-translatively stable non-trivial wide sense OU type process with parameter* $\lambda$ *and driving process* $\{K(t),\, t\in\mathbb{R}\}$

*if and only if the following conditions (vi–viii) are fulfilled:*

*vi)* $$S_c(t) = \int_{-\infty}^{t} e^{\lambda(t-s)}\,dK_c(s), \quad \text{a.s.},\ t\in\mathbb{R};$$

*vii)* $$\mathsf{E}\,S(t) = \mathsf{E}\,S(0)\,e^{\delta t}, \quad t\in\mathbb{R};$$

*viii)* $$\mathsf{E}\,S(0) = \frac{\delta}{(e^{\delta}-1)\,(\delta-\lambda)}\,\mathsf{E}\,L(1) \quad \textit{if } \delta\neq\lambda$$

*(for* $\delta = 0\neq\lambda:\ \mathsf{E}S(0) = \lim_{\delta\to 0}\mathsf{E}S(0) \equiv -\mathsf{E}L(1)/\lambda$*).*

**Proof.** The structure of the proof corresponds to the following scheme:

$$\text{(i)} \Longrightarrow \big(\text{(iv)}, \text{(ii)}, \text{(iii)}\big)$$

$$\big\{\text{(ii)}, \text{(iii)}, \text{(iv)}, \text{(v)}\big\} \Longrightarrow \big(\text{(vi)}, \text{(vii)}, \text{(viii)}\big)$$

$$\big\{\text{(ii)}, \text{(iii)}, \text{(iv)}, \text{(vi)}, \text{(vii)}, \text{(viii)}\big\} \Longrightarrow \text{(v)}$$

$$\big\{\text{(ii)}, \text{(iii)}, \text{(iv)}\big\} \Longrightarrow \text{(i)}$$

(round brackets denote also the order).

*Proof of* (i) $\Longrightarrow$ $\big($(iv), (ii), (iii)$\big)$: At first we prove (iv). By (2.5.1) and the $\delta$-translative stability we have

$$K(t+T)-K(T)=S(t+T)-S(T)-\lambda\int_T^{t+T} S(s)\,ds$$

$$\overset{\text{1d}}{\sim}\left(S(t)-S(0)-\lambda\int_0^t S(s)\right)^{\circledast\, e^{\delta T}}=\big(K(t)\big)^{\circledast\, e^{\delta T}},\quad t,\,T\in\mathbb{R}.$$

Making the substitution $T=nt$ in the above relation and summing for $n=0,\dots,N-1,$ because of the independence of the increments we obtain

$$K(Nt)\overset{\text{1d}}{\sim}K(t)^{\circledast\,\frac{e^{\delta Nt}-1}{e^{\delta}-1}},\quad t\in\mathbb{R},\ \ N=1,2,\dots. \tag{2.5.4}$$

At first for rational arguments then, due to the stochastic continuity of $\{K(t),\,t\in\mathbb{R}\}$, for any $t\in\mathbb{R}$, we get from (2.5.4) that

$$K(t)\overset{\text{1d}}{\sim}\begin{cases}K(1)^{\circledast\,\frac{e^{\delta t}-1}{e^{\delta}-1}} & \text{if } t\ge 0,\\[2ex] K(-1)^{\circledast\,\frac{e^{\delta t}-1}{e^{\delta}-1}} & \text{if } t<0.\end{cases} \tag{2.5.5}$$

- If $\delta>0$, let

$$L_1(t)\doteq K\left(\frac{1}{\delta}\log\left(1+\left(e^{\delta}-1\right)t\right)\right),\quad 0\le t,$$

$$L_2(t)\doteq -K\left(\frac{1}{\delta}\log\left(1-\left(e^{\delta}-1\right)t\right)\right),\quad 0\le t<\frac{1}{e^{\delta}-1},$$

- if $\delta<0$, let

$$L_1(t)\doteq K\left(\frac{1}{\delta}\log\left(1+\left(e^{\delta}-1\right)t\right)\right),\quad 0\le t<\frac{1}{1-e^{\delta}},$$

$$L_2(t)\doteq -K\left(\frac{1}{\delta}\log\left(1-\left(e^{\delta}-1\right)t\right)\right),\quad 0\le t,$$

• and if $\delta = 0$, let

$$L_1(t) \doteq K(t), \quad 0 \le t,$$

$$L_2(t) \doteq -K(-t), \quad 0 \le t.$$

In each of the three cases $\{L_i(t),\, t \ge 0\}$, $i = 1, 2$, are independent processes (since, $\{K(t),\, t \in \mathbb{R}\}$ is of independent increments), they are a.s. càdlàg (since so is $\{K(t),\, t \in \mathbb{R}\}$) and using (2.5.5), one can check that they are Lévy processes over the intervals given. Let us extend those which are defined only on finite intervals, to $[0, \infty)$ and let $\{L(t),\, t \in \mathbb{R}\}$ be the two-sided Lévy process given by (2.5.3). Then, in each of the three cases above we obtain (2.5.2) by resubstitution. Moreover, since $\{S(t),\, t \in \mathbb{R}\} \in \widetilde{\mathcal{I}}$, it follows from (2.5.1) and (2.5.2) that $L(1) = K(1)$ has finite moments of all orders. By Remark 2.5.9 we have

$$S(t) = e^{\lambda t}\left(S(0) + \int_0^t e^{-\lambda s} dK(s)\right), \quad t \in \mathbb{R}, \tag{2.5.6}$$

where for every $t \ge 0$ : $S(0)$ and the integral in (2.5.6) are independent. Let $2 \le m \in \mathbb{N}$ be arbitrary and let us take the cumulants of order $2m$ in (2.5.6) for $t \ge 0$:

$$\mathsf{cum}_{2m}(S(t)) = e^{2m\lambda t}\left(\mathsf{cum}_{2m}(S(0)) + \mathsf{cum}_{2m}\left(\int_0^t e^{-\lambda s} dK(s)\right)\right), \quad t \ge 0.$$

We show indirectly that $L(1)$ is non-Gaussian. Assuming that it is Gaussian, it follows that $\{K(t),\, t \in \mathbb{R}\}$ and the integral in (2.5.6) are also Gaussian. Therefore we obtain that

$$\mathsf{cum}_{2m}(S(t)) = e^{2m\lambda t}\mathsf{cum}_{2m}(S(0)), \quad t \ge 0,$$

hence, by the $\delta$-translative stability of $\{S(t),\, t \in \mathbb{R}\}$,

$$e^{\delta t} = e^{2m\lambda t}, \quad t \ge 0,$$

and since $2 \le m$ was arbitrary, $\lambda = \delta = 0$ follows. Using (2.5.1) and the $\delta$-translative stability, we get $S(0) \sim S(0) + L(t)$, where $S(0)$ and $L(t)$ are independent, $t \ge 0$. So $S \equiv S(0)$, which is a contradiction, because $\{S(t),\, t \in \mathbb{R}\}$ is non-trivial. Thus we have proved that $L(1)$ is non-Gaussian. If $\delta = \lambda \ne 0$, then taking the expectation in (2.5.6) and using the $\delta$-translative stability of $\{S(t),\, t \in \mathbb{R}\}$, then using (2.5.2), we obtain that

$$e^{\lambda t}\mathsf{E}\, S(0) = \mathsf{E}\, S(t) = e^{\lambda t}\left(\mathsf{E}\, S(0) + \int_0^t e^{-\lambda s} d\, \mathsf{E}\, K(s)\right)$$

$$= e^{\lambda t}\left(\mathsf{E}\, S(0) + \frac{\lambda}{e^{\lambda}-1}\, \mathsf{E}\, L(1) t\right), \quad t \in \mathbb{R}, \qquad (2.5.7)$$

thus $\mathsf{E} L(1) = 0$ and we have finished the proof of (iv).

Next we prove (ii) and (iii) simultaneously. Both $S(0)$ and $L(1)$ have non-Gaussian infinitely divisible distributions, therefore $\mathsf{cum}_{2m}(S(0)) > 0$ and $\mathsf{cum}_{2m}(L(1)) > 0$ for every even orders $2m$. Let $m \in \mathbb{N}$ be arbitrary and let us take the cumulants of order $2m$ in (2.5.6) for $t \geq 0$:

$$\mathsf{cum}_{2m}(S(t))$$

$$= e^{2m\lambda t}\left(\mathsf{cum}_{2m}(S(0)) + \mathsf{cum}_{2m}\left(\int_0^t e^{-\lambda s} dK(s)\right)\right)$$

$$= e^{2m\lambda t}\left(\mathsf{cum}_{2m}(S(0)) + \int_0^t e^{-2m\lambda s} d\left(\frac{e^{\delta s}-1}{e^{\delta}-1}\right) \mathsf{cum}_{2m}(L(1))\right)$$

$$= e^{2m\lambda t}\left(\mathsf{cum}_{2m}(S(0)) + \frac{\delta}{(\delta - 2m\lambda)\left(e^{\delta}-1\right)}\, \mathsf{cum}_{2m}(L(1))\left(e^{(\delta-2m\lambda)t} - 1\right)\right),$$

$$t \geq 0,$$

if $\delta \neq 2m\lambda$. Hence, by the $\delta$-translative stability of $\{S(t),\, t \in \mathbb{R}\}$ it follows that

$$\left(e^{\delta t} - e^{2m\lambda t}\right)\left(1 - \frac{\mathsf{cum}_{2m}(L(1))}{\mathsf{cum}_{2m}(S(0))}\, \frac{\delta}{\left(e^{\delta}-1\right)(\delta - 2m\lambda)}\right) = 0, \quad t \geq 0. \qquad (2.5.8)$$

If $\delta = 2m\lambda$, then instead of (2.5.8) we get

$$0 = \frac{\mathsf{cum}_{2m}(L(1))}{\mathsf{cum}_{2m}(S(0))}\, \frac{\delta}{e^{\delta}-1}\, t, \quad t > 0,$$

which is a contradiction, because $\mathsf{cum}_{2m}(L(1)) \neq 0$. Thus we have $\delta \neq 2m\lambda$ and (2.5.8) yields

$$\mathsf{cum}_{2m}(S(0)) = \mathsf{cum}_{2m}(L(1)) \frac{\delta}{(\delta - 2m\lambda)\,(e^{\delta} - 1)}\,,$$

implying that $\delta > 2m\lambda$. Since this is true for arbitrarily large values of $m$, (ii) follows. Taking $m = 1$ in the last inequality, we obtain (iii).

*Proof of* $\big\{(\mathrm{ii}), (\mathrm{iii}), (\mathrm{iv}), (\mathrm{v})\big\} \Longrightarrow \big((\mathrm{vi}), (\mathrm{vii}), (\mathrm{viii})\big)$: By (2.5.6) we have

$$S_c(t) = e^{\lambda(t-r)} S_c(r) + \int\limits_r^t e^{\lambda(t-s)}\, dK_c(s), \quad r, t \in \mathbb{R},$$

where

$$\mathsf{E}\left(e^{\lambda(t-r)} S_c(r)\right)^2 = e^{2\lambda(t-r)} \mathsf{D}^2 S(r) = e^{2\lambda t + (\delta - 2\lambda) r} \mathsf{D}^2 S(0) \underset{r \to -\infty}{\longrightarrow} 0,$$

because of the $\delta$-translative stability of $\{S(t),\, t \in \mathbb{R}\}$ and (iii). Therefore

$$S_c(t) = \lim_{r \to -\infty} \int\limits_r^t e^{\lambda(t-s)}\, dK_c(s) = \int\limits_{-\infty}^t e^{\lambda(t-s)}\, dK_c(s), \quad t \in \mathbb{R},$$

where the convergence is meant in the $L^2$ sense and so, in the a.s. sense as well (since the integral can be written as an independent sum, because of the independence of the increments of $\{K(t),\, t \in \mathbb{R}\}$). Thus we have obtained (vi).

(vii) follows from the $\delta$-translative stability of $\{S(t),\, t \in \mathbb{R}\}$.

Let us show (viii). Taking the expectation in (2.5.6) and using the $\delta$-translative stability of $\{S(t),\, t \in \mathbb{R}\}$, we obtain that

$$e^{\delta t} \mathsf{E}\, S(0) = \mathsf{E}\, S(t) = e^{\lambda t} \left( \mathsf{E}\, S(0) + \int\limits_0^t e^{-\lambda s}\, d\, \mathsf{E}\, K(s) \right), \quad t \in \mathbb{R}.$$

Hence, using (2.5.2), we have

$$e^{\delta t} \mathsf{E}\, S(0) = e^{\lambda t} \left( \mathsf{E}\, S(0) + \frac{\delta}{(e^{\delta} - 1)\,(\delta - \lambda)}\, \mathsf{E}\, L(1) \left( e^{(\delta - \lambda) t} - 1 \right) \right), \quad t \in \mathbb{R}, \tag{2.5.9}$$

if $\delta \neq \lambda,$ so

$$\left(e^{\delta t} - e^{\lambda t}\right)\left(\mathsf{E}\,S(0) - \frac{\delta}{(e^{\delta}-1)\,(\delta-\lambda)}\,\mathsf{E}\,L(1)\right) = 0, \quad t \in \mathbb{R},$$

from which (viii) follows.

*Proof of* $\{(\mathrm{ii}),(\mathrm{iii}),(\mathrm{iv}),(\mathrm{vi}),(\mathrm{vii}),(\mathrm{viii})\} \Longrightarrow (\mathrm{v})$: We have

$$\begin{aligned}
&\Big(K_c(s_2) - K_c(s_1),\, K_c(s_3) - K_c(s_2)\Big)^{\circledast e^{\delta T}} \\
&= \left(L_c\left(\frac{e^{\delta s_2}-1}{e^{\delta}-1}\right) - L_c\left(\frac{e^{\delta s_1}-1}{e^{\delta}-1}\right),\; L_c\left(\frac{e^{\delta s_3}-1}{e^{\delta}-1}\right) - L_c\left(\frac{e^{\delta s_2}-1}{e^{\delta}-1}\right)\right)^{\circledast e^{\delta T}} \\
&\sim \left(L_c\left(e^{\delta T}\frac{e^{\delta s_2}-1}{e^{\delta}-1}\right) - L_c\left(e^{\delta T}\frac{e^{\delta s_1}-1}{e^{\delta}-1}\right),\; L_c\left(e^{\delta T}\frac{e^{\delta s_3}-1}{e^{\delta}-1}\right) - L_c\left(e^{\delta T}\frac{e^{\delta s_2}-1}{e^{\delta}-1}\right)\right) \\
&\sim \left(L_c\left(\frac{e^{\delta(s_2+T)}-1}{e^{\delta}-1}\right) - L_c\left(\frac{e^{\delta(s_1+T)}-1}{e^{\delta}-1}\right),\; L_c\left(\frac{e^{\delta(s_3+T)}-1}{e^{\delta}-1}\right) - L_c\left(\frac{e^{\delta(s_2+T)}-1}{e^{\delta}-1}\right)\right) \\
&\sim \Big(K_c(s_2+T) - K_c(s_1+T),\, K_c(s_3+T) - K_c(s_2+T)\Big),
\end{aligned}$$

for $s_1 \leq s_2 \leq s_3$ and $T \in \mathbb{R}$. Similarly,

$$\begin{aligned}
&\Big(K_c(s_2) - K_c(s_1),\, \ldots,\, K_c(s_m) - K_c(s_{m-1})\Big)^{\circledast e^{\delta T}} \\
&\sim \Big(K_c(s_2+T) - K_c(s_1+T),\, \ldots,\, K_c(s_m+T) - K_c(s_{m-1}+T)\Big), \qquad (2.5.10)
\end{aligned}$$

for $m = 2, 3, \ldots$ and $s_1 \leq \ldots \leq s_m$, $T \in \mathbb{R}$. Hence, for all $r, T \in \mathbb{R}$ :

$$\left(\int_r^t e^{-\lambda s} dK_c(s)\right)^{\circledast e^{\delta T}} \overset{\mathrm{fd}}{\sim} \int_r^t e^{-\lambda s} dK_c(s+T) \;=\; e^{\lambda T}\int_{r+T}^{t+T} e^{-\lambda s} dK_c(s),$$

thus, by (vi), for every $T \in \mathbb{R}$ :

$$\left(\int_{-\infty}^t e^{\lambda(t-s)} dK_c(s)\right)^{\circledast e^{\delta T}} \overset{\mathrm{fd}}{\sim} \int_{-\infty}^{t+T} e^{\lambda(t+T-s)} dK_c(s),$$

i.e., process $\{S_c(t),\, t\in\mathbb{R}\}$ is $\delta$-translatively stable. Considering (vii), we obtain that also $\{S(t),\, t\in\mathbb{R}\}$ is $\delta$-translatively stable.

Since $S(0)=S_c(0)+\mathsf{E}S(0)$ is independent of $\{K(t),\, t\geq 0\}$, by Remark 2.5.9 there exists a unique wide sense OU type process with parameter $\lambda$, driving process $\{K(t),\, t\in\mathbb{R}\}$ and initial value $S(0)$ and it is

$$V(t)=e^{\lambda t}\left(S(0)+\int\limits_0^t e^{-\lambda s}dK(s)\right),\quad t\in\mathbb{R}. \tag{2.5.11}$$

Using that

$$S(0)=S_c(0)+\mathsf{E}S(0)=\int\limits_{-\infty}^0 e^{-\lambda s}dK_c(s)+\mathsf{E}S(0),$$

moreover (iv) and (vi–viii), we obtain from (2.5.11) that

$$\begin{aligned}V(t)=e^{\lambda t}\left(\int\limits_{-\infty}^0 e^{-\lambda s}dK_c(s)+\int\limits_0^t e^{-\lambda s}dK_c(s)+e^{-\lambda t}\mathsf{E}S(t)\right)&\\ =\int\limits_{-\infty}^t e^{\lambda(t-s)}dK_c(s)+\mathsf{E}\,S(t)=S(t),\quad t\in\mathbb{R}&\end{aligned}$$

(whether $\delta=\lambda$ or not). Thus, $\{S(t),\, t\in\mathbb{R}\}$ is a wide sense OU type process with parameter $\lambda$ and driving process $\{K(t),\, t\in\mathbb{R}\}$.

Finally, if $\{S(t),\, t\in\mathbb{R}\}$ were trivial, i.e. $S(t)\equiv S(0)$, then by (2.5.1) we should obtain that $K(t)=-\lambda t S(0)$, $t\in\mathbb{R}$, which contradicts to the fact that $\{K(t),\, t\in\mathbb{R}\}$ has independent increments, unless $\lambda=0$. On the other hand, from (vii) we should obtain $\delta=0$. Therefore $\lambda=\delta=0$, but this contradicts to (iii). So, $\{S(t),\, t\in\mathbb{R}\}$ is non-trivial and we have finished the proof of (v).

*Proof of* $\{(\text{ii}),(\text{iii}),(\text{iv})\}\implies$ (i): We will prove that under the conditions (ii), (iii), (iv), there exists a stochastic process $\{S(t),\, t\in\mathbb{R}\}$, for which (vi), (vii), (viii) hold. From this (v) will follow (see the outline at the beginning of the proof), and since (v) $\implies$ (i), the proof will be complete. Now, process

$$S_0(t)\doteq\int\limits_{-\infty}^t e^{\lambda(t-s)}dK_c(s),\quad t\in\mathbb{R},$$

exists, because its mean is zero and

$$\int_{-\infty}^{t} \left(e^{\lambda(t-s)}\right)^2 \mathsf{E}(dK_c(s))^2 = e^{2\lambda t} \int_{-\infty}^{t} e^{(\delta-2\lambda)s} ds \, \frac{\delta}{e^{\delta}-1} \, \mathsf{D}^2 L(1) < \infty.$$

Let

$$S(t) \doteq S_0(t) + \mu \, e^{\delta t}, \quad t \in \mathbb{R},$$

where

$$\mu = \begin{cases} \dfrac{\delta}{(e^{\delta}-1)\,(\delta-\lambda)} \, \mathsf{E}\, L(1) & \text{if } \delta \neq \lambda, \\ \text{arbitrary} & \text{if } \delta = \lambda. \end{cases}$$

Then (vi–viii) are fulfilled for $\{S(t),\, t \in \mathbb{R}\}$. □

**2.5.12 Remark.** In Theorem 2.5.11 the mean $\mathsf{E}S(t),\, t \in \mathbb{R},$ had to be handled separately, because in general, conditions (ii–iv) do not guarantee the finiteness of the integral

$$\int_{-\infty}^{t} e^{\lambda(t-s)} d\mathsf{E}K(s) = e^{\lambda t} \int_{-\infty}^{t} e^{(\delta-\lambda)s} ds \, \frac{\delta}{e^{\delta}-1} \, \mathsf{E}L(1).$$

**2.5.13 Remark.** Let us observe that there exist $\delta$-translatively stable nontrivial wide sense OU type processes even when $\lambda = 0$. Then necessarily $\delta > 0$ and these processes are of the form

$$S(t) = K(t) - K(-\infty) = L\left(\frac{e^{\delta t}-1}{e^{\delta}-1}\right) - L\left(\frac{-1}{e^{\delta}-1}\right), \quad t \in \mathbb{R},$$

where $\{L(t),\, t \in \mathbb{R}\}$ is the two-sided Lévy process given in condition (iv) of Theorem 2.5.11.

**2.5.14 Remark.** When $\delta = 0$, the two statements of Theorem 2.5.11 simplify to the characterization of stationary OU type processes (in $\widetilde{\mathcal{I}}$):

1) There exists in $\widetilde{\mathcal{I}}$ a stationary OU type process with parameter $\lambda$ and driving process $\{K(t),\, t \in \mathbb{R}\}$ if and only if

$$\lambda < 0$$

and $\{K(t),\, t \in \mathbb{R}\}$ is the two-sided Lévy process

$$K(t) = L(t) = \begin{cases} L_1(t) & \text{if } t \geq 0, \\ -L_2(-t) & \text{if } t < 0, \end{cases} \qquad t \in \mathbb{R},$$

where $\{L_i(t),\, t \geq 0\}$, $i = 1, 2$, are independent Lévy processes and $L_1(1) \sim L_2(1)$ is non-Gaussian with finite moments of all orders.

2) Under the above two conditions with respect to $\lambda$ and $\{K(t),\, t \in \mathbb{R}\}$, a stochastic process $\{S(t),\, t \in \mathbb{R}\}$ is a stationary OU type process in $\widetilde{\mathcal{I}}$ with parameter $\lambda$ and driving process $\{K(t),\, t \in \mathbb{R}\}$ if and only if

$$S(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} dK(s), \quad \text{a.s.}, \quad t \in \mathbb{R}.$$

## 2.6 Self-similar and dilatively stable processes with independent increments

As it was presented in Section 2.4, the Lamperti transforms connect
• self-similar processes and stationary processes,
• dilatively stable processes and translatively stable processes.
In this section it will turn out that some restrictions of these transforms connect also other significant processes:
• self-similar additive processes and stationary OU type processes,
• dilatively stable additive processes and translatively stable wide sense OU type processes.

Let $\{X(t),\, t \geq 0\}$ be a self-similar additive process on $[0, \infty)$. By Sato [30] such a process exists if and only if the distribution of $X(1)$ is self-decomposable, that is, the distribution of a stationary OU type process (see Rocha-Arteaga–Sato [28, Sect. 2.4] for the latter processes). In Jeanblanc et al. [19] it turned out that just the Lamperti transforms link these two type of processes. More precisely, stationary OU type processes with parameter $-\alpha$ and $\alpha$-self-similar additive processes correspond to each other, see Figure 2.7.1. This correspondence and its dilatively stable analogue is the subject of the following theorem.

**2.6.1 Theorem.** ***(SS)*** *(Jeanblanc et al. [19]) Let $\alpha > 0$. Then $\mathcal{S}_{\mathsf{st}} \bigcap \mathcal{I}_{\mathsf{OU}-\alpha} \neq \emptyset$, $\mathcal{S}_{\alpha\mathsf{ss}} \bigcap \mathcal{I}_{\mathsf{add}} \neq \emptyset$, and the restrictions of the Lamperti transforms $L_{ST}$, $L_{SS}$ given by (2.4.2–2.4.3):*

$$L_{ST}| : \mathcal{S}_{\mathsf{st}} \bigcap \mathcal{I}_{\mathsf{OU}-\alpha} \longrightarrow \mathcal{S}_{\alpha\mathsf{ss}} \bigcap \mathcal{I}_{\mathsf{add}} \tag{2.6.1}$$

$$L_{SS}| : \mathcal{S}_{\alpha\mathsf{ss}} \bigcap \mathcal{I}_{\mathsf{add}} \longrightarrow \mathcal{S}_{\mathsf{st}} \bigcap \mathcal{I}_{\mathsf{OU}-\alpha} \tag{2.6.2}$$

*are well-defined, that is, $L_{ST}|$ maps $\mathcal{S}_{\mathsf{st}} \bigcap \mathcal{I}_{\mathsf{OU}-\alpha}$ into $\mathcal{S}_{\alpha\mathsf{ss}} \bigcap \mathcal{I}_{\mathsf{add}}$ and $L_{SS}|$ maps $\mathcal{S}_{\alpha\mathsf{ss}} \bigcap \mathcal{I}_{\mathsf{add}}$ into $\mathcal{S}_{\mathsf{st}} \bigcap \mathcal{I}_{\mathsf{OU}-\alpha}$. Moreover, they are the inverse of each other (see Figure 2.7.1.).*

***(DS)*** *Let $\alpha > 0$, $\delta \leq 2\alpha$. Then*

$$\mathcal{I}_{\delta\mathsf{ts}} \bigcap \mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})} \neq \emptyset. \tag{2.6.3}$$

*1) If $\delta \geq 0$, then*

$$\mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \bigcap \mathcal{I}_{\mathsf{add}} \neq \emptyset \tag{2.6.4}$$

*and the restrictions of the Lamperti transforms $L_{TS}$, $L_{DS}$ given by (2.4.4–2.4.5):*

$$L_{TS}| : \mathcal{I}_{\delta\mathsf{ts}} \bigcap \mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})} \longrightarrow \mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \bigcap \mathcal{I}_{\mathsf{add}} \tag{2.6.5}$$

$$L_{DS}| : \mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \bigcap \mathcal{I}_{\mathsf{add}} \longrightarrow \mathcal{I}_{\delta\mathsf{ts}} \bigcap \mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})} \tag{2.6.6}$$

*are well-defined, that is, $L_{TS}$ maps $\mathcal{I}_{\delta\mathsf{ts}} \bigcap \mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})}$ into $\mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \bigcap \mathcal{I}_{\mathsf{add}}$ and $L_{DS}$ maps $\mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \bigcap \mathcal{I}_{\mathsf{add}}$ into $\mathcal{I}_{\delta\mathsf{ts}} \bigcap \mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})}$ and $L_{TS}$ and $L_{DS}$ are the inverse of each other (see Figure 2.7.2.).*

*2) If $\delta < 0$, then the same hold as in (1) above, but with all processes of zero mean.*

**Proof.** The proof of the (SS) part can be found in Jeanblanc et al. [19], so, only the (DS) part will be treated here.

Conditions (i–ii) in Theorem 2.5.11 are fulfilled with $\lambda \doteq -(\alpha - \delta/2)$ and condition (iii) is fulfilled trivially, hence we obtain (2.6.3).

1) Let $\delta \geq 0$. By Theorem 2.4.9 $L_{TS}$ carries $\delta$-translatively stable processes into $(\alpha, \delta)$-dilatively stable ones, thus

$$L_{TS}\left(\mathcal{I}_{\delta\mathsf{ts}} \bigcap \mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})}\right) \subseteq \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}.$$

On the other hand, using the second statement of Theorem 2.5.11, processes $\{S(t),\, t\in\mathbb{R}\}\in\mathcal{I}_{\delta\mathsf{ts}}\bigcap\mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})}$ are of the form

$$S(t)=\int\limits_{-\infty}^{t} e^{-\left(\alpha-\frac{\delta}{2}\right)(t-s)}dK(s)\ +\ \mathsf{E}\,S(0)\,e^{\delta t},\quad t\in\mathbb{R},$$

with a zero mean driving process $\{K(t),\, t\in\mathbb{R}\}$ defined by condition (iv) in Theorem 2.5.11 and $\mathsf{E}S(0)$ given by condition (viii). Hence, by (2.4.4), the transformed process is

$$\big(L_{TS}(S)\big)(t)=t^{\alpha-\frac{\delta}{2}}S(\log t)=\int\limits_{-\infty}^{\log t} e^{\left(\alpha-\frac{\delta}{2}\right)s}dK(s)+\mathsf{E}\,S(0)\,t^{\alpha+\frac{\delta}{2}},\quad t>0,\quad (2.6.7)$$

$$\big(L_{TS}(S)\big)(0)=0,$$

which has independent increments.

Since the integral term on the right-hand side of (2.6.7) is $L^2$-continuous (even at $t=0$), it is stochastically continuous as well. Moreover, the second term on the right-hand side of (2.6.7) is continuous (even at $t=0$), therefore $\{(L_{TS}(S))(t),\, t\geq 0\}$ is stochastically continuous.

Since process $\{S(t),\, t\in\mathbb{R}\}$ is càdlàg, so is the left-hand side of (2.6.7). Furthermore, it is a.s. right continuous in $t=0$, because so are both terms on the right-hand side of (2.6.7). Thus, process $\{(L_{TS}(S))(t),\, t\geq 0\}$ is càdlàg.

By the above facts, $\{(L_{TS}(S))(t),\, t\geq 0\}$ is an additive process. So, we have obtained (2.6.4–2.6.5). Since by Theorem 2.4.9 $L_{TS}$ is a one-to-one mapping and $L_{TS}$ and $L_{DS}$ are the inverse of each other, what we still have to prove, is that $L_{TS}|$ is surjective, or equivalently, that (2.6.6) holds. Since by Theorem 2.4.9 $L_{DS}$ carries $(\alpha,\delta)$-dilatively stable processes into $\delta$-translatively stable processes, we have

$$L_{DS}\left(\mathcal{I}_{(\alpha,\delta)\mathsf{ds}}\bigcap\mathcal{I}_{\mathsf{add}}\right)\subseteq\mathcal{I}_{\delta\mathsf{ts}}.$$

On the other hand, if $\{X(t),\, t\geq 0\}\in\mathcal{I}_{(\alpha,\delta)\mathsf{ds}}\bigcap\mathcal{I}_{\mathsf{add}}$, then by (2.4.5), the transformed process is

$$S(t)\doteq\big(L_{DS}(X)\big)(t)=e^{-\left(\alpha-\frac{\delta}{2}\right)t}X(e^t),\quad t\in\mathbb{R}.\qquad (2.6.8)$$

Let

$$K(t)\doteq S(t)-S(0)+\left(\alpha-\frac{\delta}{2}\right)\int\limits_{0}^{t}S(s)\,ds,\quad t\in\mathbb{R}.\qquad (2.6.9)$$

The additive process $\{X(t), t \geq 0\}$ is natural, since

$$\mathsf{E}\, X(t) = t^{\alpha+\frac{\delta}{2}}\, \mathsf{E}\, X(1), \quad t \geq 0,$$

is of locally bounded variation. Therefore process $\{X(e^t), t \in \mathbb{R}\}$ is also a natural additive process, so, one can integrate with respect to it. Integrating by parts (see Sato [31, Cor. 4.9]) the integral term in (2.6.9) and using (2.6.8) we obtain

$$\begin{aligned}\left(\alpha - \frac{\delta}{2}\right) \int_0^t S(s)\, ds &= -\int_0^t \left(e^{-(\alpha-\frac{\delta}{2})s}\right)' e^{(\alpha-\frac{\delta}{2})s} S(s)\, ds \\ &= -S(t) + S(0) + \int_0^t e^{-(\alpha-\frac{\delta}{2})s}\, dX(e^s)\,, \qquad t \in \mathbb{R},\end{aligned}$$

hence

$$K(t) = \int_0^t e^{-(\alpha-\frac{\delta}{2})s}\, dX(e^s)\,, \quad t \in \mathbb{R},$$

which is of independent increments. It is also stochastically continuous (since so is process $\{X(e^t), t \in \mathbb{R}\}$) and càdlàg (by (2.6.9) and since process $\{S(t), t \in \mathbb{R}\}$ is càdlàg). So, $\{K(t), t \in \mathbb{R}\}$ is an additive process. It is also natural, since

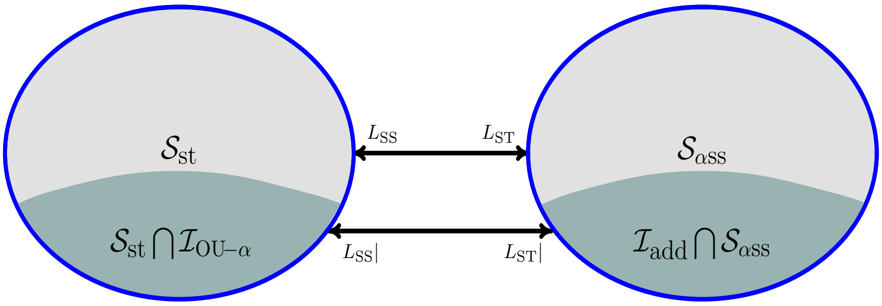
$$\begin{aligned}\mathsf{E}\, K(t) = \int_0^t e^{-(\alpha-\frac{\delta}{2})s}\, d\,\mathsf{E} X(e^s) &= \int_0^t e^{-(\alpha-\frac{\delta}{2})s}\, de^{(\alpha+\frac{\delta}{2})s}\, \mathsf{E}\, X(1) \\ &= \left(\alpha + \frac{\delta}{2}\right) \mathsf{E}\, X(1) \frac{e^{\delta t} - 1}{\delta}, \quad t \in \mathbb{R},\end{aligned}$$

is of locally bounded variation. Therefore, by (2.6.9), $\{S(t), t \in \mathbb{R}\}$ is a wide sense OU type process with parameter $-(\alpha - \delta/2)$, which means that we have proved that

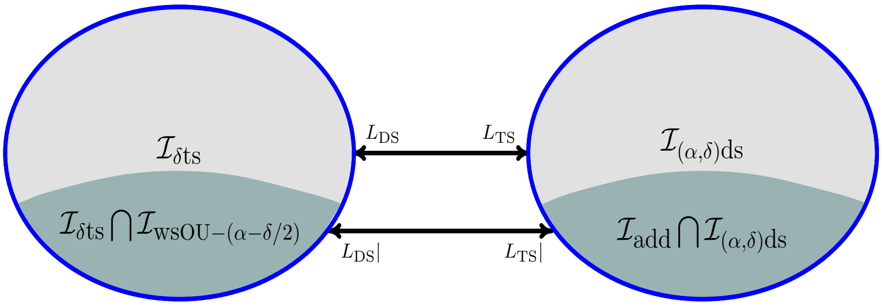
$$L_{DS}\left(\mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \bigcap \mathcal{I}_{\mathsf{add}}\right) \subseteq \mathcal{I}_{\mathsf{wsOU}-(\alpha-\frac{\delta}{2})},$$

hence (2.6.6) holds.

2) Let $\delta < 0$ and let all the processes have zero mean. Going through the steps of the above proof and adding to them that every process is of zero mean, each statement remains valid, or becomes needless. $\square$




**2.6.1 Figure.** The Lamperti transforms (2.4.2–2.4.3) and their restrictions (2.6.1–2.6.2)




**2.6.2 Figure.** The Lamperti transforms (2.4.4–2.4.5) and their restrictions (2.6.5–2.6.6). The latter (bottom) correspondence holds for $\delta \geq 0$ or for processes with zero mean.

## 2.7 Self-similar and dilatively stable processes with stationary increments

**2.7.1 Theorem.** ***(SS)*** *Let* $\alpha > 0$, $\{X(t),\, t \geq 0\}$ *a process with stationary increments and* $\mathsf{E}X^2(1) < \infty$.
*1) If* $\{X(t),\, t \geq 0\}$ *is* $\alpha$*-self-similar, then* $\alpha \leq 1$.
*2) If* $\{X(t),\, t \geq 0\}$ *is* 1*-self-similar, then* $X(t) = tX(1)$ *a.s. (degenerate).*
*3) If* $\{X(t),\, t \geq 0\}$ *is* $\alpha$*-self-similar, where* $\alpha < 1$, *then* $\mathsf{E}X(t) \equiv 0$.

***(DS)*** *Let* $\alpha > 0$, $\delta \le 2\alpha$ *and let* $\{X(t),\, t \ge 0\}$ *be a process with stationary increments.*

*1) If* $\{X(t),\, t \ge 0\}$ *is* $(\alpha,\delta)$*-dilatively stable, then* $\alpha \le 1$.

*2) If* $\{X(t),\, t \ge 0\}$ *is* $(1,\delta)$*-dilatively stable, then* $X(t) = tX(1)$ *a.s. (degenerate).*

*3) If* $\{X(t),\, t \ge 0\}$ *is* $(\alpha,\delta)$*-dilatively stable, where* $\alpha + \frac{\delta}{2} \ne 1$, *then* $\mathsf{E}X(t) \equiv 0$.

**Proof.** We prove only the dilatively stable case, the self-similar one is simpler.

1) By the stationarity of the increments we have

$$\begin{aligned} c_2(t+s) &= c_2(t) + c_2(s) + 2\mathsf{Cov}(X(t), X(t+s) - X(t)) \\ &\le c_2(t) + c_2(s) + 2\sqrt{c_2(t)}\sqrt{c_2(s)} = \left(\sqrt{c_2(t)} + \sqrt{c_2(s)}\right)^2. \end{aligned} \tag{2.7.1}$$

In the case of $(\alpha,\delta)$-dilative stability $c_2(t) = t^{2\alpha}c_2(1)$, thus the first statement of the theorem follows from (2.7.1) with $t = s \ne 0$.

2) If $X(t)$ is $(1,\delta)$-dilatively stable, then $c_2(t) = t^2 c_2(1)$. On the other hand the stationarity of the increments yields $c_1(t) = tc_1(1)$ and

$$\mathsf{Cov}(X(t), X(s)) = \frac{1}{2}\Big(c_2(t) + c_2(s) - c_2(|t-s|)\Big).$$

Hence

$$\begin{aligned} E\big(X(t) - tX(1)\big)^2 &= E\Big(\big(X(t) - c_1(t)\big) - \big(tX(1) - tc_1(1)\big)\Big)^2 \\ &= c_2(t) + t^2 c_2(1) - 2\,t\,\mathsf{Cov}\big(X(t), X(1)\big) \end{aligned}$$

$$\begin{aligned} &= c_2(t) + t^2 c_2(1) - t\Big(c_2(t) + c_2(1) - c_2(|t-1|)\Big) \\ &= t^2 c_2(1) + t^2 c_2(1) - t^3 c_2(1) - tc_2(1) + t(t-1)^2 c_2(1) \\ &= 0, \end{aligned}$$

therefore $X(t) = tX(1)$ a.s..

3) If $\{X(t),\, t \ge 0\}$ is $(\alpha,\delta)$-dilatively stable, $\alpha + \frac{\delta}{2} \ne 1$, then $c_1(t) = tc_1(1)$ and, on the other hand, $c_1(t) = t^{\alpha + \frac{\delta}{2}} c_1(1)$. Therefore $\mathsf{E}X(t) = c_1(t) = 0$. $\square$

We note that the (SS) part of Theorem 2.7.1 holds true if we assume only $\mathsf{E}|X(1)| < \infty$ instead of $\mathsf{E}X^2(1) < \infty$, see Maejima–Sato [24] and Vervaat [37].

By Theorem 2.7.1 among non-degenerate processes with stationary increments and finite second moments self-similarity and dilative stability can occur only when the parameter domain is $0<\alpha<1$ and $0<\alpha<1$, $\delta\le 2\alpha$, resp., and then, apart from some exceptions, the mean must be zero.

**2.7.2 Theorem.** *Assume one of the following conditions to be satisfied:*
***(SS)*** $0 < \alpha < 1$ *and* $\{X(t),\, t \ge 0\}$ *is an* $\alpha$*-self-similar process with* $L^2$*-stationary increments.*
***(DS)*** $0 < \alpha < 1$, $\delta \le 2\alpha$ *and* $\{X(t),\, t \ge 0\}$ *is an* $(\alpha,\delta)$*-dilatively stable process with stationary increments.*
*Then the second order cumulant structure, i.e. the covariance function of process* $\{X(t),\, t \ge 0\}$ *is the same as that of the FBM with parameter* $H = \alpha$ *(apart from a constant factor):*

$$\begin{aligned} \mathsf{Cov}\big(X(t_1), X(t_2)\big) &= \frac{1}{2}\mathsf{D}^2X(1)\Big(t_1^{2\alpha} + t_2^{2\alpha} - |t_1 - t_2|^{2\alpha}\Big) \\ &= const. \times \mathsf{Cov}\big(B^{(\alpha)}(t_1), B^{(\alpha)}(t_2)\big). \end{aligned} \tag{2.7.2}$$

**Proof.** The proofs of the two cases are the same. Because of the stationary increments, the covariance function can be expressed in terms of the variance function: $\mathsf{Cov}\big(X(t_1),X(t_2)\big) = \frac{1}{2}\Big(\mathsf{D}^2X(t_1) + \mathsf{D}^2X(t_2) - \mathsf{D}^2X(|t_1-t_2|)\Big)$. By the self-similarity or the dilative stability we have $\mathsf{D}^2X(t) = t^{2\alpha}\mathsf{D}^2X(1)$, hence we obtain (2.7.2). □

By the previous theorem in the stationary increment non-degenerate case parameter $\alpha$ has the role of a memory parameter. For this reason we denote it by $H$, i.e. $\alpha = H$ and

$$0 < H < 1, \quad \delta \le 2H. \tag{2.7.3}$$

The discrete time increment process of the FBM is

$$X_k^{(H)} \doteq B^{(H)}(k) - B^{(H)}(k-1), \quad k = 1, 2, \ldots,$$

known as the fractional Gaussian noise (FGN). It is a long memory (or long-range dependent) process if and only if $1/2 < H < 1$. Then we call also FBM itself a long memory process. Similarly, we call a process with $L^2$-stationary increments to be long memory if its discrete time increment process is long memory. The next corollary is the consequence of Theorem 2.7.2.

**2.7.3 Corollary.** ***(SS)*** *A non-degenerate $H$-self-similar process with stationary increments is long memory if and only if $1/2 < H < 1$.*

***(DS)*** *A non-degenerate $(H, \delta)$-dilatively stable process with stationary increments is long memory if and only if*

$$1/2 < H < 1, \quad \delta \leq 2H. \tag{2.7.4}$$

For every $0 < H < 1$ there exists a $H$-self-similar process with stationary increments: e.g. FBM with parameter $H$. It is open whether or not the dilatively stable counterpart is true: given a pair $(H, \delta)$ in domain (2.7.3) does there exist a $(H, \delta)$-dilatively stable process with stationary increments? So far only a few remarkable dilatively stable processes with stationary increments are known. They are given in Table 2.7.1:

| | | |
|---|---|---|
| FLP, see Example 2.1.7 (DS) | $\frac{1}{2} < H < 1$ | $\delta = 1$ |
| LISOU process, see Theorem 3.3.4 | $\frac{1}{2} < H < 1$ | $\delta = 2H - 2$ |
| LISCBI process, see Theorem 4.5.4 (LISDLG process, see Theorem 4.6.19) | $\frac{1}{2} < H < 1$ | $\delta = 2H - 2$ |

**2.7.1 Table.** Remarkable dilatively stable processes with stationary increments

The following proposition shows how to obtain further self-similar and dilatively stable processes from given ones, preserving the parameter of self-similarity and dilative stability, respectively. It indicates the corresponding semi-group structures as well. The proofs are obvious.

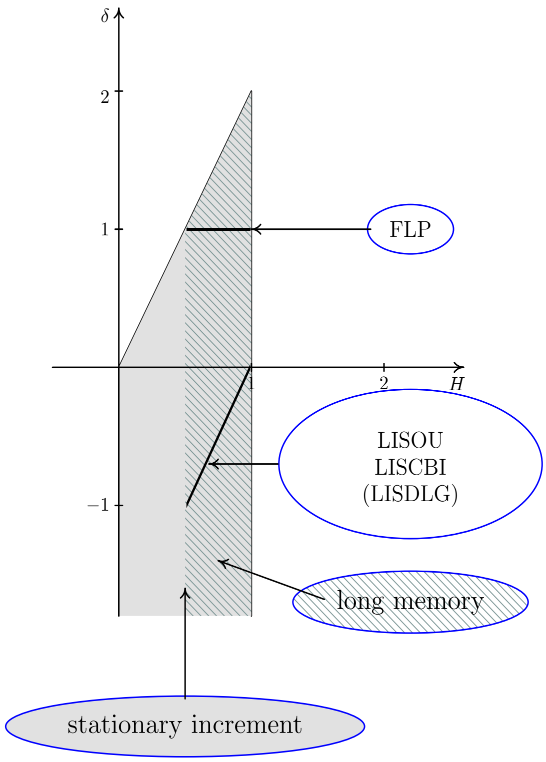
**2.7.4 Proposition.** ***(SS)*** *The sum of independent (possibly stationary increment or long memory) $\alpha$-self-similar processes is also a (stationary increment or long memory, resp.) $\alpha$-self-similar process.*

***(DS)*** *The sum of independent (possibly stationary increment or long memory) $(\alpha, \delta)$-dilatively stable processes is also a (stationary increment or long memory, resp.) $(\alpha, \delta)$-dilatively stable process.*

Another way to obtain a new dilatively stable process is the following one.

**2.7.5 Proposition.** *The sum of a (possibly stationary increment or long memory) $(H, \delta)$-dilatively stable process and an independent FBM with parameter $H$ is also a (stationary increment or long memory, resp.) $(H, \delta)$-dilatively stable process.*

The reason for the statement of Proposition 2.7.5 is that the FBM could be dilatively stable with any $\delta \in \mathbb{R}$, see the explanation after Proposition 2.1.4.




**2.7.1 Figure.** The dilatively stable, stationary increment parameter domain

**2.7.6 Remark.** It is well-known that a discrete time process has long memory with parameter $1/2 < H < 1$, if and only if its spectrum is $-2h$-regularly varying in zero, i.e. $s(f) = f^{-2h}\ell(f)$ $(0 < f < 2\pi$ is the frequency), where $h \doteq H - 1/2$. The so-called „1/f spectrum" phenomenon occurs if $h \approx 1$, and this is so in most cases of long memory timeseries arising from nature and practical applications. Now, applying Proposition 2.2.13 for processes with stationary increments, we obtain an explanation of this phenomenon.

## 2.8 Self-similar and dilatively stable renormalization

**2.8.1 Definition.** **(SS)** Let $\alpha > 0$ and let us define on $\mathcal{S}$ operators $A_T^{(\alpha)}$, $T > 0$:

$$A_T^{(\alpha)} : \mathcal{S} \longrightarrow \mathcal{S}, \quad X \mapsto A_T^{(\alpha)} X,$$
$$\left(A_T^{(\alpha)} X\right)(t) \doteq T^{-\alpha} X(Tt), \qquad t \geq 0.$$

Operators $A_T^{(\alpha)}$ are called *renormalization operators.*

**(DS)** Let $\alpha \geq 0$, $\delta \leq 2\alpha$ and let us define on $\mathcal{I}$ operators $A_T^{(\alpha,\delta)}$:

$$A_T^{(\alpha,\delta)} : \mathcal{I} \longrightarrow \mathcal{I}, \quad X \mapsto A_T^{(\alpha,\delta)} X,$$
$$\left(A_T^{(\alpha,\delta)} X\right)(t) \doteq T^{-\left(\alpha - \frac{\delta}{2}\right)} X^{\circledast T^{-\delta}}(Tt), \qquad t \geq 0.$$

Operators $A_T^{(\alpha,\delta)}$ are called *renormalization operators.*

**2.8.2 Remark.** **(SS)** The family $\mathcal{F}^{(\alpha)} \doteq \left\{A_T^{(\alpha)} : T > 0\right\}$ with the composition operation

$$\left(A_T^{(\alpha)} \circ A_S^{(\alpha)}\right) X(t) \doteq A_T^{(\alpha)} \left(A_S^{(\alpha)} X\right)(t), \quad S, T > 0,\ X \in \mathcal{S},$$

is a semigroup, where $A_T^{(\alpha)} \circ A_S^{(\alpha)} = A_{TS}^{(\alpha)}$ for all $T, S > 0$. (In fact, it is a group, but this is of no consequence.)

$\mathcal{F}^{(\alpha)}$ is called a renormalization semigroup with index $\alpha$. Its discrete time equivalent was first defined in Sinai [34].

**(DS)** The family $\mathcal{F}^{(\alpha,\delta)} \doteq \left\{A_T^{(\alpha,\delta)} : T > 0\right\}$ with the composition operation

$$\left(A_T^{(\alpha,\delta)} \circ A_S^{(\alpha,\delta)}\right) X(t) \doteq A_T^{(\alpha,\delta)} \left(A_S^{(\alpha,\delta)} X\right)(t), \quad S, T > 0,\ X \in \mathcal{I},$$

is a semigroup, where $A_T^{(\alpha,\delta)} \circ A_S^{(\alpha,\delta)} = A_{TS}^{(\alpha,\delta)}$ for all $T, S > 0$. $\mathcal{F}^{(\alpha,\delta)}$ is called a renormalization semigroup with index $(\alpha, \delta)$. (In fact, it is a group, but this is of no consequence.)

Given an $\alpha$-self-similar process, we will call $A_T^{(\alpha)}$ and $\mathcal{F}^{(\alpha)}$ the renormalization operator and the renormalization semigroup also of the process itself. Analogously, in the dilatively stable case we will call them the renormalization operator and the renormalization semigroup of the process.

**2.8.3 Example. (SS)** The renormalization operator of the FBM with parameter $0 < H < 1$ is $A_T^{(H)}$.

**(DS)** The renormalization operator of the FLP with parameter $1/2 < H < 1$ is $A_T^{(H,1)}$, see Example 2.1.7 (DS). The LISOU process and the LISCBI (particularly the LISDLG) process with parameter $1/2 < H < 1$ have the renormalization operator $A_T^{(H,2H-2)}$ (see Remarks 3.3.6 and 4.5.6).

Using this terminology, we can formulate self-similarity and dilative stability as follows.

**2.8.4 Theorem.** ***(SS)*** *(Taqqu [35]) A process in $\mathcal{S}$ is self-similar if and only if it is a fixed point of the elements of its renormalization semigroup (briefly, a fixed point of its renormalization semigroup).*

***(DS)*** *A process in $\mathcal{I}$ is dilatively stable if and only if it is a fixed point of its renormalization semigroup.*

The (SS) part of the previous theorem explains why it is the family of renormalized processes that converges in functional limit theorems on convergence to self-similar processes. In Taqqu [35] the renormalization principle is explained in the discrete time case, and demonstrated how it leads to the renormalization functional limit theorem of Davydov [7]. Analogously, the (DS) part of Theorem 2.8.4 suggests the way of obtaining renormalization functional limit theorems on convergence to dilatively stable processes. Namely, let $\{X(t),\, t \geq 0\}$ be an $(\alpha, \delta)$-dilatively stable process. Then it is a fixed point of the renormalization semigroup $\mathcal{F}^{(\alpha,\delta)}$. So, we can expect that when we apply the renormalization operators to some suitable starting process, the limit process exists when $T \to \infty$, and it is the fixed point, i.e. $\{X(t),\, t \geq 0\}$. The particular case of Theorem 2.2.7 (DS) when the slowly varying functions are constants, suggests the same. Exactly this renormalization principle will show up (also as a particular case) in the renormalization type limit Theorems 3.3.4 and 4.5.4.

# Chapter 3

# The LISOU process

In this chapter we study the superposition of stationary Ornstein–Uhlenbeck (OU) type processes, and state a dilatively stable renormalization functional limit theorem for the integrated superposition process. OU type processes are homogeneous Markov processes with a particular property called regular affinity, defined in Duffie et al. [9] and can be outlined briefly as follows. A homogeneous Markov process is called affine if the logarithm of its transition characteristic function is an affine (i.e. linear) function of the initial state, while regularity is a technical (differentiability) assumption. Regular affine processes are exactly those regular, homogeneous Markov processes, which are infinitely decomposable (roughly, the transition probability distribution is infinitely divisible). On the other hand, infinitely decomposable processes are just those processes which are suitable for being superposed. The two types of regular affine processes: OU type processes and continuous state branching processes with immigration (CBI processes). This is the reason why in this chapter we happen to superpose OU type processes. The superposition of CBI processes will be treated in the next chapter.

## 3.1 The joint cumulants of a stationary OU type process

By Definition 2.5.5 a càdlàg process $\{X(t),\ t \in \mathbb{R}\}$ is called an Ornstein–Uhlenbeck (OU) type process with parameter[1] $\alpha \in \mathbb{R}$ and driving process $\{L(t), t \in \mathbb{R}\}$ if

$$X(t) = X(0) + \alpha \int\limits_0^t X(s)\, ds + L(t), \quad t \in \mathbb{R},$$

shortly

$$dX(t) = \alpha X(t)\, dt + dL(t),$$

where $\{L(t), t \in \mathbb{R}\}$ is a two-sided Lévy process and $X(0)$ and $\{L(t), t \geq 0\}$ are independent. By Remark 2.5.14 the OU type process $\{X(t), t \in \mathbb{R}\}$ is a stationary OU type process in $\widetilde{\mathcal{I}}$ if and only if

$$X(t) = \int\limits_{-\infty}^t e^{\alpha(t-s)} dL(s), \quad t \in \mathbb{R}, \tag{3.1.1}$$

where $\alpha < 0$ and $L(1)$ is non-Gaussian with finite moments of all orders. In this chapter the value $\alpha = -1$ will be fixed. Moreover, since we will consider only processes with non-negative time parameters, from now on it is process $\{X(t), t \geq 0\}$ which we will call the stationary OU type process in $\widetilde{\mathcal{I}}$.

Before the first lemma we remind the reader of the following fact.

**3.1.1 Remark.** If $\{X(t), t \geq 0\}$ is a stochastic process, then the joint cumulant of $(X(t_1), \ldots, X(t_m))$ of order $(k_1, \ldots, k_m)$ can be written as

$$\mathsf{cum}_{k_1,\ldots,k_m}\big(X(t_1), \ldots, X(t_m)\big) = \mathsf{cum}\big(X(t_1), \ldots, X(t_n)\big), \tag{3.1.2}$$

where

$$\big(t_1, \ldots, t_n\big) = \big(\underbrace{t_1, \ldots, t_1}_{k_1}, \ldots, \underbrace{t_m, \ldots, t_m}_{k_m}\big).$$

Therefore every cumulant (either single or joint) can be given in the form of the right-hand side of (3.1.2).

[1] Hereafter we denote the parameter of the OU type process by $\alpha$.

**3.1.2 Lemma.** *Let* $\{X(t),\, t \geq 0\} \in \widetilde{\mathcal{I}}$ *be a stationary OU type process with parameter* $\alpha = -1$ *and driving process* $\{L(t),\, t \in \mathbb{R}\}$. *Then its joint cumulants are*

$$\mathsf{cum}\big(X(t_1), \ldots, X(t_m)\big) = \frac{1}{m}\,\mathsf{cum}_m(L(1))\, e^{-(t_2^* - t_1^* + \cdots + t_m^* - t_1^*)}\,, \qquad 0 \leq t_1, \ldots, t_m, \quad m \in \mathbb{N}$$

*(for* $m = 1$ *the exponent is* $0$*).*

**Proof.** The key is the informal rule

$$\mathsf{cum}\big(L(ds_1), \ldots, L(ds_m)\big) = \delta_{s_1 = \ldots = s_m} \mathsf{cum}_m(L(1)) ds_1 \tag{3.1.3}$$

($\delta$ is the Cronecker delta) and the multilinearity (i.e. linearity in each variable) of the joint cumulant. This can be proved by the standard method, using the definition of the integral, Hölders' inequality and Remark 2.4.2. Hence, using the representation (3.1.1), we obtain the statement of the lemma. □

## 3.2 Superposition of stationary OU type processes

The construction called superposition will be the following one. Independent processes, in this chapter stationary OU type processes will be summed. Each term of the sum, i.e. each process (or more precisely, its distribution) will be some transform of a basic stationary OU type process. Each transform will consist of a convolution power and a time dilation, i.e. will be of the form

$$\big\{X(t),\, t \geq 0\big\} \longmapsto \big\{X(d_j t),\, t \geq 0\big\}^{\circledast p_j}.$$

Each dilational costant $d_j$ will occur with probability $p_j$, so $p_j$, $j \in \mathbb{N}$, will be a distribution on the discrete set $\{d_j : j \in \mathbb{N}\}$. That is, the time dilation will be a discrete random variable $\delta$, with a certain property of its distribution given in the following assumption.

**Assumption SPL.** (**s**uper**p**ositional **l**aw) Let $1/2 < H < 1$ be a parameter (serving later as the Hurst parameter), $0 < d_1, d_2, \ldots$ be a sequence of constants (serving further as time dilation constants), $0 < p_1, p_2, \ldots$ be a discrete probability distribution (serving further as the superposition weight sequence) and let $\delta$ be a positive, discrete random variable[2] with the law

$$P\big(\delta = d_i\big) = p_i, \quad i \in \mathbb{N},$$

which will be called the superposition law. Let the left tail (i.e. around zero) of the distribution function of $\delta$ be regularly varying of order $2-2H$, i.e.

$$F_\delta(x) = P\big(\delta < x\big) = x^{2-2H}\ell\left(\frac{1}{x}\right), \tag{3.2.1}$$

where $\ell$ is a slowly varying function (i.e. slowly varying at infinity).

Now we detail the assumptions concerning the OU type processes, which we will superpose.

- Let $\{L(t),\, t \in \mathbb{R}\}$ be a two-sided Lévy process with the generating triplet $(\sigma^2, \gamma, \nu)$ with respect to the following Lévy–Khintchine representation of $L(1)$:

$$\log\left(\mathsf{E}\, e^{iuL(1)}\right) = -\sigma^2 u^2 \,+\, i\gamma u \,+\, \int\limits_{-\infty}^{\infty} \big(e^{iux} - 1 - iux\big)\nu(dx), \quad u \in \mathbb{R},$$

(accordingly, $\mathsf{E}\, L(1)$ exists and $\gamma = \mathsf{E}\, L(1)$) and let $\nu \not\equiv 0$. Moreover, let $\int_{-1}^{1} x^2\nu(dx) + \int_{\mathbb{R}\setminus(-1,1)} e^{u|x|}\nu(dx) < \infty$, for some $u > 0$. Equivalently, let the characteristic function of $L(1)$ be analytic in some complex neighbourhood of zero.

- Let $\{X(t),\, t \ge 0\}$ be a stationary OU type process with parameter $\alpha = -1$ and driving process $\{L(t),\, t \in \mathbb{R}\}$ and let

$$\big\{X_j(t),\, t \ge 0\big\} \doteq \big\{X(d_j t),\, t \ge 0\big\}^{\circledast p_j}, \quad j \in \mathbb{N}. \tag{3.2.2}$$

- Let processes $\{X_j(t),\, t \ge 0\}$, $j \in \mathbb{N}$, be independent.

Hereafter we consider the above assumptions (inclusive of Assumption SPL) to be fulfilled.

[2]Not to be confused with $\delta$ in the usual notation "$(\alpha,\delta)$-dilative stability".

**3.2.1 Remark.** By Lemma 3.1.2, for each $m \in \mathbb{N}$, the cumulants $\mathsf{cum}_m(X(1))$ and $\mathsf{cum}_m(L(1))$ differ only in a factor $1/m$. Therefore the assumption that the characteristic function of $L(1)$ is analytic in some complex neighbourhood of zero, can be replaced by the same assumption with respect to the characteristic function of $X(1)$ (and, naturally, of $X_j(1)$, for any $j \in \mathbb{N}$).

**3.2.2 Remark.** For each $j \in \mathbb{N}$, process $\{X_j(t),\, t \ge 0\}$ is a stationary OU type process in $\widetilde{\mathcal{I}}$ with parameter $-d_j$ and driving process $\{L(t),\, t \in \mathbb{R}\}^{\circledast p_j d_j}$. Indeed, by Lemma 3.1.2 the two processes have the same joint cumulants and since by Remark 3.2.1 the characteristic function of $X_j(0)$ is analytic in some complex neighbourhood of zero, the finite-dimensional distributions of $\{X_j(t),\, t \ge 0\}$ are uniquely determined by its joint cumulants.

**3.2.3 Theorem.** *For each $t \ge 0$ the series $\sum_j X_j(t)$ converges in $L^2$, uniformly in $t \ge 0$ and also almost surely.*

**Proof.** Using Lemma 3.1.2 we have

$$\sum_{j=1}^{\infty} \mathsf{E}\, X_j(t) = \mathsf{E}\, L(1) \sum_{j=1}^{\infty} p_j = \mathsf{E}\, L(1)\,, \tag{3.2.3}$$

$$\sum_{j=1}^{\infty} \mathsf{D}^2 X_j(t) = \frac{1}{2}\, \mathsf{D}^2 L(1) \sum_{j=1}^{\infty} p_j = \frac{1}{2}\, \mathsf{D}^2 L(1),$$

thus the series $\sum_j X_j(t)$ converges in $L^2$, uniformly in $t \ge 0$. The a.s. convergence follows by Kolmogorov's two series theorem. □

**3.2.4 Definition.** Process

$$Y(t) \doteq \sum_{j=1}^{\infty} X_j(t)\,, \quad t \ge 0,$$

is called the *superposition of OU type processes (SOU process)* (with parameters $(H,\ell)$ and $(\sigma^2,\gamma,\nu)$).

**3.2.5 Lemma.** *The joint cumulants of the SOU process $\{Y(t),\, t \ge 0\}$ are*

$$\mathsf{cum}\big(Y(t_1),\dots,Y(t_m)\big) = \frac{1}{m}\, \mathsf{cum}_m(L(1))\, \mathsf{E}\, e^{-\delta(t_2^*-t_1^*+\cdots+t_m^*-t_1^*)}\,,$$

$$0 \le t_1,\dots,t_m, \quad m \in \mathbb{N}.$$

**Proof.** Processes $\{X_j(t),\, t \geq 0\}$, $j \in \mathbb{N}$, are independent, so, by Lemma 3.1.2 we obtain

$$
\begin{aligned}
\mathsf{cum}\big(Y(t_1), \ldots, Y(t_m)\big) &= \mathsf{cum}\left(\sum_{j=1}^{\infty} X_j(t_1), \ldots, \sum_{j=1}^{\infty} X_j(t_m)\right) \\
&= \sum_{j=1}^{\infty} \mathsf{cum}\big(X_j(t_1), \ldots, X_j(t_m)\big) \\
&= \sum_{j=1}^{\infty} p_j\, \mathsf{cum}\big(X(d_j t_1), \ldots, X(d_j t_m)\big) \qquad (3.2.4) \\
&= \frac{1}{m}\, \mathsf{cum}_m(L(1)) \sum_{j=1}^{\infty} p_j\, e^{-d_j (t_2^* - t_1^* + \cdots + t_m^* - t_1^*)} \\
&= \frac{1}{m}\, \mathsf{cum}_m(L(1))\, \mathsf{E}\, e^{-\delta (t_2^* - t_1^* + \cdots + t_m^* - t_1^*)}.
\end{aligned}
$$

□

**3.2.6 Remark.** Considering the equation between the left-hand side and the third row of (3.2.4), the cumulants of $Y(1)$ are the same as those of $X(1)$. Therefore by Remark 3.2.1 the characteristic function of $Y(1)$ is analytic in some complex neighbourhood of zero.

**3.2.7 Remark.** Assumption $\alpha = -1$ is not really a restriction, since without it we would obtain

$$
\mathsf{cum}\big(Y(t_1), \ldots, Y(t_m)\big) = \frac{1}{-\alpha m}\, \mathsf{cum}_m(L(1))\, \mathsf{E}\, e^{\alpha\delta (t_2^* - t_1^* + \cdots + t_m^* - t_1^*)}
$$

and the factor $\alpha$ can be included in $\delta$ and $L(1)$.

Examining the SOU process from the point of view of continuity, we find that by Lemma 3.2.5 both the mean function (which is the constant $\gamma$) and the covariance function are continuous. Therefore, the SOU process is $L^2$-continuous, so, it is also stochastically continuous. Hence, by Doob [8, Thm. 2.6], it has a jointly measurable modification. We will consider this modification the SOU process. Because of the joint measurability, Fubini's theorem can be applied and using also the stationarity, there follows the a.s. local Lebesgue integrability of the SOU process. The latter property is implicit in the following definition.

**3.2.8 Definition.** Let $\{Y(t),\ t \geq 0\}$ be a SOU process (with parameters $(H, \ell)$ and $(\sigma^2, \gamma, \nu)$). Then process[3]

$$J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t) \doteq \int_0^t Y(s)\, ds, \quad t \geq 0, \tag{3.2.5}$$

defined almost surely, will be called the *ISOU (integrated SOU) process* (with parameters $(H, \ell)$ and $(\sigma^2, \gamma, \nu)$).

**3.2.9 Remark.** The ISOU process has stationary increments, since the SOU process is stationary.

The following lemma is a simple consequence of Lemma 3.2.5 and the multilinearity (i.e. linearity in each variable) of the joint cumulant.

**3.2.10 Lemma.** *The joint cumulants of the ISOU process are:*

$$\begin{aligned}&\mathsf{cum}\left(J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t_1), \ldots, J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t_m)\right)\\ &= \frac{\mathsf{cum}_m(L(1))}{m} \int_0^{t_1} \cdots \int_0^{t_m} \mathsf{E}\, e^{-\delta(s_2^* - s_1^* + \cdots + s_m^* - s_1^*)} d\underline{s}\,, \quad 0 \leq t_1, \ldots, t_m, \quad m \in \mathbb{N}.\end{aligned}$$

**3.2.11 Lemma.** *For each $t > 0$ the characteristic function of $J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t)$ is analytic in some complex neighbourhood of zero. Therefore, the finite-dimensional distributions of the ISOU process are uniquely determined by the joint cumulants.*

**Proof.** By Remark 3.2.6 the characteristic function of the one-dimensional distribution of the SOU process $\{Y(t),\, t \geq 0\}$ is analytic in some complex neighbourhood of zero. Equivalently,

$$\limsup_{m\to\infty} \sqrt[2m]{\frac{\mathsf{E}(Y(1))^{2m}}{(2m)!}} < \infty.$$

[3]Throughout this thesis we denote integrated processes by $J$, because the notation $I$ would be misleading, as it used to denote the identity.

Let $t > 0$. We have

$$\mathsf{E}\left(J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t)\right)^{2m} = \mathsf{E}\left(\int_0^t Y(s)\,ds\right)^{2m} \leq t^{2m}\mathsf{E}\,(Y(1))^{2m}, \quad m \in \mathbb{N},$$

hence

$$\limsup_{m\to\infty} \sqrt[2m]{\frac{\mathsf{E}\left(J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t)\right)^{2m}}{(2m)!}} \leq t \limsup_{m\to\infty} \sqrt[2m]{\frac{\mathsf{E}(Y(1))^{2m}}{(2m)!}} < \infty,$$

thus the characteristic function of $J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t)$ is analytic in some complex neighbourhood of zero. Hence, the finite-dimensional distributions of the ISOU process $\left\{J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t),\, t \geq 0\right\}$ are uniquely determined by their joint cumulants. □

The following theorem states that in the construction of the ISOU process the order of the superposition and the integration can be interchanged.

**3.2.12 Theorem.** *The sequence of finite superpositions of integrated stationary OU type processes, $\left\{\sum_{j=1}^n \int_0^t X_j(s)\,ds,\, t \geq 0\right\}$, converges pointwise, both in $L^2$ and almost surely to the ISOU process, i.e., for each $t \geq 0$ we have*

$$\sum_{j=1}^{n} \int_0^t X_j(s)\,ds \underset{n\to\infty}{\longrightarrow} J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t) \quad \text{in } L^2 \text{ and a.s..} \tag{3.2.6}$$

**Proof.** The pointwise $L^2$-convergence in 3.2.6 is a simple consequence of Theorem 3.2.3, while the a.s. convergence follows by Kolmogorov's two series theorem. □

**3.2.13 Remark.** Because of Theorem 3.2.12 the ISOU process could also be named a SIOU process.

Also the following functional limit theorem holds.

**3.2.14 Theorem.** *The sequence of distributions on $C[0,\infty)$ of finite superpositions of integrated stationary OU type processes, $\left\{\sum_{j=1}^n \int_0^t X_j(s)\,ds,\ t \geq 0\right\}$, converges weakly to the distribution of the ISOU process, i.e.*

$$\left\{\sum_{j=1}^{n} \int_0^t X_j(s)\,ds,\ t \geq 0\right\} \underset{n\to\infty}{\overset{\mathrm{w}}{\longrightarrow}} \left\{J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t),\ t \geq 0\right\} \quad \text{on } C[0,\infty). \tag{3.2.7}$$

**Proof.** The corresponding convergence of the finite-dimensional distributions is a consequence of Theorem 3.2.12. Moreover, we have

$$\mathsf{E}\left(\sum_{j=1}^{n}\int_0^t X_j(s)\,ds\right)^2 = \mathsf{D}^2\sum_{j=1}^{n}\int_0^t X_j(s)\,ds + \left(\mathsf{E}\sum_{j=1}^{n}\int_0^t X_j(s)\,ds\right)^2$$

$$\leq \mathsf{D}^2 J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(t) + t^2\,\mathsf{E}\big(Y(0)\big)^2 \leq t^2\,\mathsf{D}^2\,Y(0) + t^2\,\mathsf{E}\big(Y(0)\big)^2 \leq t^2\big(\mathsf{E}\,Y(0)\big)^2,$$

so, Kolmogorov's tightness condition is fulfilled and the statement of the theorem follows. □

## 3.3 The LISOU process

We begin with a lemma which deals with the superposition law given by Assumption SPL. At the first sight it seems to be of a purely technical nature. Actually, it is a Tauberian theorem, which gives the connection between the asymptotic behaviour of the Laplace–Stieltjes transform at ifinity and that of the distribution function at zero. This lemma is important also because it will be the key to the dilatively stable renormalization limit theorems (Theorems 3.3.4 and 4.5.4).

**3.3.1 Lemma.** *Under Assuption SPL we have*

$$\lim_{T\to\infty}\mathsf{E}\,e^{-\delta T r}\frac{T^{2-2H}}{\ell(T)} = \Gamma(3-2H)\,r^{2H-2}, \quad r>0,$$

*and*

$$\mathsf{E}\,e^{-\delta T r}\frac{T^{2-2H}}{\ell(T)} \leq const. \times r^{2H-2}, \quad r>0,\ T>0,$$

*where the constant depends neither on $T$ nor $r$.*

**Proof.** We will prove that

$$\lim_{T\to\infty}\mathsf{E}\,e^{-\delta T r}\frac{T^{2-2H}}{\ell(T)}\,r^{2-2H} = \Gamma(3-2H) \tag{3.3.1}$$

uniformly (in $r$) on $(0,\infty)$, from which the statements of the lemma follow.

The pointwise convergence in (3.3.1) is a simple consequence of the Tauberian theorem in Feller [10, Ch. XIII, § 5] (see also Bingham et al. [3, Thm. 1.7.1']). To prove that the convergence in (3.3.1) is uniform, let us integrate by parts to obtain

$$\begin{aligned}
\mathsf{E}\, e^{-\delta T r} \frac{T^{2-2H}}{\ell(T)}\, r^{2-2H} &= \int_0^\infty e^{-xTr} F_\delta(dx)\, \frac{T^{2-2H}}{\ell(T)}\, r^{2-2H} \\
&= \int_0^\infty \ell\left(\frac{1}{x}\right) x^{2-2H} e^{-xTr} dx\, \frac{T^{3-2H}}{\ell(T)}\, r^{3-2H} \\
&= \int_0^r \frac{\ell\left(\frac{Tr}{x}\right)}{\ell(T)}\, x^{2-2H} e^{-x} dx + \int_r^\infty \frac{\ell\left(\frac{Tr}{x}\right)}{\ell(T)}\, x^{2-2H} e^{-x} dx.
\end{aligned}$$

We deal with the last two integrals separately.

• Let $\rho < 0$ and define the function $f(x) \doteq \ell(x)x^\rho,\ x > 0$. Then $f$ is regularly varying at infinity of order $\rho$, thus, applying Bingham et al. [3, Thm. 1.5.2] (uniform convergence theorem for regularly varying functions) we have

$$\lim_{T\to\infty} \frac{f(Tx)}{f(T)} = x^\rho,$$

uniformly on $[1, \infty)$. Hence

$$\lim_{T\to\infty} \frac{\ell\left(\frac{Tr}{x}\right)}{\ell(T)} = \lim_{T\to\infty} \frac{f\left(T\frac{r}{x}\right)}{f(T)}\, x^\rho r^{-\rho} = 1$$

uniformly on $(0, r]$, and so

$$\lim_{T\to\infty} \int_0^r \frac{\ell\left(\frac{Tr}{x}\right)}{\ell(T)}\, x^{2-2H} e^{-x} dx = \int_0^r x^{2-2H} e^{-x} dx \tag{3.3.2}$$

uniformly (in $r$) on $(0, \infty)$.

• Similarly, let $\rho > 0$ and define the function $f(x) \doteq \ell(x)x^\rho,\ x > 0$. Then $f$ is regularly varying at infinity of order $\rho$, thus, applying again Bingham et al. [3, Thm. 1.5.2] we obtain

$$\lim_{T\to\infty} \frac{f(Tx)}{f(T)} = x^\rho,$$

uniformly on $(0,1]$. (The boundedness condition of Bingham et al. [3, Thm. 1.5.2] fulfilles because

$$f(x) = \ell(x)x^{\rho} = F_{\delta}\left(\frac{1}{x}\right)x^{2-2H+\rho}$$

is a bounded function on every intervals of the form $(0,b]$.) Hence

$$\lim_{T\to\infty}\frac{\ell\left(\frac{Tr}{x}\right)}{\ell(T)} = \lim_{T\to\infty}\frac{f\left(T\frac{r}{x}\right)}{f(T)}x^{\rho}r^{-\rho} = 1$$

uniformly on $[r,\infty)$, and so

$$\lim_{T\to\infty}\int_{r}^{\infty}\frac{\ell\left(\frac{Tr}{x}\right)}{\ell(T)}x^{2-2H}e^{-x}dx = \int_{r}^{\infty}x^{2-2H}e^{-x}dx \tag{3.3.3}$$

uniformly (in $r$) on $(0,\infty)$.

Summing (3.3.2) and (3.3.3) we obtain that the convergence in (3.3.1) is uniform (in $r$) on $(0,\infty)$. □

We note that in Iglói [14] the above lemma is stated with other parametrization and in a somewhat complicated form.

Now we are ready to deal with convergence of a family of properly renormalized ISOU processes. Indeed, it can be seen from Lemma 3.2.10 and 3.2.11 that an ISOU process is infinitely divisible, hence it is meaningful to renormalize it. Namely, we may consider the renormalized centered processes

$$\frac{1}{T}\left\{J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt) - \mathsf{E}\,J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt),\ t\geq 0\right\}^{\circledast\frac{T^{2-2H}}{\ell(T)}},\quad T>0. \tag{3.3.4}$$

The main theorem in this section, Theorem 3.3.4 will be a dilatively stable renormalization functional limit theorem, stating the convergence as $T\to\infty$, of the family (3.3.4). The limit process will be necessarily dilatively stable. It will be called the LISOU process, giving the title of this chapter. The key for the proof of the mentioned dilatively stable renormalization limit theorem will be Lemma 3.3.1 and through it, directly the next lemma.

**3.3.2 Lemma.** *The family of the joint cumulants of the renormalized centered ISOU processes (3.3.4) converge as $T \to \infty$, namely,*

$$\lim_{T\to\infty} \mathsf{cum}\left(\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt_1) - \mathsf{E}\, J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt_1)\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}, \dots, \frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt_m) - \mathsf{E}\, J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt_m)\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}\right)$$

$$= \frac{\mathsf{cum}_m(L(1))}{m}\,\Gamma(3-2H)\int_0^{t_1}\cdots\int_0^{t_m}\left(s_2^* - s_1^* + \cdots + s_m^* - s_1^*\right)^{2H-2} d\underline{s}\,,$$

$$0 \le t_1, \dots, t_m, \quad m = 2, 3, \dots . \tag{3.3.5}$$

**Proof.** First of all, the joint cumulants of order $m \ge 2$ of the centered process are the same as those of the uncentered process. Using Lemma 3.2.10 we have

$$\mathsf{cum}\left(\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,\nu}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_1), \dots, \frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,\nu}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_m)\right) \tag{3.3.6}$$

$$= \frac{T^{-m+2-2H}\mathsf{cum}_m(L(1))}{\ell(T)m}\int_0^{Tt_1}\cdots\int_0^{Tt_m}\mathsf{E}\, e^{-(s_2^*-s_1^*+\cdots+s_m^*-s_1^*)}d\underline{s}$$

$$= \frac{\mathsf{cum}_m(L(1))}{m}\int_0^{t_1}\cdots\int_0^{t_m}\mathsf{E}\, e^{-\delta(s_2^*-s_1^*+\cdots+s_m^*-s_1^*)T}\,\frac{T^{2-2H}}{\ell(T)}\,d\underline{s}.$$

Taking the limit as $T \to \infty$, by Lemma 3.3.1 we can change the order of the limit and the integral and obtain

$$\lim_{T\to\infty}\mathsf{cum}\left(\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,\nu}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_1), \dots, \frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,\nu}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_m)\right)$$

$$= \frac{\mathsf{cum}_m(L(1))}{m}\,\Gamma(3-2H)\int_0^{t_1}\cdots\int_0^{t_m}\left(s_2^* - s_1^* + \cdots + s_m^* - s_1^*\right)^{2H-2} d\underline{s}\,.$$

$\square$

**3.3.3 Lemma.** *The family of renormalized centered ISOU processes (3.3.4) satisfies Kolmogorov's tightness condition. Therefore, processes (3.3.4) are almost surely continuous, and the corresponding family of distributions on $C[0,\infty)$ is tight.*

**Proof.** Let $T>0$ be arbitrarily fixed. Since the mean of the renormalized centered processes (3.3.4) are zero, for fixed $t$ the second moment is the second order cumulant. The latter equals the right-hand side of (3.3.6) with $m=2$ and $t_1=t_2=t$ in it. Hence we have

$$\mathsf{E}\left(\frac{1}{T}\left(J_{\sigma^2,\nu}^{(H,\ell)}(Tt)-\mathsf{E}\,J_{\sigma^2,\nu}^{(H,\ell)}(Tt)\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}\right)^2=\mathsf{cum}_2\left(\frac{1}{T}\left(J_{\sigma^2,\nu}^{(H,\ell)}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt)\right)$$

$$=\frac{\mathsf{cum}_2(L(1))}{2}\int_0^t\int_0^t \mathsf{E}\,e^{-\delta|s_1-s_2|T}\,\frac{T^{2-2H}}{\ell(T)}\,ds_1ds_2.$$

Applying now the inequality of Lemma 3.3.1, we obtain

$$\mathsf{E}\left(\frac{1}{T}\left(J_{\sigma^2,\nu}^{(H,\ell)}(Tt)-\mathsf{E}\,J_{\sigma^2,\nu}^{(H,\ell)}(Tt)\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}\right)^2$$

$$=\frac{\mathsf{cum}_2(L(1))}{2}\int_0^t\int_0^t \mathsf{E}\,e^{-\delta|s_1-s_2|T}\,\frac{T^{2-2H}}{\ell(T)}\,ds_1ds_2$$

$$\le const_1\int_0^t\int_0^t |s_1-s_2|^{2H-2}ds_1ds_2=const_2\,t^{2H},$$

where the constants depend neither on $t$ nor on $T$. Considering that the processes (3.3.4) have stationary increments and start from zero, and remembering that $H>1/2$, we obtain Kolmogorov's tightness condition to be fulfilled. □

Now, we can state the main theorem.

**3.3.4 Theorem.** *The family of distributions of the renormalized centered ISOU processes (3.3.4) converges weakly on $C[0,\infty)$ to a limit distribution. We will denote the process with this distribution by $\left\{J_{\sigma^2,\nu}^{(H)}(t),\,t\ge 0\right\}$ and call it the*

*LISOU process (limit of [renormalized centered] ISOU processes) with parameters $H$ and $(\sigma^2, \nu)$:*

$$\frac{1}{T}\left\{J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt) - \mathsf{E}\, J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(Tt),\ t \geq 0\right\}^{\circledast \frac{T^{2-2H}}{\ell(T)}}$$
$$\xrightarrow[T\to\infty]{\mathrm{w}} \left\{J^{(H)}_{\sigma^2,\nu}(t),\ t \geq 0\right\} \quad \text{on } C[0,\infty). \tag{3.3.7}$$

*The distribution on $C[0,\infty)$ of the LISOU process with parameters $H$ and $(\sigma^2,\nu)$ is uniquely determined by its zero mean and its joint cumulants, which are:*

$$\mathsf{cum}\left(J^{(H)}_{\sigma^2,\nu}(t_1),\dots,J^{(H)}_{\sigma^2,\nu}(t_m)\right) = \frac{\mathsf{cum}_m(L(1))}{m}\,\Gamma(3-2H)$$
$$\times \int_0^{t_1}\!\!\cdots\!\int_0^{t_m} (s_2^* - s_1^* + \cdots + s_m^* - s_1^*)^{2H-2}\, d\underline{s}\,, \tag{3.3.8}$$

$0 \leq t_1,\dots,t_m,\ m \geq 2$. *Moreover, $\left\{J^{(H)}_{\sigma^2,\nu}(t),\ t \geq 0\right\}$ has stationary increments and it is $(H, 2H-2)$-dilatively stable.*

**Proof.** By Lemma 3.3.3 the family of distributions on $C[0,\infty)$ corresponding to the family (3.3.4) is relatively sequentially compact, so there exists a subsequence

$$\frac{1}{T_n}\left\{J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(T_n t) - \mathsf{E}\, J^{(H,\ell)}_{\sigma^2,\gamma,\nu}(T_n t),\ t \geq 0\right\}^{\circledast \frac{T_n^{2-2H}}{\ell(T_n)}}, \quad n \in \mathbb{N},$$

such that the corresponding subsequence of distributions converges weakly to a distribution on $C[0,\infty)$. Let us denote the process corresponding to this limit distribution by $\left\{J^{(H)}_{\sigma^2,\nu}(t),\ t \geq 0\right\}$. This process has zero mean and joint cumulants given by the right-hand side of (3.3.5) (hence the notation $J^{(H)}_{\sigma^2,\nu}$ is right, since parameters $\ell$ and $\gamma$ have disappeared). We have to prove that this limit process is unique, i.e. its distribution on $C[0,\infty)$ is unique. In other words, we have to prove that the joint cumulants of $\left\{J^{(H)}_{\sigma^2,\nu}(t),\ t \geq 0\right\}$, i.e. those given by the right-hand side of (3.3.5), uniquely determine the finite-dimensional distributions. For this it is enough to prove that for each $t > 0$ the characteristic function of $J^{(H)}_{\sigma^2,\nu}(t)$ is analytic in some complex neighbourhood of zero. But this is true, since we have

$$\left|\int_0^1\!\!\cdots\!\int_0^1 (s_2^* - s_1^* + \cdots + s_m^* - s_1^*)^{2H-2}\, d\underline{s}\right| \leq \frac{m}{2H-1}\,,$$

so

$$\limsup_{m\to\infty} \sqrt[m]{\frac{\left|\mathsf{cum}_m\left(J^{(H)}_{\sigma^2,\nu}(t)\right)\right|}{m!}}$$

$$= \limsup_{m\to\infty} \sqrt[m]{\frac{\left|\frac{\mathsf{cum}_m(L(1))}{m}\,\Gamma(3-2H)\,t^{m+2H-2}\int_0^{t_1}\cdots\int_0^{t_m}(s_2^*-s_1^*+\cdots+s_m^*-s_1^*)^{2H-2}d\underline{s}\right|}{m!}}$$

$$\le t\,\limsup_{m\to\infty} \sqrt[m]{\frac{|\mathsf{cum}_m(L(1))|}{m!}} < \infty.$$

The LISOU process has stationary increments, because processes on the left-hand side of (3.3.7) are such.

Now, we show that $\left\{J^{(H)}_{\sigma^2,\nu}(t),\, t\ge 0\right\}\in\mathcal{I}$ (see Notation 2.1.1). The non-Gaussianity is obvious, as well the infinite divisibility of the finite-dimensional distributions and the right-continuity of the cumulant functions

$$\mathsf{cum}_m\left(J^{(H)}_{\sigma^2,\nu}(t)\right) = \frac{\mathsf{cum}_m(L(1))}{m}\,\Gamma(3-2H)\,t^{m+2H-2}$$
$$\times\int_0^1\cdots\int_0^1(s_2^*-s_1^*+\cdots+s_m^*-s_1^*)^{2H-2}\,d\underline{s}\,, \quad (3.3.9)$$

$t\ge 0$, $m\ge 2$. The right-continuity in distribution follows from the right-continuity of the cumulant functions and from the fact that for any fixed $t>0$ the cumulants (3.3.9) and the zero mean uniquely determine the distribution of $J^{(H)}_{\sigma^2,\nu}(t)$. Therefore $\left\{J^{(H)}_{\sigma^2,\nu}(t),\, t\ge 0\right\}\in\mathcal{I}$, so we can apply Theorem 2.2.7 (DS), hence the $(H,2H-2)$-dilative stability follows. □

**3.3.5 Remark.** The distribution of the ISOU process depends also on parameters $\ell$ and $\gamma$, while that of the LISOU process does not depend on these parameters any more. Accordingly, the superscript $\ell$ and the subscript $\gamma$ disappear on the right-hand side of (3.3.7).

**3.3.6 Remark.** The LISOU process with parameter $H$ (and $(\sigma^2,\nu)$) is $(H,2H-2)$-dilatively stable, so it has the renormalization operator $A_T^{(H,\,2H-2)}$ (see Example 2.8.3 (DS)). This explains, at least when $\lim_{T\to\infty}\ell(T)$ exists and nonzero, the renormalization used in (3.3.7).

**3.3.7 Remark.** Theorem 3.3.4 can be interpreted as it details the fact that the ISOU process with parameters $(H,\ell)$ and $(\sigma^2,\nu)$ is asymptotically $(H,2H-2)$-dilatively stable (see Definition 2.2.9 (DS)).

**3.3.8 Remark.** By the last statement of Theorem 3.3.4, Theorem 2.7.2 (DS), and Corollary 2.7.3 (DS), the LISOU process with parameter $H$ has the same autocovariance function as that of the FBM with parameter $H$ (apart from a constant factor), and it is of long memory.

**3.3.9 Remark.** The superposition of OU type processes and the limit of its integral process is treated also in Barndorff-Nielsen [1]. In a representation of the type (3.1.1), instead of a two-sided Lévy process (which can be considered as an independently scattered random measure on $\mathbb{R}$) [1] uses an independently scattered random measure $z(ds,d\xi)$ on $\mathbb{R}\times\mathbb{R}_+$ (the notations are those of the paper under review), and carries out the superposition by integrating with respect to $z(\cdot,d\xi)$ (see [1, Sect. 3]). Entering into details, for every $A\in\mathbb{B}(\mathbb{R}\times\mathbb{R}_+)$ $z(A)$ has no drift, no Gaussian component (we think the latter assumption to be unnecessary), and its Lévy measure factorizes as $Q(A,du)=M(A)W(du)$, where $W$ is a Lévy measure of some infinitely decomposable probability measure, and the measure $M$ factorizes as $M(ds,d\xi)=ds\nu(d\xi)$, with some probability measure $\nu$. The "supOU" process is defined as

$$x(t)\doteq\int\limits_{\mathbb{R}_+}e^{-\xi t}\int\limits_{-\infty}^{\xi t}e^{s}z(ds,d\xi),\quad t\in\mathbb{R},$$

(see [1, (3.3–3.4)]). Then the integrated process $x^*(t)\doteq\int_0^t x(s)ds$, $t\geq 0$, the Lévy measure $W$ and the probability measure $\nu$ are admitted to depend on a parameter $\lambda$ as $x^*_\lambda(t)\doteq\lambda^{-1}x^*(\lambda t)$, $t\geq 0$, $W_\lambda(dx)\doteq\lambda^{2-2H}W(dx)$ and $\nu\sim\Gamma(2-2H,1/\lambda)$, respectively, where $1/2<H<1$, and it is proved that $\{x^*_\lambda(t),\,t\geq 0\}\xrightarrow[\lambda\to\infty]{\text{fd}}\{x^*_0(t),\,t\geq 0\}$, where the limit process is given by its cumulant (i.e. log-characteristic) functional [1, (6.6)]. Now, performing a suitable substitution in [1, (6.6)] we obtain first the cumulant (log-characteristic) function of $(x^*_0(t_1),\dots,x^*_0(t_m))$, then, by differentiation, the cumulant $\mathsf{cum}(x^*_0(t_1),\dots,x^*_0(t_m))$. It can be checked that the latter expression can be rewritten into the form (3.3.8). Consequently, Barndorff-Nielsen's limit of integrated "supOU" processes and our LISOU process are the same. In fact, even our SOU process is a particular case of Barndorff-Nielsen's "supOU" process, namely, when $\nu$ is the discrete distribution given in Assumption SPL.

# Chapter 4

# The LISCBI process

In this chapter we will superpose continuous state branching processes with immigration (CBI processes). The particular case when the CBI process is the so-called diffusion process with linear generator (DLG process), will be treated separately, in the last section (Section 4.6). We will state a dilatively stable renormalization functional limit theorem for the integrated superposition process.

## 4.1 The CBI process

CBI processes are fundamental stochastic processes. They are continuous time and continuous state analogues of discrete time branching processes with immigration (Galton–Watson processes with immigration). On the other hand, they are also generalizations of the continuous time birth and death processes with immigration: instead of

- the birth and death, governed by a discrete distribution on the set $\{-1, 0, 1\}$
- and the immigration Poisson process

there are

- the branching, governed by an infinitely divisible distribution on $\mathbb{R}$
- and the immigration compound Poisson process with a non-negative drift,

respectively. Moreover, CBI processes have one more attractive characterization (see Shiga–Watanabe [32] and Duffie et al. [9, Cor. 2.10]): they are exactly those regular, homogeneous Markov processes with state space $[0, \infty)$, which are infinitely decomposable (roughly, the transition probability distribution is infinitely divisible). On the other hand, infinitely decomposable, regular, homogeneous Markov processes with state space $\mathbb{R}$ are just the OU type processes with state space $\mathbb{R}$ (see Duffie et al. [9, Cor. 2.10]). The OU type processes with state space $[0,\infty)$ are, at the same time, particular CBI processes (see Remark 4.2.5). This coincidence enables the results of this chapter to be compared with the corresponding ones of chapter 3 and so, to be checked.

Now, let us recall some standard terminology and notation.

• For a homogeneous Markov process $\{X(t),\, t\geq 0\}$ with state space $[0,\infty)$ and for an $m\in\mathbb{N}$ the conditional $m$-dimensional distribution, the transition probability function, the transition Laplace transform and the Laplace transform of the $m$-dimensional distribution are, resp.:

$$\mathsf{P}_{X(t_1),\dots,X(t_m)|x}(B) \doteq \mathsf{P}\big(X(t_1),\dots,X(t_m)\in B\big|X(0)=x\big),$$
$$B\in\mathcal{B}([0,\infty)^m),\ \ x\in[0,\infty),$$

$$\mathsf{P}_{|x}(t,B) \doteq \mathsf{P}_{X(t)|x}(B),\quad B\in\mathcal{B}([0,\infty)),\ \ t,\,x\geq 0,$$

$$\varphi_{|x}(t,u) \doteq \mathsf{E}_{|x}\, e^{-uX(t)} = \int_0^\infty e^{-uy}\mathsf{P}_{|x}(t,dy),\quad u,\,t,\,x\geq 0,$$

and

$$\varphi_{X(t_1),\dots,X(t_m)}\big(u_1,\dots,u_m\big) \doteq \mathsf{E}\, e^{-u_1X(t_1)-\cdots-u_mX(t_m)},\quad u_1,\dots,u_m\geq 0.$$

• The Markov semigroup of $\{X(t),\, t\geq 0\}$ is $\{T_t,\, t\geq 0\}$, given by

$$T_t : B([0,\infty)) \to B([0,\infty))$$

$$T_tf(x) \doteq \mathsf{E}_{|x}f(X(t)) = \mathsf{E}\big(f(X(t))\big|\,X(0)=x\big),\quad t,\,x\geq 0,$$

where $B([0,\infty))$ denotes the space of bounded, measurable functions on $[0,\infty)$ with the supremum norm.

• The homogeneous Markov process $\{X(t),\, t\geq 0\}$ is called a Feller process with state space $[0,\infty)$ if $T_t\big(C_0([0,\infty))\big)\subset C_0\big([0,\infty)\big)$ and $\lim_{t\to 0}\|T_tf-f\|=0$, $f\in C_0\big([0,\infty)\big)$, where $C_0([0,\infty))$ denotes the space of continuous functions on $[0,\infty)$ with the supremum norm, vanishing at infinity. (See Kawazu–Watanabe [21, Prop. 1.1] for equivalent characterizations of this property.)

• If $\{X(t),\, t\geq 0\}$ is a Feller process with state space $[0,\infty)$, then a function $f\in C_0([0,\infty))$ is said to belong to the domain $\mathcal{D}_A$ of the infinitesimal generator of $\{X(t),\, t \geq 0\}$ if the limit

$$Af \doteq \lim_{t \to 0} \frac{T_t f - f}{t}$$

exists in $C_0([0,\infty))$. The operator $A : \mathcal{D}_A \longrightarrow C_0([0,\infty))$ thus defined is called the infinitesimal generator of Feller process $\{X(t),\, t \geq 0\}$ with state space $[0,\infty)$.

• The linear hull of the following set of exponential functions is denoted by $\mathcal{L} \doteq \mathcal{L}\big\{f_c(x) \doteq e^{-cx},\, 0 \leq x < \infty : c \geq 0\big\}$. By the Stone–Weierstrass theorem $\mathcal{L}$ is dense in $C_0([0,\infty))$.

• A Markov process $\{X(t),\, t \geq 0\}$ with state space $[0,\infty)$ is called conservative, if for all $x, t \geq 0$ : $\mathsf{P}_{|x}(t, [0,\infty)) = 1$. From now on we consider Markov processes to be automatically conservative.

Now come the main definitions.

**4.1.1 Definition.** (Kawazu–Watanabe [21]) A Feller process $\{X(t),\, t \geq 0\}$ with state space $[0,\infty)$ is called a *continuous state branching process with immigration (CBI process)*, if its transition Laplace transform is of the form

$$\varphi_{|x}(t,u) = e^{-\phi(t,u) - \psi(t,u)\, x}, \quad u,\, t,\, x \geq 0. \tag{4.1.1}$$

Functions $\phi$ and $\psi$ (which are obviously unique) are called the *Laplace exponent functions* of the CBI process.

**4.1.2 Theorem.** *(Kawazu–Watanabe [21, Thm. 1.1–1.1'], Filipović [11, Thm. 4.3]) Let $A$ be the infinitesimal generator of the CBI process. Then $\mathcal{L} \subseteq \mathcal{D}_A$ and*

$$Af(x) = \frac{\sigma^2}{2}\, x f''(x) + \left(\vartheta + \int_0^\infty (1 \wedge y)\, m_2(dy) + \alpha x\right) f'(x) \tag{4.1.2}$$

$$+ \int_0^\infty \Big(f(x+y) - f(x) - f'(x)(1 \wedge y)\Big)\Big(m_2(dy) + x\, m_1(dy)\Big), \quad f \in \mathcal{L},$$

*where*

$$\sigma^2 \geq 0, \quad \vartheta \geq 0, \quad -\infty < \alpha < \infty, \tag{4.1.3}$$

*and* $m_1$, $m_2$ *are Borel measures on* $(0,\infty)$ *satisfying*

$$\int_0^\infty (1\wedge y^2)\, m_1(dy) + \int_0^\infty (1\wedge y)\, m_2(dy) < \infty\,. \tag{4.1.4}$$

*Moreover, the Laplace exponent functions of the CBI process satisfy the differential equations*

$$\partial_1\psi(t,u) = R(\psi(t,u))\,, \quad \psi(0,u) = u\,, \tag{4.1.5}$$

$$\partial_1\phi(t,u) = S(\psi(t,u))\,, \quad \phi(0,u) = 0\,, \tag{4.1.6}$$

*where*

$$R(u) = -\frac{\sigma^2}{2}\,u^2 + \alpha u + \int_0^\infty \Big(1 - e^{-uy} - u(1\wedge y)\Big) m_1(dy)\,, \quad u \ge 0, \tag{4.1.7}$$

$$S(u) = \vartheta u + \int_0^\infty \big(1 - e^{-uy}\big) m_2(dy)\,, \quad u \ge 0 \tag{4.1.8}$$

*and function* $R$ *fulfills the condition (of conservativity)*

$$\int_{0^+} \frac{1}{0\vee R(u)}\, du = \infty\,, \tag{4.1.9}$$

*where* $0^+$ *denotes an arbitrary small right neighbourhood of zero.*
*Conversely, given parameters (4.1.3) and Borel measures* $m_1$, $m_2$ *on* $(0,\infty)$ *satisfying (4.1.4) such that function (4.1.7) fulfills (4.1.9), there exist a unique CBI process with the infinitesimal generator (4.1.2) and Laplace exponent functions satisfying the differential equations (4.1.5–4.1.6).*

**4.1.3 Remark.** It follows from (4.1.1) and (4.1.5–4.1.6) that for any fixed $u \ge 0$ and $x \ge 0$ the differential of the transition Laplace transform in $t = 0$ is

$$\partial_1\varphi_{|x}(0,u)dt = -e^{-ux}\big(S(u) + R(u)x\big)dt.$$

Let us consider the initial state $x = 0$. Then $\partial_1\varphi_{|0}(0,u) = -S(u)$, $u \ge 0$, hence the function $S$ characterizes the immigration. On the other hand, assume that parameters $\vartheta = 0$ and $m_2 \equiv 0$, therefore $S \equiv 0$, i.e. there is no immigration.

Then for any $x > 0$ the change of the Laplace transform is $\partial_1 \varphi_{|x}(0,u)dt = -e^{-ux}R(u)xdt$, hence the function $R$ characterizes the branching. In fact, $S$ is the Laplace exponent function (i.e. the negative logarithm of the Laplace transform) corresponding to an increasing compound Poisson process with a non-negative drift: the immigration process. $R$ in turn, is the Laplace exponent function of an infinitely divisible distribution, governing the branching. Note that the latter distribution can have a Gaussian component, agreeing with the fact that due to the branching the process can also decrease. Let us observe also that since the measure $m_1$ in (4.1.7) is concentrated to the positive half line, the Laplace transform $e^{-R(u)}$, $u \geq 0$, uniquely defines a distribution (i.e., it is not necessary to use the characteristic function), even if this distribution itself is not concentrated onto the positive half line.

**4.1.4 Definition.** The infinitely divisible distributions with Laplace exponent functions $R$ and $S$ given by (4.1.7–4.1.8) are called the *branching distribution* and the *immigration distribution*, respectively. Correspondingly, $(\sigma^2, \alpha, m_1)$ is called the *branching parameter* (vector), while $(\vartheta, m_2)$ is called the *immigration parameter* (vector).

**4.1.5 Remark.** Assume that the value of a CBI process is the instantaneous size of some population. Then equation

$$\varphi_{|x}(t,u) = e^{-\phi(t,u)} \left(e^{-\psi(t,u)}\right)^x, \quad u \geq 0,$$

means that the population size at $t$ decomposes into two independent terms. The Laplace transform of the first term is $e^{-\phi(t,u)}$. This term is the size of the population consisting of those immigrants in the interval $(0,t)$ and their progeny, who are alive at $t$. The other term, with the Laplace transform $\left(e^{-\psi(t,u)}\right)^x$ is the size of the population consisting of those natives and their progeny, who are alive at $t$, provided that the size the original population (the natives) was $x$.

**4.1.6 Remark.** If $\int_1^\infty y\, m_1(dy) < \infty$ (equivalently, the branching distribution has finite mean), then function $R$ is Lipschitz in $u = 0$, hence (4.1.9) fulfills.

The following portion as far as Proposition 4.1.10 shed light on the importance of the CBI process.

**4.1.7 Definition.** (Shiga–Watanabe [32]) A homogeneous Markov process $\{X(t),\, t \geq 0\}$ with state space $[0, \infty)$ is called *infinitely decomposable*, if for every $n \in \mathbb{N}$ there exists a homogeneous Markov process $\{X^{(n)}(t),\, t \geq 0\}$ with state space $[0,\infty)$, such that for every $m \in \mathbb{N}$, $0 < t_1 < \ldots < t_m$ and $x^{(1)}, \ldots x^{(n)} \in [0,\infty)$ for which $x = x^{(1)} + \cdots + x^{(n)}$, it holds that

$$\mathsf{P}_{X(t_1),\ldots,X(t_m)|x} \sim \mathsf{P}_{X^{(n)}(t_1),\ldots,X^{(n)}(t_m)|x^{(1)}} \circledast \cdots \circledast \mathsf{P}_{X^{(n)}(t_1),\ldots,X^{(n)}(t_m)|x^{(n)}}.$$

**4.1.8 Proposition.** *Given independent CBI processes $\{X_i(t),\, t \geq 0\}$ with immigration parameters $(\vartheta_i, m_{2,i})$, $i$=1, 2, both with branching parameter $(\sigma^2, \alpha, m_1)$, their sum $\{X_1(t)+X_2(t),\, t \geq 0\}$ is also a CBI process with immigration parameter $(\vartheta_1 + \vartheta_2, m_{2,1} + m_{2,2})$ and with the common branching parameter $(\sigma^2, \alpha, m_1)$. Briefly, the set of CBI processes with the same branching parameter is closed with respect to convolution and the immigration parameter is additive.*

**Proof.** It is a simple consequence of Theorem 4.1.2 and the fact that if two processes are independent, then they are also conditionally independent with respect to their values at zero. □

**4.1.9 Corollary.** *The CBI process is infinitely decomposable.*

The converse is also true:

**4.1.10 Proposition.** *(Shiga–Watanabe [32]) A Feller process with state space $[0,\infty)$ is infinitely decomposable if and only if it is a CBI process.*

**4.1.11 Example.** *OU type process with state space* $[0,\infty)$: Let $\sigma^2 = 0$, $\alpha \in \mathbb{R}$, $\vartheta = 0$, $m_2 \not\equiv 0$ be some $\sigma$-finite measure on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge y) m_2(dy) < \infty$ and $m_1 \equiv 0$. Then (4.1.7–4.1.8) read

$$R(u) = \alpha u, \quad u \geq 0,$$
$$S(u) = \int_0^\infty \big(1 - e^{-uy}\big) m_2(dy), \quad u \geq 0.$$

By Remark 4.1.6 condition (4.1.9) is fulfilled, therefore, there exists a unique CBI process $\{X(t),\, t \geq 0\}$ with the infinitesimal generator given on $\mathcal{L}$ by

$$Af(x) = \alpha x f'(x) + \int_0^\infty \big(f(x+y) - f(x)\big)\, m_2(dy), \quad f \in \mathcal{L},$$

where $\mathcal{L}$ is dense in $\mathcal{D}_A$. By Duffie et al. [9, Cor. 2.10] it follows that [9, Thm. 2.12] can be applied, from which we obtain that process $\{X(t),\, t \geq 0\}$ is the unique weak solution of the stochastic differential equation

$$dX(t) = \alpha X(t)\,dt + dL(t), \tag{4.1.10}$$

with initial state $X(0) = x \geq 0$, where $\{L(t),\, t \geq 0\}$ is an increasing compound Poisson process associated with the Lévy measure $m_2$. But, by Rocha-Arteaga–Sato [28, Prop. 37], the stochastic differential equation (4.1.10) has a unique strong solution: the OU type process with parameter $\alpha$ and driving process $\{L(t),\, t \geq 0\}$. Therefore, process $\{X(t),\, t \geq 0\}$ is an OU type process with state space $[0, \infty)$. The immigration distribution is the non-negative compound Poisson distribution associated to the Lévy measure $m_2$, and the immigration Lévy process is $\{L(t),\, t \geq 0\}$. The branching distribution is the distribution of the constant $\alpha$. The Laplace exponent functions of $\{X(t),\, t \geq 0\}$ are the solutions of (4.1.5–4.1.6):

$$\psi(t, u) = ue^{\alpha t}, \quad t, u \geq 0$$

$$\phi(t, u) = \int_0^t S\big(ue^{\alpha s}\big)ds,$$

in line with the expression in Rocha-Arteaga–Sato [28, Prop. 38].

**4.1.12 Example.** *DLG process*: Let $\sigma^2 > 0$, $\alpha \in \mathbb{R}$, $\vartheta \geq 0$, $m_1 \equiv m_2 \equiv 0$. Then (4.1.7–4.1.8) read

$$R(u) = -\frac{\sigma^2}{2}\,u^2 + \alpha u, \quad u \geq 0,$$

$$S(u) = \vartheta u, \quad u \geq 0.$$

By Remark 4.1.6 condition (4.1.9) is fulfilled, therefore, there exists a unique CBI process $\{X(t),\, t \geq 0\}$ with the infinitesimal generator given on $\mathcal{L}$ by

$$Af(x) = \frac{\sigma^2 x}{2}\,f''(x) + \big(\vartheta + \alpha x\big)f'(x), \quad f \in \mathcal{L},$$

where $\mathcal{L}$ is dense in $\mathcal{D}_A$. So, the third term in (4.1.2) is zero, implying that process $\{X(t),\, t \geq 0\}$ is a diffusion. Both the drift and the diffusion coefficients are linear, therefore we will call it a (non-negative) diffusion process with linear generator, shortly a DLG process (with parameter $(\sigma^2, \alpha, \vartheta)$). (There is another diffusion process with linear infinitesimal generator: the [Gaussian] OU process, but it is not non-negative.) By Duffie et al. [9, Cor. 2.10] it follows that [9, Thm. 2.12] can be applied, from which we obtain that the DLG process is the unique weak solution of the stochastic differential equation

$$dX(t) = \big(\vartheta + \alpha X(t)\big)\, dt + \sigma\sqrt{X(t)}\, dB(t), \tag{4.1.11}$$

with initial state $X(0) = x \geq 0$, where $\{B(t),\, t \geq 0\}$ is the standard BM. In fact, it is also the unique strong solution of (4.1.11), see Ikeda–Watanabe [17, Thm. IV.3.2]. The immigration distribution is the constant $\vartheta$ (so the immigration Lévy process is simply $\{\vartheta t,\, t \geq 0\}$, i.e. deterministic). The branching distribution is Gaussian with mean $\alpha$ and variance $\sigma^2$. The differential equation (4.1.5) is now of a Riccati type, its solution is

$$\psi(t,u) = \frac{u\, e^{\alpha t}}{1 + u\frac{\sigma^2}{-2\alpha}\big(1 - e^{\alpha t}\big)}, \quad u \geq 0,\ t \geq 0 \tag{4.1.12}$$

(taking $\frac{1-e^{\alpha t}}{-\alpha} = t$ if $\alpha = 0$). The differential equation (4.1.6) is a trivial one, thus

$$\phi(t,u) = \frac{2\vartheta}{\sigma^2} \log\left(1 + u\frac{\sigma^2}{-2\alpha}\big(1 - e^{\alpha t}\big)\right), \quad u \geq 0,\ t \geq 0. \tag{4.1.13}$$

Hence, the transition Laplace transform is

$$\varphi_{|x}(t,u) = \left(1 + u\frac{\sigma^2}{-2\alpha}\big(1 - e^{\alpha t}\big)\right)^{-\frac{2\vartheta}{\sigma^2}} \exp\left(\frac{-u\, e^{\alpha t} x}{1 + u\frac{\sigma^2}{-2\alpha}\big(1 - e^{\alpha t}\big)}\right), \quad u \geq 0,\ t \geq 0.$$

## 4.2 The stationary CBI process

A CBI process has a stationary distribution different from the trivial distribution concentrated to zero if and only if for every $u > 0$

$$\exists \lim_{t\to\infty} \psi(t,u) = 0, \tag{4.2.1}$$

$$\exists \lim_{t\to\infty} \phi(t,u) \in (0,\infty). \tag{4.2.2}$$

**4.2.1 Notation.** $\phi(\infty,u) \doteq \lim_{t\to\infty} \phi(t,u),\ u \geq 0.$

Using this notation, the Laplace transform of the stationary distribution is

$$\varphi(u) = e^{-\phi(\infty,u)}, \quad u \geq 0.$$

**4.2.2 Remark.** We will assume that the branching and the immigration distributions of the CBI process have finite second moments, or even finite moments of all orders. The equivalent condition formulated by the help of the parameters is that the measures $m_1$ and $m_2$ have finite second moments, or finite moments of all orders, respectively.

**4.2.3 Theorem.** *A CBI process with branching and immigration distributions of finite second moments has a stationary distribution, different from the trivial distribution concentrated to zero if and only if the following conditions are fulfilled.*

$$R'(0) = \alpha + \int_1^\infty (y-1)\, m_1(dy) < 0, \tag{4.2.3}$$

*i.e. the mean of the branching distribution is negative and*

$$S'(0) = \vartheta + \int_0^\infty y\, m_2(dy) > 0, \tag{4.2.4}$$

*i.e. the mean of the immigration distribution is positive.*

**Proof.** Before starting to prove either direction, let us observe that $S(0) = R(0) = 0$, both $R$ and $S$ are continuous functions an they are concave on $[0,\infty)$, since both are mixtures of concave functions, see (4.1.7–4.1.8). Moreover, function $S$ is non-negative valued (as it can be seen directly from (4.1.8)). $\phi$ and $\psi$ are also non-negative valued, since for each $t \geq 0$ (4.1.1) is the (conditional) Laplace transform of a non-negative r.v..

Now, if $\{X(t),\, t \geq 0\}$ has a stationary distribution, different from the trivial distribution concentrated to zero, then (4.2.1–4.2.2) hold. By (4.1.8) we have

$$S'(0) = \vartheta + \int_0^\infty y\, m_2(dy) \geq 0\,.$$

Even $S'(0) > 0$, because otherwise $\vartheta = 0$ and $m_2 \equiv 0$, thus $S \equiv 0$ and $\phi \equiv 0$, contradicting to the fact that the limit in (4.2.2) is positive.

Next we prove (4.2.3). By (4.1.7) we have

$$R'(0) = \alpha + \int_1^\infty (y-1)\, m_1(dy)\,.$$

Let $u > 0$ be arbitrarily fixed and let

$$t_0 \doteq \inf\big\{t > 0 : \psi(t,u) = 0\big\}. \tag{4.2.5}$$

The stationary distribution is not concentrated to zero, so $t_0$ is a positive real number or infinity. In either case by (4.1.5–4.1.6) and (4.2.1–4.2.2) we have

$$\psi(t,u) = 0 \ \text{ and } \ \phi(t,u) = \phi(t_0,u) \quad \text{for } t \geq t_0$$

and

$$\lim_{t \nearrow t_0} \psi(t,u) = 0, \ \text{ and } \ \lim_{t \nearrow t_0} \phi(t,u) = \phi(t_0,u)\,. \tag{4.2.6}$$

Since $S(v) \geq 0$ if $v \geq 0$ and $S(0) = 0$, it follows from $S'(0) > 0$ that $S(v) > 0$ for $v > 0$. Hence, by (4.1.6) we have $\partial_1\phi(t,u) > 0$ if $u > 0$ and $t \in (0,t_0)$, that is, $(0,t_0) \ni t \mapsto \phi(t,u)$ is strictly increasing. Furthermore, by the definition of $t_0$, $\psi(t,u) > 0$ if $t \in (0,t_0)$. Therefore, using L'Hospital rule, (4.1.5–4.1.6) and (4.2.6), we obtain the relation

$$\begin{aligned} 0 \geq \lim_{t \nearrow t_0} \frac{\phi(t,u) - \phi(t_0,u)}{\psi(t,u)} &= \lim_{t \nearrow t_0} \frac{\partial_1\phi(t,u)}{\partial_1\psi(t,u)} \\ &= \lim_{t \nearrow t_0} \frac{S(\psi(t,u))}{R(\psi(t,u))} = \lim_{v \to 0} \frac{S(v)}{R(v)} = \frac{S'(0)}{R'(0)}\,. \end{aligned}$$

$R'(0)$, as the mean of the branching distribution, is finite by the assumption of the theorem. Hence $R'(0) < 0$.

Conversely, assume that conditions (4.2.3–4.2.4) are fulfilled. Our aim is to prove that for every $u > 0$ (4.2.1–4.2.2) hold. Let $u > 0$ be fixed. Since $R(0) = 0$ and $R$ is concave, it follows from (4.2.3) that

$$\forall\, v > 0 : R(v) < 0. \tag{4.2.7}$$

Let $t_0$ be defined by (4.2.5). By (4.1.5) function $\psi(t,u)$, as a function of $t$, is not identically zero, thus $0 < t_0$ is a real number or infinity. By (4.1.5) and

(4.2.7), function $(0,t_0) \ni t \mapsto \psi(t,u)$ is strictly decreasing. Therefore we can perform the change of variables

$$\phi(t,u) = \int_0^t S\big(\psi(s,u)\big)ds = \int_{\psi(t,u)}^u \frac{S(v)}{-R(v)}\,dv\,, \quad t \in (0,t_0). \tag{4.2.8}$$

$\psi$ is also non-negative, therefore $\exists \lim_{t\to\infty} \psi(t,u) \geq 0$. On the other hand, function $-S/R$ is integrable on intervals of the type $(0,\varepsilon)$, $\varepsilon > 0$, since

$$\frac{S(v)}{-R(v)} \underset{v\to 0}{\sim} \frac{S'(v)}{-R'(v)} \underset{v\to 0}{\longrightarrow} \frac{S'(0)}{-R'(0)} \in \mathbb{R}.$$

Consequently,

$$\exists\, \phi(\infty,u) \doteq \lim_{t\to\infty} \phi(t,u) \in \mathbb{R}. \tag{4.2.9}$$

The function $S$ is non-negative, by (4.2.4) it is even positive and $-R$ is also positive (by (4.2.7)). It follows from the initial condition of (4.1.5) and the fact that $(0,t_0) \ni t \mapsto \psi(t,u)$ is strictly decreasing, that for all $t > 0 : u = \psi(0,u) > \psi(t,u)$. Hence for all $t \in (0,t_0)$ (4.2.8) is positive and since

$$\phi(t,u) = \int_0^t S\big(\psi(s,u)\big)ds = \int_0^{t\wedge t_0} S\big(\psi(s,u)\big)ds$$

holds for all $t > 0$, the limit in (4.2.9) is positive. Thus, we have proved (4.2.2).

Finally, (4.1.8), (4.2.4) and $S(0) = 0$ imply that function $S$ is strictly increasing. Hence, there hold the implications

$$\phi(\infty,u) = \int_0^\infty S\big(\psi(s,u)\big)ds < \infty \implies \lim_{s\to\infty} S\big(\psi(s,u)\big) = 0 \implies \lim_{s\to\infty} \psi(s,u) = 0,$$

the conclusion of which is (4.2.1). $\square$

In the following we will consider stationary CBI processes. We will assume that the branching and the immigration distributions have finite second moments, conditions (4.2.3–4.2.4) hold and the CBI process starts from its stationary distribution.

**4.2.4 Lemma.** *The Laplace transform of the one-dimensional distribution of a non-zero, stationary CBI process is*

$$\varphi(u) = \exp\big(\phi(\infty, u)\big) = \exp\left(\int_0^u \frac{S(v)}{-R(v)}\, dv\right), \quad u \geq 0.$$

**Proof.** We refer to the proof of Theorem 4.2.3. For any $u > 0$, whether $t_0$ is infinite or finite, (4.2.8) holds, $\lim_{t \to t_0} \psi(t, u) = 0$ and function $-S(v)/R(v)$, $0 < v < u$, is integrable, hence we obtain the statement. □

**4.2.5 Remark.** Let us consider the following sets:
• the set of stationary OU type processes with non-negative time parameter;
• the set of stationary CBI processes.
The intersection of these two sets is the set of stationary OU type processes with non-negative time parameter and non-negative state space, i.e. the set of processes in Example 4.1.11 with parameter $\alpha < 0$. In other words, a stationary CBI process with parameters

$$\sigma^2 = 0,\ \alpha < 0,\ m_1 \equiv 0 \quad \text{and} \quad \vartheta = 0,\ m_2 \not\equiv 0, \tag{4.2.10}$$

is, at the same time, a stationary OU type process with parameter $\alpha < 0$ and driving process corresponding to the Lévy measure $m_2$. This parametrization will enable the results of this chapter to be compared with the corresponding ones of Chapter 3 and so, to be checked.

**4.2.6 Remark.** We now apply the above checking method: for a non-trivial, non-negative OU type process with state space $[0, \infty)$ (see Example 4.1.11) and with a finite second moment, Theorem 4.2.3 gives the well-known equivalent condition of stationarity: $\alpha < 0$.

**4.2.7 Lemma.** *Let* $\{X(t),\, t \geq 0\}$ *be a stationary CBI process. The Laplace transforms of its 1-, 2- and 3-dimensional distributions are*

$$\varphi_{X(t_1)}(u_1) = e^{-\phi(\infty, u_1)}, \tag{4.2.11}$$

$$\begin{aligned}\varphi_{X(t_1),X(t_2)}\big(u_1, u_2\big) &= e^{-\phi(\infty, u_1 + \psi(t_2 - t_1, u_2))} \\ &\quad \times e^{-\phi(t_2 - t_1, u_2)},\end{aligned} \tag{4.2.12}$$

$$\begin{aligned}\varphi_{X(t_1),X(t_2),X(t_3)}(u_1,u_2,u_3) &= e^{-\phi(\infty,u_1+\psi(t_2-t_1,u_2+\psi(t_3-t_2,u_3)))}\\ &\quad\times e^{-\phi(t_2-t_1,u_2+\psi(t_3-t_2,u_3))}\\ &\quad\times e^{-\phi(t_3-t_2,u_3)}, \qquad (4.2.13)\end{aligned}$$

$0 \le t_1 \le t_2 \le t_3$ *and* $0 \le u_1, u_2, u_3$. *In general, the following recursion holds:*

$$\begin{aligned}&\varphi_{X(t_1),\dots,X(t_m)}(u_1,\dots,u_m)\\ &\qquad= \varphi_{X(t_1),\dots,X(t_{m-1})}\big(u_1,\dots,u_{m-2},u_{m-1}+\psi(t_m-t_{m-1},u_m)\big)\\ &\qquad\quad\times e^{\phi(t_m-t_{m-1},u_m))}, \quad m \ge 2 \qquad (4.2.14)\end{aligned}$$

$0 \le t_1 \le t_2 \le \dots$ *and* $0 \le u_1, u_2, \dots$.

**Proof.** With the help of the Markov property and the time homogeneity of $\{X(t),\, t \ge 0\}$, the recursion (4.2.14) can be obtained as follows:

$$\begin{aligned}&\varphi_{X(t_1),\dots,X(t_m)}(u_1,\dots,u_m)\\ &\quad= \mathsf{E}\, e^{-u_1X(t_1)-\dots-u_mX(t_m)}\\ &\quad= \mathsf{E}\Big(e^{-u_1X(t_1)-\dots-u_{m-1}X(t_{m-1})}\mathsf{E}\big(e^{-u_mX(t_m)}\big|X(t_{m-1})\big)\Big)\\ &\quad= \mathsf{E}\Big(e^{-u_1X(t_1)-\dots-u_{m-1}X(t_{m-1})}\varphi_{|X(t_{m-1})}(t_m-t_{m-1},u_m)\Big)\\ &\quad= \mathsf{E}\Big(e^{-u_1X(t_1)-\dots-u_{m-1}X(t_{m-1})}e^{-\phi(t_m-t_{m-1},u_m)-\psi(t_m-t_{m-1},u_m)\,X(t_{m-1})}\Big)\\ &\quad= \mathsf{E}\Big(e^{-u_1X(t_1)-\dots-(u_{m-1}+\psi(t_m-t_{m-1},u_m))X(t_{m-1})}\Big)\, e^{-\phi(t_m-t_{m-1},u_m)}\\ &\quad= \varphi_{X(t_1),\dots,X(t_{m-1})}\big(u_1,\dots,u_{m-2},u_{m-1}+\psi(t_m-t_{m-1},u_m)\big)\\ &\qquad\times e^{\phi(t_m-t_{m-1},u_m))}, \quad m \ge 2.\end{aligned}$$

The Laplace transform of the one-dimensional distribution of $\{X(t),\, t \ge 0\}$ is (4.2.11). Using the recursion (4.2.14) and the initial formula (4.2.11), the expressions (4.2.12–4.2.13) follow immediately. □

## 4.3 The joint cumulants of the stationary CBI process

**4.3.1 Lemma.** *If the branching and the immigration distributions of a CBI process have finite moments of all orders, then the following statements hold:*
*i)* $R$, $S \in C^\infty([0,\infty))$ *(i.e. infinitely differentiable) and the derivatives are*

$$R'(0) = \alpha + \int_1^\infty (y-1)\, m_1(dy)\,, \tag{4.3.1}$$

$$R''(0) = -\sigma^2 - \int_0^\infty y^2\, m_1(dy)\,,$$

$$R^{(k)}(0) = (-1)^{k+1} \int_0^\infty y^k\, m_1(dy)\,, \quad k \geq 3,$$

$$S'(0) = \vartheta + \int_0^\infty y\, m_2(dy)\,, \tag{4.3.2}$$

$$S^{(k)}(0) = (-1)^{k+1} \int_0^\infty y^k\, m_2(dy)\,, \quad k \geq 2; \tag{4.3.3}$$

*ii)* $\phi$, $\psi \in C^\infty\big([0,\infty) \times [0,\infty)\big)$.

**Proof.** i) The finiteness of the moments of the branching distribution is equivalent with the finiteness of the moments of the Lévy measure $m_1$ and the same is true for the immigration distribution and the Lévy measure $m_2$. Hence follow the statements.
ii) By Duffie et al. [9, Lemma 6.5 i)] the statement can be obtained from (i) above. □

**4.3.2 Lemma.** *If the branching and the immigration distributions of a CBI process have finite moments of all orders, then*

$$\partial_2 \psi(t, 0) = e^{R'(0)t}, \tag{4.3.4}$$

$$\partial_2^2\psi(t,0) = \frac{R''(0)}{R'(0)}\, e^{2R'(0)t} - \frac{R''(0)}{R'(0)}\, e^{R'(0)t}, \tag{4.3.5}$$

$$\begin{aligned}\partial_2^3\psi(t,0) &= \frac{R'(0)R'''(0) + 3(R''(0))^2}{2(R'(0))^2}\, e^{3R'(0)t} - 3\,\frac{(R''(0))^2}{(R'(0))^2}\, e^{2R'(0)t} \\ &\quad + \frac{3(R''(0))^2 - R'(0)R'''(0)}{2(R'(0))^2}\, e^{R'(0)t},\end{aligned} \tag{4.3.6}$$

*each equation holds for $t \geq 0$. In general, for any $k \in \mathbb{N}$ we have*

$$\partial_2^k\psi(t,0) = Q_k\big(e^{R'(0)t}\big), \quad t \geq 0, \tag{4.3.7}$$

*where $Q_k(x)$ is a polynomial of degree at most $k$, without zero order term. Polynomials $Q_k(x)$, $k \in \mathbb{N}$, depend neither on $t$, nor $(\vartheta,\, m_2)$ (but may depend on $(\sigma^2,\, \alpha,\, m_1)$) and they are invariant under the transforms $R(t) \longmapsto cR(t)$, $t \geq 0$, i.e. $(\sigma^2, \alpha, m_1) \longmapsto (c\sigma^2, c\alpha, cm_1)$, $c > 0$.*

**Proof.** By Lemma 4.3.1 $\psi$ is infinitely differentiable. Let us prove (4.3.4). Differentiating the equation $\partial_1\psi(t,u) = R(\psi(t,u))$ (i.e. (4.1.5)) at $u = 0$, we obtain

$$\partial_2\partial_1\psi(t,u) = R'(\psi(t,u))\partial_2\psi(t,u) \tag{4.3.8}$$

$$\partial_1\partial_2\psi(t,0) = R'(0)\partial_2\psi(t,0)$$

$$\partial_2\psi(t,0) = e^{R'(0)t},$$

because the initial condition is $\partial_2\psi(0,0) = \frac{d}{du}u|_{u=0} = 1$. Thus, we have proved (4.3.4).

Let us look at (4.3.5). Differentiating the equation (4.3.8) at $u = 0$, we obtain

$$\partial_2^2\partial_1\psi(t,u) = R''(\psi(t,u))\,(\partial_2\psi(t,u))^2 + R'(\psi(t,u))\partial_2^2\psi(t,u) \tag{4.3.9}$$

$$\partial_2^2\partial_1\psi(t,0) = R''(0)\,(\partial_2\psi(t,0))^2 + R'(0)\partial_2^2\psi(t,0)$$

$$\partial_1\partial_2^2\psi(t,0) = R'(0)\partial_2^2\psi(t,0) + R''(0)e^{2R'(0)t}.$$

The last row is a first order linear differential equation for $f(t) \doteq \partial_2^2\psi(t,0)$:

$$f'(t) = R'(0)f(t) + R''(0)e^{2R'(0)t}$$
$$f(0) = 0,$$

the solution of which is

$$\partial_2^2\psi(t,0) = f(t) = \frac{R''(0)}{R'(0)}\left(e^{2R'(0)t} - e^{R'(0)t}\right), \quad t \geq 0,$$

thus, we have proved (4.3.5).

The proof of (4.3.6) is similar. Differentiating the equation (4.3.9) at $u = 0$, we obtain

$$\partial_2^3\partial_1\psi(t,u) = R'''(\psi(t,u))\left(\partial_2\psi(t,u)\right)^3$$
$$+ 3R''(\psi(t,u))\partial_2\psi(t,u)\partial_2^2\psi(t,u) + R'(\psi(t,u))\partial_2^3\psi(t,u)$$

$$\partial_2^3\partial_1\psi(t,0) = R'''(0)\left(\partial_2\psi(t,0)\right)^3 + 3R''(0)\partial_2\psi(t,0)\partial_2^2\psi(t,0) + R'(0)\partial_2^3\psi(t,0)$$

$$\partial_1\partial_2^3\psi(t,0) = R'(0)\partial_2^3\psi(t,0) + \frac{R'(0)R'''(0) + 3(R''(0))^2}{R'(0)}e^{3R'(0)t} - 3\frac{(R''(0))^2}{R'(0)}e^{2R'(0)t}.$$

The last equation is a first order linear differential equation for the function $f(t) \doteq \partial_2^3\psi(t,0)$:

$$f'(t) = R'(0)f(t) + \frac{R'(0)R'''(0) + 3(R''(0))^2}{R'(0)}e^{3R'(0)t} - 3\frac{(R''(0))^2}{R'(0)}e^{2R'(0)t}$$
$$f(0) = 0,$$

the solution of which is

$$\partial_2^3\psi(t,0) = f(t) = \frac{R'(0)R'''(0) + 3(R''(0))^2}{2(R'(0))^2}e^{3R'(0)t} - 3\frac{(R''(0))^2}{(R''(0))^2}e^{2R'(0)t}$$
$$+ \frac{3(R''(0))^2 - R'(0)R'''(0)}{2(R'(0))^2}e^{R'(0)t}, \quad t \geq 0,$$

hence (4.3.6) is proved.

In the general case we use induction on the order of the derivative. For $k = 1$ the statement is (4.3.4). In the induction step from $k-1$ to $k \geq 2$ we apply Faá di Bruno's formula (the higher order derivatives of a composite function) for the right-hand side of the equation $\partial_1 \psi(t,u) = R(\psi(t,u))$:

$$\begin{aligned}\partial_2^k \partial_1 \psi(t,u) &= \partial_2^k R(\psi(t,u)) \\ &= \sum_{\underline{r}_{(k)}} \frac{k!}{r_1! \cdots r_k!} R^{(r)}(\psi(t,u)) \left(\frac{1}{1!}\,\partial_2 \psi(t,u)\right)^{r_1} \cdots \left(\frac{1}{k!}\,\partial_2^k \psi(t,u)\right)^{r_k},\end{aligned}$$

where the sum is over all partitions of the integer $k$, $\underline{r}_{(k)} \doteq (r_1, \ldots, r_k)$ and $r_j$ is the number of such sets in the actual partition, which have exactly $j$ elements, $j = 1, \ldots, k$. So, $r_1 + 2r_2 + \ldots + kr_k = k$ and $r \doteq r_1 + \ldots + r_k$ is the number of sets constituting the partition. Hence we obtain for $u = 0$:

$$\partial_2^k \partial_1 \psi(t,0) = \sum_{\underline{r}_{(k)}} \frac{k!}{r_1! \cdots r_k!} R^{(r)}(0) \left(\frac{1}{1!}\,\partial_2 \psi(t,0)\right)^{r_1} \cdots \left(\frac{1}{k!}\,\partial_2^k \psi(t,0)\right)^{r_k}. \tag{4.3.10}$$

On the left-hand side let us change the order of the derivatives and on the right-hand side let us separate the term corresponding to the trivial partition with one element:

$$\begin{aligned}&\partial_1 \partial_2^k \psi(t,0) = R'(0) \partial_2^k \psi(t,0) \\ &+ \sum_{\underline{r}_{(k-1)}} \frac{k!}{r_1! \cdots r_{k-1}!} R^{(r)}(0) \left(\frac{1}{1!}\,\partial_2 \psi(t,0)\right)^{r_1} \cdots \left(\frac{1}{(k-1)!}\,\partial_2^{k-1} \psi(t,0)\right)^{r_{k-1}},\end{aligned} \tag{4.3.11}$$

since $r_k = 0$ in each term of the sum in (4.3.10), except for one term, which we have detached from the others, into the first row of (4.3.11). In each term of the sum in the second row of (4.3.11), each factor of the form

$$\left(\frac{1}{j\,!}\,\partial_2^j \psi(t,0)\right)^{r_j}, \qquad j = 1, \ldots, k-1,$$

is, by the induction assumption, an at most $jr_j$-th degree polynomial of $e^{R'(0)t}$, without zero order term and with coefficients depending neither on $t$ nor $\vartheta$ and $m_2$. Therefore, the sum in the second row of (4.3.11) is an at most $\sum_{j=1}^{k-1} j\, r_j = \sum_{j=1}^{k} j\, r_j = k$-th degree polynomial of $e^{R'(0)t}$, say $q_k\big(e^{R'(0)t}\big)$, such that $q_k(x)$

has no zero order term and it depends neither on $t$ nor $\vartheta$ and $m_2$ (because $R'(0)$ does not depend on $\vartheta$ and $m_2$, see (4.3.1)). Let us consider for fixed $u$ the differential equation obtained from (4.3.11), as an ordinary differential equation in $t$:

$$\partial_1 \partial_2^k \psi(t,0) = R'(0) \partial_2^k \psi(t,0) + q_k\left(e^{R'(0)t}\right).$$

Under the initial condition $\partial_2^k \psi(0,0) = \frac{d^k}{du^k} u\big|_{u=0} = 0$ (since we have $k \geq 2$) its solution is

$$\begin{aligned}
\partial_2^k \psi(t,0) &= \int_0^t e^{R'(0)(t-s)} q_k\left(e^{R'(0)s}\right) ds = e^{R'(0)t} \int_0^t e^{-R'(0)s} \sum_{j=1}^k c_j\, e^{jR'(0)s} ds \\
&= \sum_{j=1}^k c_j\, e^{R'(0)t} \frac{e^{(j-1)R'(0)t} - 1}{R'(0)} = -\sum_{j=2}^k \frac{c_j}{R'(0)} e^{R'(0)t} + \sum_{j=2}^k \frac{c_j}{R'(0)} e^{jR'(0)t} \\
&= Q_k\left(e^{R'(0)t}\right), \quad t \geq 0,
\end{aligned}$$

where $Q_k(x)$ is an at most $k$-th degree polynomial, without zero order term and depending neither on $t$, nor $\vartheta$ and $m_2$. Transform $R(t) \longmapsto cR(t)$, $t \geq 0$, implies the same transform on $q_k(x)$ (because of the factor $R'(0)$ in (4.3.11)), hence polynomial $Q_k(x)$ is invariant under it. □

**4.3.3 Lemma.** *If the branching and the immigration distributions of a non-zero stationary CBI process have finite moments of all orders, then*

$$\partial_2 \phi(\infty, 0) = \frac{1}{-R'(0)} S'(0) \tag{4.3.12}$$

$$\partial_2^2 \phi(\infty, 0) = \frac{R''(0)}{2(R'(0))^2} S'(0) + \frac{1}{-2R'(0)} S''(0) \tag{4.3.13}$$

$$\begin{aligned}
\partial_2^3 \phi(\infty, 0) = &\frac{-R'(0)R'''(0) + 3(R''(0))^2}{-6(R'(0))^3} S'(0) + \frac{R''(0)}{2(R'(0))^2} S''(0) \\
&+ \frac{1}{-3R'(0)} S'''(0).
\end{aligned} \tag{4.3.14}$$

*In general, for any* $k \in \mathbb{N}$ *we have*

$$\partial_2^k \phi(\infty, 0) = \sum_{j=1}^{k} \beta_{k,j} S^{(j)}(0), \tag{4.3.15}$$

*where the coefficients* $\beta_{k,j}$, $j = 1, \ldots, k$, *do not depend on the immigration parameter* $(\vartheta, m_2)$. *Moreover, the transform* $R(t) \longmapsto cR(t)$, $t \geq 0$ *(equivalently, the transform* $(\sigma^2, \alpha, m_1) \longmapsto (c\sigma^2, c\alpha, cm_1)$*) implies the change of the* $\beta_{k,j}$*s as* $(\beta_{k,1}, \ldots, \beta_{k,k}) \longmapsto (\beta_{k,1}/c, \ldots, \beta_{k,k}/c)$, $c > 0$.

**Proof.** Consider the equation

$$\phi(\infty, u) = \int_0^\infty S(\psi(s, u))\, ds, \tag{4.3.16}$$

arising from (4.1.6). Taking the derivative of both sides at $u = 0$ and using (4.3.4), we obtain

$$\partial_2 \phi(\infty, 0) = \int_0^\infty S'(\psi(s, 0)) \partial_2 \psi(s, 0)\, ds = \int_0^\infty S'(0)\, e^{R'(0)s}\, ds = \frac{1}{-R'(0)}\, S'(0),$$

which is (4.3.12). Taking the second derivative at $u = 0$ of both sides of (4.3.16) and using (4.3.5), we obtain

$$\begin{aligned}
\partial_2^2 \phi(\infty, 0) &= \int_0^\infty \Big( S''(\psi(s, 0))\, (\partial_2 \psi(s, 0))^2 + S'(\psi(s, 0)) \partial_2^2 \psi(s, 0) \Big)\, ds \\
&= \int_0^\infty \left( S''(0)\, e^{2R'(0)s} + S'(0) \frac{R''(0)}{R'(0)} \left( e^{2R'(0)s} - e^{R'(0)s} \right) \right) ds \\
&= \frac{R''(0)}{2(R'(0))^2}\, S'(0) + \frac{1}{-2R'(0)}\, S''(0),
\end{aligned}$$

which is (4.3.13). By a similar computation, using (4.3.6), we get (4.3.14).

For an arbitrary $k \in \mathbb{N}$ we have

$$\partial_2^k S(\psi(s, 0)) = \sum_{j=1}^{k} \lambda_{k,j}(s)\, S^{(j)}(\psi(s, 0)) = \sum_{j=1}^{k} \lambda_{k,j}(s)\, S^{(j)}(0)\,,$$

where coefficients $\lambda_{k,j}(s)$, $j = 1, \ldots, k$, are are some products of the polynomials $Q_\ell\big(e^{R'(0)s}\big)$ given in Lemma 4.3.2. Hence, $\lambda_{k,j}(s)$, $j = 1, \ldots, k$, are polynomials of $e^{R'(0)s}$ without zero order term. So, by (4.2.3), the $\lambda_{k,j}(s)$-es are intagrable on $[0, \infty)$. Therefore, we can take the $k$-th order derivative at $u = 0$ of both sides of the equation $\phi(\infty, u) = \int_0^\infty S(\psi(s, u))\, ds$ (arising from (4.1.6)) and obtain

$$\partial_2^k \phi(\infty, 0) = \sum_{j=1}^{k} \int_0^\infty \lambda_{k,j}(s)\, ds\, S^{(j)}(0) = \sum_{j=1}^{k} \beta_{k,j} S^{(j)}(0)\,.$$

Each coefficient $\beta_{k,j}$ is the integral of $\lambda_{k,j}(s)$, hence the integral of some products of the polynomials $Q_\ell\big(e^{R'(0)s}\big)$. Neither the polynomials $Q_\ell(x)$, nor $R'(0)$ depend on the immigration parameter $(\vartheta, m_2)$, therefore the $\beta_{k,j}$ s do not depend on them either. Moreover, because of the integration, each $\beta_{k,j}$ is the product of two factors. The one factor is invariant under the transform $R(t) \longmapsto cR(t)$, $t \geq 0$, because of the last statement of Lemma 4.3.2 while the other factor, arising from the integration, is the fraction $\frac{1}{-R'(0)}$. Hence follows the last statement of the lemma. □

**4.3.4 Proposition.** *If $\{X(t),\, t \geq 0\}$ is a stationary CBI process with branching and immigration distributions of finite moments of all orders, then*

$$\mathsf{E}\, X(t) = \frac{1}{-R'(0)}\, S'(0), \quad t \geq 0,$$

$$\mathsf{cum}\,\big(X(t_1), X(t_2)\big) = \left(\frac{-R''(0)}{2(R'(0))^2}\, S'(0) - \frac{1}{-2R'(0)}\, S''(0)\right) e^{R'(0)(t_2^* - t_1^*)},$$

$$\begin{aligned}\mathsf{cum}\,\big(X(t_1), X(t_2), X(t_3)\big) = &\left(\frac{R'''(0)}{6(R'(0))^2}\, S'(0) + \frac{1}{-3R'(0)}\, S'''(0)\right)\\ &\qquad \times e^{2R'(0)(t_2^* - t_1^*)} e^{R'(0)(t_3^* - t_2^*)}\\ &+ \left(\frac{(R''(0))^2}{-2(R'(0))^3}\, S'(0) + \frac{R''(0)}{2(R'(0))^2}\, S''(0)\right)\\ &\qquad \times e^{R'(0)(t_2^* - t_1^*)} e^{R'(0)(t_3^* - t_2^*)},\end{aligned}$$

*$t_1, t_2, t_3 \geq 0$. In general, for any $m \geq 2$, its $m$-fold joint cumulant is some multivariate polynomial of $e^{R'(0)(t_2^*-t_1^*)}, \ldots, e^{R'(0)(t_m^*-t_{m-1}^*)}$:*

$$\mathsf{cum}\left(X(t_1), \ldots, X(t_m)\right) = P_m\left(e^{R'(0)(t_2^*-t_1^*)}, \ldots, e^{R'(0)(t_m^*-t_{m-1}^*)}\right), \qquad (4.3.17)$$

*$t_1, \ldots, t_m \geq 0$, where the polynomial*

$$P_m(\underline{x}) = P_m(x_1, \ldots x_{m-1}) = \sum_{k_1, \ldots, k_{m-1}} c_{k_1,\ldots,k_{m-1}} x_1^{k_1} \cdots x_{m-1}^{k_{m-1}} \qquad (4.3.18)$$

*has the following properties:*

*i) $c_{k_1,\ldots,k_{m-1}} = \sum_{j=1}^{m} \lambda_{m,j}(k_1, \ldots, k_{m-1}) S^{(j)}(0)$ where the $\lambda_{m,j}(k_1, \ldots, k_{m-1})$s do not contain the parameter $(\vartheta, m_2)$,*

*ii) all the coefficients $c_{k_1,\ldots,k_{m-1}}$ are positive;*

*iii) the transform $(\sigma^2, \alpha, m_1) \longmapsto (c\sigma^2, c\alpha, cm_1)$, i.e. $R(t) \longmapsto cR(t)$, $t \geq 0$, implies the change of polynomial $P_m(\underline{x})$ as $P_m(\underline{x}) \longmapsto \frac{1}{c} P_m(\underline{x})$, $c > 0$;*

*iv) the transform $(\vartheta, m_2) \longmapsto (c\vartheta, cm_2)$, i.e. $S(t) \longmapsto cS(t)$, $t \geq 0$, implies the change of polynomial $P_m(\underline{x})$ as $P_m(\underline{x}) \longmapsto cP_m(\underline{x})$, $c > 0$.*

**Proof.** By Lemma 4.2.7, 4.3.3 and 4.3.2 we obtain

$$\mathsf{E}\, X(t) = -\frac{d}{du} \log \varphi_{X(t)}(0) = \partial_2 \phi(\infty, 0) = \frac{1}{-R'(0)} S'(0),$$

$$\mathsf{cum}\left(X(t_1), X(t_2)\right) = \mathsf{cum}\left(X(t_1^*), X(t_2^*)\right)$$

$$\begin{aligned}
&= \partial_1 \partial_2 \log \varphi_{X(t_1^*), X(t_2^*)}(0, 0) \\
&= -\left.\frac{\partial^2}{\partial u_1 \partial u_2} \phi\left(\infty, u_1 + \psi(t_2^* - t_1^*, u_2)\right)\right|_{\underline{u}=\underline{0}} \\
&= -\partial_2^2 \phi(\infty, 0) \partial_2 \psi(t_2^* - t_1^*, 0) \\
&= \left(\frac{-R''(0)}{2(R'(0))^2} S'(0) - \frac{1}{-2R'(0)} S''(0)\right) e^{R'(0)(t_2^*-t_1^*)},
\end{aligned}$$

$$
\begin{aligned}
&\mathsf{cum}\,\big(X(t_1),X(t_2),X(t_3)\big) = \mathsf{cum}\,\big(X(t_1^*),X(t_2^*),X(t_3^*)\big)\\
&\qquad = -\partial_1\partial_2\partial_3 \log\varphi_{X(t_1^*),X(t_2^*),X(t_3^*)}(0,0,0)\\
&\qquad = \frac{\partial^3}{\partial u_1\partial u_2\partial u_3}\,\phi\Big(\infty, u_1+\psi\big(t_2^*-t_1^*, u_2+\psi(t_3^*-t_2^*,u_3)\big)\Big)\Big|_{\underline{u}=\underline{0}}\\
&\qquad = \partial_2^3\phi(\infty,0)\,(\partial_2\psi(t_2^*-t_1^*,0))^2\,\partial_2\psi(t_3^*-t_2^*,0)\\
&\qquad\quad + \partial_2^2\phi(\infty,0)\partial_2^2\psi(t_2^*-t_1^*,0)\partial_2\psi(t_3^*-t_2^*,0)\\
&\qquad = \left(\frac{R'''(0)}{6(R'(0))^2}\,S'(0)+\frac{1}{-3R'(0)}\,S'''(0)\right)e^{2R'(0)(t_2^*-t_1^*)}e^{R'(0)(t_3^*-t_2^*)}\\
&\qquad\quad + \left(\frac{(R''(0))^2}{-2(R'(0))^3}\,S'(0)+\frac{R''(0)}{2(R'(0))^2}\,S''(0)\right)e^{R'(0)(t_2^*-t_1^*)}e^{R'(0)(t_3^*-t_2^*)}.
\end{aligned}
$$

Proof of the general expression (4.3.17): By (4.2.14) we have

$$
\begin{aligned}
&\partial_1\ldots\partial_m \log\varphi_{X(t_1),\ldots,X(t_m)}(\underline{0})\\
&= \frac{\partial^m}{\partial u_m\ldots\partial u_1}\log\varphi_{X(t_1),\ldots,X(t_{m-1})}\big(u_1,\ldots,u_{m-2},u_{m-1}+\psi(t_m-t_{m-1},u_m)\big)\Big|_{\underline{u}=\underline{0}},
\end{aligned}
$$

$0\le t_1\le\ldots\le t_m$. Recursively substituting into the logarithm of (4.2.14), at each step it is only the first term which contains all variables $u_1,\ldots,u_m$. Retaining only this term, we obtain the following sequence of expressions:

$$
\begin{aligned}
&\log\varphi_{X(t_1),\ldots,X(t_m)}\big(u_1,\ldots,u_m\big),\\
&\log\varphi_{X(t_1),\ldots,X(t_{m-1})}\big(u_1,\ldots,u_{m-2},u_{m-1}+\psi(t_m-t_{m-1},u_m)\big),\\
&\log\varphi_{X(t_1),\ldots,X(t_{m-2})}\Big(u_1,\ldots,u_{m-3},\\
&\qquad\qquad u_{m-2}+\psi\big(t_{m-1}-t_{m-2},u_{m-1}+\psi(t_m-t_{m-1},u_m)\big)\Big),\\
&\vdots
\end{aligned}
$$

$$\vdots$$

$$\log\varphi_{X(t_1)}\Big(u_1+\psi\Big(t_2-t_1,u_2+\psi\big(t_3-t_2,\dots,u_{m-1}+\psi(t_m-t_{m-1},u_m)\dots\big)\Big)\Big)$$

$$=-\phi\bigg(\infty,\Big(u_1+\psi\Big(t_2-t_1,u_2+\psi\big(t_3-t_2,\dots,u_{m-1}+\psi(t_m-t_{m-1},u_m)\dots\big)\Big)\Big)\bigg),$$

$0\le t_1\le\dots\le t_m$. Hence we have

$$\begin{aligned}
&\mathsf{cum}\,\big(X(t_1),\dots,X(t_m)\big)=\mathsf{cum}\,\big(X(t_1^*),\dots,X(t_m^*)\big)\\
&\quad=(-1)^m\partial_1\dots\partial_m\log\varphi_{X(t_1^*),\dots,X(t_m^*)}(\underline{0})\\
&\quad=(-1)^{m+1}\frac{\partial^m}{\partial u_m\dots\partial u_1}\,\phi\bigg(\infty,\Big(u_1+\psi\Big(t_2^*-t_1^*,u_2+\psi\big(t_3^*-t_2^*,\dots,\\
&\qquad\qquad u_{m-1}+\psi(t_m^*-t_{m-1}^*,u_m)\dots\big)\Big)\Big)\bigg)\bigg|_{\underline{u}=\underline{0}}.
\end{aligned}\tag{4.3.19}$$

Performing the differentiations we obtain a sum wherein each term will be (aside from the $\pm$ sine) a product of derivatives taken at $u=0$ (because of $\psi(t,0)=0$) and in each term there will be exactly one factor of the form $\partial_2^j\phi(\infty,0)$, with some $j$, i.e. of the form (4.3.15). The other factors will be of the form $\partial_2^j\psi(t_{i+1}^*-t_i^*,0)$, with some $j$, i.e. of the form (4.3.7). So, the result of (4.3.19) can be written into some multivariate polynomial of the variables $e^{R'(0)(t_2^*-t_1^*)},\dots,e^{R'(0)(t_m^*-t_{m-1}^*)}$, where the coefficients of the polynomial are linear combinations of $S'(0),\dots,S^{(m)}(0)$ with coefficients depending only on the branching parameters. Thus, we have proved (4.3.17–4.3.18) and i).

The statements iii) and iv) follow at once, using the last statements of Lemma 4.3.2 and Lemma 4.3.3.

Proof of ii): We premise that the joint cumulants $\mathsf{cum}\,\big(X(t_1),\dots,X(t_m)\big)$ are all non-negative, because each one is the joint moment of the corresponding Lévy measure, which is concentrated on $[0,\infty)^m$ (recall that the $X(t)$s are non-negative).

• First we prove that there exists a $y_0>0$ such that statement ii) holds for each $y>y_0$ in the particular case when the immigration parameters are $\vartheta=0$ and the Lévy measure $m_2$ is the Dirac measure concentrated on $y$ with mass 1, i.e.

$$m_2(B) = \begin{cases} 1 & \text{if } y \in B, \\ 0 & \text{otherwise}, \end{cases} \qquad B \in \mathcal{B}(0,\infty). \tag{4.3.20}$$

Then $S(u) = 1 - e^{-uy}$, $u \geq 0$, by (4.1.8) and $S^{(j)}(0) = (-1)^{j+1} y^j$, $j \in \mathbb{N}$, by (4.3.2–4.3.3).

Now, combining (4.3.17), (4.3.18) and i), we have

$$\begin{aligned}\mathsf{cum}\left(X(t_1),\ldots,X(t_m)\right) &= \sum_{\underline{k}} c_{\underline{k}} \left(e^{R'(0)(t_2^*-t_1^*)}\right)^{k_1} \cdots \left(e^{R'(0)(t_m^*-t_{m-1}^*)}\right)^{k_{m-1}} \\ &= \sum_{\underline{k}} \underbrace{\sum_{j=1}^{m} \lambda_{m,j}(\underline{k})(-1)^{j+1} y^j}_{c_{\underline{k}}} \left(e^{R'(0)(t_2^*-t_1^*)}\right)^{k_1} \cdots \left(e^{R'(0)(t_m^*-t_{m-1}^*)}\right)^{k_{m-1}},\end{aligned} \tag{4.3.21}$$

where $\underline{k} \doteq (k_1,\ldots,k_{m-1})$. Sorting the terms of the multivariate polynomial $P_m$ by $\underline{k}$ in decreasing lexicographic order, the degrees of the greatest and the smallest terms will be $\underline{k} = (m-1, m-2, \ldots, 1)$ and $\underline{k} = (1,\ldots,1)$, respectively (see (4.3.19)). So, the upper equation in (4.3.21) becomes

$$\begin{aligned}&\mathsf{cum}\left(X(t_1),\ldots,X(t_m)\right) \\ &\quad= c_{m-1,m-2,\ldots,1}\left(e^{R'(0)(t_2^*-t_1^*)}\right)^{m-1}\left(e^{R'(0)(t_3^*-t_2^*)}\right)^{m-2} \cdots e^{R'(0)(t_m^*-t_{m-1}^*)} \\ &\qquad\vdots \\ &\qquad + c_{k_1,\ldots,k_{m-1}} \left(e^{R'(0)(t_2^*-t_1^*)}\right)^{k_1} \cdots \left(e^{R'(0)(t_m^*-t_{m-1}^*)}\right)^{k_{m-1}} \\ &\qquad\vdots \\ &\qquad + c_{1,\ldots,1}\, e^{R'(0)(t_2^*-t_1^*)} \cdots e^{R'(0)(t_m^*-t_{m-1}^*)}.\end{aligned} \tag{4.3.22}$$

Let $\Delta t > 0$, $t_1 \doteq 0$ and $t_{i+1} \doteq t_i + 10^{m-i}\Delta t$, $i = 1,\ldots,m-1$. Then $t_{i+1} - t_i = 10^{m-i}\Delta t$, $i = 1,\ldots,m-1$ and for every $\underline{k}$, $\underline{k}'$, $\sum_{i=1}^{m-1} k_i(t_{i+1}-t_i) > \sum_{i=1}^{m-1} k_i{}'(t_{i+1}-t_i)$ if and only if $\underline{k} > \underline{k}'$. Therefore, when $\Delta t \to \infty$, i.e. when $e^{R'(0)\Delta t} \to 0$, the dominant term (i.e., which converges to zero most slowly) is the last one, the second dominant is the last but one, etc. and the least dominant is the first term. Hence, $c_{1,\ldots,1} > 0$ (otherwise the cumulant would be negative for large enough values of $\Delta t$).

It is easily seen from (4.3.19) that $S^{(m)}(0)$ (which is now constant times $y^m$) appears only in $c_{m-1,m-2,\dots,1}$. Therefore, when $y \to \infty$, the dominant term (i.e., which diverges most quickly) in (4.3.22) is the first one. Hence follows that for large enough values of $y$ the sum of the terms in (4.3.22) but for the last one (which we have already done away with) is non-negative. Therefore, when $\Delta t \to \infty$, i.e. when $e^{R'(0)\Delta t} \to 0$, the dominant term of that sum is the last one (i.e. the second last of the original terms), hence it must be positive, as well its coefficient $c_{\underline{k}}$.

On the other hand, when $y \to \infty$, the sum of the terms in (4.3.22) but for the last two terms (which we have already done away with) is non-negative (since the dominant term is the first one). Continuing this backward procedure we obtain by induction that there exists some $y_0 > 0$ such that for every $y > y_0$, all the $c_{\underline{k}}$s corresponding to $y$ (see the underbraced expression in the third row of (4.3.21)) are positive. Thus, we have proved statement ii) for the case of $\vartheta = 0$, $m_2$ being a Dirac measure concentrated on $y > y_0$ and branching parameter $(\sigma^2, \alpha, m_1)$ being arbitrary (and, of course, under the moment conditions of the theorem).

- Now we prove ii) for stationary CBI processes (satisfying the moment conditions of the theorem) with parameters $\vartheta = 0$, $m_2$ with $\mathrm{supp}(m_2) \subseteq (y_0, \infty)$, where $y_0$ is the threshold number given above and with an arbitrary branching parameter $(\sigma^2, \alpha, m_1)$. Integrating equation (4.3.22) with respect to $m_2$ and using (4.3.2–4.3.3), we obtain ii) for this case, too.

- In the next step we prove that ii) holds if $\vartheta = 0$, $m_2$ is such that

$$\mathrm{supp}(m_2) \subseteq (y_0, \infty) \quad \text{with some } y_0 > 0 \tag{4.3.23}$$

(i.e. we omit the condition that $y_0$ is large enough) and there are no additional restrictions for the branching parameter $(\sigma^2, \alpha, m_1)$. Now, for every $c > 0$, we have

$$S(cu) = \int_0^\infty \left(1 - e^{-uy}\right) m_2\left(\tfrac{1}{c}\, dy\right), \quad u \geq 0,$$

so, with the new Lévy measure $\widetilde{m_2}(B) \doteq m_2\left(\frac{1}{c}B\right)$, $B \in \mathcal{B}(0,\infty)$, and the new immigration Laplace exponent function $\widetilde{S}(u) \doteq S(cu)$, $u \geq 0$, we have

$$\widetilde{S}(u) = \int_0^\infty \left(1 - e^{-uy}\right) \widetilde{m_2}(dy), \quad u \geq 0,$$

and $\operatorname{supp}(\widetilde{m_2}) = c \operatorname{supp}(m_2) \subseteq (cy_0, \infty)$ (the tilde accent denotes the new versions). Furthermore, define $\widetilde{R}(u) \doteq \frac{1}{c} R(cu)$, $u \geq 0$. In such a way, for $c$ small enough, we obtain a new stationary CBI process $\{\widetilde{X}(t),\, t \geq 0\}$, which already satisfies the support condition (4.3.23) (and, of course, the moment conditions of the theorem), thus

$$\widetilde{c}_{\underline{k}} > 0 \quad \text{for all } \underline{k}\,, \tag{4.3.24}$$

by the statement proved in the previous step. Furthermore, the Laplace exponent functions of $\{\widetilde{X}(t),\, t \geq 0\}$ are the solutions of (4.1.5–4.1.6) with $\widetilde{R}$ and $\widetilde{S}$ instead of $R$ and $S$. So, we find that

$$\begin{aligned} \widetilde{\psi}(t,u) &= \frac{1}{c}\,\psi(t,cu), \quad t,u \geq 0, \\ \widetilde{\phi}(t,u) &= \phi(t,cu), \quad t,u \geq 0, \end{aligned} \tag{4.3.25}$$

and taking the limit as $t \to \infty$ in (4.3.25), we have

$$\widetilde{\phi}(\infty,u) = \phi(\infty,cu), \quad u \geq 0.$$

Writing (4.3.19) for process $\{\widetilde{X}(t),\, t \geq 0\}$, we obtain that

$$\begin{aligned} &\mathsf{cum}\left(\widetilde{X}(t_1),\ldots,\widetilde{X}(t_m)\right) \\ &\quad = (-1)^{m+1} \frac{\partial^m}{\partial u_m \ldots \partial u_1}\, \widetilde{\phi}\Bigg(\infty, \Big(u_1 + \widetilde{\psi}\Big(t_2^*{-}t_1^*, u_2 + \widetilde{\psi}\big(t_3^*{-}t_2^*, \ldots, \\ &\qquad\qquad u_{m-1} + \widetilde{\psi}(t_m^*{-}t_{m-1}^*, u_m)\ldots\big)\Big)\Big)\Bigg)\Bigg|_{\underline{u}=\underline{0}} \\ &\quad = (-1)^{m+1} \frac{\partial^m}{\partial u_m \ldots \partial u_1}\, \phi\Bigg(\infty, \Big(cu_1 + \psi\Big(t_2^*{-}t_1^*, cu_2 + \psi\big(t_3^*{-}t_2^*, \ldots, \\ &\qquad\qquad cu_{m-1} + \psi(t_m^*{-}t_{m-1}^*, cu_m)\ldots\big)\Big)\Big)\Bigg)\Bigg|_{\underline{u}=\underline{0}} \end{aligned}$$

$$
\begin{aligned}
&= c^m\,(-1)^{m+1}\frac{\partial^m}{\partial u_m\ldots\partial u_1}\,\phi\biggl(\infty,\Bigl(u_1+\psi\bigl(t_2^*-t_1^*,u_2+\psi\bigl(t_3^*-t_2^*,\ldots,\\
&\qquad\qquad u_{m-1}+\psi(t_m^*-t_{m-1}^*,u_m)\ldots\bigr)\bigr)\Bigr)\biggr)\bigg|_{\underline{u}=\underline{0}}\\
&= c^m\,\mathsf{cum}\,\bigl(X(t_1),\ldots,X(t_m)\bigr).
\end{aligned}
\tag{4.3.26}
$$

Hence and since $\widetilde{R}'(0)=R'(0)$, it follows by (4.3.17) that $\widetilde{P}_m=c^mP_m$, from which we get that $\widetilde{c}_{\underline{k}}=c^m c_{\underline{k}}$, for all $\underline{k}=(k_1,\ldots,k_{m-1})$. Therefore, using (4.3.24), ii) follows in this case, too.

• Now we prove ii) when $\vartheta=0$ and $m_2\not\equiv 0$. This case can also be reduced to the previous one. Namely, for each $n\in\mathbb{N}$, let Lévy measure $m_{2,n}$ be defined as the restriction of $m_2$ to the interval $(1/n,\infty)$, i.e. $m_{2,n}(B)\doteq m_2\bigl(B\bigcap(1/n,\infty)\bigr)$, $B\in\mathcal{B}(0,\infty)$, and

$$
S_n(u)\doteq\int_0^\infty\bigl(1-e^{-uy}\bigr)m_{2,n}(dy)\,,\quad u\ge 0.
$$

Then, by i), the corresponding $c_{\underline{k},n}$s are

$$
c_{\underline{k},n}\doteq\sum_{j=1}^m\lambda_{m,j}(\underline{k})S_n^{(j)}(0),
\tag{4.3.27}
$$

where only the factors $S_n^{(j)}(0)$, $j=1,\ldots,m$, depend on $n$. Since

$$
S_n^{(j)}(0)=(-1)^{j+1}\int_0^\infty y^j\,dm_{2,n}(y)\underset{n\to\infty}{\longrightarrow}(-1)^{j+1}\int_0^\infty y^j\,dm_2(y)=S^{(j)}(0),\quad j\in\mathbb{N},
$$

(see (4.3.3)), we have $c_{\underline{k},n}\underset{n\to\infty}{\longrightarrow}c_{\underline{k}}$, for every $\underline{k}$. But the $c_{\underline{k},n}$s are positive (by the previous step), therefore so are the $c_{\underline{k}}$s.

• Next we prove ii) in the case when $\vartheta>0$ and $m_2\equiv 0$. For this, assume temporarily that $\vartheta=0$ and $m_2$ is the Dirac measure concentrated on a point $y>0$, i.e. (4.3.20) holds. Then by i) we have

$$
0<c_{\underline{k}}=\sum_{j=1}^m(-1)^{j+1}\lambda_{m,j}(\underline{k})\,y^j,
$$

for every $\underline{k}$. Letting $y \to 0$, the dominant term is $\lambda_{m,1}(\underline{k})y$, hence

$$\lambda_{m,1}(\underline{k}) \geq 0 \quad \text{for every } \underline{k}\,. \tag{4.3.28}$$

Then, assume that $\vartheta > 0$ and $m_2 \equiv 0$. By (4.3.2–4.3.3) we have $S'(0) = \vartheta$ and $S^{(k)}(0) = 0$, $k \geq 2$, so the equality in i) reads as $c_{\underline{k}} = \lambda_{m,1}(\underline{k})\vartheta$. Hence, using also (4.3.28), we obtain that $c_{\underline{k}} > 0$, for every $\underline{k}$.

- Finally, by i) and (4.3.2–4.3.3) the $c_{\underline{k}}$s are additive functions of the immigration parameter $(\vartheta, m_2)$ (of course, for a common branching parameter). Therefore, from the previous two cases ($\vartheta = 0$, $m_2 \not\equiv 0$ and $\vartheta > 0$, $m_2 \equiv 0$) we get that ii) holds also in the general case. □

**4.3.5 Remark.** Applying the checking method given in Remark 4.2.5, we obtain that for a non-negative, stationary OU type process $\{X(t),\, t \geq 0\}$ (see Example 4.1.11) with finite moments of all orders, the first three cumulants in Proposition 4.3.4 are:

$$\mathsf{E}\, X(t) = \frac{1}{-\alpha} \int_0^\infty y\, m_2(dy), \quad t \geq 0,$$

$$\mathsf{cum}\,\big(X(t_1), X(t_2)\big) = \frac{1}{-2\alpha} \int_0^\infty y^2\, m_2(dy)\, e^{-\alpha(t_2^* - t_1^*)},$$

$$\mathsf{cum}\,\big(X(t_1), X(t_2), X(t_3)\big) = \frac{1}{-3\alpha} \int_0^\infty y^3\, m_2(dy)\, e^{-\alpha(t_2^* - t_1^* + t_3^* - t_1^*)},$$

$t_1, t_2, t_3 \geq 0$. Lemma 3.1.2 (which was stated particularly for $\alpha = -1$) gives the same expressions, since the cumulants of $L(1)$ (see Example 4.1.11) are the moments of its Lévy measure $m_2$. We obtain furthermore, that in this case polynomials $P_m$ in (4.3.18) are

$$P_m(x_1, \dots, x_{m-1}) = \frac{1}{-m\alpha} \int_0^\infty y^m\, m_2(dy)\, x_1^{m-1} x_2^{m-2} \cdots x_{m-1}, \quad m \geq 2, \tag{4.3.29}$$

i.e.

$$c_{k_1,\dots,k_{m-1}} = \begin{cases} \frac{1}{-m\alpha} \int_0^\infty y^m m_2(dy) & \text{if } (k_1, k_2, \dots, k_{m-1}) = (m-1, m-2, \dots, 1), \\ 0 & \text{otherwise.} \end{cases}$$

**4.3.6 Remark.** The polynomial form (4.3.17–4.3.18) of the joint cumulants and the form i) of the coefficients, moreover, the property ii) in Proposition 4.3.4 will be the key in the proof of the dilatively stable renormalization functional limit theorem, Theorem 4.5.4.

## 4.4 Superposition of stationary CBI processes

The construction is the same as in Section 3.2, but with CBI processes instead of OU type processes: independent, stationary CBI processes will be summed. Each term of the sum, i.e. each process (or more precisely, its distribution) will be some transform of a basic stationary CBI process. Each transform will consist of a convolution power and a time dilation, i.e. will be of the form

$$\big\{X(t),\, t \geq 0\big\} \;\longmapsto\; \big\{X(d_j t),\, t \geq 0\big\}^{\circledast p_j}.$$

Each dilational costant $d_j$ will occur with probability $p_j$, so the sequence $p_j$, $j \in \mathbb{N}$, will be a distribution on the discrete set $\{d_j : j \in \mathbb{N}\}$. That is, the time dilation will be a discrete random variable $\delta$. Moreover, the left tail behaviour of the distribution of $\delta$ is given in Assumption SPL on page 57.

Now we detail the assumptions concerning the CBI processes, which we will superpose.

- Let $R$ given by (4.1.7) be the Laplace exponent function of a branching distribution with parameter $\big(\sigma^2,\, \alpha = \int_1^\infty (1-y) m_1(dy) - 1,\, m_1\big)$ and let $S$ given by (4.1.8) be the Laplace exponent function of an immigration distribution with parameter $(\vartheta,\, m_2)$. Assume that

$$\int_1^\infty e^{ux}(m_1 + m_2)(dx) < \infty \tag{4.4.1}$$

for some $u > 0$. Equivalently (by Duffie et al. [9, Lemma 5.3 iii)]) let $R$ and $S$ be analytic functions on some complex neighbourhood of zero.

- Let $\{X(t),\, t \geq 0\}$ be a stationary CBI process with branching and immigration Laplace exponent functions $R$ and $S$, i.e. with branching and immigration parameters $\big(\sigma^2, \alpha = \int_1^\infty (1-y) m_1(dy) - 1, m_1\big)$ and $(\vartheta, m_2)$, respectively. Let

$$\big\{X_j(t),\, t \geq 0\big\} \doteq \big\{X(d_j t),\, t \geq 0\big\}^{\circledast p_j}, \quad j \in \mathbb{N}.$$

- Let processes $\{X_j(t),\, t \geq 0\}$, $j \in \mathbb{N}$, be independent.

**4.4.1 Remark.** By (4.3.1), the assumption that $\alpha = \int_1^\infty (1-y)m_1(dy) - 1$ is equivalent with the equality $R'(0) = -1$.

Hereafter we consider the above assumptions and Assumption SPL at page 57 to be fulfilled.

**4.4.2 Lemma.** *The characteristic function of the random variable $X(0)$ is analytic in some complex neighbourhood of zero and so are the characteristic functions of the random variables $X_j(0)$, $j \in \mathbb{N}$.*

**Proof.** By our assumptions, $R$ and $S$ are analytic functions on some complex neighbourhood of zero. Since $R(0) = 0$ and $S(0) = 0$ (see (4.1.7–4.1.8), also $u \mapsto R(u)/u$ and $u \mapsto S(u)/u$ are analytic on some complex neighbourhood of zero. By Remark 4.4.1, $\lim_{u \to 0} \big(R(u)/u\big) = R'(0) = -1 \neq 0$, so

$$u \longmapsto \frac{S(u)}{-R(u)} = -\frac{\frac{S(u)}{u}}{\frac{R(u)}{u}}$$

is also analytic on some complex neighbourhood of zero. Therefore, by Lemma 4.2.4 we obtain the statements. $\square$

**4.4.3 Remark.** For each $j \in \mathbb{N}$, process $\{X_j(t),\, t \geq 0\}$ is a stationary CBI process with branching parameter $\big(d_j\sigma^2,\, d_j\big(\int_1^\infty (1-y)m_1(dy) - 1\big),\, d_j m_1\big)$ and immigration parameter $(p_j d_j \vartheta, p_j d_j m_2)$. Indeed, by Proposition 4.3.4 the two processes have the same joint cumulants and since by Lemma 4.4.2 the characteristic function of $X_j(0)$ is analytic in some complex neighbourhood of zero, the finite-dimensional distributions of $\{X_j(t),\, t \geq 0\}$ are uniquely determined by the joint cumulants.

**4.4.4 Theorem.** *For each $t \geq 0$ the series $\sum_j X_j(t)$ converges in $L^2$, uniformly in $t \geq 0$ and also almost surely.*

**Proof.** Using Proposition 4.3.4 we have

$$\sum_{j=1}^{\infty} \mathsf{E}\, X_j(t) = \sum_{j=1}^{\infty} p_j\, \mathsf{E}\, X(d_j t) = \sum_{j=1}^{\infty} p_j\, S'(0) = S'(0), \tag{4.4.2}$$

$$\sum_{j=1}^{\infty} \mathsf{D}^2 X_j(t) = \sum_{j=1}^{\infty} p_j\, \mathsf{D}^2 X(d_j t) = \sum_{j=1}^{\infty} p_j \left( \frac{-R''(0)}{2}\, S'(0) - \frac{1}{2}\, S''(0) \right)$$

$$= -\frac{1}{2}\big(R''(0)S'(0) + S''(0)\big),$$

thus the series $\sum_j X_j(t)$ converges in $L^2$, uniformly in $t \geq 0$. The a.s. convergence follows by Kolmogorov's two series theorem. □

**4.4.5 Definition.** Process

$$Y(t) \doteq \sum_{j=1}^{\infty} X_j(t)\,, \quad t \geq 0,$$

is called a *superposition of CBI processes* (*SCBI process*, with parameters $(H, \ell)$ and $(\sigma^2, m_1, \vartheta, m_2)$).

**4.4.6 Lemma.** *The mean of the SCBI process* $\{Y(t),\, t \geq 0\}$ *is*

$$\mathsf{E}\, Y(t) = S'(0) = \vartheta + \int_0^{\infty} y\, m_2(dy), \quad t \geq 0,$$

*and the joint cumulants are of the form*

$$\mathsf{cum}\big(Y(t_1), \ldots, Y(t_m)\big) = \mathsf{E}\, P_m\Big(e^{-\delta(t_2^* - t_1^*)}, \ldots, e^{-\delta(t_m^* - t_{m-1}^*)}\Big)\,,$$

$$0 \leq t_1, \ldots, t_m, \quad m \geq 2, \quad (4.4.3)$$

*where* $P_m$, $m \geq 2$, *are the polynomials (4.3.18), with the restriction* $R'(0) = -1$.

**Proof.** The mean follows from (4.4.2) and (4.3.2). Let $m \geq 2$ and let $0 \leq t_1, \ldots, t_m$ be arbitrarily fixed. Since processes $\{X_j(t),\, t \geq 0\}$, $j \in \mathbb{N}$, are independent, we have

$$\mathsf{cum}\big(Y(t_1), \ldots, Y(t_m)\big) = \mathsf{cum}\left( \sum_{j=1}^{\infty} X_j(t_1), \ldots, \sum_{j=1}^{\infty} X_j(t_m) \right)$$

$$= \sum_{j=1}^{\infty} \mathsf{cum}\big(X_j(t_1), \ldots, X_j(t_m)\big) = \sum_{j=1}^{\infty} p_j\, \mathsf{cum}\big(X(d_j t_1), \ldots, X(d_j t_m)\big).$$

Hence, using (4.3.17), we obtain

$$
\begin{aligned}
\mathsf{cum}\big(Y(t_1),\dots,Y(t_m)\big) &= \sum_{j=1}^{\infty} p_j\, \mathsf{cum}\big(X(d_jt_1),\dots,X(d_jt_m)\big) \\
&= \sum_{j=1}^{\infty} p_j\, P_m\Big(e^{-d_j(t_2^*-t_1^*)},\dots,e^{-d_j(t_m^*-t_{m-1}^*)}\Big) \\
&= \mathsf{E}\, P_m\Big(e^{-\delta(t_2^*-t_1^*)},\dots,e^{-\delta(t_m^*-t_{m-1}^*)}\Big).
\end{aligned} \tag{4.4.4}
$$

□

**4.4.7 Remark.** Let us consider the particular case when $\{Y(t),\, t \geq 0\}$ is the superposition of sationary, non-negative OU type processes (see Remark 4.2.5 and Example 4.1.11). Then by Lemma 3.2.5 we have

$$
\mathsf{cum}\big(Y(t_1),\dots,Y(t_m)\big) = \frac{1}{m}\,\mathsf{cum}_m(L(1))\,\mathsf{E}\, e^{-\delta(t_2^*-t_1^*+\dots+t_m^*-t_1^*)}, \qquad 0 \leq t_1,\dots,t_m, \quad m \in \mathbb{N}. \tag{4.4.5}
$$

The cumulants (in particular, the mean) of random variable $L(1)$ are exactly the moments (in particular, the mean) of the corresponding Lévy measure $m_2$. Hence, for $m = 1$, (4.4.5) coinsides with $\mathsf{E}\,Y(t)$ given in Lemma 4.4.6. Furthermore, by (4.3.29), the right-hand side of (4.4.3) is exactly the right-hand side of (4.4.5).

**4.4.8 Remark.** By the stationarity and the first equation of (4.4.4) the cumulants of $Y(0)$ are the same as those of $X(0)$. Therefore by Lemma 4.4.2 the characteristic function of $Y(0)$ is analytic in some complex neighbourhood of zero.

**4.4.9 Remark.** The assumption that parameter $\alpha$ of the CBI process $\{X(t),\, t \geq 0\}$ is $\alpha = \int_1^\infty (1-y)m_1(dy) - 1$, or equivalently $R'(0) = -1$ means, in fact, a norming $R(u) \longmapsto R(u)/|R'(0)|$. Without it we would obtain

$$
\mathsf{cum}\big(Y(t_1),\dots,Y(t_m)\big) = \mathsf{E}\, P_m\Big(e^{R'(0)\delta(t_2^*-t_1^*)},\dots,e^{R'(0)\delta(t_m^*-t_{m-1}^*)}\Big),
$$

with a polynomial $P_m(x_1,\dots,x_{m-1})$, which (by Proposition 4.3.4 iii)) differs from the polynomial denoted similarly in Lemma 4.4.6 in a factor $|R'(0)|$. However, the factor $|R'(0)|$ can be included in $\delta$ and $P_m$, so assumption $R'(0) = -1$ is not really a restriction.

Examining the SCBI process from the point of view of continuity, we find that by Lemma 4.4.6 both the mean function (which is the constant $S'(0)$) and the covariance function are continuous. Therefore, the SCBI process is $L^2$-continuous, so, it is also stochastically continuous. Hence, by Doob [8, Thm. 2.6], it has a jointly measurable modification. We will consider this modification the SCBI process. Because of the joint measurability, Fubini's theorem can be applied and using also the stationarity, there follows the a.s. local Lebesgue integrability of the SCBI process. The latter property is implicit in the following definition.

**4.4.10 Definition.** Let $\{Y(t),\ t \geq 0\}$ be an SCBI process with parameters $(H, \ell)$ and $(\sigma^2, m_1, \vartheta, m_2)$. Then the process

$$J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t) \doteq \int_0^t Y(s)\, ds, \quad t \geq 0, \tag{4.4.6}$$

is called an *ISCBI (integrated SCBI) process* (with parameters $(H, \ell)$ and $(\sigma^2, m_1, \vartheta, m_2)$).

**4.4.11 Remark.** The ISCBI process has stationary increments, since the SCBI process is stationary.

The following lemma is a simple consequence of Lemma 4.4.6 and the multilinearity (i.e. linearity in each variable) of the joint cumulant.

**4.4.12 Lemma.** *The mean and the joint cumulants of the ISCBI process are:*

$$\mathsf{E} J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t) = S'(0)\, t = \left( \vartheta + \int_0^\infty y\, m_2(dy) \right) t, \quad t \geq 0, \tag{4.4.7}$$

$$\mathsf{cum}\left( J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t_1), \dots, J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t_m) \right) = \int_0^{t_1} \cdots \int_0^{t_m} \mathsf{E}\, P_m\left( e^{-\delta(s_2^* - s_1^*)}, \dots, e^{-\delta(s_m^* - s_{m-1}^*)} \right) d\underline{s}\,, \tag{4.4.8}$$

$0 \leq t_1, \dots, t_m,\ m \geq 2,$ *where polynomials* $P_m$ *are given by Proposition 4.3.4, with the restriction* $R'(0) = -1$.

**4.4.13 Remark.** Let us consider the particular case when the basic stationary CBI processes we superposed are non-negative OU type processes (see Remark 4.2.5 and Example 4.1.11). Then by Lemma 3.2.10 we have

$$\mathsf{E}\, J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t) = \mathsf{E}\, L(1)\, t, \quad t \geq 0, \tag{4.4.9}$$

and

$$\begin{aligned}\mathsf{cum}\Big(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t_1), \ldots, J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t_m)\Big)&\\ = \frac{\mathsf{cum}_m(L(1))}{m} \int\limits_0^{t_1} \cdots \int\limits_0^{t_m} \mathsf{E}\, e^{-\delta(s_2^*-s_1^*+\cdots+s_m^*-s_1^*)} d\underline{s}\,,&\end{aligned} \tag{4.4.10}$$

$0 \leq t_1, \ldots, t_m,\ m \geq 2$. Hence, we obtain similarly as in Remark 4.4.7 that the right-hand sides of (4.4.9) and (4.4.10) are exactly the right-hand sides of (4.4.7) and (4.4.8), respectively.

**4.4.14 Lemma.** *For each $t > 0$ the characteristic function of $J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t)$ is analytic in some complex neighbourhood of zero. Therefore, the finite-dimensional distributions of the ISCBI process are uniquely determined by the joint cumulants.*

**Proof.** Completely analogous to the proof of Lemma 3.2.11. □

The following theorem states that in the construction of the ISCBI process the order of the superposition and the integration can be interchanged.

**4.4.15 Theorem.** *The sequence of finite superpositions of integrated stationary CBI processes, $\big\{\sum_{j=1}^n \int_0^t X_j(s)\, ds,\, t \geq 0\big\}$, converges pointwise, both in $L^2$ and almost surely to the ISCBI process, i.e., for each $t \geq 0$, we have*

$$\sum_{j=1}^n \int\limits_0^t X_j(s)\, ds \underset{n\to\infty}{\longrightarrow} J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t) \quad \text{in } L^2 \text{ and a.s.}. \tag{4.4.11}$$

**Proof.** The pointwise $L^2$-convergence in 4.4.11 is a simple consequence of Theorem 4.4.4, while the a.s. convergence follows by Kolmogorov's two series theorem. □

Also the following functional limit theorem holds.

**4.4.16 Theorem.** *The sequence of distributions on $C[0,\infty)$ of finite superpositions of integrated stationary CBI processes, $\big\{\sum_{j=1}^n\int_0^t X_j(s)\,ds,\ t\geq 0\big\}$, converges weakly to the distribution of the ISCBI process, i.e.*

$$\left\{\sum_{j=1}^{n}\int_0^t X_j(s)\,ds,\ t\geq 0\right\} \xrightarrow[n\to\infty]{\mathrm{w}} \left\{J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t),\ t\geq 0\right\} \qquad \text{on } C[0,\infty). \tag{4.4.12}$$

**Proof.** The proof is analogous to that of Theorem 3.2.14. The corresponding convergence of the finite-dimensional distributions is the consequence of Theorem 4.4.15. Moreover, we have

$$\mathsf{E}\left(\sum_{j=1}^{n}\int_0^t X_j(s)\,ds\right)^2 = \mathsf{D}^2\sum_{j=1}^{n}\int_0^t X_j(s)\,ds + \left(\mathsf{E}\sum_{j=1}^{n}\int_0^t X_j(s)\,ds\right)^2$$

$$\leq \mathsf{D}^2 J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t)+t^2\big(\mathsf{E}\,Y(0)\big)^2 \leq t^2\,\mathsf{D}^2\,Y(0)+t^2\big(\mathsf{E}\,Y(0)\big)^2 = t^2\,\mathsf{E}\big(Y(0)\big)^2,$$

so, Kolmogorov's tightness condition is fulfilled and the statement of the theorem follows.

We note that (4.4.12) follows also by Jacod–Shiryaev [18, Ch. VI, Thm. 3.37 (b)] (which states that in case of increasing processes and an a.s. continuous limit process, the weak convergence on $D[0,\infty)$ follows from the convergence of the finite-dimensional distributions). □

**4.4.17 Remark.** By Theorem 4.4.15 and Theorem 4.4.12 the ISCBI process could also be named a SICBI process.

## 4.5 The LISCBI process

It follows from Lemma 4.4.12, the last statement of Proposition 4.3.4 and from Lemma 4.4.14 that the ISCBI process is infinitely divisible, hence it is meaningful to renormalize it. Namely, we may consider the renormalized centered processes

$$\frac{1}{T}\left\{J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt)-\mathsf{E}\,J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt),\ t\geq 0\right\}^{\circledast\frac{T^{2-2H}}{\ell(T)}}, \qquad T>0. \tag{4.5.1}$$

The main theorem in this section, Theorem 4.5.4 will be a dilatively stable renormalization functional limit theorem. It will state that under some additional condition the family (4.5.1) converges as $T \to \infty$. The limit process will be called the LISCBI process, giving the title of this chapter. It will be necessarily dilatively stable. One of the keys to the proof of the mentioned dilatively stable renormalization limit theorem will be Lemma 4.4.12 together with Proposition 4.3.4 i–ii). The other key, similarly as in Section 3.3, will be Lemma 3.3.1 and through it, directly the following Lemma 4.5.2.

**4.5.1 Notation.** For each $H \in (1/2, 1)$ let $\varphi_H$ be the mapping defined on the set of multivariate polynomials with positive variables, acting as

$$\begin{aligned} P(x_1, \dots, x_n) &= \sum_{k_1,\dots,k_n} c_{k_1,\dots,k_n} x_1^{k_1} \cdots x_n^{k_n} \\ &\longmapsto \varphi_H(P)(x_1, \dots, x_n) \doteq \sum_{k_1,\dots,k_n} c_{k_1,\dots,k_n} \big(k_1 x_1 + \dots + k_n x_n\big)^{2H-2}, \end{aligned}$$

where the variables of the images $\varphi_H(P)$ are also positive.

E.g., the images of the polynomials $P_2$ and $P_3$ in Proposition 4.3.4 (with the restriction $R'(0) = -1$) are:

$$\varphi_H(P_2)(x) = \varphi_H\Big(-\frac{1}{2}\big(R''(0)S'(0) + S''(0)\big)x\Big) = -\frac{1}{2}\big(R''(0)S'(0) + S''(0)\big)x^{2H-2}, \tag{4.5.2}$$

$$\begin{aligned} \varphi_H(P_3)\big(x_1, x_2\big) = \varphi_H\bigg(&\frac{1}{6}\big(R'''(0)S'(0) + 2S'''(0)\big)x_1^2 x_2 \\ &+ \frac{1}{2}\big((R''(0))^2 S'(0) + R''(0)S''(0)\big)\, x_1 x_2\bigg) \\ = &\frac{1}{6}\big(R'''(0)S'(0) + 2S'''(0)\big)\big(2x_1 + x_2\big)^{2H-2} \\ &+ \frac{1}{2}\big((R''(0))^2 S'(0) + R''(0)S''(0)\big)\,\big(x_1 + x_2\big)^{2H-2}. \end{aligned} \tag{4.5.3}$$

**4.5.2 Lemma.** *The family of the joint cumulants of the renormalized centered ISCBI processes (4.5.1) converge as $T \to \infty$, to the following limit:*

$$\lim_{T\to\infty} \mathsf{cum}\Bigg(\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt_1) - \mathsf{E}\,J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt_1)\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}},\dots,$$
$$\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt_m) - \mathsf{E}\,J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt_m)\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}\Bigg)$$
$$= \Gamma(3-2H)\int_0^{t_1}\cdots\int_0^{t_m}\varphi_H(P_m)\big(s_2^*-s_1^*,\dots,s_m^*-s_{m-1}^*\big)\,d\underline{s}\,, \tag{4.5.4}$$

*$0 \le t_1,\dots,t_m$, $m \ge 2$, where $P_m$, $m \ge 2$, are the polynomials (4.3.18), with the restriction $R'(0) = -1$.*

**Proof.** First of all, the joint cumulants of order $m \ge 2$ of the centered process are the same as those of the uncentered process. Using Lemma 4.4.12 we have

$$\mathsf{cum}\left(\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_1),\dots,\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_m)\right)$$
$$= \frac{T^{-m+2-2H}}{\ell(T)}\int_0^{Tt_1}\cdots\int_0^{Tt_m}\mathsf{E}\,P_m\left(e^{-\delta(s_2^*-s_1^*)},\dots,e^{-\delta(s_m^*-s_{m-1}^*)}\right)d\underline{s}$$
$$= \frac{T^{2-2H}}{\ell(T)}\int_0^{t_1}\cdots\int_0^{t_m}\mathsf{E}\,P_m\left(e^{-\delta T(s_2^*-s_1^*)},\dots,e^{-\delta T(s_m^*-s_{m-1}^*)}\right)d\underline{s}. \tag{4.5.5}$$

Substituting the expression (4.3.18) of $P_m$ into (4.5.5) we obtain

$$\mathsf{cum}\left(\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_1),\dots,\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_m)\right)$$
$$= \frac{T^{2-2H}}{\ell(T)}\int_0^{t_1}\cdots\int_0^{t_m}\mathsf{E}\,P_m\left(e^{-\delta T(s_2^*-s_1^*)},\dots,e^{-\delta T(s_m^*-s_{m-1}^*)}\right)d\underline{s}$$
$$= \sum_{k_1,\dots,k_{m-1}} c_{k_1,\dots,k_{m-1}}\int_0^{t_1}\cdots\int_0^{t_m}\frac{T^{2-2H}}{\ell(T)}\,\mathsf{E}\,e^{-\delta T\left(k_1(s_2^*-s_1^*)+\dots+k_{m-1}(s_m^*-s_{m-1}^*)\right)}d\underline{s}.$$

Taking the limit as $T \to \infty$, by the key Lemma 3.3.1 we can change the order of the limit and the integral and obtain

$$\lim_{T\to\infty} \operatorname{cum}\left(\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_1),\ldots,\frac{1}{T}\left(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}\right)^{\circledast\frac{T^{2-2H}}{\ell(T)}}(Tt_m)\right)$$

$$= \sum_{k_1,\ldots,k_{m-1}} c_{k_1,\ldots,k_{m-1}} \int_0^{t_1}\!\!\cdots\!\int_0^{t_m} \lim_{T\to\infty}\left(\frac{T^{2-2H}}{\ell(T)} \mathsf{E}\, e^{-\delta T\left(k_1(s_2^*-s_1^*)+\cdots+k_{m-1}(s_m^*-s_{m-1}^*)\right)}\right) d\underline{s}$$

$$= \Gamma(3-2H) \sum_{k_1,\ldots,k_{m-1}} c_{k_1,\ldots,k_{m-1}} \int_0^{t_1}\!\!\cdots\!\int_0^{t_m} \left(k_1\left(s_2^*-s_1^*\right)+\cdots+k_{m-1}\left(s_m^*-s_{m-1}^*\right)\right)^{2H-2} d\underline{s}$$

$$= \Gamma(3-2H) \int_0^{t_1}\!\!\cdots\!\int_0^{t_m} \varphi_H(P_m)\left(s_2^*-s_1^*,\ldots,s_m^*-s_{m-1}^*\right) d\underline{s}. \qquad \square$$

**4.5.3 Lemma.** *The family of renormalized centered ISCBI processes (4.5.1) satisfies Kolmogorov's tightness condition. Therefore, processes (4.5.1) are almost surely continuous, and the corresponding family of distributions on $C[0,\infty)$ is tight.*

**Proof.** Completely analogous to the proof of Lemma 3.3.3, with the difference that instead of the factor $\operatorname{cum}_2(L(1))/2$ there is $-\left(R''(0)S'(0)+S''(0)\right)/2$. $\square$

Now, we can state the main theorem.

**4.5.4 Theorem.** *The family of distributions of the renormalized centered ISCBI processes (4.5.1) converges weakly on $C[0,\infty)$ to a limit distribution. We will denote a process with this distribution by $J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)$, and call it the LISCBI process (limit of [renormalized centered] ISCBI processes) with parameters $H$ and $(\sigma^2,m_1,\vartheta,m_2)$:*

$$\frac{1}{T}\left\{J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt) - \mathsf{E}\, J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(Tt),\ t\ge 0\right\}^{\circledast\frac{T^{2-2H}}{\ell(T)}}$$

$$\xrightarrow[T\to\infty]{\mathrm{w}} \left\{J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t),\ t\ge 0\right\} \quad \text{on } C[0,\infty). \tag{4.5.6}$$

*The distribution on $C[0,\infty)$ of the LISCBI process with parameters $H$ and $(\sigma^2, m_1, \vartheta, m_2)$ is uniquely determined by its zero mean and its joint cumulants, which are:*

$$\mathsf{cum}\left(J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t_1),\ldots,J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t_m)\right)$$
$$= \Gamma(3-2H)\int_0^{t_1}\!\!\cdots\int_0^{t_m}\varphi_H(P_m)\big(s_2^*-s_1^*,\ldots,s_m^*-s_{m-1}^*\big)\,d\underline{s}\,, \quad (4.5.7)$$

*$0 \le t_1,\ldots,t_m$, $m \ge 2$, where $P_m$, $m \ge 2$, are the polynomials (4.3.18), with the restriction $R'(0) = -1$. Moreover, $\big\{J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t),\, t \ge 0\big\}$ has stationary increments, and it is $(H, 2H-2)$-dilatively stable.*

**Proof.** By Lemma 4.5.3 the family of distributions on $C[0,\infty)$ corresponding to the family (4.5.1) is relatively sequentially compact, so there exists a subsequence

$$\frac{1}{T_n}\left\{J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(T_n t) - \mathsf{E}\,J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(T_n t),\; t \ge 0\right\}^{\circledast\frac{T_n^{2-2H}}{\ell(T_n)}}, \quad n \in \mathbb{N},$$

such that the corresponding subsequence of distributions converges weakly to a distribution on $C[0,\infty)$. Let us denote the process corresponding to this limit distribution by $\big\{J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t),\, t \ge 0\big\}$. This process has zero mean and joint cumulants given by the right-hand side of (4.5.4). Note that parameter $\ell$ has disappeared, so the notation $J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}$ is right. We have to prove that this limit process is unique, i.e. its distribution on $C[0,\infty)$ is unique. In other words, we have to prove that the joint cumulants of $\big\{J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t),\, t \ge 0\big\}$, i.e. those given by the right-hand side of (4.5.4) and the zero mean uniquely determine the finite-dimensional distributions. For this it is enough to show that for each $t > 0$ the characteristic function of $J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)$ is analytic in some complex neighbourhood of zero. So we prove the latter fact. We have

$$\int_0^{t}\!\!\cdots\int_0^{t}\Big(k_1\big(s_2^*-s_1^*\big)+\cdots+k_{m-1}\big(s_m^*-s_{m-1}^*\big)\Big)^{2H-2}d\underline{s}$$
$$\le t^{m+2H-2}\int_0^{1}\!\!\cdots\int_0^{1}\big(s_2^*-s_1^*\big)^{2H-2}d\underline{s} \;\le\; t^{m+2H-2}\,\frac{1}{2H-1}\,m,$$

where the first inequality is the consequence of that all $k_j$s are at least 1. From this, using Proposition 4.3.4, we obtain that

$$\left|\mathsf{cum}_m\left(J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)\right)\right| = \mathsf{cum}_m\left(J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)\right)$$

$$\leq t^{m+2H-2}\,\frac{\Gamma(3-2H)}{2H-1}\; m \sum_{k_1,\dots,k_{m-1}} c_{k_1,\dots,k_{m-1}} = t^{m+2H-2}\,\frac{\Gamma(3-2H)}{2H-1}\; m\,\mathsf{cum}_m(X(t)),$$

where $\{X(t),\, t \geq 0\}$ is the stationary CBI process given in the assumptions at the beginning of Section 4.4. Hence

$$\limsup_{m\to\infty} \sqrt[m]{\frac{\left|\mathsf{cum}_m\left(J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)\right)\right|}{m!}} \leq t\,\limsup_{m\to\infty} \sqrt[m]{\frac{|\mathsf{cum}_m(X(0))|}{m!}} < \infty,$$

the last inequality follows from Lemma 4.4.2. Thus, we have shown that the characteristic function of $J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)$ is analytic in some complex neighbourhood of zero. Thus we have finished the proof of the theorem except for the last sentence.

The LISCBI process has stationary increments, because processes on the left-hand side of (4.5.6) are such.

Now we prove that $\left\{J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t),\, t \geq 0\right\} \in \mathcal{I}$ (see Notation 2.1.1). The non-Gaussianity is obvious, as well the infinite divisibility of the finite-dimensional distributions and the right-continuity of the cumulant functions

$$\mathsf{cum}_m\left(J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)\right)$$
$$= \Gamma(3-2H)\,t^{m+2H-2}\int_0^1\!\!\cdots\!\int_0^1 \varphi_H(P_m)\big(s_2^*-s_1^*,\dots,s_m^*-s_{m-1}^*\big)\,d\underline{s}\,, \quad (4.5.8)$$

$t \geq 0$, $m \geq 2$. The right-continuity in distribution follows from the right-continuity of the cumulant functions and from the fact that for any fixed $t > 0$ the cumulants (4.5.8) and the zero mean uniquely determine the distribution of $J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t)$. Therefore $\left\{J^{(H)}_{\sigma^2,m_1,\vartheta,m_2}(t),\, t \geq 0\right\} \in \mathcal{I}$, so we can apply Theorem 2.2.7(DS), hence the $(H, 2H-2)$-dilative stability follows. $\square$

**4.5.5 Remark.** Let us consider the particular case when the basic stationary CBI processes we superposed are non-negative OU type processes (see Remark 4.2.5 and Example 4.1.11). Then by Theorem 3.3.4 we have

$$\begin{aligned}
&\operatorname{cum}\Big(J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t_1),\dots,J^{(H,\ell)}_{\sigma^2,m_1,\vartheta,m_2}(t_m)\Big)\\
&=\frac{\operatorname{cum}_m(L(1))}{m}\,\Gamma(3-2H)\int_0^{t_1}\!\!\cdots\int_0^{t_m}(s_2^*-s_1^*+\dots+s_m^*-s_1^*)^{2H-2}\,d\underline{s}\,,
\end{aligned} \tag{4.5.9}$$

$0 \le t_1,\dots,t_m$, $m \ge 2$. Since in this case polynomial $P_m$ is given by (4.3.29), we have

$$\begin{aligned}
&\varphi_H(P_m)\big(x_1,\dots,x_{m-1}\big)\\
&\qquad=\frac{1}{m}\int_0^{\infty}y^m\,m_2(dy)\,\Big((m-1)x_1+(m-2)x_2+\dots+x_{m-1}\Big)^{2H-2}\\
&\qquad=\frac{\operatorname{cum}_m(L(1))}{m}\Big((m-1)x_1+(m-2)x_2+\dots+x_{m-1}\Big)^{2H-2},
\end{aligned}$$

$m \ge 2$, from which we obtain that the right-hand side of (4.5.7) equals to the right-hand side of (4.5.9)

**4.5.6 Remark.** The LISCBI process with parameter $H$ (and $(\sigma^2, m_1, \vartheta, m_2)$) is $(H, 2H-2)$-dilatively stable, so it has the renormalization operator $A_T^{(H,\,2H-2)}$ (see Example 2.8.3(DS)). This explains, at least when $\lim_{T\to\infty}\ell(T)$ exists and nonzero, the renormalization used in (4.5.6).

**4.5.7 Remark.** Theorem 4.5.4 can be interpreted as it details the fact that the ISCBI process with parameters $(H, \ell)$ and $(\sigma^2, m_1, \vartheta, m_2)$ is asymptotically $(H, 2H-2)$-dilatively stable (see Definition 2.2.9(DS)).

**4.5.8 Remark.** By the last statement of Theorem 4.5.4, Theorem 2.7.2 (DS), and Corollary 2.7.3 (DS), the LISCBI process with parameter $H$ has the same autocovariance function as that of the FBM with parameter $H$ (apart from a constant factor), and it is of long memory.

## 4.6 The LISDLG process

In this section we restate the more important results of the previous sections particularly for the stationary DLG process (see Example 4.1.12 for the DLG process) and the processes originating from it: (the IDLG), the SDLG, ISDLG and the LISDLG processes (the DLG family). These are the most important special cases of the stationary CBI process and its descendants, the SCBI, ISCBI and the LISCBI processes (the CBI family) treated in the previous sections. While the processes in the CBI family in general, are specified indirectly, by some properties of the expressions for their joint cumulants, we will see that the processes of the DLG family can be defined directly, by explicit expressions for their joint cumulants. We will present also some new results stated only for processes of the DLG family.

The following characterizations show the importance of the DLG process:
• Among the CBI processes the DLG process is the only diffusion process (or the only a.s. continuous process; see (4.1.2)).
• Among the non-negative (or among the non-Gaussian) diffusion processes the DLG process is the only process with linear infinitesimal generator see (4.1.2)).
• Among the non-negative (or among the non-Gaussian) diffusion processes the DLG process is the only infinitely decomposable process (Shiga–Watanabe [32, Thm. 1.2] essentially states the same).

We use the new name "DLG process", because there is a lack of agreement on terminology: in the literature the DLG process is called
• a Feller diffusion (by Proposition 4.1.10 and the above characterization, among the non-negative [or non-Gaussian] Feller diffusions the DLG process is the only infinitely decomposable process),
• a Cox–Ingersoll–Ross (CIR) process (after the authors of Cox et al. [5]),
• a square-root diffusion (by (4.1.11)),
• a population growth process (see Karlin–Taylor [20, 15.13.C])
• and a squared radial OU process, Proposition 4.6.3 explains why.

By Remark 4.2.2, Theorem 4.2.3 applies for the DLG process. Combining it with (4.2.2) and (4.1.13), we obtain the following theorem:

**4.6.1 Theorem.** *The DLG process given by the stochastic differential equation*

$$dX(t) = \big(\vartheta + \alpha X(t)\big)\,dt + \sigma\sqrt{X(t)}\,dB(t),$$

*has a stationary distribution, different from the trivial distribution concentrated* *to zero if and only if* $\vartheta > 0$ *and* $\alpha < 0$. *Then we have*

$$\phi(\infty, u) = \frac{2\vartheta}{\sigma^2} \log\left(1 + \frac{\sigma^2}{-2\alpha} u\right), \quad u \geq 0, \tag{4.6.1}$$

*hence the Laplace transform of the stationary distribution is*

$$\varphi(u) = \left(1 + \frac{\sigma^2}{-2\alpha} u\right)^{-\frac{2\vartheta}{\sigma^2}}, \quad u \geq 0,$$

*so, the stationary distribution of the DLG process is the* $\Gamma\big(2\vartheta/\sigma^2,\, \sigma^2/(-2\alpha)\big)$ *distribution.*

The next statement is a corollary of Lemma 4.2.7, (4.1.12) and (4.1.13).

**4.6.2 Proposition.** *The stationary DLG process is infinitely decomposable (i.e. its finite-dimensional distributions are such) and parameter* $\vartheta > 0$ *plays the role of a shape parameter. In other words, if* $c > 0$ *and* $\{X(t),\, t \geq 0\}$ *is a stationary DLG process with parameter* $(\sigma^2, \alpha, \vartheta)$, *then the* $c$*-th convolution power process* $\{X(t),\, t \geq 0\}^{\circledast c}$ *is a stationary DLG process with parameter* $(\sigma^2, \alpha, c\vartheta)$.

Let $\{X(t),\, t \geq 0\}$ be the stationary DLG process with parameter $(\sigma^2, \alpha, \vartheta)$. The cumulants of the one-dimensional distribution can be obtained directly from (4.6.1):

$$\mathsf{cum}_m(X(t)) = (-1)^{k+1} \frac{d^k}{du^k} \phi(\infty, 0) = (k-1)!\, \frac{2\vartheta}{\sigma^2} \left(\frac{\sigma^2}{-2\alpha}\right)^k, \quad k \in \mathbb{N}, \tag{4.6.2}$$

which are the cumulants of the $\Gamma\big(2\vartheta/\sigma^2,\, \sigma^2/(-2\alpha)\big)$ distribution. Concerning the joint cumulants, however, the foregoing results do not help to find an explicit expression for them. Hence, we will use a different method, namely, the squared OU process representation of the DLG process. It will also explain why the DLG process is called also a squared radial OU process.

**4.6.3 Proposition.** *(Shreve [33, Ch. 31]) If* $d \doteq 4\vartheta/\sigma^2 \in \mathbb{N}$, *then the (stationary) DLG process with parameter* $(\sigma^2, \alpha, \vartheta)$, *is the squared radial part (i.e. the square of the Euclidean norm) of a particular* $d$*-dimensional (stationary) OU* *process. Namely, if* $\{Y_i(t),\, t \geq 0\}$, $i = 1, \ldots, d$, *are independent (stationary) OU processes satisfying the Langevin equations*

$$dY_i(t) = \frac{\alpha}{2}\, Y_i(t)\, dt + \frac{\sigma}{2}\, dB_i(t),$$

*where* $\{B_i(t),\, t \geq 0\}$, $i = 1, 2, \ldots, d$, *are standard BMs, then*

$$X(t) \doteq \sum_{i=1}^{d} Y_i^2(t), \quad t \geq 0, \tag{4.6.3}$$

*is a (stationary) DLG process with parameter* $(\sigma^2, \alpha, \vartheta)$, *i.e. the solution of the stochastic differential equation*

$$dX(t) = \big(\vartheta + \alpha X(t)\big) dt + \sigma \sqrt{X(t)}\, dB(t), \tag{4.6.4}$$

*where* $\{B(t),\, t \geq 0\}$ *is a standard BM.*

**Proof.** By Itô's formula we obtain for $X(t)$ in (4.6.3) that

$$\begin{aligned} dX(t) &= \big(\alpha X(t) + \vartheta\big) dt + \sigma \sum_{i=1}^{d} Y_i(t)\, dB_i(t) \\ &= \big(\alpha X(t) + \vartheta\big) dt + \sigma \sqrt{X(t)} \sum_{i=1}^{d} \frac{Y_i(t)}{\sqrt{X(t)}}\, dB_i(t). \end{aligned}$$

Since the independence of the given OU processes implies the independence of the BMs, one can check that

$$B(t) \doteq \sum_{i=1}^{d} \int_0^t \frac{Y_i(s)}{\sqrt{X(s)}}\, dB_i(s)$$

is a continuous square-integrable local martingale with quadratic variation $t$. So, by Lévy's theorem, it is a standard BM. Thus $\{X(t),\, t \geq 0\}$ is a weak solution of (4.6.4), so it is a DLG process. If $\{Y_i(t),\, t \geq 0\}$, $i = 1, \ldots, d$, are stationary, then so is $\{X(t),\, t \geq 0\}$. □

**4.6.4 Corollary.** *A (stationary) DLG process* $\{X(t),\, t \geq 0\}$ *with parameter* $(\sigma^2,\, \alpha,\, \vartheta = \sigma^2/4)$ *can be represented as the square of a (stationary) OU process given by the stochastic differential equation*

$$dY(t) = \frac{\alpha}{2}\, Y(t)\, dt + \frac{\sigma}{2}\, dB(t)$$

*(*$\{B(t),\, t \geq 0\}$ *is a standard BM), i.e.* $X(t) = Y^2(t)$, $t \geq 0$.

**4.6.5 Notation.** For $m \geq 2$, $\underline{t} = (t_1, \ldots, t_m)$, $0 \leq t_1, \ldots, t_m$, and permutations $\tau = (i_1, \ldots i_{m-1}) \in \mathrm{Perm}\{2, 3, \ldots, m\}$, let

$$D_\tau(\underline{t}) \doteq \left|t_{i_1} - t_1\right| + \left|t_{i_2} - t_{i_1}\right| + \cdots + \left|t_{i_{m-1}} - t_{i_{m-2}}\right| + \left|t_1 - t_{i_{m-1}}\right|.$$

Moreover, let us denote the identity permutation by $\tau_0$. Accordingly, we have

$$D_{\tau_0}(\underline{t}) = \left|t_2 - t_1\right| + \left|t_3 - t_2\right| + \cdots + \left|t_m - t_{m-1}\right| + \left|t_1 - t_m\right|.$$

**4.6.6 Remark.** The expression $\sum_{\tau \in \mathrm{Perm}\{2,3,\ldots,m\}} e^{\frac{\alpha}{2} D_\tau(\underline{t})}$ will play a key role thereinafter. For $m = 2$ and $m = 3$ it is

$$\sum_{\tau \in \mathrm{Perm}(2)} e^{\frac{\alpha}{2} D_\tau(t_1,t_2)} = e^{\alpha(t_2^* - t_1^*)}, \qquad \sum_{\tau \in \mathrm{Perm}(2,3)} e^{\frac{\alpha}{2} D_\tau(t_1,t_2,t_3)} = 2e^{\alpha(t_3^* - t_1^*)}.$$

However, for $m \geq 4$ the matter is not so simple, e.g.

$$\sum_{\tau \in \mathrm{Perm}(2,3,4)} e^{\frac{\alpha}{2} D_\tau(t_1,t_2,t_3,t_4)} = 4e^{\alpha(t_4^* - t_1^*)} + 2e^{\alpha(t_4^* - t_1^* + t_3^* - t_2^*)}.$$

Returning to the method of obtaining the joint cumulants of the stationary DLG process, we can see that Proposition 4.6.2 and Corollary 4.6.4 make it possible to reduce the problem to find the joint cumulants of the squared stationary OU process. And for the joint cumulants of the latter process there does exist an explicit expression. Hence, we obtain the following lemma.

**4.6.7 Lemma.** *(Iglói–Terdik [16, Thm. 4][1]) Let $\{X(t),\, t \geq 0\}$ be a stationary DLG process with parameter $(\sigma^2, \alpha, \vartheta)$. Its joint cumulants are*

$$\mathsf{cum}\left(X(t_1), \ldots, X(t_m)\right) = \begin{cases} \dfrac{\vartheta}{-\alpha} & \text{if } m = 1, \\[2ex] \dfrac{\vartheta}{-\alpha}\left(\dfrac{\sigma^2}{-2\alpha}\right)^{m-1} \displaystyle\sum_{\tau \in \mathsf{Perm}(2,\ldots,m)} e^{\frac{\alpha}{2} D_\tau(\underline{t})} & \text{if } m \geq 2, \end{cases}$$
$$0 \leq t_1, \ldots, t_m. \qquad (4.6.5)$$

[1]In [16] a different parametrization is used, see [16, formulas (2.2) and (2.4)].

*Consequently, for the DLG process the right-hand side of (4.3.17) is*

$$P_m\left(e^{\alpha(t_2^*-t_1^*)},\dots,e^{\alpha(t_m^*-t_{m-1}^*)}\right)=\frac{\vartheta}{-\alpha}\left(\frac{\sigma^2}{-2\alpha}\right)^{m-1}\sum_{\tau\in\mathsf{Perm}(2,\dots,m)}e^{\frac{\alpha}{2}D_\tau(\underline{t})},$$
$$0\le t_1,\dots,t_m,\ m\ge 2. \quad (4.6.6)$$

**Proof.** The case $m=1$ follows from (4.6.2) with $k=1$. Let $m\ge 2$ and let $\{Y(t),\,t\ge 0\}$ be the stationary OU process given by

$$dY(t)=\frac{\alpha}{2}Y(t)\,dt+\frac{\sigma}{2}\,dB(t) \quad (4.6.7)$$

($\{B(t),\,t\ge 0\}$ is a standard BM). Since $\{Y(t),\,t\ge 0\}$ is Gaussian with zero mean, using Terdik [36, Example 10] (or, in fact [36, formula (2.9)], which is a direct expression for the joint cumulants of multiple Wiener–Itô integrals) we obtain

$$\mathsf{cum}\left(Y^2(t_1),\dots,Y^2(t_m)\right)=\mathsf{cum}\left(H_2(Y(t_1)),\dots,H_2(Y(t_m))\right)$$
$$=2^{m-1}\sum_{(i_1,\dots,i_{m-1})\in\mathrm{Perm}(2,\dots,m)}\mathsf{Cov}\big(Y(t_1),Y(t_{i_1})\big)\mathsf{Cov}\big(Y(t_{i_1}),Y(t_{i_2})\big)\cdots$$
$$\times\,\mathsf{Cov}\big(Y(t_{i_{m-2}}),Y(t_{i_{m-1}})\big)\mathsf{Cov}\big(Y(t_{i_{m-1}}),Y(t_1)\big)$$
$$=2^{m-1}\sum_{\tau\in\mathrm{Perm}(2,\dots,m)}\left(\frac{\sigma^2}{-4\alpha}\right)^m e^{\frac{\alpha}{2}D_\tau(\underline{t})}=2^{-m-1}\left(\frac{\sigma^2}{-\alpha}\right)^m\sum_{\tau\in\mathrm{Perm}(2,\dots,m)}e^{\frac{\alpha}{2}D_\tau(\underline{t})},$$

where $H_2(Y(t))=Y^2(t)-\mathsf{E}\,Y^2(t)$ is the second order Hermite polynomial of the Gaussian random variable $Y(t)$. By Corollary 4.6.4 process $\{Y^2(t),\,t\ge 0\}$ is a stationary DLG process with parameter $(\sigma^2,\,\alpha,\,\sigma^2/4)$. So, using Proposition 4.6.2, we obtain that

$$\mathsf{cum}\left(X(t_1),\dots,X(t_m)\right)=\frac{\vartheta}{\frac{\sigma^2}{4}}\,\mathsf{cum}\left(Y^2(t_1),\dots,Y^2(t_m)\right)$$
$$=\frac{\vartheta}{-\alpha}\left(\frac{\sigma^2}{-2\alpha}\right)^{m-1}\sum_{\tau\in\mathrm{Perm}(2,\dots,m)}e^{\frac{\alpha}{2}D_\tau(\underline{t})}.$$

□

Using the above method, we can also obtain the characteristic function of the $m$-dimensional distribution of the stationary DLG process. The next lemma deals with this, but first we define the multi-dimensional $\Gamma$ distribution.

**4.6.8 Definition.** Let $m \in \mathbb{N}$, $c_1, c_2, p > 0$ and $\Sigma$ is an $m \times m$ matrix of the form

$$\Sigma \doteq \left( c_1 e^{-c_2 |t_i - t_j|} \right)_{i,j=1,\dots m},$$

where $t_1, \dots, t_m \in \mathbb{R}$. Then the *$m$-dimensional $\Gamma$ distribution with parameter* $(p, \Sigma)$ (briefly the $\Gamma_m(p, \Sigma)$ distribution) is the distribution, the characteristic function of which is

$$\varphi(\underline{u}) = \left| I - i\Sigma U \right|^{-p}, \quad \underline{u} \in \mathbb{R}^m, \tag{4.6.8}$$

where $I = I_{m \times m}$ is the identity matrix, $U = \mathsf{diag}(u_1, \dots, u_m)$ and the vertical lines denote the determinant.

That (4.6.8) is really a characteristic function, i.e. that the multi-dimensional $\Gamma$ distribution is well-defined, is the consequence of the following lemma.

**4.6.9 Lemma.** *(Iglói–Terdik [16, Prop. 7]) Let $m \in \mathbb{N}$, $0 \le t_1 \le \ldots \le t_m$ and $\{X(t), t \ge 0\}$ be a stationary DLG process with parameter $(\sigma^2, \alpha, \vartheta)$. Then the distribution of the vector $(X(t_1), \dots, X(t_m))$ is the $\Gamma_m(2\vartheta/\sigma^2, 2\Sigma)$ distribution, where*

$$\Sigma = \Sigma_{m \times m} = \left( \frac{\sigma^2}{-4\alpha} e^{\frac{\alpha}{2} |t_i - t_j|} \right)_{i,j=1,\dots m}. \tag{4.6.9}$$

**Proof.** Let $\{Y(t), t \ge 0\}$ be the stationary OU process given by (4.6.7). Then the autocovariance matrix of $\underline{Y} \doteq (Y(t_1), \dots, Y(t_m))$ is $\Sigma$ in (4.6.9). The random matrix $\underline{Y}^T \underline{Y}$ is of the Wishart distribution, i.e. $\underline{Y}^T \underline{Y} \sim W_m(1, \Sigma)$, so, its characteristic function is

$$\varphi_{\underline{Y}^T \underline{Y}}(V) = \left| I - 2i\Sigma V \right|^{-\frac{1}{2}}, \quad V = V_{m \times m} \text{ is symmetric.}$$

Using this, the characteristic function of $\underline{Y^2} \doteq (Y^2(t_1), \dots, Y^2(t_m))$, i.e. that of the diagonal of $\underline{Y}^T \underline{Y}$ is

$$\varphi_{\underline{Y^2}}(\underline{u}) = \varphi_{\underline{Y}^T \underline{Y}}(U) = \left| I - 2i\Sigma U \right|^{-\frac{1}{2}}, \quad \underline{u} \in \mathbb{R}^m.$$

Hence, by Proposition 4.6.2 we obtain

$$\varphi_{(X(t_1),\dots,X(t_m))}(\underline{u}) = \varphi_{\underline{(Y^2)}^{\circledast \frac{\vartheta}{\sigma^2/4}}}(\underline{u}) = \varphi_{\underline{Y^2}}^{\frac{\vartheta}{\sigma^2/4}}(\underline{u}) = \left|I - 2i\Sigma U\right|^{-\frac{2\vartheta}{\sigma^2}}, \quad \underline{u} \in \mathbb{R}^m,$$

which is the characteristic function of the $\Gamma_m(2\vartheta/\sigma^2, 2\Sigma)$ distribution □

**4.6.10 Remark.** For $m = 1$, Lemma 4.6.9 states that the one-dimensional distribution of the stationary DLG process with parameter $(\sigma^2, \alpha, \vartheta)$ is the $\Gamma(2\vartheta/\sigma^2,\, \sigma^2/(-2\alpha))$ distribution, agreeing with the last statement of Theorem 4.6.1.

Now, we enter upon the superposition of stationary DLG processes. The matter of Section 4.4 necessarily applies also to the DLG case, with the simplification $m_1 \equiv m_2 \equiv 0$. Accordingly, we adopt the notation used there. The first thing we have to discuss is the naming.

**4.6.11 Definition.** Process $Y(t) \doteq \sum_{j=1}^{\infty} X_j(t),\ t \geq 0,$ which exists by Theorem 4.4.4, is called a *superposition of DLG processes (SDLG process)* (with parameters $(H, \ell)$ and $(\sigma^2, \vartheta)$).

Using Lemma 4.6.7 with the replacements $\alpha := -1$ and $\underline{t} := \delta\underline{t}$, we obtain the statement of Lemma 4.4.6 particularly for the SDLG process. But first we have to introduce a notation for the symmetrization operator.

**4.6.12 Notation.** The symmetrization operator, defined on multivariate functions, will be denoted by $\mathsf{sym}$, i.e.

$$\underset{\underline{t}}{\mathsf{sym}}\ f(\underline{t}) \doteq \frac{1}{m!} \sum_{(i_1,\dots,i_m)\in \mathsf{Perm}(1,2,\dots,m)} f(t_{i_1}, \dots, t_{i_m}),$$

$\underline{t} = (t_1, \dots, t_m),\ m \in \mathbb{N}.$

**4.6.13 Lemma.** *The mean of the SDLG process* $\{Y(t),\, t \geq 0\}$ *is* $\mathsf{E}Y(t) = \vartheta$, $t \geq 0$, *and the joint cumulants are*

$$\mathsf{cum}\big(Y(t_1), \dots, Y(t_m)\big) = 2^{1-m}\, \vartheta\, \sigma^{2m-2} (m-1)!\ \underset{\underline{t}}{\mathsf{sym}} \left(\mathsf{E}\, e^{-\delta \frac{1}{2} D_{\tau_0}(\underline{t})}\right),$$

$0 \leq t_1, \dots, t_m,\ m \geq 2$ *(see Notation 4.6.5 for* $D_{\tau_0}(\underline{t})$*).*

**Assumption SPLM.** (**s**uper**p**ositional **l**aw **m**oment) Let the superposition law given by Assumption SPL at page 57, have also finite $(1+\varepsilon)$-th moment, i.e.

$$\mathsf{E}\,\delta^{1+\varepsilon} = \sum_{i=1}^{\infty} p_j\, d_j^{1+\varepsilon} < \infty \quad \text{for some } \varepsilon > 0.$$

The following theorem states that under Assumption SPLM there holds a functional limit theorem for the convergence to the SDLG process. It is new in the sense that there is nothing in Section 4.4 corresponding to it.

**4.6.14 Theorem.** *Under Assumption SPLM, the sequence of distributions on* $C[0,\infty)$ *of finite superpositions of stationary DLG processes,* $\left\{\sum_{j=1}^{n} X_j(t),\, t \geq 0\right\}$*, converges weakly to the distribution of the SDLG process* $\{Y(t),\, t \geq 0\}$ *with parameters* $(H,\ell)$ *and* $(\sigma^2,\vartheta)$*, i.e.*

$$\left\{ \sum_{j=1}^{n} X_j(t),\ t \geq 0 \right\} \xrightarrow[n\to\infty]{\text{w}} \{Y(t),\ t \geq 0\} \quad \text{on } C[0,\infty). \tag{4.6.10}$$

**Proof.** The convergence of the finite-dimensional distributions follows from Theorem 4.4.4. Furthermore, for each $j \in \mathbb{N}$, the parameter of DLG process $\{X_j(t),\, t \geq 0\}$ is $(d_j\sigma^2,\, -d_j,\, p_j d_j \vartheta)$, so, it follows from Theorem 4.6.9 that the characteristic function of $X_j(t) - X_j(0)$ is

$$\begin{aligned}
\varphi_{X_j(t)-X_j(0)}(u) &= \varphi_{(X_j(t),X_j(0))}(u,-u) \\
&= \left| I - iu\frac{\sigma^2}{2} \begin{pmatrix} 1 & e^{-\frac{d_j}{2}t} \\ e^{-\frac{d_j}{2}t} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right|^{-\frac{2p_j\vartheta}{\sigma^2}} \\
&= \left( 1 + u^2 \frac{\sigma^4}{4} \left(1 - e^{-d_j t}\right) \right)^{-\frac{2p_j\vartheta}{\sigma^2}}.
\end{aligned}$$

Calculating the moments from the characteristic function by differentiation:

$$\begin{aligned}
\mathsf{E}\big(X_j(t) - X_j(0)\big)^2 &= p_j \vartheta \sigma^2 \left(1 - e^{-d_j t}\right), \\
\mathsf{E}\big(X_j(t) - X_j(0)\big)^4 &= \frac{3}{2} p_j \vartheta \sigma^4 \left(2p_j\vartheta + \sigma^2\right) \left(1 - e^{-d_j t}\right)^2.
\end{aligned}$$

Hence, using also the independence of the processes, we obtain that

$$\begin{aligned}
&\mathsf{E}\Big(\sum_{j=1}^{n} X_j(t) - \sum_{j=1}^{n} X_j(0)\Big)^4 \\
&= \sum_{j=1}^{n} \mathsf{E}\big(X_j(t)-X_j(0)\big)^4 + 3\sum_{j=1}^{n}\sum_{\substack{k=1\\ j\neq k}}^{n} \mathsf{E}\big(X_j(t)-X_j(0)\big)^2 \mathsf{E}\big(X_k(t)-X_k(0)\big)^2 \\
&\le c_1 \sum_{j=1}^{n} p_j\, d_j^{1+\varepsilon}\, t^{1+\varepsilon} + c_2\Big(\sum_{j=1}^{n} p_j\, d_j\Big)^2 t^2 \\
&= c_1 \mathsf{E}\,\delta^{1+\varepsilon}\, t^{1+\varepsilon} + c_2\big(\mathsf{E}\,\delta\big)^2 t^2,
\end{aligned}$$

with some numbers $c_1, c_2 > 0$. Let $T > 0$ be arbitrary. There exists a number $c_3 > 0$ (depending on $T$), such that

$$c_1 \mathsf{E}\,\delta^{1+\varepsilon}\, t^{1+\varepsilon} + c_2\big(\mathsf{E}\,\delta\big)^2 t^2 \;\le\; c_3\, t^{1+\varepsilon}, \quad 0 \le t \le T.$$

From the above inequalities we obtain that

$$\mathsf{E}\Big(\sum_{j=1}^{n} X_j(t) - \sum_{j=1}^{n} X_j(0)\Big)^4 \;\le\; c_3\, t^{1+\varepsilon}, \quad 0 \le t \le T,$$

and, since processes $\big\{\sum_{j=1}^{n} X_j(t),\, t \ge 0\big\}$, $n \in \mathbb{N}$, have stationary increments, Kolmogorov's moment condition for tightness on $C[0,T]$ is fulfilled. Thus, we have

$$\Big\{\sum_{j=1}^{n} X_j(t),\, 0 \le t \le T\Big\} \xrightarrow[n\to\infty]{\mathrm{w}} \big\{Y(t),\; 0 \le t \le T\big\} \quad \text{on } C[0,T],$$

and, since $T > 0$ was chosen arbitrarily, (4.6.10) results, too. □

Now we continue with the integral processes. Definition 4.4.10 particularly in the SDLG case reads as follows:

**4.6.15 Definition.** Let $\{Y(t),\; t \ge 0\}$ be an SDLG process with parameters $(H, \ell)$ and $(\sigma^2, \vartheta)$. Then process

$$J_{\sigma^2,\vartheta}^{(H,\ell)}(t) \doteq \int_0^t Y(s)\, ds, \quad t \ge 0, \tag{4.6.11}$$

is called an *ISDLG (integrated SDLG) process* (with parameters $(H,\ell)$ and $(\sigma^2,\vartheta)$).

**4.6.16 Remark.** The ISDLG process has stationary increments, since the SDLG process is stationary.

Merging Theorem 4.4.15 and Theorem 4.4.12 into a single one and stating it for the DLG family we obtain the next theorem.

**4.6.17 Theorem.** *The sequence of finite superpositions of integrated stationary DLG processes,* $\big\{\sum_{j=1}^{n}\int_0^t X_j(s)\,ds,\ t\geq 0\big\}$, *converges pointwise, both in* $L^2$ *and almost surely to the ISDLG process, i.e., for each* $t\geq 0$, *it holds that*

$$\sum_{j=1}^{n}\int_0^t X_j(s)\,ds \underset{n\to\infty}{\longrightarrow} J_{\sigma^2,\vartheta}^{(H,\ell)}(t) \quad \text{in } L^2 \text{ and a.s.}. \tag{4.6.12}$$

*There also holds the weak convergence of the distributions on* $C[0,\infty)$*:*

$$\left\{\sum_{j=1}^{n}\int_0^t X_j(s)\,ds,\ t\geq 0\right\} \underset{n\to\infty}{\overset{w}{\longrightarrow}} \left\{J_{\sigma^2,\vartheta}^{(H,\ell)}(t),\ t\geq 0\right\} \quad \text{on } C[0,\infty). \tag{4.6.13}$$

**4.6.18 Remark.** By Theorem 4.6.17 the ISDLG process could also be named a SIDLG process.

Now comes the main theorem of this section, the particular case of Theorem 4.5.4, stated for the ISDLG→LISDLG case. It includes the definition of the LISDLG process.

**4.6.19 Theorem.** *The family of distributions of the renormalized centered ISDLG processes*

$$\frac{1}{T}\left\{J_{\sigma^2,\vartheta}^{(H,\ell)}(Tt) - \mathsf{E}\,J_{\sigma^2,\vartheta}^{(H,\ell)}(Tt),\ t\geq 0\right\}^{\circledast\frac{T^{2-2H}}{\ell(T)}}, \quad T>0,$$

*converges weakly on* $C[0,\infty)$ *to a limit distribution. We will denote the process with this distribution by* $J_{\sigma^2,\vartheta}^{(H)}(t)$ *and call it the LISDLG process (limit of* *[renormalized centered] ISDLG processes) with parameters* $H$ *and* $(\sigma^2, \vartheta)$*:*

$$\frac{1}{T}\left\{J^{(H,\ell)}_{\sigma^2,\vartheta}(Tt) - \mathsf{E}\,J^{(H,\ell)}_{\sigma^2,\vartheta}(Tt),\ t \geq 0\right\}^{\circledast \frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T\to\infty]{\mathrm{w}} \left\{J^{(H)}_{\sigma^2,\vartheta}(t),\ t \geq 0\right\} \quad \text{on } C[0,\infty). \tag{4.6.14}$$

*The distribution on* $C[0,\infty)$*, of the LISDLG process with parameters* $H$ *and* $(\sigma^2, \vartheta)$ *is uniquely determined by its zero mean and its joint cumulants, which are:*

$$\mathsf{cum}\left(J^{(H)}_{\sigma^2,\vartheta}(t_1), \ldots, J^{(H)}_{\sigma^2,\vartheta}(t_m)\right) = 2^{3-2H-m}\,\Gamma(3-2H)\,\vartheta\,\sigma^{2m-2}(m-1)!\ \underset{\underline{t}}{\mathsf{sym}} \int_0^{t_1}\!\cdots\!\int_0^{t_m} \left(D_{\tau_0}(\underline{s})\right)^{2H-2} d\underline{s}\,, \tag{4.6.15}$$

$0 \leq t_1, \ldots, t_m,\ m \geq 2$*. Moreover,* $\left\{J^{(H)}_{\sigma^2,\vartheta}(t),\ t \geq 0\right\}$ *has stationary increments and it is* $(H, 2H-2)$*-dilatively stable.*

We note that the right-hand side of (4.6.15) was obtained from that of (4.5.7) using Notation 4.5.1 and (4.6.6).

For the LISDLG process as a particular LISCBI process, the statement of Remark 4.5.8 reads as follows:

**4.6.20 Remark.** The LISLG process with parameter $H$ has the same autocovariance function as that of the FBM with parameter $H$ (apart from a constant factor), and it is of long memory.

# Chapter 5

# Cox processes and dilative stability

According to Grandell [13, Thm. 4.2.2], if a self-similar renormalization functional limit theorem holds for the intensity process of a Cox process, then a similar self-similar renormalization functional limit theorem holds also for the Cox process itself. In this chapter we will present the analogous statement for dilatively stable renormalization functional limit theorems. As an application we will obtain a renormalization functional limit theorem for the superposition of death counting (SDC) processes of stationary birth and death processes with immigration, with a LISDLG limit process.

## 5.1 Renormalization functional limit theorems for Cox processes

Before coming to the subject of this section, we fix some notations and terminology.

**5.1.1 Notation.** $\big\{D_0[0,\infty) \doteq \big\{f \in D[0,\infty) : f(0) = 0,\ \ f \text{ is non-decreasig}\big\}$

**5.1.2 Notation.** $C^+[0,\infty) \doteq \big\{f \in C[0,\infty) : f \text{ is non-negative}\big\},\ C_c^+[0,\infty) \doteq \big\{f \in C^+[0,\infty) : f \text{ has compact support}\big\}$.

Because of the one-to-one correspondence between the elements of $D_0[0,\infty)$ and Radon measures on $[0,\infty)$, we can identify them. So, there are two topologies on $D_0[0,\infty)$: the subspace Skorokhod topology and the vague topology, i.e. the topology corresponding to the following concept of convergence. A sequence $F_n : \mathbb{N} \to D_0[0,\infty)$ is said to converge to $F \in D_0[0,\infty)$, if for all $f \in C_c^+[0,\infty)$,

$$\lim_{n\to\infty} \int\limits_0^\infty f(t)\, dF_n(t) = \int\limits_0^\infty f(t)\, dF(t).$$

The two Borel $\sigma$-algebras on $D_0[0,\infty)$, one arising from the subspace Skorokhod topology and the other arising from the vague topology are the same and both coincide with the cylinder $\sigma$-algebra on $D_0[0,\infty)$. Therefore, stochastic processes with sample paths in $D_0[0,\infty)$ are automatically random elements with values in the measurable space $D_0[0,\infty)$. So, they are called random measures. On the other hand, the subspace Skorokhod topology and the vague topology on $D_0[0,\infty)$ are different. Namely, the subspace Skorokhod topology is the finer one (see Grandell [13, Sec. 1.3]).

**5.1.3 Notation.** $\{N(t),\, t \geq 0\}$ denotes the standard Poisson process.

**5.1.4 Definition.** (Grandell [13, Sec. 1.3]) If $\{Y(t),\, t \geq 0\}$ is a random measure, independent of $\{N(t),\, t \geq 0\}$, then process

$$\big\{N \triangleleft Y(t),\, t \geq 0\big\} \doteq \big\{N(Y(t)),\, t \geq 0\big\}$$

(which is also a random measure), is called a *Cox process with intensity process* $\{Y(t),\, t \geq 0\}$.

**5.1.5 Definition.** (Grandell [13, Def. A1.1]) The functional

$$L_Y(f) \doteq \mathsf{E}\exp\left(-\int\limits_0^\infty f(t)\, dY(t)\right), \quad f \in C_c^+[0,\infty),$$

is called the *Laplace functional* of random measure $\{Y(t),\, t \geq 0\}$ (or, of its distribution on $D_0[0,\infty)$).

**5.1.6 Remark.** The distribution of a random measure is uniquely determined by its Laplace functional (see Grandell [13, Thm. A1.3]).

**5.1.7 Example.** The Laplace functional of the standard Poisson process:

$$L_N(f) = \exp\left(-\int_0^\infty \left(1 - e^{-f(t)}\right) dt\right), \quad f \in C_c^+[0,\infty).$$

The Laplace functional of the Cox process with intensity process $\{Y(t),\, t \geq 0\}$:

$$L_{N \triangleleft Y}(f) = L_Y\left(1 - e^{-f}\right), \quad f \in C_c^+[0,\infty). \tag{5.1.1}$$

**5.1.8 Remark.** By Remark 5.1.6 and (5.1.1), the distribution on $D_0[0,\infty)$ of a random measure $\{Y(t),\, t \geq 0\}$ and of the Cox process $\{N \triangleleft Y(t),\, t \geq 0\}$ mutually determine each other.

**5.1.9 Remark.** By Remark 5.1.6 and (5.1.1), if a random measure $\{Y(t),\, t \geq 0\}$ is infinitely divisible, then the Cox process $\{N \triangleleft Y(t),\, t \geq 0\}$ is such as well.

The reason for the following definition is that the weak convergence of distributions on some topological space depends also on the topology of the space.

**5.1.10 Definition.** We say that a sequence $\{Y_n(t),\, t \geq 0\}$, $n \in \mathbb{N}$, of random measures *converges* to a random measure $\{Y(t),\, t \geq 0\}$ *in the weak vague sense (in the weak [Skorokhod] sense)* and use the notation $\xrightarrow[n\to\infty]{\mathrm{wv}}$ ( $\xrightarrow[n\to\infty]{\mathrm{wS}}$ ), if for all bounded functionals $f : D_0[0,\infty) \longrightarrow \mathbb{R}$, which are continuous with respect to the vague topology (with respect to the Skorokhod topology, resp.) of $D_0[0,\infty)$, the convergence $\lim_{n\to\infty} \mathsf{E}\, f(\{Y_n(t),\, t \geq 0\}) = \mathsf{E}\, f(\{Y(t),\, t \geq 0\})$ holds.

**5.1.11 Remark.** Since the subspace Skorokhod topology on $D_0[0,\infty)$ is finer than the vague topology, the weak convergence of a sequence of random measures implies the weak vague convergence, but not conversely, in general. However, in the case of simple point processes the weak vague convergence does imply the weak convergence, according to Daley–Vere-Jones [6, Thm. 9.1.VI], along with Jacod–Shiryaev [18, Thm. VI.3.37 (b)].

**5.1.12 Remark.** By Grandell [13, Thm. A1.6], the weak vague convergence

$$\left\{Y_n(t),\, t \geq 0\right\} \xrightarrow[n\to\infty]{\mathrm{wv}} \left\{Y(t),\, t \geq 0\right\} \quad \text{on } D_0[0,\infty),$$

is equivalent with the pointwise convergence of the corresponding Laplace functionals:

$$\lim_{n\to\infty} L_{Y_n}(f) = L_Y(f), \quad f \in C_c^+[0,\infty).$$

The subject of this section is a dilatively stable renormalization functional limit theorem stating the weak convergence on $D[0,\infty)$ of certain renormalized Cox processes (i.e. the weak convergence of their distributions on $D[0,\infty)$). Therefore, we have to explain what we mean by dilatively stable renormalization of a Cox process. The difficulty is that nothing ensures that the dilatively stable renormalization operator $A_T^{(\alpha,\delta)} : \mathcal{I} \longrightarrow \mathcal{I}$ (see Definition 2.8.1 (DS)) carries càdlàg processes into càdlàg processes. The reason is that the operation $+ : D[0,\infty) \times D[0,\infty) \longrightarrow D[0,\infty)$ is not continuous, there is no continuous convolution semigroup in the set of distributions on $D[0,\infty)$ and thus, the $c$-th convolution power operation is not defined for every $c > 0$. Fortunately, we are actually interested only in convolution powers of Cox processes and of their intensity processes and these processes are also random measures on $[0,\infty)$. And an infinitely divisible random measure $\{Y(t),\, t \geq 0\}$ (on $[0,\infty)$) does induce a convolution semigroup, which is continuous in the vague topology (and even a Lévy-Khintchine type representation holds, see Daley–Vere-Jones [6, Prop. 9.2.VII]), hence the $c$-th convolution power $\{Y(t),\, t \geq 0\}^{\circledast c}$ is well-defined for any $c > 0$, and thus, $\{Y(t),\, t \geq 0\}^{\circledast c}$ is a càdlàg process as well. So, by Remark 5.1.9 and (5.1.1), for any $c > 0$ we have

$$\big\{N \triangleleft Y(t),\, t \geq 0\big\}^{\circledast c} \sim \big\{N \triangleleft Y^{\circledast c}(t),\, t \geq 0\big\}, \quad \text{on } D[0,\infty). \tag{5.1.2}$$

The convolution powers of translations of random measures and even of certain linear transforms of them can also be naturally defined as follows:

**5.1.13 Definition.** For any constants $c > 0$, $b \neq 0$, function $a \in D[0,\infty)$ and infinitely divisible random measure $\{Y(t),\, t \geq 0\}$ let us define the $c$-th convolution power of the linearly transformed random measure by

$$\begin{aligned}\left\{\left(\frac{Y(t)-a(t)}{b}\right)^{\circledast c},\ t \geq 0\right\} &= \left\{\frac{Y(t)-a(t)}{b},\ t \geq 0\right\}^{\circledast c} \\ &\doteq \left\{\frac{1}{b}\, Y^{\circledast c}(t) - \frac{c}{b}\, a(t),\ t \geq 0\right\} \\ &= \frac{1}{b}\,\big\{Y(t),\, t \geq 0\big\}^{\circledast c} - \frac{c}{b}\,\big\{a(t),\, t \geq 0\big\}. \end{aligned} \tag{5.1.3}$$

It is clear that (5.1.3) is a càdlàg process.

Coming to the point, it follows from Theorem 2.2.7 (SS) that if there exist functions $a : [0,\infty) \to \mathbb{R}$, $f : (0,\infty) \to (0,\infty)$ and process $\{X(t),\, t \geq 0\} \in \mathcal{S}$, such that

$$\left\{ \frac{Y(Tt) - a(Tt)}{f(T)},\ t \geq 0 \right\} \xrightarrow[T\to\infty]{\text{fd}} \left\{ X(t),\, t \geq 0 \right\}, \tag{5.1.4}$$

then there exists an $\alpha > 0$ such that $f$ is regularly varying of order $\alpha$ (briefly $\alpha$-varying) and $\{X(t),\, t \geq 0\}$ is $\alpha$-self-similar. Analogously, it follows from Theorem 2.2.7 (DS) that if $\{Y(t),\, t \geq 0\}$ is a process with finite moments of all orders and there exist functions $a : [0,\infty) \to \mathbb{R}$, $f : (0,\infty) \to (0,\infty)$, $g : (0,\infty) \to (0,\infty)$ and a process $\{X(t),\, t \geq 0\} \in \mathcal{I}$, such that

$$\left\{ \frac{\sqrt{g(T)}}{f(T)} \big(Y(Tt) - a(Tt)\big),\, t \geq 0 \right\}^{\circledast \frac{1}{g(T)}} \xrightarrow[T\to\infty]{\text{fd}} \{X(t),\, t \geq 0\},$$

then, under some natural moment conditions, there exists $\alpha > 0$, $\delta \leq 2\alpha$, such that $f$ is $\alpha$-varying, $g$ is $\delta$-varying and $\{X(t),\, t \geq 0\}$ is $(\alpha,\delta)$-dilatively stable. This is the motivation for the notions of self-similar and dilatively stable renormalization functional limit theorems.

**5.1.14 Definition.** **(SS)** If $\{Y(t),\, t \geq 0\}$ is a càdlàg process, then we say that a *self-similar renormalization functional limit theorem* holds for $\{Y(t),\, t \geq 0\}$, if there exists a càdlàg function $a$, a number $\alpha > 0$, an $\alpha$-varying function $f : (0,\infty) \to (0,\infty)$ and an $\alpha$-self-similar càdlàg process $\{X(t),\, t \geq 0\}$, such that

$$\left\{ \frac{Y(Tt) - a(Tt)}{f(T)},\, t \geq 0 \right\} \xrightarrow[T\to\infty]{\text{w}} \left\{ X(t),\, t \geq 0 \right\} \quad \text{on } D[0,\infty). \tag{5.1.5}$$

**(DS)** If $\{Y(t),\, t \geq 0\}$ is an infinitely divisible random measure, then we say that a *dilatively stable renormalization functional limit theorem* holds for $\{Y(t),\, t \geq 0\}$, if there exists a càdlàg function $a$, numbers $\alpha > 0$, $\delta \leq 2\alpha$, an $\alpha$-varying function $f : (0,\infty) \to (0,\infty)$, a $\delta$-varying function $g : (0,\infty) \to (0,\infty)$, for which $\lim_{t\to\infty} \big(f(t)/\sqrt{g(t)}\big) = \infty$ and an $(\alpha,\delta)$-dilatively stable càdlàg process $\{X(t),\, t \geq 0\}$, such that

$$\left\{ \frac{\sqrt{g(T)}}{f(T)} \big(Y(Tt) - a(Tt)\big),\, t \geq 0 \right\}^{\circledast \frac{1}{g(T)}} \xrightarrow[T\to\infty]{\text{w}} \left\{ X(t),\, t \geq 0 \right\} \quad \text{on } D[0,\infty). \tag{5.1.6}$$

Observe that (5.1.4) and (5.1.6) allow an additive renormalization function $a$. This function was not included in the renormalization used in Chapter 2. Indeed, such an additive renormalization was unnecessary, because we could consider directly the process $\{Y(t)-a(t),\, t\geq 0\}$ instead of $\{Y(t),\, t\geq 0\}$. However, it was essential in Chapter 3 and 4 and it is essential in this chapter, too, since process $\{Y(t),\, t\geq 0\}$ is the intensity process of a Cox process, therefore necessarily non-negative. Namely, the (SS) part (resp. the (DS) part) of the following theorem states that, if $\{Y(t),\, t\geq 0\}$ is the intensity process of a Cox process and the self-similar (resp. dilatively stable) renormalization functional limit theorem (5.1.4) (resp. (5.1.6)) holds for $\{Y(t),\, t\geq 0\}$ with $a\in D_0[0,\infty)$, then a self-similar (resp. dilatively stable) renormalization functional limit theorem holds also for the Cox process itself, with an additional term in the limit process.

**5.1.15 Theorem.** ***(SS)*** *(Grandell [13, Thm. 4.2.2]) Let $\{Y(t),\, t\geq 0\}$ be a random measure for which the self-similar renormalization functional limit theorem (5.1.4) holds, where $a\in D_0[0,\infty)$. If there exists the limit*

$$\lim_{T\to\infty}\frac{a(T)}{f^2(T)}=\kappa\in[0,\infty),$$

*then for the Cox process $\{N\triangleleft Y(t),\, t\geq 0\}$ the following self-similar renormalization functional limit theorem holds:*

$$\left\{\frac{N\triangleleft Y(Tt)-a(Tt)}{f(T)},\ t\geq 0\right\}\xrightarrow[T\to\infty]{\mathrm{w}}\left\{X(t)+B\left(\kappa t^{2\alpha}\right),\, t\geq 0\right\}\quad \text{on } D[0,\infty),$$

*where $\{B(t),\, t\geq 0\}$ is a standard Brownian motion, independent of $\{X(t),\, t\geq 0\}$.*
***(DS)*** *Let $\{Y(t),\, t\geq 0\}$ be an infinitely divisible random measure for which the dilatively stable renormalization functional limit theorem (5.1.6) holds, where $a\in D_0[0,\infty)$. If there exists the limit*

$$\lim_{T\to\infty}\frac{a(T)}{f^2(T)}=\kappa\in[0,\infty),$$

*then for the Cox process $\{N\triangleleft Y(t),\, t\geq 0\}$ the following dilatively stable renormalization functional limit theorem holds:*

$$\left\{\frac{\sqrt{g(T)}}{f(T)}\Big(N\triangleleft Y(Tt)-a(Tt)\Big),\, t\geq 0\right\}^{\circledast\frac{1}{g(T)}}\xrightarrow[T\to\infty]{\mathrm{w}}\{X(t)+B(\kappa t^{2\alpha}),\, t\geq 0\}\quad \text{on } D[0,\infty),\qquad (5.1.7)$$

*where $\{B(t),\, t\geq 0\}$ is a standard Brownian motion, independent of $\{X(t),\, t\geq 0\}$.*

**Proof.** We adapt the idea of the proof of Grandell [13, Thm. 4.2.2]. Let us introduce the notations

$$N_0(t) \doteq N(t) - t, \qquad N_{0,T}(t) \doteq \frac{N_0(Tt)}{\sqrt{T}},$$

$$Y_T(t) \doteq \frac{\sqrt{g(T)}}{f(T)} \big(Y(Tt) - a(Tt)\big)^{\circledast \frac{1}{g(T)}}, \qquad U(t) \doteq N \triangleleft Y(t),$$

$t \geq 0,\ T > 0$. Using (5.1.2) we have

$$\begin{aligned}
&\frac{\sqrt{g(T)}}{f(T)} \big(U(Tt) - a(Tt)\big)^{\circledast \frac{1}{g(T)}} \\
&\quad = \frac{\sqrt{g(T)}}{f(T)} (N \triangleleft Y)^{\circledast \frac{1}{g(T)}} (Tt) - \frac{a(Tt)}{f(T)\sqrt{g(T)}} \\
&\quad \sim \frac{\sqrt{g(T)}}{f(T)} N \triangleleft Y^{\circledast \frac{1}{g(T)}} (Tt) - \frac{a(Tt)}{f(T)\sqrt{g(T)}} \\
&\quad = \frac{\sqrt{g(T)}}{f(T)} \left( N_0\Big(Y^{\circledast \frac{1}{g(T)}} (Tt)\Big) + Y^{\circledast \frac{1}{g(T)}} (Tt) \right) - \frac{a(Tt)}{f(T)\sqrt{g(T)}} \\
&\quad = \frac{\sqrt{g(T)}}{f(T)} N_0\Big(Y^{\circledast \frac{1}{g(T)}} (Tt)\Big) + \frac{\sqrt{g(T)}}{f(T)} \left( Y^{\circledast \frac{1}{g(T)}} (Tt) - \frac{a(Tt)}{g(T)} \right) \\
&\quad = \frac{\sqrt{g(T)}}{f(T)} N_0\left( \frac{f(T)}{\sqrt{g(T}} Y_T(t) + \frac{a(Tt)}{g(T)} \right) + Y_T(t) \\
&\quad = N_{0,\, \frac{f^2(T)}{g(T)}}(Z_T(t)) + Y_T(t), \qquad (5.1.8)
\end{aligned}$$

where $\sim$ means the equivalence in distribution on $D[0,\infty)$ and

$$Z_T(t) \doteq \frac{\sqrt{g(T)}}{f(T)} Y_T(t) + \frac{a(Tt)}{f^2(T)}. \qquad (5.1.9)$$

We know that

$$\big\{Y_T(t),\ t \geq 0\big\} \xrightarrow[T\to\infty]{\mathrm{w}} \big\{X(t),\ t \geq 0\big\} \quad \text{on } D[0,\infty) \qquad (5.1.10)$$

and since $\lim_{T\to\infty}\big(f(T)/\sqrt{g(T)}\big)=\infty$ (because of the dilatively stable renormalization functional limit theorem (5.1.6)), we have

$$\left\{\frac{\sqrt{g(T)}}{f(T)}\,Y_T(t),\,t\ge 0\right\}\xrightarrow[T\to\infty]{\mathrm{P}}0\quad\text{in }\;D[0,\infty).$$

Since function $f^2$ is $2\alpha$-varying,

$$\frac{a(Tt)}{f^2(T)}=\frac{a(Tt)}{f^2(Tt)}\,\frac{f^2(Tt)}{f^2(T)}\xrightarrow[T\to\infty]{}\kappa t^{2\alpha}\tag{5.1.11}$$

pointwise for all $t\in[0,\infty)$. Since $f^2(t)=t^{2\alpha}L(t)$ with some slowly varying function $L$, by Bingham et al. [3, Thm. 1.2.1], the convergence $\lim\limits_{T\to\infty}\big(f^2(Tt)/f^2(T)\big)=t^{2\alpha}$ is uniform on each closed subinterval $[\varepsilon,c]$ of $(0,\infty)$. Also the convergence $\lim\limits_{T\to\infty}\big(a(Tt)/f^2(Tt)\big)=\kappa$ is uniform on each such interval $[\varepsilon,c]$, therefore (5.1.11) holds uniformly on each closed subinterval $[\varepsilon,c]$ of $(0,\infty)$. Since function $a$ is non-decreasing and $a(0)=0$ (this is where the assumption $a\in D_0[0,\infty)$ is exploited), both the left-hand side and the limit in (5.1.11) are small enough on intervals $[0,\varepsilon]$ small enough. Thus, (5.1.11) holds uniformly on closed intervals with zero left endpoints as well, therefore it holds uniformly on each closed nonnegative interval, and this means that (5.1.11) holds in $D[0,\infty)$. Note also that function $\kappa t^{2\alpha}$ is in $C[0,\infty)$, which is a subspace in $D[0,\infty)$ and the addition operation $+:D[0,\infty)\times D[0,\infty)\to D[0,\infty)$ (though it is not continuous on its whole domain) is continuous on $D[0,\infty)\times C[0,\infty)$ (see Grandell [13, Lemma 4.2.2]). Therefore we have

$$\big\{Z_T(t),\,t\ge 0\big\}\xrightarrow[T\to\infty]{\mathrm{P}}\big\{\kappa t^{2\alpha},\,t\ge 0\big\}\quad\text{in }\;D[0,\infty).\tag{5.1.12}$$

From (5.1.10) and (5.1.12), using Billingsley [2, Thm. 4.4] we obtain

$$\Big(\big\{Y_T(t),\,t\ge 0\big\},\ \big\{Z_T(t),\,t\ge 0\big\}\Big)\xrightarrow[T\to\infty]{\mathrm{w}}\Big(\big\{X(t),\,t\ge 0\big\},\ \big\{\kappa t^{2\alpha},\,t\ge 0\big\}\Big)$$
$$\text{on }\;D[0,\infty)\times D[0,\infty).\tag{5.1.13}$$

Since $\lim_{T\to\infty}\big(f(T)/\sqrt{g(T)}\big)=\infty$, it follows that

$$\Big\{N_{0,\,\frac{f^2(T)}{g(T)}}(t),\,t\ge 0\Big\}\xrightarrow[T\to\infty]{\mathrm{w}}\big\{B(t),\,t\ge 0\big\}\quad\text{on }\;D[0,\infty)\tag{5.1.14}$$

(see Grandell [13, Lemma 4.2.3]). Since for each $T > 0$ the processes on the left-hand sides of (5.1.13) and (5.1.14) are independent, it follows by Billingsley [2, Thm. 3.2], that

$$\left(\left\{N_{0,\,\frac{f^2(T)}{g(T)}}(t),\ t\geq 0\right\},\ \left\{Y_T(t),\ t\geq 0\right\},\ \left\{Z_T(t),\ t\geq 0\right\}\right)$$
$$\xrightarrow[T\to\infty]{\mathrm{w}} \left(\left\{B(t),\, t\geq 0\right\},\ \left\{X(t),\, t\geq 0\right\},\ \left\{\kappa t^{2\alpha},\, t\geq 0\right\}\right)$$
$$\text{on } D[0,\infty)\times D[0,\infty)\times D[0,\infty). \tag{5.1.15}$$

From Grandell [13, Lemma 4.2.2] it follows that

$$\left(\left\{N_{0,\,\frac{f^2(T)}{g(T)}}(Z_T(t)),\ t\geq 0\right\},\ \left\{Y_T(t),\ t\geq 0\right\}\right)$$

is a process with sample paths in $D[0,\infty)\times D[0,\infty)$. Moreover, $\{B(t),\, t\geq 0\}$ is a process with sample paths in $C[0,\infty)$ and $\{\kappa t^{2\alpha},\, t\geq 0\}\in D_0[0,\infty)$, so

$$\left(\left\{N_{0,\,\frac{f^2(T)}{g(T)}}(Z_T(t)),\ t\geq 0\right\},\ \left\{Y_T(t),\ t\geq 0\right\}\right)$$
$$\xrightarrow[T\to\infty]{\mathrm{w}} \left(\left\{B\left(\kappa t^{2\alpha}\right),\, t\geq 0\right\},\ \left\{X(t),\, t\geq 0\right\}\right) \quad \text{on } D[0,\infty)\times D[0,\infty). \tag{5.1.16}$$

Since $\{B(\kappa t^{2\alpha}),\, t\geq 0\}$ is a process with sample paths in $C[0,\infty)$, (5.1.16) and Grandell [13, Lemma 4.2.2] imply that

$$\left\{N_{0,\,\frac{f^2(T)}{g(T)}}(Z_T(t))+Y_T(t),\ t\geq 0\right\} \xrightarrow[T\to\infty]{\mathrm{w}} \left\{B\left(\kappa t^{2\alpha}\right)+X(t),\, t\geq 0\right\} \quad \text{on } D[0,\infty).$$

Considering (5.1.8), the last convergence is just the statement we had to prove. Finally, $\{B(t),\, t\geq 0\}$ and $\{X(t),\, t\geq 0\}$ are independent because of (5.1.15) and the independence of $\left\{N_{0,\,f^2(T)/g(T)}(t),\, t\geq 0\right\}$ and $\{Y_T(t),\, t\geq 0\}$. □

**5.1.16 Example.** Let $\alpha > 0$, $\delta \leq 2\alpha$ and $\{X(t),\, t\geq 0\}$ be an $(\alpha,\delta)$-dilatively stable random measure. (There exists such a process, e.g. if $\{L(t),\, t\geq 0\}$ is a non-negative Lévy process with all moments finite, then the process defined by $X(t) \doteq t^{\alpha-\delta/2}L(t^\delta)$, $t>0$, $X(0)\doteq 0$, will do, see Example 2.1.6 (DS).) Moreover, let $\kappa>0$, $a:[0,\infty)\to[0,\infty)$ a function in $D_0[0,\infty)$ for which $\lim_{t\to\infty}\left(a(t)/t^{2\alpha}\right)=\kappa$, and let $Y(t)\doteq X(t)+a(t)$. Then $\{Y(t),\, t\geq 0\}$ is a random measure and $\{U(t),\, t \geq 0\} \doteq \{N \triangleleft Y(t),\, t \geq 0\}$ is a Cox process. Moreover, process $\{Y(t),\, t \geq 0\}$ is infinitely divisible and

$$\Big\{T^{\frac{\delta}{2}-\alpha}\,\big(Y(Tt) - a(Tt)\big),\ t \geq 0\Big\}^{\circledast T^{-\delta}} \sim \big\{X(t),\, t \geq 0\big\},$$

hence the weak convergence

$$\Big\{T^{\frac{\delta}{2}-\alpha}\,\big(Y(Tt) - a(Tt)\big),\ t \geq 0\Big\}^{\circledast T^{-\delta}} \xrightarrow[T\to\infty]{\mathrm{w}} \big\{X(t),\, t \geq 0\big\} \quad \text{on } D[0,\infty),$$

trivially holds. Therefore Theorem 5.1.15 (DS) implies

$$\Big\{T^{\frac{\delta}{2}-\alpha}\big(U(Tt){-}a(Tt)\big),\ t \geq 0\Big\}^{\circledast T^{-\delta}} \xrightarrow[T\to\infty]{\mathrm{w}} \Big\{X(t){+}B(\kappa t^{2\alpha}),\ t \geq 0\Big\} \quad \text{on } D[0,\infty),$$

where the Brownian motion $\{B(t),\, t \geq 0\}$ is independent of $\{X(t),\, t \geq 0\}$.

The next example is less trivial.

**5.1.17 Example.** Let $\big\{J^{(H,\ell)}_{\sigma^2,\vartheta}(t),\, t \geq 0\big\}$ be the ISDLG process,

$$\big\{U(t),\, t \geq 0\big\} \doteq \Big\{N \triangleleft J^{(H,\ell)}_{\sigma^2,\vartheta}(t),\ t \geq 0\Big\} \tag{5.1.17}$$

and let us consider the functional limit Theorem 5.1.15 (DS). The norming functions in it: $a(t) = \vartheta t,\ g(t) = t^{2H-2}\ell(t)$ and $f(t) = \sqrt{g(t)}\; t = \sqrt{\ell(t)}\; t^H$, $t > 0$, and the limit process is the LISDLG process $\big\{J^{(H)}_{\sigma^2,\vartheta}(t),\, t \geq 0\big\}$. Hence

$$\kappa = \lim_{T\to\infty} \frac{a(T)}{f^2(T)} = \lim_{T\to\infty} \frac{\vartheta\, T}{\ell(T) T^{2H}} = 0$$

and by Theorem 5.1.15 (DS), we obtain that

$$\Big\{\frac{1}{T}\big(U(Tt) - Tt\,\vartheta\big),\ t \geq 0\Big\}^{\circledast \frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T\to\infty]{\mathrm{w}} \Big\{J^{(H)}_{\sigma^2,\vartheta}(t),\, t \geq 0\Big\} \quad \text{on } D[0,\infty). \tag{5.1.18}$$

In the next section we will show that Cox process (5.1.17) is, in fact, a superposition of certain well-known stochastic processes.

## 5.2 The SDC process

In Section 3.2 and 4.4, the superposition of OU type processes, resp. of CBI processes, was considered. Now, we study a further such construction. The processes we superpose in this section are the death counting (DC) processes of stationary birth and death processes with immigration. We show that the superposition process we obtain (the SDC process), is just Cox process (5.1.17).

Let us denote by $\{M(t),\, t \geq 0\}$ the birth and death process with immigration (IBD process) with parameter $(\nu, \lambda, \mu)$, where $\nu > 0$, $\lambda > 0$ and $\mu > 0$ are the rate of immigration, birth and death, respectively. That is, $\{M(t),\, t \geq 0\}$ is a continuous-time homogeneous Markov chain with state space $\{0, 1, 2, \ldots\}$ and infinitesimal transition probabilities

$$P\big(M(t{+}h) = n{+}m | M(t) = n\big) = \begin{cases} (n\lambda + \nu)\, h + o(h) & \text{if} \quad m = 1, \\ n\mu h + o(h) & \text{if} \quad m = -1, \\ 1 - (n\lambda + \nu + n\mu)h + o(h) & \text{if} \quad m = 0, \\ o(h) & \text{if} \quad |m| > 1. \end{cases}$$

We assume that $\lambda < \mu$ and $\{M(t),\, t \geq 0\}$ is stationary.

**5.2.1 Definition.** Let $\{M(t),\, t \geq 0\}$ be a stationary IBD process with parameter $(\nu, \lambda, \mu)$. The process that counts the deaths in $\{M(t),\, t \geq 0\}$ during the interval $[0, t]$ is called a *death counting (DC) process* with parameter $(\nu, \lambda, \mu)$ and it is denoted by $\{D(t),\, t \geq 0\}$.

**5.2.2 Definition.** Let $\{X(t),\, t \geq 0\}$ be a stationary DLG process with parameter $(\sigma^2,\, \alpha, \vartheta)$. Then process

$$Y(t) \doteq \int_0^t X(s)\, ds, \quad t \geq 0, \tag{5.2.1}$$

is called an *IDLG (integrated DLG) process* (with parameter $(\sigma^2,\, \alpha, \vartheta)$).

The key to the results of this section is mostly the one-to-one relationship between the IDLG processes and the DC processes, given in Wei et al. [38] and in Clifford–Wei [4]: the DC processes are Cox processes with IDLG intensity processes. The exact correspondence is the following:

**5.2.3 Lemma.** *(Wei et al. [38, Thm. 1]) Let* $\{Y(t),\, t \geq 0\}$ *be an IDLG process with parameter* $(\sigma^2, \alpha, \vartheta)$ *and* $\{D(t),\, t \geq 0\}$ *a DC process with parameter* $(\nu, \lambda, \mu)$*, where the correspondence between the parameters is:*

$$\vartheta = \nu\mu, \quad \alpha = \lambda - \mu, \quad \sigma^2 = 2\lambda\mu, \tag{5.2.2}$$

*or, equivalently,*

$$\nu = \frac{2\vartheta}{\sqrt{\alpha^2 + 2\sigma^2} - \alpha}, \quad \lambda = \frac{\sqrt{\alpha^2 + 2\sigma^2} + \alpha}{2}, \quad \mu = \frac{\sqrt{\alpha^2 + 2\sigma^2} - \alpha}{2}. \tag{5.2.3}$$

*Then*

$$\{D(t),\, t \geq 0\} \sim \big\{N \triangleleft Y(t),\, t \geq 0\big\} \quad \text{on } D_0[0, \infty),$$

*i.e. the two processes have the same distributions on* $D_0[0, \infty)$*.*

**5.2.4 Corollary.** *Using the notations of Lemma 5.2.3, since for every* $c > 0$*,* $\{Y(t),\, t \geq 0\}^{\circledast c}$ *is an IDLG process with parameter* $(\sigma^2, \alpha, c\vartheta)$*, by Lemma 5.2.3 and (5.1.2) we obtain for every* $c > 0$ *that*

$$\big\{D(t),\, t \geq 0\big\}^{\circledast c} \sim \big\{N \triangleleft Y(t),\, t \geq 0\big\}^{\circledast c} \sim \Big\{N \triangleleft Y^{\circledast c}(t),\, t \geq 0\Big\} \quad \text{on } D_0[0, \infty).$$

By Remark 5.1.8 the following definition is well-founded.

**5.2.5 Definition.** For a Cox process $\{N \triangleleft Y(t),\, t \geq 0\}$ and for any $c > 0$, Cox process $\{N \triangleleft cY(t),\, t \geq 0\}$ (more precisely, its distribution) is called a *$c$-thickening* of $\{N \triangleleft Y(t)),\, t \geq 0\}$. Accordingly, the transform

$$T_c(N \triangleleft Y) \doteq T_c\big(\{N \triangleleft Y(t),\, t \geq 0\}\big) \doteq \big\{N \triangleleft cY(t),\, t \geq 0\big\} = N \triangleleft cY,$$

defined on the set of distributions of Cox processes is called a *$c$-thickening transform.*

**5.2.6 Remark.** The $c$-thickening transform is, in fact, a $c$-thinning if $c < 1$ and a $c$-thickening if $c > 1$. For $c < 1$ transform $T_c$ is also called a $c$-rarefaction, defined for all point processes, while for $c > 1$, $T_c = T_{1/c}^{-1}$ is defined for Cox processes only (see Gnedenko–Korolev [12, p. 219]).

Now we are ready to superpose DC processes. That is, we take the sum of independent processes. Each term of the sum, i.e. each process (or more precisely, its distribution) will be some transform of a basic DC process (and, in fact, will itself be a DC process). Each transform will be of the form

$$\big\{D(t),\, t \geq 0\big\} \;\longmapsto\; \Big\{T_{\frac{1}{d_j}} D(d_j t),\, t \geq 0\Big\}^{\circledast p_j}.$$

Each dilational (and also thinning) costant $d_j$ will occur with probability $p_j$, so $p_j$, $j \in \mathbb{N}$, will be a distribution on the discrete set $\{d_j : j \in \mathbb{N}\}$. That is, the time dilation (and the thinning, too) will be a discrete random variable $\delta$. Moreover, the left tail behaviour of the distribution of $\delta$ is given in Assumption SPL on page 57.

Now we detail the assumptions concerning the DC processes, which we will superpose.

- Let $\{D(t),\, t \geq 0\}$ be a DC process with parameter $(\nu,\, \lambda,\, \mu = \lambda + 1)$. Let

$$\big\{D_j(t),\, t \geq 0\big\} \doteq \Big\{T_{\frac{1}{d_j}} D(d_j t),\, t \geq 0\Big\}^{\circledast p_j}, \quad j \in \mathbb{N}.$$

- Let processes $\{D_j(t),\, t \geq 0\}$, $j \in \mathbb{N}$, be independent.

Hereafter we consider the above two assumptions to be fulfilled.

**5.2.7 Proposition.** *For each $j \in \mathbb{N}$, $\{D_j(t),\, t \geq 0\}$ is a DC process with parameter $(\nu_j,\, \lambda_j,\, \mu_j)$, where*

$$\nu_j = \frac{2p_j\nu(\lambda+1)}{g_j+1}, \qquad \lambda_j = d_j\,\frac{g_j-1}{2}, \qquad \mu_j = d_j\,\frac{g_j+1}{2}, \tag{5.2.4}$$

*and $g_j = \sqrt{1 + 4\lambda(\lambda+1)/d_j}$.*

**Proof.** By Lemma 5.2.3, $D(t) = N \triangleleft Y(t)$, $t \geq 0$, where $\{Y(t),\, t \geq 0\}$ is an IDLG process with parameter $(\sigma^2 = 2\lambda(\lambda+1),\, \alpha = -1,\, \vartheta = \nu(\lambda+1))$. Therefore, processes $\big\{T_{1/d_j} D(t),\, t \geq 0\big\}$, $j \in \mathbb{N}$, are well-defined. Moreover, for each $j \in \mathbb{N}$, we have

$$\Big\{T_{\frac{1}{d_j}} D(d_j t),\, t \geq 0\Big\}^{\circledast p_j} \sim \Big\{N \triangleleft \tfrac{1}{d_j} Y(d_j t),\, t \geq 0\Big\}^{\circledast p_j} \sim \Big\{N \triangleleft \tfrac{1}{d_j} Y^{\circledast p_j}(d_j t),\, t \geq 0\Big\},$$

the second relation follows from Corollary 5.2.4. Since $\big\{\frac{1}{d_j} Y^{\circledast p_j}(d_j t),\, t \geq 0\big\}$ is an IDLG process with parameter $(2d_j\lambda(\lambda+1),\, -d_j,\, p_j d_j \nu(\lambda+1))$, we obtain that $\{D_j(t),\, t \geq 0\}$ is a DC process with the parameter given by (5.2.4). $\square$

The following theorem is the main result in this section. It is the analogue of Theorem 4.6.17.

**5.2.8 Theorem.** $\{\sum_{j=1}^n D_j(t),\, t \ge 0\}$, *the sequence of finite superpositions of DC processes converges pointwise, both in* $L^2$ *and almost surely to a counting process* $\{U(t),\, t \ge 0\}$, *i.e., for each* $t \ge 0$, *it holds that*

$$\sum_{j=1}^n D_j(t) \underset{n\to\infty}{\longrightarrow} U(t) \quad \text{in } L^2 \text{ and a.s..} \tag{5.2.5}$$

*There also holds the weak convergence of the distributions on* $D[0,\infty)$*:*

$$\left\{\sum_{j=1}^n D_j(t),\ t \ge 0\right\} \underset{n\to\infty}{\overset{w}{\longrightarrow}} \{U(t),\, t \ge 0\} \quad \text{on } D[0,\infty). \tag{5.2.6}$$

*Moreover,* $\{U(t),\ t \ge 0\}$ *is a Cox process with an ISDLG intensity process* $\{J^{(H,\ell)}_{2\lambda(\lambda+1),\,\nu(\lambda+1)}(t),\, t \ge 0\}$, *i.e.*

$$\{U(t),\, t \ge 0\} = \left\{N \lhd J^{(H,\ell)}_{2\lambda(\lambda+1),\,\nu(\lambda+1)}(t),\ t \ge 0\right\}. \tag{5.2.7}$$

**Proof.** It follows from Proposition 5.2.7 and Lemma 5.2.3 that for each $j \in \mathbb{N}$,

$$\{D_j(t),\, t \ge 0\} \sim \{N \lhd Y_j(t),\, t \ge 0\} \quad \text{on } D_0[0,\infty), \tag{5.2.8}$$

where $\{Y_j(t),\, t \ge 0\}$ is an IDLG process with parameter $(2d_j\lambda(\lambda+1),\, -d_j,\, p_j d_j \nu(\lambda+1))$. Hence, using also Lemma 4.6.7, we have

$$\mathsf{E}\, D_j(t) = \mathsf{E}\, Y_j(t) = t p_j \nu(\lambda+1),$$

$$\begin{aligned}\mathsf{D}^2 D_j(t) = \mathsf{D}^2 Y_j(t) + \mathsf{E}\, Y_j(t) &\le \int_0^t\!\!\int_0^t p_j \nu\lambda(\lambda+1)^2\, ds_1 ds_2 + \mathsf{E}\, Y_j(t) \\ &= p_j\nu(\lambda+1)t\big(t\lambda(\lambda+1)+1\big).\end{aligned}$$

Thus, for each $t \ge 0$, the series $\sum_j D_j(t)$ converges in $L^2$ and by Kolmogorov's two series theorem, a.s. as well. Let us denote the limit process by $\{U(t),\, t \ge 0\}$.

Let us choose the intensity IDLG processess $\{Y_j(t),\, t \ge 0\}$, $j \in \mathbb{N}$, to be independent. Then, by (5.2.8) and (5.1.1), we have

$$\left\{\sum_{j=1}^n D_j(t),\ t \ge 0\right\} \sim \left\{N \lhd \sum_{j=1}^n Y_j(t),\ t \ge 0\right\} \quad \text{on } D_0[0,\infty),$$

and by (4.6.13) and Remark 5.1.11, the weak vague convergence

$$\left\{\sum_{j=1}^{n} Y_j(s),\ t \geq 0\right\} \xrightarrow[n\to\infty]{\text{wv}} \left\{J^{(H,\ell)}_{2\lambda(\lambda+1),\nu(\lambda+1)}(t),\ t \geq 0\right\} \quad \text{on } D_0[0,\infty)$$

also holds. Hence we can apply Grandell [13, Thm. 1.5.2] (which states that on $D_0[0,\infty)$ the weak vague convergence of a sequence of Cox processes is equivalent to the weak vague convergence of the sequence of the corresponding intensity processes) to obtain that

$$\left\{\sum_{j=1}^{n} D_j(t),\ t \geq 0\right\} \xrightarrow[n\to\infty]{\text{wv}} \left\{N \triangleleft J^{(H,\ell)}_{2\lambda(\lambda+1),\nu(\lambda+1)}(t),\ t \geq 0\right\} \quad \text{on } D_0[0,\infty). \tag{5.2.9}$$

Intensity processes $\left\{\sum_{j=1}^{n} Y_j(s),\, t \geq 0\right\}$, $n \in \mathbb{N}$, and $\left\{J^{(H,\ell)}_{2\lambda(\lambda+1),\nu(\lambda+1)}(t),\, t \geq 0\right\}$ are a.s. continuous, and so, by Grandell [13, Thm. 1.5.3], Cox processes in (5.2.9) are simple point processes. Thus, by Daley–Vere-Jones [6, Thm. 9.1.VI] along with Jacod–Shiryaev [18, Thm. VI.3.37 (b)], the weak vague convergence in (5.2.9) implies, through the convergence of the finite-dimensional distributions, the weak convergence on $D_0[0,\infty)$ and so, on $D[0,\infty)$. That is, we have obtained (5.2.6) and (5.2.7). □

**5.2.9 Definition.** The limit process $\{U(t),\, t \geq 0\}$ in Theorem 5.2.8 is called a *superposition of DC processes (SDC process)* (with parameter $(H,\ell)$ and $(\nu,\lambda)$).

**5.2.10 Remark.** The name "superposition of DC processes" is correct, as the superposed processes $\{D_j(t),\, t \geq 0\}$, $j \in \mathbb{N}$, are DC processes, by Proposition 5.2.7.

Finally, we restate, using the terminology of Definition 5.2.9 and after the fashion of Theorem 3.3.4, 4.5.4 and 4.6.19, the dilatively stable renormalization functional limit theorem (5.1.18) of Example 5.1.17:

**5.2.11 Theorem.** *The family of distributions of the renormalized centered versions of an SDC process $\{U(t),\, t \geq 0\}$ with parameters $(H,\ell)$ and $(\nu,\lambda)$, converges weakly on $D[0,\infty)$ to the LISDLG process with parameters $H$ and $(2\lambda(\lambda+1),\, \nu(\lambda+1))$:*

$$\frac{1}{T}\Big\{U(Tt) - \mathsf{E}U(Tt),\ t \geq 0\Big\}^{\circledast \frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T\to\infty]{\text{w}} \left\{J^{(H)}_{2\lambda(\lambda+1),\,\nu(\lambda+1)}(t),\ t \geq 0\right\} \quad \textit{on } D[0,\infty).$$

# Summary

In the dissertation we define and study a scaling property of stochastic processes which we call dilative stability. For those processes for which both properties are defined, dilative stability is a generalization of the well-known self-similarity.

Chapter 2 draws a parallel between self-similarity and dilative stability. Chapters 3 and 4 deal with the superposition of stationary Ornstein–Uhlenbeck (OU) type processes and the superposition of stationary continuous state branching processes with immigration (CBI processes), respectively and with the dilatively stable renormalization functional limit theorems giving the so-called LISOU and LISCBI (particularly LISDLG) limit processes, respectively. In Chapter 5 we present a dilatively stable renormalization functional limit theorem for Cox processes.

Let $\mathcal{I}$ denote the set of non-Gaussian, infinitely divisible stochastic processes starting from zero and having right-continuous cumulant functions of all orders. Let functions $f,\, g : (0,\infty) \to (0,\infty)$. We call a process $\{X(t),\, t \geq 0\} \in \mathcal{I}$ $(f,g)$-dilatively stable if

$$\forall\, T > 0 : X(Tt) \overset{\text{fd}}{\sim} \frac{f(T)}{\sqrt{g(T)}}\, X^{\circledast g(T)}(t).$$

In Chapter 2 we present the dilatively stable analogues of the famous self-similarity theorems of Lamperti [22] and we show that many other results can be transferred from the self-similar to the dilatively stable case. Our most important theorem in this context is that if a process $\{X(t),\, t \geq 0\}$ is $(f,g)$-dilatively stable, then there exists a unique $(\alpha,\delta)$ such that $\alpha > 0$, $\delta \leq 2\alpha$ (which we call an admissible $(\alpha,\delta)$) and $f(t) = t^{\alpha}$, $g(t) = t^{\delta}$. Therefore "$(t^{\alpha}, t^{\delta})$-dilative stability" can be called simply "$(\alpha,\delta)$-dilative stability". By our next result the processes in $\mathcal{I}$ which are the limit processes in renormalization limit theorems, are exactly the dilatively stable processes. That is, if $\{X(t),\, t \geq 0\} \in \mathcal{I}$ is a

process such that there exist an infinitely divisible process $\{Y(t),\, t \geq 0\}$ with finite moments of all orders and functions $f, g : (0,\infty) \to (0,\infty)$ for which

$$\frac{\sqrt{g(T)}}{f(T)}\, Y^{\circledast \frac{1}{g(T)}}(Tt) \xrightarrow[T\to\infty]{\text{fd}} X(t)$$

and the corresponding convergences hold also for every cumulants of the one-dimensional distributions, then there exists an admissible $(\alpha, \delta)$ such that process $\{X(t),\, t \geq 0\}$ is $(\alpha, \delta)$-dilatively stable and $f, g$ are regularly varying functions of orders $\alpha$, $\delta$, respectively.

Next we give the connection between $\mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}}$, the set of $(\alpha-\delta/2)$-self-similar processes and $\mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$, the set of $(\alpha, \delta)$-dilatively stable processes, for any admissible $(\alpha, \delta)$. Namely, using the so-called $t^{\delta}$ function-th and $t^{-\delta}$ function-th convolution power operations (which we establish by a lemma), the mappings

$$T_S : \mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I} \longrightarrow \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}, \qquad T_S\big(\{Y(t),\, t \geq 0\}\big) \doteq \{Y(t),\, t \geq 0\}^{\circledast\, t^{\delta}},$$

$$T_D : \mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \longrightarrow \mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I}, \qquad T_D\big(\{X(t),\, t \geq 0\}\big) \doteq \{X(t),\, t \geq 0\}^{\circledast\, t^{-\delta}},$$

are well-defined, they are injections, but each of them is a bijection if and only if $\delta = 0$. So, dilative stability is not simply the combination of self-similarity and a power function-th convolution power, i.e. it is not a redundant concept.

Then we study the Lamperti transform. It is a known fact that this transform gives a one-to-one correspondence between self-similar and stationary processes. The analogous statement we prove is that the Lamperti transform connects dilatively stable processes and the so-called translatively stable processes in a one-to-one manner. We pay particular attention to the correspondence between dilatively stable processes with independent increments and translatively stable, so-called wide sense OU type processes, because this is the analogue of the known correspondence (see Jeanblanc et al. [19]) between self-similar processes with independent increments and stationary OU type processes.

There is a section about dilatively stable processes with stationary increments. The importance of that processes lies in the fact that the autocovariance function of such a process is the same as that of an appropriate self-similar process with $L^2$-stationary increments, i.e. as the autocovariance function of a fractional Brownian motion. This fact provides a new possibility for non-Gaussian long memory modelling.

The next section treats dilatively stable renormalization operators. We show that the analogy with the self-similar case holds true in this regard as well. That is, for any fixed admissible $(\alpha, \delta)$ the set $\{A_T^{(\alpha,\delta)} : T > 0\}$ of the dilatively stable renormalization operators

$$A_T^{(\alpha,\delta)} : \mathcal{I} \longrightarrow \mathcal{I}\,, \qquad \big(A_T^{(\alpha,\delta)} X\big)(t) \doteq T^{-\left(\alpha-\frac{\delta}{2}\right)} X^{\circledast T^{-\delta}}(Tt)\,, \quad t \geq 0,$$

endowed with the composition operation constitutes a semigroup. Hence, dilatively stable renormalization operators can be iterated. Furthermore, for any admissible $(\alpha, \delta)$, a process in $\mathcal{I}$ is $(\alpha, \delta)$-dilatively stable if and only if it is a fixed point of $A_T^{(\alpha,\delta)}$ for every $T > 0$. This explains, at least heuristically, why exactly the renormalized processes happen to converge in dilatively stable (functional) limit theorems.

The developments of Chapter 3 and 4 are parallel. In chapter 3 we deal with the superposition of stationary OU type processes and we state a dilatively stable renormalization functional limit theorem for the integrated superposition process. The construction we call superposition means that we sum independent processes, where each term of the sum, i.e. each process (more precisely, its distribution) is some transform of a basic stationary OU type process. Each transform is of the form

$$\big\{X(t),\, t \geq 0\big\} \;\longmapsto\; \big\{X(d_j t),\, t \geq 0\big\}^{\circledast p_j},$$

where each dilational costant $d_j$ occurs with probability $p_j$, so $p_j$, $j \in \mathbb{N}$, is a distribution on the discrete set $\{d_j : j \in \mathbb{N}\}$. We call this distribution the superpositional law and we assume that its distribution function is regularly varying at zero of order $2-2H$, i.e. $F(x) = \sum_{j:\, d_j < x} p_j = x^{2-2H}\ell(1/x)$, where $\ell$ is a function slowly varying at $\infty$ and $1/2 < H < 1$ is a parameter (being later the Hurst parameter). Under these conditions, for each $t \geq 0$ the series $\sum_j X_j(t)$ converges in $L^2$ (uniformly in $t$) and also almost surely, and we call the limit process $Y^{(H,\ell)}(t) \doteq \sum_{j=1}^{\infty} X_j(t)$, $t \geq 0$, the (infinite) superposition of OU type processes (SOU process). Its integral process

$$J^{(H,\ell)}(t) \doteq \int_0^t Y^{(H,\ell)}(s)\, ds, \quad t \geq 0,$$

defined almost surely, is called the ISOU (integrated SOU) process. The order of the superposition and the integration can be interchanged and the family of distributions on $C[0, \infty)$ of the finite superpositions of the integrated processes $\big\{\int_0^t X_j(s)ds,\, t \geq 0\big\}$, $j \in \mathbb{N}$, converges weakly to the distribution of the ISOU process. The main theorem in Chapter 3 is a so-called dilatively stable renormalization functional limit theorem which states that the family of distributions of a renormalized centered ISOU process converges weakly on $C[0,\infty)$ to a limit distribution. We denote the process with this limit distribution by $\{J^{(H)}(t),\, t\geq 0\}$ and we call it the LISOU process (limit of [renormalized centered] ISOU processes):

$$\frac{1}{T}\left\{J^{(H,\ell)}(Tt)-\mathsf{E}J^{(H,\ell)}(Tt),\ t\geq 0\right\}^{\circledast\frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T\to\infty]{\mathrm{w}} \left\{J^{(H)}(t),\, t\geq 0\right\} \quad \text{on } C[0,\infty).$$

The distribution on $C[0,\infty)$ of the LISOU process is uniquely determined by (its zero mean and) its joint cumulants, which we give explicitely. Moreover, the LISOU process has stationary increments and it is $(H, 2H-2)$-dilatively stable.

The main results of Chapter 4 are completely anlogous to those of Chapter 3, and they can be obtained replacing the initial word "OU" by "CBI" (and of course, replacing the corresponding formulas). However, to prove these theorems, we needed some new results with respect to stationary CBI processes, and these results are also contained in Chapter 4. The particular case when the CBI process is the so-called diffusion process with linear generator (DLG process), is enlarged upon in a separate section. In this case the dilatively stable limit process is called the LISDLG process and we give again an explicit expression for the joint cumulants of this process.

According to Grandell [13, Thm. 4.2.2], if a self-similar renormalization functional limit theorem holds for the intensity process of a Cox process, then a similar self-similar renormalization functional limit theorem holds also for the Cox process itself. In Chapter 5 we present the analogous statement for dilatively stable renormalization functional limit theorems. As a corollary we obtain the following dilatively stable renormalization functional limit theorem for the superposition of death counting (SDC) processes of stationary birth and death processes with immigration: the family of distributions of the renormalized centered versions of an SDC process $\{U^{(H,\ell)}(t),\, t\geq 0\}$ with parameter $(H,\ell)$ converges weakly on $D[0,\infty)$ to the LISDLG process with parameter $H$:

$$\frac{1}{T}\left\{U^{(H,\ell)}(Tt)-\mathsf{E}U^{(H,\ell)}(Tt),\ t\geq 0\right\}^{\circledast\frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T\to\infty]{\mathrm{w}} \left\{J^{(H)}(t),\ t\geq 0\right\} \quad \text{on } D[0,\infty).$$

# Összefoglaló (Hungarian summary)

A disszertációban sztochasztikus folyamatok egy dilatív stabilitásnak nevezett skálázási tulajdonságát definiáljuk és vizsgáljuk. Azon folyamatok esetén, amelyekre mindkét tulajdonság definiált, a dilatív stabilitás a közismert önhasonlóság általánosítása.

A 2. fejezetben összehasonlítjuk az önhasonlóságot és a dilatív stabilitást. A 3. és a 4. fejezetben stacionárius Ornstein–Uhlenbeck (OU) típusú folyamatok szuperpozíciójával, stacionárius, folytonos állapotterű, elágazó immigrációs (CBI) folyamatok szuperpozíciójával és az ún. LISOU és LISCBI (speciálisan LISDLG) folyamatokat előállító dilatív stabil renormalizációs funkcionális határeloszlás-tételekkel foglalkozunk. Az 5. fejezetben a Cox-folyamatra vonatkozó dilatív stabil renormalizációs funkcionális határeloszlás-tételt mondjuk ki.

Jelölje $\mathcal{I}$ azon nem Gauss, korlátlanul osztható, 0-ból induló sztochasztikus folyamatok halmazát, amelyek összes rendű kumuláns függvénye létezik és jobbról folytonos. Legyen $f,\, g : (0,\infty) \to (0,\infty)$. Egy $\{X(t),\, t \geq 0\} \in \mathcal{I}$ folyamatot $(f, g)$-dilatív stabilnak nevezünk, ha

$$\forall\, T > 0 : X(Tt) \stackrel{\text{vd}}{\sim} \frac{f(T)}{\sqrt{g(T)}}\, X^{\circledast g(T)}(t).$$

A 2. fejezetben megadjuk Lamperti [22] híres tételeinek dilatív stabil megfelelőit, és megmutatjuk, hogy sok más eredmény is átvihető az önhasonló esetről a dilatív stabil esetre. Ezzel kapcsolatban a legfontosabb tételünk azt mondja ki, hogy ha az $\{X(t),\, t \geq 0\}$ folyamat $(f, g)$-dilatív stabil, akkor létezik pontosan egy $(\alpha, \delta)$ úgy, hogy $\alpha > 0$, $\delta \leq 2\alpha$ (az ilyen $(\alpha, \delta)$-t megengedhetőnek nevezzük) és $f(t) = t^{\alpha}$, $g(t) = t^{\delta}$. Így a „$(t^{\alpha}, t^{\delta})$-dilatív stabil" kifejezés helyett egyszerűen az „$(\alpha,\delta)$-dilatív stabil” kifejezést használhatjuk. A következő eredményünk szerint azon $\mathcal{I}$-beli folyamatok, amelyek renormalizációs határfolyamatok, pontosan a dilatív stabil folyamatok. Ez azt jelenti, hogy ha $\{X(t),\, t\geq 0\} \in \mathcal{I}$ úgy, hogy létezik egy korlátlanul osztható $\{Y(t),\, t\geq 0\}$ folyamat, amelynek az összes momentuma véges, továbbá $f, g : (0,\infty) \to (0,\infty)$ függvények, amelyekre

$$\frac{\sqrt{g(T)}}{f(T)}\, Y^{\circledast \frac{1}{g(T)}}(Tt) \xrightarrow[T\to\infty]{\text{vd}} X(t),$$

és a megfelelő konvergencia fennáll az egydimenziós eloszlások összes kumulánsaira is, akkor létezik egy megengedhető $(\alpha,\delta)$ úgy, hogy az $\{X(t),\, t\geq 0\}$ folyamat $(\alpha,\delta)$-dilatív stabil, és $f$ és $g$ $\alpha$ ill. $\delta$ rendű regulárisan változó függvények.

Ezután tetszőleges megengedhető $(\alpha,\delta)$ esetén megadjuk az $(\alpha-\delta/2)$-önhasonló folyamatok halmaza, $\mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}}$ és az $(\alpha,\delta)$-dilatív stabil folyamatok halmaza, $\mathcal{I}_{(\alpha,\delta)\mathsf{ds}}$ közötti kapcsolatot. Nevezetesen, az ún. $t^{\delta}$ ill. $t^{-\delta}$ függvény rendű konvolúcióhatvány műveletet használva (amit először egy külön lemmában alapozunk meg) azt állítjuk, hogy a

$$T_S : \mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I} \longrightarrow \mathcal{I}_{(\alpha,\delta)\mathsf{ds}}, \qquad T_S\big(\{Y(t),\, t\geq 0\}\big) \doteq \{Y(t),\, t\geq 0\}^{\circledast\, t^{\delta}},$$

$$T_D : \mathcal{I}_{(\alpha,\delta)\mathsf{ds}} \longrightarrow \mathcal{S}_{(\alpha-\frac{\delta}{2})\mathsf{ss}} \bigcap \mathcal{I}, \qquad T_D\big(\{X(t),\, t\geq 0\}\big) \doteq \{X(t),\, t\geq 0\}^{\circledast\, t^{-\delta}},$$

leképezések jól definiáltak, injektívek, viszont akkor és csak akkor bijektívek, ha $\delta = 0$. Így a dilatív stabilitás nem egyszerűen az önhasonlóság és a hatványfüggvény rendű konvolúcióhatvány kombinációja, tehát nem fölösleges a dilatív stabilitással külön foglalkozni.

A következő alfejezetben a Lamperti-transzformációt általánosítjuk. Ismert, hogy ez a transzformáció kölcsönösen egyértelmű kapcsolatot ad meg az önhasonló és a stacionárius folyamatok között. A megfelelő állítás, amit bizonyítunk az, hogy a Lamperti-transzformáció a dilatív stabil folyamatokat az ún. transzlatív stabil folyamatoknak felelteti meg kölcsönösen egyértelműen. Külön foglalkozunk a dilatív stabil, független növekményű folyamatok és a transzlatív stabil, ún. tágabb értelemben OU típusú folyamatok kapcsolatával, mert ez a megfelelője az önhasonló, független növekményű folyamatok és a stacionárius, OU típusú folyamatok közötti kapcsolatnak (ld. Jeanblanc et al. [19]).

Van egy alfejezet a stacionárius növekményű dilatív stabil folyamatokról. Ezen folyamatok azért fontosak, mert egy ilyen folyamat autokovariancia függvénye ugyanaz, mint egy alkalmas $L^2$-stacionárius növekményű önhasonló folyamaté, azaz ugyanaz, mint egy megfelelő paraméterű frakcionális Brown-mozgás autokovariancia függvénye. Ez a tény új lehetőséget ad nem Gauss, hosszú memóriájú folyamatok modellezésére is.

A következő alfejezet a dilatív stabil renormalizációs operátorokról szól. Megmutatjuk, hogy az önhasonló esettel való analógia ebből a szempontból is érvényes. Nevezetesen, tetszőleges megengedhető $(\alpha,\delta)$ esetén az

$$A_T^{(\alpha,\delta)}:\mathcal{I}\longrightarrow\mathcal{I}\,,\qquad \big(A_T^{(\alpha,\delta)}X\big)(t)\doteq T^{-\left(\alpha-\frac{\delta}{2}\right)}X^{\circledast T^{-\delta}}(Tt)\,,\quad t\geq 0,$$

dilatív stabil renormalizációs operátorok $\big\{A_T^{(\alpha,\delta)}:T>0\big\}$ halmaza a kompozíció műveletre nézve félcsoportot alkot. Következésként a dilatív stabil renormalizációs operátorok iterálhatók. Továbbá tetszőleges megengedhető $(\alpha,\delta)$ esetén egy $\mathcal{I}$-beli folyamat akkor és csak akkor $(\alpha,\delta)$-dilatív stabil, ha minden $T>0$-ra fixpontja $A_T^{(\alpha,\delta)}$-nak. Ez a tény — legalábbis heurisztikus — magyarázatot ad arra, hogy miért éppen a renormalizált folyamatok konvergálnak a dilatív stabil (funcionális) határeloszlás-tételekben.

A 3. és a 4. fejezet felépítése párhuzamos egymással. A 3. fejezetben stacionárius OU típusú folyamatok szuperpozíciójával foglalkozunk, és kimondjuk az integrált szuperpozíció folyamatra vonatkozó dilatív stabil renormalizációs funkcionális határeloszlás-tételt. A szuperpozíciónak nevezett konstrukció azt jelenti, hogy független folyamatok összegét tekintjük, ahol az összeg minden tagja, azaz minden folyamat (pontosabban az eloszlása) egy bizonyos transzformáltja egy stacionárius OU típusú alapfolyamatnak. Az egyes transzformációk

$$\big\{X(t),\,t\geq 0\big\}\;\longmapsto\;\big\{X(d_jt),\,t\geq 0\big\}^{\circledast p_j}$$

alakúak, ahol mindegyik $d_j$ (dilatációs) konstans $p_j$ valószínűséggel fordul elő, tehát $p_j$, $j\in\mathbb{N}$, egy eloszlás a $\{d_j:j\in\mathbb{N}\}$ diszkrét halmazon. Ezt az eloszlást szuperpozíciós eloszlásnak nevezzük, és feltesszük, hogy az eloszlásfüggvénye 0-ban $2-2H$ rendű regulárisan változó, azaz $F(x)=\sum_{j:\,d_j<x}p_j=x^{2-2H}\ell(1/x)$, ahol $\ell$ egy ($\infty$-ben) lassan változó függvény és $1/2<H<1$ egy paraméter (később ez lesz a Hurst-paraméter). Ezen feltételek mellett tetszőleges $t\geq 0$-ra a $\sum_j X_j(t)$ sor $L^2$-ben konvergens ($t$-ben egyenletesen) és 1 valószínűséggel is konvergens, és az $Y^{(H,\ell)}(t)\doteq\sum_{j=1}^{\infty}X_j(t)$, $t\geq 0$, határfolyamatot SOU folyamatnak (superposition of OU type processes) nevezzük. A

$$J^{(H,\ell)}(t)\doteq\int_0^t Y^{(H,\ell)}(s)\,ds,\quad t\geq 0,$$

1 valószínűséggel definiált integrálfolyamatot pedig ISOU (integrated SOU) folyamatnak nevezzük. A szuperpozíció és az integrálás sorrendje felcserélhető, és az $\left\{\int_0^t X_j(s)ds,\, t \geq 0\right\}$, $j \in \mathbb{N}$, integrálfolyamatok véges szuperpozíciói eloszlásainak halmaza $C[0,\infty)$-en gyengén konvergál az ISOU folyamat eloszlásához. A 3. fejezet fő tétele, ami egy ún. dilatív stabil renormalizációs funkcionális határeloszlás-tétel, azt mondja ki, hogy a renormalizált centralizált ISOU folyamatok eloszlásainak családja $C[0,\infty)$-en gyengén konvergens. Azt a folyamatot, aminek ez a határeloszlás az eloszlása, $\left\{J^{(H)}(t),\, t \geq 0\right\}$-vel jelöljük és LISOU folyamatnak (limit of [renormalized centered] ISOU processes) nevezzük:

$$\frac{1}{T}\left\{J^{(H,\ell)}(Tt) - \mathsf{E}J^{(H,\ell)}(Tt),\ t \geq 0\right\}^{\circledast \frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T\to\infty]{\mathrm{w}} \left\{J^{(H)}(t),\, t \geq 0\right\} \quad C[0,\infty)\text{-en.}$$

A LISOU folyamat eloszlását $C[0,\infty)$-en egyértelműen meghatározzák (a 0 várhatóértéke és) az együttes kumulánsai, amelyeket explicite megadunk. Továbbá a LISOU folyamat stacionárius növekményű és $(H, 2H-2)$-dilatív stabil.

A 4. fejezet fő eredményei a 3. fejezetéinek a megfelelői, csak az ,,OU" rövidítést kell a ,,CBI"-re cserélni (és természetesen a megfelelő képleteket is). Azonban ahhoz, hogy ezeket megkapjuk ill. bizonyítsuk, szükségünk van bizonyos, a stacionárius CBI folyamatra vonatkozó eredményekre. A 4. fejezet ez utóbbiakat is tartalmazza. Ezenkívül kiemelten, egy külön alfejezetben tárgyaljuk azt a speciális esetet, amikor a CBI folyamat az ún. DLG folyamat (diffusion process with linear generator). Ekkor a dilatív stabil határfolyamatot LISDLG folyamatnak nevezzük, és ennek az együttes kumulánsait megint explicite adjuk meg.

Grandell [13, Thm. 4.2.2] szerint ha egy önhasonló renormalizációs funkcionális határeloszlás-tétel érvényes egy Cox-folyamat intenzitásfolyamatára, akkor hasonló önhasonló renormalizációs funkcionális határeloszlás-tétel érvényes magára a Cox-folyamatra is. Az 5. fejezetben az ennek dilatív stabil esetben megfelelő állítást mondjuk ki. Ezen eredmény következményeként kapjuk a következő, születési-halálozási immigrációs folyamatok halálozásszámláló folyamatainak szuperpozíciójára (SDC folyamat) vonatkozó dilatív stabil renormalizációs funkcionális határeloszlás-tételt: Legyen $\{U(t),\, t \geq 0\}$ $(H,\ell)$ paraméterű SDC folyamat. Ekkor az $\{U(t),\, t \geq 0\}$ renormalizált centralizáltjai eloszlásainak családja $D[0,\infty)$-en gyengén konvergál a $H$ paraméterű LISDLG folyamat eloszlásához:

$$\frac{1}{T}\left\{U^{(H,\ell)}(Tt) - \mathsf{E}U^{(H,\ell)}(Tt),\ t \geq 0\right\}^{\circledast \frac{T^{2-2H}}{\ell(T)}} \xrightarrow[T\to\infty]{\mathrm{w}} \left\{J^{(H)}(t),\ t \geq 0\right\} \quad D[0,\infty)\text{-en.}$$

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# Appendix B

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# Appendix C

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