ON WAGNER CONNECTIONS AND WAGNER MANIFOLDS

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ABSTRACT. Let (M, E) be a Finsler manifold. A triplet $(\overline{D}, \overline{h}, \alpha)$ is said to be a *Wagner connection* on M if $(\overline{D}, \overline{h})$ is a Finsler connection, $\alpha \in C^\infty(M)$ and the axioms $(W1) - (W4)$, formulated originally by M. Hashiguchi, are satisfied. Then $\overline{h}$ is called a *Wagner endomorphism* on M. We establish an explicit relation between the (canonical) Barthel endomorphism of (M, E) and a Wagner endomorphism $\overline{h}$. We show that the second Cartan tensors $\overline{\mathcal{C}}'$, $\overline{\mathcal{C}}'_b$ belonging to $\overline{h}$ are symmetric and totally symmetric, respectively. An explicit relation between the "canonical" tensors $\mathcal{C}'$, $\mathcal{C}'_b$ and the "Wagnerian" ones is also derived. We can conclude that the rules of calculation with respect to a Wagner connection are *formally* the same as those with respect to the classical Cartan connection. We establish some basic curvature identities concerning a Wagner connection, including Bianchi identities. Finally, we present a new, intrinsic definition as well as several tensorial characterizations of Wagner manifolds.

0. Introduction

A *Wagner connection* on a Finsler manifold is just a Cartan-type connection with non-vanishing $(h)h$-torsion. Such kind of Finsler connections were first constructed and used by V.V. WAGNER [8]. With the help of this seemingly strange connection Wagner introduced the concept of *generalized Berwald manifolds*, in which the h-connection part of the Berwald connection depends only on the position. The class of these manifolds is quite rich: Wagner himself showed that any two-dimensional Finsler manifold with cubic metric is a generalized Berwald manifold. The next important steps in the extension of the theory of Wagner connections and generalized Berwald manifolds were taken by M. HASHIGUCHI [3]. He successfully carried over Wagner's ideas to the arbitrary (but finite) dimensional case, characterizing the Wagner connections by elegant geometrical axioms. One of the most important observations, due to Hashiguchi and Y. ICHIJŌ [4] is that Wagner connections are at the heart of the theory of conformal change of Finsler metrics. Among others it turned out that the class of Wagner manifolds is closed under a conformal change of the metric. These results confirm M. MATSUMOTO's remarkable principle: "there should be existing a *best* Finsler connection for *every* theory of Finsler spaces".

The main stimulus for this paper was Hashiguchi's work. In this article we shall demonstrate that the Frölicher-Nijenhuis formalism provides a perfectly adequate conceptual and technical framework for the study even of such complicated objects as Wagner connections. Our intrinsically formulated and proved results not only cover the classical local results but give a much more precise and transparent picture and open new perspectives. In a forthcoming paper, synthetizing our previous work [6] and the present considerations, we shall also show, why the Wagner connections are (in some sense) the "best" Finsler connections for the theory of conformal changes. (Not the "absolutely best" because we have no conformally invariant curvature tensor field!)

1. Preliminaries

1.1. Throughout the paper we use the terminology and conventions described in [6]. Now we briefly summarize the basic notation.

(i) M is an $n(>1)$-dimensional, C^∞, connected, paracompact manifold, $C^\infty(M)$ is the ring of real-valued smooth functions on M.

(ii) $\pi : TM \to M$ is the tangent bundle of M, $\pi_0 : \mathcal{T}M \to M$ is the bundle of nonzero tangent vectors.

(iii) $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M.

(iv) $\Omega^k(M)$ ($k \in \mathbb{N}^+$) is the module of (scalar) k-forms on M, $\Omega^0(M) := C^\infty(M)$.

(v) $\Psi^k(M)$ ($k \in \mathbb{N}^+$) is the $C^\infty(M)$-module of vector k-forms on M, $\Psi^0(M) := \mathfrak{X}(M)$.

(vi) i_X, $\mathcal{L}_X$ ($X \in \mathfrak{X}(M)$) and d are the *insertion operator*, the *Lie-derivative* (with respect to X) and the *exterior derivative*, respectively.

1.2. We shall apply some simple facts of the Frölicher-Nijenhuis calculus of vector-valued forms. Recall that if $K \in \Psi^1(M)$, $Y \in \mathfrak{X}(M)$ then their *Frölicher-Nijenhuis bracket* $[K, Y] \in \Psi^1(M)$ acts as follows:

$$[K, Y](X) = [K(X), Y] - K[X, Y] \qquad (X \in \mathfrak{X}(M)). \tag{1}$$

For the derivation d_K induced by K we have

$$d_K f = df \circ K \qquad (f \in C^\infty(M)). \tag{2}$$

The next identities (see [7]) will also be useful:

$$[fK, X] = f[K, X] - (Xf)K, \tag{3}$$

$$[K, fL] = f[K, L] + d_K f \wedge L - df \wedge K \circ L, \tag{4}$$

$$[X, \omega \otimes Y] = \mathcal{L}_X \omega \otimes Y + [X, Y] \otimes \omega, \tag{5}$$

$$[K, \omega \otimes X] = d_K \omega \otimes Y - d\omega \otimes K(X) + (-1)^k \omega \wedge [K, X] \tag{6}$$

$$(K, L \in \Psi^1(M); \quad \omega \in \Omega^k(M); \quad X, Y \in \mathfrak{X}(M)).$$

1.3. Vertical apparatus. Semispray, spray. Let us consider the tangent bundle $\pi : TM \to M$. $\mathfrak{X}^v(TM)$ denotes the $\mathcal{C}^\infty(TM)$-module of vertical vector fields on TM, $C \in \mathfrak{X}^v(TM)$, $J \in \Psi^1(TM)$ are the *Liouville vector field* and *vertical endomorphism*, respectively. We have

$$\begin{gathered} \operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(TM), \quad J^2 = 0, \\ [C, J] = -J \quad \text{(i.e. } J \text{ is homogeneous of degree 0)}, \quad N_J := \frac{1}{2}[J, J] = 0. \end{gathered} \tag{7}$$

Definition. A mapping $S : v \in TM \to S(v) \in T_vTM$ is said to be a *semispray* on M if it satisfies the conditions

(Spr1) S is smooth on $\mathcal{T}M$,

(Spr2) $JS = C$.

A semispray is called a *spray* if it has the following two properties:

(Spr3) S is a vector field of class C^1 on M,

(Spr4) $[C, S] = S$ (i.e. S is homogeneous of degree 2).

Let S be an arbitrary semispray on M. The *vertical* and *complete lifts* of a function $f \in C^\infty(M)$ are given by

$$f^v := f \circ \pi, \qquad f^c := S(f^v), \tag{8}$$

respectively. We have

$$\begin{aligned} &C(f^v) = 0 && \text{(i.e. } f^v \text{ is homogeneous of degree 0)}, \\ &C(f^c) = f^c && \text{(i.e. } f^c \text{ is homogeneous of degree 1)}. \end{aligned} \tag{9}$$

X^v denotes the *vertical lift* of a vector field $X \in \mathfrak{X}(M)$ while its *complete lift* is given by

$$X^c(f^c) = (Xf)^c \qquad (f \in C^\infty(M)). \tag{10}$$

Local basis property 1. If $(X_1, \dots, X_n)$ is a local basis of $\mathfrak{X}(M)$ then $(X_1^v, \dots, X_n^v, X_1^c, \dots, X_n^c)$ is a local basis for $\mathfrak{X}(TM)$.

Remark 1. In the sequel we shall consider forms over TM or $\mathcal{T}M$. *Differentiability of vector (and scalar) k-forms will be required only over* $\mathcal{T}M$, unless otherwise stated.

1.4. Horizontal endomorphisms and Finsler connections. ([1], [2], and see also [5]).

Definition. A vector 1-form $h \in \Psi^1(TM)$ is said to be a *horizontal endomorphism* on M if the following conditions are satisfied:

(He1) h is smooth over $\mathcal{T}M$,

(He2) h is a projector, i.e. $h^2 = h$,

(He3) $\operatorname{Ker} h = \mathfrak{X}^v(TM)$.

The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ (with respect to h) is $X^h := h(X^c)$.

$H := [h, C]$ is the *tension* of h,

$t := [J, h]$ is the *weak torsion* of h,

$T := i_{S_0} t + H$ is the *strong torsion* of h (S_0 is an arbitrary semispray on M),

$R := -\dfrac{1}{2}[h, h]$ is the *curvature tensor* of h.

Except for (13), we get immediately from the definitions that $\forall X, Y \in \mathfrak{X}(M)$:

$$H(X^c) = [X^h, C], \tag{11}$$

$$t(X^c, Y^c) = [X^h, Y^v] - [Y^h, X^v] - [X, Y]^v, \tag{12}$$

$$T(X^c) = v[S, X^v] + X^h - X^c \qquad (S := h(S_0),\ v := 1 - h), \tag{13}$$

$$R(X^c, Y^c) = -v[X^h, Y^h]. \tag{14}$$

J and h are related as follows:

$$h \circ J = 0, \qquad J \circ h = J \tag{15}$$

and, furthermore, any horizontal endomorphism h determines an *almost complex structure* $F \in \Psi^1(TM)$ ($F^2 = -1$, F is smooth on $\mathcal{T}M$) such that

$$F \circ J = h, \qquad F \circ h = -J. \tag{16}$$

This almost complex structure can be given by the explicit formula

$$F = h[S, h] - J \tag{17}$$

where $S := h(S_0)$ is called the *semispray associated with* h (see e.g. [7]).

Local basis property 2. If $(X_1, \dots, X_n)$ is a local basis of $\mathfrak{X}(M)$ then $(X_1^v, \dots, X_n^v, X_1^h, \dots, X_n^h)$ is a local basis for $\mathfrak{X}(TM)$.

Definition [5]. Suppose that h is a horizontal endomorphism on M and D is a linear connection on the tangent manifold TM or on the manifold $\mathcal{T}M$. The pair (D, h) is said to be a *Finsler connection* if it satisfies the following two conditions:

(Finsl1) D is *reducible*: $Dh = 0$,

(Finsl2) D is *almost complex*: $DF = 0$ (F is the almost complex structure associated with h).

The map

$$h^*(DC) : X \in \mathfrak{X}(TM) \to DC(hX) := D_{hX}C$$

is called the *h-deflection* of the Finsler connection (D, h).

Remark 2. Consider the so-called vertical projector $v := 1 - h$ belonging to h. The conditions (Finsl1), (Finsl2) imply that the torsion tensor $\mathbb{T}$ of a Finsler connection (D, h) is completely determined by the tensor fields

$$\mathbb{A}(X, Y) := h\mathbb{T}(hX, hY) - (h)h\text{-torsion},$$

$$\mathbb{B}(X, Y) := h\mathbb{T}(hX, vY) - (h)hv\text{-torsion},$$

$$\mathbb{R}^1(X, Y) := v\mathbb{T}(hX, hY) - (v)h\text{-torsion},$$

$$\mathbb{P}^1(X, Y) := v\mathbb{T}(hX, vY) - (v)hv\text{-torsion},$$

$$\mathbb{S}^1(X, Y) := v\mathbb{T}(vX, vY) - (v)v\text{-torsion},$$

and, furthermore, the curvature tensor $\mathbb{K}$ of (D, h) can be described by the following three mappings:

$$\mathbb{R}(X,Y)Z := \mathbb{K}(hX, hY)JZ - h\text{-curvature},$$

$$\mathbb{P}(X,Y)Z := \mathbb{K}(hX, JY)JZ - hv\text{-curvature},$$

$$\mathbb{Q}(X,Y)Z := \mathbb{K}(JX, JY)JZ - v\text{-curvature}.$$

1.5. Finsler manifolds.

Definition. Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E), or simply M, is said to be a *Finsler manifold* with *energy function* E if the following conditions are satisfied:

(F0) $\forall v \in \mathcal{T}M : E(v) > 0$, $E(0) = 0$,

(F1) E is of class C^1 on TM and smooth on $\mathcal{T}M$,

(F2) $C(E) = 2E$ (i.e. E is homogeneous of degree 2),

(F3) the *fundamental form* $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$ is symplectic.

The mapping

$$g : \mathfrak{X}^v(\mathcal{T}M) \times \mathfrak{X}^v(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \quad (JX, JY) \to g(JX, JY) := \omega(JX, Y) \tag{18}$$

is a well-defined, nondegenerate symmetric bilinear form (over $C^\infty(TM)$) which is called the *Riemann-Finsler metric* of the Finsler manifold (M, E).

We have the following important identities:

$$i_J\omega = 0, \qquad i_C\omega = d_J E, \tag{19}$$

$$\mathcal{L}_C\omega = \omega, \quad \mathcal{L}_C d_J E = d_J E \quad \text{(i.e. the forms } \omega \text{ and } d_J E \text{ are homogeneous of degree 1)}, \tag{20}$$

$$E = \frac{1}{2} g(C, C). \tag{21}$$

On any Finsler manifold there is a spray $S : TM \to TTM$, which is uniquely determined on $\mathcal{T}M$ by the formula

$$i_S\omega = -dE. \tag{22}$$

This spray is called the *canonical spray* of the Finsler manifold.

The fundamental lemma of Finsler geometry [2]. On a Finsler manifold (M, E) there is a unique horizontal endomorphism h such that

(B1) $d_h E = 0$ (i.e. h is conservative),

(B2) the strong torsion of h vanishes.

Explicitely

$$h = \frac{1}{2}(1 + [J, S]),$$

where S is the canonical spray. h is called the *Barthel endomorphism* of the Finsler manifold (M, E).

Let h be an *arbitrary* horizontal endomorphism on M. The mapping

$$\begin{gathered} g_h : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \\ (X, Y) \to g_h(X, Y) := g(JX, JY) + g(vX, vY) \end{gathered} \tag{23}$$

is a well-defined pseudo-Riemannian metric on $\mathcal{T}M$ which is called the *prolongation* of g *along* h.

Definition. Let h be a horizontal endomorphism on the Finsler manifold (M, E). The tensor fields $\mathcal{C}$, $\mathcal{C}'$ satisfying the conditions

$$\omega(\mathcal{C}(X, Y), Z) = \frac{1}{2}\left(\mathcal{L}_{JX} J^* g_h\right)(Y, Z), \tag{Car1}$$

$$\omega(\mathcal{C}'(X, Y), Z) = \frac{1}{2}\left(\mathcal{L}_{hX} g_h\right)(JY, JZ) \tag{Car2}$$

are called the *first* and the *second Cartan tensor* belonging to h, respectively.

Remark 3. It is easy to check that $\mathcal{C}$ is independent of the choice of h and

(i) it is semibasic,
(ii) its lowered tensor $\mathcal{C}_b(X, Y, Z) := g(\mathcal{C}(X, Y), JZ)$ is totally symmetric,
(iii) $\mathcal{C}^0 := i_{S_0}\mathcal{C} = 0$ (S_0 is an arbitrary semispray on M).

Consider a smooth function $\varphi : TM \to \mathbb{R}$. Since the fundamental form ω is symplectic, there exists a unique vector field $\operatorname{grad}\varphi \in \mathfrak{X}(\mathcal{T}M)$ such that

$$i_{\operatorname{grad}\varphi}\omega = d\varphi;$$

this vector field is called the *gradient* of φ.

Proposition 1. Let (M, E) be a Finsler manifold and suppose that $\varphi \in C^\infty(TM)$ is a vertical lift: $\varphi = f \circ \pi$ ($f \in C^\infty(M)$). Then the gradient vector field of φ has the following properties:

(i) $\operatorname{grad}\varphi \in \mathfrak{X}^v(\mathcal{T}M)$,
(ii) $[C, \operatorname{grad}\varphi] = -\operatorname{grad}\varphi$ (i.e. $\operatorname{grad}\varphi$ is homogeneous of degree 0),
(ii) $\operatorname{grad}\varphi(E) = f^c$,
(iv) $i_{F\operatorname{grad}\varphi}\mathcal{C} = -\frac{1}{2}[J, \operatorname{grad}\varphi]$ (F is an almost complex structure, associated with an arbitrary horizontal endomorphism),
(v) if $\operatorname{grad}\varphi = \mu C$ ($\mu \in C^\infty(\mathcal{T}M)$) then $\mu = 0$ and, consequently, the function $f \in C^\infty(M)$ is constant.

Proof. For a proof of (i)–(iii) and (v) we refer to [6]. To verify (iv), let $Y, Z \in \mathfrak{X}(M)$ be arbitrary vector fields. Then, applying some well-known identities concerning the vertical and horizontal lifts of a vector field (see e.g. [6]), we get:

$$2g(\mathcal{C}(F \operatorname{grad}\varphi, Y^c), Z^v) \overset{\text{Remark 3/(ii)}}{=} 2g(\mathcal{C}(Y^c, F \operatorname{grad}\varphi), Z^v) \overset{(\text{CAR1})}{=}$$
$$= Y^v g(\operatorname{grad}\varphi, Z^v) - g(J[Y^v, F \operatorname{grad}\varphi], Z^v)$$
$$= Y^v((Zf)^v) - g([J, Y^v]F \operatorname{grad}\varphi, Z^v) - g([Y^v, \operatorname{grad}\varphi], Z^v)$$
$$= -g([Y^v, \operatorname{grad}\varphi], Z^v) = -g([J, \operatorname{grad}\varphi]Y^c, Z^v).$$ □

1.6. The Cartan connection on a Finsler manifold [5]. Let a Finsler manifold (M, E) be given and suppose that h is a horizontal endomorphism on M with vanishing weak torsion. There is a unique Finsler connection (D, h) on M such that

(M1) D is metrical with respect to $g_h : Dg_h = 0$,

(M2) the $(v)v$-torsion $\mathbb{S}^1$ of D vanishes: $\mathbb{S}^1 = 0$,

(M3) the $(h)h$-torsion $\mathbb{A}$ of D vanishes: $\mathbb{A} = 0$,

(M4) the h-deflection $h^*(DC)$ vanishes: $h^*(DC) = 0$.

The covariant derivatives with respect to D can explicitely be calculated by the following formulas:

(C1) $D_{JX}JY = J[JX, Y] + \mathcal{C}(X, Y)$,

(C2) $D_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y)$,

(C3) $D_{JX}hY = h[JX, Y] + F\mathcal{C}(X, Y)$,

(C4) $D_{hX}hY = hF[hX, JY] + F\mathcal{C}'(X, Y)$.

If, in addition, h is homogeneous of degree 1, i.e. the tension of h vanishes, then it coincides with the Barthel endomorphism. In this special case (D, h) is called the *Cartan connection* of the Finsler manifold (M, E).

Remark 4. If (D, h) is the Cartan connection and $\mathcal{C}'$ is the second Cartan tensor belonging to h, then

(i) $D_S\mathcal{C} = -\mathcal{C}'$, $D_C\mathcal{C} = -\mathcal{C}$, $D_C\mathcal{C}' = 0$ (S is the canonical spray),

(ii) $\forall X, Y, Z \in \mathfrak{X}(TM) : (D_{JX}\mathcal{C})(Y, Z) = (D_{JY}\mathcal{C})\ (X, Z)$ and, consequently, the lowered tensor $(D_{JX}\mathcal{C})_b(Y, Z, W) := g((D_{JX}\mathcal{C})(Y, Z), JW)$ is totally symmetric.

(For a proof, see [2].)

2. Wagner connections on a Finsler manifold

Definition. Let (M, E) be a Finsler manifold. The triplet $(\overline{D}, \overline{h}, \alpha)$ is said to be a *Wagner connection* on M if it satisfies the following conditions:

(W0) $(\overline{D}, \overline{h})$ is a Finsler connection on $M, \alpha \in C^\infty(M)$,

(W1) $\overline{D}$ is metrical with respect to $g_{\overline{h}}$: $\overline{D} g_{\overline{h}} = 0$,

(W2) the $(v)v$-torsion $\overline{\mathbb{S}}^1$ of $\overline{D}$ vanishes: $\overline{\mathbb{S}}^1 = 0$,

(W3) $\overline{D}$ is $(h)h$-semisymmetric, i.e. the $(h)h$-torsion $\overline{\mathbb{A}}$ of $\overline{D}$ has the following form:

$$\overline{\mathbb{A}} = d\alpha^v \otimes \overline{h} - \overline{h} \otimes d\alpha^v,$$

(W4) the h-deflection $\overline{h}^*(DC)$ vanishes: $\overline{h}^*(DC) = 0$.

Then $\overline{h}$ is called a *Wagner endomorphism* on M.

Proposition 2. Any Wagner endomorphism is a conservative horizontal endomorphism, i.e. $d_{\overline{h}} E = 0$.

Proof. $\forall X \in (TM)$:

$$2d_{\overline{h}}E(X) \overset{(2)}{=} 2\overline{h}(X)E \overset{(21)}{=} \overline{h}(X)g(C,C) \overset{\text{(W1)}}{=} g\left(\overline{D}_{\overline{h}X}C, C\right) + g\left(C, \overline{D}_{\overline{h}X}C\right) \overset{\text{(W4)}}{=} 0. \qquad \square$$

Theorem 1. The Wagner endomorphism $\overline{h}$ and the Barthel endomorphism h of a Finsler manifold are related as follows:

$$\overline{h} = h + \alpha^c J - E[J, \operatorname{grad} \alpha^v] - d_J E \otimes \operatorname{grad} \alpha^v. \tag{24}$$

Proof. Due to the 2nd local basis property, we can restrict ourselves to vertically and horizontally lifted vector fields, so let $X, Y, Z \in \mathfrak{X}(M)$ be arbitrary. From (W1) we get:

$$\begin{cases} X^{\overline{h}} g(Y^v, Z^v) = g\left(\overline{D}_{X^{\overline{h}}} Y^v, Z^v\right) + g\left(Y^v, \overline{D}_{X^{\overline{h}}} Z^v\right), \\ Y^{\overline{h}} g(Z^v, X^v) = g\left(\overline{D}_{Y^{\overline{h}}} Z^v, X^v\right) + g\left(Z^v, \overline{D}_{Y^{\overline{h}}} X^v\right), \\ -Z^{\overline{h}} g(X^v, Y^v) = -g\left(\overline{D}_{Z^{\overline{h}}} X^v, Y^v\right) - g\left(X^v, \overline{D}_{Z^{\overline{h}}} Y^v\right). \end{cases} \tag{25}$$

Adding now both sides of (25) it follows that

$$\begin{aligned} 2g\left(\overline{D}_{X^{\overline{h}}} Y^v, Z^v\right) &= X^{\overline{h}} g(Y^v, Z^v) + Y^{\overline{h}} g(Z^v, X^v) - Z^{\overline{h}} g(X^v, Y^v) \\ &+ g\left(X^v, \overline{D}_{Z^{\overline{h}}} Y^v - \overline{D}_{Y^{\overline{h}}} Z^v\right) + g\left(Y^v, \overline{D}_{Z^{\overline{h}}} X^v - D_{X^{\overline{h}}} Z^v\right) \\ &+ g\left(Z^v, \overline{D}_{X^{\overline{h}}} Y^v - \overline{D}_{Y^{\overline{h}}} X^v\right) = X^{\overline{h}} g(Y^v, Z^v) + Y^{\overline{h}} g(Z^v, X^v) \\ &- Z^{\overline{h}} g(X^v, Y^v) + g\left(X^v, \overline{F}\,\overline{\mathbb{A}}\left(Y^{\overline{h}}, Z^{\overline{h}}\right)\right) + g\left(Y^v, \overline{F}\,\overline{\mathbb{A}}\left(X^{\overline{h}}, Z^{\overline{h}}\right)\right) \\ &+ g\left(Z^v, \overline{F}\,\overline{\mathbb{A}}\left(Y^{\overline{h}}, X^{\overline{h}}\right)\right) - g(X^v, [Y,Z]^v) - g(Y^v, [X,Z]^v) \\ &- g(Z^v, [Y,X]^v) \overset{\text{(W3)}}{=} X^{\overline{h}} g(Y^v, Z^v) + Y^{\overline{h}} g(Z^v, X^v) - Z^{\overline{h}} g(X^v, Y^v) \\ &+ 2g(X^v, Y^v) g(\operatorname{grad} \alpha^v, Z^v) - 2g(\operatorname{grad} \alpha^v, Y^v) g(X^v, Z^v) \\ &- g(X^v, [Y,Z]^v) - g(Y^v, [X,Z]^v) - g(Z^v, [Y,X]^v). \end{aligned}$$

Applying an analogous "Christoffel process" to the Cartan connection (D, h) we get:

$$g\left(\overline{D}_{X^{\overline{h}}}Y^v - D_{X^h}Y^v, Z^v\right) = g\left(\mathcal{C}(Y^c, Z^c), X^{\overline{h}} - X^h\right)$$
$$+ g\left(\mathcal{C}(X^c, Z^c), Y^{\overline{h}} - Y^h\right) - g\left(\mathcal{C}(X^c, Y^c), Z^{\overline{h}} - Z^h\right)$$
$$+ g(X^v, Y^v)g(\operatorname{grad}\alpha^v, Z^v) - g(\operatorname{grad}\alpha^v, Y^v)g(X^v, Z^v).$$

From this follows that $\forall X, Y, Z \in \mathfrak{X}(TM)$:

$$\begin{aligned} g\left(\overline{D}_{\overline{h}X}JY - D_{\overline{h}X}JY, JZ\right) &= g\left(\mathcal{C}(X, Z), (\overline{h} - h)Y\right) \\ &- g\left(\mathcal{C}(X, Y), \left(\overline{h} - h\right)Z\right) + g(JX, JY)g(\operatorname{grad}\alpha^v, JZ) \\ &- g(\operatorname{grad}\alpha^v, JY)g(JX, JZ). \end{aligned} \tag{26}$$

By the substitution $Y := S_0$ (S_0 is an arbitrary semispray on M), we obtain

$$\begin{aligned} g\left(\left(\overline{h} - h\right)X, JZ\right) &= \alpha^c g(JX, JZ) - g(\mathcal{C}(X, Z), \overline{S} - S) \\ &-g(JX, C)g(\operatorname{grad}\alpha^v, JZ). \end{aligned} \tag{27}$$

If $X := S_0$, (27) implies the relation

$$g(\overline{S} - S, JZ) = \alpha^c g(C, JZ) - 2Eg(\operatorname{grad}\alpha^v, JZ).$$

Hence the semispray $\overline{S}$ associated with $\overline{h}$ and the canonical spray S are related as follows:

$$\overline{S} = S + \alpha^c C - 2E\operatorname{grad}\alpha^v. \tag{28}$$

Substituting this into (27) and applying the total symmetry of $\mathcal{C}_b$, we get the relation

$$\begin{aligned} (\overline{h} - h)X &= \alpha^c JX + 2E\mathcal{C}(F\operatorname{grad}\alpha^v, X) - d_J E(X)\operatorname{grad}\alpha^v \overset{\text{Prop. 1/(iv)}}{=} \\ &= \alpha^c JX - E[J, \operatorname{grad}\alpha^v]X - d_J E(X)\operatorname{grad}\alpha^v. \end{aligned}$$

□

Corollary 1. The tension of a Wagner endomorphism vanishes.

Proof. Applying the formulas (3) and (5), a routine calculation shows that

$$\overline{H} = \mathcal{L}_C d_J E \otimes \operatorname{grad}\alpha^v + [C, \operatorname{grad}\alpha^v] \otimes d_J E \overset{\text{Prop. 1/(ii), (20)}}{=} 0.$$

□

Corollary 2. The weak torsion and the strong torsion of a Wagner endomorphism can be given as follows:

$$\bar{t} = d\alpha^v \otimes J - J \otimes d\alpha^v, \qquad \overline{T} = \alpha^c J - d\alpha^v \otimes C.$$

Proof. Applying the formulas (4) and (6), we get:

$$\bar{t} = d_J\alpha^c \wedge J - E[J, [J, \operatorname{grad}\alpha^v]].$$

From the graded Jacobi identity

$$[J, [J, \operatorname{grad}\alpha^v]] = [J, [\operatorname{grad}\alpha^v, J]] - [\operatorname{grad}\alpha^v, [J, J]] \overset{(7)}{=}$$
$$= [J, [\operatorname{grad}\alpha^v, J]] = -[J, [J, \operatorname{grad}\alpha^v]],$$

therefore

$$[J, [J, \operatorname{grad}\alpha^v]] = 0.$$

Thus we have

$$\bar{t} = d_J\alpha^c \wedge J = d\alpha^v \otimes J - J \otimes d\alpha^v.$$

Finally, $\forall X \in \mathfrak{X}(M)$:

$$\overline{T}(X^c) := (i_{S_0}\bar{t} + \overline{H})X^c \overset{\text{Cor. 1.}}{=} \bar{t}(S_0, X^c) = (\alpha^c J - d\alpha^v \otimes C)X^c. \qquad \square$$

Corollary 3. Let $\overline{h}$ be a Wagner endomorphism on M. Then

$$d_{\overline{h}}\omega = \omega \wedge d\alpha^v.$$

Proof. We start from (24). Since $d_h\omega = 0$, we have only to check the relation

$$d_{\alpha^c J - E[J, \operatorname{grad}\alpha^v] - d_J E \otimes \operatorname{grad}\alpha^v}\omega = \omega \wedge d\alpha^v.$$

Here

$$d_{\alpha^c J}\omega = (i_{\alpha^c J} \circ d - d \circ i_{\alpha^c J})\,\omega = -d(\alpha^c i_J \omega) \overset{(19)}{=} 0.$$

On the other hand, $\forall X, Y \in \mathfrak{X}(TM)$:

$$-\frac{1}{2}\left(i_{[J, \operatorname{grad}\alpha^v]}\omega\right)(X, Y) = -\frac{1}{2}\Big(\omega([J, \operatorname{grad}\alpha^v]X, Y)$$
$$+ \omega(X, [J, \operatorname{grad}\alpha^v]Y)\Big) \overset{\text{Prop. 1./(iv)}}{=} \omega(\mathcal{C}(F\operatorname{grad}\alpha^v, X), Y)$$
$$- \omega(\mathcal{C}(F\operatorname{grad}\alpha^v, Y), X) \overset{(18)}{=} g(\mathcal{C}(F\operatorname{grad}\alpha^v, X), JY)$$
$$- g(\mathcal{C}(F\operatorname{grad}\alpha^v, Y), JX) \overset{\text{Remark 3/(ii)}}{=} 0,$$

therefore

$$d_{E[J, \operatorname{grad}\alpha^v]}\omega = 0.$$

Finally, $\forall X, Y \in \mathfrak{X}(TM)$:

$$\left(i_{d_J E \otimes \operatorname{grad}\alpha^v}\omega\right)(X, Y) = \omega(JX(E)\operatorname{grad}\alpha^v, Y)$$
$$+ \omega(X, JY(E)\operatorname{grad}\alpha^v) = (d_J E \otimes d\alpha^v)(X, Y)$$
$$- (d_J E \otimes d\alpha^v)(Y, X) = (d_J E \wedge d\alpha^v)(X, Y),$$

so we get

$$d_{\overline{h}}\omega = -d_{d_J E \otimes \mathrm{grad}\,\alpha^v}\omega = d\left(i_{d_J E \otimes \mathrm{grad}\,\alpha^v}\omega\right) = d(d_J E \wedge d\alpha^v) = \omega \wedge d\alpha^v. \qquad \square$$

Proposition 3. The second Cartan tensor $\overline{\mathcal{C}}'$ of a Wagner endomorphism $\overline{h}$ has the following properties;

(i) it is semibasic,

(ii) its lowered tensor $\overline{\mathcal{C}}'_b$ is totally symmetric,

(iii) $\overline{\mathcal{C}}'^{\circ} := i_{S_0}\overline{\mathcal{C}}' = 0$ (S_0 is an arbitrary semispray on M).

Proof. From the formula (Car1) we get immediately the property (i) and it is also clear that $\overline{\mathcal{C}}'_b$ is symmetric in its 2nd and 3rd arguments.

Evaluating the form $d_{\overline{h}}\,\omega$ on the vector fields $X^{\overline{h}}$, Y^v, $Z^{\overline{h}}$ ($X, Y, Z \in \mathfrak{X}(M)$) it follows that

$$\begin{aligned} d_{\overline{h}}\omega\left(X^{\overline{h}}, Y^v, Z^{\overline{h}}\right) &= 2\overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, Y^{\overline{h}}, X^{\overline{h}}\right) - 2\overline{\mathcal{C}}'_b\left(X^{\overline{h}}, Y^{\overline{h}}, Z^{\overline{h}}\right) \\ &\quad + (\omega \wedge d\alpha^v)\left(X^{\overline{h}}, Y^{\overline{h}}, Z^{\overline{h}}\right) \overset{\text{Cor. 3.}}{\Longrightarrow} \\ &\quad \overline{\mathcal{C}}'_b\left(X^{\overline{h}}, Y^{\overline{h}}, Z^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, Y^{\overline{h}}, X^{\overline{h}}\right), \end{aligned}$$

i.e. $\overline{\mathcal{C}}'_b$ is symmetric in its 1st and 3rd arguments.

According to the above symmetry properties, $\forall X, Y, Z \in \mathfrak{X}(M)$:

$$\overline{\mathcal{C}}'_b\left(X^{\overline{h}}, Y^{\overline{h}}, Z^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, Y^{\overline{h}}, X^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, X^{\overline{h}}, Y^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Y^{\overline{h}}, X^{\overline{h}}, Z^{\overline{h}}\right),$$

i.e. $\overline{\mathcal{C}}'_b$ is totally symmetric.

Finally, let S_0 be an arbitrary semispray on M. Then $\forall Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned} 2g(\overline{\mathcal{C}}'(S_0, Y^{\overline{h}}), Z^v) &\overset{\text{(ii)}}{=} 2g(\overline{\mathcal{C}}'(Y^{\overline{h}}, S_0), Z^v) := Y^{\overline{h}} g(C, Z^v) \\ &- g([Y^{\overline{h}}, C], Z^v) - g(C, [Y^{\overline{h}}, Z^v]) \overset{(18),(19),\text{Cor. 1.}}{=} \\ &= Y^{\overline{h}}(Z^v(E)) - [Y^{\overline{h}}, Z^v](E) \overset{\text{Prop. 2.}}{=} 0. \end{aligned} \qquad \square$$

Proposition 4. Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). The covariant derivatives with respect to $\overline{D}$ can explicitely be calculated by the following formulas:

(W5) $$\overline{D}_{JX} JY = J[JX, Y] + \mathcal{C}(X, Y),$$

(W6) $$\overline{D}_{\overline{h}X} JY = \overline{v}[\overline{h}X, JY] + \overline{\mathcal{C}}'(X, Y),$$

(W7) $$\overline{D}_{JX} \overline{h}Y = \overline{h}[JX, Y] + \overline{F}\mathcal{C}(X, Y),$$

(W8) $$\overline{D}_{\overline{h}X} \overline{h}Y = \overline{h}\,\overline{F}[\overline{h}X, JY] + \overline{F}\,\overline{\mathcal{C}}'(X, Y).$$

Proof. Applying the usual "Christoffel process" it can easily be seen that a Wagner connection is uniquely determined by the conditions (W1)–(W4).

Consider now the Wagner endomorphism $\overline{h}$ and let us *define* a Finsler connection $(\overline{D}, \overline{h})$ by the formulas (W5)–(W8). It is easy to check that $(\overline{D}, \overline{h})$ satisfies the conditions (W0)–(W4) and, consequently, (W5)–(W8) are just the rules of calculation with respect to the Wagner connection $(\overline{D}, \overline{h}, \alpha)$. □

Remark 5. Comparing the formulas (W5)–(W8) with (C1)–(C4) we can say that a Wagner connection is a *"Cartan connection with nonvanishing* $(h)h$*-torsion"*, i.e. it is a *generalized Cartan connection.*

Our next Proposition emphasizes the strict analogy between the Cartan connection and a Wagner connection.

Lemma 1. $\forall X, Y \in \mathfrak{X}(TM)$:

$$\begin{aligned}\overline{D}_{\overline{h}X} JY - D_{\overline{h}X} JY = g(JX, JY) \operatorname{grad} \alpha^v - g(\operatorname{grad} \alpha^v, JY) JX \\ + \mathcal{C}_b(F \operatorname{grad} \alpha^v, X, Y) C - JY(E) \mathcal{C}(F \operatorname{grad} \alpha^v, X) \\ + 2E\mathbb{Q}(F \operatorname{grad} \alpha^v, X) Y,\end{aligned} \tag{29}$$

$$\begin{aligned}\overline{\mathcal{C}}'(X, Y) = \mathcal{C}'(X, Y) + \alpha^c \mathcal{C}(X, Y) + JX(E) \mathcal{C}(F \operatorname{grad} \alpha^v, X) \\ + JY(E) \mathcal{C}(F \operatorname{grad} \alpha^v, X) + \mathcal{C}_b(F \operatorname{grad} \alpha^v, X, Y) C \\ + 2E(D_{\operatorname{grad} \alpha^v} \mathcal{C})(X, Y).\end{aligned} \tag{30}$$

Since these relations can be obtained by an easy calculation from (24) we omit the proof. (Note that (30) also implies Proposition 3.)

Proposition 5. Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection. Then the covariant differentials $\overline{D}\mathcal{C}$, $\overline{D}\,\overline{\mathcal{C}}'$ have the following properties:

$$\overline{D}_{\overline{S}} \mathcal{C} = -\overline{\mathcal{C}}', \qquad \overline{D}_C \mathcal{C} = -\mathcal{C}, \qquad \overline{D}_C \overline{\mathcal{C}}' = 0. \tag{31}$$

Proof. By Remark 4, (31) immediately follows from the relations (29), (30). □

Proposition 6. Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). Then the following assertions are equivalent:

(i) $d\alpha^v = 0$ (i.e. $\alpha \in C^\infty(M)$ is constant),

(ii) $\overline{S} = S$ (i.e. the semispray $\overline{S}$ associated with $\overline{h}$ coincides with the canonical spray),

(iii) the Wagner endomorphism arises from a semispray, i.e. there is a semispray $\overline{S}$ on M such that
$$\overline{h} = \frac{1}{2}(1 + [J, \overline{S}]),$$

(iv) $\overline{h} = h$ (i.e. the Wagner endomorphism coincides with the Barthel endomorphism),

(v) the Finsler connection $(\overline{D}, \overline{h})$ coincides with the Cartan connection (D, h).

Proof. *(i)⇒(v)* If $d\alpha^v = 0$ then it follows, by Corollary 2, that the weak torsion of $\overline{h}$ vanishes. Therefore the Wagner endomorphism is a conservative horizontal endomorphism on M with vanishing strong torsion (cf. Corollary 2). Thus $\overline{h} = h$ and (v) is an immediate consequence of (W5)–(W8).

The implications *(v)⇒(iv)* and *(iv)⇒(iii)* are evident.

(iii)⇒(ii) It is easy to check that the hypothesis (iii) implies the vanishing of the weak torsion $\overline{t}$. Hence, as above, $\overline{h} = h$ and consequently $\overline{S} = S$.

(ii)⇒(i) If (ii) holds then (28) implies the relation

$$\operatorname{grad} \alpha^v = \mu C \qquad \left(\mu = \frac{\alpha^c}{2E} \in C^\infty(\mathcal{T}M)\right).$$

In view of Proposition 1/(v) this means that $\operatorname{grad} \alpha^v = 0$, proving the implication *(ii)⇒(i)*. □

3. Curvature identities concerning a Wagner connection and some technical observations

Lemma 2. Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). There is a unique Finsler connection $(\overset{\circ}{\overline{D}}, \overline{h})$ on M such that:

(O1) the $(v)hv$-torsion $\overset{\circ}{\overline{\mathbb{P}}}{}^1$ of $\overset{\circ}{\overline{D}}$ vanishes: $\overset{\circ}{\overline{\mathbb{P}}}{}^1 = 0$,

(O1) the $(h)hv$-torsion $\overset{\circ}{\overline{\mathbb{B}}}$ of $\overset{\circ}{\overline{D}}$ vanishes: $\overset{\circ}{\overline{\mathbb{B}}} = 0$.

The covariant derivatives with respect to $\overset{\circ}{\overline{D}}$ can explicitely be calculated by the formulas

(Brw1) $$\overset{\circ}{\overline{D}}_{JX} JY = J[JX, Y],$$

(Brw2) $$\overset{\circ}{\overline{D}}_{\overline{h}X} JY = \overline{v}[\overline{h}X, JY],$$

(Brw3) $$\overset{\circ}{\overline{D}}_{JX} \overline{h}Y = \overline{h}[JX, Y],$$

(Brw4) $$\overset{\circ}{\overline{D}}_{\overline{h}X} \overline{h}Y = \overline{h}\,\overline{F}[\overline{h}X, JY].$$

In addition, $(\overset{\circ}{\overline{D}}, \overline{h})$ has the following two properties:

(Brw5) the h-deflection $\overline{h}^*(\overset{\circ}{\overline{D}}C)$ of $\overset{\circ}{\overline{D}}$ vanishes: $\overline{h}^*(\overset{\circ}{\overline{D}}C) = 0$,

(Brw6) $\overset{\circ}{\overline{D}}$ is $(h)h$-semisymmetric, i.e. the $(h)h$-torsion $\overset{\circ}{\overline{\mathbb{A}}}$ of $\overset{\circ}{\overline{D}}$ has the following form:

$$\overset{\circ}{\overline{\mathbb{A}}} = d\alpha^v \otimes \overline{h} - \overline{h} \otimes d\alpha^v.$$

Proof. We can argue as in the proof of Theorem 1 in [5]. □

Remark 6. It is easy to check (see e.g. [2], [5]) that if $\overline{h}$ coincides with the Barthel endomorphism then $(\overset{\circ}{\overline{D}}, \overline{h})$ is the well-known Berwald connection on the Finsler manifold (M, E). In general we can say that $(\overset{\circ}{\overline{D}}, \overline{h})$ is a *"Berwald connection with nonvanishing* $(h)h$*-torsion"*, i.e. it is a *"generalized Berwald connection"*.

Proposition 7. Under the conditions of Lemma 2 the curvature tensors of $\overline{D}$ and $\overset{\circ}{\overline{D}}$ are related as follows:

(i)
$$\overline{\mathbb{R}}(X,Y)Z = \overset{\circ}{\overline{\mathbb{R}}}(X,Y)Z + \left(\overline{D}_{\overline{h}X}\overline{\mathcal{C}}'\right)(Y,Z) - \left(\overline{D}_{\overline{h}Y}\overline{\mathcal{C}}'\right)(X,Z) + \overline{\mathcal{C}}'(\overline{F}\,\overline{\mathcal{C}}'(X,Z),Y) - \overline{\mathcal{C}}'(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z)) + \overline{\mathcal{C}}'\left(\overline{F}\,\overline{t}(X,Y),Z\right) + \overline{\mathcal{C}}'\left(\overline{F}\,\overline{R}(X,Y),Z\right),$$

(ii)
$$\overline{\mathbb{P}}(X,Y)Z = \overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z + \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) - \left(\overline{D}_{JY}\overline{\mathcal{C}}'\right)(X,Z) + \mathcal{C}(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z) - \overline{\mathcal{C}}'\left(X,\overline{F}\mathcal{C}(Y,Z)\right) + \mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z)) - \overline{\mathcal{C}}'\left(\overline{F}\mathcal{C}(X,Y),Z\right),$$

(iii)
$$\overline{\mathbb{Q}}(X,Y)Z = \mathcal{C}\left(\overline{F}\mathcal{C}(X,Z),Y\right) - \mathcal{C}\left(X,\overline{F}\mathcal{C}(Y,Z)\right),$$
$$\overset{\circ}{\overline{\mathbb{Q}}} = 0 \qquad (X,Y,Z \in \mathfrak{X}(TM)).$$

The *proof* is a straigthforward but lengthy calculation.

(Note that $\overline{\mathcal{C}}'\,(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z)$, $\overline{\mathcal{C}}'(\overline{F}\,\overline{t}(X,Y),Z)\dots$ are independent of the choice of the almost complex structure $\overline{F}$.)

Corollary 4. Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection. Then its curvature tensors have the following properties:

(i) $\overline{\mathbb{R}}(X,Y)S_0 = \overline{R}(X,Y)$,

(ii) $\overline{\mathbb{P}}(X,Y)S_0 = \overline{\mathcal{C}}'(X,Y)$, $\overline{\mathbb{P}}(X,S_0)Y = \overline{\mathbb{P}}(S_0,X)Y = 0$,

(iii) $\overline{\mathbb{Q}}(X,Y)S_0 = \overline{\mathbb{Q}}(X,S_0)Y = \overline{\mathbb{Q}}(S_0,X)Y = 0$
$(X,Y \in (TM)$, S_0 is an arbitrary semispray on $M)$.

Proof. We deduce only the less trivial third relation of (ii). Let $X,Y \in \mathfrak{X}(M)$ be arbitrary vector fields on M. Then

$$\overline{\mathbb{P}}\left(S_0, X^{\overline{h}}\right)Y^{\overline{h}} \overset{\text{Prop. 7/(ii)}}{=} \overset{\circ}{\overline{\mathbb{P}}}\left(S_0, X^{\overline{h}}\right)Y^{\overline{h}} \overset{\text{(BRW1), (BRW2)}}{=}$$
$$= -\overset{\circ}{\overline{D}}_{X^v}\left(\overline{v}[\overline{S}, Y^v]\right) - \overset{\circ}{\overline{D}}_{\overline{v}[\overline{S},X^v]}Y^v - \overset{\circ}{\overline{D}}_{\overline{h}[\overline{S},X^v]}Y^v$$
$$\overset{(13),\,(16),\,\text{(BRW1)}}{=} \overset{\circ}{\overline{D}}_{X^v}\left(Y^{\overline{h}} - Y^c - \overline{T}(Y^c)\right) - \overset{\circ}{\overline{D}}_{\overline{F}J[\overline{S},X^v]}Y^v$$
$$= \overset{\circ}{\overline{D}}_{X^v}\left(Y^{\overline{h}} - Y^c\right) - \overset{\circ}{\overline{D}}_{X^v}(\overline{T}(Y^c)) - \overset{\circ}{\overline{D}}_{\overline{F}J[\overline{S},X^v]}Y^v.$$

Here

$$\overset{\circ}{\overline{D}}_{X^v}\left(Y^{\overline{h}}-Y^c\right)=[X^v,Y^{\overline{h}}-Y^c],\quad \text{since the } (v)v\text{-torsion of } \overset{\circ}{\overline{D}} \text{ vanishes,}$$

$$\begin{aligned}\overset{\circ}{\overline{D}}_{X^v}(\overline{T}(Y^c)) &\overset{\text{Cor. 2, (BRW1)}}{=} (X\alpha)^vY^v-(Y\alpha)^v\overset{\circ}{\overline{D}}_{X^v}C \overset{\text{(BRW1)}}{=}\\ &= (X\alpha)^vY^v-(Y\alpha)^vJ[X^v,\overline{S}]\\ &= (X\alpha)^vY^v-(Y\alpha)^vX^v \overset{\text{Cor. 2}}{=} \overline{t}\left(X^{\overline{h}},Y^{\overline{h}}\right),\\ \overset{\circ}{\overline{D}}_{\overline{F}J[\overline{S},X^v]}Y^v &= \overset{\circ}{\overline{D}}_{-\overline{F}X^v}Y^v=\overset{\circ}{\overline{D}}_{-X^{\overline{h}}}Y^v \overset{\text{(BRW2)}}{=} -[X^{\overline{h}},Y^v].\end{aligned}$$

Thus we have:

$$\mathbb{P}\left(S_0,X^{\overline{h}}\right)Y^{\overline{h}}=[X^v,Y^{\overline{h}}-Y^c]-\overline{t}\left(X^{\overline{h}},Y^{\overline{h}}\right)+[X^{\overline{h}},Y^v]\overset{(12)}{=}0. \qquad \square$$

Lemma 3. – $\forall X,Y,Z\in\mathfrak{X}(TM)$:

$$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{\mathbb{A}}\left(\overline{\mathbb{A}}(X,Y),Z\right)-\left(\overline{D}_{\overline{h}X}\overline{\mathbb{A}}\right)(Y,Z)\right)=0.$$

Proof. Omitting the troublesome details we note that $\forall X,Y,Z\in\mathfrak{X}(M)$:

$$\begin{aligned}&\underset{X^{\overline{h}},Y^{\overline{h}},Z^{\overline{h}}}{\mathfrak{S}}\left(\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(X^{\overline{h}},Y^{\overline{h}}\right),Z^{\overline{h}}\right)+\left(\overline{D}_{X^{\overline{h}}}\overline{\mathbb{A}}\right)\left(Y^{\overline{h}},Z^{\overline{h}}\right)\right):=\\ &=\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(X^{\overline{h}},Y^{\overline{h}}\right),Z^{\overline{h}}\right)+\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(Y^{\overline{h}},Z^{\overline{h}}\right),X^{\overline{h}}\right)+\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(Z^{\overline{h}},X^{\overline{h}}\right),Y^{\overline{h}}\right)\\ &+\left(\overline{D}_{X^{\overline{h}}}\overline{\mathbb{A}}\right)\left(Y^{\overline{h}},Z^{\overline{h}}\right)+\left(\overline{D}_{Y^{\overline{h}}}\overline{\mathbb{A}}\right)\left(Z^{\overline{h}},X^{\overline{h}}\right)+\left(\overline{D}_{Z^{\overline{h}}}\overline{\mathbb{A}}\right)\left(X^{\overline{h}},Y^{\overline{h}}\right)\\ &=(Y\alpha)^v\overline{\mathbb{A}}\left(X^{\overline{h}},Z^{\overline{h}}\right)-(X\alpha)^v\overline{\mathbb{A}}\left(Y^{\overline{h}},Z^{\overline{h}}\right)-(Z\alpha)^v\overline{\mathbb{A}}\left(X^{\overline{h}},Y^{\overline{h}}\right)\\ &=-(d\alpha^v\wedge\overline{\mathbb{A}})\left(X^{\overline{h}},Y^{\overline{h}},Z^{\overline{h}}\right)\overset{\text{(W3)}}{=}0.\end{aligned} \qquad \square$$

Corollary 5. (Bianchi identities) $-\forall X,Y,Z\in\mathfrak{X}(TM)$:

(I) $$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{D}_{\overline{h}X}\overline{R}\right)(Y,Z)=\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{\mathcal{C}}'\left(\overline{F}\,\overline{R}(X,Y),Z\right)-\overline{R}\left(\overline{\mathbb{A}}(X,Y),Z\right)\right),$$

(II) $$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{D}_{\overline{h}X}\overline{\mathbb{R}}\right)(Y,Z)=\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{\mathbb{P}}\left(X,\overline{F}\,\overline{R}(Y,Z)\right)-\overline{\mathbb{R}}\left(\overline{\mathbb{A}}(X,Y),Z\right)\right),$$

(III) $$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{D}_{JX}\overline{\mathbb{Q}}\right)(Y,Z)=0,$$

(IV) $$\left(\overline{D}_{\overline{h}X}\overline{\mathbb{P}}\right)(Y,Z)-\left(\overline{D}_{\overline{h}Y}\overline{\mathbb{P}}\right)(X,Z)+\left(\overline{D}_{JZ}\overline{\mathbb{R}}\right)(X,Y)=$$
$$=\overline{\mathbb{P}}(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z))-\overline{\mathbb{P}}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z))-\overline{\mathbb{R}}(X,\overline{F}\mathcal{C}(Y,Z))$$
$$+\overline{\mathbb{R}}(Y,\overline{F}\mathcal{C}(X,Z))-\overline{\mathbb{P}}\left(\overline{\mathbb{A}}(X,Y),Z\right)-\overline{\mathbb{Q}}\left(\overline{F}\,\overline{R}(X,Y),Z\right),$$

(V) $$\left(\overline{D}_{\overline{h}X}\overline{\mathbb{Q}}\right)(Y,Z)-\left(\overline{D}_{JY}\overline{\mathbb{P}}\right)(X,Z)+\left(\overline{D}_{JZ}\overline{\mathbb{P}}\right)(X,Y)=$$
$$=\overline{\mathbb{P}}\left(\overline{F}\mathcal{C}(X,Y),Z\right)-\overline{\mathbb{P}}(\overline{F}\mathcal{C}(Z,X),Y)-\overline{\mathbb{Q}}(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z)$$
$$+\overline{\mathbb{Q}}(\overline{F}\,\overline{\mathcal{C}}'(Z,X),Y).$$

Proof. Let $X,Y,Z\in\mathfrak{X}(TM)$ be arbitrary. By Lemma 3, the "usual" first Bianchi identity

$$\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\overline{\mathbb{K}}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z=\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\left(\overline{\mathbb{T}}\left(\overline{\mathbb{T}}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)+\left(\overline{D}_{\overline{h}X}\overline{\mathbb{T}}\right)\left(\overline{h}Y,\overline{h}Z\right)\right)$$

gives the relation

$$\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\overline{\mathbb{K}}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z=\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\Big(\left(\overline{D}_{\overline{h}X}\overline{R}\right)\left(\overline{h}Y,\overline{h}Z\right)$$
$$+\overline{R}\left(\overline{\mathbb{A}}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)+\overline{F}\mathcal{C}\left(\overline{F}\,\overline{R}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z\right)$$
$$-\overline{\mathcal{C}}'\left(\overline{F}\,\overline{R}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)\Big),$$

since

$$\overline{\mathbb{T}}\left(\overline{h}X,\overline{h}Y\right)=\overline{\mathbb{A}}(X,Y)+\overline{R}(X,Y), \tag{32}$$

$$\overline{\mathbb{T}}\left(\overline{h}X,JY\right)=\overline{\mathcal{C}}'(X,Y)-\overline{F}\mathcal{C}(X,Y). \tag{33}$$

From this it follows that

$$0=\overline{v}\left(\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\overline{\mathbb{K}}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z\right)$$
$$=\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\Big(\left(\overline{D}_{\overline{h}X}\overline{R}\right)\left(\overline{h}Y,\overline{h}Z\right)+\overline{R}\left(\overline{\mathbb{A}}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)$$
$$-\overline{\mathcal{C}}'\left(\overline{F}\,\overline{R}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)\Big),$$

which proves the relation (I).

Applying the second Bianchi identity, the other relations can also be obtained by a direct calculation. For example we derive (IV). Since

$$0=\underset{\overline{h}X,\overline{h}Y,JZ}{\mathfrak{S}}\left(\overline{\mathbb{K}}\left(\overline{\mathbb{T}}\left(\overline{h}X,\overline{h}Y\right),JZ\right)+\left(\overline{D}_{\overline{h}X}\overline{\mathbb{K}}\right)\left(\overline{h}Y,JZ\right)\right),$$

we get

$$0=\overline{\mathbb{P}}\left(\overline{\mathbb{A}}(X,Y),Z\right)+\overline{\mathbb{Q}}\left(\overline{F}\,\overline{R}(X,Y),Z\right)+\overline{\mathbb{P}}(\overline{F}\,\overline{\mathcal{C}}'(Y,Z),X)$$
$$-\overline{\mathbb{R}}\left(\overline{F}\mathcal{C}(Y,Z),X\right)+\overline{\mathbb{P}}(Y,\overline{F}\overline{\mathcal{C}}'(X,Z))+\overline{\mathbb{R}}\left(\overline{F}\mathcal{C}(X,Z),Y\right)$$
$$+\left(\overline{D}_{\overline{h}X}\overline{\mathbb{P}}\right)(Y,Z)-\left(\overline{D}_{\overline{h}Y}\overline{\mathbb{P}}\right)(X,Z)+\left(\overline{D}_{JZ}\overline{\mathbb{R}}\right)(X,Y),$$

proving the relation (IV). □

Corollary 6. $\forall X, Y, Z \in \mathfrak{X}(TM)$:

(i)
$$\left(\overline{D}_C \overline{\mathbb{R}}\right)(X,Y)Z = 0, \qquad \left(\overline{D}_C \overline{\mathbb{P}}\right)(X,Y)Z = -\overline{\mathbb{P}}(X,Y)Z,$$
$$\left(\overline{D}_C \overline{\mathbb{Q}}\right)(X,Y)Z = -2\overline{\mathbb{Q}}(X,Y)Z,$$

(ii)
$$\left(\overline{D}_{\overline{S}} \overline{\mathbb{Q}}\right)(X,Y)Z = \mathcal{C}(X, \overline{F}\,\overline{\mathcal{C}}'(Y,Z)) - \mathcal{C}(Y, \overline{F}\,\overline{\mathcal{C}}'(X,Z)) + \overline{\mathcal{C}}'\left(X, \overline{F}\mathcal{C}(Y,Z)\right) - \overline{\mathcal{C}}'\left(Y, \overline{F}\mathcal{C}(X,Z)\right).$$

Proof. Substituting $Z := \overline{S}$ in the Bianchi identity (IV), we have

$$\overline{D}_C \overline{\mathbb{R}} = 0.$$

In the same way, consider the vector field $Y := \overline{S}$. From the Bianchi identity (V) it follows that

$$\overline{D}_C \overline{\mathbb{P}} = -\overline{\mathbb{P}}.$$

The relation $\overline{D}_C \overline{\mathbb{Q}} = -2\overline{\mathbb{Q}}$ is an immediate consequence of the Bianchi identity (III).

Finally, by Proposition 5 and Proposition 7/(iii), an easy direct calculation shows that (ii) holds. □

Corollary 7. $\forall X, Y, Z, W \in \mathfrak{X}(TM)$:

(i)
$$g\left(\overline{\mathbb{P}}(X,Y)Z, JW\right) = -g\left(\overline{\mathbb{P}}(X,Y)W, JZ\right),$$

(ii)
$$\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(Z,Y)X = \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) - \left(\overline{D}_{\overline{h}Z}\mathcal{C}\right)(X,Y) + \mathcal{C}(\overline{F}\,\overline{\mathcal{C}}'(X,Y), Z) - \mathcal{C}(X, \overline{F}\,\overline{\mathcal{C}}'(Z,Y)),$$

(iii)
$$\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(X,Z)Y = \left(\overline{D}_{JZ}\overline{\mathcal{C}}'\right)(X,Y) - \left(\overline{D}_{JY}\overline{\mathcal{C}}'\right)(X,Z) + \overline{\mathcal{C}}'\left(\overline{F}\mathcal{C}(Z,X), Y\right) - \overline{\mathcal{C}}'\left(\overline{F}\mathcal{C}(X,Y), Z\right),$$

(iv)
$$\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(Y,X)Z = \mathcal{C}(Y, \overline{F}\,\overline{\mathcal{C}}'(X,Z)) - \mathcal{C}(X, \overline{F}\,\overline{\mathcal{C}}'(Y,Z)) + \overline{\mathcal{C}}'\left(Y, \overline{F}\mathcal{C}(X,Z)\right) - \overline{\mathcal{C}}'\left(X, \overline{F}\mathcal{C}(Y,Z)\right).$$

Proof. It is easy to check that the first identity holds for the curvature tensors of an arbitrary metrical connection.

Let now $X, Y, Z \in \mathfrak{X}(M)$ be arbitrary vector fields on M. Applying the formulas (W5)–(W8) we get:

$$\overline{\mathbb{P}}\left(X^{\overline{h}}, Y^{\overline{h}}\right) Z^{\overline{h}} - \overline{\mathbb{P}}\left(Z^{\overline{h}}, Y^{\overline{h}}\right) X^{\overline{h}} = \left(\overline{D}_{X^{\overline{h}}}\mathcal{C}\right)\left(Y^{\overline{h}}, Z^{\overline{h}}\right) - \left(\overline{D}_{Z^{\overline{h}}}\mathcal{C}\right)\left(X^{\overline{h}}, Y^{\overline{h}}\right) + \mathcal{C}\left(\overline{F}\,\overline{\mathcal{C}}'\left(X^{\overline{h}}, Y^{\overline{h}}\right), Z^{\overline{h}}\right) - \mathcal{C}\left(X^{\overline{h}}, \overline{F}\,\overline{\mathcal{C}}'\left(Y^{\overline{h}}, Z^{\overline{h}}\right)\right).$$

Evaluating the Bianchi identity (V) on $\overline{S}$, it follows by Corollary 4 that (iii) holds.

Finally, from the Bianchi identity (V), $\forall X, Y, Z \in \mathfrak{X}(TM)$:

$$\left(\overline{D}_{\overline{S}}\mathbb{Q}\right)(X,Y)Z - \left(\overline{D}_{JX}\mathbb{P}\right)(\overline{S},Y)Z + \left(\overline{D}_{JY}\mathbb{P}\right)(\overline{S},X)Z = 0,$$

so we have

$$\begin{aligned}\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(Y,X)Z &= -\left(\overline{D}_{\overline{S}}\mathbb{Q}\right)(X,Y)Z \overset{\text{Cor. 6/(ii)}}{=} \\ &= \mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z)) - \mathcal{C}(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z)) \\ &\quad + \overline{\mathcal{C}}'\left(Y,\overline{F}\mathcal{C}(X,Z)\right) - \overline{\mathcal{C}}'\left(X,\overline{F}\mathcal{C}(Y,Z)\right). \qquad \square\end{aligned}$$

4. Wagner manifolds

Definition. Let (M,E) be a Finsler manifold endowed with a Wagner connection $(\overline{D},\overline{h},\alpha)$. (M,E) is said to be a *Wagner manifold* (with respect to $(\overline{D},\overline{h},\alpha)$) if there is a linear connection ∇ on M such that

$$\forall X,Y \in \mathfrak{X}(M) : \overline{D}_{X^{\overline{h}}}Y^v = \left(\nabla_X Y\right)^v. \tag{34}$$

Then ∇ is called the *linear connection of the Wagner manifold.*

Proposition 8. If (M,E) is a Wagner manifold (with respect to $(\overline{D},\overline{h},\alpha)$) then the second Cartan tensor $\overline{\mathcal{C}}'$ of $\overline{h}$ vanishes.

Proof. Since (M,E) is a Wagner manifold, it follows that $\overline{D}_{X^{\overline{h}}}Z^v$ $(X,Z \in \mathfrak{X}(M))$ is a vertically lifted vector field to the manifold TM and consequently $\forall X,Y,Z \in \mathfrak{X}(M)$:

$$\begin{aligned}0 &= [\overline{D}_{X^{\overline{h}}}Z^v, Y^v] \overset{\text{(W2)}}{=} \overline{D}_{\left(\overline{D}_{X^{\overline{h}}}Z^v\right)}Y^v - \overline{D}_{Y^v}\overline{D}_{X^{\overline{h}}}Z^v \overset{\text{(W5)}}{=} \\ &= \mathcal{C}\left(\overline{F}\,\overline{D}_{X^{\overline{h}}}Z^v, Y^{\overline{h}}\right) + \overline{\mathbb{P}}\left(X^{\overline{h}},Y^{\overline{h}}\right)Z^{\overline{h}} - \overline{D}_{X^{\overline{h}}}\left(\mathcal{C}\left(Y^{\overline{h}},Z^{\overline{h}}\right)\right) \\ &\quad + \mathcal{C}\left(\overline{F}[X^{\overline{h}},Y^v],Z^{\overline{h}}\right) \overset{\text{(W8)}}{=} \overline{\mathbb{P}}\left(X^{\overline{h}},Y^{\overline{h}}\right)Z^{\overline{h}} - \left(\overline{D}_{X^{\overline{h}}}\mathcal{C}\right)\left(Y^{\overline{h}},Z^{\overline{h}}\right) \\ &\quad - \mathcal{C}\left(\overline{F}\,\overline{\mathcal{C}}'\left(X^{\overline{h}},Y^{\overline{h}}\right),Z^{\overline{h}}\right),\end{aligned}$$

therefore (34) is equivalent to the relation

$$\overline{\mathbb{P}}(X,Y)Z - \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) - \mathcal{C}(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z) = 0, \tag{35}$$

where $X,Y,Z \in \mathfrak{X}(TM)$.

By the substitution $Z := \overline{S}$ we obtain that

$$0 = \overline{\mathbb{P}}(X,Y)\overline{S} \overset{\text{Cor. 4/(ii)}}{=} \overline{\mathcal{C}}'(X,Y). \qquad \square$$

Theorem 2. Let $(\overline{D},\overline{h},\alpha)$ be a Wagner connection on the Finsler manifold (M,E). Then the following assertions are equivalent:
(i) (M,E) is a Wagner manifold (with respect to $(\overline{D},\overline{h},\alpha)$),
(ii) the hv-curvature tensor $\overset{\circ}{\overline{\mathbb{P}}}$ of the Finsler connection $(\overset{\circ}{\overline{D}},\overline{h})$ vanishes.

Proof. *(i)⇒(ii)* In view of (35) and Proposition 7/(ii), we have

$$\overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z=\left(\overline{D}_{JY}\overline{\mathcal{C}}'\right)(X,Z)+\overline{\mathcal{C}}'(X,\overline{F}\mathcal{C}(Y,Z))+\overline{\mathcal{C}}'(\overline{F}\mathcal{C}(X,Y),Z)$$
$$-\,\mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z))\overset{\text{Prop. 8}}{=}0.$$

(ii)⇒(i) Since $\forall X,Y,Z\in\mathfrak{X}(M)$: $\overset{\circ}{\overline{\mathbb{P}}}\left(X^{\overline{h}},Y^{\overline{h}}\right)Z^{\overline{h}}=[[X^{\overline{h}},Y^v],Z^v]$, the vanishing of $\overset{\circ}{\overline{\mathbb{P}}}$ implies that $[X^{\overline{h}},Y^v]$ is a vertical lift.

On the other hand,

$$2g\left(\overline{\mathcal{C}}'\left(X^{\overline{h}},Y^{\overline{h}}\right),Z^v\right)=X^{\overline{h}}g(Y^v,Z^v)-g\left([X^{\overline{h}},Y^v],Z^v\right)$$
$$-\,g\left(Y^v,[X^{\overline{h}},Z^v]\right)\overset{(18)}{=}X^{\overline{h}}(Y^v(Z^vE))-[X^{\overline{h}},Y^v](Z^vE)$$
$$-\,Y^v\left([X^{\overline{h}},Z^v]E\right)\overset{\text{Prop. 2}}{=}0,$$

so the second Cartan tensor $\overline{\mathcal{C}}'$ of $\overline{h}$ vanishes, and consequently $\forall X,Y\in\mathfrak{X}(M)$:

$$\overline{D}_{X^{\overline{h}}}Y^v=\overset{\circ}{\overline{D}}_{X^{\overline{h}}}Y^v\overset{\text{(BRW2)}}{=}[X^{\overline{h}},Y^v].$$

Finally, if we define a linear connection ∇ on M by the formula

$$(\nabla_XY)^v:=[X^{\overline{h}},Y^v],\qquad(36)$$

then ∇ clearly satisfies the condition (34). □

Proposition 9. Let $(\overline{D},\overline{h},\alpha)$ be a Wagner connection. Then the following assertions are equivalent:

(i) the hv-curvature tensor $\overline{\mathbb{P}}$ of $\overline{D}$ vanishes: $\overline{\mathbb{P}}=0$,
(ii) the second Cartan tensor $\overline{\mathcal{C}}'$ of $\overline{h}$ vanishes: $\overline{\mathcal{C}}'=0$,
(iii) $\forall X,Y,Z\in\mathfrak{X}(TM):\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z)=\left(\overline{D}_{\overline{h}Z}\mathcal{C}\right)(X,Y)$,
(iv) $\overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z=-\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z)$.

Proof. From Corollary 4/(ii) we immediately get the implication *(i)⇒(ii)*.

This implies by Corollary 7/(ii) that *(i)⇒(iii)* is also valid.

(i)⇒(iv) In view of Proposition 7/(ii), we have

$$\overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z+\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z)=\left(\overline{D}_{JY}\overline{\mathcal{C}}'\right)(X,Z)-\mathcal{C}(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z)$$
$$+\,\overline{\mathcal{C}}'(X,\overline{F}\mathcal{C}(Y,Z))-\mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z))+\overline{\mathcal{C}}'(\overline{F}\mathcal{C}(X,Y),Z)\overset{(i)\Rightarrow(ii)}{=}0.$$

(iv)⇒(i) We get from (iv) by the substitution $X := \overline{S}$ the relation

$$0 = \overset{\circ}{\overline{\mathbb{P}}}(\overline{S}, Y)Z = -\left(\overline{D}_{\overline{S}}\mathcal{C}\right)(Y, Z) \overset{\text{Prop. } 5}{=} \overline{\mathcal{C}}'(Y, Z).$$

Hence, by Proposition 7/(ii),

$$\overline{\mathbb{P}}(X, Y)Z = 0.$$

(iii)⇒(i) Let $X := \overline{S}$ in (iii). Then

$$\left(\overline{D}_{\overline{S}}\mathcal{C}\right)(Y, Z) = \left(\overline{D}_{\overline{h}Z}\mathcal{C}\right)(\overline{S}, Y) = 0, \text{ and consequently } \overline{\mathcal{C}}' = 0.$$

By Corollary 7/(ii)–(iv), this means that the curvature tensor $\overline{\mathbb{P}}$ is totally symmetric. Since $\forall X, Y, Z \in \mathfrak{X}(TM)$:

$$\overline{\mathbb{P}}(X, Y)Z = \frac{1}{2}\Big(\overline{\mathbb{P}}(X + Z, Y)X + Z - \overline{\mathbb{P}}(X, Y)X - \overline{\mathbb{P}}(Z, Y)Z\Big)$$

and

$$-g(\overline{\mathbb{P}}(X, Y)X, JZ) \overset{\text{Cor. } 7/(\text{i})}{=} g(\overline{\mathbb{P}}(X, Y)Z, JX) = g(\overline{\mathbb{P}}(Z, Y)X, JX) \overset{\text{Cor. } 7/(\text{i})}{=} 0,$$

it follows that

$$\overline{\mathbb{P}} = 0.$$

(ii)⇒(i) From the assumption $\overline{\mathcal{C}}' = 0$ and Corollary 7/(iii), (iv) we get immediately that $\forall X, Y, Z \in \mathfrak{X}(TM)$:

$$\overline{\mathbb{P}}(X, Y)Z \overset{\text{Cor. } 7/(\text{iii})}{=} \overline{\mathbb{P}}(X, Z)Y \overset{\text{Cor. } 7/(\text{iv})}{=} \overline{\mathbb{P}}(Z, X)Y \overset{\text{Cor. } 7/(\text{iii})}{=} \overline{\mathbb{P}}(Z, Y)X,$$

i.e. $\overline{\mathbb{P}}$ is totally symmetric. So, repeating the preceding reasoning, we infer that $\overline{\mathbb{P}} = 0$. □

Theorem 3. Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). Then the following assertions are equivalent:
(i) (M, E) is a Wagner manifold (with respect to $(\overline{D}, \overline{h}, \alpha)$),
(ii) $\forall X, Y, Z \in \mathfrak{X}(TM) : \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z) = 0$.

Proof. (i)⇒(ii) We know from Proposition 8 that $\overline{\mathcal{C}}' = 0$ and so

$$\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z) \overset{\text{Prop. } 9}{=} -\overset{\circ}{\overline{\mathbb{P}}}(X, Y)Z \overset{\text{Th. } 2}{=} 0.$$

(ii)⇒(i) Our assumption $\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z) = 0$ implies by Proposition 5 that

$$0 = -\left(\overline{D}_{\overline{S}}\mathcal{C}\right)(Y, Z) = \overline{\mathcal{C}}'(Y, Z).$$

Applying Proposition 9, this yields the relation

$$\overset{\circ}{\overline{\mathbb{P}}}(X, Y)Z = -\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z),$$

therefore

$$\overset{\circ}{\overline{\mathbb{P}}}(X, Y)Z = 0,$$

so (M, E) is a Wagner manifold. □

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