THE SUMMARY OF THE PHD DISSERTATION

The Informal Background of Mathematics

BY

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I. THE PRECEDENTS AND THE AIM OF THE THESIS

My dissertation considers the philosophical problems of mathematical modeling. I have been interested in this topic for long years from both merely theoretical viewpoint (as a graduated I wrote a prize-winner paper on the philosophy of mathematics **[1]**) and because an important question is always brought up during my interdisciplinary research in Rational Choice Theory (in which I wrote another PhD Thesis at Eötvös Loránd University): what relevance is there between the mathematical models and the topic what we consider with the models in question?

Example 1 It was already recognized at the beginning of its foundation that Game Theory could be suitable to model economic and social situations. In his mathematical lecture on board games in 1926, the father of Game Theory, John von Neumann, emphasizes that "in many-player situation each player's lot depends on the actions of his partners", and in this case, the question is "how they have to play to get the best result they can […] hardly can imagine a situation in ordinary life where this problem is not relevant."[1]

Free-rider problem can be regarded as Prisoner's Dilemma from a mathematical point of view, which we often meet in everyday life. The specific logic of this situation is that though each actor is better off if they are ready to cooperate, they can get a higher payoff as a result of social defecting, and hence a free-riding problem can be expected. This is a scientific explanation for the phenomena of prisoner's dilemma in several areas of life including skulking behavior, corruption and pollution. But, what if we misidentify the preferences of the actors and thus we are modeling the situation in a wrong way? The Cubanian Missile Crisis in 1962 was considered as a Game of Chicken Dilemma by the experts requested by the Kennedy's administration, with the expectation from the Soviets to retreat, however the situation would have convert into a dilemma of the burglar caught out: "what if he guesses I want to shoot him and therefore he shoots me?". I presented an example like this in one of my publications [**6**], the conflict over Falkland Islands, which

[1]See Von Neumann, J. (1928): *Zur Theorie der Gesellschaftsspiele*. Mathematische Annalen, 100, p. 295-320.

results in a war between Great Brittan and Argentina in 1982, as a consequence of the misidentification of the situation by the Argentinean military junta.

The game-theoretical reasoning goes along a strict logical way by using “if… then” inferences. However, if they are applied for a social situation, we do and arrive here in a “backward” way. After estimating the possible strategies of players, we look for a set of games to different strategies, a set in which the possible strategies are equilibratory. This is called as *game engineering* in the literature of Game Theory. Game engineering cannot build on strict “if… then” reasoning, because if we try to find a modeling game to the strategies of the actors in the situation, this could include eventuality. So here, I’d ten times rather the form of reasoning was “if… then sometimes/it is possible/why not be…”.

Example 2 touches both the philosophy of time and the relation between mathematics and physics. As it is known, Aristotle associates the concept of time with the notion of change in his *Physics*. From this there occurs a metaphysical position that time can be measured by changing. To carry on this idea, Leibniz came to two important conclusions: i) motion and rest are relative; ii) space and time are inseparable. The other metaphysical position is the idea of absolute time (Galileo and Newton) and that time can exist in a world where there is no change (Shoemaker). So there exists absolute time but we cannot measure it. Newton argues against Lebniz as a representative of the other view that if all motions are relative, acceleration must also be relative, however one can physically observe the differences between accelerating and non-accelerating mechanical systems, i.e. there is *experimental* distinction between them. The argument against the relativity of motion cannot be replied by seventeenth century physics. The Theory of General Relativity was needed to answer Newton’s objection. Still the question cannot be cleared up. I present mathematical-philosophical arguments in the thesis that it is impossible to choose between the two metaphysical positions by experience, since there is no experience by which it is decidable where to draw a line between physics and the geometry of spacetime. As Newtonian physics could not beat down the Leibnizian metaphysical position, the same is true for the Einsteinian physics that it cannot shake the metaphysical position of Galileo and Newton.

To the more exhaustive discussion of the problem, we should investigate the relation between the informal mathematics with pre-analytic notions and the formal mathematics working with definitions and axioms. It has also an increasing significance in the “inner matters” of mathematics. The traditional paradigm of mathematical research and education is as follows: one defines some notions, he accepts some fundamental truth based upon them, which are called thematic (non-logical) axioms as Peano's axioms in arithmetic, then he claims a theorem and proves it by using certain logical rules. However, the expansion of mathematical research and results, the development of "empiric mathematics" induced by Wigner, Chaitin and Hamming (application and effectiveness), and Lakatos and Putman (informal aspects in proof), which is enabled by computers, the systematic considerations of conjectures and the engineering of algorithms open a new perspective. But it also means a challenge: when is a conjecture good? How can be reported and built them in mathematical art?

Famous conjectures as Fermat's conjecture[2], which became a theorem after 330 years, in 1995, mean great drifts in mathematical art. Paul Erdős, the famous Hungarian mathematician, was reputed by his several conjectures, and he considered them as a part of his mathematical work. The American mathematician, Siemion Fajtlowitz developed a computer program, called GRAFFITI, with which he could generate graph-theoretical conjectures; and one of them was turned out to be connected to a key issue of the Theory of Complexity.

We can treat mathematical objects, applied mathematical and logical methods in two different ways, which, for example, is manifested in the fact that several mathematical theorems have both existential and constructive proofs. The first one is shorter and more elegant many times, the other one serves additional information (e.g. an algorithm) to the theorem. These two programs, the traditional and the constructive program of mathematics, evoke another old philosophical problem. The Ancient Greeks recognized the crisis of mathematics in the parting of extension (measure) and numeral: they could not consider $\sqrt{2}$ as number, however they had to come up to it as measure (e.g. the diagonal of the unit square). How did they mean? If it is discrete and we have to count with it, it is unspeakable (*ἄρρητος*), but if it is a line, it is an existing (geometric) entity? In this way we arrive at the sensitive problem of the relationship between continuous and discrete mathematical structures, which went through the whole history of mathematics, and mathematical application could not completely get rid of it up to now.

Today a researcher of a certain scientific area has to work both kinds of mathematical structures. The traditional area of mathematics is physics where the applied mathematical

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[2] Wiles's theorem or Fermat's Last Theorem: no three positive integers, *a*, *b* and *c,* each differs from zero, can satisfy the equation $a^n + b^n = c^n$ for any integer value of *n* greater than two.

device is calculus of functions, which deals with continuous structures, but even a physicist meets discrete mathematical structures, e.g. in Quantum Field Theory. The success of linear programming in Economics and in Operational Research builds on the presumption that processes of production can be unrestrictedly divided and be described by convex functions; the consideration of indivisibility in decisions however leads to discrete programming and other combinatorial models.

This kind of problems brought up the analysis of the foundations of mathematics, mathematical philosophy as thematic philosophy, which formulates presumptions and theories about mathematical approach and entities, mathematical truth and proof, eventually about the expressive power and limits of mathematics, by basing upon philosophical considerations. The traditional paradigm mentioned above acts in the spirit of mathematical realism (PLATONISM), it assigns substantive existence to classic mathematical concepts independently of whether or not they are fancied, and by it, we can discover the truth of mathematical theorems merely by analyzing concepts in a logical way. The other paradigm is partly constructive dealing with the finite construction of mathematical entities, and is partly needed to adopt the ideas of Imre Lakatos's philosophy of mathematics.

The aimed topic of my thesis, the informal background of mathematics, can be considered as a reply to these traditional and current problems.

II. THE SKETCH OF APPLIED METHOD

The development of the thesis is essentially thematic: metamathematical, logical, set-theoretical, arithmetic and geometric topics are considered, which are joined to mathematically relevant questions whose historical background is discussed in Chapter 6, and the summary of the author's conception can be found here, too.

The view of Ancient Greeks essentially builds on geometry, though there was other alternative in that time (the Pythagoreans). They considered chaos as a world without geometry. Anaximander said that the world had come into being from the "unlimited" or "unbounded" (*ἄπειρον*) and its end is also going to be over there. Existence is that if matters get their form, they will have bound, and thus they will be able to be measured. In his *Elements* Euclid defines line as to be made up of points with no part; however, a line has a measure: its length. In order to become a line from points, there must be infinite points. But how is it possible to get something being from non-beings? A point with no part is nothing or something? And if it is nothing, how can it constitute something, the line. Straight lines constructing a square with an area in (positive) integer but no in square number, have no mutual measure with the numeral of the length (e.g. $\sqrt{2}$ or $\sqrt{5}$). Plato coped with this problem in the dialogues of *Theaetetus*, but its complete discussion can be found in the tenth book of Euclid's *Elements*. Greeks did not consider the "square irrationals" as numerals because they thought that they could not be measured in and be expressed by numbers.

We can claim the inverse of the problem before in a more general form. How do we grasp space (and time): whether can it be divisible without restraint or is it made up of indivisible little "atoms"? Zeno tried to point out that both assumptions leaded to contradictions. The problem of the infinite divisibility of space brings up the question of continuity. Both varicolored myths of creation of mankind and the different views of the world seem to share in one fact: they accept the postulate of the intensive infinity of the world, that is, their common presumption is the continuity of the physical world. And, in fact, it is really difficult to imagine a world in which there are "time gaps", though the possibility of this case cannot be *a priori* ruled out (Shoemaker). Up to the end of the nineteenth century mathematics was governed by a principle occurred in Aristotle's *Physics*: *infinitum actu non datur*. It proves to be reliable means to save our reasoning against the paradoxes arisen from the problems of this kind.

In connection with existence and continuity, there are three "archaic" problems of mathematical thinking: irrational quantities, infinite and the problem of self-reference. It was a serious difficulty for ancient mathematics to cope with irrational numbers, and for ancient philosophy to handle infinity. The previous one is called as unspeakable by Pythagoreans, the consequence of the latter one was that mathematics was governed by the Aristotelian principle „*infinitum actu non datur*". However, it was necessary to give up the taboo of actual infinite in mathematics, because the notion of "infinitely little quantity" was proved to be practically unmanageable. Calculus proposed by Newton and Leibniz is very effective device in science, but infinitesimal could be mathematically used only in objectionable fashion. Long efforts of outstanding mathematicians (Cauchy, Heine, Weierstrass) were needed to clear up the concept of limit and the continuity of functions, but it arrived at their final, exact foundation in Cantor's theory of infinite sets.

Although it was finally succeeded in overcoming the mess of infinitesimals by using the concepts of sets and actual infinite, difficulties remained on the agenda. In Set Theory the problem of irrational quantities welcomes back in Continuum Hypothesis, and the problem of infinite in Skolem's Paradox. If it is an undecidable question whether or not real numbers take their places on the number line without "gap", the basis cannot eventually be fixed on which the existence of the limit of regular sequences and the theorems about the continuum can be proved. The notion of infinity breaks down the intuitive sense of existence. One of the criteria of being may be the principle of coherence: all with no including inner contradictions exist. Another criterion based upon "naïve realism" is constructivism. The question of Skolem Paradox, i.e. how can fit a system of non-countable sets over a countable domain of a model, beats down both criteria.

A couple of years after the solution of Cantor, which brought back the actual infinite into mathematics, paradoxes also occurred in Set Theory, which cannot be exiled, just can be ruled out from mathematics. The paradox of the School of Megara presented the trouble of self-reference in the background of which we can find expressions that refer to themselves. Self-reference appears in Set Theory, too: whether the set of all sets is an element of itself, which are not elements of themselves? The obvious answer is it is element of itself if and only if it is not element of itself. The application of self-reference is also in the background of Gödel's incompleteness theorem in which he proved that mathematical theories could not be closed systems, as the foundations of such system cannot be provable in the system.

There is a famous claim in epistemology, which is called Munchhausen trilemma: any theory depends upon principles. Now, they are considered either (1) as true, though they are not founded; or (2) they build on other principles and these principles depend on further principles, *ad infinitum*; or (3) they are considered as founded in a holistic way via their consequences (so no in absolute fashion). Scientific evidences in the trilemma recall the story of the baron Munchhausen, the legendary viscount, who was able to pull him out of the bog in which he had got stuck. If one chooses the first option of the trilemma, we have the fidelistic theory of knowledge: apart from the fact that one is able to recognize that the theory in question is not built on demonstrative principle, one grasp them as stable and valid.

The conventional approach of mathematical true, the platonic and constructive schools of mathematical philosophy are considered as fidelistic theories. By the second option, since any theory can be objectionable and we have no consequent and transparent evidences for founding their principles, mathematician must be skeptic. Apart from the arisen difficulties, however, he could try to apply his subject to use up the results. This is the typical character of pragmatic approach to mathematical true, and formalists shared this opinion, too, until the activity of Gödel who proved that one must be skeptic also in that mathematics could be made instrumental in this fashion. The third option appreciates the arguments of the preceding two, but it presents a minimalist, axiological way of truth. We have this strategy with the Tarskian reasoning of correspondence and coherence theory discussed in Chapter 3.

Not being accessible to the truth, we tackle the semantic-pragmatic background of mathematics in no static, but in dynamic way by using Imre Lakatos`s “proof and refutation” method. Our starting point is an informal background from which concepts and proofs are generated. New theorems and theories are created, which are not stable and definitive, because neither do their concepts and proofs. The meaning of the concepts of the theory is sketched outside the language, in a cognitive manner, taking contents in application, and then influences on the informal background outside the language. And this there and back moving is going on and on. This is “living mathematics”. The “proof and refutation” method represents a position in epistemology whose central significant is to investigate mathematics in progress. There are no static concepts in mathematical theories, and mathematical discourse cannot coincide with a comprehensive view of the world or form of communication.

III. THE SUMMARY OF THE NEW RESULTS AND IDEAS

The investigation of “space” presented in Chapter 4 demonstrates the consequence of the above told well. “Space” is a type of structures in mathematics, which is characterized either by a system of surroundings or by its equivalent system of opened sets. We have a formal space structure called *topology* whose basic part, the point, is a non-defined notion in topology. To put it in another way, topological structure can be added to any set. Our physical space can be considered as topological structure, but not only the structure of this space, but also the set supporting this structure have empirical content, which can be considered as the existential presumptions of the concept of physical space. This concept of space can never be definitive, because the features of physical body in space (matter, position, time, motion) act and are in connection with one another in each characteristic of the world. The existential presumptions inputted in the axioms of geometry and the existential presumptions of physical theories explicitly expressed in postulates can influence on each other that there cannot be drawn an empirical distinction, and thus we have equivalent epistemic representations of the events of the world. However, the views of the world characterized by the existential presumptions of different physical theories are obviously incompatible. That's why physicists have to canvass the different interpretations, but it is possible only on criteria including external (metaphysical) elements from the viewpoint of theories. They are not evidences, but explaining facts from the informal background of the theory.

> **Theorem: The following conditions are inconsistent:**
> **(1) K is a consistent system of mathematical knowledge;**
> **(2) K is deductively closed for consequence relation;**
> **(3) K includes - arithmetic (first-order Peano axioms)**
> **- the rule of Excluded Middle;**
> **(4) K is a Gödel-coded system of claims (methodological principle of constructivism)**
> **(5) Positive introspectivity: the interpersonality of mathematical view (Lucas-Penrose argument)**

In chapter 5, we introduce a theorem that alloys certain ideas of Gödel, Tarski and Lakatos, and which suggests that several standpoints of mathematics such as mathematical entities (numerals, structures, infinite), mathematical and logical methods (e.g. proving methods), the basic hypothesis of computation (Church thesis) and even the adequacy of first-order logic, are, in Quine's words, ontological dependant. This means that certain standpoints cannot be hold under the "consistent mathematical knowledge" if a mathematician does not clear, at least as for the used mathematical and logical methods, that he is either a Platonist or a Constructivist. However, we present examples that our intuition leads to paradoxes as a result of this choice, and it is questionable whether definitions and axioms at the starting point can be considered as something to be true.

To resolve contradictions, under the "proof and refutation" method, we suggest a metaphysical change of attitude, changing for a metaphysically monist interpretation that better meets the requirements of mathematics to be permanently formed. A mathematical theory seems to be mental and lingual construction, which is bounded and ontological committed. That's why, the part of the world addressed by our theories will receive a human face, and the rest, in disguise of paradoxes, will remain a myth in our “approached” world.

This myth might be interpreted as Kantian *Ding an sich* [dualistic metaphysical interpretation]. However, it might be interpreted also in another way that it is a complex whose each part (*Gestalt*) is a certain relation to one another, and which is uprooted from the whole either by its concepts (Wittgeinstein) or by its forms (Spencer Brown) [monistic metaphysical interpretation]. In this idea “living mathematics” is a potential infinite, autopoietic system in which also reflects that occurred outside the system (e.g. “non-intended” model, paradoxes).

In dualist attitude mathematical intention is due to the spiritual activity of mathematician, therefore to tackle exceptions and antinomies, it applies “exception-expelling” method and restrictions in problem-solving and formalization. In monist attitude the use of the “proof and refutation” method is needed, and thus mathematician of this kind is always in the problem during the problem-solving. An axiomatic system is contingent for him: it does not express logical necessity, but it works as if it was something necessary. The formal systems of mathematics do not require existential presumptions, because if a formal system is adequately prepared, it automatically produces its own complex patterns. System cannot be supported nor refuted, since these are not only the conditions of proof, but also that of reasoning. However, just here there occur informal existential presumptions that the formal construction depend on, and they cannot be separated from the informal background including, in Putnam`s word, the “noumenal goods” of mathematics. Mathematician is in a dilemma of Platonism and Constructivism, because he is obliged to grasp the subject in a constructive way inside the theory and in a platonic way outside the theory, and paradoxes indicate the fact that he has crossed the theory’s threshold. Mathematics is a self-developing system characterized by content, form and liaisons, and it overgrows the imagination of any of its creators.

IV. THE AUTHOR'S PUBLICATIONS RELATED TO THE TOPIC OF THE THESIS

1. *The mathematical view of the world, 1997, p. 24. (*A prize-winner paper in the National Scientific Competition on Philosophy of Mathematics organized by Eötvös Loránd University, Budapest).

2. *Thought, Language, Science* (with a co-author). Gond, 1999/21-22, p. 143-170.

3. *A Game-Theoretical Characterization of the Liberal Social View*. Magyar Tudomány, 2005/7, p. 869-882.

4. *Analytic Social Theory (The Principles of Rational Choice and Game Theory)*. Magánkiadás, Budapest, 2006 (book, p. 356, ISBN 963 06 0315 2).

5. *Platonism or Intuitionism: An Impossibility Theorem*, 2009, p. 14 (A conference paper at *Cambridge Graduate Conference on the Philosophy of Logic and Mathematics*).

6. *Game-Theoretical Modeling of Political Conflicts*. Századvég, 2009/ 54, p. 27-48.

All the publications in the list, except **5**, are originally in Hungarian.