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Letter

Time evolution and thermal renormalization group flow in cosmology

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ABSTRACT

Time-evolution of the Universe as described by the Friedmann equation can be coupled to equations of motion of matter fields. Quantum effects may be incorporated to improve these classical equations of motion by the renormalization group (RG) running of their couplings. Since temporal and thermal evolutions are linked to each other, astrophysical and cosmological treatments based on zero-temperature RG methods require the extension to finite-temperatures. We propose and explore a modification of the usual finite-temperature RG approach by relating the temperature parameter to the running RG scale as $T \equiv k_T = \tau k$ (in natural units), where k_T is acting as a running cutoff for thermal fluctuations and the momentum k can be used for the quantum fluctuations. In this approach, the temperature of the expanding universe is related to the dimensionless quantity τ (and not to k_T). We show that by this choice dimensionless RG flow equations have no explicit k-dependence, as it is convenient. We also discuss how this modified thermal RG is used to handle high-energy divergences of the RG running of the cosmological constant and to "solve the triviality" of the ϕ^4 model by a thermal phase transition in terms of τ in $d = 4$ Euclidean dimensions.

1. Introduction

There is an increasing interest in astrophysics and cosmology to incorporate quantum effects by renormalization group (RG) running [1] such as in the asymptotically safe gravity approach [2], for a recent review see e.g. [3–5]. Application to cosmology [6–10] and to black-hole physics have been extensively discussed [11,12]. In these approaches, the running RG cutoff k is identified with a typical energy scale of the system [11] and the RG running is used to connect the physics of different energy scales and at different times. The RG running of the parameters (couplings) is then incorporated into the classical equations of motion, i.e., the Friedmann equations describing the time-evolution of a homogeneous and isotropic universe.

Indeed, particle physics and cosmology are intimately connected, however, in accelerators a limited number of colliding particles is taken into account, while the early Universe must be seen as a hot dense plasma. Thus, the zero temperature quantum field theory which is the usual theoretical framework for particle processes must be extended to finite temperatures for cosmological applications. Thermal field theory is a very well-developed framework for finite temperature applications [14]. However, when applied to cosmological problems one has to decide whether setting the temperature is to be equal to the running RG momentum scale or link it to a fixed momentum. The latter choice appears to be the correct one since the quantized theory must be obtained in the physical limit where the RG scale is sent to zero, nevertheless it has a serious drawback, as we are going to argue in the following. One finds an explicit RG scale dependence in the dimensionless RG flow equations, and therefore is no room for non-trivial fixed points. This is the problem we address in the present paper.

Several types of cutoff identifications [11] have been used in astrophysics and cosmology. By using units in which $c = \hbar = k_B = 1$, one can relate the RG scale k to the energy density, $k \propto \rho^{1/4}$, or to the temperature of the cosmic plasma, $k \propto T$ [7]. In the radiation dominated epoch one finds $\rho \propto T^4$, and thus these are identical to each other. In addition, the RG scale k can be connected to the time-dependent Hubble-parameter $H(t)$ which is compatible with the Bianchi identities, see e.g. [7]. Then one can introduce the Hubble (or expansion) time, $t_{\rm H} = H^{-1} = \sqrt{3/(8\pi G\rho)} \propto T^{-2}$, where G is the Newton constant and we used the relation $\rho \propto T^4$ holding in the radiation dominated epoch. Thus, the temporal and the thermal evolution of the Universe

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are linked to each other, confirming that astrophysical and cosmological applications of the RG approach may require the extension to finite-temperatures.

In this paper we decided to explore the finite temperature generalization of the RG equations written in Euclidean spacetime (say, in d dimensions) via the following modification for the integrals entering the action: $\int d^d x \to \int_0^\beta d\tilde{t} \int d^{d-1}x$, where $\beta = 1/T$ is the inverse temperature parameter and one sees that for $T = 0$ the full integration on d variables, including the imaginary time $\tilde{t}$, is restored. It is employed in many-body physics [13,14] and relates a quantum system in $d-1$ (spatial) dimensions at $T = 0$ and the corresponding classic system in d dimensions at finite temperature [15]. It generates the discretization of the imaginary time-integral in momentum space, i.e., summation on Matsubara frequencies [13,14].

To complete the finite-temperature RG description one has to relate the temperature parameter to a momentum scale. In the perturbative RG, the standard choice is $\mu = 2\pi T$ where μ is the RG scale. In the Wilsonian RG one has to take the limit $k \to 0$, and thus in [16–18] the temperature is linked to the ultraviolet (UV) initial value Λ of the running momentum cutoff by putting $T = \tau\Lambda$. More precisely, the couplings are defined at an arbitrary but fixed intermediate scale $k_\star = 2\pi T = 2\pi\tau\Lambda$ (so $T = \tau\Lambda$) and the zero-temperature flow equation is integrated from $k_\star$ up to the UV cutoff Λ. Then starting from these bare parameters one can follow the RG flow down from $k = \Lambda$ to $k = 0$ with the temperature T turned on. The physical quantities are then obtained at $k = 0$. However, this choice has a consequence that the explicit k-dependence cannot be removed from the dimensionless RG flow equations.

Finite temperature formalism requires an introduction of a “finite volume” for the (imaginary) time integral. However, if one follows the Wilsonian approach, this “finite volume” has to be rescaled in every blocking step, which means it has to be linked to the running momentum cutoff k. Since in the RG approach the infrared (IR) limit $k \to 0$ has to be taken, this “finite volume” cannot be considered as the (inverse) temperature, but it can rather be used as a running momentum cutoff: $T \equiv k_T$. Thus, we suggest and explore the following identification

$$T \equiv k_T = \tau k\,. \tag{1}$$

In this case, the temperature of the expanding Universe is related to the dimensionless quantity τ (and not to k_T) where k_T is the running cutoff for thermal fluctuations and k is for the quantum ones. In the Wilsonian approach fluctuations are taken into account by the successive elimination (integration) of degrees of freedom above these running cutoffs which are chosen to be different in order to highlight the different contributions of thermal and quantum fluctuations. For $k_T \sim k$ (i.e., for $\tau \sim 1$) one cannot distinguish between thermal and quantum fluctuations.

In this work we discuss the possibility of a thermal RG method with the special choice (1) $T = \tau k$, where we will argue that τ plays the role of the temperature of the cosmic plasma. We show that in this case the dimensionless RG flow equations have no explicit k-dependence. In addition, we show how this modified thermal RG can be used to handle UV divergences of the running of the cosmological constant, and to “solve the triviality”, i.e., to generate a thermal phase transition in terms of τ for the ϕ^4 theory in $d = 4$ Euclidean dimensions.

It is important to note that all computations of the present paper are performed in the Euclidean spacetime. In order to connect the results with cosmology and particle physics, one could speculate that there is the need for justification of the use of Euclidean metric. Indeed, there is an increasing interest in the literature to discuss whether an analytic continuation to Minkowski spacetime is possible in the framework of the Wilsonian RG method, see e.g., [45]. However, let us note that lattice calculations are also performed in the Euclidean spacetime with huge amount of phenomenological applications, which serve as a strong support for the use of Euclidean instead of the Minkowski spacetime.

2. Thermal RG equation

The RG concept plays an important role in the description of physical systems across different scales. The Wilsonian RG method has been constructed to perform the renormalization non-perturbatively [1]. In general, RG allows to treat the effects of fluctuations via the successive elimination of the degrees of freedom which lie above a running momentum cutoff k and generates the momentum-shell functional RG flow equation, i.e., the Wetterich equation [19]. At zero-temperature it is formulated in Euclidean spacetime and stands for the running effective action $\Gamma_k[\phi]$ with its Hessian $\Gamma_k^{(2)}[\phi]$ where k is the running momentum cutoff (i.e., the RG scale). The Wetterich equation [19] contains, an appropriately chosen regulator function R_k, which is introduced to decouple slow modes with low momenta while leaving high momentum modes unaffected. The regulator ensures that the effective action captures all relevant fluctuations.

In the finite-temperature formalism, momentum integrals of the zero-temperature RG equation [19] are modified by using Matsubara frequency summation

$$\int d^d p \to T \sum_{\omega_n} \int d^{d-1} p$$

which is a summation over discrete imaginary frequencies – for bosonic degrees of freedom: $\omega_n = 2\pi n T$. Typically, one assumes that the regulator function $R_k(p)$ of the zero-temperature RG method is independent of the Matsubara frequency, but otherwise the same as for the zero-temperature case, where a usual choice is the Litim regulator $R_k(p) = (k^2 - p^2)\Theta(k^2 - p^2)$ [20]. In the local potential approximation (LPA) when one does not consider any wave function renormalization and the couplings of the scaling potential $V_k(\phi)$ carry the RG scaling, the RG equation at finite temperature T with the (frequency-independent) Litim regulator is written as [16]:

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1}\, k^{d-1} T \sum_{n=-\infty}^{\infty} \frac{k^2}{k^2 + \omega_n^2 + \partial_\phi^2 V_k(\phi)}\,, \tag{2}$$

where k is the RG scale, $\alpha_d = \Omega_d/(2(2\pi)^d)$, and $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d-dimensional solid angle. The summation can be performed [16], which results in

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1}\, k^{d+1} \frac{\coth\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{2T}\right)}{2\sqrt{k^2 + \partial_\phi^2 V_k(\phi)}}\,, \tag{3}$$

where we used the identity $\coth(x/2) = \frac{\exp(x)+1}{\exp(x)-1}$. Eq. (3) has been derived in [16], where the authors performed a detailed analysis of the scalar polynomial field theory by the thermal functional RG method which was also discussed in [17] with the inclusion of pressure and in [18] by taking into account volume fluctuations too.

If one relates the temperature to the UV cutoff $T = \tau\Lambda$ the dimensionless flow equations obtained from (3) have an explicit k-dependence which is not the case for the zero-temperature RG equation [19]. Using the identification (1) we suggest the following thermal RG equation:

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1}\, k^{d+1} \frac{\coth\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{2\tau k}\right)}{2\sqrt{k^2 + \partial_\phi^2 V_k(\phi)}}\,. \tag{4}$$

The thermal RG (4) has a singularity structure identical to its zero-temperature counterpart and determined by $k^2 + \partial_\phi^2 V_k = 0$. In Appendix A we show that the dimensionless RG flow equations derived from (4) have no explicit k-dependence. This is not the case if T is fixed, i.e., not k-dependent. This implies that the choice (1) can lead to non-trivial fixed points, unlike the choice of a non-running temperature. This observation is one of the main results of the paper.

These results can be used, among other possible applications, to study the critical behavior: *(i)* around the Wilson-Fisher fixed point of the polynomial quantum field theory in lower dimensions; and *(ii)* around the Coleman fixed point of the periodic quantum field theory in $d = 2$ dimensions. Our goal here is to consider the ϕ^4 quantum field theory in $d = 4$ dimensions.

3. Inflationary cosmology

The origin and the precise mechanism of cosmic inflation is one of the most pressing questions in modern cosmology [21–25], for reviews see [26,27]. In its simplest form, a scalar field, called the inflation, is assumed to roll down slowly from a potential hill towards its minimum which explains the cosmic microwave background radiation (CMBR) anisotropy and the seeds of the large-scale structure. Particle physics can provide us with candidates for the inflaton field, however, the complete understanding of particle physics origin remains open problem.

The observation of the Higgs boson renewed research activity where the inflaton is associated with the SM Higgs field [28–32] introduces a large non-minimal coupling ξ between the Higgs boson and the Ricci curvature scalar [33–36]. In case of non-minimal coupling, the Einstein frame where slow-roll study is performed and the Jordan-frame where the RG flow is considered are related to each other via a non-linear transformation. RG transformations generate an additional, R^2 term. However, we do not consider here quantum effects for gravity, only for the scalar matter field, and thus we choose

$$S = \int d^4x\sqrt{-g}\left[\frac{m_p^2+\xi\phi^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

where $m_p = 1/\sqrt{8\pi G}$ is the Planck mass, ξ is the non-minimal coupling and ϕ is the scalar field where the field independent term is the cosmological constant:

$$V(\phi=0) \equiv m_p^2\Lambda_{\rm cosmo}, \qquad (5)$$

where $\Lambda_{\rm cosmo}$ denotes the cosmological constant and not the UV momentum cutoff. We assume an expanding homogeneous and isotropic Universe (with a flat curvature), and with the Friedmann–Lemaître–Robertson–Walker metric, $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})} = a^3$ where the time-dependence of the scalar factor $a(t)$ is given by the Friedmann equation coupled to the equation of motion of the scalar field. This scenario is the an "economical" one for the theoretical framework for slow-roll inflation where the usual form for the Higgs-like potential is

$$V_{\rm Higgs}(\phi) = \frac{g_4}{4}\left(\phi^2 - v^2\right)^2 \qquad (6)$$

which is the SM symmetry breaking potential where v is the vacuum expectation value (VeV) and g_4 is the quartic coupling (dimensionless in $d = 4$ dimensions). The slow-roll study of (6) has to confronted to observations, e.g., to most recent Planck data [38] on CMBR anisotropy. If one assumes a non-minimal coupling to gravity, i.e., $\xi \neq 0$, the potential has to be written in the Einstein frame to perform the slow-roll study. This transformation is non-linear and has a simplified form for large non-minimal coupling ($\xi \gg 1$): $\varphi \approx m_p\sqrt{\frac{3}{2}}\ln(F)$ and $U(\varphi) \equiv m_p^2 V(\phi)/F^2(\phi)$ where $F(\phi) = 1 + \xi\phi^2/m_p^2$. The Higgs-like potential (6) reads in the Einstein frame:

$$U_{\rm Higgs}(\varphi) = \frac{m_p^4 g_4}{4\xi^2}\left[1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\varphi}{m_p}\right)\right]^2, \qquad (7)$$

where the VeV is neglected because it is assumed to be small compared to the field [37]. The validity of the large ξ approximation is characterized by a critical value φ_c. Below this critical scale, Higgs inflation coincides, with the SM minimally coupled to gravity [37]. The large difference between the electroweak and the transition scales allows us to neglect the VeV [37], however it plays an important role in the electroweak phase transition.

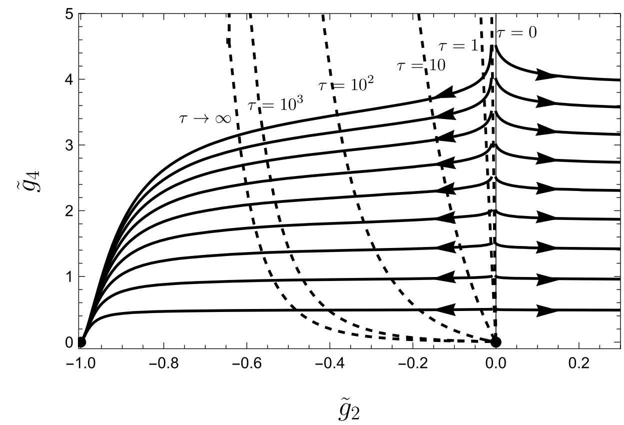

Fig. 1. Thermal RG flow of the ϕ^4 model in $d = 4$ dimensions. Dashed lines show how the separatrix changes with τ. The separatrix belongs to $\tau = 0$ (most right dashed line) almost overlaps with the vertical axis (thin solid line).

4. Thermal RG and triviality in cosmology

We apply the thermal RG equation (4) to the Higgs-like potential (6) in $d = 4$ dimensions by using

$$V_k(\phi) = \sum_{n=0}^{\rm NCUT}\frac{g_{2n,k}}{(2n)!}\phi^{2n} \to \tilde{V}_k(\tilde{\phi}) = \sum_{n=0}^{\rm NCUT}\frac{\tilde{g}_{2n,k}}{(2n)!}\tilde{\phi}^{2n},$$

where dimensionless quantities denoted by the tilde superscript. Thermal RG flow equations for the first three (dimensionful) couplings are given in Appendix A where we show that the dimensionless RG flow equations have no explicit k-dependence and discuss also the subtraction method [39] for the field-independent coupling $g_{0,k}$ which is related to the cosmological constant.

It is well-known that the Wilson-Fisher fixed point of the ϕ^4 model coincides with the Gaussian one in $d = 4$ dimensions [40]. Although the model has two phases, RG trajectories bifurcate from the vertical line at the vanishing mass (when $\tilde{g}_{2,k} = 0$), see solid lines on Fig. 1. The classical analysis gives exactly the same result: If the squared mass is negative the potential has a double-well structure which signals the broken phase where the Z_2 reflection symmetry is broken spontaneously. If the squared mass is positive the potential has a single minimum which is the symmetric phase. The numerical solution of the (dimensionless) thermal RG flow equations (see Appendix A) modifies the flow diagram. With non-vanishing value for τ, the RG trajectory which separates the phases is no longer a vertical line, see dashed lines on Fig. 1. For increasing values of τ, the broken phase is decreased and for $\tau \to \infty$ it is not possible to find any starting point in the vicinity of the Gaussian fixed point from which an RG trajectory can run into the broken phase. Since τ measures how thermal fluctuations are important compared to quantum fluctuations, one can say that by changing τ a thermal phase transition occurs. One can always determine a critical value τ_c for any starting point close to the Gaussian fixed point, see the black cross on Fig. 2. If $\tau < \tau_c$ the RG trajectory from that starting point runs into the broken (low-temperature) phase and for $\tau > \tau_c$ the RG trajectory ends up in the symmetric (high-temperature) phase. Thus, the RG flow diagram is no longer "trivial" and a thermal phase transition is observed in terms of τ. The temperature of the Universe at the scale of cosmic inflation is many magnitudes higher than the temperature taken at the electroweak scale. The VeV is zero above and non-vanishing below the electroweak transition temperature. Although the SM symmetric phase requires the use of an SU(2) doublet Higgs field, in a simplified scenario one can use the thermal phase transition of the 4-dimensional ϕ^4 model to switch between the broken and the symmetric phases. In other words, the zero

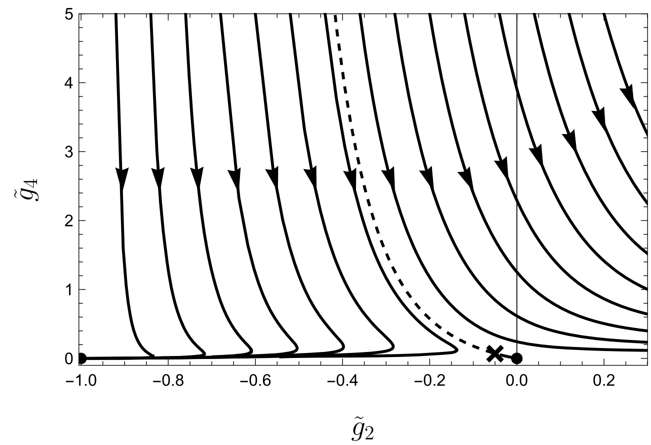

Fig. 2. Thermal RG flow of the ϕ^4 model in $d = 4$ dimensions with $\tau = 10^2$. Black cross denotes an initial condition which lies on the separatrix with $\tau = 10^2$. For $\tau < 10^2$ the RG trajectory from that starting point runs into the broken (low-temperature) phase and for $\tau > 10^2$ the RG trajectory ends up in the symmetric (high-temperature) phase.

or non-zero value of the VeV must distinguish between the low and the high temperature phases. Below the electroweak phase transition temperature, it is valid to use the SM symmetry breaking potential (6) and consequently the 4-dimensional ϕ^4 model. Time and the thermal evolution of the Universe are linked to each other, so, the couplings of the model are assumed to depend on the temperature which is encoded in their dependence on the parameter τ. So, for an arbitrary but fixed starting point taken from the vicinity of the Gaussian fixed point, see again the black cross on Fig. 2, one can determine a critical value τ_c which can be associated to the critical temperature of the electroweak phase transition. The temperature at the scale of inflation is many magnitudes higher, $\tau \gg \tau_c$, thus the RG trajectory from that starting point runs into the symmetric (high-temperature) phase and consequently a vanishing VeV must be chosen in Eq. (6) which supports the use of Eq. (7) in the slow-roll study of Higgs inflation.

5. Conclusions

A modification of the usual finite-temperature renormalization group (RG) approach is proposed by relating the temperature parameter to the running RG scale: $T \equiv k_T = \tau k$. This choice is convenient since with it the dimensionless RG equation has no explicit k-dependence, see Appendix A, and this is crucial to have fixed points. We applied this thermal RG to handle UV divergences in the RG running of the cosmological constant, see Appendix A, and to solve the triviality ϕ^4 theory in $d = 4$. These results suggest to associate the parameter τ with the temperature of the expanding universe, so $g_2 = g_2(\tau)$ and $g_4 = g_4(\tau)$ are assumed. The quantum effects are taken into account by the thermal RG running of the couplings at a given temperature, i.e., one has to take the limit $k \to 0$ by moving along the thermal RG trajectory with fixed value of τ and with given initial conditions. Then τ can be decreased and the thermal RG procedure repeated, producing the quantum effective action at a given temperature. Time and thermal evolutions of the Universe are linked to each other, so the change in τ connects the scales of the cosmic inflation and the electroweak phase transition and the thermal phase transition of the ϕ^4 theory validates the vanishing VeV in Eq. (6) when the slow-roll study of Higgs inflation is performed by Eq. (7).

We observe that since perturbative and non-perturbative RG equations are related to each other [41–44], one expects that our modified thermal RG method with $T \equiv k_T = \tau k$ finds application in particle cosmology both in perturbative and non-perturbative approaches. However, let us note that it is a valid question to ask whether the Wilsonian RG running scale parameter k can or cannot be identified with physical properties of the system. In order to compute observables using the effective action one has to take the physical limit $k \to 0$ which is in general not yet possible in reasonable approximations. Although, there are RG studies of cosmological problems where the Wilsonian flow with the scale "k" is used as an approximation to the physical flow with respect to the physical momentum "p", in this letter we rely on the effective action taken in the physical limit, i.e., $k \to 0$ from where physical quantities can be extracted. In other words, we do not consider the k-flow as an approximation to catch the main physical properties of the system at a given range of energy.

Finally, let us mention that an interesting direction of research would be to discuss $T \to \infty$ considering quantum gravity effects that should become relevant at the Planck scale, in order to understand how does the inclusion of gravity affects the proposed scheme. It would be also useful to apply the new thermal RG approach proposed here with the introduction of the dimensionless temperature τ to investigate in this context the QPT-CPT phase diagram of quantum field theories [46], such as the ϕ^4 and the sine-Gordon models in lower dimensions. The so called QPT-CPT diagram represents graphically the interplay between the classical phase transitions (CPT) and the quantum phase transitions (QPT) which has also been investigated in connection to the Naturalness/Hierarchy problem [47–50]. For example, in [49,50] it was shown that the hierarchy problem as well as the metastability of the electroweak vacuum can be understood as the Higgs potential being near-critical, i.e., close to a QPT.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Thermal RG flow equations of the ϕ^4 model in $d = 4$ Euclidean dimensions

By using $T \equiv k_T = \tau k$ the following thermal RG equation can be written, see also (4)

$$k\partial_k V_k(\phi) = \frac{2\alpha_{d-1}}{d-1}\, k^{d+1} \frac{\coth\left(\frac{\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}{2\tau k}\right)}{2\sqrt{k^2+\partial_\phi^2 V_k(\phi)}}. \tag{A.1}$$

It can be rewritten for the dimensionless potential $\tilde{V}_k(\tilde{\phi})$ with dimensionless field variable $\tilde{\phi}$ which are related to their dimensionful counterparts as $\phi = k^{(d-2)/2}\tilde{\phi}$, $V_k = k^d \tilde{V}_k$ with $\partial_\phi^2 V_k(\phi) = k^2 \partial_{\tilde{\phi}}^2 \tilde{V}_k(\tilde{\phi})$. The resulting thermal RG equation for dimensionless quantities reads as,

$$\left(d - \frac{(d-2)}{2}\tilde{\phi}\partial_{\tilde{\phi}} + k\partial_k\right)\tilde{V}_k(\tilde{\phi}) = \frac{2\alpha_{d-1}}{d-1} \frac{\coth\left(\frac{\sqrt{1+\partial_{\tilde{\phi}}^2\tilde{V}_k(\tilde{\phi})}}{2\tau}\right)}{2\sqrt{1+\partial_{\tilde{\phi}}^2\tilde{V}_k(\tilde{\phi})}}, \tag{A.2}$$

which has no explicit k-dependence. However, if one takes the usual choice and relates the temperature parameter to a fixed momentum scale, i.e., $T = \tau\Lambda$ one finds a dimensionless thermal RG equation with explicit k-dependence,

$$\left(d - \frac{(d-2)}{2}\tilde{\phi}\partial_{\tilde{\phi}} + k\partial_k\right)\tilde{V}_k(\tilde{\phi}) = \frac{2\alpha_{d-1}}{d-1}\frac{\coth\left(\frac{k}{\Lambda}\frac{\sqrt{1+\partial^2_{\tilde{\phi}}\tilde{V}_k(\tilde{\phi})}}{2\tau}\right)}{2\sqrt{1+\partial^2_{\tilde{\phi}}\tilde{V}_k(\tilde{\phi})}}, \tag{A.3}$$

which makes no room for non-trivial fixed point solutions, i.e., by setting $k\partial_k\tilde{V}_k \equiv 0$, the remaining algebraic equation can have only k-dependent solution.

Eq. (A.1) can be applied to a Higgs-like potential, in $d = 4$ dimensions by using a general scale-dependent polynomial potential:

$$V_k(\phi) = \sum_{n=0}^{\mathrm{NCUT}} \frac{g_{2n,k}}{(2n)!}\phi^{2n} \to \tilde{V}_k(\tilde{\phi}) = \sum_{n=0}^{\mathrm{NCUT}} \frac{\tilde{g}_{2n,k}}{(2n)!}\tilde{\phi}^{2n},$$

where dimensionless quantities denoted by the tilde superscript. Thermal RG flow equations for the first three (dimensionful) couplings, i.e., for NCUT = 2 are the following in $d = 4$ dimensions:

$$k\partial_k g_{0,k} = \frac{1}{6\pi^2}k^5\frac{\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{2\sqrt{k^2+g_{2,k}}}, \tag{A.4}$$

$$k\partial_k g_{2,k} = \frac{1}{6\pi^2}k^5\left[-\frac{g_{4,k}\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{4\left(k^2+g_{2,k}\right)^{3/2}} - \frac{g_{4,k}\,\mathrm{csch}^2\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{8\tau k\left(k^2+g_{2,k}\right)}\right], \tag{A.5}$$

$$k\partial_k g_{4,k} = \frac{1}{6\pi^2}k^5\left[\frac{9g^2_{4,k}\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{8(k^2+g_{2,k})^{5/2}} + \frac{9g^2_{4,k}\mathrm{csch}^2\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{16\tau k(k^2+g_{2,k})^2} + \frac{3g^2_{4,k}\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)\mathrm{csch}^2\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{16\tau^2k^2(k^2+g_{2,k})^{3/2}}\right] \tag{A.6}$$

where g_0 is the field independent term. For dimensions $d = 1$ this is associated with the ground-state energy, for $d = 4$ it plays the role of the cosmological constant.

To perform the so called subtraction method [39] for the finite temperature RG approach one has to consider the RG flow equation of the field-independent term in the UV limit ($k^2 \gg g_{2,k}$), i.e., the Taylor expansion of (A.4) with respect to $g_{2,k}$ around zero:

$$k\partial_k g_{0,k} \approx \frac{k^4}{12\pi^2}\coth\frac{1}{2\tau} + \frac{k^2}{24\pi^2}\frac{1+\tau\sinh\frac{1}{\tau}}{\tau - \tau\cosh\frac{1}{\tau}}g_{2,k} + \ldots,$$

where the first (second) term has a k^4 (k^2) divergence, so they have to be subtracted in order to restore the Gaussian fixed point for the dimensionless couplings. The correct form of the thermal RG flow equation for $g_{0,k}$ is

$$k\partial_k g_{0,k} = \frac{1}{6\pi^2}k^5\frac{\coth\left(\frac{\sqrt{k^2+g_{2,k}}}{2\tau k}\right)}{2\sqrt{k^2+g_{2,k}}} - \frac{k^4}{12\pi^2}\coth\frac{1}{2\tau} - \frac{k^2}{24\pi^2}\frac{1+\tau\sinh\frac{1}{\tau}}{\tau-\tau\cosh\frac{1}{\tau}}g_{2,k}, \tag{A.7}$$

where the subtracted terms do not violate the IR behavior since they vanish for $k \to 0$. The RG flow for the cosmological constant can be derived from (A.7) based on the relation

$$V(\phi = 0) \equiv m_p^2\Lambda_{\mathrm{cosmo}}, \tag{A.8}$$

where Λ_{cosmo} denotes the cosmological constant and not the UV momentum cutoff, which gives

$$k\partial_k\lambda_k = 8\pi\left(g_k k\partial_k\tilde{g}_{0,k} + \tilde{g}_{0,k}k\partial_k g_k\right) \tag{A.9}$$

with $\lambda_k = \Lambda_{\mathrm{cosmo},k}k^{-2}$, $g_k = Gk^2$ is the dimensionful Newton constant, and G is scale-dependent due to the absence of quantum gravity, so $k\partial_k g_k = 2g_k$. By using dimensionless couplings, $g_{0,k} = \tilde{g}_{0,k}k^4$ and $g_{2,k} = \tilde{g}_{2,k}k^2$ the flow equation for the field independent term is substituted into (A.9) and the thermal RG flow equation for the cosmological constant is obtained.

As a final step, let us rewrite the flow equations for dimensionless couplings $g_{0,k} = \tilde{g}_{0,k}k^4$, $g_{2,k} = \tilde{g}_{2,k}k^2$ and $g_{4,k} = \tilde{g}_{4,k}$ by using the subtracted form (A.7). In this case one finds,

$$(4 + k\partial_k)\tilde{g}_{0,k} = \frac{1}{6\pi^2}\frac{\coth\left(\frac{\sqrt{1+\tilde{g}_{2,k}}}{2\tau}\right)}{2\sqrt{1+\tilde{g}_{2,k}}} - \frac{1}{12\pi^2}\coth\frac{1}{2\tau} - \frac{1}{24\pi^2}\frac{1+\tau\sinh\frac{1}{\tau}}{\tau-\tau\cosh\frac{1}{\tau}}\tilde{g}_{2,k}, \tag{A.10}$$

$$(2 + k\partial_k)\tilde{g}_{2,k} = \frac{1}{6\pi^2}\left[-\frac{\tilde{g}_{4,k}\coth\left(\frac{\sqrt{1+\tilde{g}_{2,k}}}{2\tau}\right)}{4\left(1+\tilde{g}_{2,k}\right)^{3/2}} - \frac{\tilde{g}_{4,k}\,\mathrm{csch}^2\left(\frac{\sqrt{1+\tilde{g}_{2,k}}}{2\tau}\right)}{8\tau\left(1+\tilde{g}_{2,k}\right)}\right], \tag{A.11}$$

$$k\partial_k\tilde{g}_{4,k} = \frac{1}{6\pi^2}\left[\frac{9\tilde{g}^2_{4,k}\coth\left(\frac{\sqrt{1+\tilde{g}_{2,k}}}{2\tau}\right)}{8(1+\tilde{g}_{2,k})^{5/2}} + \frac{9\tilde{g}^2_{4,k}\mathrm{csch}^2\left(\frac{\sqrt{1+\tilde{g}_{2,k}}}{2\tau}\right)}{16\tau(1+\tilde{g}_{2,k})^2} + \frac{3\tilde{g}^2_{4,k}\coth\left(\frac{\sqrt{1+\tilde{g}_{2,k}}}{2\tau}\right)\mathrm{csch}^2\left(\frac{\sqrt{1+\tilde{g}_{2,k}}}{2\tau}\right)}{16\tau^2(1+\tilde{g}_{2,k})^{3/2}}\right]. \tag{A.12}$$

Thus, we demonstrated that by using our proposal $T \equiv k_T = \tau k$ the dimensionless thermal RG flow equations have no explicit RG scale-dependence, so one can find non-trivial fixed points. This is not the case if T is fixed, not k-dependent. Therefore, with $T = \tau k$ the usual RG flow diagram method can be used to study the critical behavior.

We finally observe that perturbative and non-perturbative RG equations are related to each other [41–44], thus, one expects that our modified thermal RG method with $T \equiv k_T = \tau k$ finds application in particle cosmology both in perturbative and non-perturbative approaches.

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