Short thesis for the degree of
Doctor of Philosophy (PhD)

Conditional and Quantitative Strong Laws of Large Numbers

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Debrecen, 2026

Introduction

The laws of large numbers (LLN) form one of the most fundamental pillars of probability theory and mathematical statistics. They formalize the intuitive principle that empirical averages of repeated random experiments stabilize around their expected values when the number of observations increases. Beyond their intrinsic theoretical importance, LLNs provide the mathematical foundation of statistical inference, stochastic modeling, information theory, Monte Carlo methods, econometrics, and modern data sciences.

Historically, the theory evolved from early studies of Bernoulli trials into a broad and sophisticated framework involving different modes of convergence, dependence structures, infinite-dimensional settings, random fields, conditional frameworks, and quantitative convergence rates. These developments reflect the growing complexity of modern stochastic models, where observations may exhibit dependence, multidimensional indexing, or partial information.

The classical form of the Strong Law of Large Numbers (SLLN) was established by Kolmogorov. It states that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $\mathbb{E}|X_1| < \infty$, then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow \mathbb{E}X_1 \quad \text{almost surely.}$$

A major research direction has been to determine the weakest dependence assumptions under which a strong law still holds. One important example is the case of pairwise independence, where any two variables are independent although the sequence as a whole may not be mutually independent. Etemadi [3] showed that if $\{X_n, n \geq 1\}$ is a sequence of pairwise independent and identically distributed random variables with $\mathbb{E}|X_1| < \infty$, then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow \mathbb{E}X_1 \quad \text{almost surely.}$$

Later, Csörgő, Tandori and Totik [2] extended these results to non-identically distributed sequences while maintaining pairwise independence. Other extensions include conditioning, mult-indexing, vector valued random variables and non-additive framework.

The main contributions of this thesis can be summarized as follows. First, the thesis develops a general approach to conditional strong laws of large numbers. Significantly, this method does not impose any restriction on underlying dependence structure of the random variables. Second, the work contributes to the quantitative theory of strong laws for multi-indexed random fields. In Chapter 2, the result of Neri [11] is extended to random variable with double indices. This method yields explicit probability bounds and quantitative SLLNs for pairwise independent and quasi-uncorrelated random fields.

Finally, the dissertation extends strong laws of large numbers to nonlinear probabilistic frameworks based on sub-additive probabilities and sublinear expectations. In particular, conditional strong laws are established without classical independence assumptions, and convergence results are proved for φ-sub-Gaussian random variables under sublinear expectations using exponential-type tail control.

Chapter 1

A General Approach to Conditional Strong Laws of Large Numbers

This chapter is based on our article [6]. It develops a general method for establishing conditional strong laws of large numbers (SLLNs) by means of conditional expectation and conditional probability inequalities. It is shown that a conditional Kolmogorov type inequality implies a conditional Hájek-Rényi type inequality and this implies strong law of large numbers. This method does impose any restriction on the underlying dependence structure of random variables.

1.1 Main Results

First we present our results on conditional moments.

Theorem 1.1.1 (Fazekas & Masasila [6]). *Let $\{X_k, 1 \leq k \leq n\}$ be a sequence of random variables, let $S_k = X_1 + \cdots + X_k$. Let $\mathcal{F}$ be a σ-subalgebra, $\alpha_1, \ldots, \alpha_n$ be nonnegative $\mathcal{F}$-measurable random variables, $r > 0$ real number. Assume that the general conditional Kolmogorov's type inequality is true, that is*

$$\mathbb{E}\left(\left[\max_{1\leq l\leq m} |S_l|\right]^r |\mathcal{F}\right) \leq \sum_{l=1}^{m} \alpha_l \qquad \textit{for all} \quad 1 \leq m \leq n. \tag{1.1.1}$$

Then the conditional Hájek-Rényi inequality is true, that is

$$\mathbb{E}\left(\left[\max_{1\le l\le n}\left|\frac{S_l}{\beta_l}\right|\right]^r|\mathcal{F}\right)\le 4\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r} \tag{1.1.2}$$

for $\mathcal{F}$-measurable random variables $\beta_1 \le \beta_2 \le \ldots \le \beta_n$ with $\beta_1 \ge \beta_0$, where β_0 is a positive constant.

Theorem 1.1.2 (Fazekas and Masasila [6]). *Let $\{X_n, n \ge 1\}$ be random variables, $S_n = X_1 + \cdots + X_n$ for any n. Let $b_0 \le b_1 \le b_2 \le \ldots$ be $\mathcal{F}$-measurable random variables with $b_n \longrightarrow \infty$ a.s., where b_0 is a positive constant. Let $\alpha_1, \alpha_2, \ldots$ be nonnegative $\mathcal{F}$-measurable random variables. Let $r > 0$ be a fixed number. Assume that for any $n \ge 1$*

$$\mathbb{E}\left(\left[\max_{1\le l\le n}|S_l|\right]^r|\mathcal{F}\right)\le \sum_{l=1}^{n}\alpha_l. \tag{1.1.3}$$

If $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ *a.s., then*

$$\lim_{n\longrightarrow\infty}\frac{S_n}{b_n}=0 \qquad a.s. \tag{1.1.4}$$

The following results are based on conditional probabilities.

Theorem 1.1.3 (Fazekas and Masasila [6]). *Let $\{X_k, 1 \le k \le n\}$ be random variables, $S_k = X_1+\cdots+X_k$. Let $\mathcal{F}$ be a σ-subalgebra. Let r be a positive real number. Let $\beta_1 \le \beta_2 \le \cdots \le \beta_n$ be $\mathcal{F}$-measurable, $\alpha_1, \ldots, \alpha_n$ nonnegative $\mathcal{F}$-measurable random variables. Assume that $\beta_1 \ge \beta_0 > 0$, where β_0 is non random. If*

$$\mathbb{P}\left(\max_{1\le l\le m}|S_l|\ge\varepsilon|\mathcal{F}\right)\le\frac{1}{\varepsilon^r}\sum_{l=1}^{m}\alpha_l \qquad \textit{for all} \quad 1\le m\le n \tag{1.1.5}$$

and for all $\varepsilon > 0$, then

$$\mathbb{P}\left(\max_{1\le l\le n}\left|\frac{S_l}{\beta_l}\right|\ge\varepsilon|\mathcal{F}\right)\le\frac{4}{\varepsilon^r}\sum_{k=1}^{n}\frac{\alpha_k}{\beta_k^r} \tag{1.1.6}$$

for all $\varepsilon > 0$.

Theorem 1.1.4 (Fazekas and Masasila [6]). *Let $\{X_n, n \ge 1\}$ be random variables, $S_k = X_1 + \cdots + X_k$. Let $\mathcal{F}$ be a σ-subalgebra. Let $b_0 \le b_1 \le b_2 \ldots$ be $\mathcal{F}$-measurable random variables with $b_n \longrightarrow \infty$ a.s., where b_0 is a positive constant.*

Let $\alpha_1, \alpha_2, \ldots$ *be nonnegative* $\mathcal{F}$*-measurable random variables. Let* $r > 0$ *be a fixed number. Assume that for any* $n \geq 1$

$$\mathbb{P}\left(\max_{1\leq l\leq n} |S_l| \geq \varepsilon|\mathcal{F}\right) \leq \frac{1}{\varepsilon^r}\sum_{l=1}^{n}\alpha_l \qquad \textit{for all} \quad \varepsilon > 0. \tag{1.1.7}$$

If $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ *a.s., then*

$$\lim_{n\longrightarrow\infty}\frac{S_n}{b_n} = 0 \qquad \textit{a.s.} \tag{1.1.8}$$

1.2 Applications

Theorem 1.2.1 (Majerek, Nowak & Zięba [8]). *Let* $\{X_n, n \geq 1\}$ *be a sequence of* $\mathcal{F}$*-independent random variables such that* $\sum_{k=1}^{\infty}\frac{\sigma_{\mathcal{F}}^2 X_k}{k^2} < \infty$ *a.s. Let* $S_n = X_1 + \cdots + X_n, n = 1, 2, \ldots$ *Then*

$$\lim_{n\longrightarrow\infty}\frac{S_n - \mathbb{E}(S_n|\mathcal{F})}{n} = 0 \qquad \textit{a.s.} \tag{1.2.1}$$

Theorem 1.2.2 (Prakasa Rao [12]). *If* $\{X_n, n \geq 1\}$ *is a sequence of* $\mathcal{F}$*-independent random variables such that*

$$\sum_{n=1}^{\infty}\frac{\mathbb{E}\left(|X_n - \mathbb{E}(X_n|\mathcal{F})|^{2r}|\mathcal{F}\right)}{n^{r+1}} < \infty \quad \textit{a.s.,} \tag{1.2.2}$$

for some $r \geq 1$*. Then,*

$$\frac{S_n - \mathbb{E}(S_n|\mathcal{F})}{n} \longrightarrow 0 \quad \textit{a.s. as} \quad n \longrightarrow \infty. \tag{1.2.3}$$

Chapter 2

Quantitative Strong Laws of Large Numbers for Random Variables with Double Indices

Chapter 2 is based on our new results obtained in [7]. The main objective of this chapter is to obtain an explicit rate of convergence in the strong law of large numbers for double-indexed families of random variables, using proof-mining techniques as employed by Neri in [11]. We show that a rate of convergence along a suitable subsequence yields a rate of convergence for the entire sequence. This result is then applied to derive rates of convergence in the strong law of large numbers for double-indexed families of random variables that are pairwise independent, quasi-uncorrelated, and negatively dependent.

2.1 General results

We consider sequences of random variables with double indices. $\boldsymbol{n} = (n_1, n_2) \in \mathbb{N}^2$ will denote the indices. Here $\mathbb{N}$ denotes the set of positive integers. We will denote by $\mathbb{N}_0$ the set of non-negative integers. We say that the double sequence of random variables $\eta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, converges almost surely to η, if $\eta_{n_1,n_2}(\omega) \longrightarrow \eta(\omega)$ as $n_1, n_2 \longrightarrow \infty$ for $\omega \in A$, $\mathbb{P}(A) = 1$. We say that the double sequence of random variables $\eta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2$, converges almost surely strongly to η, if $\eta_{n_1,n_2}(\omega) \longrightarrow \eta(\omega)$ as $\max\{n_1, n_2\} \longrightarrow \infty$ for $\omega \in A$, $\mathbb{P}(A) = 1$.

Let $\alpha > 1$ and let $\boldsymbol{p} = (p_1, p_2) \in \mathbb{N}_0^2$ be fixed. Define the following set of indices

$$C_{\alpha,\boldsymbol{p}} = \{\boldsymbol{n} : \boldsymbol{n} \in \mathbb{N}^2, \alpha^{p_1} \le n_1 < \alpha^{p_1+1}, \alpha^{p_2} \le n_2 < \alpha^{p_2+1}\}. \tag{2.1.1}$$

These sets are (possibly empty) rectangles of integer lattice points.

Let $\boldsymbol{k}^-(\boldsymbol{p}) = \min C_{\alpha,\boldsymbol{p}}$ and $\boldsymbol{k}^+(\boldsymbol{p}) = \max C_{\alpha,\boldsymbol{p}}$, if $C_{\alpha,\boldsymbol{p}} \neq \emptyset$, where min and max is defined coordinate-wise. Then $\boldsymbol{k}^-(\boldsymbol{p}) \le \boldsymbol{n} \le \boldsymbol{k}^+(\boldsymbol{p})$ for any $\boldsymbol{n} \in C_{\alpha,\boldsymbol{p}}$ ($\le$ is defined coordinate-wise). When $C_{\alpha,\boldsymbol{p}} = \emptyset$, let $\boldsymbol{k}^-(\boldsymbol{p}) = \boldsymbol{k}^+(\boldsymbol{p}) = \boldsymbol{0} = (0,0)$. We shall use notation $\boldsymbol{k}^\pm(\boldsymbol{p})$ if a relation is true both for $\boldsymbol{k}^-(\boldsymbol{p})$ and $\boldsymbol{k}^+(\boldsymbol{p})$.

Let $|\boldsymbol{n}| = n_1 \cdot n_2$ if $\boldsymbol{n} = (n_1, n_2)$.

Proposition 2.1.1 (Fazekas & Masasila [7]). Let $\{\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2\}$, be non-negative random variables, and let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \le \boldsymbol{n}} \xi_{\boldsymbol{k}}$, $\mathbb{E}\xi_{\boldsymbol{n}} = \mu$ for all $\boldsymbol{n} \in \mathbb{N}^2$. If for each $\alpha > 1$,

$$\sum_{\boldsymbol{p} \in \mathbb{N}_0^2, \boldsymbol{k}^\pm(\boldsymbol{p}) \neq \boldsymbol{0}} \mathbb{P}\left(\left| \frac{S_{\boldsymbol{k}^\pm(\boldsymbol{p})}}{|\boldsymbol{k}^\pm(\boldsymbol{p})|} - \mu \right| > \varepsilon \right) < \infty, \quad \text{for all} \quad \varepsilon > 0,$$

then

$$\frac{S_{\boldsymbol{n}}}{|\boldsymbol{n}|} \to \mu \quad \text{a.s. strongly as} \quad \boldsymbol{n} \to \infty. \tag{2.1.2}$$

Proposition 2.1.2 (Fazekas & Masasila [7]). Let $\{\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2\}$, be non-negative random variables, and let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \le \boldsymbol{n}} \xi_{\boldsymbol{k}}$, $\mathbb{E}\xi_{\boldsymbol{n}} = \mu$ for all $\boldsymbol{n}$. Assume that for any $\varepsilon > 0$ and any $\alpha > 1$

$$\sum_{\boldsymbol{p} \ge \boldsymbol{l}, \boldsymbol{k}^\pm(\boldsymbol{p}) \neq \boldsymbol{0}} \mathbb{P}\left(\left| \frac{S_{\boldsymbol{k}^\pm(\boldsymbol{p})}}{|\boldsymbol{k}^\pm(\boldsymbol{p})|} - \mu \right| > \varepsilon \right) \le \lambda \quad \text{if} \quad \boldsymbol{l} \ge \boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda), \tag{2.1.3}$$

where $\{\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda) \ : \ \lambda > 0\}$ is a rate. Then

$$\mathbb{P}\left(\sup_{\boldsymbol{m} \ge \boldsymbol{n}} \left| \frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|} - \mu \right| > \varepsilon \right) \le \lambda \quad \text{if} \quad \boldsymbol{n} \ge \boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda},\alpha}(\lambda),$$

where $\boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda},\alpha}(\lambda) = \alpha^{\lceil \boldsymbol{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda) \rceil}$ is the rate of convergence of $\frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|} \to \mu$ a.s., and $\alpha^2 = \frac{\varepsilon}{2\mu} + 1$. Here $\lceil x \rceil$ denotes the smallest integer being not smaller than x and $\alpha^{\boldsymbol{\Gamma}} = (\alpha^{\Gamma_1}, \alpha^{\Gamma_2})$ if $\boldsymbol{\Gamma} = (\Gamma_1, \Gamma_2)$.

2.2 Pairwise independent random variables

Lemma 2.2.1 (Fazekas & Masasila [7]). Let $\{\xi_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{N}^2\}$, be pairwise independent random variables with $\mathbb{E}\xi_{\boldsymbol{k}} = \mu$ and $\mathrm{Var}\xi_{\boldsymbol{k}} \le \sigma^2$ for all $\boldsymbol{k} \in \mathbb{N}^2$, $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \le \boldsymbol{n}} \xi_{\boldsymbol{k}}$. Let $\alpha > 1$. Then

$$\sum_{\boldsymbol{n} \ge \boldsymbol{p}, \boldsymbol{k}^{\pm}(\boldsymbol{n}) \ne \boldsymbol{0}} \mathbb{P}\left(\left| \frac{S_{\boldsymbol{k}^{\pm}(\boldsymbol{n})}}{|\boldsymbol{k}^{\pm}(\boldsymbol{n})|} - \mu \right| > \varepsilon \right) \le \lambda, \tag{2.2.1}$$

if $p_1 + p_2 \ge \log_\alpha(\frac{\sigma^2\alpha^2}{\varepsilon^2\lambda(\alpha-1)^2}) = \rho_{\varepsilon,\alpha}(\lambda)$, that is the rate of convergence is given by $\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda)$, where $\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda)$ is determined by the curve $x + y = \rho_{\varepsilon,\alpha}(\lambda)$.

Lemma 2.2.2 (Fazekas & Masasila [7]). Let $\{\xi_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{N}^2\}$, be pairwise independent non-negative random variables with $\mathbb{E}\xi_{\boldsymbol{k}} = \mu$ and $\mathrm{Var}\xi_{\boldsymbol{k}} \le \sigma^2$ for all $\boldsymbol{k} \in \mathbb{N}^2$. Let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \le \boldsymbol{n}} \xi_{\boldsymbol{k}}$, $\alpha^2 = \frac{\varepsilon}{2\mu} + 1$. Then for all $\varepsilon > 0$, $\lambda > 0$,

$$\mathbb{P}\left(\sup_{\boldsymbol{m} \ge \boldsymbol{n}} \left| \frac{S_{\boldsymbol{m}}}{|\boldsymbol{m}|} - \mu \right| > \varepsilon \right) \le \lambda, \tag{2.2.2}$$

if $\boldsymbol{n} \ge \boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda}}(\lambda)$, where $\boldsymbol{\Phi}_{\varepsilon,\boldsymbol{\Lambda}}(\lambda) = \alpha^{\lceil \boldsymbol{\Lambda}_{\frac{\varepsilon}{2\alpha^2},\alpha}(\lambda) \rceil}$ and $\boldsymbol{\Lambda}_{\varepsilon,\alpha}(\lambda)$ is determined by the curve $x + y = \rho_{\varepsilon,\alpha}(\lambda)$ with $\rho_{\varepsilon,\alpha}(\lambda) = \log_\alpha\left(\frac{\sigma^2\alpha^2}{\lambda\varepsilon^2(\alpha-1)^2}\right)$. Inequality (2.2.2) is satisfied if $n_1 \cdot n_2 \ge \frac{4\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2}$.

Proposition 2.2.3 (Fazekas & Masasila [7]). Let $\{\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2\}$, be pairwise independent random variables with $\mathbb{E}\xi_{\boldsymbol{n}} = 0$, $\mathbb{E}|\xi_{\boldsymbol{n}}| = \tau > 0$ and $\mathrm{Var}\xi_{\boldsymbol{n}} \le \sigma^2 > 0$ for all $\boldsymbol{n} \in \mathbb{N}^2$. Let $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \le \boldsymbol{n}} \xi_{\boldsymbol{k}}$. Then for all $\varepsilon > 0$, $\lambda > 0$,

$$\mathbb{P}\left(\sup_{\boldsymbol{m} \ge \boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon \right) \le \lambda \quad \text{if} \quad |\boldsymbol{n}| \ge \frac{32\sigma^2\alpha^8}{\lambda\varepsilon^2(\alpha-1)^2} \quad \text{and} \quad \alpha^2 = \frac{\varepsilon}{2\tau} + 1.$$

Lemma 2.2.4. Let $\tau > 0$ and $\alpha^2 = \frac{\varepsilon}{2\tau} + 1$ be the values from Proposition 2.2.3. Then for any $C > 4$, we have

$$\frac{\alpha^8}{(\alpha-1)^2} \le \frac{4\tau^2}{\varepsilon^2} C \tag{2.2.3}$$

for small enough $\varepsilon > 0$. More precisely, for any $C > 4$ there exists an $a \in (0, 1/2)$ for which $\left(\frac{(1-a)^4}{a^5}\right)^2 = C$, and with $b = (1-2a)/a^2$ the inequality (2.2.3) is satisfied for $0 < \varepsilon < 2\tau b$.

Theorem 2.2.5 (Fazekas & Masasila [7]). *Let* $\{\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2\}$*, be pairwise independent random variables with* $\mathbb{E}\xi_{\boldsymbol{n}} = 0$*,* $\mathbb{E}|\xi_{\boldsymbol{n}}| = \tau > 0$ *and* $\mathrm{Var}\xi_{\boldsymbol{n}} \le \sigma^2 > 0$ *for all* $\boldsymbol{n} \in \mathbb{N}^2$*. Let* $S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \le \boldsymbol{n}} \xi_{\boldsymbol{k}}$*. Then*

$$\mathbb{P}\left(\sup_{\boldsymbol{m} \ge \boldsymbol{n}} \frac{|S_{\boldsymbol{m}}|}{|\boldsymbol{m}|} > \varepsilon\right) \le \frac{K\sigma^2\tau^2}{\varepsilon^4|\boldsymbol{n}|} \tag{2.2.4}$$

if $K > 512$ *and* $\varepsilon > 0$ *is small enough.*

Chapter 3

Strong Laws of Large Numbers for General Random Variables under Conditional Sub-additive Expectation and Capacity

This chapter is based on our new results in [10]. In this chapter, we study strong laws of large numbers in a non-linear framework based on conditional sub-additive expectations and conditional sub-additive capacities. Using an axiomatic approach to conditional sub-additive expectation, we establish a conditional Hájek-Rényi-type maximal inequality assuming a general conditional Kolmogorov-type maximal inequality but without imposing any weak dependence structure on the underlying sequence. As a consequence, we derive a general conditional strong law of large numbers. Finally, we introduce a notion of conditional negative dependence under sub-additive expectations and obtain the corresponding conditional Kolmogorov-type maximal inequality, leading to a conditional strong law of large numbers for conditionally negatively dependent random variables. The results of this chapter extend some results of [5] and [4] to conditional sub-additive expectations and conditional sub-additive probabilities.

In this chapter, the key properties of conditional sub-additive expectations and conditional capacities are summarized and formulated as axioms.

Definition 3.0.1. $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ is called a sub-additive conditional expectation operator if $\hat{\mathbb{E}}[X|\mathcal{F}]$ is an $\mathcal{F}$-measurable random variable for any $X \in \mathcal{H}$, which satisfies the following properties.

1. *Non-triviality:* $-\infty < \hat{\mathbb{E}}[0|\mathcal{F}] < +\infty$;
2. *Monotonicity:* $\hat{\mathbb{E}}[X|\mathcal{F}] \geq \hat{\mathbb{E}}[Y|\mathcal{F}]$ if $X \geq Y$;
3. *Sub-additivity:* $\hat{\mathbb{E}}[X+Y|\mathcal{F}] \leq \hat{\mathbb{E}}[X|\mathcal{F}] + \hat{\mathbb{E}}[Y|\mathcal{F}]$;
4. *Measurability:* $\hat{\mathbb{E}}[X+Y|\mathcal{F}] = X + \hat{\mathbb{E}}[Y|\mathcal{F}]$ if X is $\mathcal{F}$-measurable;
5. *Positive homogeneity:* $\hat{\mathbb{E}}[cX|\mathcal{F}] = c\hat{\mathbb{E}}[X|\mathcal{F}]$ if $c \geq 0$ is a constant;
6. *Monotone convergence:* If $X_n \uparrow X$, and $X_1 \geq 0$, then $\hat{\mathbb{E}}[X_n|\mathcal{F}] \uparrow \hat{\mathbb{E}}[X|\mathcal{F}]$.

Definition 3.0.2. $\hat{\mathbb{V}}[\cdot|\mathcal{F}]$ is called a sub-additive conditional probability operator if $\hat{\mathbb{V}}[A|\mathcal{F}]$ is an $\mathcal{F}$-measurable random variable for any $A \in \mathcal{A}$, which satisfies the following properties.

1. *Normalized:* $\hat{\mathbb{V}}[\emptyset|\mathcal{F}] = 0$, $\hat{\mathbb{V}}[\Omega|\mathcal{F}] = 1$;
2. *Monotonicity:* $\hat{\mathbb{V}}[A|\mathcal{F}] \leq \hat{\mathbb{V}}[B|\mathcal{F}]$ if $A \subseteq B$;
3. *Sub-additivity:* $\hat{\mathbb{V}}[A \cup B|\mathcal{F}] \leq \hat{\mathbb{V}}[A|\mathcal{F}] + \hat{\mathbb{V}}[B|\mathcal{F}]$;
4. *Lower continuity:* $\hat{\mathbb{V}}[A_n|\mathcal{F}] \uparrow \hat{\mathbb{V}}[A|\mathcal{F}]$ if $A_n \uparrow A$.

Furthermore, Finiteness and Recursivity axioms are established from the definitions 3.0.1 and 3.0.2.

1. *Finiteness:* If $\hat{\mathbb{E}}[X|\mathcal{F}] < \infty$ $\hat{\mathbb{V}}$-quasi-surely, then $X < \infty$ $\hat{\mathbb{V}}$-quasi-surely.
2. *Recursivity:* If $\hat{\mathbb{V}}[A|\mathcal{F}] = 0$ $\hat{\mathbb{V}}$-quasi-surely, then $\hat{\mathbb{V}}(A) = 0$.

Based on these axioms, we then establish our main results.

3.1 Main Results

The first two theorems ensure that the conditional Kolmogorov inequality for sub-additive expectation implies the conditional Hájek-Rényi inequality both for sub-additive expectation and capacities.

Theorem 3.1.1 (Masasila & Fazekas [10]). *Let* $\{X_k, 1 \leq k \leq n\}$ *be random variables belonging to the space* $\mathcal{H}$*. Assume that the conditional expectation operator* $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ *on space* $\mathcal{H}$ *satisfies the monotonicity, sub-additivity and positive homogeneity axioms of Definition 3.0.1. Let* $\alpha_1, \ldots, \alpha_n$ *be non-negative* $\mathcal{F}$*-measurable random variables, and* $r > 0$ *be real number. Assume that the general conditional Kolmogorov-type inequality is true, that is,*

$$\hat{\mathbb{E}}\left[\left(\max_{1\leq l\leq m} |S_l|\right)^r \Big|\mathcal{F}\right] \leq \sum_{l=1}^{m} \alpha_l \qquad \textit{for all} \quad 1 \leq m \leq n. \tag{3.1.1}$$

Then the conditional Hájek-Rényi inequality is true, that is,

$$\hat{\mathbb{E}}\left[\left(\max_{1\leq l\leq n} \left|\frac{S_l}{\beta_l}\right|\right)^r \Big|\mathcal{F}\right] \leq 4\sum_{l=1}^{n} \frac{\alpha_l}{\beta_l^r} \tag{3.1.2}$$

for $\mathcal{F}$*-measurable random variables* $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ *with* $\beta_1 \geq \beta_0$*, where* β_0 *is a positive constant.*

Theorem 3.1.2 (Masasila & Fazekas [10]). *Let* $\{X_n, 1 \leq k \leq n\}$ *be random variables,* $S_k = X_1 + \cdots + X_k$*. Let* $\hat{\mathbb{V}}[\cdot|\mathcal{F}]$ *be a conditional sub-additive probability satisfying the axioms normalization, monotonicity, and sub-additivity of Definition 3.0.2. Let* r *be a positive real number. Let* $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ *be* $\mathcal{F}$*-measurable,* $\alpha_1, \ldots, \alpha_n$ *non-negative* $\mathcal{F}$*-measurable random variables. Assume that* $\beta_1 \geq \beta_0 > 0$*, where* β_0 *is non random. If*

$$\hat{\mathbb{V}}\left[\max_{1\leq l\leq m} |S_l| \geq \varepsilon \big|\mathcal{F}\right] \leq \frac{1}{\varepsilon^r}\sum_{l=1}^{m} \alpha_l \qquad \textit{for all} \quad 1 \leq m \leq n \tag{3.1.3}$$

and for all $\varepsilon > 0$*, then*

$$\hat{\mathbb{V}}\left[\max_{1\leq l\leq n} \left|\frac{S_l}{\beta_l}\right| \geq \varepsilon \big|\mathcal{F}\right] \leq \frac{4}{\varepsilon^r}\sum_{k=1}^{n} \frac{\alpha_k}{\beta_k^r} \tag{3.1.4}$$

for all $\varepsilon > 0$.

Next, we present strong laws of large numbers in terms of conditional sub-additive expectations and capacities

Theorem 3.1.3 (Masasila & Fazekas [10]). *Let $\{X_n, n \geq 1\}$ be random variables, $S_n = X_1 + \cdots + X_n$ for any n. Let $b_1, b_2, \ldots$ be q.s. finite, $\mathcal{F}$-measurable random variables with $b_0 \leq b_1 \leq b_2 \leq \ldots$ q.s., $b_n \to \infty$ q.s., where b_0 is a positive constant. Let $\alpha_1, \alpha_2, \ldots$ be non-negative $\mathcal{F}$-measurable random variables. Assume that for the conditional expectation $\hat{\mathbb{E}}[\,\cdot\,|\mathcal{F}]$ the axioms in Definition 3.0.1 are satisfied, where all relations among random variables are understood in the $\hat{\mathbb{V}}$-quasi-sure sense. Assume further that the finiteness axiom holds. Let $r > 0$ be a fixed number and suppose that, for any $n \geq 1$*

$$\hat{\mathbb{E}}\left[\left(\max_{1\leq l\leq n}|S_l|\right)^r \Big|\mathcal{F}\right] \leq \sum_{l=1}^{n}\alpha_l \quad q.s. \tag{3.1.5}$$

If $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ q.s., then

$$\lim_{n\longrightarrow\infty}\frac{S_n}{b_n} = 0 \quad q.s. \tag{3.1.6}$$

and $\frac{S_n}{b_n} = \mathrm{O}\left(\frac{\beta_n}{b_n}\right)$, where β_n is defined by $\beta_n = \max_{1\leq k\leq n} b_k v_k^{\frac{1}{2r}}$.

Theorem 3.1.4 (Masasila & Fazekas [10]). *Let $\{X_n, n \geq 1\}$ be random variables, $S_n = X_1 + \cdots + X_n$ for any n. Let $b_1, b_2, \ldots$ be q.s. finite, $\mathcal{F}$-measurable random variables with $b_0 \leq b_1 \leq b_2 \leq \ldots$ q.s., $b_n \to \infty$ q.s., where b_0 is a positive constant. Let $\alpha_1, \alpha_2, \ldots$ be non-negative $\mathcal{F}$-measurable random variables. Assume that the conditional sub-additive probability $\hat{\mathbb{V}}[\,\cdot\,|\mathcal{F}]$ satisfies the axioms in Definition 3.0.2, where all relations among random variables are understood in the $\hat{\mathbb{V}}$-quasi-sure sense. Assume further that the recursivity axiom holds. Let $r > 0$ be a fixed number and suppose that, for all $n \geq 1$ and all $\varepsilon > 0$,*

$$\hat{\mathbb{V}}\left[\max_{1\leq l\leq n}|S_l| \geq \varepsilon\Big|\mathcal{F}\right] \leq \frac{1}{\varepsilon^r}\sum_{l=1}^{n}\alpha_l \quad q.s. \tag{3.1.7}$$

If $\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty$ q.s., then

$$\lim_{n\to\infty}\frac{S_n}{b_n} = 0 \quad q.s. \tag{3.1.8}$$

with the convergence rate $\frac{S_n}{b_n} = \mathrm{O}\left(\frac{\beta_n}{b_n}\right)$ q.s.

Chapter 4

Strong law of large numbers for φ-sub-Gaussian random variables under sub-linear expectation spaces

This chapter is based on our new results in [9]. In this chapter, we introduce the notions of sub-Gaussian and φ-sub-Gaussian random variables in sub-linear expectation spaces. To avoid the problem caused by the existence of two different expectations, i.e., the upper expectation and the lower expectation, we divide the definition of the sub-Gaussian property into an upper part and a lower part. It turns out that this approach fits well to the sub-linear setting; it provides a proper framework for extending the general result of Zajkowski [13] to sublinear expectation spaces. Within our framework, we establish a strong law of large numbers for sub-Gaussian sequences.

4.1 The Main Result

The following definition plays important role in our main result.

Definition 4.1.1. For a φ-sub-Gaussian random variable ξ with fixed $\overline{m}$ and $\underline{m}$ let

$\tau_\varphi(\xi)$ be defined as

$$\tau_\varphi(\xi) = \inf\Big\{a \geq 0 : \hat{\mathbb{E}}e^{\lambda(\xi-\overline{m})} \leq e^{\varphi(a\lambda)},\ \text{for}\ \lambda > 0,\\ \text{and}\ \hat{\mathbb{E}}e^{\lambda(\xi-\underline{m})} \leq e^{\varphi(a\lambda)},\ \text{for}\ \lambda < 0\Big\}.$$

Theorem 4.1.2 (Masasila & Fazekas [9])**.** *For some $p > 1$, let $\{Z_n, n \geq 1\}$, be a sequence of φ_p-sub-Gaussian random variables with parameters $\overline{m}$ and $\underline{m}$. Let $\tau_\varphi(Z_n)$ be defined according to Definition 4.1.1. If there exist positive numbers c and α such that for every natural number n, the condition $\tau_{\varphi_p}(Z_n) \leq cn^{-\alpha}$ is satisfied, then*

$$\hat{\mathbb{V}}\left(\left\{\liminf_{n\to\infty} Z_n < \underline{m}\right\} \cup \left\{\limsup_{n\to\infty} Z_n > \overline{m}\right\}\right) = 0 \tag{4.1.1}$$

and

$$v\left(\underline{m} \leq \liminf_{n\to\infty} Z_n \leq \limsup_{n\to\infty} Z_n \leq \overline{m}\right) = 1. \tag{4.1.2}$$

4.2 Strong law of large numbers for independent sub-Gaussian variables

We shall use the independence notion given in [1]. The sequence of random variables $\xi_1, \xi_2, \ldots$ is called independent if for each $n = 1, 2, \ldots$ and each non-negative measurable functions $f_1, f_2, \ldots$ we have

$$\hat{\mathbb{E}}(f_1(\xi_1)f_2(\xi_2)\cdots f_n(\xi_n)) = \hat{\mathbb{E}}(f_1(\xi_1))\hat{\mathbb{E}}(f_2(\xi_2))\cdots\hat{\mathbb{E}}(f_n(\xi_n)).$$

If $\xi_1, \xi_2, \ldots$ are independent, then they satisfy the following property

$$\hat{\mathbb{E}}\prod_{i=1}^{k}\exp\left(\lambda(\xi_i - m)\right) \quad \leq \quad \prod_{i=1}^{k}\hat{\mathbb{E}}\exp\left(\lambda(\xi_i - m)\right)$$
$$\text{for any real } \lambda,\ m, \text{ and positive integer } k. \tag{4.2.1}$$

Now, let $\xi_1, \xi_2, \ldots$ be random variables satisfying the sub-Gaussian property (that is they are φ_2-sub-Gaussian): for fixed constants $\sigma > 0$, $\overline{m}$, and $\underline{m}$

$$\hat{\mathbb{E}}(e^{\lambda(\xi_i-\overline{m})}) \leq e^{(\sigma^2\lambda^2/2)}, \quad \text{for} \quad \lambda > 0, \quad \text{and} \quad \hat{\mathbb{E}}(e^{\lambda(\xi_i-\underline{m})}) \leq e^{(\sigma^2\lambda^2/2)}, \quad \text{for} \quad \lambda < 0. \tag{4.2.2}$$

Theorem 4.2.1. *Let* $\xi_1, \xi_2, \ldots$ *be random variables satisfying* (4.2.1) *and* (4.2.2) *with* S_n *defined above. Then*

$$\hat{\mathbb{V}}\Big(\Big\{\liminf_{n\to\infty}\frac{S_n}{n} < \underline{m}\Big\} \cup \Big\{\limsup_{n\to\infty}\frac{S_n}{n} > \overline{m}\Big\}\Big) = 0. \tag{4.2.3}$$

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Registry number: DEENK/103/2026.PL
Subject: PhD Publication List

Candidate: Nyanga Honda Masasila
Doctoral School: Doctoral School of Informatics
MTMT ID: 10100089

List of publications related to the dissertation

Foreign language scientific articles in Hungarian journals (1)

1. Fazekas, I., **Masasila, N. H.**: Quantitative strong laws of large numbers for random variables with double indices.
   Ann. Math. Inform. 62, 55-65, 2025. ISSN: 1787-5021.
   DOI: http://dx.doi.org/10.33039/ami.2025.09.002
   IF: 0.3 (2024)

Foreign language scientific articles in international journals (2)

2. **Masasila, N. H.**, Fazekas, I.: Strong Laws of Large Numbers for General Random Variables Under Conditional Sub-Additive Expectation and Capacity.
   Mathematics. 14 (5), 1-19, 2026. EISSN: 2227-7390.
   DOI: http://dx.doi.org/10.3390/math14050775
   IF: 2.2 (2024)

3. Fazekas, I., **Masasila, N. H.**: A general approach to conditional strong laws of large numbers.
   Annales Universitat. Mariae Curie-Sklodowska, A - Math. 78 (1), 27-35, 2024. ISSN: 0365-1029.
   DOI: http://dx.doi.org/10.17951/a.2024.78.1.27-35

Foreign language abstracts (1)

4. Fazekas, I., **Masasila, N. H.**: Explicit rate of convergence in strong laws of large numbers for random variables with double indices.
   In: The Mediterranean International Conference of Pure & Applied Mathematics and Related Areas : Proceedings book of MICOPAM2025. Szerk.: Mustafa Alkan, Irem Kucukoglu, Ortaç Öneş, et. al, Josip Juraj Strossmayer University of Osijek, Osijek, 200, 2025. ISBN: 9789534691502

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List of other publications

Foreign language scientific articles in international journals (4)

5. **Masasila, N. H.**, Fazekas, I.: Strong law of large numbers for [phí]-sub-Gaussian random variables under sub-linear expectation spaces.
   Probab. Uncertain. Quant. Risk. [Submitted] 1-10, 2026. ISSN: 2095-9672.

6. Kigodi, O. J., **Masasila, N. H.**, Faisal, M., Badruddin, I. A., Zedan, A. S. A. H., Chacha, C. S.: Thermal and Entropic Analysis of Viscous Fluid Flow in a Porous Channel With Convective Heat Transfer and Magnetic Field Aspects.
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   Heat Trans. 1, 1-14, 2025. ISSN: 2688-4534.
   DOI: http://dx.doi.org/10.1002/htj.70087
   IF: 2.6 (2024)

Total IF of journals (all publications): 7,7
Total IF of journals (publications related to the dissertation): 2,5

The Candidate's publication data submitted to the Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

10 March, 2026