



# The holonomy group of locally projectively flat Randers two-manifolds of constant curvature <sup>☆</sup>



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## ARTICLE INFO

### Article history:

Received 18 August 2019

Received in revised form 2 June 2020

Accepted 5 August 2020

Available online 21 August 2020

Communicated by Z. Shen

### MSC:

53C29

53B40

22E65

17B66

### Keywords:

Finsler geometry

Randers metric

Zermelo navigation

Holonomy

Diffeomorphism groups

Infinite-dimensional Lie group

## ABSTRACT

In this paper, we investigate the holonomy structure of the most accessible and demonstrative 2-dimensional Finsler surfaces, the Randers surfaces. Randers metrics can be considered as the solutions of the Zermelo navigation problem. We give the classification of the holonomy groups of locally projectively flat Randers two-manifolds of constant curvature. In particular, we prove that the holonomy group of a simply connected non-Riemannian projectively flat Finsler two-manifold of constant non-zero flag curvature is maximal and isomorphic to the orientation preserving diffeomorphism group of the circle.

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## 1. Introduction

The holonomy group of a Riemann or Finsler manifold is a natural geometric object: it is the transformation group generated by parallel translations along loops with respect to the associated canonical (linear, resp. homogeneous) connection. The Riemannian holonomy groups have been extensively studied during the second half of the last century but, maybe because of the computational difficulties, little attention was paid to the Finslerian case. Although there are similarities, the holonomy properties of Finsler manifolds can be very different from the Riemannian ones, as recent results show. Indeed, the fundamental result of Borel and

<sup>☆</sup> This work is partially supported by the EFOP-3.6.2-16-2017-00015 and EFOP-3.6.1-16-2016-00022 project, by the 307818 TKA-DAAD exchange project, and by the ÚNKP-19-3-I-DE-531 New National Excellence Program of the Ministry for Innovation and Technology.

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Lichnerowicz [3] from 1952 claims that the holonomy group of a simply connected Riemannian manifold is a (finite dimensional) closed Lie subgroup of the special orthogonal group  $SO(n, \mathbb{R})$ . In contrast with this, in [8] it has been proven that the holonomy group of an at least three-dimensional non-Riemannian Finsler manifold of nonzero constant curvature is not a compact Lie group. In [11] it has been shown that there exist large families of projectively flat Finsler manifolds of constant curvature such that their holonomy groups are not finite dimensional Lie groups. In [9] projective Finsler manifolds of constant curvature having infinite dimensional holonomy group have been characterized. The proofs in the above-mentioned papers [8,9,11] give estimates for the dimension of tangent Lie algebras of the holonomy group but unfortunately, they do not give direct information on the structure of the holonomy groups.

In contrast to the Riemannian case where the complete classification is already known, Finsler holonomy groups are described only for few, very special classes of metrics: the holonomy of Landsberg manifolds have been investigated in [7] and the holonomy of Berwald manifolds have been characterized in [15]. First examples describing infinite dimensional Finslerian holonomy groups can be found in [10]. To achieve further significant progress, it would be very important to investigate systematically the holonomy structure of different classes of Finsler metrics.

In this article, we consider one of the most accessible and demonstrative examples for non-Riemannian Finsler manifolds, the Randers manifolds [1,12]. In the Randers case, the Finsler function is a Riemann norm deformed by a 1-form. As it has been proven in [2], Randers metrics describe the Zermelo navigation problem on Riemannian manifolds. This fact may suggest that the holonomy structure of Randers manifolds are similar to that of Riemann manifolds but, on the contrary: they can be very different, as the results of this paper show. We focus our attention on the holonomy properties of simply connected, locally projectively flat Randers two-manifolds of constant flag curvature  $\lambda$ . This class of manifolds was already considered in [11] where it has been proved that the holonomy group of such a manifold is finite dimensional if  $\lambda = 0$  or the metric is Riemannian, and *infinite dimensional* if  $\lambda \neq 0$  and the metric is non-Riemannian. The finite dimensional holonomy structures are already well known, but nothing was known about the infinite dimensional case: the results reveal no information about the structure of the holonomy group.

The goal of this paper is to complete the results of [11] by describing the infinite dimensional holonomy structure of projectively flat Randers surfaces of non-zero constant curvature. The main result (Theorem 3) shows that the holonomy group of a such manifold is maximal and its closure is isomorphic to  $Diff_+(\mathbb{S}^1)$ , the orientation preserving diffeomorphism group of the circle  $\mathbb{S}^1$ . This result is surprising because it shows that even in the case when the geodesic structure is simple (the geodesics are straight lines), the holonomy group can still be a very large group. Finally, we obtain the classification of the holonomy groups of projectively flat Randers surfaces (Corollary 4).

## 2. Preliminaries

Throughout this article,  $M$  is a  $C^\infty$  smooth manifold,  $\mathfrak{X}^\infty(M)$  denotes the Lie algebra of smooth vector fields on  $M$  and  $Diff^\infty(M)$  is the group of  $C^\infty$ -diffeomorphisms of  $M$ . The first and the second tangent bundles of  $M$  are denoted by  $(TM, \pi, M)$  and  $(TTM, \tau, TM)$ , respectively.

### 2.1. Finsler manifolds

A *Finsler manifold* is a pair  $(M, \mathcal{F})$ , where the norm function  $\mathcal{F}: TM \rightarrow \mathbb{R}_+$  is continuous, smooth on  $\hat{TM} := TM \setminus \{0\}$ , its restriction  $\mathcal{F}_x = \mathcal{F}|_{T_x M}$  is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y}: (u, v) \mapsto g_{ij}(x, y)u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s \partial t} \Big|_{t=s=0}$$

is positive definite at every  $y \in \hat{T}_x M$ .

Geodesics of  $(M, \mathcal{F})$  are determined by a system of 2nd order ordinary differential equation  $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$ ,  $i = 1, \dots, n$  in a usual local coordinate system  $(x^i, y^i)$  of  $TM$ , where  $G^i(x, y)$  are given by

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y) \left( 2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \right) y^j y^k. \quad (1)$$

A vector field  $X(t) = X^i(t) \frac{\partial}{\partial x^i}$  along a curve  $c(t)$  is said to be parallel with respect to the associated homogeneous (nonlinear) connection if it satisfies

$$D_c X(t) := \left( \frac{dX^i(t)}{dt} + G_j^i(c(t), X(t)) \dot{c}^j(t) \right) \frac{\partial}{\partial x^i} = 0, \quad (2)$$

where  $G_j^i = \frac{\partial G^i}{\partial y^j}$ .

The horizontal Berwald covariant derivative  $\nabla_X \xi$  of  $\xi(x, y) = \xi^i(x, y) \frac{\partial}{\partial y^i}$  by the vector field  $X(x) = X^i(x) \frac{\partial}{\partial x^i}$  is expressed locally by

$$\nabla_X \xi = \left( \frac{\partial \xi^i(x, y)}{\partial x^j} - G_j^k(x, y) \frac{\partial \xi^i(x, y)}{\partial y^k} + G_{jk}^i(x, y) \xi^k(x, y) \right) X^j \frac{\partial}{\partial y^i}, \quad (3)$$

where we denote  $G_{jk}^i(x, y) := \frac{\partial G_j^i(x, y)}{\partial y^k}$ .

### 2.1.1. Curvatures of Finsler manifolds

The curvature tensor field  $R = R_{jk}^i dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^i}$  has the expression

$$R_{jk}^i = \frac{\partial G_j^i(x, y)}{\partial x^k} - \frac{\partial G_k^i(x, y)}{\partial x^j} + G_j^m(x, y) G_{km}^i(x, y) - G_k^m(x, y) G_{jm}^i(x, y). \quad (4)$$

The Riemannian curvature tensor is  $R_y := R(\cdot, y)$ , its components can be obtained as  $R_j^i = R_{jk}^i y^k$ . The Ricci curvature  $Ric(y)$  is defined to be the trace of  $R_y$ ,  $Ric(y) := R_m^m(x, y)$ . For a tangent plane  $P = \text{Span}\{y, u\} \subset T_x M$ , the flag curvature is defined as

$$\mathbf{K}(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$

If a manifold has constant flag curvature  $\mathbf{K} = \lambda \in \mathbb{R}$ , then the Ricci curvature is constant in the sense that  $Ric(y) = (n-1)\lambda F^2$  and the local expression of the coefficients of the curvature is

$$R_{jk}^i = \lambda (\delta_k^I N G_{jm}(x, y) y^m - \delta_j^i g_{km}(x, y) y^m),$$

where  $\delta_j^i$  is the Kronecker delta symbol. Assume that the Finsler manifold  $(M, \mathcal{F})$  is locally projectively flat. Then for every point  $x \in M$  there exists an adapted local coordinate system, that is a mapping  $(x^1, \dots, x^n)$  on a neighborhood  $U$  of  $x$  into the Euclidean space  $\mathbb{R}^n$ , such that the straight lines of  $\mathbb{R}^n$  correspond to the geodesics of  $(M, \mathcal{F})$ . Then the geodesic coefficients and the connection coefficients are of the form

$$G^i = \mathcal{P} y^i, \quad G_k^i = \frac{\partial \mathcal{P}}{\partial y^k} y^i + \mathcal{P} \delta_k^i, \quad G_{kl}^i = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l} y^i + \frac{\partial \mathcal{P}}{\partial y^k} \delta_l^i + \frac{\partial \mathcal{P}}{\partial y^l} \delta_k^i \quad (5)$$

where  $\mathcal{P}(x, y)$  is a 1-homogeneous function in  $y$ , called the *projective factor* of  $(M, \mathcal{F})$ . According to Lemma 8.2.1 in [4, p. 155], if  $(M \subset \mathbb{R}^n, \mathcal{F})$  is a projectively flat manifold, then its projective factor can be computed using the formula

$$\mathcal{P}(x, y) = \frac{1}{2\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x^i} y^i. \quad (6)$$

### 2.1.2. Projectively flat Randers manifolds with constant curvature

Projectively flat Randers manifolds with constant flag curvature were classified by Z. Shen in [13]. He proved that any projectively flat Randers manifold  $(M, \mathcal{F})$  with non-zero constant flag curvature has negative curvature. These metrics can be normalized by a constant factor so that the curvature is  $\lambda = -1/4$ . In this case  $(M, \mathcal{F})$  is isometric to the Finsler manifold defined by the Finsler function

$$\mathcal{F}_a(x, y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} + \epsilon \left( \frac{\langle x, y \rangle}{1 - |x|^2} + \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} \right) \quad (7)$$

on the unit ball  $\mathbb{D}^n \subset \mathbb{R}^n$ , where  $a \in \mathbb{R}^n$  is any constant vector with  $|a| < 1$  and  $\epsilon = \pm 1$  ([13, Theorem 1.1]).  $\mathcal{F}_a$  is called generalized Funk metric (see [14]).

We note that the restriction of any orthogonal transformation  $\phi \in SO(n, \mathbb{R})$  on  $\mathbb{D}^n$  does not change the Finsler function (7), therefore one can assume that  $a \in \mathbb{R}^n$  has the form  $a = (a_1, 0, \dots, 0)$ . We can consider the generalized Funk disc  $(\mathbb{D}^n, \mathcal{F}_a)$  as the standard model of projectively flat Randers manifolds with non-zero constant flag curvature.

We remark that the computation of the coefficients of the associated connection is relatively easy: according to Lemma 8.2.1 of [4, p. 155], the projective factor  $\mathcal{P}(x, y)$  can be computed by the formula (6) which gives in the case (7)

$$\mathcal{P}(x, y) = \frac{1}{2} \left( \frac{\epsilon \sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)} + \langle x, y \rangle}{1 - |x|^2} - \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} \right). \quad (8)$$

The geodesic coefficients and the connection coefficients can be computed from (8) by using (5).

## 2.2. Holonomy group and its tangent Lie algebras

The *holonomy group*  $\text{Hol}_x(M)$  of the Finsler manifold  $(M, \mathcal{F})$  at a point  $x \in M$  is the subgroup of the diffeomorphism group  $\text{Diff}^\infty(T_x M)$  generated by parallel translations along piece-wise differentiable closed curves initiated and ended at the point  $x \in M$ . Since the canonical parallel translation is homogeneous and preserves the Finsler function, the holonomy group can be considered on the *indicatrix*  $\mathcal{I}_x := \{y \in T_x M \mid \mathcal{F}(y) = 1\}$ . That way  $\text{Hol}_x(M)$  is a subgroup of the diffeomorphism group  $\text{Diff}^\infty(\mathcal{I}_x)$ . The topological closure  $\overline{\text{Hol}_x(M)}$  of the holonomy group in the Fréchet topology of  $\text{Diff}^\infty(\mathcal{I}_x)$  is called the *closed holonomy group*.

### 2.2.1. The curvature algebra

A vector field  $\xi \in \mathfrak{X}(TM)$  is called a *curvature vector field* of  $(M, \mathcal{F})$  if it is in the image of the curvature tensor, that is  $\xi = R(X, Y)$  for some  $X, Y \in \mathfrak{X}(M)$ . The *curvature algebra*  $\mathfrak{R}(M)$  is the Lie algebra generated by curvature vector fields. Its restriction

$$\mathfrak{R}_x(M) := \{ \xi|_{\mathcal{I}_x} \mid \xi \in \mathfrak{R}(M) \}$$

is the curvature algebra at the point  $x \in M$ .  $\mathfrak{R}_x(M)$  is a Lie subalgebra of  $\mathfrak{X}^\infty(\mathcal{I}_x)$ .

### 2.2.2. The infinitesimal holonomy algebra

The *infinitesimal holonomy algebra*, denoted by  $\mathfrak{hol}^*(M)$  is the smallest Lie algebra generated by curvature vector fields and by the horizontal Berwald covariant derivation. More precisely,  $\mathfrak{hol}^*(M)$  is the smallest Lie algebra of vector fields on  $TM$  satisfying the following two conditions:

- (1) for any curvature vector field  $\xi$  we have  $\xi \in \mathfrak{hol}^*(M)$ ,
- (2) if  $\xi \in \mathfrak{hol}^*(M)$  and  $X \in \mathfrak{X}^\infty(M)$  then  $\nabla_X \xi \in \mathfrak{hol}^*(M)$ .

By considering the restriction of  $\mathfrak{hol}^*(M)$  on the indicatrix  $\mathcal{I}_x$  we obtain the infinitesimal holonomy algebra at the point  $x \in M$ :

$$\mathfrak{hol}_x^*(M) := \{ \xi|_{\mathcal{I}_x} \mid \xi \in \mathfrak{hol}^*(M) \}.$$

We remark that if the manifold is two-dimensional, then the curvature algebra  $\mathfrak{R}_x(M)$  is at most one-dimensional, but the infinitesimal holonomy algebra  $\mathfrak{hol}_x^*(M)$  can be higher, even infinite dimensional:  $\mathfrak{hol}_x^*(M)$  is generated by the restriction of the curvature vector field and its covariant derivatives. Using the notation  $\xi_0 = R(\partial_{x_1}, \partial_{x_2})|_{\mathcal{I}_x}$  and  $\nabla_{i_1, \dots, i_k} \xi_0 := (\nabla_{\partial_{x_{i_1}}} \dots \nabla_{\partial_{x_{i_k}}} \xi)|_{\mathcal{I}_x}$  we get

$$\mathfrak{hol}_x^*(M) = \left\langle \xi_0, \nabla_1 \xi_0, \nabla_2 \xi_0, \nabla_{11} \xi_0, \dots \right\rangle_{\mathcal{L}ie} \quad (9)$$

**Property** (Theorem 4, [8]). *At any point  $x \in M$  the infinitesimal holonomy algebra  $\mathfrak{hol}_x^*(M)$  is tangent to the holonomy group  $\text{Hol}_x(M)$ .*

### 3. Holonomy of projectively flat Randers two-manifolds of non-zero constant curvature

Our aim is to describe the holonomy structure of projectively flat non-Riemannian Randers two-manifolds with non-zero constant flag curvature. As a first step, we investigate the holonomy of the standard model described in Subsection 2.1.2. We have the following

**Proposition 1.** *The holonomy group of the generalized Funk disk  $(\mathbb{D}^2, \mathcal{F}_a)$  is maximal and it is diffeomorphic to the orientation preserving diffeomorphism group of  $\mathbb{S}^1$ , that is*

$$\overline{\text{Hol}(\mathbb{D}^2, \mathcal{F}_a)} \cong \text{Diff}_+^\infty(\mathbb{S}^1).$$

**Proof.** Let  $a = (a_1, 0) \in \mathbb{R}^2$  be a constant nonzero vector with  $|a_1| < 1$ . We consider the case when  $\epsilon = +1$  in the expression (7) of  $\mathcal{F}_a$ . The computation when  $\epsilon = -1$  is analogous. The projective factor  $\mathcal{P}$  and the geodesic coefficients  $G_j^i$  can be easily computed by formula (8) and (5). In particular, the connection coefficients at  $0 \in \mathbb{D}^2$  are

$$\begin{aligned} G_1^1 &= \frac{1}{2} \frac{(y_1)^2 + |y|^2 - 2a_1 y_1 |y|}{|y|} & G_1^2 &= \frac{1}{2} \frac{y_1 y_2}{|y|} \\ G_2^1 &= -\frac{1}{2} \frac{(a_1 |y| - y_1) y_2}{|y|} & G_2^2 &= \frac{1}{2} \frac{(y_2)^2 + |y|^2 - a_1 y_1 |y|}{|y|}. \end{aligned} \quad (10)$$

The expression of the curvature vector field  $\xi = R(\partial_{x_1}, \partial_{x_2})$  at the point  $0 \in \mathbb{R}^2$  is

$$\xi = R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = \frac{1}{4} \frac{y_2 (a_1 y_1 + |y|)}{|y|} \frac{\partial}{\partial y_1} - \frac{1}{4} (|y| + a_1 y_1) \left(\frac{y_1}{|y|} + a_1\right) \frac{\partial}{\partial y_2}. \quad (11)$$

Since the Minkowski norm at  $0 \in \mathbb{D}^2$  is  $\mathcal{F}_a(0, y) = |y| + \langle a, y \rangle$ , the indicatrix  $\mathcal{I}_0 \subset T_0M$  at 0 is defined by the equation

$$|y| + a_1 y_1 = 1. \quad (12)$$

Using polar coordinates  $(r, t)$  on  $T_0\mathbb{R}^2$ , the equation of the indicatrix  $\mathcal{I}_0$  is  $r(1 + a_1 \cos t) = 1$ . A parametrization of  $\mathcal{I}_0$  is given by

$$\phi(t) = ((y_1(t), y_2(t))) = \left( \frac{\cos t}{1 + a_1 \cos t}, \frac{\sin t}{1 + a_1 \cos t} \right), \quad (13)$$

in terms of the parameter  $t$ . Using this parametrization the coordinate expression of the restriction  $\xi_0 := \xi|_{\mathcal{I}_0}$  of the curvature vector field (11) on the indicatrix  $\mathcal{I}_0$  is

$$\xi_0 := \omega(t) \frac{d}{dt} \quad (14)$$

where

$$\omega(t) := -\frac{1}{4}(1 + a_1 \cos t)^2. \quad (15)$$

Let us introduce the notation

$$\Sigma_n := \text{Span}_{\mathbb{R}} \left\{ \xi_0^{l,m} \mid l + m \leq n \right\}, \quad (16)$$

where

$$\xi_0^{l,m} := (\sin^l t \cos^m t) \cdot \xi_0 \in \mathfrak{X}(\mathcal{I}_0) \quad (17)$$

are vector fields on the indicatrix  $\mathcal{I}_0$  defined as functional multiples of (14). We have the following

**Lemma 1.** *For any  $n \in \mathbb{N}$  we have  $\Sigma_n \subset \mathfrak{hol}_0^*(\mathbb{D}^2)$ .*

**Remark 2.** Using the Pythagorean trigonometric identity  $\sin^2 t + \cos^2 t = 1$ , every elements of  $\Sigma_n$  can be expressed as a linear combination of the elements  $\xi_0^{0,m} = \cos^m t \xi_0$  and  $\xi_0^{1,m-1} = \sin t \cos^{m-1} t \xi_0$ ,  $0 \leq m \leq n$ , that is

$$\Sigma_n = \text{Span}_{\mathbb{R}} \left\{ \xi_0^{0,m}, \xi_0^{1,m-1} \mid 0 \leq m \leq n \right\}. \quad (18)$$

**Proof of the Lemma.** Taking into account Remark 2 we prove the lemma by mathematical induction by showing that the generating elements (18) of  $\Sigma_n$  are elements of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ .

• *First step.* From the definition of the infinitesimal holonomy algebra (see subsection 2.2.2) we know that  $\xi_0^{0,0} = \xi_0$  given by (14) is an element of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . Moreover, as (9) shows, the restriction of successive covariant derivatives of the curvature vector field (11) on  $\mathcal{I}_0$  are also elements of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . They can be expressed in terms of multiples of (14). Computing the first covariant derivatives at  $0 \in \mathbb{D}^2$  we get

$$\begin{aligned} \nabla_1 \xi &= -\frac{3}{8} \frac{y_2 (a_1 y_2^2 + (a_1^2 - 1)|y|y_1)}{|y|^2} \frac{\partial}{\partial y_1} + \frac{3}{8} \frac{(a_1^2 |y|^2 - y_1^2)(a_1 y_1 + |y|)}{|y|^2} \frac{\partial}{\partial y_2} \\ \nabla_2 \xi &= \frac{3}{8} \frac{y_2^2 (a_1 y_1 + |y|)}{|y|^2} \frac{\partial}{\partial y_1} - \frac{3}{8} \frac{(a_1 (y_1^2 + |y|^2) + (a_1^2 + 1)y_1 |y|) y_2}{|y|^2} \frac{\partial}{\partial y_2} \end{aligned} \quad (19)$$

Using (12), we get on the indicatrix  $\mathcal{I}_0$ :

$$(\nabla_1 \xi)|_{\mathcal{I}_0} = -\frac{3}{2}(a_1 - \cos t) \xi_0, \quad (20a)$$

$$(\nabla_2 \xi)|_{\mathcal{I}_0} = \frac{3}{2} \sin t \xi_0, \quad (20b)$$

Using a linear combination of (20a) and (20b) we get that  $\xi_0^{1,0} = \sin t \xi_0$  and  $\xi_0^{0,1} = \cos t \xi_0$  are element of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . Therefore we have

$$\Sigma_1 = \{ \xi_0, \sin t \xi_0, \cos t \xi_0 \} \subset \mathfrak{hol}_{x_0}^*(\mathbb{D}^2), \quad (21)$$

that is the statement of the Lemma is correct for  $n = 1$ .

• *Second step.* By definition, the second covariant derivatives of the curvature vector field are also elements of the infinitesimal holonomy algebra. Using similar computation to that of the first step one can find that

$$(\nabla_1 \nabla_1 \xi)|_{\mathcal{I}_0} = \frac{3}{4}(5a_1^2 - a_1 \cos^3 t - 5a_1 \cos t + 4 \cos^2 t + 1) \xi_0, \quad (22a)$$

$$(\nabla_1 \nabla_2 \xi)|_{\mathcal{I}_0} = -\frac{3}{4}(a_1 \cos^2 t + 3a_1 - 4 \cos t) \sin t \xi_0, \quad (22b)$$

$$(\nabla_2 \nabla_2 \xi)|_{\mathcal{I}_0} = \frac{3}{4}(a_1 \cos^3 t + 5 - 4 \cos^2 t) \xi_0. \quad (22c)$$

Using linear combinations of the elements (21) of  $\Sigma_1$  and (22a)–(22c) we get that  $\{ \cos^2 t \xi_0, \sin t \cos t \xi_0 \} \subset \mathfrak{hol}_0^*(\mathbb{D}^2)$ . Completing this set with the elements of (21) we get that

$$\Sigma_2 \subset \mathfrak{hol}_0^*(\mathbb{D}^2).$$

• *Third step.* Let us suppose that the statement of the lemma is true for some  $n \in \mathbb{N}$ , that is  $\Sigma_n \subset \mathfrak{hol}_0^*(\mathbb{D}^2)$ . We will show that  $\Sigma_{n+1} \subset \mathfrak{hol}_0^*(\mathbb{D}^2)$  too. According to the Remark,  $\Sigma_{n+1}$  is generated by the elements

$$\left\{ \xi_0^{0,m}, \xi_0^{1,m-1} \mid 0 \leq m \leq n \right\} \cup \left\{ \xi_0^{0,n+1}, \xi_0^{1,n} \right\} \quad (23)$$

One can observe that the vector fields of the first set are elements of  $\Sigma_n$ , and by the inductive hypotheses, they are elements of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . Hence, to prove the lemma we have to show that  $\xi_0^{0,n+1}$  and  $\xi_0^{1,n}$  are elements of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . We have

$$\begin{aligned} [\xi_0, \xi_0^{0,n-1}] &= [\xi_0, \cos^{n-1} t \xi_0] = (\mathcal{L}_{\xi_0}(\cos^{n-1} t)) \xi_0 \\ &= -(n-1)\omega(t)(\sin t \cos^{n-2} t) \xi_0 \stackrel{(15)}{=} \\ &= \frac{n-1}{4}(1 + 2a_1 \cos t + a_1^2 \cos^2 t)(\sin t \cos^{n-2} t) \xi_0 \\ &= \frac{n-1}{4} \xi_0^{1,n-2} + \frac{a_1(n-1)}{2} \xi_0^{1,n-1} + \frac{a_1^2(n-1)}{4} \xi_0^{1,n}. \end{aligned}$$

By the inductive hypothesis, the Lie bracket on the left hand side and the first two terms in the last line are elements of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . Consequently the last one must be also an element of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . Moreover, the coefficient of  $\xi_0^{1,n}$  is a nonzero constant, therefore we get that  $\xi_0^{1,n} \in \mathfrak{hol}_0^*(\mathbb{D}^2)$ .

Similarly, computing the Lie bracket of the elements  $\xi_0$  and  $\xi_0^{1,n-2}$  of the Lie algebra  $\mathfrak{hol}_0^*(\mathbb{D}^2)$  we get

$$\begin{aligned} [\xi_0, \xi_0^{1,n-2}] &= [\xi_0, \sin t \cos^{n-2} t \xi_0] = \mathcal{L}_{\xi_0}(\sin t \cos^{n-2} t) \xi_0 \\ &= \omega(t)(\cos t \cos^{n-2} t - (n-2) \sin^2 t \cos^{n-3} t) \xi_0 \stackrel{(15)}{=} \\ &= \frac{n-3}{4} \xi_0^{0,n-3} + \frac{a_1(n-2)}{2} \xi_0^{0,n-2} + \frac{a_1^2(n-2)-(n-1)}{4} \xi_0^{0,n-1} - \frac{a_1(n-1)}{2} \xi_0^{0,n-1} - \frac{a_1^2(n-1)}{4} \xi_0^{0,n+1} \end{aligned}$$

From the inductive hypothesis we know that the Lie bracket on the left hand side and the first four terms in the last line on the right hand side are elements of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ , therefore the last one must be also an element of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . Since the coefficient of  $\xi_0^{0,n+1}$  is nonzero we get that  $\xi_0^{0,n+1} \in \mathfrak{hol}_0^*(\mathbb{D}^2)$ . Consequently, the vector fields (23) generating  $\Sigma_{n+1}$  are all elements of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$  and  $\Sigma_{n+1} \subset \mathfrak{hol}_0^*(\mathbb{D}^2)$ .  $\square$

**Proof of Proposition 1.** From the multiple-angle formulas of the sine and cosine functions

$$\sin nt = \sum_{k=0}^n \binom{n}{k} \cos^k t \sin^{n-k} t \sin\left(\frac{n-k}{2}\pi\right), \quad \cos nt = \sum_{k=0}^n \binom{n}{k} \cos^k t \sin^{n-k} t \cos\left(\frac{n-k}{2}\pi\right),$$

we get that the vector fields  $\sin nt \xi_0, \cos nt \xi_0 \in \mathfrak{X}(\mathcal{I}_0)$  can be expressed as a linear combination of elements of  $\Sigma_n$ :

$$\sin nt \xi_0 = \sum_{k=0}^n \binom{n}{k} \sin\left(\frac{n-k}{2}\pi\right) \xi_0^{k,n-k}, \quad (24a)$$

$$\cos nt \xi_0 = \sum_{k=0}^n \binom{n}{k} \cos\left(\frac{n-k}{2}\pi\right) \xi_0^{k,n-k}. \quad (24b)$$

On the other side, Lemma 1 shows that the vector fields of  $\Sigma_n$  are elements of the holonomy algebra  $\mathfrak{hol}_0^*(\mathbb{D}^2)$ . Therefore, the vector fields (24a) and (24b),  $n \in \mathbb{N}$ , are element of  $\mathfrak{hol}_0^*(\mathbb{D}^2)$  and

$$\text{Span}\{\sin nt \xi_0, \cos nt \xi_0 \mid n = 0, 1, \dots\} \subset \mathfrak{hol}_0^*(\mathbb{D}^2). \quad (25)$$

Moreover, any  $2\pi$  periodic smooth function can be approximated uniformly by the arithmetical means of the partial sums of its Fourier series [6, Theorem 2.12]. In particular the functions  $(\sin nt)/\omega(t)$  and  $(\cos nt)/\omega(t)$  can be approximated uniformly by their Fourier sums. Hence we get

$$\left\{\frac{d}{dt}, \cos nt \frac{d}{dt}, \sin nt \frac{d}{dt}\right\}_{n \in \mathbb{N}} \subset \overline{\left\{\text{Span}\{\xi_0, \cos nt \xi_0, \sin nt \xi_0\}_{n \in \mathbb{N}}\right\}} \subset \mathfrak{hol}_0^*(\mathbb{D}^2). \quad (26)$$

The Lie algebra generated by the left hand side of (26) is diffeomorphic to the *Fourier algebra*  $F(\mathbb{S}^1)$  on  $\mathbb{S}^1$ . Therefore from (26) we get that the infinitesimal holonomy algebra contains a Lie algebra diffeomorphic to the *Fourier algebra*  $F(\mathbb{S}^1)$  on  $\mathbb{S}^1$ . Hence from Proposition 5.1 of [10] we get that the holonomy group  $\text{Hol}_0(\mathbb{D}^2)$  is maximal and  $\overline{\text{Hol}_0(\mathbb{D}^2)}$  is diffeomorphic to  $\text{Diff}_+^\infty(\mathbb{S}^1)$ .  $\square$

Using Z. Shen's classification theorem on Randers manifolds we can get the following

**Theorem 3.** *The holonomy group of a simply connected non-Riemannian projectively flat Randers two-manifold of constant non-zero flag curvature is maximal and  $\overline{\text{Hol}(M)}$  is diffeomorphic to the orientation preserving diffeomorphism group of  $\mathbb{S}^1$ , that is*

$$\overline{\text{Hol}(M)} \cong \text{Diff}_+^\infty(\mathbb{S}^1).$$

**Proof.** Let  $(M, \mathcal{F})$  be a simply connected non-Riemannian projectively flat Randers two-manifold of constant non-zero flag curvature and  $x_0 \in M$ . Since rescaling the metric by a constant factor does not change the connection and the parallel translation, it does not change the holonomy group either. Hence we can suppose that the metric is normalized so that the curvature is  $\lambda = -\frac{1}{4}$ . Using Shen's results,  $\mathcal{F}$  can be locally expressed in the form  $\mathcal{F}_a$  given in (7) where  $a = (a_1, 0) \in \mathbb{R}^2$  is a nonzero constant vector with  $|a_1| < 1$ . From Proposition 1 we get, that the closed holonomy group of the generalized Funk disc  $(\mathbb{D}^2, \mathcal{F}_a)$  is maximal



and diffeomorphic to  $\text{Diff}_+^\infty(\mathbb{S}^1)$ , therefore the same is true for the closed holonomy group  $\overline{\text{Hol}_x(M)}$  of  $(M, \mathcal{F})$  at  $x_0 \in M$ .  $\square$

We can obtain the following classification:

**Corollary 4.** *The closure of the holonomy group  $\text{Hol}(M)$  of a simply connected, locally projectively flat Randers two-manifold of constant flag curvature  $\lambda$  is*

- (1) *the trivial group  $\{id\}$ , when  $\lambda = 0$ ;*
- (2) *the rotation group  $SO(2, \mathbb{R})$ , when  $\lambda \neq 0$  and the metric is Riemannian;*
- (3) *the orientation preserving diffeomorphism group of the circle  $\text{Diff}_+^\infty(\mathbb{S}^1)$ , when  $\lambda \neq 0$  and the metric is non-Riemannian.*

**Proof.** The holonomy structure of projectively flat Finsler manifolds was investigated in [11]: It has been proved that the holonomy group of projectively flat Finsler manifold is *a)* finite dimensional if  $\lambda = 0$  or the metric is Riemannian, and *b)* infinite dimensional if  $\lambda \neq 0$  and the metric is non-Riemannian. It is clear that the holonomy structures listed in (1) and (2) correspond to the (already well known) finite dimensional holonomy cases. Moreover, when  $\lambda \neq 0$  and the metric is non-Riemannian we get (3) from Theorem 3.  $\square$

We remark that many interesting geometric properties are related to the properties of the holonomy group. In the Riemannian case, several papers are dedicated to such problems, but in the Finslerian settings, there are very few results. For example, in [5] S. Ishihara and Y. Tashiro proved that if a complete, non-flat Riemannian manifold admits a nontrivial concircular transformation, then its local homogeneous holonomy group at any point is the special orthogonal group  $SO(n, \mathbb{R})$ . The investigation of such problems in the Finslerian settings would be very fruitful bringing new lights on the geometric structure.

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