



Clusterization and its symmetries in light nuclei

Thesis for the Degree of Doctor of Philosophy (PhD)

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Debrecen, 2026

Hereby I declare that I prepared this thesis within the Doctoral Council for Natural Sciences and Engineering, Doctoral School of Physics, University of Debrecen in order to obtain a PhD Degree in Natural Sciences at Debrecen University.

The results published in the thesis are not reported in any other PhD theses.

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Clusterization and its symmetries in light nuclei

Dissertation submitted in partial fulfilment of the requirements for the doctoral (PhD)
degree in Physics

Written by Chandra Sekhar Panda certified physicist

Prepared in the framework of the Doctoral School of Physics of the University of Debrecen

Nuclear Physics programme

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Dedicated to,

My beloved Maa, Bapa, Indu, Mausi and Nana for their
endless love, sacrifices, patience, and encouragement.

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Chapter 1

Introduction

“Ekam sat viprā bahudhā vadanti.”

“Truth is one; the wise call it by many names.”

— *Rig Veda (1.164.46)*

Nucleus is a many-body quantum mechanical system. And it is not possible to solve the many-body quantum mechanical system exactly. It is a complex system in which all the fundamental interactions act together to govern its behaviour. Because of this, nuclei show a very rich variety of phenomena. These phenomena are so diverse that no single model can describe all of them. Instead, there are models based on different physical pictures. For example, the first ever structure model considered it as a microscopic liquid drop (liquid drop model). In the shell model, it is considered as a miniature atom, as a molecular-like system in the cluster model. The situation is to the ancient Indian parable of the blind men and the elephant. Each man touches, in which each blind man touches a different part of the elephant and describes it differently. Likewise, the fundamental structure models reflect only a few aspects of the same physical reality. To explain many nuclear phenomena within a model, we need to understand the connection between these fundamental models.

Symmetry plays a central role in connecting the fundamental structure models of atomic nuclei. A common intersection of these models was found in 1958 in terms of $U(3)$ (dynamical) symmetry. This was possible when Elliot showed that the quadrupole deformation and collective rotation can be obtained from the spherical shell model: the states corresponding to a collective band are determined by their specific $SU(3)$ symmetry [1, 2]. Wildermuth and Kanellopoulos established the relation between the shell and cluster models by demonstrating

the exact equivalence of their Hamiltonians within the harmonic oscillator approximation [3]. Later, this relationship was interpreted by Bayman and Bohr in terms of $SU(3)$ symmetry [4]. However, this connection was established only for a single-shell problem, as the original Elliott model based on single valence shell. Since many nuclear phenomena extend beyond a single-shell framework, it is essential to generalize this relation to multi-major shells.

In the following decades, substantial theoretical effort was devoted to extending the connection to include major shell excitations, through which many aspects of the problem were clarified. ATOMKI has also made significant contributions to this development, continuing the advancement of algebraic approaches in the nuclear structure model.

My supervisor and his collaborator developed and applied a cluster model called semi-microscopic algebraic model (SACM) [5–7]. The SACM describes nuclear cluster structures while incorporating the Pauli exclusion principle. A connecting symmetry for different cluster configurations was introduced [8]. It was initially introduced to connect different cluster configurations. Later on, it was realised that this connecting symmetry is a unifying symmetry that connects the fundamental structure models of atomic nuclei [9–11]. This connecting symmetry is known as Multiconfigurational dynamical symmetry (MUSY).

Before my involvement in this research, my supervisors and colleagues had already applied MUSY to several nuclear systems [12–14], and many of its fundamental features had been explored [15–18]. Although MUSY is still a relatively young concept, several of its predictions have already received experimental confirmation [19, 20], thereby strengthening its theoretical foundation. In recent years, MUSY has begun to gain wider recognition and is sometimes regarded as part of the renaissance of symmetry principles in nuclear physics [21].

One of the main objectives of my research was to apply MUSY to the complex spectra of various nuclei and to provide a microscopic and unified description of their structure.

MUSY enables us to determine the stable quadrupole shapes of nuclei directly from the symmetry properties of the shell model, providing an alternative

to conventional energy-surface calculations. In particular, shape isomers can be identified by analyzing the stability and consistency of the connecting $U(3)$ symmetry, a procedure known as the Stability and Consistency of $SU(3)$ -symmetry (SCS) method [22]. This method has already been applied to investigate shape isomers in several nuclei [23–25].

A long-standing question in nuclear structure concerns how nuclear shells evolve, and whether clusterization exhibits a shell-like or a rigid, molecule-like character. The discussion so far has largely focused on qualitative aspects. The Semimicroscopic Algebraic Cluster Model (SACM), however, offers the potential to provide quantitative insights into this issue.

The SACM has been used to explore phase transitions in nuclear clustering [26–29]. These works mainly focus on the limit of very large numbers of particles (or excitation quanta), where the real phase transitions can emerge. In finite nuclear systems, some remnants of such transitions can be seen. Another important question is how experimentally observed cluster structures can be positioned within the corresponding phase diagram of clustering.

As an initial step in this research, the objective was to investigate the spectra of $U(3)$, $O(4)$, and $O(3)$ dynamical symmetry limits of SACM. Among these, only the $U(3)$ dynamical limit fully incorporates the Pauli exclusion principle, while the $O(3)$ limit does so only partially, and the $O(4)$ limit does not include it at all.

The computer program for generating the $U(3)$ dynamical limit model space was already available. My task was to develop additional programs to construct the model spaces corresponding to the $O(4)$ and $O(3)$ limits.

Chapter 2

Background

“Beauty is bound up with symmetry.”

— *Hermann Weyl*

2.1 The fundamental structure models

The structure of the atomic nucleus presents two fascinating yet challenging difficulties. One, the nature of the nuclear force, which is not fully understood. Second, the complex nature of the atomic nucleus, which is a many-body quantum mechanical system. One needs to solve the many-body Schrödinger equation to know the nuclear structure, and it is analytically impossible.

Therefore, we require simplified approaches, i.e., models to describe the nature of the nucleus that have a well-defined mathematical formalism and have a clear physical interpretation. A brief overview of fundamental structure models: the shell, the collective model, and cluster models, is presented below.

2.1.1 Shell model

The shell model assumes the atomic nucleus as miniature atom where the nucleus is described as a system of nucleons moving in discrete orbits within an average potential, in a way similar to electrons moving around the nucleus in an atom. In this picture, each nucleon moves independently in a mean field generated by all other nucleons, reducing the complex many-body problem to an effective single-particle description.

An appropriate form of the nuclear potential is required to describe the motion of nucleons. A simple starting point is the three-dimensional harmonic os-

cillator potential, because with this potential the three dimensional Schrödinger equation can be solved and a well defined shell structure. However, this potential alone is not enough to reproduce experimental observations. A more realistic description includes additional terms, most importantly a strong spin-orbit interaction; $\mathbf{L}\cdot\mathbf{S}$, where $\mathbf{L}$ is the angular momentum and $\mathbf{S}$ is spin of the nucleon. This interaction plays a crucial role in determining the ordering of energy levels.

The spin-orbit interaction splits the energy levels that have the same orbital angular momentum but different total angular momentum. This modification of the level structure helps to explain the experimentally observed magic numbers, 2, 8, 20, 28, 50, 82, and 126, which correspond to closed shells.

In addition to explaining magic numbers, the shell model provides a useful framework to determine the ground-state spin-parity of a nucleus from the configuration of the valence nucleons, magnetic dipole moments and transition probabilities.

2.1.2 Collective model

There are nuclei for which the shell model provides only a limited description, since most of its predictions are based on unpaired nucleons. In such cases, the nucleus shows collective behavior arising from all its constituent nucleons. In particular, for nuclei with even numbers of protons and neutrons, experimental observations often indicate patterns that are better understood in terms of collective vibrations and rotations of a deformed nuclear shape rather than independent-particle motion.

The collective model of the atomic nucleus is based on the liquid drop picture, in which the nucleus is treated as a electrically charged liquid drop. The properties of nucleus come from the collective motion of all its constituent nucleons rather than from the behavior of individual particles.

A key assumption of the collective model is that the nucleus can deviate from a spherical shape and adopt deformed configurations. Depending on the nature of this deformation, the nucleus can exhibit different types of collective motion, i.e., vibrational and rotational motion.

In the vibrational limit, the average shape of the nucleus is spherical, but the instantaneous shape is not. The nucleus undergoes surface vibrations, such as dipole, quadrupole, and octupole modes, which give rise to vibrational spectra. On the other hand, for deformed nuclei, rotational motion becomes dominant. In such cases, the energy levels follow a rotational pattern, proportional to $\mathbf{L}(\mathbf{L}+1)$

where L is the angular momentum.

2.1.3 Cluster model

In cluster model, nucleons are not treated as completely independent particles, but rather as forming tightly bound groups (clusters) which interact with each other within the nucleus.

One of the most common example cluster is α -cluster, consisting of two protons and two neutrons, which is especially stable due to its closed-shell structure. The nucleus is then described as a system of such clusters arranged in specific geometric configurations.

The degrees of freedom of clusters can be divided into two types: one describes the internal structure of the clusters, while the other accounts for the relative motion of the clusters with respect to each other. The interaction between clusters gives rise to characteristic energy spectra and decay properties.

The cluster model has been particularly successful in describing phenomena such as α -decay, or other processes involving the emission of exotic, heavier nuclei (e.g., carbon, molybdenum).

2.2 Connection of the models

2.2.1 Single shell problem

Elliott defined an effective Hamiltonian [1, 2] for spherical shell model with harmonic oscillator Hamiltonian and quadrupole-quadrupole interaction:

$$H = H_{HO} + \chi QQ. \quad (2.1)$$

The quadrupole-quadrupole interaction QQ is expressed with algebraic quadrupole operators and it can be expressed in terms of second order casimir invariant of $SU(3)$ and $SO(3)$ groups and therefore, diagonal in $SU(3) \supset SO(3)$ basis. As the harmonic oscillator Hamiltonian can be defined by first order invariant of $U(3)$ group, the expression 2.1 can be rewritten as

$$H = C_{U(3)}^{(1)} + \alpha C_{SU(3)}^{(2)} + \beta C_{SO(3)}^{(2)} \quad (2.2)$$

Here, $C^{(i)}$ represents the invariant operator of order i corresponding to the algebra specified in the subscript. Since the expression is constructed using Casimir operators associated with a single algebraic chain,

$$\begin{aligned} U(3) \supset SU(3) \supset SO(3) \\ |[n_1, n_2, n_3], (\lambda, \mu), K, L \rangle, \end{aligned} \quad (2.3)$$

the corresponding eigenvalue problem can be solved analytically. The $U(3)$ quantum number $[n_1, n_2, n_3]$ related to $SU(3)$ labels (λ, μ) as $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$. The quantum number K connects the $SU(3)$ labels (λ, μ) to the orbital angular momentum L as follows [1, 2]:

$$K = \min(\lambda, \mu), \min(\lambda, \mu) - 2, \dots, 1 \text{ or } 0 \quad (2.4)$$

$$L = K, K + 1, K + 2, \dots, K + \max(\lambda, \mu) \quad (2.5)$$

However, for $K = 0$, the allowed values of L are given by

$$L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, \dots, 1 \text{ or } 0. \quad (2.6)$$

In group chain 2.3 only space part of the model space is shown. The spin-isospin sector is defined by Wigner's supermultiplet theory [30]

$$\begin{array}{ccccc} U^{ST}(4) & \supset & SU^S(2) & \otimes & SU^T(2) \\ |[f_1, f_2, f_3, f_4], & & S, & & T \end{array} \rangle. \quad (2.7)$$

Thus, the model has $U^{ST}(4) \otimes U(3)$ group structure.

The nuclear quadrupole deformation is governed by the $SU(3)$ quantum numbers (λ, μ) . the irreducible representations $(0, 0)$, $(\lambda, 0)$, $(0, \mu)$, and arbitrary (λ, μ) describe spherical, prolate, oblate, and triaxial shapes, respectively. Furthermore, rotational bands are associated with states exhibiting a definite $SU(3)$ symmetry.

The relationship between shell model and cluster model was established by Wildermuth and Kanellopoulos [3]. Within the harmonic oscillator approximation, they showed that the shell-model Hamiltonian can be rewritten exactly in the form of the cluster-model Hamiltonian, i.e., $H_{SM} = H_{CM}$. This equivalence establishes a direct correspondence between the wave functions of the two descriptions. Specifically, a cluster-model state can be expressed as a superposition of shell-model basis states that share the same energy.

Bayman and Bohr [4] later reformulated the connection between the cluster and shell models within the framework of $U(3)$ symmetry. As a result, collective and cluster bands could be systematically identified among the shell-model states through their well-defined $SU(3)$ symmetry properties.

Hence, the common intersection of shell, collective, and cluster model was found in terms of $U(3)$ (dynamical) symmetry.

2.2.2 MUSY: the general connection

Considerable efforts have been made to construct symmetry-based models for the multi-major-shell problem within the shell [31, 32], collective [33, 34], and cluster [5–7] models. Ultimately, for many major shell a unified description of the fundamental structure model is given by the group chain:

$$\begin{array}{ccccc} U_s(3) & \otimes & U_e(3) & \supset & U(3) \supset SU(3) \supset SO(3) \\ |[n_1^s, n_2^s, n_3^s], & & [n_1^e, n_2^e, n_3^e], & \rho, & [n_1, n_2, n_3], (\lambda, \mu), K, L \end{array} \rangle \quad (2.8)$$

Here, ρ distinguishes the multiple occurrences of the irreducible representation $[n_1, n_2, n_3]$ that arise in the direct product $[n_1^s, n_2^s, n_3^s] \otimes [n_1^e, n_2^e, n_3^e]$. The

script s represent the band-head or valence shell for shell of collective model, where as for cluster model, it represents internal cluster structure. The superscript e indicates major-shell excitation in each case.

This connecting symmetry is known as multiconfigurational dynamical symmetry (MUSY) [9, 11]. It is a composite symmetry wherein each configuration possesses a standard $U(3)$ dynamical symmetry, and an extra symmetry that connects different configurations. The latter corresponds to invariance under transformations in the pseudo-space of particle indices. Through this unifying role, MUSY offers a common framework capable of describing various configurations, encompassing both shell and cluster structures.

In MUSY the model space is constructed microscopically, i.e., the Pauli principle is treated properly.

For the shell model space we use an L-S coupling scheme. To obtain a complete anti-symmetric many-body wave function, first it is necessary to determine the $U^{ST}(4)$ irreps of the different major shells. The anti-symmetrization is satisfied by the associate representation of $U(N)$ (N denotes the total number of orbitals within a single-major shell of the harmonic oscillator) and $U^{ST}(4)$, where their Young patterns are related by reflection: interchanging the rows and columns [35]. Then, the $U(3)$ representations associated with the relevant $U(N)$ and $UST(4)$ representations are used in constructing the model space. Finally, the spurious center of mass must be removed, e.g., according to the procedure of [36].

For cluster model we use the strongly coupled $U(3)$ scheme, where the clusters are characterized by $U(3):[n_1, n_2, n_3]$ quantum numbers. The Wildermuth condition minimum provides the minimum number of oscillator quanta in relative motion. The $U(3)$ representations are then coupled with a wide range of allowed relative-motion configurations. The Pauli-allowed cluster space, free from spurious center-of-mass excitations, can be obtained by matching it with the shell-model space.

One of the interesting property of MUSY is that it shows dual symmetry breaking [17]. The $U(3)$ and $SU(3)$ symmetries are dynamically broken due presence of symmetry-breaking interaction represented by the $SO(3)$ -invariant, while the rotational ($SO(3)$) symmetry is spontaneously broken in the eigenvalue problem of the intrinsic Hamiltonian [18]. The non-symmetric ground state of a symmetric Hamiltonian is due the spontaneous symmetry breaking. In fact this type of dual symmetry breaking is common in the algebraic structure models,

e.g. Elliot model.

The connecting nature of MUSY allows it to describe, within a single framework, a range of nuclear phenomena: quadrupole deformation, shell structure, clusterization, and relationships to reaction channels. Thus, MUSY has a broad range of applications.

Energy spectrum

In MUSY, a simple dynamically symmetric Hamiltonian that invariant with respect to transformation between different configuration can be applied. It expressed using the invariant of second part of group chain (3); $U(3) \supset SU(3) \supset SO(3)$:

$$\hat{H} = \hat{C}_{U(3)}^{(1)} + a\hat{C}_{SU(3)}^{(2)} + b\hat{C}_{SU(3)}^{(3)} + d\hat{C}_{SO(3)}^{(2)}. \quad (2.9)$$

This Hamiltonian allows the analytical expression of energies for various configurations within a unified framework. In the above expression, the effects of shell excitation $U_e(3)$ and internal structure $U_s(3)$ arise only through their coupled $[U(3)]$ representation. The Casimir operator $\hat{C}_{U(3)}^{(1)}$ is associated with the Harmonic oscillator Hamiltonian $\{(\hbar\omega)\hat{n}\}$, whereas $\hat{C}_{SU(3)}^{(2)}$ and $\hat{C}_{SO(3)}^{(2)}$ represent the quadrupole-quadrupole and angular momentum interactions, respectively. In the third term $\hat{C}_{SU(3)}^{(3)}$ distinguishes between prolate and oblate shape of nucleus [13]. The eigen values of $\hat{C}_{SU(3)}^{(2)}$ and $\hat{C}_{SU(3)}^{(3)}$ for $SU(3)$ quantum numbers are $\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)$ and $(\lambda - \mu)(\lambda + 2\mu + 3)(2\lambda + \mu + 3)$, respectively. And the expectation value of $\hat{C}_{SO(3)}^{(2)}$ is $L(L+1)$.

Electromagnetic transition

The reduced transition probability $B(\omega l; \alpha_i L_i \rightarrow \alpha_f L_f)$ is defined using the reduced matrix element of the corresponding multipole transition operator, for both electric and magnetic transitions, as

$$B(\omega l; \alpha_i L_i \rightarrow \alpha_f L_f) = \frac{2L_f + 1}{2L_i + 1} \left| \langle \alpha_f L_f \| T^{(\omega l)} \| \alpha_i L_i \rangle \right|^2. \quad (2.10)$$

Here, $T^{(\omega l)}$ denotes the transition operator, where ω specifies the type of transition and l indicates the multipolarity. The quantum numbers L_i and L_f correspond to the angular momenta of the initial and final states, respectively, while α_i and α_f represent any additional quantum numbers required to fully describe these states.

In algebraic models, electromagnetic transition operators are constructed as Hermitian combinations of group generators, chosen to have the proper tensorial properties [37]. In most practical applications, it is sufficient to include only one-body contributions. However, higher-order terms may also be incorporated when required by specific physical considerations.

In my investigations, I calculated the $B(E2)$ values. The exact expressions and the selection rules for the $T^{(E2)}$ operator used in the calculations are explained when they appear, before presenting the results.

Shape isomers

The applicability of exact SU(3) symmetry is confined to the low-energy domain of light nuclei. At higher energy and in heavier nuclei, it is not valid as a result of symmetry breaking interactions. However, a generalized framework known as quasi-dynamical (or effective) SU(3) symmetry can be utilized in both situation [38–40].

The quasi-dynamical generalization enables the investigation of the stability and self-consistency of the SU(3) symmetry, called the SCS method [22]. Therefore, this approach can be used to identify shape isomers and determine their corresponding quasi-dynamical SU(3) quantum numbers. This method serves as an alternative to usual energy minimum calculation. The details procedure are as follows:

1. In the first step, the Nilsson states are calculated by solving the eigenvalue equation a cylindrically symmetric deformed Hamiltonian:

$$H = -\frac{\hbar^2}{2M} \Delta + \frac{M}{2} [\omega_{\perp}^2 (x^2 + y^2) + \omega_z^2 z^2] - c (\vec{l} \cdot \vec{s}) - D \vec{l}^2, \quad (2.11)$$

which consists of deformed harmonic oscillator potential, along with spin-orbit and angular momentum-dependent terms. The elongation parameter ϵ is defined through the relations

$$\omega_z = \omega_0(1 - \frac{2}{3}\epsilon), \omega_{\perp} = \omega_0(1 + \frac{1}{3}\epsilon), \epsilon = \frac{\omega_{\perp} - \omega_z}{\omega_0}. \quad (2.12)$$

The deformation parameter β related to ϵ , with $\epsilon \approx 0.95\beta$ [22].

2. The Nilsson orbitals are then determined as functions of the quadrupole deformation parameters. Using these orbitals, the many-particle state is built by filling them with nucleons according to the energy-minimum and the Pauli

exclusion principle.

3. Diagonalize the triaxially deformed Hamiltonian in cylindrical coordinated using the deformed harmonic oscillator potential

$$V = \frac{M}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \quad (2.13)$$

The Nilsson orbitals obtained for a specific deformation (ϵ, γ) are then expressed as linear combinations of the previously determined asymptotic Nilsson states [41].

4. The effective $SU(3)$ quantum numbers (λ, μ) are determined from the linear combinations of the single-particle orbitals and together with the relations of the large deformation [40].

5. The quadrupole deformation parameters (β, γ) can be determined from (λ, μ) using following relations [42]:

$$\beta^2 = \frac{16\pi}{5N_0^2}(\lambda^2 + \mu^2 + \lambda\mu), \gamma = \arctan\left(\frac{\sqrt{3}\mu}{2\lambda + \mu}\right). \quad (2.14)$$

Here $N_0 = n_1 + n_2 + n_3 + \frac{3}{2}(A - 1)$ is the total number of oscillator quanta, where n_1, n_2, n_3 denote $U(3)$ quantum numbers and A represents the nuclear mass number.

In this way, the stability and self-consistency of the effective $SU(3)$ quantum numbers and deformation parameters can be studied. More detailed procedures are described in [22].

Clusterizations and reaction channels

The shape isomers are characterized by their space ($U(3)$) and spin-isospin ($U^{ST}(4)$) symmetry. These symmetries impose selection rules that determine which binary clusterizations are allowed. Since cluster configurations are directly associated with reaction channels, they provide insight into the possible binary reactions through which shape isomers can be populated or decay.

For a binary cluster configuration, the $U(3)$ selection rule is expressed as

$$[n_1, n_2, n_3] = [n_1^{(1)}, n_2^{(1)}, n_3^{(1)}] \otimes [n_1^{(2)}, n_2^{(2)}, n_3^{(2)}] \otimes [n^{(R)}, 0, 0], \quad (2.15)$$

where $[n_1, n_2, n_3]$ denotes the $U(3)$ quantum numbers of the parent nucleus. The superscripts (1) and (2) refer to the two clusters, while (R) corresponds to the relative motion between them. The clusters are considered in their intrinsic

ground states, although collective excitations within the same irreducible representation are included. The clusters may exhibit quadrupole deformations, and no restriction is imposed on their relative orientations.

In addition to the spatial symmetry, the spin-isospin degrees of freedom are governed by a corresponding $U^{ST}(4)$ selection rule,

$$[f_1, f_2, f_3, f_4] = [f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, f_4^{(1)}] \otimes [f_1^{(2)}, f_2^{(2)}, f_3^{(2)}, f_4^{(2)}]. \quad (2.16)$$

If a given cluster configuration is not allowed by these selection rules, its degree of forbiddenness can be quantified. This is achieved by defining the minimal distance between the $U(3)$ representation of the parent nucleus and those associated with the cluster configuration:

$$\min \sqrt{(\Delta n_1)^2 + (\Delta n_2)^2 + (\Delta n_3)^2}, \quad (2.17)$$

where $\Delta n_i = |n_i - n_{i,k}^c|$. Here, n_i corresponds to the parent nucleus, while $n_{i,k}^c$ represents the $U(3)$ quantum numbers of the cluster configuration c , with k distinguishing different resulting representations obtained from the triple product of Eq. 2.15.

A quantity called reciprocal forbiddenness, denoted by S , is introduced and defined as

$$S = \frac{1}{1 + \min [(\Delta n_1)^2 + (\Delta n_2)^2 + (\Delta n_3)^2]^{1/2}}, \quad (2.18)$$

such that $0 \leq S \leq 1$. The values $S \approx 0$ and $S \approx 1$ indicate completely forbidden and allowed cluster configuration, respectively.

Spectroscopic factor

The spectroscopic amplitude $A_{nl}(y' \rightarrow y + x)$, describing the breakup of y' into $y + x$ with relative motion $\phi_{nlm}(r_{xy})$, is expressed using the internal-coordinate wave functions $\phi(\xi)$ of the participating nuclei. It is given by

$$A_{nl}(y', y) = \left(\frac{y'}{y} \right)^{1/2} \int [\phi_y(\xi_y) \phi_x(\xi_x) \phi_{nlm}(r_{xy})]^* \phi_{y'}(\xi_{y'}) d\xi_{y'}. \quad (2.19)$$

The binomial coefficient is due to antisymmetrization [43].

The spectroscopic strength associated with the $A(a, b)B$, $a = b + x, B = A + x$

reaction is defined as

$$A_{nl}^{n'l'} = \sum A_{nl}(B \rightarrow A + x) A_{n'l'}^*(a \rightarrow b + x), \quad (2.20)$$

where the sum runs over all possible intermediate configurations generated by the transferred nucleon(s) x . If the x transferred particles occupy a single, well-defined configuration (for example, $(0s)^4$ for an α particle), only one term remains in the sum. In this case, the structure factor is given by the product of the two corresponding spectroscopic amplitudes.

For harmonic-oscillator eigenstates, Elliott and Skyrme [44] derived a straightforward connection between wave functions expressed in internal coordinates (ϕ) and those defined in a space-fixed frame (ψ). Using this result, the spectroscopic amplitude can be written as

$$A_{nl}(y' \rightarrow y + x) = \left(\frac{y'}{y}\right)^{1/2} \left(\frac{y'}{y}\right)^{n/2} \times \sum \langle \psi_y; \psi_x | \psi_{y'} \rangle \langle \phi_x(\xi_x) \phi_{nlm}(r) | \psi_x \rangle. \quad (2.21)$$

Here, $\langle \psi_y; \psi_x | \psi_{y'} \rangle$ denotes the shell-model coefficient of fractional parentage (c.f.p.) associated with removing x nucleons in an eigenstate $|\psi_x\rangle$ from the state $|\psi_{y'}\rangle$, leaving the remaining $y = y' - x$ nucleons in the state $|\psi_y\rangle$. The factor $\langle \phi_x(\xi_x) \phi_{nlm}(r) | \psi_x \rangle$ (the "G" factor of Ref. [45]) represents the overlap between the separated x nucleons in the state $|\psi_x\rangle$ and a configuration characterized by the internal wave function $\phi_x(\xi_x)$ and relative motion $\phi_{nlm}(r)$ in a space-fixed coordinate system. The summation in 2.21 runs over all possible states $|\psi_x\rangle$ consistent with $|\psi_y\rangle$, $|\psi_{y'}\rangle$, and $\phi_x(\xi_x)$ and $\phi_{nlm}(r)$. Within a single major oscillator shell, the sum reduces to one term when the transferred nucleons occupy $(0s)^x$ configuration with internal SU(3) symmetry $(0, 0)$.

The SU(3) LS -coupled shell-model formalism can be employed here. Let us consider that the initial and final states of the target and residual nuclei are characterized by pure symmetry labels $[f]\alpha(\lambda\mu)\beta ST$, where $[f]$ denotes the spatial symmetry of the major shell, $(\lambda\mu)$ are the SU(3) quantum numbers, α and β are multiplicity indices, and S and T represent the spin and isospin, respectively.

Under this assumption, the expression which involves a summation over all

quantum numbers $[f]\alpha(\lambda\mu)\kappa_L L\beta ST$, simplifies and can be written as

$$\begin{aligned}
A_{nlsj}(y' \rightarrow y + x) &= G \left(\frac{y'}{y} \right)^{1/2} \left(\frac{y'}{y} \right)^{n/2} \\
&\times \langle [f]\alpha(\lambda\mu)\beta ST; [x](n0)lst | \rangle [f']\alpha'(\lambda'\mu')\beta' S'T' \rangle \\
&\times \langle TM_T; tm_t | T'M'_T \rangle \\
&\times \sum_{\kappa_L L \kappa_{L'} L'} cc' X \left\{ \begin{matrix} L & S & J \\ l & s & j \\ L' & S' & J' \end{matrix} \right\} \\
&\times \langle (\lambda\mu)\kappa_L L; (n0)l | \rangle \langle (\lambda'\mu')\kappa'_L L' \rangle.
\end{aligned} \tag{2.22}$$

In this expression, $X\{\dots\}$ denotes a unitary $9j$ -symbol, which governs the transformation between the jj - and LS -coupling schemes [46]. The quantity $\langle (\lambda\mu)\kappa_L L; (n0)l | \rangle \langle (\lambda'\mu')\kappa'_L L' \rangle$ is a reduced $SU(3) \supset R(3)$ Wigner coefficient [47], while c and c' are the expansion coefficients of the target and residual nuclear wave functions in the $SU(3)$ - LS coupled basis. For $x = 4$, this factor takes the simplified form [48],

$$G = \left[\frac{Q!}{4^Q q_1! q_2! q_3! q_4!} \right]^{1/2} \left[\frac{4!}{a! b! c! d!} \right]^{1/2} \delta_{[f][4]} \delta_{S0} \delta_{T0} \delta_{Q, 2N+L} \delta_{\lambda Q} \delta_{\mu 0}. \tag{2.23}$$

Here, $q_i = 2n_i + l_i$ is the number of oscillator quanta associated with the i -th transferred particle, and $Q = q_1 + q_2 + q_3 + q_4 = 2N + L$ is the total number of oscillator quanta of the transferred cluster.

The factor $\langle [f]\alpha(\lambda\mu)\beta ST; [x](n0)lst | \rangle [f']\alpha'(\lambda'\mu')\beta' S'T' \rangle$ represents an x -particle coefficient of fractional parentage (c.f.p.) [49].

A reduced (or relative) spectroscopic amplitude is expressed as the product of the last three factors appearing in Eq. (2.22). These factors can be evaluated within the $SU(3)$ basis, for which the coefficients c and c' are equal to unity.

$$\begin{aligned}
A_{nlsj}^R(y' \rightleftharpoons y + x) &= \sum_{\kappa_L L \kappa_{L'} L'} C_{\kappa_L L} C'_{\kappa_{L'} L'} \\
&\times X \left\{ \begin{matrix} L & S & J \\ l & s & j \\ L' & S' & J' \end{matrix} \right\} \\
&\times \langle (\lambda\mu)\kappa_L L; (n0)l | \rangle \langle (\lambda'\mu')\kappa'_L L' \rangle.
\end{aligned} \tag{2.24}$$

The quantity A_{nlsj}^R contains the same spin dependence as the full spectroscopic amplitude A_{nlsj} given in Eq. (2.22), therefore, relative to specific transfer (e.g., ground-state to ground-state), both expressions are identical.

The complete spectroscopic factor can be obtained by multiplying square of $G\left(\frac{y'}{y}\right)^{\frac{1}{2}}\left(\frac{y'}{y}\right)^{\frac{1}{2}n} \times \langle [f]\alpha(\lambda\mu)\beta ST; [x](n0)lst | \rangle [f']\alpha'(\lambda'\mu')\beta' S'T' \rangle \times \langle TM_T; tm_t | T'M'_T \rangle$ with $(A_{nlsj}^R)^2$.

2.3 Phases of nuclei

Atomic nuclei exhibit both thermal and non-thermal phases, along with their transitions. The liquid–gas transition in finite nuclear matter [50] and the transition from quark–gluon plasma [51] are well-known examples of thermal phase transitions. In contrast, for cold phases the temperature is not a relevant parameter; instead, their properties and transitions can be described in close analogy with thermodynamical phase concepts. These non-thermal phases, often referred to as shape or zero temperature phases. The cold phases can be studied using dynamical symmetry limits of algebraic models of nuclear structure [52–55]. Instead of temperature and pressure, the control parameters are determined by the relative strengths of the Hamiltonian components associated with specific dynamical symmetries, which characterize the corresponding phase(s).

A model is regarded as fully algebraic if both the basis states and the physical operators are defined in terms of underlying group symmetries. Within such a framework, dynamical symmetries arise as particular limits in which the energy eigenvalue problem becomes analytically solvable. These limits correspond to different phases of the system, while the relative strengths of interactions associated with different dynamical symmetries serve as control parameters.

2.3.1 Phases of shell and collective models

Phases in the nuclear shell model correspond independent particle, the $SU(2)$ pairing and the $SU(3)$ limits [56]. The last two limits allow analytical solutions of manybody problem. But they differ in their type of interactions. The $SU(2)$ limit corresponds to the jj -coupling scheme, where as $SU(3)$ limit corresponds to LS -coupling scheme. The phases of shell model can be shown in a triangle (shown in left part of Fig. 2.1).

In collective models, phases are associated with different geometrical shapes and types of motion of the nucleus. Algebraic collective models, such as the interacting boson model (IBM) [53], possess several dynamical-symmetry limits that correspond to characteristic structures: the $U(5)$ limit represents spherical vibrational nuclei, the $SU(3)$ limit corresponds to axially deformed rotational nuclei, and the $O(6)$ limit describes γ -soft nuclei.

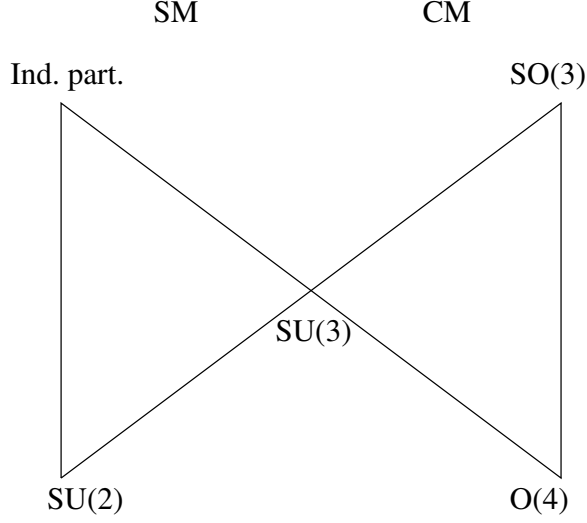


Figure 2.1: Relationship between phases of the shell model and the cluster model.

2.3.2 Phases of clusterization

One of the earliest frameworks developed to describe nuclear structure is the cluster model. It was first introduced as the alpha-particle model in the pioneering works of Gamow [57, 58], before the discovery of the neutron. Since then, a large number of cluster models have been developed [59], most of which are founded on the idea that nuclear clusters are composite objects made up of nucleons.

In the simplest binary cluster system consisting of closed-shell clusters, the phase space is determined by their relative motion. It can be described by the vibron model [60, 61], which is an algebraic framework for dipole-type collectivity based on a $U(4)$ group structure. Here $U(4)$ is a dynamical symmetry group, that contains spectrum-generating component alongside the $U(3)$ symmetry group associated with three-dimensional motion. It has two dynamical symmetry limits, in which $U(4)$ group reduced to two group chains:

$$U_R(4) \supset U_R(3) \supset SU_R(3) \supset SO_R(3) \quad (2.25)$$

$$U_R(4) \supset O_R(4) \supset SO_R(4) \supset SO_R(3) \quad (2.26)$$

Here, R denotes the relative motion. These limits correspond to two phases and can be represented on a single line [62]. By including the internal degrees

of freedom of the clusters, three dynamical-symmetry limits arise, which correspond to three phases of clusterization. The three phases of clusterization can be explained by the semi-microscopic algebraic cluster model (SACM) [5–7].

Dynamical symmetry limits of SACM

Within the semimicroscopic algebraic cluster model (SACM), the intrinsic structure of each cluster is treated using Elliott’s SU(3) shell-model framework [1, 2], characterized by the group structure $U^{ST}(4) \otimes U(3)$. The relative motion between the clusters is described by the modified vibron model based on the $U_R(4)$ symmetry [60, 61] (Modification means to the removal of the Pauli-forbidden states and the spurious excitation of the centre of mass). The resulting model space becomes identical to that used in fully microscopic cluster approaches, when it is expressed in the $U(3)$ basis.

In the case of binary clusterization, the model is characterized by the algebraic structure $U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_R(4) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3)$. The group can be reduced to three different group chains:

$$\begin{aligned} U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4) &\supset U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes U_C(3) \otimes U_R(3) \supset U_C^S(2) \otimes U_C^T(2) \otimes U(3) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes O(3) \supset SU(2) \otimes SU_C^T(2) \end{aligned} \quad (2.27)$$

$$\begin{aligned} U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4) &\supset U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes O_C(3) \otimes O_R(4) \supset U_C^S(2) \otimes U_C^T(2) \otimes O_C(3) \otimes O_R(3) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes O(3) \supset SU(2) \otimes SU_C^T(2) \end{aligned} \quad (2.28)$$

$$\begin{aligned} U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4) &\supset U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes U_C(3) \otimes U_R(3) \supset U_C^S(2) \otimes U_C^T(2) \otimes O_C(3) \otimes O_R(3) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes O(3) \supset SU(2) \otimes SU_C^T(2) \end{aligned} \quad (2.29)$$

However, when the clusters have zero spin and isospin the groups $U^{ST}C_i(4)$ contribute only to the construction of the model space and do not affect the physical

operators. In this situation, the algebraic structure simplifies to $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$ [6, 7]. Then the group can be reduced to three subgroup chains that contain angular momentum ($SO(3)$) group:

$$\begin{aligned}
& U_{C_1}(3) \quad \otimes \quad U_{C_2}(3) \quad \otimes \quad U_R(4) \\
& \left| [n_1^{C_1}, n_2^{C_1}, n_3^{C_1}], \quad [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}], \quad [N, 0, 0, 0], \right. \\
& \supset U_C(3) \quad \otimes \quad U_R(3) \supset U(3) \supset SU(3) \supset SO(3) \\
& \left. [n_1^C, n_2^C, n_3^C], \quad [n_\pi, 0, 0], \quad [n_1, n_2, n_3], \quad (\lambda, \mu), \quad K, L \right\rangle
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
& U_{C_1}(3) \quad \otimes \quad U_{C_2}(3) \quad \otimes \quad U_R(4) \\
& \left| [n_1^{C_1}, n_2^{C_1}, n_3^{C_1}], \quad [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}], \quad [N, 0, 0, 0], \right. \\
& \supset U_C(3) \quad \otimes \quad O_R(4) \supset SU_C(3) \quad \otimes \quad SO_R(4) \\
& [n_1^C, n_2^C, n_3^C], \quad (\omega, 0), \quad (\lambda_C, \mu_C), \quad \omega, \\
& \supset SO_C(3) \quad \otimes \quad SO_R(3) \supset SO(3) \\
& K_C, \quad L_C, \quad L_R, \quad L \rangle
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
& U_{C_1}(3) \quad \otimes \quad U_{C_2}(3) \quad \otimes \quad U_R(4) \\
& \left| [n_1^{C_1}, n_2^{C_1}, n_3^{C_1}], \quad [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}], \quad [N, 0, 0, 0], \right. \\
& \supset U_C(3) \quad \otimes \quad U_R(3) \supset SU_C(3) \quad \otimes \quad SU_R(3) \\
& [n_1^C, n_2^C, n_3^C], \quad [n_\pi, 0, 0], \quad (\lambda_C, \mu_C), \quad (n_\pi, 0), \\
& \supset SO_C(3) \quad \otimes \quad SO_R(3) \supset SO(3) \\
& K_C, \quad L_C, \quad L_R, \quad L \rangle
\end{aligned} \tag{2.32}$$

The group chain 2.30 represents the $U(3)$ dynamical symmetry, which is associated with a shell-like cluster configuration. The group chain 2.31 corresponds to the $O(4)$ limit, describing a rigid-rotor or rigid-molecule-like cluster structure, whereas the group chain 2.32 represents the weak-coupling $O(3)$ limit. The earlier equations (2.27)-(2.29) also correspond to the shell-like, rigid-molecule-like, and weak-coupling clusterization limits, analogous to these equations. They contain the spin and isospin degrees of freedom, with a spin-orbit coupling for the angular momentum.

In Eqs. 2.30-2.32, $[n_1^{C_i}, n_2^{C_i}, n_3^{C_i}]$ refers to the $U(3)$ quantum numbers of the i^{th} cluster C_i . The $U_C(3)$: $[n_1^C, n_2^C, n_3^C]$ is obtained from the coupling of the

individual cluster representations: $[n_1^{C_1}, n_2^{C_1}, n_3^{C_1}] \otimes [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}]$. The $U_R(4)$ symmetry is represented by $[N, 0, 0, 0]$, with N representing the total boson numbers. As N is conserved, a single $U_R(4)$ representation is sufficient to calculate the energy eigenstates. The size of the model space can be adjusted by varying N .

The outer product $[n_1^C, n_2^C, n_3^C] \otimes [n_\pi, 0, 0]$ give the combined $U(3)$ representations, $[n_1, n_2, n_3]$. $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$ are the $SU(3)$ quantum numbers (λ, μ) derived from $U(3)$. The same procedure is also used to derive the quantum numbers for various $SU_\gamma(3)$ groups, where γ stands for C_1 , C_2 , C , or R . The $SO(3)$ group is labelled by the angular momentum quantum number, L . It is related to $SU(3)$ quantum number (λ, μ) through K as given in Eqs. 2.4, 2.5, and 2.6. Within a given $SU(3)$ irreps (λ, μ) , the quantum number K distinguish states that share the same angular momentum L . Similarly, L_C and K_C can be determined. For the relative angular momentum L_R , associated with $SO_R(3)$, is uniquely determined within the $(n_\pi, 0)$ representation and therefore does not require K_R .

The label ω denotes the $SO_R(4)$ related to N and L_R as (Eqs. (5.128) and (5.129) of Ref. [63]):

$$\begin{aligned}\omega &= N, N-2, \dots, 1 \text{ or } 0, \\ L_R &= 0, 1, \dots, \omega.\end{aligned}\tag{2.33}$$

The angular momentum L for the case of $O(4)$ and $O(3)$ limit is obtained from coupling of L_C and L_R .

Using only the relevant quantum numbers, the basis states corresponding to these symmetries may be written as follows:

$$|[n_1^C, n_2^C, n_3^C], n_\pi, [n_1, n_2, n_3], (\lambda, \mu), K, L\rangle\tag{2.34}$$

$$|[n_1^C, n_2^C, n_3^C], (\lambda_C, \mu_C), \omega, K_C, L_C, L_R, L\rangle\tag{2.35}$$

$$|[n_1^C, n_2^C, n_3^C], (\lambda_C, \mu_C), n_\pi, K_C, L_C, L_R, L\rangle\tag{2.36}$$

The dynamically symmetric Hamiltonians for each limit can be expressed using invariant operators, which have analytical solutions in their respective bases. The most relevant Casimir operators are listed below (their analytical forms can be obtained from Chapter 5 of [64], while some are also available in Ref. [7].):

$$\langle \hat{C}_{U(3)}^1 \rangle = n_1 + n_2 + n_3 \quad (2.37)$$

$$\langle \hat{C}_{U(3)}^2 \rangle = n_1^2 + n_2^2 + n_3^2 + 2(n_1 - n_3) \quad (2.38)$$

$$\langle \hat{C}_{SU(3)}^2 \rangle = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu) \quad (2.39)$$

$$\langle \hat{C}_{SO(3)}^2 \rangle = L(L + 1) \quad (2.40)$$

$$\langle \hat{C}_{SO(4)}^2 \rangle = \omega(\omega + 2) \quad (2.41)$$

To investigate phase transitions, a general Hamiltonian can be written using control parameters x and y : $H = xH_{U(3)} + (1 - x - y)H_{O(4)} + yH_{O(3)}$. When $x = y = 0$ it belongs to the $O(4)$ limit, while $x = 1, y = 0$ leads to the $U(3)$ limit, and to the $O(3)$ limit when $x = 0, y = 1$.

Since the binary cluster model has three dynamical symmetry limits, its phase diagram is two-dimensional and can be illustrated by a triangle [62]. The ls -coupling limit establishes a connection between the phase diagrams of the shell and cluster models, and in this limit Elliott's $SU(3)$ symmetry appears naturally. It is illustrated in Figure 2.1.

Chapter 3

New investigations

“It is more important to have beauty in one’s equations than to have them fit experiment.”

— *Paul Dirac*

3.1 ^{18}O spectrum and MUSY

The interplay between cluster and shell-model degrees of freedom remains a central topic in nuclear structure studies, it was mainly focused on α -like ($N=Z=\text{even}$) nuclei, e.g., see in [65]. But more recently the study extended to neutron-rich systems [66], including the development of covalent neutron configurations [67].

The ^{18}O nucleus provides a particularly interesting example, as within a cluster description the centers of its mass and charge distributions are spatially separated. It was argued in Ref. [68] that the ^{14}C nucleus can be regarded as a suitable cluster, analogous to ^{16}O , due to its similar properties.

Here, I applied multiconfigurational dynamical symmetry (MUSY) to the ^{18}O nucleus [P1]. The present work shares some similarities with the previous semi-microscopic algebraic cluster mode(SACM) study [6]; still, several important differences should be noted. First, the current analysis benefits from richer experimental data and a well-defined band structure. Second, whereas Ref. [6] considered only the $^{14}\text{C} + \alpha$ cluster configuration, the present work also incorporates shell-model aspects as well as the $^{12}\text{C} + 2n + \alpha$ configuration, including the interplay between these structures. In particular, interactions invariant under transformations between different configurations are employed. Third, the Hamiltonian used here involves only three parameters, in contrast to the seven parameters adopted in Ref. [6]. Moreover, the parameters of the energy

functional have clear physical interpretations, such as the harmonic oscillator component, quadrupole-quadrupole interaction, and the L^2 term. Finally, additional features are explored, including the occurrence of shape isomers and the evaluation of α -spectroscopic factors.

3.1.1 Experiment and previous interpretations

The multi-nucleon transfer reaction $^{12}\text{C}(^7\text{Li},p)$ at $E_{lab}=44$ MeV identified 30 new states of ^{18}O nucleus. A comprehensive survey of available experimental and theoretical studies led to the identification of six rotational bands [68] and classified as shell-like, core-plus- α , and $^{12}\text{C} + 2n + \alpha$ cluster structures. However, this interpretation remains uncertain, as newer $^{12}\text{C}(^7\text{Li},p)$ measurements [69], aimed at extracting α -widths, raise questions about the significance of α -cluster configurations.

$^{14}\text{C} + \alpha$ elastic scattering were performed to investigate alpha cluster structure and several resonances have been observed [70,71]. It should be noted that a state can be populated through elastic α -scattering itself suggests, to some extent, the existence of a core-plus- α configuration in its structure.

The ^{18}O nucleus has been investigated using different theoretical models to explore its possible α -cluster structure. Both Generator Coordinate Method (GCM) calculations [72] and the Coupled-Channel Orthogonality Condition Model (CCOCM) [73] predicted α -cluster rotational bands. A phenomenological algebraic model [74] produced a molecular band with alternating parity. An antisymmetrized molecular dynamics (AMD) calculation and combined with GCM [75], revealed two rotational bands displaying a prominent $^{14}\text{C} + \alpha$ structure in ^{18}O . Semi-microscopic algebraic cluster model (SACM) was used to study the ^{18}O in [6].

3.1.2 MUSY Analysis of ^{18}O

Model space

The L-S coupled scheme is used for the construction of no-core shell model space, where the spatial part of the wave function is described by

$$U_s(3) \otimes U_e(3) \supset U(3) \supset SU(3) \supset SO(3) \quad (3.1)$$

$$|[n_1^s, n_2^s, n_3^s], [n_1^e, n_2^e, n_3^e], \rho, [n_1, n_2, n_3], (\lambda, \mu), K, L \rangle.$$

group chain. The spin-isospin part is defined by Wigner's supermultiplet theory [30]

$$\begin{aligned} U^{ST}(4) &\supset SU^S(2) \otimes SU^T(2) \\ &|[f_1, f_2, f_3, f_4], \quad S, \quad T \rangle. \end{aligned} \quad (3.2)$$

The model space is free from the Pauli exclusion principle and also free from spurious center of mass excitation. The detail method is described in Ref. [11].

For the cluster model space the strongly coupled U(3) scheme is used. The clusters ^{14}C and α , characterized by the U(3) symmetries $[4,4,2]$ and $[0,0,0]$, respectively. In this framework, strong coupling corresponds to the inclusion of the ground-state rotational band ($J^\pi = 0^+, 2^+$) of the ^{14}C cluster. As per the Wildermuth condition, the minimum value of the relative-motion quantum number permitted by the Pauli principle is six. The Pauli-allowed cluster space, free from spurious center-of-mass excitations, is obtained by matching it to the shell-model space.

Regarding the $^{12}\text{C} + 2n + ^4\text{He}$ configuration, it is allowed in all $^{14}\text{C} + \alpha$ states, since the $^{12}\text{C} + 2n$ system reproduces the same $[4,4,2]$ symmetry as the ^{14}C cluster.

The basis states up to $7\hbar\omega$ excitation are presented in Table 3.1. Only the relevant shell-model basis states are included, while the cluster model space is complete.

When the multiplicity in the shell-model space is equal to one, the different configurations (shell, core-plus- α , $^{12}\text{C} + 2n + ^4\text{He}$, etc.) correspond to identical wave functions due to antisymmetrization. This is the case, for the most deformed states of $1\hbar\omega$, $2\hbar\omega$, and $3\hbar\omega$ excitation.

Figure 3.1 shows the unification of these three ($0[8,4,4](4,0)$, $1[10,4,3](6,1)$, $2[12,4,2](8,2)$) cluster configurations. The left side of the figure shows merging of two clusters in Harvey's prescription [76]. Even though the ^{14}C cluster has the same symmetry, i.e., $[4,4,2]$ in each case, the corresponding states differ in their spatial orientation of the cluster relative to the molecular z-axis. For example in the $[8,4,4]$ configuration, the symmetry axis of the oblate ^{14}C is along along the z-direction, whereas in the $[12,4,2]$ case it is oriented perpendicular to it, along the y-axis. In the $[10,4,3]$ configuration, the orientation lies between the z and y directions. The middle part of the figure illustrates the density distribution before antisymmetrization, while the right-hand side shows the shell-model wave function, which becomes identical to the cluster wave function after antisymmetrization.

Table 3.1: Model space of ^{18}O nucleus up to $7\hbar\omega$ excitations. n represents the major shell excitation, $C^{(2)}$ is the eigenvalue of the second order Casimir invariant of $\text{SU}(3)$. The states are listed in descending order of $C^{(2)}$ values. The multiplicities in both the shell-model (^{18}O) and cluster-model ($^{14}\text{C} + \alpha$) spaces are denoted by m . Some shell model basis are omitted, and it is indicated in the last column.

n	$\text{U}(3)$	$\text{SU}(3)$	$C^{(2)}$	$m_{Sh.}$	$m_{Cl.}$	left out
$0\hbar\omega$	[8,4,4]	(4,0)	28	1	1	
	[6,6,4]	(0,2)	10	1		
$1\hbar\omega$	[10,4,3]	(6,1)	64	1	1	
	[9,5,3]	(4,2)	46	1		
	[9,4,4]	(5,0)	40	2	1	
	[8,6,3]	(2,3)	34	2		
						4
$2\hbar\omega$	[12,4,2]	(8,2)	114	1	1	
	[11,5,2]	(6,3)	90	1		
	[11,4,3]	(7,1)	81	3	1	
						4
	[10,4,4]	(6,0)	54	9	1	
						8
$3\hbar\omega$	[13,5,1]	(8,4)	148	1		
	[13,4,2]	(9,2)	136	2	1	
						4
	[12,4,3]	(8,1)	100	12	1	
						4
	[11,4,4]	(7,0)	70	20	1	
						11
$4\hbar\omega$	[15,4,1]	(11,3)	205	1		
						4
	[14,4,2]	(10,2)	160	9	1	
						6
	[13,4,3]	(9,1)	121	35	1	
						4
	[12,4,4]	(8,0)	88	61	1	
						15

Table 3.1: (continued)

n	U(3)	SU(3)	$C^{(2)}$	$m_{Sh.}$	$m_{Cl.}$	left out
$5\hbar\omega$	[16,5,0]	(11,5)	249	1		5
	[15,4,2]	(11,2)	186	24	1	7
	[14,4,3]	(10,1)	144	91	1	4
	[13,4,4]	(9,0)	108	146	1	19
$6\hbar\omega$	[18,4,0]	(14,4)	322	1		7
	[16,4,2]	(12,2)	214	75	1	6
	[15,4,3]	(11,1)	169	240	1	6
	[14,4,4]	(10,0)	130	378	1	23
$7\hbar\omega$	[19,4,0]	(15,4)	358	2		7
	[17,4,2]	(13,2)	244	173	1	5
	[16, 4, 3]	(12, 1)	196	542	1	7
	[15, 4, 4]	(11, 0)	154	822	1	27

The density is calculated using the expression: [77].

$$\rho(x, y, z, (n_x, n_y, n_z)) = \sum_i^A |\psi_i(x, n_x) \psi_i(y, n_y) \psi_i(z, n_z)|^2 \quad (3.3)$$

Where

$$\psi(i, n_i) = \sqrt{\frac{1}{2^{n_i} n_i!}} \left(\frac{m\omega_i}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega_i}{2\hbar} i^2} H_{n_i} \left(\sqrt{\frac{m\omega_i}{\hbar}} i \right); \quad (3.4)$$

$i = x, y, z$

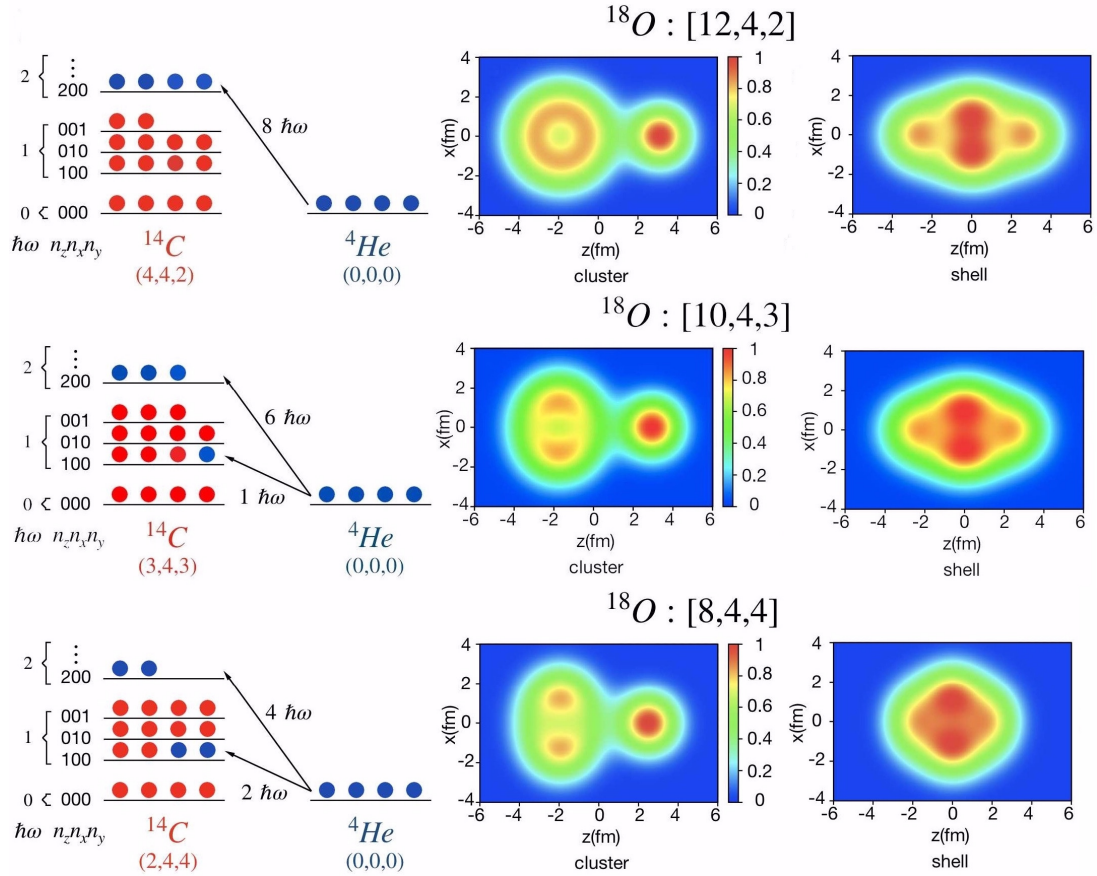


Figure 3.1: Harvey's prescription for combining two clusters (left), the density distributions of the clusters before antisymmetrization (middle), and the shell-model density (right). After antisymmetrization, the cluster and shell-model wave functions become identical. The separation between the cluster centers for the configurations $[8, 4, 4]$, $[10, 4, 3]$, and $[12, 4, 2]$ are 4.436, 4.691, and 4.997 fm, respectively.

is the harmonic oscillator wave function of a single nucleon along i -direction [78]. Here, n_i denotes the number of oscillator quanta, ω_i is the oscillator frequency in the i -direction, and H_{n_i} represents the Hermite polynomial of order n_i . A denotes the mass number of the nucleus.

The oscillator frequencies along the Cartesian directions are taken from Ref. [78] as

$$\omega_i = \omega_0 \frac{R_0}{a_i}, \quad i = x, y, z. \quad (3.5)$$

Here, $R_0 = \sqrt[3]{abc}$ and $r_0 = 1.2 \text{ fm}$ is the radius parameter. The value of ω_0 is given by $\hbar\omega_0 = 45A^{-1/3} - 25A^{-2/3}$ [79].

The semi-axis lengths (a,b,c) are determined using self-consistency argument and volume conservation [80]:

$$\frac{a}{c} = \frac{n_1 + \frac{A}{2}}{n_3 + \frac{A}{2}}, \quad \frac{b}{c} = \frac{n_2 + \frac{A}{2}}{n_3 + \frac{A}{2}} \quad (3.6)$$

$$c = r_0 \sqrt[3]{A \frac{(n_3 + \frac{A}{2})^2}{(n_1 + \frac{A}{2})(n_2 + \frac{A}{2})}}. \quad (3.7)$$

The density of two clusters are calculated considering their internal structure of the clusters (represented by shell model wave functions), while the relative separation is determined from the touching configuration of classical ellipsoids.

Shape isomers

The shape isomers of ^{18}O are determined from the stability and self-consistency of the SU(3) symmetry (SCS) method [22]. The detailed method is explained in Section 2.2.2. Since this method is based on the U(3) symmetry, it allows one to examine, through selection rules, whether a given reaction channel can be populated or can decay.

Study of shape isomers is of relevance in general, but here the main purpose is to determine whether stable configurations exist in 2 and $4\hbar\omega$ and exhibit α -cluster structure, as suggested in [68]. The shape isomers appear as horizontal plateau regions in the $\beta_{in} - \beta_{out}$ plots, as illustrated in Fig. 3.2. The stable shapes are presented in Table 3.2. The effective U(3) quantum numbers obtained in the calculations carry some uncertainties [22] (for example, the resulting values are real numbers rather than integers [40]). For this reason, a closely related simple shell-model configuration (denoted by U(3)) is also indicated in Table 3.2.

Additionally, the ratios of the major axes of the ellipsoid and the corresponding deformation parameters are mentioned.

Table 3.2: Shape isomers of the ^{18}O nucleus. The notations used are: GS for ground state, lin-ch for linear chain, Pro for prolate, SD for super-deformed, HD for hyper-deformed, and Ex.D for extra-deformed states. The parameters (β, γ) describe the quadrupole deformation (γ in degrees), while $a : b : c$ represents the ratios of the ellipsoid's principal axes.

State	$\hbar\omega$	U(3)	$U_{eff}(3)$	a:b:c	(β, γ)
GS	0	[8,4,4]	[8,4,4]	1.3:1:1	(0.31,0)
			[7,5,4]	1.2:1.1:1	(0.2,19.1)
SD	2	[12,4,2]	[12,4,2]	1.9:1.2:1	(0.67,10.9)
			[12,4,3]	1.75:1.1:1	(0.61,5.8)
			[11,4,3]	1.7:1.1:1	(0.55,6.6)
$HD_{ob.}$	4	[12,8,0]	[12,8,0]	2.3:1.9:1	(0.74,40.9)
$HD_{pr.}$	6	[18,4,0]	[18,3,1]	2.7:1.2:1	(1.07,6.18)
Ex.D	10	[24,2,0]	[24,2,0]	3.7:1.2:1	(1.42,4.3)
lin-ch	16	[32,0,0]	[32,0,0]	4.6:1:1	(1.76,0)

The binary clusterization in the shape isomers are investigated using U(3) and $U^{ST}(4)$ selection rules. The reciprocal forbiddenness as defined in Eq. (2.18) is used to indicate whether a cluster configuration is structurally allowed.

$S = 1$ correspond to fully allowed structures, while lower values indicate configurations that are strongly forbidden. The reciprocal forbiddenness, plotted as a function of the mass number of the lighter cluster, is shown for the shape isomers of ^{18}O in Figure 3.3.

From the perspective of the structural interpretation in Ref. [68], as well as in light of present analysis, the following observations can be made. The $2\hbar\omega$ α -cluster state appears to belong to the first excited shape isomer listed in Table 3.2, labeled as Pro. In contrast, no stable $4\hbar\omega$ state supports an α -cluster configuration. Therefore, these states do not constitute the ground band of a new shape isomer (local minimum in the potential surface), but instead correspond to excited states. Although a shape isomer with $4\hbar\omega$ excitation is predicted, it is an triaxial shape and does not exhibit an α -cluster structure.

A prolate superdeformed state with $6\hbar\omega$ excitation is obtained from this calculation, that permits α clusterisation.

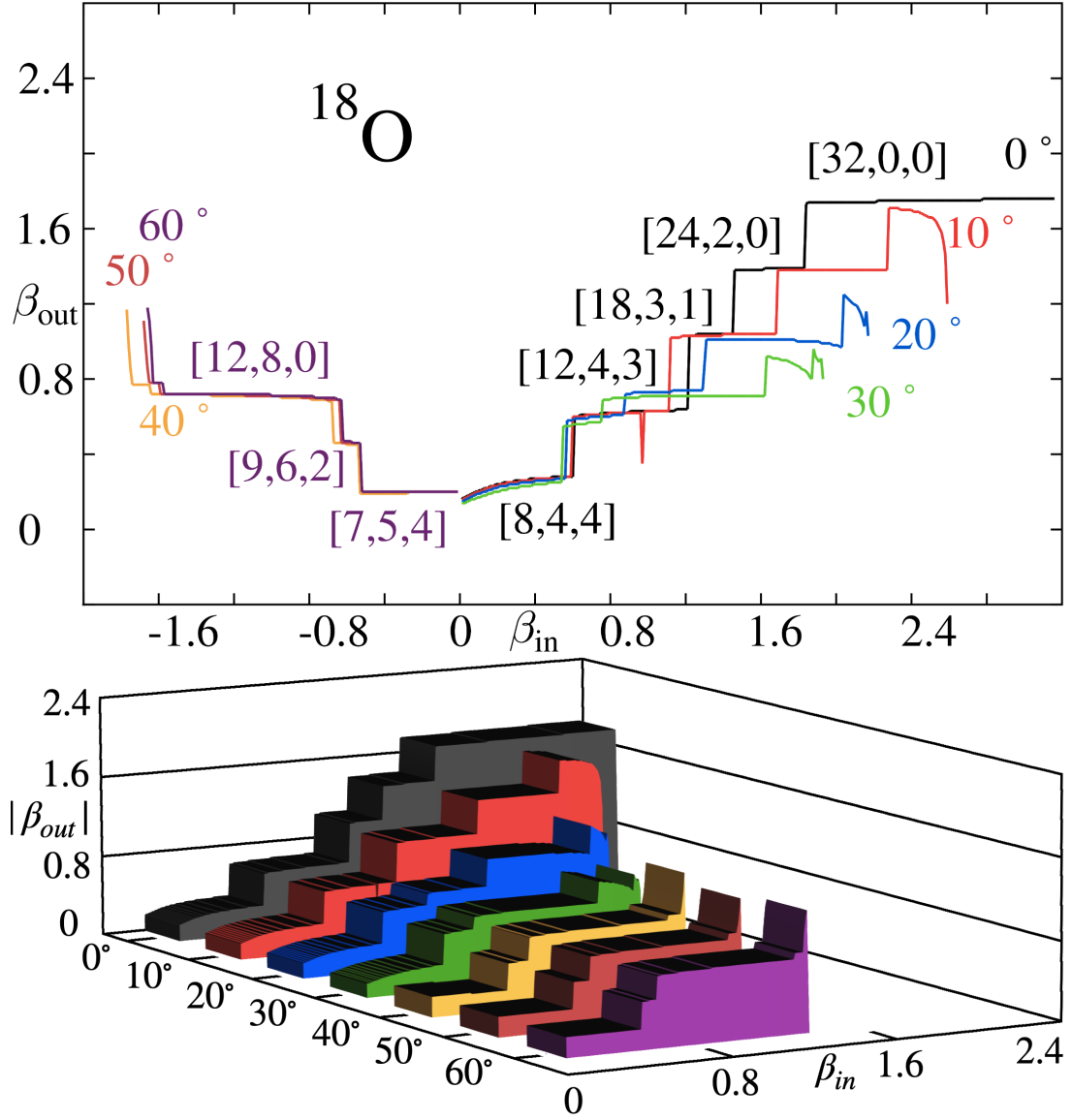


Figure 3.2: Shape isomers of the ^{18}O nucleus. In the upper part, β_{in} and β_{out} are shown in horizontal and vertical axes, respectively. The γ parameter is given in degrees with steps of 10. The lower section presents the same data in a three-dimensional view.

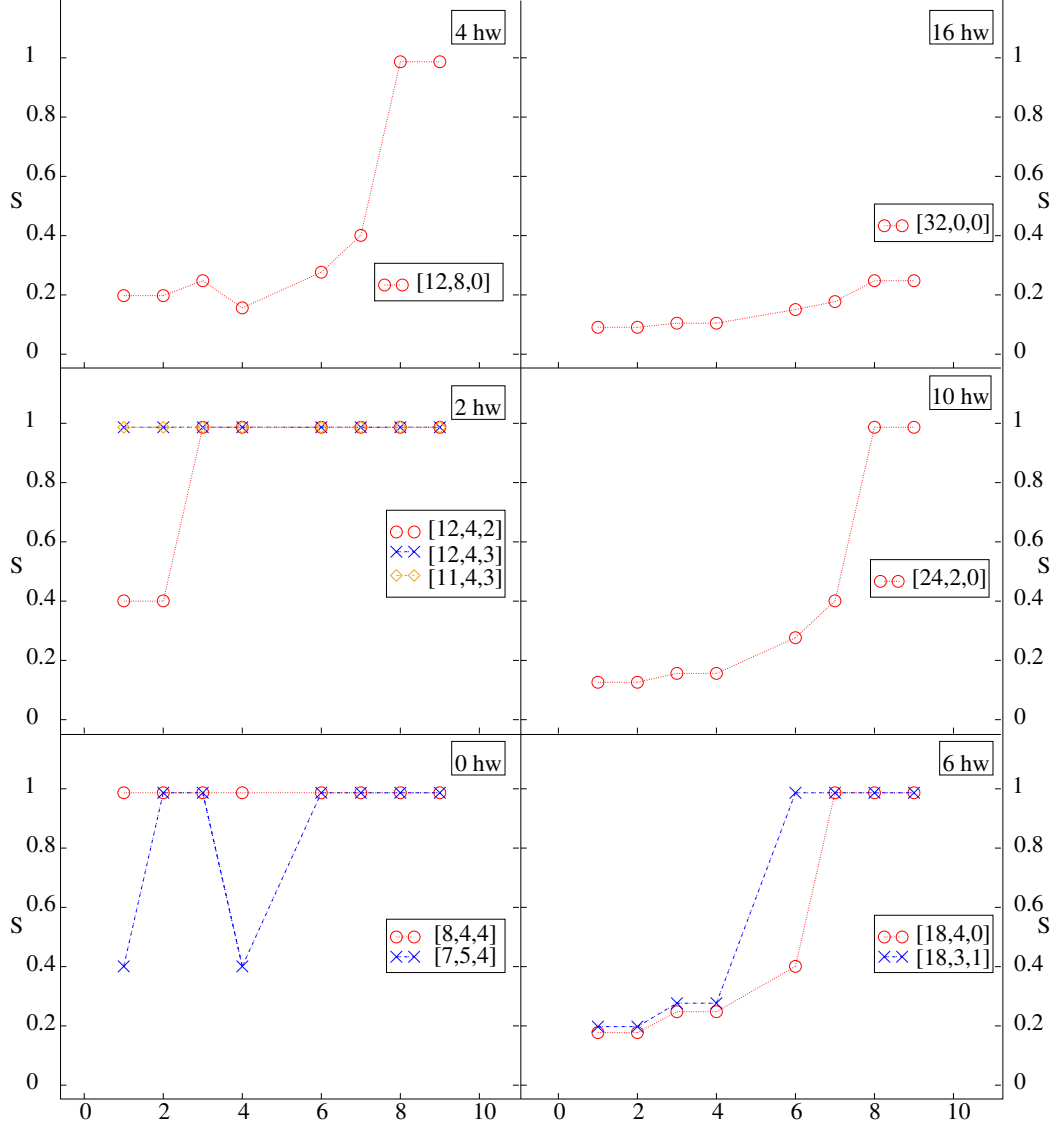


Figure 3.3: Reciprocal forbiddenness is shown as a function of the mass number of the lighter cluster for the shape isomers in ^{18}O . Only cluster configurations that contain at least one stable cluster are considered. The lines are drawn simply to guide the eye.

Spectrum calculation

A dynamically symmetric Hamiltonian, written by using invariant operators corresponding to the group chain $U(3) \supset SU(3) \supset SO(3)$:

$$\hat{H} = (\hbar\omega)\hat{n} + a\hat{C}_{SU(3)}^{(2)} + \frac{d}{2\theta}\hat{L}^2, \quad (3.8)$$

is used for the calculation of spectrum. Where $\hat{n}$ is the number operator, $\hat{C}_{SU(3)}^{(2)}$ is the second-order Casimir operator of $SU(3)$, and $\hat{L}^2$ is the angular momentum squared operator. The moment of inertia θ is calculated assuming a cylindrically symmetric ellipsoid with major axes a and $b = c$, which rotates about an axis perpendicular to its symmetry axis a :

$$\theta = \frac{1}{5}m(a^2 + b^2) \quad (3.9)$$

Its unit is $\hbar^2/\text{MeV}$. If the nucleus has a triaxial shape with semi-axis lengths a , b , and c , it can be approximated by a cylindrically symmetric shape with semi-axis lengths A , B , and C using the following approximation: if $a/b \geq b/c$, then $A=a$ and $B=C=bc$; otherwise, $A=B=ab$ and $C=c$ [81].

The Hamiltonian (3.8) is similar to Elliott model [1, 2]. But there are differences. Here the strengths of the quadrupole and rotational terms are not fixed, and the moment of inertia is different for each band.

The parameters of the Hamiltonian are obtained from the fitting with observed experimental data. For the fitting procedure, the bands Ref. [68] ($K^\pi = 0_1^+, 1_1^-, 0_2^+, 0_2^-, 0_4^+, 0_4^-$ of Figure 3.4) were used as a starting point. States with well-defined spin and parity were given unit weight, while those with uncertain spin-parity were weighted 0.5. In addition, the data from the compilation in Ref. [82] and the resonances observed in α -scattering experiments [71] were considered to arrange potential band candidates. In particular, states that show an approximately linear trend in the E - $J(J+1)$ diagram, with three or more members, were identified as potential band candidates. These states were included in the fitting procedure with a weight of 0.1. The resulting band structure obtained through this approach is presented in the lower part of Figure 3.4.

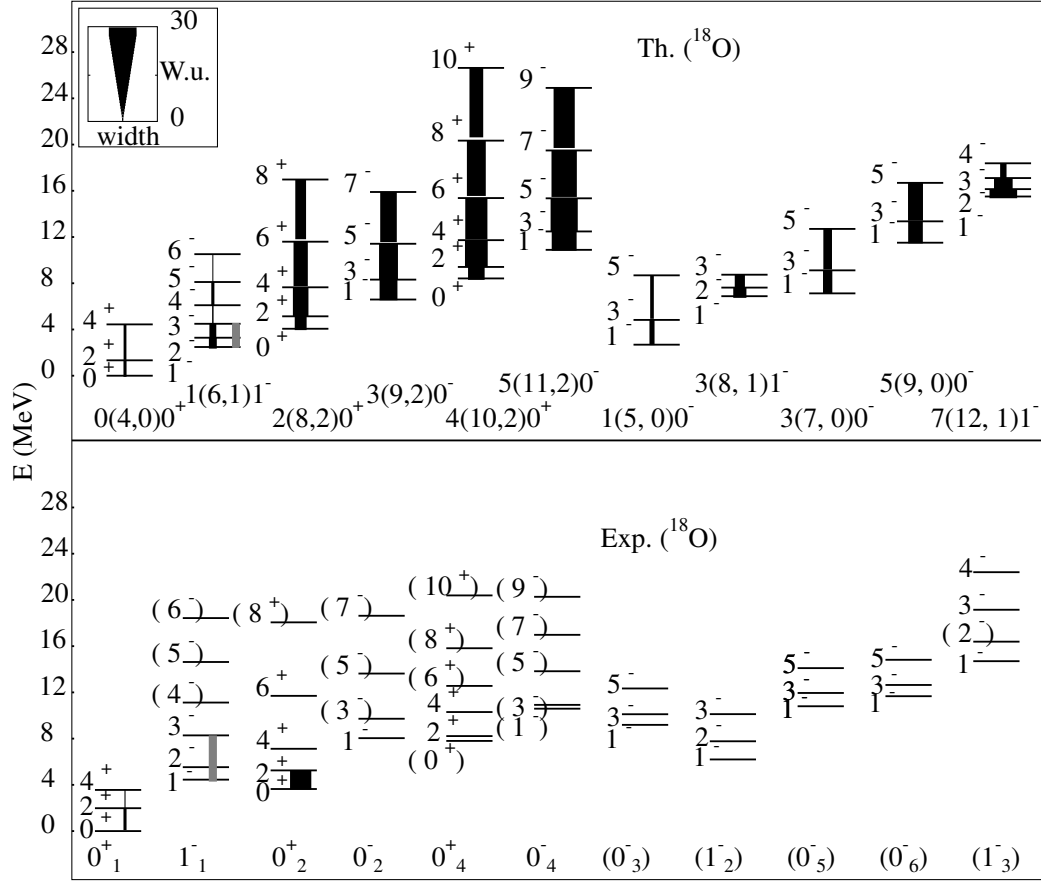


Figure 3.4: Comparison between the MUSY spectrum (top) and the experimental ^{18}O spectrum (bottom). The bands reported in Ref. [68] are denoted by K^π quantum numbers, whereas the arranged bands are indicated by (K^π) . The theoretical bands are characterized by the quantum numbers $n(\lambda, \mu)K^\pi$. The (0^-) bands are arranged using the resonances of Ref. [71], while the (1^-) bands are arranged from the compiled data in Ref. [82]. The thickness of the arrows connecting the states reflects the relative strength of the $E2$ transitions, and the gray arrow corresponds to the $3^- \rightarrow 1^-$ transition.

The fitted parameter values are $\hbar\omega = 2.344$ MeV, $a = -0.007257$ MeV, and $d = 0.8572$.

In Figure 3.5, the distribution of band heads is presented.

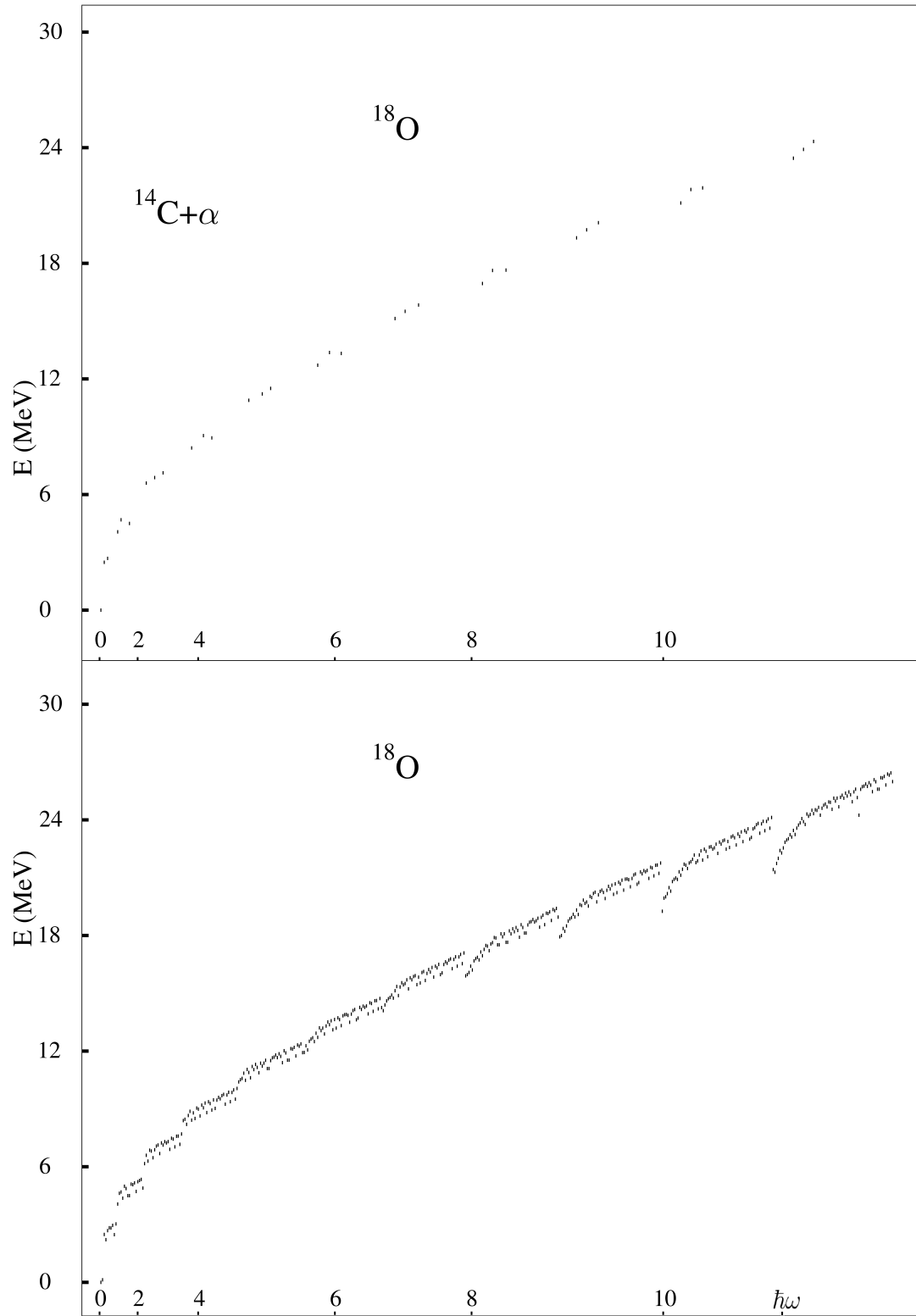


Figure 3.5: Distribution of shell and cluster band heads in the ^{18}O nucleus.

The moment of inertia calculated for the model bands using Eq. 3.9 are compared with experimental ones in Table 3.3.

Table 3.3: The moment of inertia (θ) calculated from Eq. 3.9 is compare with the experimental values of ref. [68].

Exp. Bands [68]		Th. Bands	
Bands (K^π)	$\theta(\hbar^2/\text{MeV})$	n(λ, μ)	$\theta(\hbar^2/\text{MeV})$
0_1^+	3.00	0(4,0)	1.93
1_1^-	1.41	1(6,1)	2.13
0_2^+	2.55	2(8,2)	2.39
0_2^-	2.52	3(9,2)	2.48
0_4^+	4.38	4(10,2)	2.58
0_4^-	4.34	5(11,2)	2.69
		1(5,0)	2.00
		3(8,1)	2.30
		3(7,0)	2.15
		5(9,0)	2.31
		7(12,1)	2.68

By fixing the parameters of Hamiltonian (3.8) obtained from the fit to the experimental band, the $^{14}\text{C}+\alpha$ cluster model spectrum can be calculated and can be compared with the alpha-scattering resonances. In particular, one can investigate the population of states with specific spin-parity within a certain energy window. They are shown in Figure 3.6 and Table 3.4. The model states are presented within the experimental energy window, extending from 0.5 MeV below the lowest-lying resonance to 0.5 MeV above the highest-lying one.

Table 3.4: Number of $^{14}\text{C} + \alpha$ resonance states with given spin-parity (L^π) within the experimental energy window. The values labeled N_{exp} correspond to experimentally observed states, while N_{mod} denote the corresponding number of states predicted by the model. .

L^π	0^+	2^+	4^+	6^+	1^-	3^-	5^-
N_{exp}	1	11	7	2	10	13	6
N_{mod}	0	8	4	3	3	8	4

Electromagnetic transitions

- **E2 transitions:**

The $B(E2)$ value for the intra-band transition from L_i to L_f was calculated

The corresponding transition strengths are illustrated in Fig. 3.4 of the energy spectrum.

Spectroscopic factor

The microscopic spectroscopic factor can be evaluated using the SU(3) LS-coupled basis states. A detailed description of the calculation is provided in Chapter 2.

Here, the relative α -spectroscopic factors are obtained as the squares of the corresponding relative spectroscopic amplitudes, A_{nlsj}^R (see Eq.(2.24)). The calculated values are listed in Table 3.5.

Table 3.5: $[A_{nlsj}^R(^{18}\text{O} \leftrightarrow ^{14}\text{C} + \alpha)]^2$ corresponding to different states of ^{18}O .

^{14}C		^{18}O											
$(\lambda, \mu)\kappa_L$	L	$n\hbar\omega$	$(\lambda', \mu')\kappa_{L'}$	L'									
	0			2	4	6	8	10					
$(0, 2)0$	0	$0\hbar\omega$	$(4, 0)0$	0.25	0.214	0.131							$[A_{nlsj}^R]^2$
		$2\hbar\omega$	$(8, 2)0$	0.74	0.714	0.67	0.57	0.39					
		$4\hbar\omega$	$(10, 2)0$	0.72	0.71	0.68	0.62	0.52	0.344				
$(0, 2)0$	0			L'									
				1	2	3	4	5	6	7	8	9	
		$1\hbar\omega$	$(6, 1)1$	0.017		0.1		0.25					
			$(5, 0)0$	0.25		0.2		0.12					
		$3\hbar\omega$	$(9, 2)0$	0.72		0.70		0.641	0.544		0.37		
			$(8, 1)1$	0.01		0.067		0.167	0.31				
			$(7, 0)0$	0.27		0.24		0.181	0.103				
		$5\hbar\omega$	$(11, 2)0$	0.714		0.695		0.66	0.595		0.495		
	$(9, 0)0$	0.28		0.26		0.22	0.162	0.09					
	$7\hbar\omega$	$(12, 1)1$	0.006		0.035		0.09	0.17	0.27				

Summary of ^{18}O

I applied multiconfigurational dynamical symmetry (MUSY) for the unified description of ^{18}O spectrum. I used a simple dynamically symmetric Hamiltonian which is invariant under transformations between the different configurations, contains only harmonic oscillator, quadrupole, and angular momentum interactions. Therefore, all three previously proposed configurations (the shell, $^{14}\text{C} + ^4\text{He}$, and $^{12}\text{C} + 2n + ^4\text{He}$ clusters) contribute to every state in the model spectrum.

MUSY could describe the experimentally established band structure [68] and the high-lying alpha scattering resonances [71] within a unified framework. In the states of the three lowest-lying bands, the overlap of the three relevant configurations is 100%. Besides the ground state, a shape isomer associated with a $2\hbar\omega$ excitation is found to contribute to the core-plus-alpha spectrum. In addition to the ground state, a prolate superdeformed configuration characterized by the $[12,4,2](8,2)$ quantum numbers is predicted, which is likely analogous to the $[12,4,0](8,4)$ superdeformed state of ^{16}O [22]. Some of the high-lying states identified in alpha-scattering experiments may be associated with this second minimum in the energy surface.

With a simple dynamically symmetric Hamiltonian, $^{14}\text{C} + ^4\text{He}$ cluster spectrum reproduces the gross features of the experimental energy spectrum of ^{18}O .

3.2 ^{40}Ca spectrum and MUSY

Exotic nuclear shapes have been a subject of significant interest in both experimental and theoretical studies; see Refs. [84–89], and references therein. In light nuclei, several superdeformed bands have been observed experimentally. And heavy-ion resonance studies even hint at possible hyperdeformed configurations. Though the evidence is not yet conclusive, nevertheless they are very important observations.

Theoretical descriptions of such shape isomers face several challenges. One needs reliable information about the spectroscopic characteristics of the states, such as, energies, spin-parities, deformations, electromagnetic transitions, coupling to reaction channels, etc. A further question is whether these highly deformed, high-excitation states can be understood within the same framework used for the ground-state region. In many cases, models developed for heavy-ion resonances are not directly connected to standard nuclear-structure approaches such as the shell model.

The multiconfigurational dynamical symmetry (MUSY) has been used for the unified description of the spectrum of the ^{36}Ar nucleus [12]. It was found that the gross features of the spectrum could be obtained in a unified way for the first (ground), second (superdeformed), and third (hyperdeformed) minima of the potential energy surface.

Here, I applied (MUSY) to the nucleus ^{40}Ca to investigate whether MUSY can provide a unified description of its energy spectrum [P2].

3.2.1 Experiment and previous interpretations

The ^{40}Ca nucleus has attracted much attention in both theoretical and experimental nuclear structure studies because it is a doubly magic nucleus with spherical ground state; yet it shows deformed bands in the low-energy region. The first excited 0^+ at 3.35 MeV and the second excited 0^+ at 5.21 MeV were suggested to be 4particle-4hole (4p-4h) and 8particle-8hole (8p-8h) configurations, respectively [90]. These 0^+ states serve as the band heads of the low-lying positive-parity rotational band and the superdeformed (SD) band, observed in the $^{28}\text{Si}(^{20}\text{Ne}, 2\alpha)^{40}\text{Ca}$ reaction [91]. The same experimental study also reported two further bands, with band heads at 8^+ (8.103 MeV) and 3^+ (6.03 MeV).

Except the last two highest-spin states, all the member of the SD band were observed in the $^{24}\text{Mg}(^{28}\text{Si}, 3\alpha)^{40}\text{Ca}$ reaction [92]. The states of the SD band were

also observed in the $^{24}\text{Mg}(^{24}\text{Mg}, 2\alpha)^{40}\text{Ca}$ reaction [93]. The first three members of SD band were identified through the alpha transfer reaction $^{36}\text{Ar}(^6\text{Li}, d)^{40}\text{Ca}$ [94].

After the confirmation of SD band many theoretical investigation has been carried out using several models. Large-scale shell-model calculations [95] revealed that the superdeformed structure is predominantly characterized by multi-particle–multi-hole excitations across the $N = Z = 20$ shell closure. Specifically, the leading contribution is associated with an 8p–8h configuration, which generates a strongly deformed intrinsic state. The cranking covariant density functional theory, combined with a shell-model-like treatment of pairing correlations, was used to describe the energy spectra, moments of inertia, and transition quadrupole moments of the SD band in ^{40}Ca [96]. Using the symmetry-unrestricted cranked Skyrme–Hartree–Fock approach, the SD band in ^{40}Ca was shown to be highly soft with respect to both axially symmetric and non-axial octupole deformations [97]. Within the covariant density functional theory framework, a detailed investigation of the rotational spectra and structural properties of ^{40}Ca was also carried out [98]. The SD states in ^{40}Ca were characterized as excitations involving multiple-particle-multi-hole (e.g., 8p–8h) within a semi-empirical Nilsson-model framework [99]. Calculations using a self-consistent mean-field approach suggested that the evolution of shape in SD states of ^{40}Ca at high spin is influenced by signature-dependent triaxiality [100]. The SD structure of ^{40}Ca has been explored through beyond-mean-field methodologies [101]. The cluster structures are investigated using antisymmetrized molecular dynamics (AMD) and the generator coordinate method (GCM) [102,103]. It was found that the band built on the first excited 0^+ state contains an α -cluster component, while the SD band exhibits a $^{12}\text{C} + ^{28}\text{Si}$ cluster structure.

A negative parity band proposed to be inversion doublet of lower lying positive parity band in [92] and its properties were described within the local potential cluster model [104].

Experimental candidate states for the hyperdeformed (HD) band in ^{40}Ca were proposed in [105], including several high-spin states with both even and odd angular momenta. Among them, four states: 8^+ , 10^+ , 13^- , and 15^- at excitation energies of 32.4, 34.0, 35.3, and 37.0 MeV, have certain spin–parity assignments. In addition, seven $^{12}\text{C} + ^{28}\text{Si}$ [106] and six $^{16}\text{O} + ^{24}\text{Mg}$ [107] resonances with even and odd angular momenta but uncertain parity have been proposed.

Several theoretical investigations have examined hyperdeformed (HD) states in ^{40}Ca . According to fixed-configuration Hartree-Fock calculations, there ex-

ists a stable 12p–12h HD state with $\beta \approx 0.97$ [108]. This finding aligns with results from macroscopic–microscopic calculations [109]. Furthermore, cranked Skyrme–Hartree–Fock calculations also identify an HD configuration corresponding to this state [97]. Using a U(3) symmetry-based approach, two HD candidate configurations were identified [110], associated with 12p–12h and 16p–16h excitations and characterized by the U(3) irreducible representations [48,12,12] and [52,12,12], respectively. Within the framework of covariant density functional theory, ^{36}Ar and ^{40}Ca were identified as favorable candidates for extremely deformed rotational structures (super-, hyper-, megadeformed) at high spin [98]. In particular, the HD configuration in ^{40}Ca was associated with the occupation of the Nilsson orbital $1/2[440]$. Energy-density functional calculations also indicated the appearance of HD configurations at high angular momentum when both a proton and a neutron occupied this orbital, leading to a deformation of $\beta \approx 0.97$ [100]. Furthermore, a study of shape isomers [24] based on quadrupole deformation stability and self-consistency [22] conditions predicted a stable HD state together with possible binary cluster configurations. This state is associated with the U(3) irreducible representation [56, 12, 8] of $16\hbar\omega$ excitation.

3.2.2 MUSY Analysis of ^{40}Ca

Model space

A no-core shell-model space was constructed in which all nucleon degrees of freedom were taken into account for the application of MUSY. In building this model space, spurious center-of-mass excitations were removed, while the Pauli exclusion principle was fully appreciated.

For the cluster-model space, the leading ground-state U(3) irreducible representations (irreps) associated with the internal structures of the clusters were considered. These irreps were [20,20,12] for ^{36}Ar , [0,0,0] for α , [20,8,8] and [16,16,4] for the prolate and oblate configurations of ^{28}Si , respectively, [4,4,0] for ^{12}C , [16,8,4] for ^{24}Mg , [4,4,4] for ^{16}O , and [12,4,4] for ^{20}Ne . The relative motion between the clusters was restricted by the Wildermuth condition [3], which determines the minimum number of oscillator quanta, n_π , required to fulfill the Pauli principle. The resulting U(3) representations were coupled with a wide range of allowed relative-motion [6]. However, not all resulting configurations were physically allowed, as some were excluded by the Pauli principle. These forbidden components were identified and removed by matching the cluster-model basis

with the corresponding shell-model space. In the case of the identical-cluster configuration an additional selection rule should be considered, that is $n_\pi + \lambda_C + \mu_C$ must be even [7]. Here, the quantum numbers (λ_C, μ_C) denote the SU(3) representation obtained from coupling the U(3) irreps of the clusters. I listed the relevant representations for shell and cluster configurations in Table 3.6.

Table 3.6: The relevant quartet and cluster model spaces of the ^{40}Ca nucleus are shown for excitations ranging from 0 to $8\hbar\omega$. Here, n denotes the number of major-shell excitations, while $C^{(2)}$ represents the expectation value of the second-order SU(3) Casimir operator. The states are ordered according to decreasing values of $C^{(2)}$, that is corresponding to decreasing deformation. The columns labeled C1–C5 give the multiplicities of the SU(3) representations in the cluster model spaces corresponding to the channels: C1 = $\alpha + ^{36}\text{Ar}$, C2 = $^{12}\text{C} + ^{28}\text{Si}_{\text{pro.}}$, C3 = $^{12}\text{C} + ^{28}\text{Si}_{\text{ob.}}$, C4 = $^{16}\text{O} + ^{24}\text{Mg}$, and C5 = $^{20}\text{Ne} + ^{20}\text{Ne}$.

n	U(3)	SU(3)	$C^{(2)}$	quartet	C1	C2	C3	C4	C5
0	[20,20,20]	(0,0)	0	1	1		1		
1	[23,20,18]	(3,2)	34	1	1		1		
	[22,20,19]	(2,1)	16	1	1		1		
2	[26,20,16]	(6,4)	106	1	1		1		
	[26,18,18]	(8,0)	88	1			1		
	[24,22,16]	(2,6)	76	1					
	[25,20,17]	(5,3)	73	2	1		2		
	[25,19,18]	(6,1)	64	2			2		
	[24,21,17]	(3,4)	58	2					
	[23,22,17]	(1,5)	49	2					
	[24,20,18]	(4,2)	46	6	1		3		
3	[29,20,14]	(9,6)	216	1	1		1		
	[29,18,16]	(11,2)	186	1			1		
	[27,22,14]	(5,8)	168	1					
	[28,20,15]	(8,5)	168	2	1		2		
	[26,23,14]	(3,9)	153	1					
	[28,19,16]	(9,3)	153	3			3		

Table 3.6: (continued)

n	U(3)	SU(3)	$C^{(2)}$	quartet	C1	C2	C3	C4	C5
4	[32,20,12]	(12,8)	364	1	1	1	1	1	1
	[32,18,14]	(14,4)	322	1			1		1
	[32,16,16]	(16,0)	304	1			1		1
	[31,20,13]	(11,7)	301	2	1		2		
	[30,22,12]	(8,10)	298	1		1			
	[31,19,14]	(12,5)	280	3			3		
5	[34,20,11]	(14,9)	472	1		1	1	1	1
	[34,19,12]	(15,7)	445	2		2	2	1	
	[33,21,11]	(12,10)	430	2		2			
	[34,18,13]	(16,5)	424	2			2		1
	[34,17,14]	(17,3)	409	2			2		
	[33,20,12]	(13,8)	400	6	1	5	5	1	
	[34,16,15]	(18,1)	400	1			1		1
	[32,22,11]	(10,11)	394	3		3			
	[33,19,13]	(14,6)	376	11			4		
	[31,23,11]	(8,12)	364	4		2			
	[32,21,12]	(11,9)	361	12		4			
	[33,18,14]	(15,4)	358	14			3		
6	[36,20,10]	(16,10)	594	1		1	1	1	1
	[36,19,11]	(17,8)	564	3		3	3	1	
	[35,21,10]	(14,11)	546	2		2			
	[36,18,12]	(18,6)	540	6		5	3	1	2
	[36,17,13]	(19,4)	522	4			2		
	[35,20,11]	(15,9)	513	12		4	5	1	1
7	[38,20,9]	(18,11)	730	1		1	1	1	1
	[38,19,10]	(19,9)	697	3		3	3	1	
	[37,21,9]	(16,12)	676	2		2			
	[38,18,11]	(20,7)	670	7		4	3	1	2
	[38,17,12]	(21,5)	649	10		5	2	1	
	[37,20,10]	(17,10)	640	13		3	5	1	1
	[38,16,13]	(22,3)	634	8			1		2

Table 3.6: (continued)

n	U(3)	SU(3)	$C^{(2)}$	quartet	C1	C2	C3	C4	C5
8	[40,20,8]	(20,12)	880	1		1	1	1	1
	[40,19,9]	(21,10)	844	2		2	2	1	
	[39,21,8]	(18,13)	820	1		1			
	[40,18,10]	(22,8)	814	7		3	3	1	3
	[40,17,11]	(23,6)	790	11		4	2	1	
	[39,20,9]	(19,11)	781	10		2	5	1	2
	[40,16,12]	(24,4)	772	16		5	1	1	3
	[38,22,8]	(16,14)	766	4		1			2
	[40,15,13]	(25,2)	760	8					

If a U(3) representation has multiplicity one in the shell-model basis, then after antisymmetrization the shell-model and cluster descriptions coincide. One interesting case is the $4\hbar\omega$ [32,20,12] configuration, where all five possible cluster configurations are allowed (see Fig. 3.7).

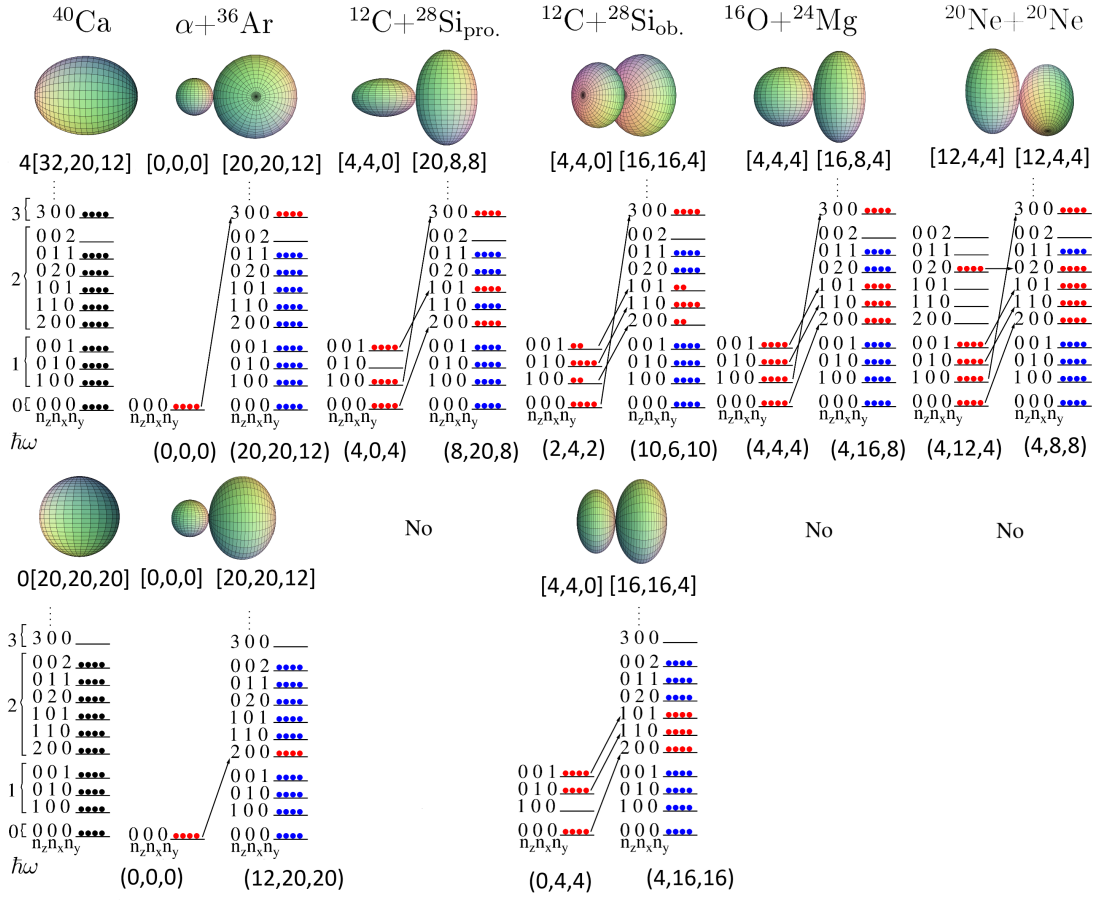


Figure 3.7: The shapes of the ground state and the most deformed state in the $4\hbar\omega$ excitation of ^{40}Ca are compared with the corresponding binary cluster configurations. Harvey's prescription for the fusion of clusters is shown below each configuration. The U(3) labels are given in square brackets, while specific basis states are indicated in parentheses. The overlap between the cluster and shell-model wave functions is 100%.

Shape isomer

The shape isomers were identified using the symmetry-guided Stability and Self-Consistency of the SU(3) Symmetry (SCS) approach [22], discussed in Section (2.2.2).

In the $\beta_{\text{in}} - \beta_{\text{out}}$ diagram, the shape isomers appeared as horizontal plateau-like structures, as illustrated in Fig. 3.8. A similar figure is present in [24].

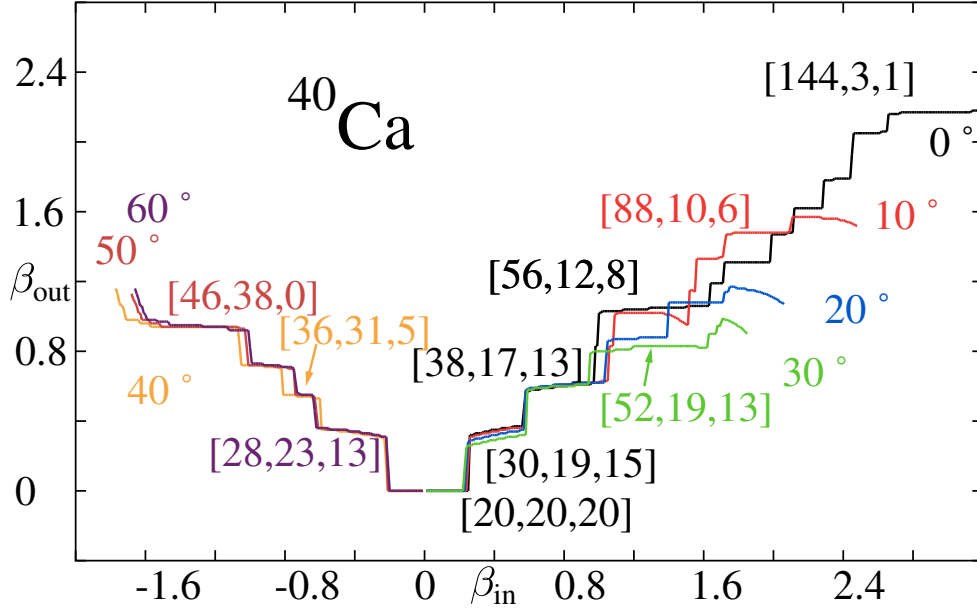


Figure 3.8: The shape isomers of the ^{40}Ca nucleus are illustrated using the effective U(3) quantum numbers obtained from the SCS method. The horizontal and vertical axes represent the β_{in} and β_{out} values, respectively. The corresponding γ deformation is given in degrees with intervals of 10° .

Experimental data

As mentioned in the previous section, four positive-parity rotational bands were reported in [91]. One of these bands extends from the 0^+ (3.352 MeV) up to the 16^+ (20.578 MeV). In addition, two further positive-parity bands were identified, covering the spin ranges from 8^+ (8.103 MeV) to 15^+ (19.195 MeV) and from 3^+ (6.03 MeV) to 13^+ (16.579 MeV). The SD band, which starts from the 0^+ at 5.21 MeV and extends up to spin 16^+ at 22.06 MeV. A negative parity band was proposed in [92].

All the bands discussed above are listed in the compilation data [111, 112]. However, the negative-parity band differs between the compilations and Ref. [92]. In the compilation data, the band extends from the (7^-) state at 9.033 MeV to the (15^-) state at 18.215 MeV. In contrast, Ref. [92] proposed that the band begins with the 1^- state at 5.902 MeV and includes two 3^- states at 6.28 and 6.58 MeV, as well as a spin-5 state with uncertain parity at 7.399 MeV. Since this spin-5 state was previously assigned to the positive-parity band built on the 3^+ state at 6.03 MeV [91], I did not consider here as a member of the negative-parity band.

Using the compilation data [111, 112], four positive-parity bands and five

negative-parity bands were arranged in the low-energy region between 0 and 10.04 MeV. A sequence of states was considered to form a rotational band when at least three experimental states with well-defined spin-parity nearly followed a linear behavior in the energy versus $J(J+1)$ plot. The resulting band structures are presented in Fig. 3.9.

Eighteen candidate states for the HD band were proposed in Ref. [105]. These states were separated into positive- and negative-parity groups based on model calculations.

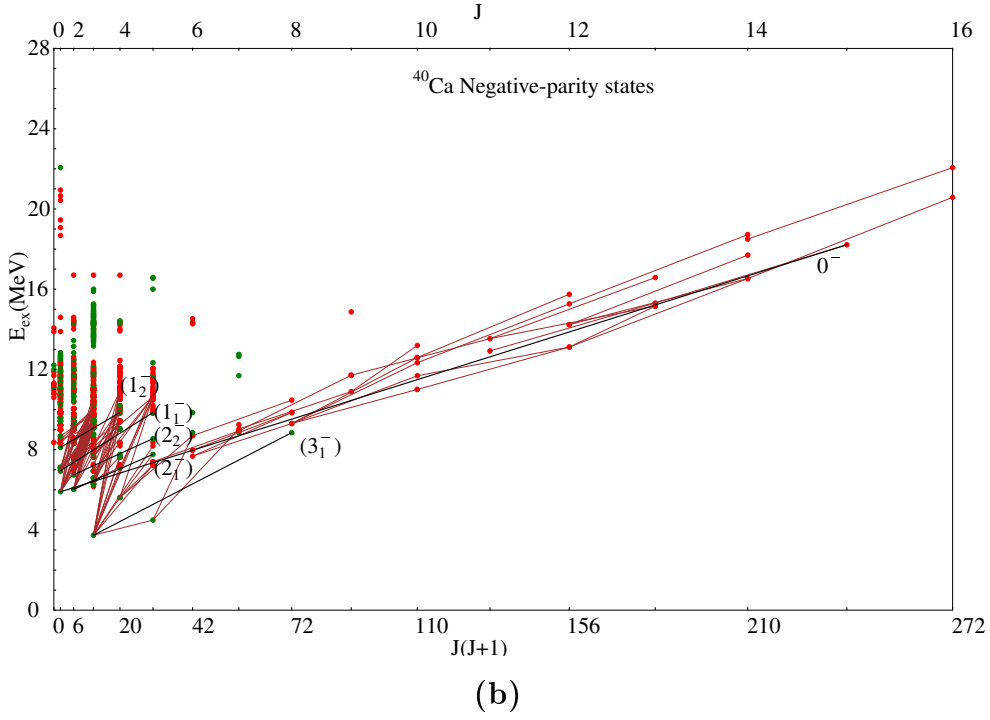
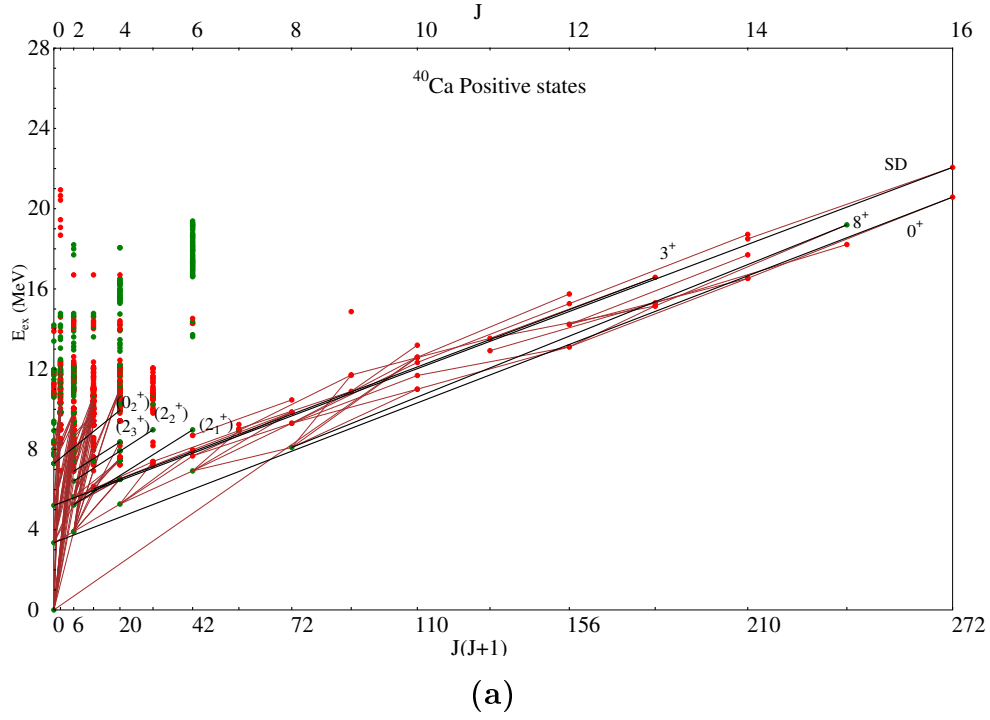


Figure 3.9: Positive (a) and negative-parity (b) states [111,112] of ^{40}Ca nucleus as a function of $J(J+1)$. Green and red circles indicate states with well-established and uncertain spin-parities, respectively. Electromagnetic transitions are indicated by brown connecting lines, and rotational bands by black lines. Bands are labeled according to the spin-parity J^π of their band head, except for the 0^- band, which was named as $K^\pi=0^-$ band in [92]. The superdeformed band is indicated as SD. 0^+ , 8^+ , 3^+ , 0^- and SD bands were previously known [91,92,111,112]. Bands labeled by (J^π) correspond to the arranged bands.

Spectrum calculation

To calculate the energy spectrum, the theoretical bands were assigned with the corresponding experimental ones. The quantum numbers for the positive-parity part of the HD band were taken from the SCS calculations [24]. For the negative-parity HD band, the quantum number was assigned based on configurations that are closest in deformation to the positive-parity band and that also has appropriate spin-parity content.

Cranked relativistic mean-field calculations [91] suggest the SD band as arising from an 8p–8h configuration. In Ref. [25], a quasi-dynamical SU(3) symmetry characterized by (22,4) in the $8\hbar\omega$ was proposed. Here, the configuration (24,4) is adopted, which lies closest in deformation to (22,4), within the same $8\hbar\omega$ excitation.

The lower-lying positive-parity band had previously been interpreted as arising from a 4p–4h configuration [90,91]. I associated it with the most deformed representation available in the $4\hbar\omega$ model space that satisfied the required spin-parity properties. The negative-parity band suggested as its partner in Ref. [92], had been described as a $K^\pi = 0^-$ band. It was assigned here to the most deformed SU(3) quantum number of the $5\hbar\omega$ model space, i.e. (15,4), which has the appropriate K^π value and spin-parity characteristics.

The experimentally observed 8^+ and 3^+ bands were assigned to the most deformed SU(3) representations within the model space, chosen such that their K^π values match the lowest angular momentum states of the respective experimental bands and have the well spin-parity content. Using the same approach, all candidate bands were associated with the most deformed representations in the $1\hbar\omega$, $2\hbar\omega$, and $3\hbar\omega$ model spaces, with each set of SU(3) quantum numbers employed only once in the overall assignment.

I applied the MUSY Hamiltonian, expressed in terms of the invariant operators of the group chain: $U(3) \supset SU(3) \supset SO(3)$. The corresponding energy functional is:

$$E = (\hbar\omega)n + a(\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)) + b(\lambda - \mu)(\lambda + 2\mu + 3)(2\lambda + \mu + 3) + \frac{d}{2\theta}L(L+1). \quad (3.11)$$

The physical meaning of all the terms is explained in Chapter 2 (see subsection 2.2.2). The parameters $\hbar\omega$, a, b, d are fitted to experimental data. θ represents the moment of inertia, which can be evaluated either geometrically for an axi-

ally symmetric ellipsoid defined by the U(3) quantum numbers, or alternatively expressed through the SU(3) invariant operator as $\theta = a_0 + a_1 \hat{C}_{SU(3)}^{(2)}$ [81]. In this calculation, the former approach was adopted.

The quality of the fit is evaluated by

$$F = \sum_i w(i) \frac{(E_i^{exp.} - E_i^{Th.})^2}{(E_i^{exp.})^2}, \quad (3.12)$$

and the parameters of the Hamiltonian are determined by minimizing this expression.

In the fitting process, states in experimentally established bands (including the SD band) with certain spin-parity assignments were given unit weight, whereas states with uncertain spin-parity were weighted by 0.5. For the HD band, the well-known spin-parity states were given a weight of 0.5, and those with uncertain spin-parity were assigned a weight of 0.25.

Two different fitting calculations were performed. They differ by the treatment of arranged bands in weighting procedure:

- A. States of arranged bands with well-defined spin-parity were assigned a weight of 0.1, while a weight of 0.05 was given to states with uncertain assignments (Figure 3.10).
- B. All the states of the arranged bands were excluded from the fit by assigning them zero weight (Figure 3.11).

The parameters of Eq. (3.11) obtained from these calculations are given in Table 3.7.

Table 3.7: The fitted parameters of Eq. (3.11) and the corresponding fit quality are characterized by $f = \frac{F}{m}$, where m denotes the total number of states included in the fit. The two rows represent the results of calculations performed with and without the inclusion of the newly arranged bands (see the discussion in the text for the explanation of A and B). For a meaningful comparison between the two approaches, the values of F were recomputed using the same set of weights $w(i)$ as in the calculation where the arranged bands were excluded; the resulting quantity is denoted by f' .

	$\hbar\omega$ (MeV)	a (MeV)	b (MeV)	d	f	f'
A.	3.18366	-0.03732	0.00025	1.21304	0.01779	0.01779
B.	2.238319	-0.02379	0.00018	1.18467	0.01562	0.01926

The reduced transition probability [B(E2)] for the intra-band transitions from L_i to L_f were obtained by using

$$B(E2, L_i \longrightarrow L_f) = \frac{2L_f + 1}{2L_i + 1} \alpha^2 |\langle (\lambda\mu)KL_i, (11)2 || (\lambda\mu)KL_f \rangle|^2 C_{SU(3)}^{(2)}. \quad (3.13)$$

Where $\langle (\lambda\mu)KL_i, (11)2 || (\lambda\mu)KL_f \rangle$ is a $SU(3) \supset SO(3)$ Wigner coefficient [83], and $\alpha^2 (= 0.798)$ was obtained by fitting to the experimental value of the 4^+ (5.278 MeV) $\rightarrow 2^+$ (3.904 MeV) transition strength of 67 W.u.

The spectrum and the B(E2) are shown in Figures 3.10 and 3.11

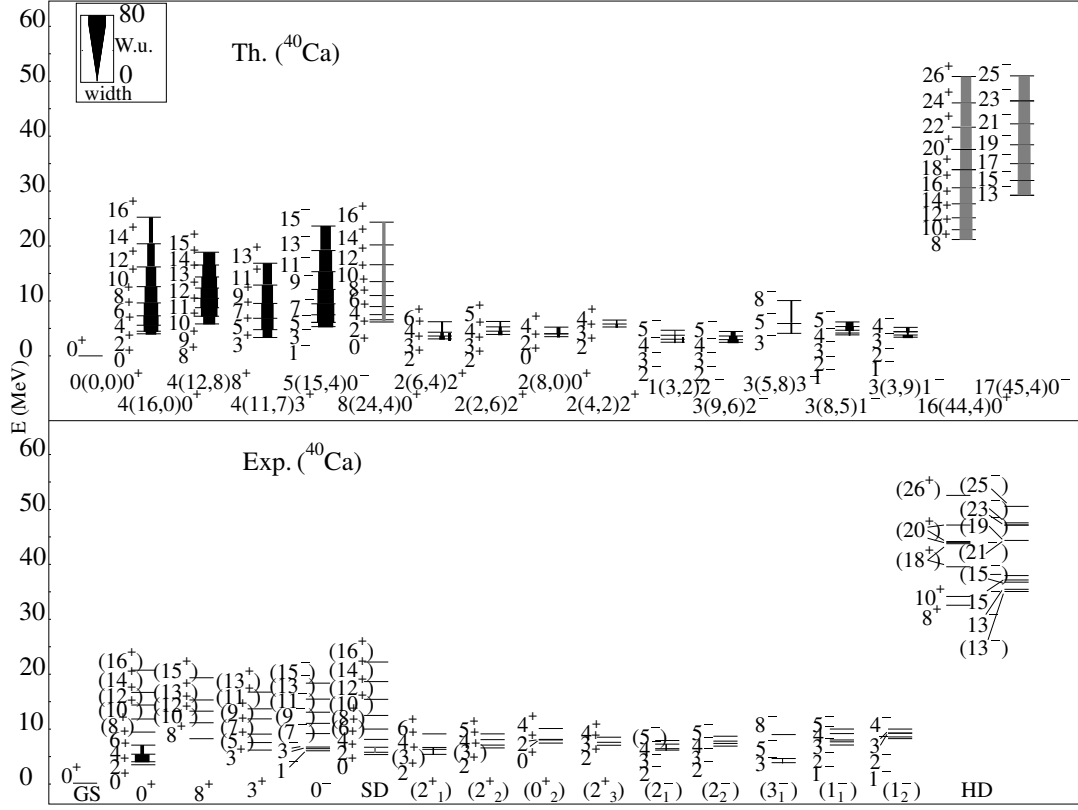


Figure 3.10: The MUSY spectrum (top) is displayed together with the experimental spectrum of ^{40}Ca (bottom). The ground-state band, the low-lying experimental (and arranged) bands, as well as the superdeformed and hyperdeformed bands, are denoted by GS, K^π , SD, and HD, respectively. The theoretical bands are labeled by $n(\lambda, \mu)K^\pi$. The widths of the arrows indicate the relative $E2$ transition strengths; for the SD and HD bands, the actual strengths corresponding to the gray arrows are ten times larger than those shown. Transitions are drawn from each state to the preceding level with the next lower angular momentum, except for the $(2_1^+)\{4^+ \rightarrow 2^+\}$ and $(2_1^-)\{4^- \rightarrow 2^-\}$ transitions. For better visibility, the theoretical transitions are shifted slightly to the right. This figure corresponds to the calculation in which states belonging to the arranged bands with well-defined spin-parity assignments were weighted 0.1, while states with uncertain spin-parity were assigned a weight of 0.05.

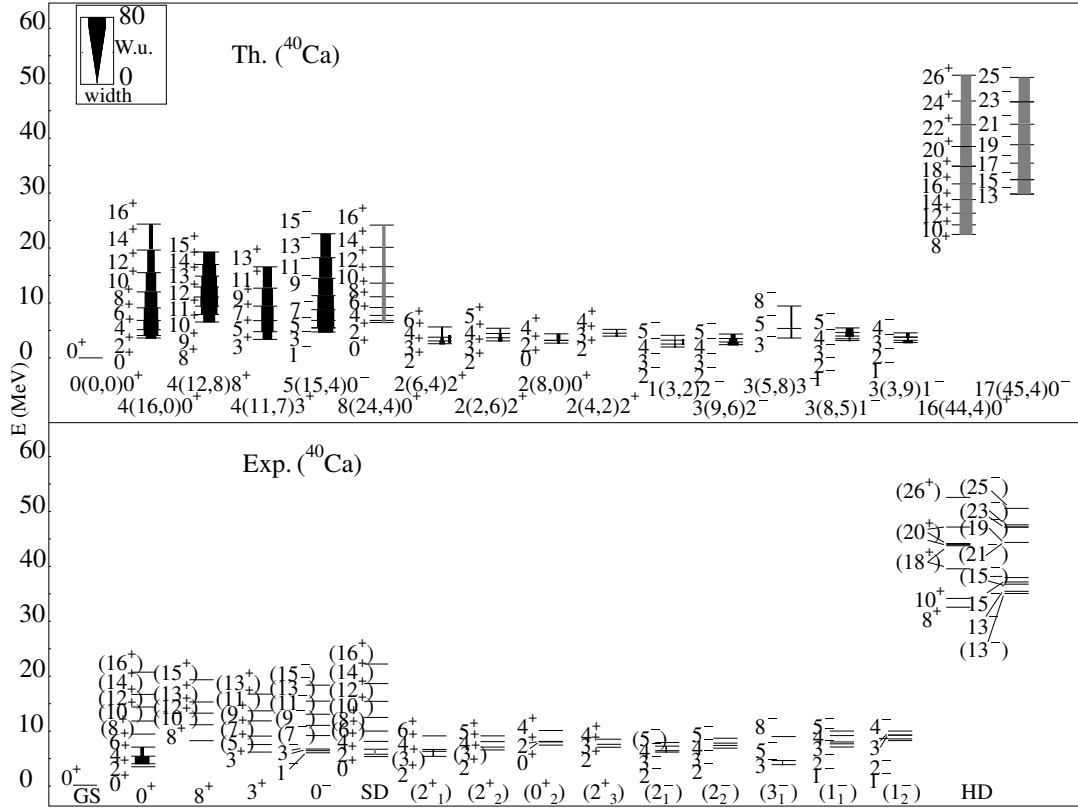


Figure 3.11: Same as Figure 3.10, but corresponding to the calculation in which the states of the arranged bands were assigned zero weight.

Summary of ^{40}Ca

I applied the multiconfigurational dynamical symmetry for the unified description of the spectrum of the ^{40}Ca nucleus. This symmetry represents the common intersection of the shell, collective, and cluster models for the general multi-major-shell problem. As a result, it can predict shape isomers through the behavior of their connecting $U(3)$ symmetry and describe the energy spectra of different reaction channels in a wide range of energies and deformations within a single framework.

In the low-energy region, the nucleus exhibits a well-defined band structure, and its superdeformed band has been observed experimentally. High-resolution heavy-ion resonance measurements also provide indications of hyperdeformation. Based on the available experimental compilations, I arranged additional candidate bands. The results demonstrate that MUSY provides a unified description in good agreement with the experiment. A key finding is that the arranged bands were predicted from the well-known bands; the agreement between the experimental states and the model states was nearly as good as in the fitted case, differing by only about 9%.

3.3 Phases of binary cluster configurations

The question of whether nuclear clustering is more like a rigid molecule or a shell-like configuration remains a long-standing topic of debate. Various models have addressed different aspects of this problem, yet much of the discussion has either remained qualitative or relied on computationally demanding methods that cannot be applied systematically; see, for example, Refs. [113–115]. In the present work I took an initial step toward a systematic and quantitative investigation of the nature of clustering in light nuclei.

The study of phases in quadrupole collective models [53] and in the shell model [54] has been successfully carried out using algebraic models. For the cluster model, the possible phases have also been identified [62], revealing three distinct phases characterized by their dynamical symmetries: the shell-like (soft-vibrator) limit with $U(3)$ symmetry, the rigid molecule-like (rigid-rotor) limit with $O(4)$ symmetry, and the weakly coupled limit associated with $O(3)$ symmetry. A connection between the phases of the shell and cluster models is also revealed (Fig. 2.1), in which their common intersection is the $U(3)$ limit. However, a thorough investigation and comparison of the spectra of these three phases have not yet been carried out, which is done in this work.

I applied the semimicroscopic algebraic cluster model (SACM) [5–7] for binary cluster configurations [P3]. In particular, I discuss the role of the Pauli principle and construct the Hamiltonians for each limit. Later, I consider the $^{12}\text{C}+\alpha$ cluster system to investigate the influence of the Pauli principle and analyze the spectra in the three limiting cases.

3.3.1 Pauli principle

Since the nucleus is a multi-fermion system, the Pauli exclusion principle should be taken into account while constructing the model states. In SACM, the internal wave function of the clusters is completely antisymmetric and is described by Elliott’s model [1, 2] ($U^{ST}(4)\otimes U(3)$ group structure). However, when the relative motion ($U_R(4)$) is coupled with the wave function of the two clusters, the resulting total wave function may contain Pauli-forbidden states, since the antisymmetrization associated with the exchange of nucleons between the clusters has not yet been considered. Therefore, a proper treatment of Pauli principle is required to get microscopic model space.

In the following, the influence of the Pauli principle and how it is treated in

the different dynamical symmetry limits are discussed.

- U(3) dynamical symmetry:

In the U(3) dynamical symmetry limit, the implementation of the Pauli exclusion principle is relatively straightforward. The Wildermuth condition [116] determines the minimum number of oscillator quanta n_π required for the relative motion between two clusters ($n_\pi \geq n_0$). The minimum number of oscillator quanta required for the relative motion of two clusters is determined by the Wildermuth condition [116]. Within the shell-model, n_π corresponds to a major shell. As a consequence, this condition determines whether an entire major shell is allowed or forbidden based on whether the n_π value lies above or below the threshold value n_0 . For clusters without internal excitation, i.e., closed shell clusters, this requirement already fulfills the Pauli principle. For example, $^{16}\text{O}+^4\text{He}$ system, the minimum value is $n_0=8$.

The situation becomes more complicated when the clusters possess internal degrees of freedom that couple with the relative motion. In such cases, not all U(3) irreps within a given major shell are allowed. The allowed ones can be identified by comparing the U(3) basis associated with the cluster configuration with the corresponding shell-model basis [7].

- O(4) dynamical symmetry:

In this limit, U(3) quantum numbers are not good quantum numbers for either the nuclear states or their relative motion. Therefore, the Pauli-forbidden states cannot be removed directly. The O(4) basis states are contaminated with Pauli forbidden states. Nevertheless, these O(4) states can be written as linear combinations of U(3) basis states [63].

- O(3) dynamical symmetry:

In this limit, n_π remains a good quantum number. Therefore, the Wildermuth condition eliminates the lowest major shells below n_0 that consist entirely of Pauli-forbidden states. However, above the n_0 limit, it cannot be distinguished between allowed and forbidden states. This is because the states are not labeled by U(3) quantum numbers but by the angular momenta associated with the internal structure of the clusters (L_C) and their relative motion (L_R). As a result, some Pauli-forbidden states are removed in O(3) basis, although not all of them. Nevertheless, the O(3) basis states can be expressed as linear combinations of the U(3) basis states.

3.3.2 Hamiltonian

For each dynamical-symmetry limit (see Eqs. 2.30-2.32), the Hamiltonian can be written in terms of the invariant operators of the corresponding group chain, considering contributions up to two-body terms

$$H_{U(3)} = a\hat{C}_{U_C(3)}^{(1)} + b\hat{C}_{U_C(3)}^{(2)} + c\hat{C}_{U_R(3)}^{(1)} + d\hat{C}_{U_R(3)}^{(2)} + e\hat{C}_{U(3)}^{(1)} + f\hat{C}_{SU(3)}^{(2)} + g\hat{C}_{SO(3)}^{(2)} \quad (3.14)$$

$$H_{O(4)} = a\hat{C}_{U_C(3)}^{(1)} + b\hat{C}_{U_C(3)}^{(2)} + h\hat{C}_{SO_R(4)}^{(2)} + j\hat{C}_{SO_C(3)}^{(2)} + k\hat{C}_{SO_R(3)}^{(2)} + g\hat{C}_{SO(3)}^{(2)} \quad (3.15)$$

$$H_{O(3)} = a\hat{C}_{U_C(3)}^{(1)} + b\hat{C}_{U_C(3)}^{(2)} + c\hat{C}_{U_R(3)}^{(1)} + d\hat{C}_{U_R(3)}^{(2)} + j\hat{C}_{SO_C(3)}^{(2)} + k\hat{C}_{SO_R(3)}^{(2)} + g\hat{C}_{SO(3)}^{(2)} \quad (3.16)$$

The invariant operators of $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$ should also present in the expressions. However, when the clusters are described by single internal structure, these operators contribute equally to the energy of every state in the spectrum. When only the excitation spectrum is considered, they can be neglected. In $O(4)$ limit, the Casimir operator of $O_R(4)$ is not written explicitly, because the orthogonal and special orthogonal groups share the same algebra [64].

In principle, Eqs. 3.14–3.16 require the invariant operators of $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$ as well. However, their contribution is identical for all states within a given spectrum when the clusters are characterized by single internal $U(3)$ symmetries. Therefore, they can be omitted when we are interested in the excitation spectrum, i.e. energies relative to the ground state. They become relevant, for example, when the binding energy of the ground state is considered.

The operator $\hat{C}_{U(3)}^{(2)}$ does not appear in $H_{U(3)}$, since it can be written using $\hat{C}_{U(3)}^{(1)}$ and $\hat{C}_{SU(3)}^{(2)}$. For the same reason, $\hat{C}_{SU_C(3)}^{(2)}$ in $H_{O(4)}$, as well as $\hat{C}_{SU_C(3)}^{(2)}$ and $\hat{C}_{SU_R(3)}^{(2)}$ in $H_{O(3)}$, are omitted. These operators can be expressed through the first- and second-order Casimir operators of $U_C(3)$ (or $U_R(3)$), which will be discussed later. The operators $\hat{C}_{U(3)}^{(1)}$, $\hat{C}_{U(3)}^{(2)}$, and $\hat{C}_{SU(3)}^{(2)}$ are therefore not independent but are uniquely related.

From a practical perspective, including dependent operators would lead to an overparameterized Hamiltonian, making it difficult to determine a unique set of parameters. In such a case, different parameter sets may reproduce the same energy spectrum. For this reason, only $\hat{C}_{U(3)}^{(1)}$ and $\hat{C}_{SU(3)}^{(2)}$ are retained in the Hamiltonian, since they have clear physical interpretations. Specifically, $\hat{C}_{U(3)}^{(1)}$ represents the oscillator quanta number, whereas $\hat{C}_{SU(3)}^{(2)}$ is associated with the quadrupole–quadrupole interaction.

The linear ($\hat{C}^{(1)}$) and quadratic ($\hat{C}^{(2)}$) Casimir operators can be written in terms of physical operators (Eqs. (5.32), (5.38), (8.39), (5.43), and (5.44) of Ref. [63]):

$$\hat{C}_{U(3)}^1 = \hat{n} \quad (3.17)$$

$$\hat{C}_{U(3)}^2 = \frac{1}{3}\hat{n}^2 + \frac{1}{2}\hat{L}^2 + \hat{Q}^2 \quad (3.18)$$

$$\hat{C}_{SU(3)}^2 = \hat{Q}^2 + \frac{1}{2}\hat{L}^2 \quad (3.19)$$

$$\hat{C}_{SO(3)}^2 = \hat{L}^2 \quad (3.20)$$

$$\hat{C}_{SO(4)}^2 = \hat{L}^2 + \hat{D}^2 \quad (3.21)$$

Here, $\hat{n}$, $\hat{L}$, $\hat{Q}$, and $\hat{D}$ denote the number operator, the angular momentum operator, the quadrupole operator, and the dipole operator, respectively. These operators can also be represented using alternative physical operators, some of which are given below:

$$\begin{aligned} \hat{C}_{U(3)}^2 &= \frac{1}{3}(\hat{C}_{U(3)}^1)^2 + \hat{C}_{SU(3)}^2, \\ \text{or, } \frac{1}{3}(\hat{C}_{U(3)}^1)^2 &+ \frac{1}{2}\hat{C}_{SO(3)}^2 + \hat{Q}^2, \\ \text{or, } \frac{1}{3}(\hat{C}_{U(3)}^1)^2 &+ \frac{1}{2}\hat{C}_{SO(4)}^2 + \hat{Q}^2 - \frac{1}{2}\hat{D}^2. \end{aligned} \quad (3.22)$$

$$\begin{aligned} \hat{C}_{SU(3)}^2 &= \hat{C}_{U(3)}^2 - \frac{1}{3}(\hat{C}_{U(3)}^1)^2, \\ \text{or, } \frac{1}{2}\hat{C}_{SO(3)}^2 &+ \hat{Q}^2, \\ \text{or, } \frac{1}{2}\hat{C}_{SO(4)}^2 &+ \hat{Q}^2 - \frac{1}{2}\hat{D}^2. \end{aligned} \quad (3.23)$$

$$\begin{aligned} \hat{C}_{SO(3)}^2 &= 2\hat{C}_{U(3)}^2 - \frac{2}{3}(\hat{C}_{U(3)}^1)^2 - 2\hat{Q}^2, \\ \text{or, } 2\hat{C}_{SU(3)}^2 &- 2\hat{Q}^2, \\ \text{or, } \hat{C}_{SO(4)}^2 &- \hat{D}^2. \end{aligned} \quad (3.24)$$

The expectation values of the Casimir operators are listed below. They can be derived from Chapter 5 of Ref. [64], and some are also provided in Ref. [7]:

$$\langle \hat{C}_{U(3)}^1 \rangle = n_1 + n_2 + n_3 \quad (3.25)$$

$$\langle \hat{C}_{U(3)}^2 \rangle = n_1^2 + n_2^2 + n_3^2 + 2(n_1 - n_3) \quad (3.26)$$

$$\langle \hat{C}_{SU(3)}^2 \rangle = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu) \quad (3.27)$$

$$\langle \hat{C}_{SO(3)}^2 \rangle = L(L + 1) \quad (3.28)$$

$$\langle \hat{C}_{SO(4)}^2 \rangle = \omega(\omega + 2) \quad (3.29)$$

It should be noted that the expression for $\langle \hat{C}_{SU(3)}^{(2)} \rangle$ differs from the form given in Ref. [64] by a factor of 6/9; the convention of Ref. [7] is followed here.

The eigenvalues of the three dynamical-symmetry Hamiltonians are calculated within their respective bases, omitting contributions that are identical for all states.

$$E_{U(3)} = c n_\pi + d n_\pi(n_\pi + 2) + e (n_1 + n_2 + n_3) + f (\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)) + g L(L + 1), \quad (3.30)$$

$$E_{O(4)} = h \omega(\omega + 2) + j L_C(L_C + 1) + k L_R(L_R + 1) + g L(L + 1), \quad (3.31)$$

$$E_{O(3)} = c n_\pi + d n_\pi(n_\pi + 2) + j L_C(L_C + 1) + k L_R(L_R + 1) + g L(L + 1). \quad (3.32)$$

3.3.3 Schematic spectra

I consider $^{12}\text{C} + \alpha$ cluster configuration of ^{16}O . To illustrate the spectra corresponding to the three phases of the SACM, the model space and qualitative spectra are presented.

• Model space

The model spaces presented here are presented in three font styles: italics for completely Pauli-forbidden states, bold for configurations contaminated by Pauli-forbidden states, and roman type for completely Pauli-allowed states.

The $U_{C_1}(3)$ representation of ^{12}C is taken as $[n_1^{C_1}, n_2^{C_1}, n_3^{C_1}] = [4, 4, 0]$, while the α particle is a scalar $U_{C_2}(3)$: $[n_1^{C_2}, n_2^{C_2}, n_3^{C_2}] = [0, 0, 0]$. The coupled $U_C(3)$ representation for the $^{12}\text{C} + \alpha$ cluster is $[n_1^C, n_2^C, n_3^C] = [4, 4, 0]$, corresponding to $(\lambda_C, \mu_C) = (0, 4)$. For the ^{16}O nucleus, the total number of oscillator quanta is 12. In the shell-model description, ^{12}C contains eight oscillator quanta, while the α -particle has none; therefore, the two clusters together contain eight oscillator

quanta. According to the Wildermuth condition, minimum four oscillator quanta must be appeared to the relative motion ($n_0=4$) in order to satisfy the Pauli exclusion principle. The cluster-model space becomes fully Pauli allowed by comparing the $U(3)$ basis with the corresponding shell-model basis.

Table 3.8 presents the complete model space of the $U(3)$ dynamical-symmetry limit.

Table 3.8: Complete model space of the $^{12}\text{C} + \alpha$ system in the $U(3)$ dynamical-symmetry limit for $N = 5$ bosons (up to $n_\pi = 5$). Quantum numbers shown in *italics* correspond to Pauli-forbidden configurations, while those in roman type denote Pauli-allowed states.

U(3) Limit			
n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu) \mathbf{K}^\pi$	$\mathbf{L}^\pi$
0	$[4, 4, 0]$	$(0, 4) 0^+$	$0^+, 2^+, 4^+$
1	$[5, 4, 0]$	$(1, 4) 1^-$	$1^-, 2^-, 3^-, 4^-, 5^-$
	$[4, 4, 1]$	$(0, 3) 0^-$	$1^-, 3^-$
2	$[6, 4, 0]$	$(2, 4) 0^+$ $(2, 4) 2^+$	$0^+, 2^+, 4^+$ $2^+, 3^+, 4^+, 5^+, 6^+$
	$[5, 4, 1]$	$(1, 3) 1^+$	$1^+, 2^+, 3^+, 4^+$
	$[4, 4, 2]$	$(0, 2) 0^+$	$0^+, 2^+$
	$[7, 4, 0]$	$(3, 4) 1^-$ $(3, 4) 3^-$	$1^-, 2^-, 3^-, 4^-, 5^-$ $3^-, 4^-, 5^-, 6^-, 7^-$
3	$[6, 4, 1]$	$(2, 3) 0^-$ $(2, 3) 2^-$	$1^-, 3^-$ $2^-, 3^-, 4^-, 5^-$
	$[5, 4, 2]$	$(1, 2) 1^-$	$1^-, 2^-, 3^-$
	$[4, 4, 3]$	$(0, 1) 0^-$	1^-

Table 3.8: (continued)

U(3) Limit			
n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu) \mathbf{K}^\pi$	$\mathbf{L}^\pi$
4	$[8, 4, 0]$	$(4, 4) 0^+$	$0^+, 2^+, 4^+$
		$(4, 4) 2^+$	$2^+, 3^+, 4^+, 5^+, 6^+$
		$(4, 4) 4^+$	$4^+, 5^+, 6^+, 7^+, 8^+$
	$[7, 4, 1]$	$(3, 3) 1^+$	$1^+, 2^+, 3^+, 4^+$
		$(3, 3) 3^+$	$3^+, 4^+, 5^+, 6^+$
	$[6, 4, 2]$	$(2, 2) 0^+$	$0^+, 2^+$
		$(2, 2) 2^+$	$2^+, 3^+, 4^+$
	$[5, 4, 3]$	$(1, 1) 1^+$	$1^+, 2^+$
	$[4, 4, 4]$	$(0, 0) 0^+$	0^+
	$[9, 4, 0]$	$(5, 4) 0^-$	$1^-, 3^-, 5^-$
		$(5, 4) 2^-$	$2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
		$(5, 4) 4^-$	$4^-, 5^-, 6^-, 7^-, 8^-, 9^-$
	$[8, 4, 1]$	$(4, 3) 1^-$	$1^-, 2^-, 3^-, 4^-, 5^-$
		$(4, 3) 3^-$	$3^-, 4^-, 5^-, 6^-, 7^-$
	$[7, 4, 2]$	$(3, 2) 0^-$	$1^-, 3^-$
		$(3, 2) 2^-$	$2^-, 3^-, 4^-, 5^-$
	$[6, 4, 3]$	$(2, 1) 1^-$	$1^-, 2^-, 3^-$
	$[5, 4, 4]$	$(1, 0) 0^-$	1^-

The $O(4)$ limit does not incorporate the Pauli principle. For a fixed number of bosons N , the corresponding model space can be constructed directly. The model space for the $O(4)$ limit is presented in Table 3.9 for $N = 5$ bosons. The total

angular momentum L is obtained by coupling the cluster angular momentum L_C with the relative motion angular momentum L_R , while the total parity is determined from the product of corresponding parities, $\pi = \pi_C \pi_R$.

Table 3.9: Model space of the $^{12}\text{C} + \alpha$ system for $N = 5$ bosons in the $O(4)$ dynamical-symmetry limit. The quantum number ν is defined as $(N - \omega)/2$. Since Pauli-forbidden configurations cannot be excluded in this limit, the basis states contain contributions from both Pauli-allowed and Pauli-forbidden $U(3)$ configurations.

O(4) Limit				
ω	ν	$\mathbf{L}_C^\pi$	$\mathbf{L}_R^\pi$	$\mathbf{L}^\pi$
1	2	0^+	0^+	0^+
		0^+	1^-	1^-
		2^+	0^+	2^+
		2^+	1^-	$1^-, 2^-, 3^-$
		4^+	0^+	4^+
		4^+	1^-	$3^-, 4^-, 5^-$
		0^+	0^+	0^+
		0^+	1^-	1^-
3	1	0^+	2^+	2^+
		0^+	3^-	3^-
		2^+	0^+	2^+
		2^+	1^-	$1^-, 2^-, 3^-$
		2^+	2^+	$0^+, 1^+, 2^+, 3^+, 4^+$
		2^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-$
		4^+	0^+	4^+
		4^+	1^-	$3^-, 4^-, 5^-$
		4^+	2^+	$2^+, 3^+, 4^+, 5^+, 6^+$
		4^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
		0^+	0^+	0^+
		0^+	1^-	1^-
		0^+	2^+	2^+
		0^+	3^-	3^-
		2^+	0^+	2^+
		2^+	1^-	$1^-, 2^-, 3^-$
		2^+	2^+	$0^+, 1^+, 2^+, 3^+, 4^+$
		2^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-$
		4^+	0^+	4^+
		4^+	1^-	$3^-, 4^-, 5^-$
		4^+	2^+	$2^+, 3^+, 4^+, 5^+, 6^+$
		4^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$

Table 3.9: (continued)

O(4) Limit				
ω	ν	$\mathbf{L}_C^\pi$	$\mathbf{L}_R^\pi$	$\mathbf{L}^\pi$
5	0	0⁺	0⁺	0⁺
		0⁺	1⁻	1⁻
		0⁺	2⁺	2⁺
		0⁺	3⁻	3⁻
		0⁺	4⁺	4⁺
		0⁺	5⁻	5⁻
		2⁺	0⁺	2⁺
		2⁺	1⁻	1⁻, 2⁻, 3⁻
		2⁺	2⁺	0⁺, 1⁺, 2⁺, 3⁺, 4⁺
		2⁺	3⁻	1⁻, 2⁻, 3⁻, 4⁻, 5⁻
		2⁺	4⁺	2⁺, 3⁺, 4⁺, 5⁺, 6⁺
		2⁺	5⁻	3⁻, 4⁻, 5⁻, 6⁻, 7⁻
		4⁺	0⁺	4⁺
		4⁺	1⁻	3⁻, 4⁻, 5⁻
		4⁺	2⁺	2⁺, 3⁺, 4⁺, 5⁺, 6⁺
		4⁺	3⁻	1⁻, 2⁻, 3⁻, 4⁻, 5⁻, 6⁻, 7⁻
		4⁺	4⁺	0⁺, 1⁺, 2⁺, 3⁺, 4⁺, 5⁺, 6⁺, 7⁺, 8⁺
		4⁺	5⁻	1⁻, 2⁻, 3⁻, 4⁻, 5⁻, 6⁻, 7⁻, 8⁻, 9⁻

In O(3) limit, Pauli-forbidden $^{12}\text{C}+\alpha$ model states can be removed below $n_0 = 4$. However, for $n_\pi \geq n_0$, the states are mixed with Pauli-forbidden components. As like the previous case, the total angular momentum L is obtained coupling of L_C and L_R , and the total parity is given by $\pi = \pi_C \pi_R$. The complete model space is presented in Table 3.10.

Table 3.10: Model space of the $^{12}\text{C} + \alpha$ system for $N = 5$ bosons (up to $n_\pi = 5$) in the $O(3)$ dynamical-symmetry limit. For $n_\pi < 4$, all configurations are Pauli-forbidden (indicated in italics). For $n_\pi \geq 4$, the states are mixed and contain contributions from both Pauli-forbidden and Pauli-allowed components (shown in bold).

O(3) Limit			
n_π	$\mathbf{L}_C^\pi$	$\mathbf{L}_R^\pi$	$\mathbf{L}^\pi$
<i>0</i>	<i>0^+</i>	<i>0^+</i>	<i>0^+</i>
	<i>2^+</i>	<i>0^+</i>	<i>2^+</i>
	<i>4^+</i>	<i>0^+</i>	<i>4^+</i>
<i>1</i>	<i>0^+</i>	<i>1^-</i>	<i>1^-</i>
	<i>2^+</i>	<i>1^-</i>	<i>$1^-, 2^-, 3^-$</i>
	<i>4^+</i>	<i>1^-</i>	<i>$3^-, 4^-, 5^-$</i>
<i>2</i>	<i>0^+</i>	<i>0^+</i>	<i>0^+</i>
	<i>0^+</i>	<i>2^+</i>	<i>2^+</i>
	<i>2^+</i>	<i>0^+</i>	<i>2^+</i>
	<i>2^+</i>	<i>2^+</i>	<i>$0^+, 1^+, 2^+, 3^+, 4^+$</i>
	<i>4^+</i>	<i>0^+</i>	<i>4^+</i>
	<i>4^+</i>	<i>2^+</i>	<i>$2^+, 3^+, 4^+, 5^+, 6^+$</i>
<i>3</i>	<i>0^+</i>	<i>1^-</i>	<i>1^-</i>
	<i>0^+</i>	<i>3^-</i>	<i>3^-</i>
	<i>2^+</i>	<i>1^-</i>	<i>$1^-, 2^-, 3^-$</i>
	<i>2^+</i>	<i>3^-</i>	<i>$1^-, 2^-, 3^-, 4^-, 5^-$</i>
	<i>4^+</i>	<i>1^-</i>	<i>$3^-, 4^-, 5^-$</i>
	<i>4^+</i>	<i>3^-</i>	<i>$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$</i>

Table 3.10: (continued)

O(3) Limit			
n_π	$\mathbf{L}_C^\pi$	$\mathbf{L}_R^\pi$	$\mathbf{L}^\pi$
4	0^+	0^+	0^+
	0^+	2^+	2^+
	0^+	4^+	4^+
	2^+	0^+	2^+
	2^+	2^+	$0^+, 1^+, 2^+, 3^+, 4^+$
	2^+	4^+	$2^+, 3^+, 4^+, 5^+, 6^+$
	4^+	0^+	4^+
	4^+	2^+	$2^+, 3^+, 4^+, 5^+, 6^+$
	4^+	4^+	$0^+, 1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+$
	0^+	1^-	1^-
	0^+	3^-	3^-
	0^+	5^-	5^-
	2^+	1^-	$1^-, 2^-, 3^-$
	2^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-$
	2^+	5^-	$3^-, 4^-, 5^-, 6^-, 7^-$
5	4^+	1^-	$3^-, 4^-, 5^-$
	4^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
	4^+	5^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-, 8^-, 9^-$

• Spectra

The qualitative spectra associated with the three dynamical-symmetry is shown Figure 3.12. The spectra are calculated using Eqs. 3.30, 3.31, and 3.32.

Three different lines have used to distinguish completely Pauli-forbidden (dotted), partially forbidden (dashed), and completely Pauli-allowed states (solid). The figure demonstrates the essential role of the Pauli principle in the construction of the model space. The resulting spectra differ qualitatively among the three symmetry limits.

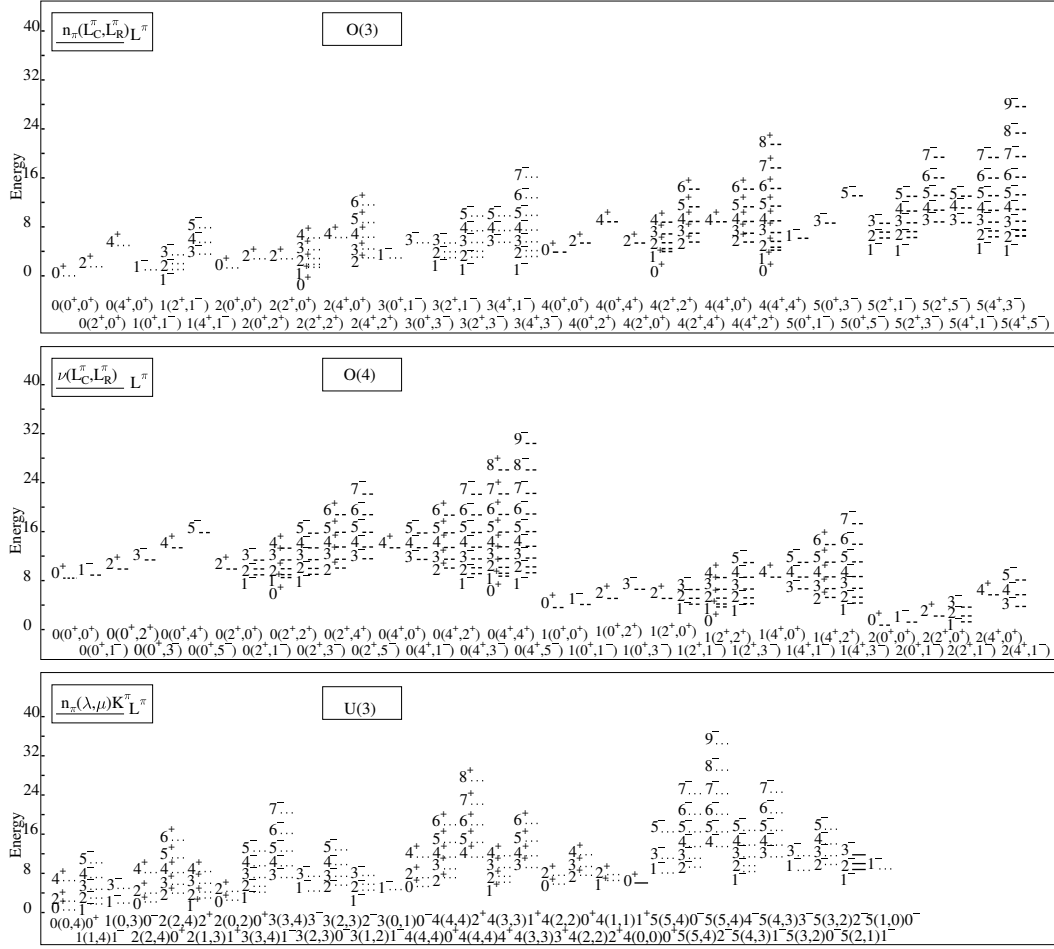


Figure 3.12: Spectra corresponding to the $U(3)$ (bottom), $O(4)$ (middle), and $O(3)$ (top) dynamical-symmetry limits are displayed. The energies are calculated from expressions $E_{U(3)}$, $E_{O(4)}$, and $E_{O(3)}$ given in Eqs. 3.30, 3.31, and 3.32. For the $^{12}\text{C} + \alpha$ system, the parameters are chosen as $d = 0.2$, $e = 0.1$, $f = -0.01$, $g = 0.3$, $h = 0.3$, and $c = j = k = 0.01$. The eigenstates in the $U(3)$, $O(4)$, and $O(3)$ limits are characterized by $n_\pi(\lambda, \mu)K^\pi$, $\nu(L_C^\pi, L_R^\pi)$, and $n_\pi(L_C^\pi, L_R^\pi)$ quantum numbers, respectively. Here, $\nu = (N - \omega)/2$, where N denotes the boson number. The spectra are shown for $N = 5$ bosons, and energies are given in arbitrary units. Different line styles indicate the Pauli character of the states: dotted lines represent completely Pauli-forbidden states, dashed lines indicate states containing Pauli-forbidden components, and solid lines represent fully Pauli-allowed states.

3.3.4 Realistic Spectra

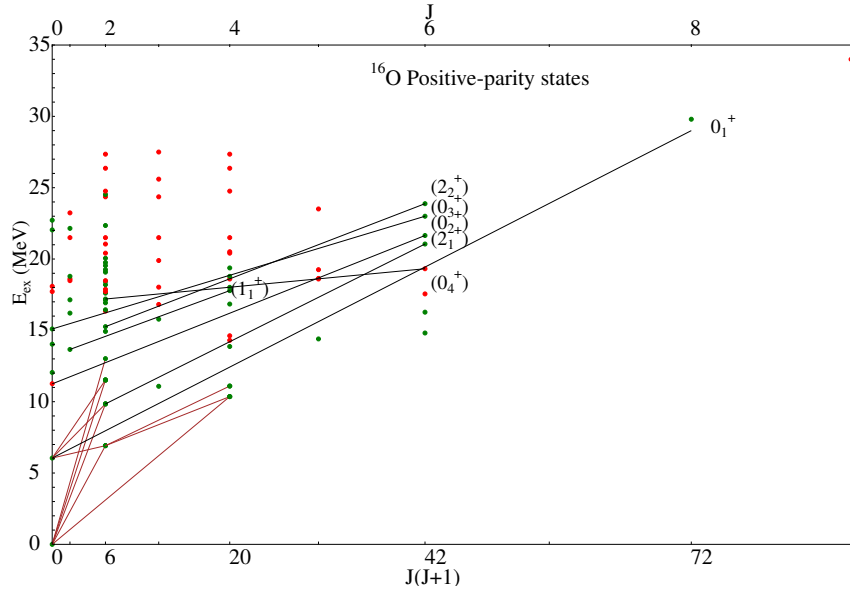
As within $U(3)$ dynamical symmetry limit of semimicroscopic algebraic model the Pauli principle can be treated properly, this limit is used to calculate α -cluster band of ^{16}O .

• Experimental Data

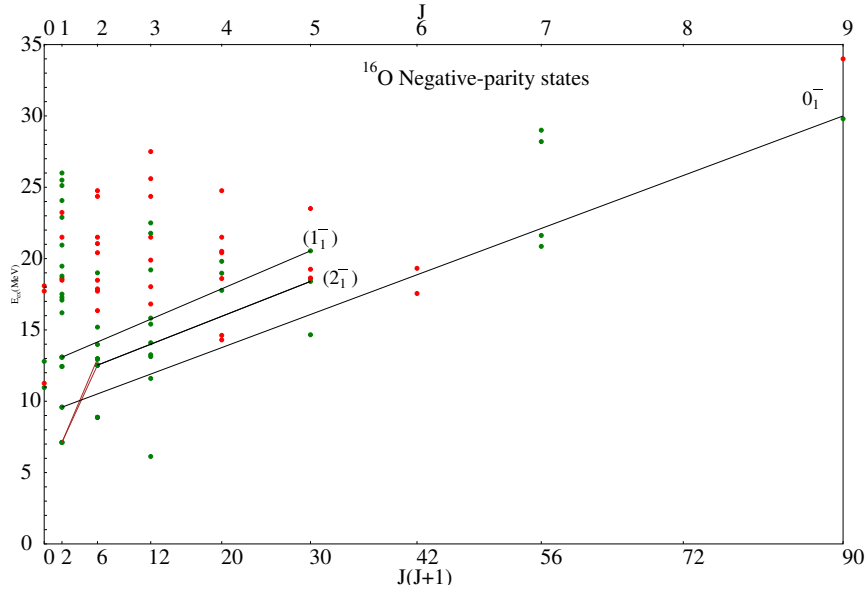
Two rotation bands with positive and negative parities are presented in [117,118]. The positive parity band starts with first excited 0^+ state (at 6.05 MeV), and has the members 2^+ (6.92 MeV), 4^+ (10.36 MeV), and 6^+ (16.28 MeV). An 8^+ state was proposed as a possible member of this band. Based on an extrapolation [118], its excitation energy was predicted to be $E_x \approx 20$ MeV, while elastic-scattering studies located it around $E_x = 29$ MeV [117]. The negative parity band consists of the states 1^- (9.59 MeV), 3^- (11.6 MeV), 5^- (14.66 MeV), and 7^- (20.86 MeV). A 9^- member at approximately 30 MeV was later suggested by Katsuma [117].

In the α -cluster model study [119], several rotational bands were arranged. The experimentally established bands are also appear in this arrangement, except the highest-spin members (8^+ and 9^-). The bands (2_1^+) , (2_2^+) , (0_2^+) , (0_4^+) , and (2_1^-) shown in Figure 3.13 belong to these arrangements. The excitation energies of the states were determined by comparing Fig. 7 of Ref. [119] with the experimental compilation [120]. Although a 0^+ state is listed as the band head of the (0_4^+) band in the model, no such state has been reported in the experimental compilation.

The experimental spectrum is shown in Figure 3.14.



(a)



(b)

Figure 3.13: Positive- (a) and negative-parity (b) states [120] of ^{16}O nucleus as a function of $J(J+1)$. Green and red circles indicate states with well-established and uncertain spin-parities, respectively. Brown lines represent electromagnetic transitions, while the black lines indicate rotational bands (labeled by K^π). The 0_1^+ , 0_1^- bands were reported in [117, 118], whereas the (2_1^+) , (0_2^+) , (2_2^+) , (0_4^+) , (2_1^-) bands were arranged by Bauhoff *et al.* [119]. The (1_1^+) , (0_3^+) , (1_1^-) bands are arranged in the present work.

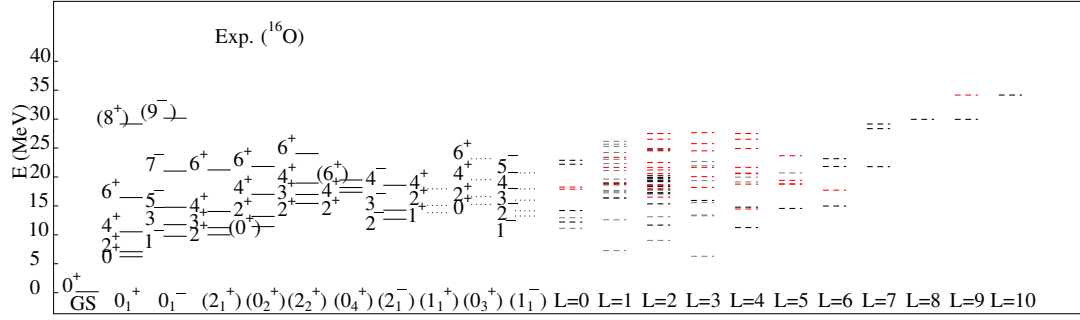


Figure 3.14: The complete experimental spectrum of the ^{16}O nucleus is presented, with rotational bands labeled according to their K^π quantum numbers. The bands represented in solid lines were previously reported [117–119]. The 0_1^- and 0_1^+ bands were presented in experimental work [117, 118], while the (2_1^+) , (0_2^+) , (2_2^+) , (0_4^+) , and (2_1^-) bands were organized by Bauhoff *et al.* [119]. The dotted lines represent the bands arranged in the present work. The states with angular momentum $L = 0$ to $L = 10$ that are not assigned to any rotational band are shown in dashed lines. Black, grey, and red dashed lines denote positive-parity, negative-parity, and uncertain spin-parity states, respectively.

• Model space

The fully Pauli-allowed $U(3)$ model space is presented in Table 3.11.

Table 3.11: Pauli-allowed model space in the $U(3)$ dynamical-symmetry limit for the $^{12}\text{C} + \alpha$ cluster configuration (up to $n_\pi = 10$).

n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu)\mathbf{K}^\pi$	$\mathbf{L}^\pi$
4	[4,4,4]	(0,0)0 ⁺	0 ⁺
5	[6,4,3]	(2,1)1 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻
6	[8,4,2]	(4,2)0 ⁺ (4,2)2 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺ 2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺
	[7,4,3]	(3,1)1 ⁺	1 ⁺ , 2 ⁺ , 3 ⁺ , 4 ⁺
	[6,4,4]	(2,0)0 ⁺	0 ⁺ , 2 ⁺
7	[10,4,1]	(6,3)1 ⁻ (6,3)3 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ , 8 ⁻ , 9 ⁻
	[9,4,2]	(5,2)0 ⁻ (5,2)2 ⁻	1 ⁻ , 3 ⁻ , 5 ⁻ 2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻
	[8,4,3]	(4,1)1 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻
	[7,4,4]	(3,0)0 ⁻	1 ⁻ , 3 ⁻
8	[12,4,0]	(8,4)0 ⁺ (8,4)2 ⁺ (8,4)4 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺ , 6 ⁺ , 8 ⁺ 2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺ , 9 ⁺ , 10 ⁺ 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺ , 9 ⁺ , 10 ⁺ , 11 ⁺ , 12 ⁺
	[11,4,1]	(7,3)1 ⁺ (7,3)3 ⁺	1 ⁺ , 2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺ 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺ , 9 ⁺ , 10 ⁺
	[10,4,2]	(6,2)0 ⁺ (6,2)2 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺ , 6 ⁺ 2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺
	[9,4,3]	(5,1)1 ⁺	1 ⁺ , 2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺
	[8,4,4]	(4,0)0 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺

Table 3.11: (continued)

n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu)\mathbf{K}^\pi$	$\mathbf{L}^\pi$
9	$[13, 4, 0]$	$(9, 4)0^-$	$1^-, 3^-, 5^-, 7^-, 9^-$
		$(9, 4)2^-$	$2^-, 3^-, 4^-, 5^-, 6^-, 7^-, 8^-, 9^-, 10^-, 11^-$
		$(9, 4)4^-$	$4^-, 5^-, 6^-, 7^-, 8^-, 9^-, 10^-, 11^-, 12^-, 13^-$
	$[12, 4, 1]$	$(8, 3)1^-$	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-, 8^-, 9^-$
		$(8, 3)3^-$	$3^-, 4^-, 5^-, 6^-, 7^-, 8^-, 9^-, 10^-, 11^-$
	$[11, 4, 2]$	$(7, 2)0^-$	$1^-, 3^-, 5^-, 7^-$
		$(7, 2)2^-$	$2^-, 3^-, 4^-, 5^-, 6^-, 7^-, 8^-, 9^-$
	$[10, 4, 3]$	$(6, 1)1^-$	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
	$[9, 4, 4]$	$(5, 0)0^-$	$1^-, 3^-, 5^-$
10	$[14, 4, 0]$	$(10, 4)0^+$	$0^+, 2^+, 4^+, 6^+, 8^+, 10^+$
		$(10, 4)2^+$	$2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+, 11^+, 12^+$
		$(10, 4)4^+$	$4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+, 11^+, 12^+, 13^+, 14^+$
	$[13, 4, 1]$	$(9, 3)1^+$	$1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+$
		$(9, 3)3^+$	$3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+, 11^+, 12^+$
	$[12, 4, 2]$	$(8, 2)0^+$	$0^+, 2^+, 4^+, 6^+, 8^+$
		$(8, 2)2^+$	$2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+$
	$[11, 4, 3]$	$(7, 1)1^+$	$1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+$
	$[10, 4, 4]$	$(6, 0)0^+$	$0^+, 2^+, 4^+, 6^+$

• Spectrum Calculation

For the calculation of spectrum the multiconfigurational dynamical symmetry (MUSY) Hamiltonian :

$$\hat{H} = \hbar\omega\hat{n} + a\hat{C}_{SU(3)}^{(2)} + bK^2 + \frac{1}{2\theta}L(L+1), \quad (3.33)$$

is used instead of Eq. 3.30. Both Eqs. 3.30 and 3.33 belong to the $U(3)$

dynamical-symmetry limit and contain some common terms: the harmonic-oscillator term (the linear invariant of $U(3)$), the quadrupole–quadrupole ($\hat{C}_{SU(3)}^{(2)}$), and the angular-momentum. Despite these similarities, there are also notable differences. The energy functional given in Eq. 3.30 contains five parameters, whereas 3.33 has only three. An additional advantage of the Hamiltonian in 3.30 is that it is unaffected by transformations between different configurations [11, 16, 121]. Furthermore, it provides good physical interpretation: the K^2 term removes the degeneracy among states that share the same angular momentum within a single $SU(3)$ irrep. The parameter θ is the moment of inertia (in units $\hbar^2/\text{MeV}$) of a cylindrically symmetric ellipsoid which can be evaluated by using the $U(3)$ quantum numbers [81].

In the realistic calculation, the parameters of Eq. 3.33 are obtained by fitting the theoretical spectrum to the experimental data. For this purpose, the model space is assigned to the corresponding experimental bands.

In the qualitative shell model description [122], the lowest excited state with positive parity (0^+ at 6.05 MeV) originates from the leading 4p-4h $SU(3)$ irreducible representation. The 0_1^+ positive-parity band (shown in Fig. 3.13) is built on this state. Here, it is associated with the leading representation of the $4\hbar\omega$ configuration, characterized by quantum numbers $n_\pi(\lambda, \mu)K^\pi = 8(8, 4)0^+$.

According to Katsuma’s calculations [117], the relative motion between the α particle and ^{12}C system for this band is characterized by a total harmonic-oscillator quantum number $N = 8$, while the 0_1^- band is assigned $N = 9$. Here, the 0_1^+ and 0_1^- bands are assigned to the leading $SU(3)$ representations of the $4\hbar\omega$ and $5\hbar\omega$ configurations, respectively, corresponding to $n_\pi = 8$ and $n_\pi = 9$.

The rotational bands arranged by Bauhoff *et al.* [119] are associated with the lowest-lying, most deformed configurations consistent with K^π and spin-parity. The bands defined in this work follow the same assignment principle.

The 0_4^+ band is assigned to the configuration $10(8, 2)0^+$. Although representations at lower lying configurations could also be considered, but they contain both even- and odd-spin members. Since the 0_4^+ band consists of only even-spin states, the $10(8, 2)0^+$ representation is chosen from the available model space.

Furthermore, the 2_1^- and 1_1^- bands are described by the $7(5, 2)2^-$ and $7(6, 3)1^-$ configurations, respectively. The less deformed representation $7(5, 2)2^-$ is associated with the lower 2_1^- band, as its minimum angular momentum ($L = 2$) is consistent with the observed experimental band-head. Where as the more deformed representation $7(6, 3)1^-$ is associated with the higher-lying 1_1^- band.

The function minimized to obtain the Hamiltonian parameters is:

$$F = \sum_i w(i) \frac{(E_i^{exp} - E_i^{Th.})^2}{(E_i^{exp.})^2}. \quad (3.34)$$

In the fitting procedure, states of two experimentally established bands [117, 118] with well-defined spin-parity were given a unit weight, while uncertain spin-parity states were assigned a weight of 0.5. For both the bands arranged by Bauhoff *et al.* and those arranged in the present work, states with well-determined spin-parity were given an equal weight of 0.2, while uncertain spin-parity states were assigned a considerably lower weight of 0.01.

The calculation was carried out in three steps:

- i) First, only the two experimentally established bands (est.) [117, 118] were considered.
- ii) Next, the bands arranged by Bauhoff *et al.* (est.+Bauhoff) [119] were included.
- iii) Lastly, the bands arranged in this work were incorporated (est.+Bauhoff+present).

The parameters obtained from the calculation are listed in Table 3.12. A quantitative comparison between the model and experimental spectrum is presented in Figure 3.15 and 3.16.

Table 3.12: The Hamiltonian parameters (Equation (3.33)) $\hbar\omega$, a , and b , are given in MeV. The quantity F represents the minimization function used to determine these parameters, while $N_{exp.}$ is the number of experimental states considered in the fit. See the explanation in the text for the meaning of est., est.+Bauhoff, and est.+Bauhoff+present.

	$\hbar\omega$	a	b	$\frac{F}{N_{exp.}}$
est.	5.27635	-0.10213	0	0.00378
est.+Bauhoff	3.75356	-0.05886	1.04696	0.00409
est.+Bauhoff+present	4.08735	-0.06776	1.18675	0.00783

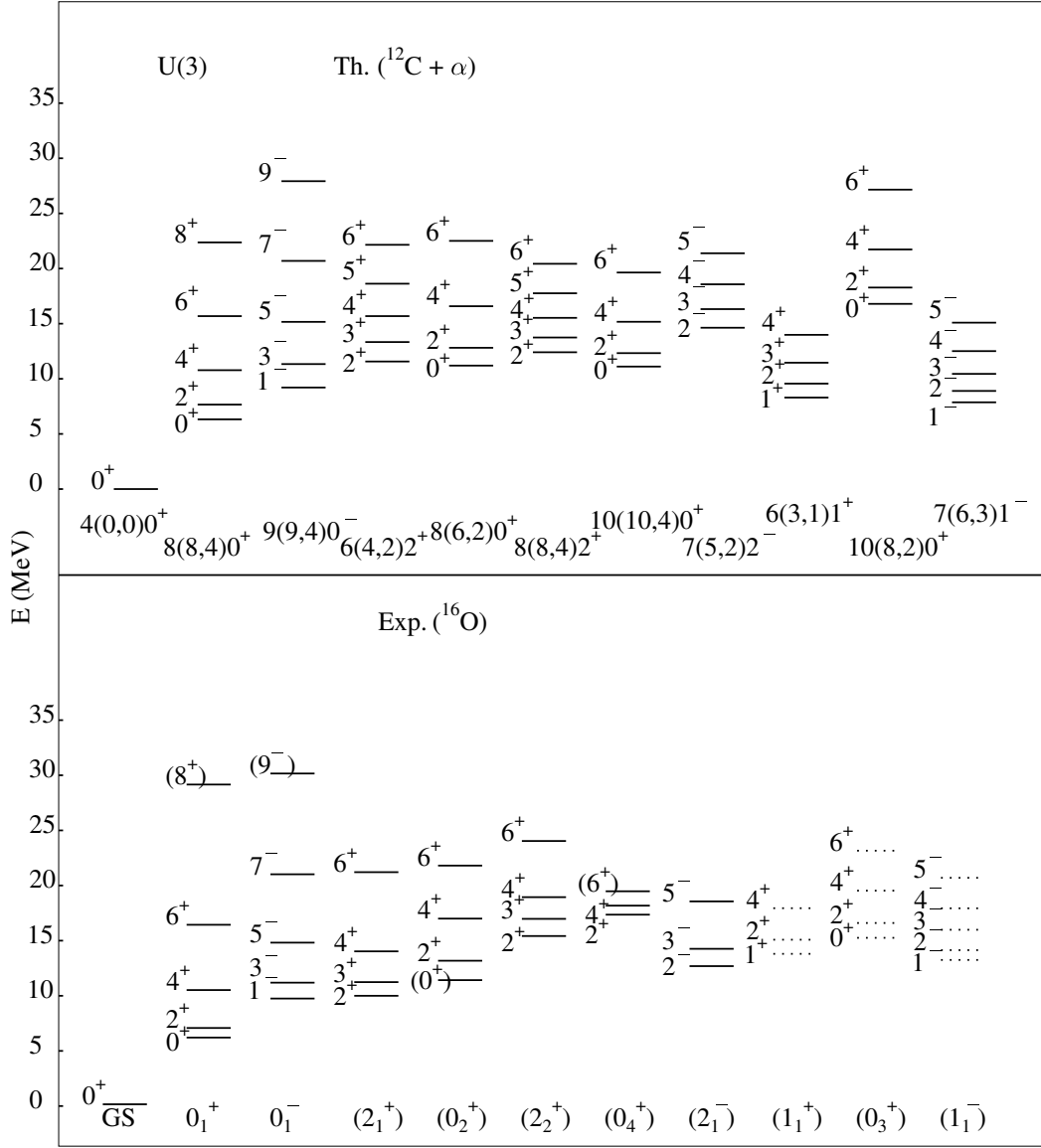


Figure 3.15: The U(3) dynamically-symmetry spectrum of the $^{12}\text{C} + \alpha$ system (top) is compared with the experimental spectrum of ^{16}O (bottom). Bands taken from [117–119] are drawn with solid lines, whereas those arranged in the present work appear as dotted lines. Experimental bands are labeled by K^π , whereas the model bands are labeled as $n_\pi(\lambda, \mu)K^\pi$ quantum numbers. The figure corresponds to the fit in which the experimentally established bands [117, 118], the bands proposed by Bauhoff *et al.* [119], and the arranged bands of the present work were all included.

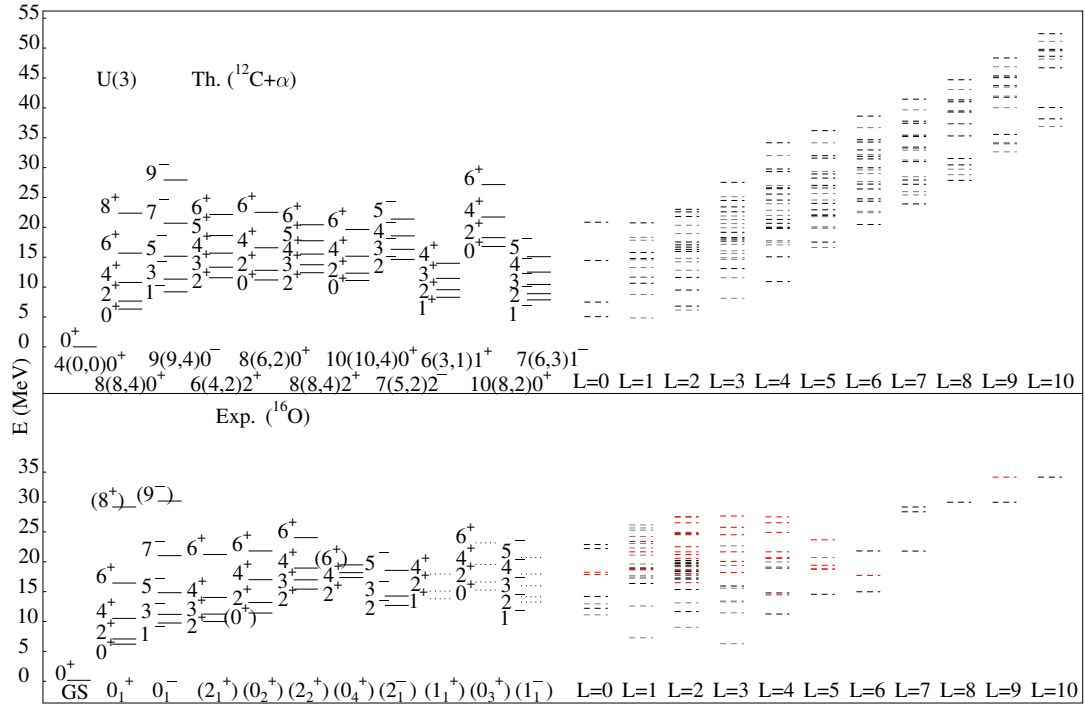


Figure 3.16: Comparison of the $U(3)$ dynamical-symmetric spectrum of the $^{12}\text{C}+\alpha$ cluster system (top) (calculated using the Hamiltonian in Eq. (3.33)) with the experimental spectrum (bottom) of ^{16}O . The bands reported in [117–119] are indicated by solid lines, while those proposed in the present work are shown in dotted lines. Experimental bands are labeled by quantum numbers K^π , whereas the theoretical bands are labeled as $n_\pi(\lambda, \mu)K^\pi$. The figure presents results obtained by including, in the fitting procedure, the experimentally established bands [117, 118], the bands arranged by Bauhoff *et al.* [119], and those proposed in this work. Dashed lines represent states with angular momentum $L = 0$ to $L = 10$ that are not assigned to any rotational band; black, grey, and red dashed lines denote positive-parity, negative-parity, and uncertain spin-parity states, respectively. The respective theoretical states (top) are calculated within a model space up to $n_\pi = 10$, using fixed parameters of model Hamiltonian. In this case, black and grey dashed lines indicate positive- and negative-parity states, respectively.

3.3.5 Summary of the phases

I carried out a systematic investigation of the spectra of the phases clustering phases in light nuclei. In the case of binary cluster configurations, three phases defined by the dynamical symmetry limits: to shell-like, rigid-molecule-like, and weakly-coupled phases, respectively. These phases are associated to the $U(3)$, $O(4)$, and $O(3)$ symmetries, respectively. Therefore, the phase structure can be described within a two-dimensional phase diagram represented by a triangle.

To illustrate the characteristic spectra of the three clustering phases, I applied the semimicroscopic algebraic cluster model (SACM). The spectra show pronounced differences. A significant difference arises from the treatment of the Pauli exclusion principle, which is exactly fulfilled in the $U(3)$ limit, only partially included in $O(3)$ limit, and for $O(4)$ basis all the states are partially Pauli allowed and partially forbidden.

Since the shell-like ($U(3)$ limit) provides a fully microscopic and physically consistent description when combined with a $U(3)$ dynamically symmetric interaction. I used the $^{12}\text{C}+\alpha$ system as an illustrative example. The resulting $U(3)$ dynamical spectrum of $^{12}\text{C}+\alpha$ was compared with the experimental spectrum of ^{16}O . This comparison shows that the band structure of ^{16}O , together with the ground state, can be well reproduced within the $U(3)$ dynamically symmetric model. In addition, several experimentally observed states that do not belong to any band seem to be compatible with possible $^{12}\text{C}+\alpha$ cluster configurations. However, further experimental investigations are needed for a definitive confirmation.

Chapter 4

Summary and conclusion

In this thesis, the structure of light nuclei has been investigated with particular interest on clusterization and its underlying symmetries. Since nuclear systems are complex many-body quantum systems, simplified but physically meaningful approaches are essential. In this context, algebraic models provide a powerful tool, offering both a clear mathematical formulation and an intuitive physical interpretation.

The present work focused on the application of the multiconfigurational dynamical symmetry (MUSY) and the semimicroscopic algebraic cluster model (SACM). These models were developed and applied to many nuclear systems by my supervisor and his collaborators.

MUSY is able to connect different configurations due to its composite nature. A major objective of this thesis was to apply MUSY to investigate the nature of clusterization in light nuclei and to reveal how different configurations contribute to the observed spectra.

I used the multiconfigurational dynamical symmetry (MUSY) to provide a unified description of the ^{18}O spectrum [P1]. I used a simple dynamically symmetric Hamiltonian that contains only harmonic oscillator, quadrupole, and angular momentum interactions. It is invariant under transformations between the different configurations. As a result, the shell, $^{14}\text{C}+^4\text{He}$, and $^{12}\text{C}+2n+^4\text{He}$ cluster configurations contribute to every state in the calculated spectrum. These three configurations have been proposed previously for the interpretation of different parts of the spectrum. I considered the experimentally established band

structure and, in addition, constructed candidate bands using compilation data and alpha scattering resonances to determine the Hamiltonian parameters. Then I gave a parameter-free prediction for the distribution of α -cluster states with specific spin-parities within a certain energy window and compared them with alpha scattering resonances. I found that the states of the three lowest-lying bands have identical wave functions for different configurations due to antisymmetrization.

In addition, a symmetry-governed method, called Stability and self-Consistency of the SU(3) Symmetry (SCS) was applied to determine the stable shape isomers. I predicted reaction channels in which the shape isomer states can be populated or through which they can decay. With a simple dynamically symmetric Hamiltonian, the $^{14}\text{C}+^4\text{He}$ cluster spectrum reproduces the gross features of the experimental energy spectrum of ^{18}O .

I have applied the multiconfigurational dynamical symmetry to provide a unified description of the spectrum of the ^{40}Ca nucleus [P2]. In the low-energy region, it shows a well-established band structure, and its superdeformed band has been confirmed experimentally. Furthermore, there are indications of hyperdeformation from high-resolution heavy-ion resonance experiments. Using the experimental compilations, I have arranged additional candidate bands. The results show that MUSY provides a unified description with good agreement.

One of the interesting findings is that the arranged bands were predicted based on the description of well-known bands. The agreement between the experimental and calculated energy levels was nearly as good as that obtained from the fitting procedure, with only about a 9% difference.

Atomic nuclei exhibit both thermal and non-thermal (cold) phases and their transitions. In cold phases, temperature is not a relevant variable, yet many thermodynamic phase-transition concepts can be treated in close analogy. These cold phases can be investigated within algebraic nuclear-structure models, where the cold phases are defined by the dynamical-symmetry limits of the model.

I systematically investigated the spectra of the phases of clustering in light nuclei [P3]. For binary clusterization, there are three phases correspond to the dynamical symmetry limits: the shell-like (or soft-vibrator), rigid-molecule-like

(or rotator), and weakly-coupled limits. These correspond to the $U(3)$, $O(4)$, and $O(3)$ symmetries, respectively. The phases of clustering can be represented within a two-dimensional phase diagram, which has a triangular shape.

I applied the semimicroscopic algebraic cluster model (SACM) to demonstrate the spectra of the three phases. The spectra exhibit notable differences. The most significant differences arise from the role of the Pauli exclusion principle. By using the (strongly coupled) $U(3)$ basis, the effect of antisymmetrization can be treated exactly. This means that all Pauli forbidden states can be eliminated, and all allowed states are retained. In the $O(3)$ limit it can be incorporated only partially through truncation of the relative-motion quantum numbers, whereas in the $O(4)$ limit the basis states are mixtures of Pauli allowed and Pauli forbidden components.

My analysis shows that when clustering is described within a dynamically symmetric (pure-phase) framework, the shell-like ($U(3)$) configuration emerges as the most appropriate choice, because of the crucial role of the Pauli principle. When symmetry-breaking effects are considered, the exact location within the phase diagram can be identified by performing the calculation in the $U(3)$ basis, while employing an interaction that contains contributions from the $U(3)$, $O(4)$, and $O(3)$ dynamical symmetries. The relative strengths of these components (obtained through the best fit to experimental data) serve as the control parameters that determine the location in the phase diagram.

As the shell-like phase (i.e. $U(3)$ limit) provides a fully microscopic and physically meaningful description when combined with a $U(3)$ dynamically symmetric interaction, I considered the $^{12}\text{C}+\alpha$ system for illustration [P3]. I compared the $U(3)$ dynamical spectrum of $^{12}\text{C}+\alpha$ with the experimental energy spectrum of ^{16}O . The comparison showed that the band structure of ^{16}O , together with the ground-state, can be reproduced with good accuracy within the $U(3)$ dynamically symmetric $^{12}\text{C}+\alpha$ model. Moreover, several experimentally observed states without definite band assignment were found to be consistent with possible $^{12}\text{C}+\alpha$ cluster configurations, although additional experimental investigations are required for a definitive identification.

Chapter 5

Acknowledgment

I would like to sincerely thank my PhD supervisor, Prof. József Cseh, for his constant support, encouragement, and guidance throughout my doctoral studies.

I would also like to thank Dr. Gábor Riczu for always being available for discussions whenever I faced problems with calculations during my research work.

I extend my heartfelt thanks to Dr. Judit Darai for her assistance during the preparation of my thesis.

I would also like to express my sincere thanks to Dr. Géza Lévai for his valuable explanations and for patiently answering questions related to my research work.

I am thankful to my former office-mate, Dr. Gábor Drótos, for his friendship and for serving as a local guide, helping me learn and appreciate many aspects of the local culture and daily life.

I would like to acknowledge my friends and colleagues Dr. Malek, Dr. Nanda, Dr. Ram, Dr. Wafikul, and Dr. István for the enjoyable moments we shared and for their friendship, support, and positive wishes throughout my stay.

I would like to thank my M.Sc. friends—Guru, Debu, Anitya, Prateek, Pratyasa, and Subham—for their friendship, encouragement, and the memorable experiences we shared over the years.

My special thanks go to Priya for her kindness and care, especially for always checking on my health and well-being.

I would also like to acknowledge support provided by the Stipendium Hungaricum Scholarship Programme of the Hungarian Government, University of Debrecen, and HUN-REN ATOMKI during my doctoral studies.

Last but not least, I express my sincere gratitude to all my teachers and well-wishers for their constant support, encouragement, and kind wishes.

Chapter 6

Publications

- P1. G. Riczu, **C. S. Panda**, G. Lévai, and J. Cseh, The spectrum of ^{18}O in terms of the multiconfigurational dynamical symmetry, to be submitted to PRC, arXiv:2606.02270.
- P2. **C. S. Panda**, J. Darai, G. Riczu, and J. Cseh, Multiconfigurational Dynamical Symmetry Description of ^{40}Ca Spectrum, *Symmetry* (2026), 18, 1193.
- P3. **C. S. Panda**, G. Riczu, J. Cseh, Spectra of Phases of Clusterized Nuclei, *Symmetry* (2026), 18, 923.

Conference proceedings:

- CP1. **C. S. Panda**, G. Riczu, J. Darai, and J. Cseh, A unified description of the spectrum of ^{18}O and ^{40}Ca , XXVIII. Spring Wind Conference (2025), 12.
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Appendix A

Computer Programs

Here, I present the programs that I wrote for my research work. All of these were written in Fortran. The program task, example of input and output are explained for each program.

A.1 O(4) and O(3) model space

The construction procedure for the O(4) and O(3) model space is explained in section 2.3 of Chapter 2.

A.1.1 O(4) model space

Program description:

The program "O(4)_modelspace.f" produces model spaces in the O(4) dynamical symmetry limit of the semi-microscopic algebraic cluster model (SACM).

Structure of the input file:

1. The first row specifies the total number of bosons.
2. The second row gives the U(3) representation of the clusters, i.e., $[n_1^C, n_2^C, n_3^C]$.
3. The third row takes the value 1 or 0. It is set to 1 if the U(3) cluster corresponds to positive-parity angular momenta, and 0 if it corresponds to negative-parity angular momenta. In general, the ground-state representation of the clusters is considered, for which the parity is positive for all angular momenta. However, if excited cluster configurations are used, the parity of the angular momenta may vary.

Example: The following example corresponds to the $^{12}\text{C} + \alpha$ cluster system for $N = 8$ bosons.

$$\begin{array}{c} 8 \\ 4 \quad 4 \quad 0 \\ 1 \end{array}$$

Structure of output file:

The output is presented in the following order:

$$(\text{lamC}, \text{muC}) \quad \omega \quad K_C \quad L_C \quad L_R \quad L^{\wedge}p$$

where:

- ω : quantum number of $O(4)$,
- $(\text{lamC}, \text{muC})$: $SU(3)$ representation of the clusters,
- K_C, L_C : projection and angular momentum quantum numbers corresponding to the $SU(3)$ representation of the clusters,
- L_R : angular momentum derived from ω ,
- $L^{\wedge}p$: total angular momentum (L) and parity (p), obtained from the coupling of L_C and L_R .

Example: The following output corresponds to the sample input.

$(\text{lamC}, \text{muC})$	ω	KC	LC	LR	$L^{\wedge}p$
(0,4)	0	0	0	0	0^+
(0,4)	0	0	2	0	2^+
(0,4)	0	0	4	0	4^+
(0,4)	2	0	0	0	0^+
(0,4)	2	0	0	1	1^-
(0,4)	2	0	0	2	2^+
(0,4)	2	0	2	0	2^+
(0,4)	2	0	2	1	1^-
(0,4)	2	0	2	1	2^-
(0,4)	2	0	2	1	3^-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

A.1.2 O(3) model space

Program description:

The program "O(3)_modelspec.f" produce model spaces for O(3) dynamical symmetry limit of the semi-microscopic algebraic cluster model (SACM).

Structure of the input file:

1. The first row specifies the total number of bosons.
2. The second row gives the minimum number of oscillator quanta in the relative motion, as determined by the Wildermuth condition.
3. The third row represents the U(3) structure of the clusters, i.e., $[n_1^C, n_2^C, n_3^C]$.
4. The fourth row takes the value 1 or 0. It is set to 1 if the U(3) cluster corresponds to positive-parity angular momenta, and 0 if it corresponds to negative-parity angular momenta. In most cases, the ground-state representation of the clusters is considered, for which the parity is positive for all angular momenta. However, if excited cluster configurations are used, the parity of the angular momenta may vary.

Example: The example is present for $^{12}\text{C}+\alpha$ clusters for N=8 bosons.

```
8
4
4 4 0
1
```

Structure of output file:

The output is presented in the following order:

(lamC,muC) npi KC LC LR L^p

where:

- npi: number of oscillator quanta in the relative motion,
- (lamC,muC): SU(3) representation of the clusters,
- KC, LC: projection and angular momentum quantum numbers corresponding to the SU(3) representation of the cluster,

- LR: angular momentum derived from ω ,
- $L^{\wedge}p$: total angular momentum (L) and parity (p), obtained from the coupling of L_C and L_R .

Example: The following output corresponds to the sample input.

(lamC,muC)	npi	K_C	L_C	L_R	$L^{\wedge}p$
(0,4)	4	0	0	0	0^+
(0,4)	4	0	0	2	2^+
(0,4)	4	0	0	4	4^+
(0,4)	4	0	2	0	2^+
(0,4)	4	0	2	2	0^+
(0,4)	4	0	2	2	1^+
(0,4)	4	0	2	2	2^+
(0,4)	4	0	2	2	3^+
(0,4)	4	0	2	2	4^+
(0,4)	4	0	2	4	2^+
(0,4)	4	0	2	4	3^+
(0,4)	4	0	2	4	4^+
(0,4)	4	0	2	4	5^+
(0,4)	4	0	2	4	6^+
(0,4)	4	0	4	0	4^+
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$