

D-optimal designs for complex Ornstein-Uhlenbeck processes

Sándor Baran^{a,*}, Csilla Szák-Kocsis^a, Milan Stehlík^b

^a*Faculty of Informatics, University of Debrecen, Kassai street 26, H-4028 Debrecen, Hungary*

^b*Department of Applied Statistics and Linz Institute of Technology, JKU, Altenberger Straße 69, A-4040, Austria
Institute of Statistics, University of Valparaíso, Gran Bretaña 1111, Valparaíso, Chile*

Abstract

Complex Ornstein-Uhlenbeck (OU) processes have various applications in statistical modelling. They play role e.g. in the description of the motion of a charged test particle in a constant magnetic field or in the study of rotating waves in time-dependent reaction diffusion systems, whereas Kolmogorov used such a process to model the so-called Chandler wobble, small deviation in the Earth's axis of rotation. In these applications parameter estimation and model fitting is based on discrete observations of the underlying stochastic process, however, the accuracy of the estimation strongly depend on the observation points.

This paper studies the properties of D-optimal designs for estimating the parameters of a complex OU process with a trend. In special situations we show that in contrast with the case of the classical real OU process, a D-optimal design exists not only for the trend parameter, but also for joint estimation of the covariance parameters, moreover, these optimal designs are equidistant.

Keywords: Chandler wobble, complex Ornstein-Uhlenbeck process, D-optimality, optimal design, parameter estimation

1. Introduction

Random processes have various applications in statistical modelling in different areas of science such as physics, chemistry, biology or finance, where one usually cannot observe continuous trajectories. In these situations parameter estimation and model fitting is based on discrete observations of the underlying stochastic process, however, the accuracy of the results strongly depend on the observation points. The theory of optimal experimental designs, dating back to the late 50s of the twentieth century (see e.g. Hoel, 1958; Kiefer, 1959), deals with finding design sets $\xi = \{t_1, t_2, \dots, t_n\}$ of distinct time points (or locations in space) where the process under study is observed, which are optimal according to some previously specified

*Correspondence address: Phone: +36 52 512900/75212, Fax: +36 52 512996
Email address: baran.sandor@inf.unideb.hu (Sándor Baran)

criterion (Müller, 2007). In parameter estimation problems the most popular criteria are based on the Fisher information matrix (FIM) of the observations. D-, E- and T-optimal designs maximize the determinant, the smallest eigenvalue and the trace of the FIM, respectively, an A-optimal design minimizes the trace of the inverse of the FIM (for an overview see Pukelsheim, 1993), whereas K-optimality refers to the minimization of the condition number of the FIM (see e.g. Ye and Zhou, 2013; Baran, 2017). In the last decades information based criteria have intensively been studied both in the uncorrelated setup (see e.g. Silvey, 1980) and in the more difficult correlated situation (Dette *et al.*, 2015, 2016).

In the present paper we derive D-optimal designs for parameter estimation of complex (or vector) Ornstein-Uhlenbeck (OU) processes with trend (see e.g. Arató, 1982), defined in detail in Section 2. A complex OU process describes e.g. the motion of a charged test particle in a constant magnetic field (Balescu, 1997), it is used in the description of the rotation of a planar polymer (Vakeroudis *et al.*, 2011) or in the study of rotating waves in time-dependent reaction diffusion systems (Beyn and Lorenz, 2008; Otten, 2015), and it also has several applications in financial mathematics (see e.g. Barndorff-Nielsen and Shephard, 2001). Further, Kolmogorov proposed to model the so-called Chandler wobble, small deviation in the Earth's axis of rotation (Lambeck, 1980), by the model

$$Z(t) = Z_1(t) + iZ_2(t) = me^{i2\pi t} + Y(t), \quad t > 0, \quad (1.1)$$

where $Z_1(t)$ and $Z_2(t)$ are the coordinates of the deviation of the instantaneous pole from the North Pole and $Y(t)$ is a complex OU process (Arató *et al.*, 1962). We remark that most of our results correspond to the special case of a constant trend, however, even this simple situation gives a nice insight into the behavior of D-optimal designs for complex OU processes, highlighting the differences between the real and complex models.

Note that the properties of D-optimal designs for classical one-dimensional OU processes have already investigated by Kiseřák and Stehlík (2008) and later by Zagoraiou and Baldi Antognini (2009), where the authors proved that there is no D-optimal design for estimating the covariance parameter, whereas the D-optimal design for trend estimation is equidistant and larger distances resulting in more information. Later these results were generalized for OU sheets under various sampling schemes (Baran and Stehlík, 2015; Baran *et al.*, 2013, 2015).

The paper is organized as follows. In Section 2 we introduce the model to be studied, Section 3 contains our results on D-optimal designs, whereas in Section 4 some applications are presented. The paper ends with the concluding remarks of Section 4. To maintain the continuity of the explanation, the proofs are given in the Appendix.

2. Complex Ornstein-Uhlenbeck process with a trend

Consider the complex stochastic process $Z(t) = Z_1(t) + iZ_2(t)$, defined as

$$Z(t) = mf(t) + Y(t), \quad t \geq 0, \quad (2.1)$$

with design points taken from the non-negative half-line \mathbb{R}_+ , where $m = m_1 + im_2$, $m_1, m_2 \in \mathbb{R}$, $f(t) = f_1(t) + if_2(t)$ with $f_1(t), f_2(t) : \mathbb{R}_+ \mapsto \mathbb{R}$ and $Y(t) = Y_1(t) + iY_2(t)$, $t \geq 0$, is a complex valued stationary OU process. The process $Y(t)$ can be defined by the stochastic differential equation (SDE)

$$dY(t) = -\gamma Y(t)dt + \sigma dW(t), \quad Y(0) = \xi, \quad (2.2)$$

where $\gamma = \lambda - i\omega$ with $\lambda > 0$, $\omega \in \mathbb{R}$, $\sigma > 0$, $W(t) = W_1(t) + iW_2(t)$, $t \geq 0$, is a standard complex Brownian motion, that is $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions, and $\xi = \xi_1 + i\xi_2$, where ξ_1 and ξ_2 are centered normal random variables that are chosen according to stationarity (Arató, 1982).

Instead of the complex process $Y(t)$ defined by (2.2) one can consider the two-dimensional real valued stationary OU process $(Y_1(t), Y_2(t))^\top$ defined by the SDE

$$\begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = A \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} dt + \sigma \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}, \quad \text{where } A := \begin{bmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{bmatrix} \quad (2.3)$$

We remark that in physics (2.3) is called A-Langevin equation, see e.g. (Balescu, 1997). If $(Y_1(t), Y_2(t))^\top$ satisfies (2.3) then $Y_1(t) + iY_2(t)$ is a complex OU process which solves (2.2), and conversely, the real and imaginary parts of a complex OU process form a two-dimensional real OU process satisfying (2.3). Obviously, $EY_1(t) = EY_2(t) = 0$, whereas the covariance matrix function of the process $(Y_1(t), Y_2(t))^\top$ is given by

$$\mathcal{R}(\tau) := E \begin{bmatrix} Y_1(t+\tau) \\ Y_2(t+\tau) \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}^\top = \frac{\sigma^2}{2\lambda} e^{A\tau} = \frac{\sigma^2}{2\lambda} e^{-\lambda\tau} \begin{bmatrix} \cos(\omega\tau) & -\sin(\omega\tau) \\ \sin(\omega\tau) & \cos(\omega\tau) \end{bmatrix}, \quad \tau \geq 0. \quad (2.4)$$

This results in a complex covariance function of the complex OU process $Y(t)$ defined by (2.2) of the form

$$\mathcal{C}(\tau) := EY(t+\tau)\overline{Y(t)} = \frac{\sigma^2}{\lambda} e^{-\lambda\tau} (\cos(\omega\tau) + i\sin(\omega\tau)), \quad \tau \geq 0,$$

behaving like a damped oscillation with frequency ω .

In the present study the damping parameter λ , frequency ω and standard deviation σ are assumed to be known. However, a valuable direction for future research will be the investigation of models where these parameters should also be estimated. Note that the estimation of σ can easily be done on the basis of a single realization of the complex process, see e.g. Arató (1982, Chapter 4). Now, without loss of generality,

one can set the variances of $Y_1(t)$ and $Y_2(t)$ to be equal to one, that is $\sigma^2/(2\lambda) = 1$, which reduces $\mathcal{R}(\tau)$ to a correlation matrix function. Further results on the maximum-likelihood estimation of the covariance parameters can be found e.g. in Arató *et al.* (1999).

3. D-optimal designs

Suppose the complex process $Z(t)$ is observed in design points $0 \leq t_1 < t_2 < \dots < t_n$ resulting in the $2n$ -dimensional real vector $(Z_1(t_1), Z_2(t_1), Z_1(t_2), Z_2(t_2), \dots, Z_1(t_n), Z_2(t_n))^\top$, where

$$Z_1(t) = m_1 f_1(t) - m_2 f_2(t) + Y_1(t), \quad Z_2(t) = m_2 f_1(t) + m_1 f_2(t) + Y_2(t). \quad (3.1)$$

As it has mentioned in the Introduction, a D-optimal design maximizes the determinant of the FIM on the unknown parameters corresponding to the observations. Here we consider optimal designs for estimating the trend parameter m and the damping parameter λ and frequency ω , as well.

3.1. Estimation of the trend parameter

According to the results of e.g. Xia et al. (2006) or Pázman (2007) the FIM on parameter vector $(m_1, m_2)^\top$ based on observations $\{(Z_1(t_j), Z_2(t_j)), j = 1, 2, \dots, n\}$ equals

$$\mathcal{I}_{m_1, m_2}(n) = H(n)C(n)^{-1}H(n)^\top,$$

where

$$H(n) := \begin{bmatrix} f_1(t_1) & f_2(t_1) & f_1(t_2) & f_2(t_2) & \cdots & f_1(t_n) & f_2(t_n) \\ -f_2(t_1) & f_1(t_1) & -f_2(t_2) & f_1(t_2) & \cdots & -f_2(t_n) & f_1(t_n) \end{bmatrix},$$

and $C(n)$ is the $2n \times 2n$ covariance matrix of the observations.

Lemma 3.1 *The FIM on trend parameters $(m_1, m_2)^\top$ of the two-dimensional analogue of the complex model (2.1) based on the real observation vector $\{(Z_1(t_j), Z_2(t_j)), j = 1, 2, \dots, n\}$ equals $\mathcal{I}_{m_1, m_2}(n) = Q(n)\mathbb{I}_2$, where \mathbb{I}_k , $k \in \mathbb{N}$, denotes the k -dimensional unit matrix and*

$$Q(n) := f_1^2(t_n) + f_2^2(t_n) + \sum_{j=1}^{n-1} \frac{1}{1 - e^{-2\lambda d_j}} \left(f_1^2(t_j) + f_2^2(t_j) + e^{-2\lambda d_j} (f_1^2(t_{j+1}) + f_2^2(t_{j+1})) \right. \\ \left. + 2e^{-\lambda d_j} ((f_1(t_j)f_2(t_{j+1}) - f_2(t_j)f_1(t_{j+1}))\sin(\omega d_j) - (f_1(t_j)f_1(t_{j+1}) + f_2(t_j)f_2(t_{j+1}))\cos(\omega d_j)) \right), \quad (3.2)$$

with $d_j := t_{j+1} - t_j$, $j = 1, 2, \dots, n-1$.

Consider now the special case of a constant trend, that is the model

$$Z(t) = m + Y(t). \quad (3.3)$$

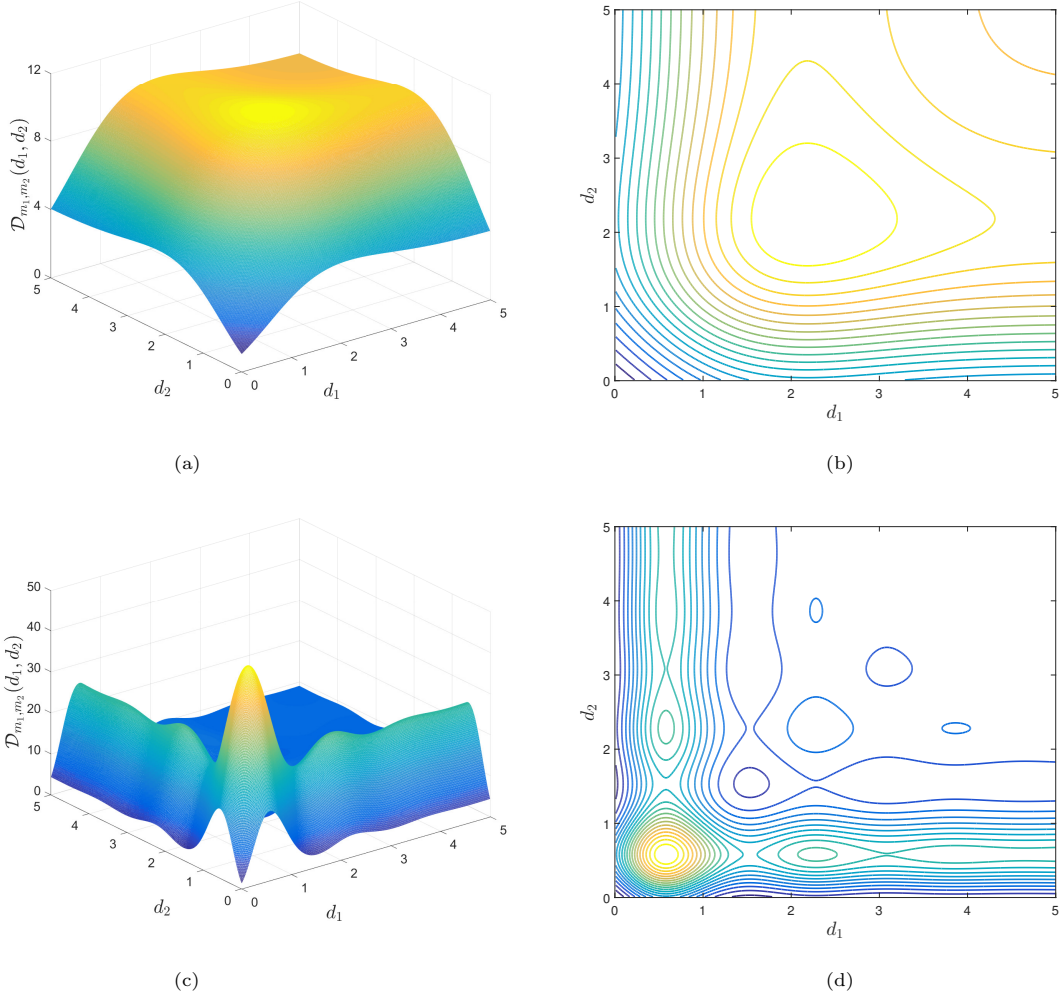


Figure 1: Bivariate objective function \mathcal{D}_{m_1, m_2} for a three-point design $\{0, d_1, d_1 + d_2\}$ for (a) $\lambda = 1$, $\omega = 1$ and (c) $\lambda = 1$, $\omega = 4$, together with the corresponding contour plots (b) and (d), respectively.

In this situation $f_1(t) \equiv 1$ and $f_2(t) \equiv 0$, so

$$Z_1(t) = m_1 + Y_1(t), \quad Z_2(t) = m_2 + Y_2(t),$$

and the expression in (3.2) reduces to

$$Q(n) = 1 + \sum_{\ell=1}^{n-1} g(d_\ell), \quad \text{where} \quad g(x) := \frac{1 - 2e^{-\lambda x} \cos(\omega x) + e^{-2\lambda x}}{1 - e^{-2\lambda x}}, \quad x > 0, \quad \text{and} \quad g(0) := 0. \quad (3.4)$$

Hence, in order to obtain the D-optimal design, one has to find the maximum in $\mathbf{d} = (d_1, d_2, \dots, d_{n-1})$ of

$$\mathcal{D}_{m_1, m_2}(\mathbf{d}) := \det(\mathcal{I}_{m_1, m_2}(n)) = \left(1 + \sum_{\ell=1}^{n-1} g(d_\ell)\right)^2. \quad (3.5)$$

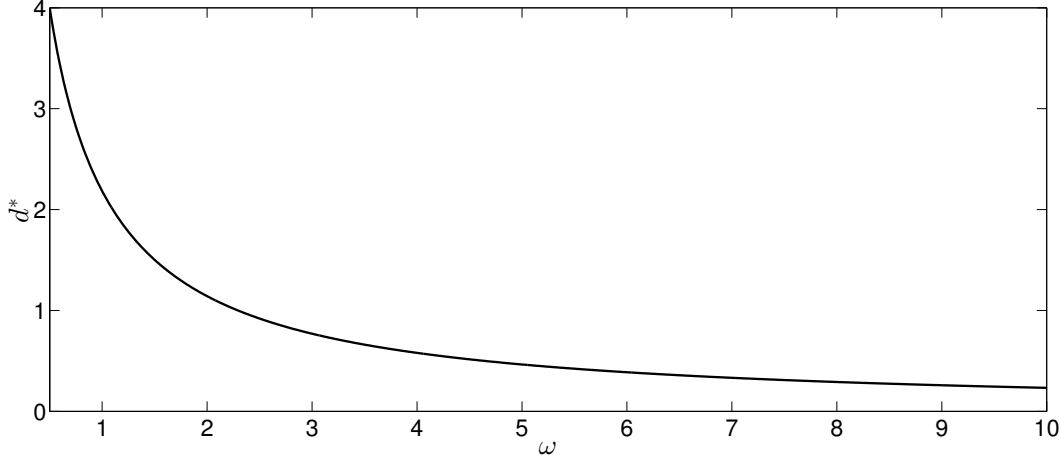


Figure 2: Location d^* of the global maximum of $g(x)$ as a function of the frequency ω for $\lambda = 1$.

Theorem 3.2 Consider the complex model (3.3) with $\omega \neq 0$ observed in design points $0 \leq t_1 < t_2 < \dots < t_n$. The D-optimal design for estimating the trend parameter is equidistant with $d_1 = d_2 = \dots = d_{n-1} = d^*$, where d^* is the (existing) global maximum point of $g(x)$.

Observe that for $\omega = 0$ we have

$$Q(n) = 1 + \sum_{\ell=1}^{n-1} \frac{1 - e^{-\lambda d_\ell}}{1 + e^{-\lambda d_\ell}},$$

which is exactly the Fisher information on the constant trend parameter of a shifted real valued stationary OU process with covariance parameter λ (Kiseřák and Stehlík, 2008; Zagoraiou and Baldi Antognini, 2009). In this case the D-optimal design on trend is also equidistant, however, with the increase of this equal distance the information is also increasing. According to the statement of Theorem 3.2 this is not the case for the complex OU process as there exists an optimal distance which provides the highest information.

Example 3.3 As an illustration consider a three-point design $\{0, d_1, d_1 + d_2\}$. Figures 1a and 1c show the bivariate objective function \mathcal{D}_{m_1, m_2} for $\lambda = 1$, $\omega = 1$ and $\lambda = 1$, $\omega = 4$ together with the corresponding contour plots (Figures 1b and 1d, respectively).

To get a better insight on the behavior of the optimal design, consider the function $g(x)$ defined by (3.4). Figure 2 shows the location d^* of the global maximum of $g(x)$ as a function of the frequency ω for $\lambda = 1$. The general case can obviously be obtained by rescaling both axes by the value of the damping parameter λ .

3.2. Estimation of the covariance parameters

Consider now the problem of estimating the damping parameter λ and frequency ω . Recalling again the results of Xia et al. (2006) and Pázman (2007), the FIM on these parameters based on observations $\{(Z_1(t_j), Z_2(t_j)), j = 1, 2, \dots, n\}$ has the form

$$\mathcal{I}_{\lambda, \omega}(n) = \begin{bmatrix} \mathcal{I}_\lambda(n) & I_{\lambda, \omega}(n) \\ I_{\lambda, \omega}(n) & \mathcal{I}_\omega(n) \end{bmatrix}, \quad (3.6)$$

where

$$\begin{aligned} \mathcal{I}_\lambda(n) &:= \frac{1}{2} \operatorname{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} \right\}, \\ \mathcal{I}_\omega(n) &:= \frac{1}{2} \operatorname{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \omega} C^{-1}(n) \frac{\partial C(n)}{\partial \omega} \right\}, \\ I_{\lambda, \omega}(n) &:= \frac{1}{2} \operatorname{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} C^{-1}(n) \frac{\partial C(n)}{\partial \omega} \right\}. \end{aligned} \quad (3.7)$$

Note, that here $\mathcal{I}_\lambda(n)$ and $\mathcal{I}_\omega(n)$ are Fisher information on parameters λ and ω , respectively, taking the other parameter as a nuisance.

Theorem 3.4 *Consider the two-dimensional analogue of the complex model (3.3) with $\omega \neq 0$ observed in design points $0 \leq t_1 < t_2 < \dots < t_n$. Then*

$$\mathcal{I}_\lambda(n) = \sum_{\ell=1}^{n-1} \varphi(d_\ell), \quad \mathcal{I}_\omega(n) = \sum_{\ell=1}^{n-1} \psi(d_\ell) \quad \text{and} \quad I_{\lambda, \omega}(n) = 0, \quad (3.8)$$

where $d_j = t_{j+1} - t_j$, $j = 1, 2, \dots, n-1$,

$$\varphi(x) := \frac{x^2 e^{-\lambda x} \cosh(\lambda x)}{\sinh^2(\lambda x)}, \quad \psi(x) := \frac{x^2 e^{-\lambda x}}{\sinh(\lambda x)}, \quad x > 0, \quad \text{and} \quad \varphi(0) := \frac{1}{\lambda^2}, \quad \psi(0) := 0. \quad (3.9)$$

Observe that none of the entries of the FIM $\mathcal{I}_{\lambda, \omega}(n)$ depends on the frequency parameter. Further, $\mathcal{I}_\lambda(n)/2$ coincides with the Fisher information on the covariance parameter of a real valued OU process given by Zagoraiou and Baldi Antognini (2009). Note that here we consider two-dimensional OU processes, which justifies the halving of the information, however, due to this connection the first statement of the following theorem is a direct consequence of Zagoraiou and Baldi Antognini (2009, Theorem 4.2).

Theorem 3.5 *Consider the two-dimensional analogue of the complex model (3.3) with $\omega \neq 0$ observed in design points $0 \leq t_1 < t_2 < \dots < t_n$.*

- a) *The D-optimal design for the damping parameter λ maximizing the Fisher information $\mathcal{I}_\lambda(n)$ does not exist within the class of admissible designs.*

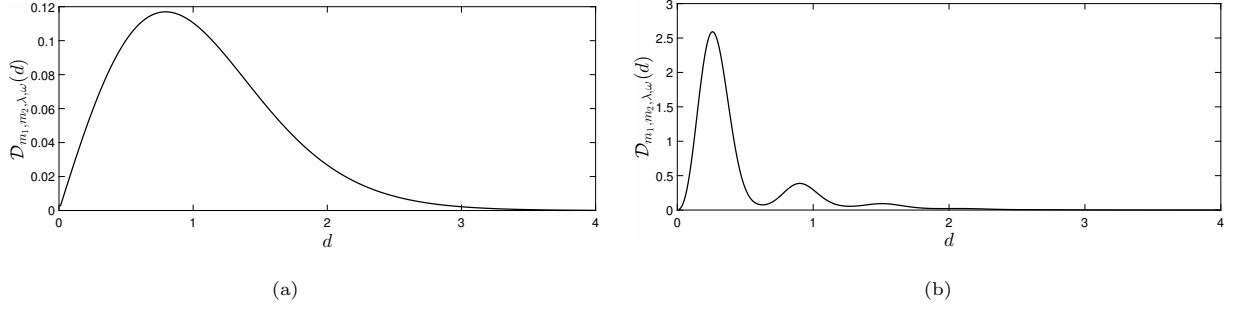


Figure 3: Objective function $\mathcal{D}_{m_1, m_2, \lambda, \omega}$ for a two-point design $\{0, d\}$ for (a) $\lambda = 1, \omega = 1$ and (b) $\lambda = 1, \omega = 4$.

b) The D -optimal design for the frequency ω maximizing the Fisher information $\mathcal{I}_\omega(n)$ is equidistant with

$$d_1 = d_2 = \dots = d_{n-1} = d_\lambda := \frac{1}{\lambda} \left(W(-2e^{-2})/2 + 1 \right) \approx \frac{0.7968}{\lambda},$$

where $W(x)$ denotes the Lambert W -function (Corless et al., 1996).

c) The D -optimal design for both covariance parameters of the complex OU process maximizing $\det(\mathcal{I}_{\lambda, \omega}(n))$ is equidistant with

$$d_1 = d_2 = \dots = d_{n-1} = d^\circ / \lambda,$$

where $d^\circ \approx 0.4930$ is the unique positive solution of

$$1 - d - 2de^{-2d} - e^{-4d} = 0. \quad (3.10)$$

We remark that in the case of one-dimensional OU processes an admissible design for the covariance parameter λ can be obtained by introducing the nugget effect, see e.g. Stehlík et al. (2017).

3.3. Estimation of all parameters

In the most general case one has to estimate both the components m_1, m_2 of the mean and the covariance parameters λ and ω . The FIM on these parameters equals

$$\mathcal{I}_{m_1, m_2, \lambda, \omega}(n) = \begin{bmatrix} \mathcal{I}_{m_1, m_2}(n) & \mathbf{0}_{2,2} \\ \mathbf{0}_{2,2} & \mathcal{I}_{\lambda, \omega}(n) \end{bmatrix}, \quad (3.11)$$

where $\mathcal{I}_{m_1, m_2}(n)$ and $\mathcal{I}_{\lambda, \omega}(n)$ are the information matrices defined in Lemma 3.1 and by (3.6), respectively, and $\mathbf{0}_{k, \ell}$ denotes the $k \times \ell$ matrix of zeros. Hence, according to the results of Lemma 3.1 and Theorem 3.4, the D -optimal design for all four parameters maximizes in $\mathbf{d} = (d_1, d_2, \dots, d_{n-1})$

$$\mathcal{D}_{m_1, m_2, \lambda, \omega}(\mathbf{d}) := \det(\mathcal{I}_{m_1, m_2, \lambda, \omega}(n)) = \left(1 + \sum_{\ell=1}^{n-1} g(d_\ell) \right)^2 \left(\sum_{\ell=1}^{n-1} \varphi(d_\ell) \right) \left(\sum_{\ell=1}^{n-1} \psi(d_\ell) \right), \quad (3.12)$$

where the functions g , φ and ψ are defined by (3.4) and (3.9).

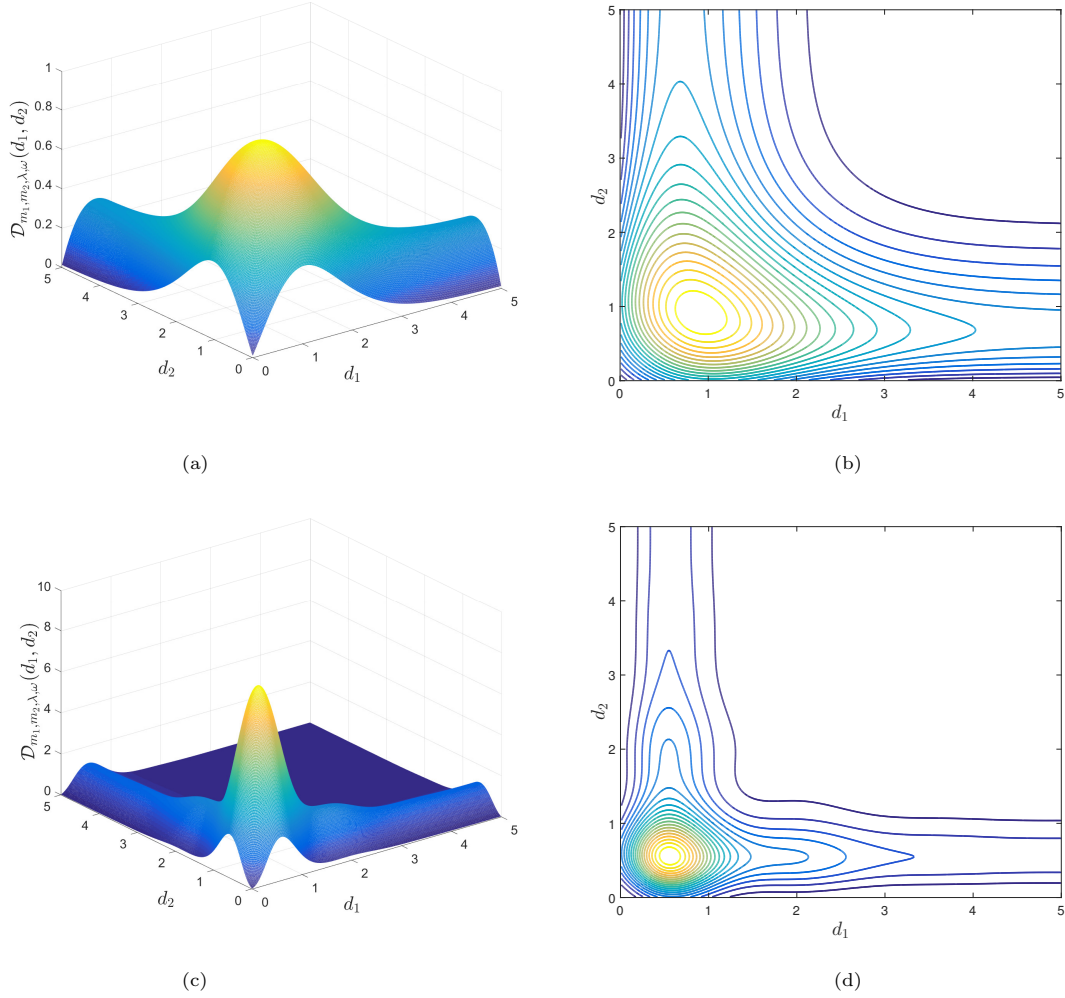


Figure 4: Bivariate objective function $\mathcal{D}_{m_1, m_2, \lambda, \omega}$ for a three-point design $\{0, d_1, d_1 + d_2\}$ for (a) $\lambda = 1, \omega = 1$ and (c) $\lambda = 1, \omega = 4$, together with the corresponding contour plots (b) and (d), respectively.

Example 3.6 Consider first the simplest case, that is a two-point design $\{0, d\}$, when the objective function (3.12) is univariate. Figures 3a and 3b, showing the graph of $\mathcal{D}_{m_1, m_2, \lambda, \omega}(d)$ for $\lambda = 1, \omega = 1$ and $\lambda = 1, \omega = 4$, respectively, clearly illustrate the existence of an optimal design.

Example 3.7 Similar to Example 3.3, consider again a three-point design $\{0, d_1, d_1 + d_2\}$. In Figures 4a and 4c the bivariate objective function $\mathcal{D}_{m_1, m_2, \lambda, \omega}$ is plotted for $\lambda = 1, \omega = 1$ and $\lambda = 1, \omega = 4$, together with the corresponding contour plots (Figures 4b and 4d, respectively).

Although the objective function $\mathcal{D}_{m_1, m_2, \lambda, \omega}(\mathbf{d})$ is too complicated for treating it analytically, numerical results in higher dimensions show that there exists a D-optimal design and it is equidistant. In Figure 5 the

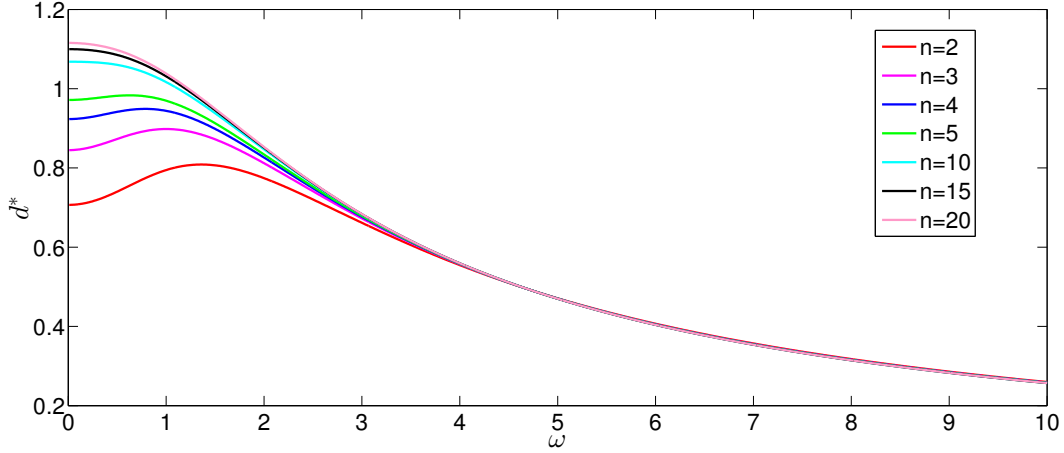


Figure 5: Optimal distance d^* of the equidistant n -point D-optimal design for estimation of all parameters as a function of the frequency ω for $\lambda = 1$.

optimal distance d^* of the equidistant n -point D-optimal design is plotted as a function of the frequency ω for $\lambda = 1$. Again, the general case can easily be obtained by rescaling both axes by the damping λ . Note that the larger the frequency, the smaller the effect of the number of design points on the optimal design.

4. Application

The obtained designs, especially the ones given in Theorem 3.5 can be applied in the assessment of the quality of parameter estimation for damping parameter λ and frequency parameter ω in Kolmogorov's model (1.1) of the Chandler wobble. The maximum likelihood estimator and sufficient statistics for λ are given in Arató (1968, 1982). However, as noted after Theorem 3.5, for the drift parameter there exist no admissible design, unless we consider a model with nugget effect.

For the model of Arató *et al.* (1962) two different estimates of the damping parameter are given by Pančenko (1960) ($\hat{\lambda} = 0.3$) and Walker and Young (1955) ($\hat{\lambda} = 0.01$). By the second statement of Theorem 3.5 the D-optimal design for frequency ω is equidistant with an optimal lag of $d_\lambda \approx \frac{0.7968}{\lambda}$. However, the various estimates for λ give a broad range of optimal equidistant times for measurements.

Further, due to the difference in the estimated values of the damping parameter λ , one might be interested in the sensitivity of the standardized setup ($\lambda = \omega = 1$) with respect to the D-optimality criterion for estimation of the trend parameter. For the n -point equidistant design this means the evaluation of $R(d, \lambda, 1; n)$, where

$$R(d, \lambda, \omega; n) := \frac{(1 + (n-1)g(d, 1, 1))^2}{(1 + (n-1)g(d, \lambda, \omega))^2}, \quad \text{with} \quad g(x, \lambda, \omega) := \frac{1 - 2e^{-\lambda x} \cos(\omega x) + e^{-2\lambda x}}{1 - e^{-2\lambda x}}, \quad (4.1)$$

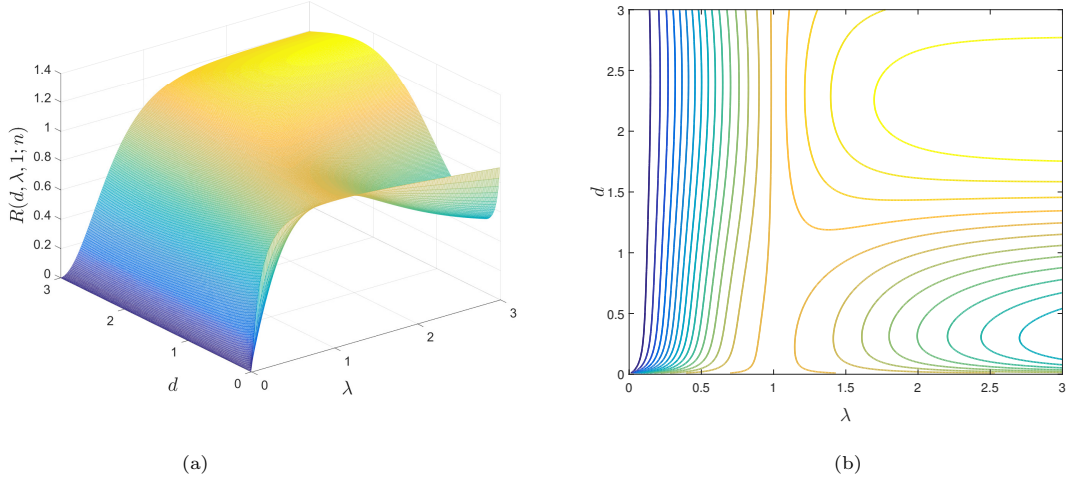


Figure 6: Surface $R(d, \lambda, 1; n)$ (a) and the corresponding contour plot (b) for a ten-point equidistant design.

see also (3.4). Here we analyze the situation for arbitrary d , since experimenter usually does not have a free choice of the lag, although for $\lambda = \omega = 1$ the optimal value equals $d^* \approx 2.1835$ (see Theorem 3.2). Thus, this analysis incorporates all possible design spaces of form $[0, T_{\max}]$ where T_{\max} denotes the *upper bound* of the design space.

The 3rd order Taylor expansion of $R(d, \lambda, 1; n)$ around the origin $\lambda = 0$ results in

$$R(d, \lambda, 1; n) = \frac{d^2 \left(1 + (n-1) \frac{(1-2e^{-d} \cos(d) + e^{-2d})}{1-e^{-2d}} \right)^2}{(n-1)(\cos(d)-1)^2} \lambda^2 + \mathcal{O}(\lambda^3),$$

which shows a very high sensitivity of the efficiency for the standardized design with respect to small values of the damping parameter λ . The same phenomenon can be observed on Figure 6 showing the surface $R(d, \lambda, 1; n)$ and the corresponding contour plot for a ten-point equidistant design.

If one can use the optimal lag $d^* \approx 2.1835$ for the standardized design (i.e. has the possibility of choosing an arbitrary lag), then the 3rd order Taylor expansion of $R(d, \lambda, 1; n)$ around the origin $\lambda = 0$ reduces to

$$R(d^*, \lambda, 1; n) = \frac{1.9218 \cdot (1 + 1.1569 \cdot (n-1))^2}{(n-1)^2} \lambda^2 + \mathcal{O}(\lambda^3),$$

that is for $\lambda \approx 0$ we have

$$\lim_{n \rightarrow \infty} R(d^*, \lambda, 1; n) \approx 2.5723 \lambda^2.$$

This also confirms the sensitivity of the efficiency for the standardized design at small damping values λ .

Consider now the sensitivity of efficiency for the standardized design with respect to frequency ω . For the n -point equidistant design one has to evaluate $R(d, 1, \omega; n)$ defined again by (4.1), which for $\omega \approx 0$

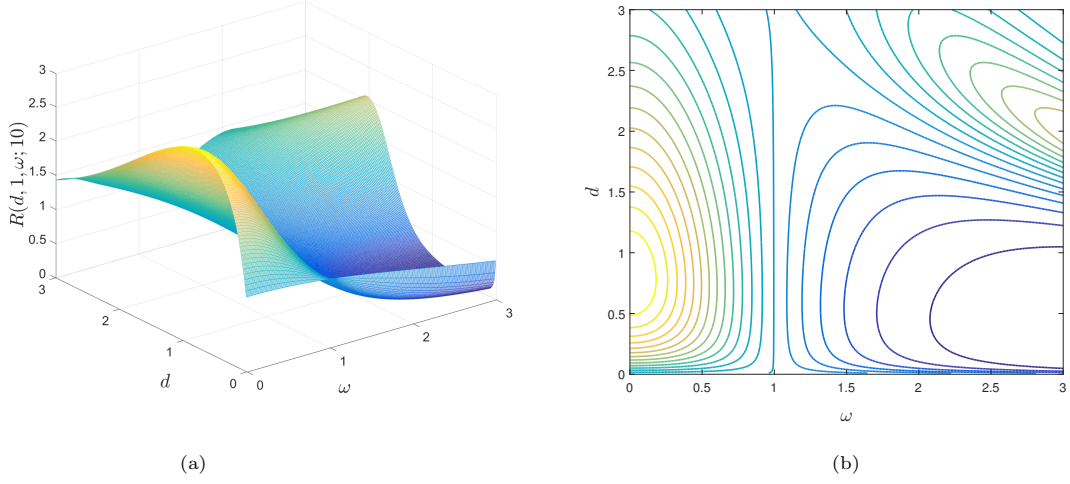


Figure 7: Surface $R(d, 1, \omega; n)$ (a) and the corresponding contour plot (b) for a ten-point equidistant design.

indicates even higher sensitivity. This can be derived from the form

$$R(d, 1, \omega; n) = \frac{((n-2)e^{-2d} - 2(n-1)e^{-d}\cos(d) + n)^2}{((n-2)e^{-2d} - 2(n-1)e^{-d} + n)^2} - \frac{2(n-1)d^2e^{-d}((n-2)e^{-2d} - 2(n-1)e^{-d}\cos(d) + n)^2}{((n-2)e^{-2d} - 2(n-1)e^{-d} + n)^3}\omega^2 + \mathcal{O}(\omega^4).$$

of the 4th order Taylor expansion around $\omega = 0$, and one can also observe it on Figure 7, where $R(d, 1, \omega; n)$ and the corresponding contour plot are shown for a ten-point equidistant design.

Again, for the optimal lag $d^* \approx 2.1835$ the 4th order Taylor expansion of $R(d, 1, \omega; n)$ around the origin $\omega = 0$ reduces to

$$R(d^*, 1, \omega; n) = \frac{2.0455(9.0024n - 1.2211)^2(6.2058n + 1.5755 - 8.4655(n-1)\omega^2)}{(7.8777n + 2)^3} + \mathcal{O}(\omega^4).$$

Hence, for $\omega \approx 0$ we have

$$\lim_{n \rightarrow \infty} R(d^*, 1, \omega; n) \approx 1.9476 - 2.3318\omega^2,$$

which also confirms the sensitivity of the efficiency at small frequencies ω .

Recently Malkin and Miller (2010) discovered that besides the well-known Chandler wobble phase jump in the 1920s, two other large phase jumps have been identified in the 1850s and 2000s. As in the 1920s, these phase jumps occurred contemporary with a sharp decrease in the Chandler wobble amplitude. However, sharp decrease of amplitude can drastically change the optimal design, as confirmed by Theorem 3.5 for the case of a complex OU process. This underpins the importance of further research on stochastic approach to Chandler wobble. Moreover, substantial relation of large seismic events to Chandler wobble excitation

(O’Connell and Dziewonski, 1976) justify further studies of optimal designs for both damping and frequency parameters.

5. Conclusions

We derive our results with the help of the two-dimensional real valued OU process, which directly relates to the complex OU process. Compared to the one-dimensional situation the design strategies for D-optimality are much more convenient for the case of complex OU process, e.g. there exist admissible designs for estimation of both location and scale parameters.

In this paper we focus on D-optimality, since estimation of the parameters is the main practical problem in most of the considered applications, such as the Chandler wobble. Here our results on optimal designs for estimation of damping and frequency parameters in Kolmogorov’s model correspond to the empirically observed facts in the geophysical literature (see e.g. Pančenko (1960)). In particular, the results of Theorem 3.5 show interesting but intrinsic dependence between a lag for D-optimal design for estimation of the frequency parameter and the damping parameter. The latter was observed empirically in experimental physics.

We have demonstrated the high sensitivity of the efficiency of the standardized design with respect to small values of both damping and frequency parameters. From practical point of view the results given in this paper justify the importance of complex OU processes and their optimal design strategies, not only for the applications, but also for the theory.

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Appendix A. Appendix

Appendix A.1. Proof of Lemma 3.1

Similar to the real valued case (Zagoraïou and Baldi Antognini, 2009), the correlation matrix of observations $\{(Z_1(t_j), Z_2(t_j)), j = 1, 2, \dots, n\}$ of the two-dimensional process (3.1) equals

$$C(n) = \begin{bmatrix} \mathbb{I}_2 & e^{Ad_1} & e^{A(d_1+d_2)} & e^{A(d_1+d_2+d_3)} & \dots & \dots & e^{A(\sum_{j=1}^{n-1} d_j)} \\ e^{A^\top d_1} & \mathbb{I}_2 & e^{Ad_2} & e^{A(d_2+d_3)} & \dots & \dots & e^{A(\sum_{j=2}^{n-1} d_j)} \\ e^{A^\top(d_1+d_2)} & e^{A^\top d_2} & \mathbb{I}_2 & e^{Ad_3} & \dots & \dots & e^{A(\sum_{j=3}^{n-1} d_j)} \\ e^{A^\top(d_1+d_2+d_3)} & e^{A^\top(d_2+d_3)} & e^{A^\top d_3} & \mathbb{I}_2 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & e^{Ad_{n-1}} \\ e^{A^\top(\sum_{j=1}^{n-1} d_j)} & e^{A^\top(\sum_{j=2}^{n-1} d_j)} & e^{A^\top(\sum_{j=3}^{n-1} d_j)} & \dots & \dots & e^{A^\top d_{n-1}} & \mathbb{I}_2 \end{bmatrix}.$$

Short calculation shows that the inverse of $C(n)$ is given by

$$C^{-1}(n) = \begin{bmatrix} U_1 & -e^{Ad_1}U_1 & 0 & 0 & \dots & \dots & 0 \\ -e^{A^\top d_1}U_1 & V_2 & -e^{Ad_2}U_2 & 0 & \dots & \dots & 0 \\ 0 & -e^{A^\top d_2}U_2 & V_3 & -e^{Ad_3}U_3 & \dots & \dots & 0 \\ 0 & 0 & -e^{A^\top d_3}U_3 & V_4 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & V_{n-1} & -e^{Ad_{n-1}}U_{n-1} \\ 0 & 0 & 0 & \dots & \dots & -e^{A^\top d_{n-1}}U_{n-1} & U_{n-1} \end{bmatrix},$$

where $U_k := [\mathbb{I}_2 - e^{(A+A^\top)d_k}]^{-1}$, $k = 1, 2, \dots, n-1$, and $V_k := U_k + e^{(A+A^\top)d_{k-1}}U_{k-1}$, $k = 2, 3, \dots, n-1$, which is a direct generalization of the result of Kiselák and Stehlík (2008) for the classical OU process.

Now, using (2.4) one can easily obtain

$$U_k = \frac{1}{1 - e^{-2\lambda d_k}} \mathbb{I}_2, \quad e^{Ad_k}U_k = \frac{e^{-\lambda d_k}}{1 - e^{-2\lambda d_k}} \begin{bmatrix} \cos(\omega d_k) & -\sin(\omega d_k) \\ \sin(\omega d_k) & \cos(\omega d_k) \end{bmatrix}, \quad k = 1, 2, \dots, n-1,$$

$$V_k = \left(\frac{1}{1 - e^{-2\lambda d_k}} + \frac{e^{-2\lambda d_{k-1}}}{1 - e^{-2\lambda d_{k-1}}} \right) \mathbb{I}_2, \quad k = 2, 3, \dots, n-1,$$

which together with (2.4) specify both $C(n)$ and $C^{-1}(n)$. Finally, tedious but straightforward calculations lead us to (3.2). \square

Appendix A.2. Proof of Theorem 3.2

Consider first the function $g(x)$ defined by (3.4). As

$$g'(x) = \frac{r(x)}{\sinh^2(\lambda x)} \quad \text{with} \quad r(x) := \lambda \cosh(\lambda x) \cos(\omega x) + \omega \sinh(\lambda x) \sin(\omega x) - \lambda, \quad x > 0, \quad (\text{A.1})$$

the critical points of $g(x)$ are the roots of $r(x)$. Short calculation shows that $r(x) < 0$ if and only if

$$\lambda \left[\tanh\left(\frac{\lambda x}{2}\right) \cot\left(\frac{\omega x}{2}\right) \right]^2 + 2\omega \left[\tanh\left(\frac{\lambda x}{2}\right) \cot\left(\frac{\omega x}{2}\right) \right] - \lambda < 0, \quad (\text{A.2})$$

and the quadratic polynomial in (A.2) has two distinct roots

$$\varrho_- := \frac{-\omega - \sqrt{\omega^2 + \lambda^2}}{\lambda} < 0 \quad \text{and} \quad \varrho_+ := \frac{-\omega + \sqrt{\omega^2 + \lambda^2}}{\lambda} > 0.$$

Now, if $\omega \neq 0$, by the properties of the cotangent and hyperbolic tangent functions there exist points $\{x_n^-, n \in \mathbb{N}\}$ such that $0 < x_n^- < x_{n+1}^-$ and

$$\varrho_- < \tanh\left(\frac{\lambda x_n^-}{2}\right) \cot\left(\frac{\omega x_n^-}{2}\right) < \varrho_+, \quad \text{that is} \quad r(x_n^-) < 0, \quad n \in \mathbb{N}.$$

Similar arguments prove the existence of points $\{x_n^+, n \in \mathbb{N}\}$ satisfying $0 < x_n^+ < x_{n+1}^+$ and $r(x_n^+) > 0$, $n \in \mathbb{N}$. Thus, $r(x)$ (and $g'(x)$) has an infinite number of sign changes, moreover, since $\lim_{x \rightarrow 0} g'(x) = \frac{\omega^2}{2\lambda} + \frac{\lambda}{2} > 0$, the first change is from positive to negative. Denote by $\{x_n^\circ, n \in \mathbb{N}\}$ the roots of $r(x)$, that is the critical points of $g(x)$, where again, $0 < x_n^\circ < x_{n+1}^\circ$, $n \in \mathbb{N}$, and x_1° is a local maximum of $g(x)$. Further, we have

$$g''(x_n^\circ) = (\lambda^2 + \omega^2) \frac{\cos(\omega x_n^\circ)}{\sinh(\lambda x_n^\circ)}.$$

Assume that $g''(x_n^\circ) = 0$, that is $\omega x_n^\circ = \frac{\pi}{2} + \pi k_n$ for some $k_n \in \mathbb{Z}$. In this case

$$0 = r(x_n^\circ) = \pm \omega \sinh\left(\frac{\lambda}{\omega} \left(\frac{\pi}{2} + \pi k_n\right)\right) - \lambda, \quad (\text{A.3})$$

where the positive sign will apply if k_n is odd, and the negative sign if it is even. However, the Taylor series expansion of the hyperbolic sine implies

$$\pm \omega \sinh\left(\frac{\lambda}{\omega} \left(\frac{\pi}{2} + \pi k_n\right)\right) - \lambda = \pm \lambda \left(\frac{\pi}{2} \mp 1 + \pi k_n\right) \pm \sum_{\ell=1}^{\infty} \frac{\lambda^{2\ell+1}}{(2\ell+1)! \omega^{2\ell}} \left(\frac{\pi}{2} + \pi k_n\right)^{2\ell+1} \neq 0,$$

which contradicts to (A.3). Hence, for the critical points $\{x_n^\circ, n \in \mathbb{N}\}$ of $g(x)$ either $g''(x_n^\circ) < 0$ or $g''(x_n^\circ) > 0$, so they are either local maxima or local minima, respectively. This ensures the existence of a global maximum of g as well at some point $d^* \in \{x_n^\circ, n \in \mathbb{N}\}$, where $g'(d^*) = 0$, and $g''(d^*) < 0$.

Obviously, instead of \mathcal{D}_{m_1, m_2} defined by (3.5) it suffices to maximize

$$F(d_1, d_2, \dots, d_{n-1}) := Q(n) = 1 + \sum_{\ell=1}^{n-1} g(d_\ell)$$

Now, consider the $(n-1)$ -dimensional vector $\mathbf{d}^* := (d^*, d^*, \dots, d^*)^\top$. As

$$\frac{\partial F}{\partial d_k}(d^*, d^*, \dots, d^*) = g'(d^*) = 0, \quad k = 1, 2, \dots, n-1,$$

\mathbf{d}^* is a critical point of F . Further, as

$$\frac{\partial^2 F}{\partial d_\ell \partial d_k}(d^*, d^*, \dots, d^*) = 0 \quad \text{if } k \neq \ell, \quad k, \ell = 1, 2, \dots, n-1,$$

and

$$\frac{\partial^2 F}{\partial d_k^2}(d^*, d^*, \dots, d^*) = g''(d^*) < 0, \quad k = 1, 2, \dots, n-1,$$

the Hessian of F at \mathbf{d}^* is negative definite. Hence, \mathbf{d}^* is a maximum point of F and since the components $g(d^*)$ are the largest possible, this maximum is global. \square

Appendix A.3. Proof of Theorem 3.4

Using the expressions of the covariance matrix $C(n)$ of observations and its inverse $C^{-1}(n)$ given in the proof of Lemma 3.1 and (3.6), the formulae of (3.8) can be verified by induction, similar to the proof of Theorem 2 of Baran and Stehlík (2015). As an example consider the Fisher information on damping parameter λ , where one has to show

$$\mathcal{I}_\lambda(n) := \frac{1}{2} \operatorname{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} \right\} = \sum_{\ell=1}^{n-1} \frac{2d_\ell^2 q_\ell^2 (1+q_\ell^2)}{(1-q_\ell^2)^2}, \quad (\text{A.4})$$

with $q_k := e^{-\lambda d_k}$, $k = 1, 2, \dots, n-1$.

For $n=2$ equation (A.4) holds trivially. Assume also that (A.4) is true for some n , and we are going to show it for $n+1$. Let

$$\Delta(n) := \left(-(d_1 + d_2 + \dots + d_n) q_1 q_2 \dots q_n B_{1,\dots,n}, -(d_2 + d_3 + \dots + d_n) q_2 q_3 \dots q_n B_{2,\dots,n}, \dots, -d_n q_n B_n \right)^\top,$$

where

$$B_{k,\dots,n} := \begin{bmatrix} \cos(\omega(d_k + \dots + d_n)) & -\sin(\omega(d_k + \dots + d_n)) \\ \sin(\omega(d_k + \dots + d_n)) & \cos(\omega(d_k + \dots + d_n)) \end{bmatrix}, \quad k = 1, 2, \dots, n.$$

Using the representations of $C(n)$ and $C^{-1}(n)$ given in Section Appendix A.1, one can easily see

$$\frac{\partial C(n+1)}{\partial \lambda} = \begin{bmatrix} \frac{\partial C(n)}{\partial \lambda} & \Delta(n) \\ \Delta(n)^\top & \mathbf{0}_{2,2} \end{bmatrix} \quad \text{and} \quad C^{-1}(n+1) = \begin{bmatrix} C^{-1}(n) & \Lambda_{1,2}(n) \\ \Lambda_{1,2}(n)^\top & (1-q_n^2)^{-1} \mathbb{I}_2 \end{bmatrix} + \begin{bmatrix} \Lambda_{1,1}(n) & \mathbf{0}_{2n,2} \\ \mathbf{0}_{2,2n} & \mathbf{0}_{2,2} \end{bmatrix},$$

where $\mathbf{0}_{k,\ell}$ denotes the $k \times \ell$ matrix of zeros and

$$\Lambda_{1,1}(n) := \begin{bmatrix} \mathbf{0}_{2n-2,2n-2} & \mathbf{0}_{2n-2,2} \\ \mathbf{0}_{2,2n-2} & \frac{q_n^2}{1-q_n^2} \mathbb{I}_2 \end{bmatrix} \quad \text{and} \quad \Lambda_{1,2}(n) := \begin{bmatrix} \mathbf{0}_{2n-2,2} \\ -\frac{q_n}{1-q_n^2} B_n \end{bmatrix}.$$

In this way

$$C^{-1}(n+1)\frac{\partial C(n+1)}{\partial \lambda} = \left[\begin{array}{c|c} C^{-1}(n)\frac{\partial C(n)}{\partial \lambda} + \mathcal{K}_{1,1}(n) & C^{-1}(n)\Delta(n) \\ \hline \mathcal{K}_{1,2}(n) & \frac{d_n q_n^2}{1-q_n^2} \mathbb{I}_2 \end{array} \right] + \left[\begin{array}{c|c} \mathcal{K}_{2,1}(n) & \mathcal{K}_{2,2}(n) \\ \hline \mathbf{0}_{2,2n} & \mathbf{0}_{2,2} \end{array} \right],$$

with

$$\begin{aligned} \mathcal{K}_{1,1}(n) &:= \left[\begin{array}{c|c} \mathbf{0}_{2n-2,2n} & \\ \hline -\frac{q_n}{1-q_n^2} B_n \Delta^\top(n) & \end{array} \right], & \mathcal{K}_{1,2}(n) &:= \frac{1}{1-q_n^2} \Delta^\top(n) - \frac{q_n}{1-q_n^2} B_n^\top \left[\begin{array}{c|c} \Delta^\top(n-1) & \\ \hline \mathbf{0}_{2,2} \end{array} \right], \\ \mathcal{K}_{2,1}(n) &:= \left[\begin{array}{c|c} \mathbf{0}_{2n-2,2n-2} & \mathbf{0}_{2n-2,2} \\ \hline \frac{q_n^2}{1-q_n^2} \Delta^\top(n-1) & \mathbf{0}_{2,2} \end{array} \right], & \mathcal{K}_{2,2}(n) &:= \left[\begin{array}{c|c} \mathbf{0}_{2n-2,2} & \\ \hline -\frac{d_n q_n^3}{1-q_n^2} B_n & \end{array} \right]. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{I}_\lambda(n+1) &= \mathcal{I}_\lambda(n) + \text{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} \mathcal{K}_{1,1}(n) \right\} + \text{tr} \left\{ C^{-1}(n) \Delta(n) \mathcal{K}_{1,2}(n) \right\} + \frac{1}{2} \text{tr} \left\{ \mathcal{K}_{1,1}^2(n) \right\} \\ &\quad + \frac{d_n^2 q_n^4}{(1-q_n^2)^2} + \text{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \alpha} \mathcal{K}_{2,1}(n) \right\} + \text{tr} \left\{ \mathcal{K}_{1,2}(n) \mathcal{K}_{2,2}(n) \right\}. \end{aligned} \quad (\text{A.5})$$

After long but straightforward calculations one can get

$$\begin{aligned} \text{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} \mathcal{K}_{1,1}(n) \right\} &= - \sum_{\ell=1}^{n-1} \frac{2d_\ell^2 q_\ell^2 q_{\ell+1}^2 \cdots q_n^2}{(1-q_\ell^2)(1-q_n^2)} = - \text{tr} \left\{ C^{-1}(n) \frac{\partial C(n)}{\partial \lambda} \mathcal{K}_{2,1}(n) \right\}, \\ \text{tr} \left\{ C^{-1}(n) \Delta(n) \mathcal{K}_{1,2}(n) \right\} &= \frac{2d_n^2 q_n^2}{(1-q_n^2)^2} - \frac{2d_n^2 q_n^4}{(1-q_n^2)^2}, \\ \frac{1}{2} \text{tr} \left\{ \mathcal{K}_{1,1}^2(n) \right\} &= \frac{d_n^2 q_n^4}{(1-q_n^2)^2}, & \text{tr} \left\{ \mathcal{K}_{1,2}(n) \mathcal{K}_{2,2}(n) \right\} &= \frac{2d_n^2 q_n^4}{(1-q_n^2)^2}, \end{aligned}$$

so (A.5) implies

$$\mathcal{I}_\lambda(n+1) = \mathcal{I}_\lambda(n) + \frac{2d_n^2 q_n^2 (1+q_n^2)}{(1-q_n^2)^2},$$

which completes the proof of (A.4). The other two statements of Theorem 3.4 can be verified in a similar way. \square

Appendix A.4. Proof of Theorem 3.5

As it has been mentioned, statement a) directly follows from Theorem 4.2 of Zagoraiou and Baldi Antognini (2009).

Now, consider the function $\psi(x)$ defined by (3.8). As

$$\psi'(x) = \frac{4xe^{-2\lambda x}(1-\lambda x - e^{-2\lambda x})}{(1-e^{-2\lambda x})^2},$$

$\psi(x)$ has a single critical point at $d^* := \frac{1}{\lambda}(W(-2e^{-2})/2 + 1)$, moreover, $\psi''(d^*) < 0$, so d^* is a global maximum point. Hence, statement b) can be verified using the same arguments as in the proof of Theorem 3.2.

According to (3.8), the D-optimal design on λ and ω maximizes

$$G(d_1, d_2, \dots, d_{n-1}) := \left(\sum_{\ell=1}^{n-1} \varphi(d_\ell) \right) \left(\sum_{\ell=1}^{n-1} \psi(d_\ell) \right) = \left(\sum_{\ell=1}^{n-1} \frac{2d_\ell^2 e^{-2\lambda d_\ell} (1 + e^{-2\lambda d_\ell})}{(1 - e^{-2\lambda d_\ell})^2} \right) \left(\sum_{\ell=1}^{n-1} \frac{2d_\ell^2 e^{-2\lambda d_\ell}}{(1 - e^{-2\lambda d_\ell})} \right),$$

and obviously, it suffices to consider the case $\lambda = 1$. The critical points of G are solutions of the system

$$\frac{\partial G}{\partial d_k}(d_1, d_2, \dots, d_{n-1}) = \varphi'(d_k) \left(\sum_{\ell=1}^{n-1} \psi(d_\ell) \right) + \psi'(d_k) \left(\sum_{\ell=1}^{n-1} \varphi(d_\ell) \right) = 0, \quad (\text{A.6})$$

$k = 1, 2, \dots, n-1$, where

$$\varphi'(x) = \frac{4xe^{-2\lambda x}(1 - \lambda x - 3\lambda xe^{-2\lambda x} - e^{-4\lambda x})}{(1 - e^{-2\lambda x})^3} < 0, \quad x > 0.$$

Hence, for $\lambda = 1$ the system of equations (A.6) is equivalent to

$$\kappa(d_k) = \left(\sum_{\ell=1}^{n-1} \psi(d_\ell) \right) / \left(\sum_{\ell=1}^{n-1} \varphi(d_\ell) \right), \quad k = 1, 2, \dots, n-1, \quad (\text{A.7})$$

where

$$\kappa(x) := \frac{1 - x + (x - 2)e^{-2x} + e^{-4x}}{x - 1 + 3xe^{-2x} + e^{-4x}}, \quad x > 0.$$

Since $\kappa(x)$ is strictly monotone decreasing and continuous with $\lim_{x \rightarrow 0} \kappa(x) = 1$ and $\lim_{x \rightarrow \infty} \kappa(x) = -1$, the solution of (A.7) should satisfy $d_1 = d_2 = \dots = d_{n-1} =: d$. Hence, the critical points of G have equal coordinates and at a critical point (A.7) reduces to $\kappa(d) = (1 - e^{-2d})/(1 + e^{-2d})$, which is equivalent to (3.10). The latter equation has a unique positive solution d° , so $d_1 = d_2 = \dots = d_{n-1} = d^\circ$ is the only critical point of G . Short calculation shows that at this point the Hessian of G equals

$$\begin{aligned} H &= (n-1)[\varphi''(d^\circ)\psi(d^\circ) + \psi''(d^\circ)\varphi(d^\circ)]\mathbb{I}_{n-1} + 2[\varphi'(d^\circ)\psi'(d^\circ)]\mathbf{1}_{n-1}\mathbf{1}_{n-1}^\top \\ &\approx -0.5083 \cdot (n-1)\mathbb{I}_{n-1} - 0.2754 \cdot \mathbf{1}_{n-1}\mathbf{1}_{n-1}^\top, \end{aligned}$$

where $\mathbf{1}_k$, $k \in \mathbb{N}$, denotes the k -dimensional vector of ones. This means that for all $\mathbf{0} \neq \mathbf{v} = (v_1, v_2, \dots, v_{n-1})^\top \in \mathbb{R}^{n-1}$ we have

$$\mathbf{v}^\top H \mathbf{v} = (n-1)[\varphi''(d^\circ)\psi(d^\circ) + \psi''(d^\circ)\varphi(d^\circ)]\mathbf{v}^\top \mathbf{v} + 2[\varphi'(d^\circ)\psi'(d^\circ)] \left(\sum_{\ell=1}^{n-1} v_\ell \right)^2 < 0,$$

so the Hessian is negative definite. Hence, the unique critical point of G is a global maximum, which completes the proof. \square

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