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Review

Schrödinger Potentials with Polynomial Solutions of Heun-Type Equations

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Abstract: The present review discusses the solution of the Heun, confluent, biconfluent, double confluent, and triconfluent equations in terms of polynomial expansions, and applies the results to generate exactly solvable Schrödinger potentials. Although there are more general approaches to solve these differential equations in terms of the expansions of certain special functions, the importance of polynomial solutions is unquestionable, as most of the known potentials are solvable in terms of the hypergeometric and confluent hypergeometric functions; i.e., Natanzon-class potentials possess bound-state solutions in terms of classical orthogonal polynomials, to which the (confluent) hypergeometric functions can be reduced. Since some of the Heun-type equations contain the hypergeometric and/or confluent hypergeometric differential equations as special limits, the potentials generated from them may also contain Natanzon-class potentials as special cases. A power series expansion is assumed around one of the singular points of each differential equation, and recurrence relations are obtained for the expansion coefficients. With the exception of the triconfluent Heun equations, these are three-term recurrence relations, the termination of which is achieved by prescribing certain conditions. In the case of the biconfluent and double confluent Heun equations, the expansion coefficients can be obtained in the standard way, i.e., after finding the roots of an $(N+1)$th-order polynomial in one of the parameters, which, in turn, follows from requiring the vanishing of an $(N+1) \times (N+1)$ determinant. However, in the case of the Heun and confluent Heun equations, the recurrence relation can be solved directly, and the solutions are obtained in terms of rationally extended X_1-type Jacobi and Laguerre polynomials, respectively. Examples for solvable potentials are presented for the Heun, confluent, biconfluent, and double confluent Heun equations, and alternative methods for obtaining the same potentials are also discussed. These are the schemes based on the rational extension of Bochner-type differential equations (for the Heun and confluent Heun equation) and solutions based on quasi-exact solvability (QES) and on continued fractions (for the biconfluent and double confluent equation). Possible further lines of investigations are also outlined concerning physical problems that require the solution of second-order differential equations, i.e., the Schrödinger equation with position-dependent mass and relativistic wave equations.

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1. Introduction

Physical models rely heavily on differential equations, because they describe the spatial and temporal variations of physical quantities. An especially important example is the Schrödinger equation, which accounts for the dynamics of quantum systems in the non-relativistic setting. In its most widely used form, this equation is written in coordinate representation, and contains a kinetic term, which encompasses up to second-order differential terms of the coordinate, and a potential term that is the function of the coordinate. The potential describes the interaction of a particle with its environment. The Schrödinger equation can take on rather diverse forms, depending on the physical system: it can describe a many-body or one-body problem in various physical dimensions in flat or curved spaces, containing potentials that are local or non-local, real or complex, time dependent or time independent; furthermore, the mass of the particle can also be postulated to be constant or coordinate dependent. Here, we consider the one-body Schrödinger equation with real, local, time-independent potentials, and with constant mass. Furthermore, we consider the one-dimensional Schrödinger equation defined either on the full real axis ($x \in (-\infty, \infty)$) or on a finite domain ($x \in [a, b]$), or the radial Schrödinger equation ($x \in [0, \infty)$) derived from higher-dimensional problems after separating the variables. The solutions can then be searched in forms containing special functions of mathematical physics satisfying a second-order ordinary differential equation. A number of special functions of this kind are known, and efforts to find and classify solvable Schrödinger equations, or potentials, has attracted the attention of physicists in the past 100 years, i.e., since the introduction of the Schrödinger equation. The goal was to derive closed expressions for the solutions of the Schrödinger equation and the corresponding energy eigenvalues. These efforts were helped by the extensive mathematical literature accumulated on special functions before and, in particular, on various forms of the hypergeometric function.

A number of exactly solvable potentials have been found in the 1920s and 1930s, e.g., the harmonic oscillator, Coulomb, Morse, Pöschl–Teller, and Eckart potentials (see, e.g., Ref. [1] for a compilation) by transforming the Schrödinger equation into the differential equation of the hypergeometric and confluent hypergeometric function by a well-chosen variable transformation. The bound-state solutions of these potentials are furthermore reduced to the classical orthogonal polynomials (Jacobi, generalized Laguerre, and Hermite; see, e.g., Ref. [2]). Later, a reversed approach has been introduced: rather than searching for the variable transformation that transforms the Schrödinger equation with a specific potential into the differential equation of some special function, the focus was placed on the question, what kind of potentials can be derived from those differential equations [3]? This approach was implemented in a systematic way by Natanzon, who developed the formalism by which the most general potentials admitting solutions in terms of the (confluent) hypergeometric function could be constructed [4]. Practically all known exactly solvable potentials could be identified as a member of the Natanzon potential class. (Among the few exceptions there are the exponential potential and the particle enclosed in a sphere, in which case, the solutions are written in terms of Bessel functions, and piecewise constant potentials with solutions composed of exponential and trigonometric functions matched at the boundaries; see, e.g., Ref. [5])

Another line of investigations was based on the factorization method, originally developed by Schrödinger [6] and implemented systematically in Ref. [1]. This method, in which the Schrödinger equation is factorized as the product of two first-order differential operators, was found to be an ideal tool to generate new exactly solvable potentials from known ones. It was later reinterpreted in a supersymmetric framework as supersymmetric quantum mechanics, SUSYQM [7,8]. This method also has its roots in the 19th century, when the Darboux transformation was developed [9]. The factorization method also allows associating group theoretical and algebraic structures to special functions [10]. A particularly useful concept brought about by SUSYQM was the shape invariance of potentials [11]. When the SUSYQM transform of a potential has the same mathematical form with only their parameters differing, it is called shape invariant. It turned out that the most well-known solvable potentials found in the early years of quantum mechanics have this property [12]. This finding follows from the differential and recurrence properties of the classical orthogonal polynomials [2].

Natanzon class potentials are well understood (see, e.g., Refs. [13,14] and Chapter 7 of Ref. [15]), so it is straightforward to search for more general classes of solvable potentials with solutions containing more general special functions satisfying second-order differential equations. As a possible attempt, the generalization of Bochner-type differential equations has been proposed. It was known that the classical orthogonal polynomials solve, and are the only solutions of Bochner-type differential equations, as follows:

$$p(z)\frac{\mathrm{d}^2 P_N}{\mathrm{d}z^2} + q(z)\frac{\mathrm{d}P_N}{\mathrm{d}z} + r(z)P_N(z) = \lambda_N P_N(z)\ , \tag{1}$$

in which $p(z)$, $q(z)$, and $r(z)$ are polynomials of degrees 2, 1 and 0, respectively [16]. As a generalization of this finding, differential equations with rational, rather than polynomial, coefficients have been introduced [17]. The rationally extended X_1-Laguerre and X_1-Jacobi polynomials share most features of their classical counterparts; however, one of their zeros falls outside their interval of orthogonality, meaning also that their sequence starts with a first-order polynomial, rather than a constant (zeroth-order polynomial). It was also shown that the X_1-type Jacobi and Laguerre polynomials can be expressed in terms of two classical orthogonal polynomials of the same type. To state it in a more general form, exceptional orthogonal polynomials can be obtained by applying a finite sequence of Darboux transformations to classical orthogonal polynomials [18].

These new polynomials were soon employed to construct new types of exactly solvable potentials, and the rational extension of the harmonic oscillator and the Scarf I potential was introduced [19]. It was also found that these potentials can be obtained from their ordinary counterparts by SUSYQM transformations. It was also realized that these potentials cannot belong to the Natanzon class, as their solutions contain expressions of two, rather than a single, orthogonal polynomial. As further generalizations, the X_m-Laguerre and X_m-Jacobi polynomials have also been introduced [20,21], and have been used to generate further extensions of known exactly solvable potentials.

An alternative approach to extending the range of exactly solvable potentials is transforming the Schrödinger equation into the Heun equation and its four confluent (confluent, biconfluent, double confluent, and triconfluent) versions [22]. This is an especially promising path, because some of these equations recover the (confluent) hypergeometric differential equation by specific choices of their parameters, so direct generalizations of already known exactly solvable potentials are possible.

In the theory of these differential equations, their singular points play a central role. The Heun equation was introduced towards the end of the 19th century [23], but its theory is far less elaborated than that of the hypergeometric differential equation. The first effort to generate solvable potentials from the Heun-type equations has been made in Ref. [24], where a classification of possible potentials is also presented, without a detailed analysis of the bound-state solutions and the bound-state energy eigenvalues. (Before that, another effort to derive exactly solvable potentials from rather general classes of differential equations has been given in Ref. [25], without explicitly referring to the Heun and related equations).

More recently, a systematic study of solvable potentials related to the Heun equation and its confluent variants has been carried out. See Ref. [26] for the Heun, [27] for the confluent Heun, and [28] for the biconfluent Heun equation. See also Ref. [29]. The solutions of these potentials are usually expanded in terms of simpler special functions, such as the hypergeometric [30] of Hermite functions [28].

Inspired by the experience accumulated on exactly solvable potentials and, in particular, on Natanzon class potentials, here, we focus on the polynomial solution of the Heun-type equations. The bound-state solutions on Natanzon class potentials are generally expressed in terms of the classical orthogonal polynomials, and both the wave functions and the corresponding energy eigenvalues are typically written in closed analytical form. The same applies to potentials solved by the rationally extended exceptional polynomials, which fall outside the Natanzon class. Polynomial solutions of the Heun-type equations have been discussed in the mathematical literature [22]. The general form of the solution is written in terms of a power series expansion around a singular point, and the expansion coefficients are determined by a recurrence relation. The termination of the recurrence can be achieved by the specific choice of the parameters. This generally leads to an algebraic system of equations, the solution of which requires the diagonalization of a tridiagonal matrix. This procedure is not especially promising when the goal is constructing exact solutions of a physical problem. Nevertheless, here, we review the status of the field, and also connect results from diverse approaches in order to place the problem in a broader perspective.

Before closing this section, we briefly recall further concepts used in the characterization of potentials from the point of view of their solvability. Conditional solvability means that a potential is solvable under the condition that the coupling coefficient of some of its terms is not arbitrary, but fixed. In the case of quasi-exact solvability (QES) [31], only part of the bound-state solutions (typically those of the lowest few states in energy) can be written in closed form. A collection of QES potentials is presented in Ref. [32] in the context of the hidden $sl(2,\mathbf{R})$ Lie algebra. Some of these potentials are present among the ones derived from Heun-type equations. Although the solutions are not discussed in much detail, we are going to mention these potentials in the subsections discussing the relevant Heun-type differential equation.

The arrangement of the present work is as follows: The basic properties of Heun-type differential equations are recalled in Section 2, together with a general method for transforming the second-order differential equation of a special function into the Schrödinger equation. Section 3 discusses the power series expansion of the solutions of Heun-type equations, together with the conditions under which the solutions reduce to polynomial form. Specific potentials are presented in each case as examples. The results are summarized in Section 4, where possible further directions of the present study are also outlined.

2. Heun-Type Differential Equations and Exactly Solvable Schrödinger Potentials Derived from Them

The general form of the Heun-type differential equations can be written as

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2}+Q(z)\frac{\mathrm{d}F}{\mathrm{d}z}+R(z)F(z)=0 \tag{2}$$

where the actual forms of the $Q(z)$ and $R(z)$ functions are displayed in Table 1. Here, the notation used generally in the literature [22,33] is modified in order to avoid confusing situations, which arise due to the simultaneous discussion of several differential equations, and also in order to allow the discussion of special limiting cases.

Table 1. Expressions for the key quantities for the Heun (H), confluent Heun (Hc), biconfluent Heun (Hb), double confluent Heun (Hd), and triconfluent Heun (Ht) differential equations. In the first column, the symbolic form of the actual function is displayed, while the second and third columns contain the $Q(z)$ and $R(z)$ functions. In the fourth column, an exponential expression is presented, which will be useful later on. The fifth column indicates whether the given equation can be reduced to the hypergeometric (${}_2F_1$) or the confluent hypergeometric (${}_1F_1$) differential equation.

$F(z)$	$Q(z)$	$R(z)$	$\exp\left(\frac{1}{2}\int^z Q(z)\mathrm{d}z\right)$	Spec. Limit
$H(\alpha,\beta,\gamma,\delta,d,q;z)$	$\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-d}$	$\frac{\alpha\beta z-q}{z(z-1)(z-d)}$	$z^{\gamma/2}(z-1)^{\delta/2}(z-d)^{\epsilon/2}$	${}_2F_1$
$Hc(\alpha,\beta,\gamma,\delta,\sigma;z)$	$\alpha+\frac{\gamma}{z}+\frac{\delta}{z-1}$	$\frac{\alpha\beta z-\sigma}{z(z-1)}$	$z^{\gamma/2}(z-1)^{\delta/2}\exp(\alpha z/2)$	${}_2F_1$, ${}_1F_1$
$Hb(\alpha,\gamma,\delta,\epsilon,q;z)$	$\frac{\gamma}{z}+\delta+\epsilon z$	$\alpha-\frac{q}{z}$	$z^{\gamma/2}\exp\left(\delta z/2+\epsilon z^2/4\right)$	${}_1F_1$
$Hd(\alpha,\gamma,\delta,\epsilon,q;z)$	$\frac{\gamma}{z^2}+\frac{\delta}{z}+\epsilon$	$\frac{\alpha}{z}-\frac{q}{z^2}$	$z^{\delta/2}\exp(-\gamma/(2z)+z\epsilon/2)$	${}_1F_1$
$Ht(\alpha,\gamma,\delta,\epsilon,q;z)$	$\gamma+\delta z+\epsilon z^2$	$\alpha z-q$	$\exp\left(\gamma z/2+\delta z^2/4+\epsilon z^3/6\right)$	

The three regular singularities of the general Heun equation could appear at three arbitrary finite values, but one can replace these with $z=0,1$, and $d\neq 0$, $d\neq 1$ without the loss of generality. Its solutions are written in terms of the six-parameter $H(\alpha,\beta,\gamma,\delta,d,q;z)$ functions [22]. Note that $Q(z)$ and $R(z)$ depend on seven parameters; however, only the six parameters appearing in the argument of $H(\alpha,\beta,\gamma,\delta,d,q;z)$ are independent, because the first four parameters determine ϵ due to the prescription

$$\gamma+\delta+\epsilon=\alpha+\beta+1\,. \tag{3}$$

The Heun equation reduces to the hypergeometric differential equation with the solutions ${}_2F_1(a,b;c;z)$ [2] for $\alpha=a$, $\beta=b$, $\gamma=c$, $\delta=a+b+1-c$, $\epsilon=0$, $d=0$l, and $q=0$ (which recovers Equation (3)).

The confluent Heun equation has regular singularities at $z=0$ and $z=1$ and an irregular singularity of rank 1 at infinity [22,33]. The regular singularities could be moved to arbitrary finite z values, but this essentially means only a reparametrization of the problem. Its solutions are formally written in terms of the $Hc(\alpha,\beta,\gamma,\delta,\sigma;z)$ functions that depend the five parameters appearing in Equation (8) [22]. It is known that special forms of this equation coincide with the hypergeometric and confluent hypergeometric differential equation; i.e., one obtains ${}_2F_1(a,b;c;z)$ and ${}_1F_1(a;c;z)$ [2] with $\alpha=0$, $\gamma=c$, $\delta=a+b+1-c$, $\sigma=-ab$; and $\alpha=-1$, $\gamma=c$, $\delta=0$, $\beta=a$, $\sigma=-a$, respectively.

The biconfluent Heun equation has one regular singularity and an irregular singularity of rank 2 [22,33], the former placed conventionally at $z=0$ and the latter at infinity. Its solutions $Hb(\alpha,\gamma,\delta,\epsilon,q;z)$ depend on five parameters [22] and reduce to the confluent hypergeometric function for $\alpha=0$, $\gamma=c$, $\delta=-1$, $\epsilon=0$, and $q=a$.

The double confluent Heun equation has two rank 1 irregular singularities at $z=0$ and at infinity. With a scaling transformation, its five-parameter version can be brought to a four-parameter form with $\epsilon=1$. However, its reduction to the confluent hypergeometric equation is possible only for the five-parameter case: $\alpha=0$, $\gamma=c$, $\delta=a+b+1-c$, $\sigma=-ab$; and $\alpha=-1$, $\gamma=0$, $\delta=c$, $\epsilon=-1$, $q=0$.

The triconfluent Heun equation has one irregular singularity of rank 3 at infinity. In contrast with the other Heun-type equations, this one cannot be reduced to the confluent hypergeometric differential equation.

The transformation of these equations to the one-dimensional Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+(E-V(x))\psi(x)=0 \tag{4}$$

(in units of $2m = 1$ and $\hbar = 1$) can be performed by standard procedures involving variable transformations $z(x)$ (see, e.g., Refs. [12,13] and references). With the choice of $z(x)$, one has to take into consideration the domain of definition of the special function $F(z)$ and that of the Schrödinger equation, which can be defined on the full x axis: $x \in (-\infty, \infty)$, on the positive semi-axis: $x \in [0, \infty)$ or on a finite domain of it: $x \in [a, b]$, depending on the nature of the actual physical problem.

Assuming that the solution of the Schrödinger equation is written in the form

$$\psi(x) = f(x)F(z(x)) \,, \tag{5}$$

one can determine

$$\begin{aligned} E - V(x) \quad &= \quad \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ &\quad + (z'(x))^2 \left[R(z(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z} - \frac{1}{4}Q^2(z(x))\right], \end{aligned} \tag{6}$$

and the solutions of the Schrödinger equation

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{z(x)} Q(z)\mathrm{d}z\right) F(z(x)) \,. \tag{7}$$

The integral expression appearing in Equation (7) is displayed in Table 1 for each Heun-type equation. The structure of the coordinate-dependent (potential) terms in Equation (6) is generally instructive in choosing suitable $z(x)$ variable transformation functions (see, e.g., Ref. [13]).

This transformation procedure has been applied systematically to generate exactly solvable potentials from the hypergeometric and confluent hypergeometric functions, i.e., Natanzon (confluent) potentials [4]. For reviews, see, e.g., Refs. [13–15]. It was found that their bound-state solutions are conveniently written in terms of Jacobi, generalized Laguerre, and Hermite polynomials, to which the hypergeometric and confluent hypergeometric functions reduce for a certain choice of their parameters [2]. It is thus worthwhile to study situations in which the solutions of the Heun-type equations reduce to polynomial forms. This option is especially inspiring considering that these equations also reduce to the hypergeometric and confluent hypergeometric equations for specific parameter combinations.

3. Polynomial Solutions of Heun-Type Differential Equation

In this Section, we expand the solutions in terms of a power series, and identify conditions under which the infinite series reduces to a finite polynomial form. The coefficients of the polynomial generally satisfy a (typically three-term) recurrence relation, so the solution of the problem amounts to finding solutions of the recurrence relation. In some cases, expansion coefficients can be obtained in explicit form, while generally, they can be determined by diagonalizing an $(N+1) \times (N+1)$ matrix. This usually leads to an $(N+1)$th-order algebraic equation in one of the parameters, and solving it, the coefficients of solutions, i.e., the polynomial solutions themselves, and the corresponding eigenvalues can be generated with $n \leq N$. Even in this case, exact analytic results can be obtained up to $N = 2$.

Each subsection is devoted to one of the Heun-type equations. After the presentation of the general formalism, illustrative examples are given. Previously published works with Schrödinger potentials and polynomial solutions are also cited, pointing out how they fit into the general scheme reviewed in the present work. We generally do not mention works discussing specific potentials related to Heun-type equations, unless they analyze the solutions too.

3.1. The Confluent Heun Equation

The non-symmetrical canonical form of the confluent Heun equation is

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2}+\left(\alpha+\frac{\gamma}{z}+\frac{\delta}{z-1}\right)\frac{\mathrm{d}F}{\mathrm{d}z}+\frac{\alpha\beta z-\sigma}{z(z-1)}F(z)=0\,. \tag{8}$$

Following the discussion in Ref. [34], we expand the $Hc(\alpha,\beta,\gamma,\delta,\sigma;z)$ solutions into a power series around the singular point $z=0$ as

$$Hc(\alpha,\beta,\gamma,\delta,\sigma;z)=\sum_{k=0}^{\infty}C_kz^k\,. \tag{9}$$

We note that the more general Frobenius expansion could also be used here, with a prefactor of the type z^{ω}. However, from the point of view of our goal, i.e., to find solutions of the Schrödinger Equation (4), this does not imply a generalization, because with the exception of the triconfluent Heun equation, such a factor (originating from the z^{-1} term in $Q(z)$) appears in the solutions (7), as can be seen in Table 1. From the substitution of Equation (9) into Equation (8), a three-term recursion relation follows for the C_k coefficients [22], as follows:

$$(k+1)(\gamma+k)C_{k+1}=\alpha(k-1+\beta)C_{k-1}+[k(k-1)+k(\gamma+\delta-\alpha)-\sigma]C_k\,. \tag{10}$$

The boundary conditions $C_{-1}=0$ and C_{-2} are prescribed here.

Assuming that $\gamma\neq-N$ and $\gamma\neq-N-1$, the termination of the recursion is achieved at $k=N$ provided that the following conditions apply:

$$0=(N+1)(\gamma+N)C_{N+1}^{(N)}=\alpha(N-1+\beta)C_{N-1}^{(N)}+[N(N-1)+N(\gamma+\delta-\alpha)-\sigma]C_N^{(N)}\,, \tag{11}$$

$$0=(N+2)(\gamma+N+1)C_{N+2}^{(N)}=\alpha(N+\beta)C_N^{(N)}+[(N+1)N+(N+1)(\gamma+\delta-\alpha)-\sigma]C_{N+1}^{(N)} \tag{12}$$

The first Equation (11) defines a relation between $C_{N-1}^{(N)}$ and $C_N^{(N)}$ that sets $C_{N+1}^{(N)}=0$. With this, and prescribing $\beta=-N$, Equation (12) is also satisfied, i.e., $C_{N+2}^{(N)}=0$, so the termination of the series is reached. Provided that $\alpha\neq0$, the conditions can be summarized as follows:

$$\begin{aligned}\beta&=-N,\\ C_{N-1}^{(N)}&=\tfrac{1}{\alpha}[N(N-1)+N(\gamma+\delta-\alpha)-\sigma]C_N^{(N)}\,.\end{aligned} \tag{13}$$

Under these conditions, the confluent Heun function reduces to the following polynomial form:

$$Hc(\alpha,\beta=-N,\gamma,\delta,\sigma;z)=\sum_{k=0}^{N}C_k^{(N)}z^k\,. \tag{14}$$

It has to be stressed that a constant function, i.e., a polynomial of the order $N=0$, cannot be the solution of Equation (8) unless $R(z)=0$ holds for $N=0$. Contrary to the case of the classical orthogonal polynomials, the Jacobi, generalized Laguerre, and Hermite polynomials [2], this requirement is not fulfilled automatically. It holds only if $\sigma=0$ is valid for $N=0$.

A constructive method has been introduced in Ref. [34] to determine $C_k^{(N)}$ coefficients that satisfy the recursion relation (10), as well as the conditions for the termination. Four different solutions have been found. One of them corresponded to the following relation for the parameters:

$$\gamma = \alpha + 1\,, \qquad \delta = -2\,, \qquad \sigma = (2-N)\alpha \tag{15}$$

and resulted in the explicit form

$$C_k^{(N)} = \frac{\alpha^{k-1}(N-1)!\alpha!}{k!(N-k)!(\alpha+k)!}[\alpha(N-2k) - k(k-1)]\,. \tag{16}$$

This $C_k^{(N)}$ is normalized such that $C_0^{(N)} = 1$ holds.

It can be seen that, for positive α values, $\alpha(N-2k) - k(k-1)$ determines the sign of $C_k^{(N)}$. This expression is $\alpha N > 0$ for $k = 0$, while it is $-\alpha N - N(N-1) < 0$ for $k = N$, while it is a monotonously decreasing function of k. This implies that $C_k^{(N)}$ changes sign exactly once as k proceeds from 0 to N, so the polynomial has one zero on the positive real z axis.

The actual form of the confluent Heun equation with the parameter set (15) is

$$z(z-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + [\alpha z(z-1) + (\alpha+1)(z-1) - 2z]\frac{\mathrm{d}F}{\mathrm{d}z} + \alpha(-Nz + N - 2)F(z) = 0\,. \tag{17}$$

This equation depends on a single parameter, α, and it also depends on N, which is the order of the polynomial solution.

Applying the variable transformation $z = -y/\alpha$, one is led to the differential equation

$$-y(y+\alpha)\frac{\mathrm{d}^2F}{\mathrm{d}y^2} + (y-\alpha)(y+\alpha+1)\frac{\mathrm{d}F}{\mathrm{d}y} + [-Ny + \alpha(2-N)]F(z) = 0\,, \tag{18}$$

which can be recognized as the differential equation of the X_1-type exceptional Laguerre polynomials $F(y) = \hat{L}_N^{(\alpha)}(y)$ [19,35]. This means that, up to a normalization constant, these exceptional polynomials represent a special case of the confluent Heun function [34]

$$\hat{L}_N^{(\alpha)}(y) = Hc(\alpha, \beta = -N, \gamma = \alpha+1, \delta = -2, \sigma = \alpha(2-N); z = -y/\alpha)\,. \tag{19}$$

It is known that the sequence of the X_1-type exceptional Laguerre polynomials starts with $N = 1$ [19,35], a finding that has a natural explanation in the framework based on the confluent Heun equation. It is also known that one of its zeros falls into the negative domain $y < 0$. This result is also explained by the present scheme: the variable transformation modifies the coefficients of the polynomials by a factor of $(-1)^k$, implying that the coefficients follow an alternating sequence, except for one step, i.e., when the expression $\alpha(N-2k) - k(k-1)$ changes signs. The coefficients of the X_1-type exceptional Laguerre polynomials are related to Equation (16) as

$$\begin{aligned} \hat{C}_k^{(N)} &= \tfrac{(-1)^k(\alpha+N-2)!(\alpha+N)}{k!(N-k)!(\alpha+k)!}[\alpha(2k-N) + k(k-1)] \\ &= \tfrac{(-1)^{k+1}(\alpha+N)!}{\alpha^{k-1}\alpha!(N-1)!(\alpha+N-1)}C_k^{(N)}\,. \end{aligned} \tag{20}$$

The extra terms in (20) are due to the factor introduced in the variable transformation and to the different normalization used in the two cases. The (20) coefficients can also be obtained [34] from those of the generalized Laguerre polynomials [2], taking into account the relation between these polynomials and the X_1-type exceptional Laguerre polynomials [17], as follows:

$$\hat{L}_N^{(\alpha)}(y) = -(y+\alpha+1)L_{N-1}^{(\alpha)}(y) + L_{N-2}^{(\alpha)}(y)\,. \tag{21}$$

Polynomial solutions of the confluent Heun equation have been discussed in Ref. [22]; however, they are based on the solution of $(N+1)$th-order algebraic equations, which follow from the diagonalization of an $(N+1)\times(N+1)$ tridiagonal matrix derived from the recursion relation. The connection to the generalized Laguerre polynomials through the X_1-type exceptional Laguerre polynomials, which allow explicit construction of the $C_k^{(N)}$ coefficients, was not known at that time.

It can be noted that the systematic constructive approach in Ref. [34] also recovered the generalized Laguerre polynomials as a special polynomial solution of the confluent Heun equation, as well as two further polynomial systems. The latter two examples had features typical for semi-classical orthogonal polynomials [36]: their polynomial order N also appeared in the first-order derivative term of the differential equation $(Q(y))$.

The study in Ref. [34] was also extended to the investigation of variable transformations by which the confluent Heun equation can be transformed into the Schrödinger equation with exactly solvable quantum potentials. It was found that, among the possible variable transformations, the most natural choice is considering

$$y(x)=\frac{\omega}{2}x^2,\qquad \alpha=l+1/2\,, \tag{22}$$

which recovers the rationally extended harmonic oscillator [19,35], as follows:

$$V(x)=\frac{\omega^2}{4}x^2+\frac{l(l+1)}{x^2}+\frac{4\omega}{\omega x^2+2l+1}-\frac{8\omega(2l+1)}{(\omega x^2+2l+1)^2}\,, \tag{23}$$

$$E_N=\omega(2N+l-1/2)$$

$$\psi_N(x)\sim\frac{x^{l+1}}{\omega x^2+2l+1}\exp(-\frac{\omega}{4}x^2)\hat{L}_N^{(l+1/2)}(\frac{\omega}{2}x^2)\,.$$

Here, N corresponds to $N=\nu+1$ in the notation of Ref. [19], where $\nu=0,\ 1,\ldots$ labels the actual degree of the exceptional Laguerre polynomial. This potential can also be obtained from the conventional harmonic oscillator by a supersymmetric transformation with broken supersymmetry, and also fulfills the criteria for shape invariance [11]. For a pedagogical review, see Ref. [37]. It may be mentioned that potential (23) was derived [38] by SUSY transformations from the radial harmonic oscillator a decade before the concept of rationally extended potentials was introduced. However, its importance as a new shape-invariant potential class was not realized at that time.

In another study [39], potentials were derived from the symmetrical canonical form of the confluent Heun equation. This choice was more suited to the formalism of $\mathcal{PT}$-symmetric quantum mechanics, because the $y(x)$ functions had definite parity with respect to the space reflection operator $\mathcal{P}$, which facilitated identifying the parity properties of the potentials and the solutions. General expressions have been derived for five different variable transformations to generate five different potentials, and conditions under which the potentials admitted $\mathcal{PT}$ symmetry have been identified. However, the solutions were left in their general (non-polynomial) form.

Finally, we note that a systematic study of generating solvable Schrödinger potentials from the confluent Heun equation has been presented in Ref. [27], although without discussing the bound-state solutions in detail. Nine independent potentials arising from nine variable transformation functions $z(x)$ have been identified, among them those discussed in the earlier work [24]. Six of the nine potentials were found to be generalizations of Natanzon (confluent) class potentials. It can be shown that the one denoted by $(m_1,m_2)=(1/2,0)$ can be identified with the rationally extended harmonic oscillator discussed above. The potential with the terms $\cosh^4(ax)$, $\cosh^2(ax)$, and $\cosh^{-2}(ax)$ appearing in Equation (1.3.33) in Ref. [32] and its analogous trigonometric version with the terms $\cos^4(ax)$, $\cos^2(ax)$, $\tan^2(ax)$ and $\cot^2 ax$ from Equation (3) of Ref. [40] correspond to the potential denoted with $(1/2, 1/2)$ in [27]. (The first potential misses the fourth $\sinh^{-2}$ term of the general expression, while the last two terms in the latter potential can be written into the expected $\cos^{-2}(ax)$ and $\sin^{-2}(ax)$ terms plus a constant). Similarly, potential (1.3.41) in Ref. [32] and potential (4) in Ref. [40] with the terms $\cosh^{-6}(ax)$, $\cosh^{-4}(ax)$, and $\cosh^{-2}(ax)$ correspond to a limited three-term hyperbolic version of the potential denoted with $(1, 1/2)$ in [27]. These potentials, similar to two other ones appearing in Equations (2) and (5) in Ref. [40], which are trivial transforms or special cases of the two mentioned potentials, have been obtained within the quasi-exactly solvable (QES) setting [31]. Furthermore, the periodic $\mathcal{PT}$-symmetric potential $V(x) = -[\mathrm{i}a\sin(bx) + c]^2$ in Ref. [41] can also be rewritten into another form containing the $\cos^4(bx/2)$ and $\cos^2(bx/2)$ terms.

3.2. The Heun Equation

The Heun equation is written as [22,33]

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-d}\right)\frac{\mathrm{d}F}{\mathrm{d}z} + \frac{ABz - q}{z(z-1)(z-d)}F(z) = 0\,, \tag{24}$$

where we changed the notation used in Table 1 ($\alpha \to A$ and $\beta \to B$) in order to avoid confusion in the later stages of the discussion.

Let us expand the Heun function into a power series around the singular point $z = 1$, as follows:

$$H(A, B, \gamma, \delta, d, q; z) = \sum_{k=0}^{\infty} C_k \left(\frac{z-1}{2}\right)^k\,, \tag{25}$$

where the factor of $1/2$ was introduced for practical reasons. Substituting this expansion into the Heun equation and eliminating ϵ using Equation (3), once again, obtains a three-term recursion relation, as follows:

$$\begin{aligned}(k+1)(\gamma+k)(d-1)C_{k+1} &= [k(k-1)(3-d) + k(\gamma(1-d) - \delta(1+d) + 2A + 2B + 2) + AB - q]C_k \\ &\quad +2[(k-1)(k-2) + (k-1)(A+B+1) + AB]C_{k-1}\,.\end{aligned} \tag{26}$$

The boundary conditions $C_{-1} = 0$ and $C_{-2} = 0$ are prescribed here too. Assuming again that $\gamma \neq -N$ and $\gamma \neq -N-1$, the conditions for termination at $k = N$ are as follows:

$$\begin{aligned}&(N+1)(\gamma+N)(d-1)C_{N+1} \\ = \; &[N(N-1)(3-d) + N(\gamma(1-d) - \delta(1+d) + 2A + 2B + 2) + AB - q]C_N \\ &+2[(N-1)(N-2) + (N-1)(A+B+1) + AB]C_{N-1}\end{aligned} \tag{27}$$

and

$$\begin{aligned}&(N+2)(\gamma+N+1)(d-1)C_{N+2} \\ = \; &[N(N+1)(3-d) + (N+1)(\gamma(1-d) - \delta(1+d) + 2A + 2B + 2) + AB - q]C_{N+1} \\ &+2[N(N-1) + N(A+B+1) + AB]C_N\end{aligned} \tag{28}$$

If the right-hand side of Equation (27) is zero, i.e., if C_N is proportional to C_{N-1}, then $C_{N+1} = 0$ has to hold. Furthermore, if the coefficient of C_N is zero in Equation (28), then $C_{N+2} = 0$ also holds; i.e., the termination of the recursion is achieved. The latter condition is

$$AB = -N(N+A+B)\,. \tag{29}$$

This relation is valid for either $A = -N$ or $B = -N$. Note that the situation is analogous to the condition under which the hypergeometric function ${}_2F_1(a, b; c; z)$ reduces to a polynomial: there also, a and b are interchangeable, and either of them can be chosen as $-N$. The conditions for termination are the following:

$$\begin{aligned} A &= -N\,, \\ C_{N-1}^{(N)} &= \tfrac{1}{2(N-B-1)}\left[N^2(1-d)+N(\gamma(1-d)-\delta(1+d)+B-1)-q\right]C_N^{(N)}\,. \end{aligned} \tag{30}$$

The polynomial solutions of the Heun equation are then written as

$$H(A=-N,B,\gamma,\delta,d,q;z)=\sum_{k=0}^{N}C_k^{(N)}\left(\frac{z-1}{2}\right)^k\,. \tag{31}$$

As it has been mentioned earlier, these equations are equivalent with those obtained from the $A \leftrightarrow B$ replacement.

The Heun equation can be matched with the differential equation of the X_1-type exceptional Jacobi polynomials $\hat{P}_N^{(\alpha,\beta)}(z)$ [19,35]. These polynomials are defined for $\alpha \neq \beta$, and they satisfy the differential equation

$$(z^2-1)\frac{\mathrm{d}^2F}{\mathrm{d}y^2}+2a\frac{1-dz}{d-z}(z-c)\frac{\mathrm{d}F}{\mathrm{d}y}+\left[-2a\frac{1-dz}{d-z}-(N-1)(\alpha+\beta+N)\right]F(z)=0\,, \tag{32}$$

where

$$a=\frac{1}{2}(\beta-\alpha),\qquad d=\frac{\beta+\alpha}{\beta-\alpha},\qquad c=\frac{\beta+\alpha+2}{\beta-\alpha}\,. \tag{33}$$

It can be shown that with the substitutions

$$\begin{gathered} A=-N,\quad B=N-1+\alpha+\beta,\quad \gamma=\alpha+1\quad \delta=\beta+1,\quad \epsilon=-2,\quad d=\tfrac{\beta+\alpha}{\beta-\alpha} \\ q=-\tfrac{1}{\beta-\alpha}\left[N^2(\alpha+\beta)+N(\alpha+\beta)(\alpha+\beta-1)-4\alpha\beta\right] \end{gathered} \tag{34}$$

the Heun equation takes the form (32). This also means that up to a constant normalization factor,

$$\hat{P}_N^{(\alpha,\beta)}(z)=H(A=-N,B=N-1+\alpha+\beta,\gamma=\alpha+1,\delta=\beta+1,d,q;z) \tag{35}$$

holds, where b and q are related to α and β as in Equation (34). The relation between the solutions of the Heun equation, the X_1-type exceptional Jacobi polynomials, and the generalized hypergeometric functions has been discussed in Ref. [42].

Since the X_1-type exceptional Jacobi polynomials can be expressed in terms of classical Jacobi polynomials, the $C_k^{(N)}$ expansion coefficient appearing in Equation (31) can also be expressed in terms of the expansion coefficients of the classical Jacobi polynomials [17].

$$\begin{aligned} \hat{P}_N^{(\alpha,\beta)}(z) &= -f_N P_N^{(\alpha,\beta)}(z)+2dg_N P_{N-1}^{(\alpha,\beta)}(z)-h_N P_{N-2}^{(\alpha,\beta)}(z) \\ f_N &= \tfrac{N(\alpha+\beta+N)}{(\alpha+\beta+2N-1)(\alpha+\beta+2N)} \\ g_N &= \tfrac{(\alpha+N)(\beta+N)}{(\alpha+\beta+2N-2)(\alpha+\beta+2N)} \\ h_N &= \tfrac{(\alpha+N)(\beta+N)}{(\alpha+\beta+2N-2)(\alpha+\beta+2N-1)}\,. \end{aligned} \tag{36}$$

The classical Jacobi polynomials are expressed in terms of the expansion

$$P_N^{(\alpha,\beta)}(z)=\frac{\Gamma(\alpha+N+1)}{n!\Gamma(\alpha+\beta+N+1)}\sum_{k=0}^{N}\binom{n}{k}\frac{\Gamma(\alpha+\beta+N+k+1)}{\Gamma(\alpha+k+1)}\left(\frac{z-1}{2}\right)^k\,. \tag{37}$$

Since the solutions of the Heun equation have been expanded in the same form in Equation (25), the expansion coefficients will be obtained from the combination of the expansion coefficients in Equation (37) with f_N, g_N and h_N in Equation (36).

Polynomial solutions of the Heun equation have also been discussed in Ref. [22], where conditions for the termination of the infinite power series have also been analyzed. However, there, the connection with the Jacobi polynomials via the X_1-type exceptional Jacobi polynomials was not mentioned, which is understandable, as the latter were introduced later [17].

The zeros of the X_1-type exceptional Jacobi polynomials are known to fall within the domain $z \in [-1, 1]$ with the exception of one. The first member of the sequence $\hat{P}_1^{(\alpha,\beta)}(z)$ is a first-order polynomial, similar to the case of the X_1-type exceptional Laguerre polynomials.

There are several exactly solvable potentials with bound-state solutions containing X_1-type exceptional Jacobi polynomials. These can be obtained from variable transformations governed by some $z(x)$ functions. All of them are rational extensions of some well-known shape-invariant potentials.

One is the rational extension of the Scarf I potential [19]

$$\begin{aligned}\hat{V}(x) \quad = \quad & \frac{a^2}{\sin^2(ax)}\left[\left(\frac{\alpha+\beta}{2}\right)^2+\left(\frac{\alpha-\beta}{2}\right)^2-\frac{1}{4}\right] \\ & +\frac{2a^2\cos(ax)}{\sin^2(x)}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) \\ & +\frac{2a^2(\alpha+\beta)}{(\alpha-\beta)\cos(ax)+\alpha+\beta}-\frac{2a^2\left[(\alpha+\beta)^2-(\alpha-\beta)^2\right]}{[(\alpha-\beta)\cos(ax)+\alpha+\beta]^2}\,,\end{aligned} \tag{38}$$

which is obtained for $z(x) = \cos(ax)$. (Note that, here, a is a simple parameter that scales the coordinate, and is different from the a appearing in Equation (33).) The bound-state energy eigenvalues are

$$\hat{E}_N = a^2\left(N - 1 + \frac{\alpha+\beta+1}{2}\right)^2, \tag{39}$$

while the corresponding wavefunctions are

$$\hat{\psi}_N(\alpha,\beta;x) = \hat{C}_n^{(\alpha,\beta)}(1-\cos(ax)-1)^{\frac{\alpha}{2}+\frac{1}{4}}(1+\cos(ax))^{\frac{\beta}{2}+\frac{1}{4}}[(\alpha-\beta)\cos(ax)+\alpha+\beta]^{-1}\hat{P}_N^{(\alpha,\beta)}(\cos(x))\,. \tag{40}$$

Another choice is $z(x) = \cosh(ax)$, which results in the rational extension of the generalized Pöschl–Teller potential [43]

$$\begin{aligned}\hat{V}(x) \quad = \quad & \frac{a^2}{\sinh^2(ax)}\left[\left(\frac{\alpha+\beta}{2}\right)^2+\left(\frac{\alpha-\beta}{2}\right)^2-\frac{1}{4}\right] \\ & +\frac{2a^2\cosh(ax)}{\sinh^2(x)}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) \\ & -\frac{2a^2(\alpha+\beta)}{(\alpha-\beta)\cosh(ax)+\alpha+\beta}-\frac{2a^2\left[(\alpha+\beta)^2-(\alpha-\beta)^2\right]}{[(\alpha-\beta)\cosh(ax)+\alpha+\beta]^2}\,.\end{aligned} \tag{41}$$

The bound-state eigenfunctions are

$$\hat{\psi}_N(\alpha,\beta;x) = \hat{C}_N^{(\alpha,\beta)}(\cosh(ax)-1)^{\frac{\alpha}{2}+\frac{1}{4}}(\cosh(ax)+1)^{\frac{\beta}{2}+\frac{1}{4}}[(\alpha-\beta)\cosh(ax)+\alpha+\beta]^{-1}\hat{P}_N^{(\alpha,\beta)}(\cosh(x))\,, \tag{42}$$

while the energy eigenvalues take the form

$$\hat{E}_N = -a^2\left(N - 1 + \frac{\alpha+\beta+1}{2}\right)^2. \tag{43}$$

In both examples, the parametrization guarantees that the singularity of the potential appears outside the domain of definition of the potentials ($[0, \pi/a]$ and $[0, \infty)$ in the two cases. There is also a third example with $z(x) = \mathrm{i}\sinh(ax)$, resulting in the rationally extended Scarf II Potential [43], which is normally defined on the full real x axis. In order to avoid singularities, this potential is defined only in the $\mathcal{PT}$-symmetric setting, where singularities on the real x axis can be avoided by applying an imaginary coordinate shift $x \to x + \mathrm{i}d$. All three examples are discussed in Ref. [44] in the context of supersymmetric transformations connecting the Scarf I, generalized Pöschl–Teller, and Scarf II potentials to their rational extensions. See Ref. [45] for a review on the subtleties of infinite and finite orthogonal polynomial systems related to Jacobi polynomials and their connection to Romanovski polynomials.

In a systematic study, solvable Schrödinger potentials have been constructed from the Heun equation [26]. Eleven independent potentials have each corresponded to a specific coordinate transformation. Four of them have already been identified in the earlier study in Ref. [24]. Nine potentials can be considered generalizations of Natanzon-class potentials. In the solutions, the Heun function is expanded in terms of hypergeometric functions. The rational extensions of Scarf I and the generalized Pöschl–Teller potentials discussed here are not mentioned, but they can be identified as that denoted with $(m_1, m_2, m_3) = (1, 1/2, 0)$ in Table 1 in Ref. [26].

3.3. The Biconfluent Heun Equation

The biconfluent Heun equation is written as

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \delta + \epsilon z\right)\frac{\mathrm{d}F}{\mathrm{d}z} + \left(\alpha - \frac{q}{z}\right)F(z) = 0 \tag{44}$$

Expanding the Hb function into a power series around the singular point $z = 0$

$$Hb(\alpha, \gamma, \delta, \epsilon, q; z) = \sum_{k=0}^{\infty} C_k z^k \tag{45}$$

and substituting it into Equation (44), again, a three-term recursion relation follows:

$$(k+1)(\gamma + k)C_{k+1} = -(\delta k - q)C_k - [\alpha + \epsilon(k-1)]C_{k-1}\,. \tag{46}$$

$C_{-1} = 0$ is prescribed in the calculations.

Assuming again that $\gamma \neq -N$ and $\gamma \neq -N-1$, the the recursion terminates at $k = N$ under the following conditions:

$$0 = (N+1)(\gamma + N)C_{N+1} = -(\delta N - q)C_N - [\alpha + \epsilon(N-1)]C_{N-1}\,, \tag{47}$$

$$0 = (N+2)(\gamma + N + 1)C_{N+2} = -[\delta(N+1) - q]C_{N+1} - [\alpha + \epsilon N]C_N\,. \tag{48}$$

These equations are satisfied if the following conditions apply:

$$\begin{aligned} \alpha &= -N\epsilon\,, \\ C_{N-1}^{(N)} &= \tfrac{1}{\epsilon}(\delta N - q)C_N^{(N)}\,. \end{aligned} \tag{49}$$

The first equation cancels the coefficient of C_N in Equation (48), while the second one (combining it with Equation (49) sets $C_{N+1}^{(N)} = 0$ in Equation (47). These two results lead to $C_{N+2}^{(N)} = 0$, and to the termination of the recursion meaning that Hb can be written into a finite, polynomial form as

$$Hb(\alpha = -N\epsilon, \gamma, \delta, \epsilon, q; z) = \sum_{k=0}^{N} C_k^{(N)} z^k\,. \tag{50}$$

This equation has to be considered together with Equation (49). In the case of the confluent Heun and the Heun equations, the analogous condition was automatically satisfied, because the solutions of the confluent Heun and the Heun equations reduced to X_1-type Laguerre and Jacobi polynomials, with coefficients satisfying the required relation, including the one analogous to Equation (49). This is not the case now, so this condition has to be enforced.

There are five different variable transformations that lead to (in principle) solvable quantum potentials from the biconfluent Heun equation [14]. When $z(r) = r^2/4$ is applied, the sextic oscillator

$$V(r) = \left(2s - \frac{1}{2}\right)\left(2s - \frac{3}{2}\right)\frac{1}{r^2} + \left[b^2 - 4a\left(s + N + \frac{1}{2}\right)\right]r^2 + 2abr^4 + r^6 \tag{51}$$

is recovered if the parameters are chosen as

$$\gamma = 2s\,, \qquad \delta = -4b\,, \qquad \epsilon = -16a\,, \tag{52}$$

while the energy eigenvalues are related to q through

$$E = 4sb - q\,. \tag{53}$$

The wave functions are written as

$$\psi(r) = Cr^{2s-1/2}\exp\left(-\frac{ar^4}{4} - \frac{br^2}{2}\right)P_N\left(\frac{r^2}{4}\right)\,, \tag{54}$$

where $P_N(z = r^2/4)$ is the polynomial solution (50). The coefficients of the polynomial are determined by the recursion (46), while the termination condition (49) amounts to solving an $(N+1)$th-order algebraic equation for q, which arises from diagonalizing an $(N + 1 \times (N+1)$ tridiagonal matrix, which, in turn, follows from the recursion (46). This potential and its solutions have been applied to describe certain collective excitations of nuclei. Closed expressions for the wavefunctions and the corresponding energy eigenvalues have been determined first for $N = 0$ and 1 [46] and then also for $N = 2$, i.e., for a cubic algebraic equation for q [47].

The sextic oscillator either as a one-dimensional problem or as a radial one, as it stands in Equation (51), is the archetype of quasi-exactly solvable (QES) potentials [31]. At the same time, it has been known for a long time [24] that it can be obtained from the biconfluent Heun equation. The equivalence of the two approaches has been presented in Ref. [48], where the solutions of the biconfluent Heun equation have been expressed in terms of Hermite functions. That approach recovered the QES results when the Hermite functions reduced to Hermite polynomials $H_N(z)$. In the present approach, the solutions are expanded in terms of polynomials of $z = r^2/4$; nevertheless, the final results are the same, as expected.

Before closing this subsection, we note that polynomial solutions of the biconfluent Heun equation (with different parametrization) have been discussed in Ref. [22] to some detail. It was established that these solutions arise if one of the parameters is equal to a non-positive integer, while another one follows from the roots of a polynomial. This is rather similar to the results obtained in Ref. [22] for the Heun and confluent Heun equations, and are practically equivalent with the formalism of quasi-exactly solvable potential models [31]. It is remarkable that these two approaches have been connected only recently [48]. Somewhat later, a further independent approach to the sextic oscillator has been presented in terms of the extended Nikiforov–Uvarov method [49].

A systematic study of solvable potentials arising from the biconfluent Heun equation has been carried out in Ref. [28]; however, the solutions there are expanded in terms of Hermite functions, rather than in power series. That expansion also terminates under certain conditions, leading to a finite sum of Hermite functions in the solutions. Several potentials are identified, depending on the number of Hermite functions involved in the sum. The five potentials found in Ref. [24] have also been recovered, including the sextic oscillator and some potentials found earlier independently [50–52], which include terms with fractional powers of x. These potentials are identified in Ref. [28] with the variable transformations $z(x) = x^2/4$, $z(x) = (2x)^{1/2}$, and $z(x) = (3x/2)^{2/3}$. The one with $z(x) = x$, containing the terms x^{-2}, x^{-1}, x, and x^2 and polynomial solutions has been discussed in Ref. [53] and in Ref. [32] (see Equation (1.3.66)). The latter work applied the quasi-exactly solvable (QES) formalism [31], with no reference to the biconfluent Heun equation, and also mentioned the fifth potential with $z(x) = e^{-x}$ and with the terms e^{-kx}, $k = 1, 2, 3,$ and 4 (see Equations (1.3.29) and (1.3.38) with the $x \to -x$ choice).

3.4. The Double Confluent Heun Equation

The double confluent Heun equation can be written in the general form

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \left(\frac{\gamma}{z^2} + \frac{\delta}{z} + \epsilon\right)\frac{\mathrm{d}F}{\mathrm{d}z} + \left(\frac{\alpha}{z} - \frac{q}{z^2}\right)F(z) = 0\,. \tag{55}$$

Following the steps analogous to those applied to the other equations, the solutions of Equation (55) can be expanded into a power series around the singular point $z = 0$ as

$$Hd(\alpha,\gamma,\delta,\epsilon,q;z) = \sum_{k=0}^{\infty} C_k z^k\,. \tag{56}$$

Substituting it into Equation (55) leads to a three-term recursion relation, as follows:

$$(k+1)\gamma C_{k+1} = -[k(k-1) + \delta k - q]C_k - [\alpha + \epsilon(k-1)]C_{k-1}\,. \tag{57}$$

The conditions for termination at $k = N$ do not require any assumptions on γ in this case:

$$(N+1)\gamma C_{N+1} = -[N(N-1) + \delta N - q]C_N - [\alpha + \epsilon(N-1)]C_{N-1}\,, \tag{58}$$

$$(N+2)\gamma C_{N+2} = -[N(N+1) + \delta(N+1) - q]C_{N+1} - [\alpha + \epsilon N]C_N\,. \tag{59}$$

The conditions for termination are satisfied if the following equations hold:

$$\begin{aligned} \alpha &= -N\epsilon\,, \\ C_{N-1}^{(N)} &= \tfrac{1}{\epsilon}(N(N-1) + \delta N - q)C_N^{(N)}\,. \end{aligned} \tag{60}$$

These equations guarantee that $C_k^{(N)} = 0$ for $k > N$, so Hd can be written into the finite, polynomial form as

$$Hd(\alpha = -N\epsilon,\gamma,\delta,\epsilon,q;z) = \sum_{k=0}^{N} C_k^{(N)} z^k\,. \tag{61}$$

This equation is formally similar to Equation (50), but they are essentially different. However, there is a similarity with the case of the biconfluent Heun equation in that the explicit forms of the $C_k^{(N)}$ coefficients cannot be expressed in closed form; rather, they have to be determined from the roots of an $(N+1)$th-order algebraic equation.

It is known [14] that there are three different variable transformations leading to relatively simple solvable quantum potentials. Selecting $z(r) = 4/r^2$ as an example, the potential

$$V(r) = \frac{\gamma^2}{16}r^2 + \left(4q + \frac{3}{4} - 2\delta + \delta^2 + 2\gamma\epsilon\right)\frac{1}{r^2} + 8\epsilon(\delta + 2N)\frac{1}{r^4} + 16\epsilon^2\frac{1}{r^6} \tag{62}$$

is obtained, and the energy eigenvalues are written as

$$E = \gamma\left(1 - \frac{\delta}{2}\right). \tag{63}$$

The wave functions take the form

$$\psi(r) = Cr^{-\delta+3/2}\exp\left(-\frac{\gamma r^2}{8} - \frac{2\epsilon}{r^2}\right)P_N(\frac{4}{r^2}) \tag{64}$$

where $P_N(z)$ is the polynomial solution (61). The coefficients of the polynomial are, again, determined by the recursion (67), with a termination condition (60), leading to an $(N+1)$th-order algebraic equation for q.

The potential (62) is a "mirror image" of the sextic oscillator in the sense that its potential terms (r^{2k}) are the inverse of those (r^{-2k}) appearing in the sextic oscillator (51). Another of the three potentials related to the double confluent Heun equation with the transformation function $z(x) = x$ and the potential terms x^k, $k = -1, -2, -3$, and -4 has been discussed in Ref. [32] in the quasi-exactly solvable (QES) setting [31], without referring to the double confluent Heun equation (see Equation (1.3.79)). In the same work, the third potential with $z(x) = e^{-x}$ and with the terms e^{-kx}, $k = -2, -1, 1$, and 2 is also mentioned (see Equation (1.3.19)).

Before closing this subsection, we mention that quasi-polynomial solutions of another parametrization of the double confluent Heun function have been studied in Ref. [22]. A more detailed comparison of the two approaches seems worthwhile, but we do not consider it here.

3.5. The Triconfluent Heun Equation

The polynomial solutions of the triconfluent Heun equation

$$\frac{d^2F}{dz^2} + \left(\gamma + \delta z + \epsilon z^2\right)\frac{dF}{dz} + (\alpha z - q)F(z) = 0 \tag{65}$$

can be written in form of the standard series expansion around $z = 0$.

$$Ht(\alpha,\gamma,\delta,\epsilon,q;z) = \sum_{k=0}^{\infty} C_k z^k \tag{66}$$

now results in a four-term recursion relation, as follows:

$$(k+1)(k+2)C_{k+2} = -(k-1)\gamma C_{k-1} - (\delta k - q)C_k - [\alpha + \epsilon(k-1)]C_{k-1}\,. \tag{67}$$

This is because the substitution of Equation (66) into Equation (65) now produces four different powers, i.e., z^j, $j = k-2$, $k-1$, k, and $k+1$. The termination of this recursion requires a more complicated set of conditions than what was found in the case of the other four equations, so we do not discuss the case of the triconfluent Heun equation any further. Note also that Equation (65) cannot be reduced to the hypergeometric or to the confluent hypergeometric differential equation, so it is not expected that the generalization of

Natanzon-class potentials can be generated from it. Polynomial solutions of the triconfluent Heun equation (with another parametrization) are briefly analyzed in Ref. [22].

4. Summary and Outlook

The present study focuses on polynomial solutions of the Heun equation and its confluent versions in order to generate exactly solvable solutions of the Schrödinger equation. It was inspired by the experience accumulated in the past century, according to which, the solution of the majority of such problems can be written in terms of the hypergeometric or confluent hypergeometric functions, which reduce to one of the classical orthogonal polynomials (Jacobi, generalized Laguerre, Hermite) for bound states. These potentials form the Natanzon (confluent) class.

The Heun equation and its confluent variants (confluent, biconfluent, double confluent, and triconfluent) offer a natural generalization of Natanzon-class potentials, as some of these differential equations reduce to the hypergeometric and confluent hypergeometric differential equation for specific choices of their parameters, so the potentials derived from them may contain Natanzon-class potentials as special cases. However, there is a serious problem: in contrast with the hypergeometric and related functions, the theory of which has been well established since the 19th century, the solution of the Heun equation and its confluent variants is far less elaborated. This also means that, after the introduction of the Schrödinger equation a hundred years ago, the mathematical formalism was ready to be used to derive exactly solvable quantum potentials solved by the hypergeometric and confluent hypergeometric functions, but the same did not apply to Heun-type equations.

There were early efforts to write the solutions of the Heun and related differential equations as expansions in terms of power series, as well as in terms of well-known special functions [54,55]. See Refs. [28,30] for more recent developments for expansions in terms of hypergeometric and Hermite functions [28], for example. A general pattern of solution is to derive recurrence relations for the expansion coefficients and to search for conditions under which the recurrence could be terminated. With this, a finite number of solutions (N) can be generated in a polynomial form, or as a finite linear combination of special functions. The expansion coefficients can be determined from the requirement that an $(N+1)\times(N+1)$ determinant vanishes, which typically leads to an Nth-order algebraic equation in one of the parameters.

These solutions were used to generate exactly solvable Schrödinger potentials. It turned out that there were parallel efforts to find solvable potentials using different methods, not referring to the Heun equation and its confluent variants, which often led to the same potentials. One such effort was using the rational extension of Bochner-type differential equations. These recovered the rational extensions of the Jacobi and generalized Laguerre polynomials, i.e., the X_1-type Jacobi and Laguerre polynomials [17]. These functions supply the solutions to the rational extension of certain Natanzon-class potentials. Furthermore, these potentials were found to be the supersymmetric partners of these Natanzon-class potentials. It is remarkable, that the X_1-type Jacobi and Laguerre polynomials are special cases of the solutions of the Heun and the confluent Heun equation. As we have shown, they are polynomial solutions of these equations, for which the expansion coefficients can be expressed in closed form, so there is no need for solving the recurrence relation by the traditional methods. The finding that the expansion coefficients can be expressed in closed form is also related to the fact that the X_1-type Jacobi and Laguerre polynomials can be expressed in terms of Jacobi and generalized Laguerre polynomials. This, in turn, is ultimately the consequence of supersymmetric quantum mechanics, or interpreting it in a wider context, that of the Darboux transformation.

The situation is different in the case of the biconfluent Heun equation. In this case, there are no classical orthogonal polynomials that can be related to the differential equation, so the strategy of finding explicit solutions of the biconfluent Heun has to rely on expanding it in terms of some functions. Hermite functions were found to be a suitable choice in this case [28]. Some of the potentials derived within this framework have also been discussed in terms of quasi-exactly solvable (QES) potentials. This was the case for the sextic oscillator, for which polynomial solutions have been found [31]. It was proven that these polynomial solutions can be obtained from the approach based on the expansion of the solutions of the biconfluent Heun equation in terms of Hermite functions: the QES solutions are recovered when the Hermite functions are reduced to Hermite polynomials. An alternative way of solving the recurrence relations is using continued fractions. See, e.g., Ref. [56] on the relation of the two methods.

Technically, the case of the double confluent Heun equation is similar, although potentials related to this equation are less widely studied. We presented here an example that can be considered the "mirror image" of the sextic oscillator in the sense that its terms are the inverse of those appearing in the latter potential.

The situation is more complex in the case of the triconfluent Heun equation. Then the recurrence relation is four-term, rather than three, so the conditions for its termination are more complicated. This equation was mentioned for the sake of giving a complete picture of Heun-type equations, without presenting any example for a solvable Schrödinger potential. Furthermore, the triconfluent Heun equation differs from the other Heun-type equations in that it cannot be reduced to the hypergeometric or to the confluent Heun differential equation, so any potential derived from it is not expected to be the generalization of a Natanzon-class potential.

Polynomial solutions of the Heun equation and its confluent variants represent a specific approach, which has both advantages and disadvantages compared with solutions making use of expansions in terms of special functions, such as hypergeometric or Hermite functions. The latter approach can also be reduced to polynomial expansions, because the named special functions also contain classical orthogonal polynomials as special cases. This was seen in the case of the biconfluent Heun equation and the sextic oscillator [48], for example, where a combination of Hermite functions was reduced to a polynomial. The X_1-type Jacobi and Laguerre polynomials can be composed as linear combinations of the corresponding classical orthogonal polynomials, so they show similarity to the solutions of the Heun and confluent Heun equation that are expanded in terms of a finite number of special functions.

One advantage of expansions in terms of special functions is that the range of (in principle) solvable potentials obtained this way is wider: they formally contain more parameters. In addition to the parameters of the transformation function $z(x)$, which set the scale and the starting point of the coordinate, and to an arbitrary finite value setting the zero point of the energy scale, the coupling coefficients of the potential terms can depend on the parameters of the actual Heun-type equation. In contrast with the latter group of parameters, the former are less important from the physical point of view. The rationally extended versions of potentials obtained from the Heun equation, (38) and (41), depend on two essential parameters, α and β, while the Heun equation depends on six independent parameters. One of these has to be chosen as a negative integer in order to obtain polynomial solutions; see Equation (35). The confluent Heun equation depends on five parameters, of which only one independent parameter, α, remains in the polynomial solution Equation (19), and another one has to be chosen as a negative integer. However, in the case of non-polynomial expansions, the formalism becomes more involved, and the determination of the energy eigenvalues and bound-state wavefunctions may ultimately require numerical techniques. In the case of polynomial expansions, these quantities can be expressed in closed analytical form. (For the sake of completeness, we mention here a special approach to polynomial solutions of the Heun equation [57]. In that work, the solutions are expressed in implicit forms containing polynomial forms of differential operators).

The results surveyed here may be useful in deriving exact solutions of further differential equations occurring in various fields of physics. One possibility is considering the generalization of the one-dimensional Schrödinger equation to position-dependent mass, $m(x)$. In this case, coordinate-dependent terms appear also in the kinetic (differential) term of the Schrödinger equation, which requires the modification of the way this equation is transformed into the differential equation of special functions (see, e.g., Ref. [58]). For a discussion on the conceptual aspects of position-dependent mass problems, see, e.g., Refs. [59,60]. The results can also be used in solving relativistic wave equations (see, e.g., Ref. [61]). The Klein–Gordon equation also contains second-order derivative terms with respect to the coordinate, but the energy E appears in a different way, which also requires the modification of the formalism. Preliminary work has been carried out in the preprint [62], where a classification of Klein–Gordon potentials solvable in terms of Heun equation has been carried out; however, explicit solutions have been given only for some special cases in terms of hypergeometric functions. The Dirac equation is a system of coupled first-order differential equations; however, second-order ordinary differential equations can be derived from it for its components. The solutions of some of the Schrödinger potentials can be adapted to this situation. Two potentials have been discussed in the preprint [63], where they have been transformed into Schrödinger potentials solvable in terms of the biconfluent and the double confluent Heun equation; however, the explicit solutions have not been presented.

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