

The $SU(2) \times SU(2)$ chiral spin model in terms of $SO(3)$ and Z_2 variables. Vortices and disorder^{*}

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We rewrite the two-dimensional $SU(2) \times SU(2)$ chiral spin model in terms of $SO(3)$ and Z_2 degrees of freedom. The transformation, which is motivated by a similar representation of the corresponding lattice gauge theory in higher dimensions, exhibits the presence of dynamical $SO(3)$ vortices and associated strings. We present arguments that (pairs of) $SO(3)$ vortices with long strings play a crucial role in disordering the spin system at arbitrarily low temperatures.

Two dimensional spin models are well known to have properties analogous to four dimensional gauge theories. For gauge theories on the lattice, a convenient framework for a systematic comparison of the $SU(N)$, $SU(N)/Z_N$ and mixed action models is provided by the rewriting of the $SU(N)$ theory in terms of $SU(N)/Z_N$ and Z_N variables. It has been known for some time that by means of such a transformation the $SU(N)$ theory can be represented as a Z_N gauge theory coupled to dynamical $SU(N)/Z_N$ monopole currents [1]. The effect of these monopoles and associated vortices on the phase diagram as determined by bulk properties, as well as on certain long-distance quantities such as the magnetic disorder parameter ('t Hooft loop) has been studied fairly extensively, mostly for $N = 2$, in a variety of situations [2,3]. More recently, this representation of the $SU(2)$ theory has been shown to be useful for addressing the problem of confinement at arbitrarily weak coupling [4]. It is well-known that the 4D Z_2 gauge theory has a weak coupling deconfined phase, so we can-

not expect the Z_2 part of the $SU(2)$ model to produce confinement in itself. However, the coupling to the $SU(2)/Z_2 \cong SO(3)$ monopoles and their Dirac sheets sufficiently disorders the Z_2 system to avoid a transition, provided monopole current correlations obey certain bounds. Thus the confinement problem is reduced to estimates on monopole expectations at large β .

The Monte Carlo study of 2D $SU(N)$ and $SU(N)/Z_N$ chiral spin models suggests a physical picture completely analogous to the results obtained for $SU(N)$ gauge theories [5,7]. In particular, $SU(N)/Z_N$ vortices, the analogues of monopoles, seem to play an important role in disordering the system. In view of these results, it is somewhat surprising that the explicit isolation of the vortices and their couplings in the partition function measure has not been performed before. In this paper we derive the representation of the partition function and the 2pt correlation function of the 2D $SU(2) \times SU(2)$ chiral spin model in terms of Z_2 and $SO(3)$ variables, and explicitly exhibit the vortices as part of the measure. Using this representation we give arguments supporting the role of $SO(3)$ vortices and their strings (the analogs of monopoles and Dirac strings in gauge theory) in disordering the system at large β .

We shall work on a finite two-dimensional square

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lattice Λ with free boundary conditions. The elementary constituents of the lattice are simplices of various dimensions and will be denoted by p (plaquettes), l (links) and s (sites). It will be convenient to use the notations of (co)homology theory. In this language abelian group valued configurations can be described by chains (a k -chain is an assignment of group elements to all k -simplices). The (exterior) differential and codifferential operators are denoted by d and δ respectively. The $SU(2) \times SU(2)$ chiral spin model is defined by the action

$$\mathcal{A} = - \sum_{l \in \Lambda} \text{tr } U_l, \quad (1)$$

where $SU(2)$ elements U_s are attached to lattice sites and $U_l = U_s^\dagger U_{s'}$ with $[ss'] = \partial l$ (∂l denotes the boundary of l). The model is analogous to an $SU(2)$ gauge theory, the only difference being that in the gauge model each corresponding object lives on simplices one dimension higher than in the spin model. The analogy makes it possible to translate the method described in [1] to the language of the spin model and rewrite the partition function in a similar fashion.

The partition function (PF) of the spin model is given by

$$Z = \prod_{s \in \Lambda} \int dU_s \exp \left[\beta \sum_{l \in \Lambda} \text{tr } U_l \right], \quad (2)$$

where dU_s is the Haar-measure on $SU(2)$ normalised to unity and the product runs over all lattice sites. Throughout this paper all group integrations will be performed using the invariant measure normalised to unity, regardless of the discrete or continuous nature of the group. The invariance of the group measure guarantees that a shift of the variables $U_s \rightarrow \gamma_s U_s$, where $\gamma_s \in Z_2$ at each site does not affect the PF. Since the PF does not depend on the γ configuration, we can integrate on all these γ variables. Thus

$$Z = \prod_{s \in \Lambda} \int dU_s \int d\gamma_s \exp \left[\beta \sum_{l \in \Lambda} (d\gamma)_l \text{tr } U_l \right], \quad (3)$$

where

$$(d\gamma)_l = \prod_{s \in \partial l} \gamma_s. \quad (4)$$

Introducing the notations

$$K(U_l) = \beta |\text{tr } U_l|, \quad \eta_l = \text{sign tr } U_l, \quad (5)$$

the PF can be written as

$$Z = \prod_{s \in \Lambda} \int dU_s \int d\gamma_s \exp \left[\sum_{l \in \Lambda} K(U_l) \eta_l (d\gamma)_l \right]. \quad (6)$$

The Z_2 -valued 1-chain, η is determined by the spin configuration $[U]$ through (5) and $\eta_l = -1$ means that the bond l is highly excited. Now it is convenient to introduce a new Z_2 -valued 1-chain σ_l with the definition

$$\sigma_l = \eta_l (d\gamma)_l. \quad (7)$$

The constraint (7) is enforced by a delta function on each link, giving

$$Z = \prod_{s \in \Lambda} \int dU_s \int d\gamma_s \prod_{l \in \Lambda} \int d\sigma_l \delta(\sigma_l^{-1} \eta_l (d\gamma)_l) \times \exp \left[\sum_{l \in \Lambda} K(U_l) \sigma_l \right]. \quad (8)$$

Since the γ variables are contained only in the delta functions, the γ -integrations can be carried out, resulting in the constraint that the 1-chain $\sigma^{-1} \eta$ is closed. In this way the PF is obtained in the following form:

$$Z = \prod_{s \in \Lambda} \int dU_s \prod_{l \in \Lambda} \int d\sigma_l \prod_{p \in \Lambda} \delta((d\sigma^{-1})_p (d\eta)_p) \times \exp \left[\sum_{l \in \Lambda} K(U_l) \sigma_l \right]. \quad (9)$$

The remarkable property of this form of the PF is that the integrand depends on the $SU(2)$ degrees of freedom U_s only through the $SU(2)/Z_2$ cosets. In other words, it has a $U_s \rightarrow -U_s$ "gauge" symmetry and effectively the spins can be regarded $SU(2)/Z_2 \cong SO(3)$ variables rather than $SU(2)$ ones. Of course, the price that is paid for this extra symmetry is the appearance of the new Z_2 degrees of freedom (σ_l) attached to links of the lattice.

There are two different couplings between the $SO(3)$ and the Z_2 degrees of freedom. Firstly, σ_l contributes an extra sign to the coupling between

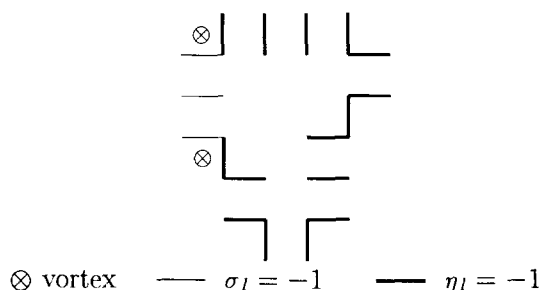


Fig. 1. Two vortices connected by a $\sigma = -1$ and an $\eta = -1$ string.

the $\text{SO}(3)$ spins residing on the two ends of the link l . Secondly, there is a coupling through the delta function constraint.

The physical meaning of the quantities $d\sigma$ and $d\eta$ appearing in the constraint is the following. If the σ -field is viewed as a \mathbb{Z}_2 gauge field, then $d\sigma$ is the associated curvature (recall that $(d\sigma)_p = \prod_{l \in \partial p} \sigma_l$). $(d\eta)_p = -1$ means that there is an odd number of “negative” ($\eta_l = -1$) links around p , i.e. the plaquette p carries an $\text{SO}(3)$ “vortex”. The “charge” of $\text{SO}(3)$ vortices in two dimensions is characterised by elements of \mathbb{Z}_2 , the fundamental group of $\text{SO}(3)$. Notice that in terms of the $\text{SO}(3)$ variables, the location of negative- η links is ambiguous. They can be moved around by changing the representative elements of cosets. On the other hand the position of vortices is gauge invariant, because a $U_s \rightarrow -U_s$ transformation flips the sign of the two η_l around the affected plaquettes. It is also clear from the construction that each $\text{SO}(3)$ vortex is attached to a string of negative- η links which terminates in another vortex or on the boundary of the lattice. These strings can be deformed by changing coset representatives but their end-points, the vortices are fixed. The situation is analogous to gauge theories, where one has Dirac-strings attached to monopoles and these strings can be deformed by gauge transformations. The delta function in (9) constrains the σ -curvature to be equal to the $\text{SO}(3)$ vortex number on each plaquette. In this way vortices are also connected by strings of negative σ links (fig. 1).

It should be noted that there is a substantial difference between σ and η strings. To see this, let us look at a closed contour of links enclosing exactly one vortex. Both the σ and the η string attached to the vor-

tex have to pierce the contour somewhere. The link l , where the σ string intersects the contour is unambiguously given by the $[\sigma]$ configuration and l carries an energy $2K(U_l)$. On the other hand, the location where the η string crosses the contour (l') has no physical meaning in terms of the $\text{SO}(3)$ variables; it can be moved by choosing different representative elements of the cosets at some sites. It follows that the energy of the η string is not necessarily localised on the $\eta_l = -1$ links. Indeed, for most of the configurations the $\text{SO}(3)$ spins change slowly along the contour. This is more favourable both in terms of energy and entropy, than having an abrupt change somewhere. The analogous mechanism in lattice gauge theories is called flux spreading, and it is believed to play an important role in confinement [6,2]. It can be seen from the above argument that the energy cost of a σ string is necessarily proportional to its length. On the other hand, many $\text{SO}(3)$ configurations can be produced for which the energy cost of an η string is constant, independent of its length. As a consequence, at low temperatures the σ string connecting two given vortices is very likely to be of minimal length, but the corresponding η string can fluctuate considerably (cp. fig. 1).

It is now instructive to perform a duality transformation on the \mathbb{Z}_2 degrees of freedom, which amounts to trading the σ_l link variables for plaquette variables $\omega_p \in \mathbb{Z}_2$. In this way the expression of the PF becomes similar to that of an Ising model on the dual lattice with fluctuating couplings. Technically the duality transformation is done by expanding the Boltzmann weights in (9) in characters of \mathbb{Z}_2 and carrying out the σ integrations. The dual form of the PF is then obtained in the form

$$Z = \prod_{s \in \Lambda} \int dU_s \exp \left[\sum_{l \in \Lambda} \widehat{M}(U_l) \right] \times \prod_{p \in \Lambda} \int d\omega_p \chi_{(d\eta)_p}(\omega_p) \exp \left[\sum_{l \in \Lambda} \widehat{K}(U_l) (\delta\omega)_l \right], \quad (10)$$

where

$$\widehat{K}(U_l) = \frac{1}{2} \ln \coth K(U_l), \quad \widehat{M}(U_l) = \frac{1}{2} \ln (\cosh K(U_l) \sinh K(U_l)), \quad (11)$$

$(\delta\omega)_l = \prod_{p \ni l} \omega_p$ and χ_q are the characters of \mathbb{Z}_2 . This form of the partition function contains \mathbb{Z}_2 spins (ω_p),

attached to plaquettes. Spins on the two plaquettes sharing the link l interact via the fluctuating coupling $K(U_l)$. The group characters couple these Z_2 spins to $SO(3)$ vortices.

Let us now consider the 2pt correlation function. We follow the same procedure as in the case of the PF, but now take periodic b.c. to within elements of Z_2 , i.e. periodic b.c. for the coset variables. Then the expectation $\langle \text{tr}(U_x^\dagger U_{x'}) \rangle$ in terms of the new variables is

$$Z \langle \text{tr}(U_x^\dagger U_{x'}) \rangle = \prod_{s \in \Lambda} \int dU_s \exp \left[\sum_{l \in \Lambda} \widehat{M}(U_l) \right] \\ \times \prod_{p \in \Lambda} \int d\omega_p \chi_{(d\eta)_p}(\omega_p) \eta_C \text{tr}(U_x^\dagger U_{x'}) \\ \times \exp \left[\sum_{l \in \Lambda} \widehat{K}(U_l) (\delta\omega)_l (-1)^{E(C)_l} \right], \quad (12)$$

where C is an arbitrary path of links connecting the sites x and x' , $\eta_C = \prod_{l \in C} \eta_l$ and $E(C)_l$ is the characteristic function of C , i.e. it is 1 if $l \in C$, 0 otherwise. In the r.h.s. of (12) the U_s 's now obey periodic, the ω_p 's free b.c. A deformation of the path C is equivalent to an irrelevant shift of the ω integration variables, so the correlation function is independent of C . It will be convenient to choose x and x' on opposite edges of the lattice along the "1" direction (figs. 2a,2b). Then by the periodic b.c. the $\text{tr}(U_x^\dagger U_{x'})$ factor on the r.h.s. reduces to unity and (12) becomes the 2D spin analog of the familiar electric flux free energy order parameter of gauge theories.

Let us now choose some fixed "background" configuration of the $SO(3)$ variables and integrate out all the Z_2 degrees of freedom. To understand the behaviour of the 2pt function, we have to describe those $SO(3)$ configurations that can give considerably different contribution to the partition function with a $\text{tr}(U_x^\dagger U_{x'})$ insertion, eq. (12), than without it, eq. (10). For a fixed $[U]$ configuration there are two places where the two integrands differ:

–There is a $(-1)^{E(C)_l}$ "twist" along the curve C , that changes the sign of the couplings between spins on different sides of C .

– η_C measures the number (mod 2) of η strings piercing C .

The effect of the twist can be compensated by flipping all the spins on one side (say above) of C . In

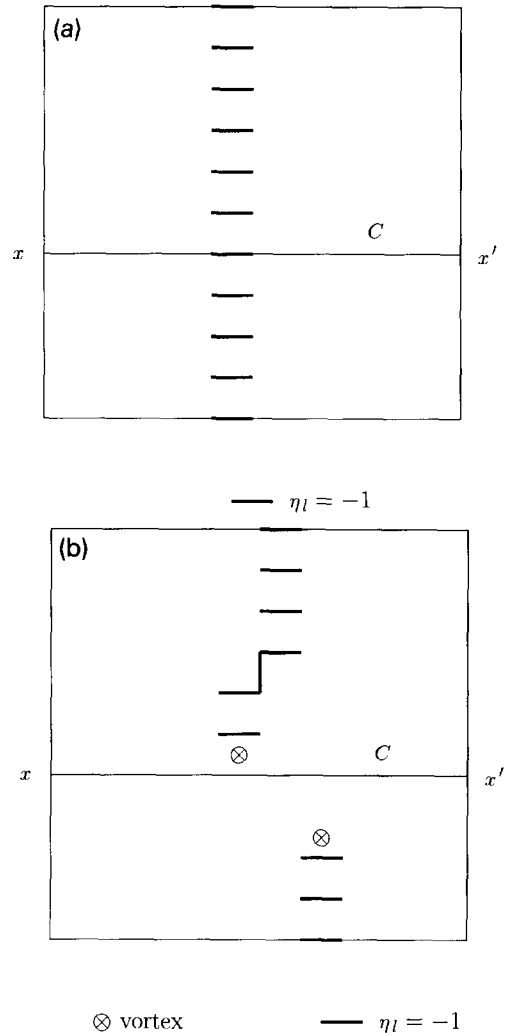


Fig. 2. (a) An $\eta = -1$ string running all the way through the lattice intersecting the path C . C is an arbitrary path connecting the points x and x' . (b) A pair of vortices residing on different sides of C with their attached strings terminating on the edge of the lattice.

other words, for a given configuration $[\omega]$ there is another $[\tilde{\omega}]$ such that the exponent without the twist evaluated at $[\omega]$ is equal to the exponent with the twist at $[\tilde{\omega}]$. Since the Z_2 spin measure is invariant, $SO(3)$ configurations with no strings piercing C and no vortices give the same contribution to (10) and (12).

Now let us look at an $SO(3)$ configuration that contains V vortices above C and S string crossing

points on C . Upon changing from $[\omega]$ to $[\tilde{\omega}]$ each vortex above C contributes an extra -1 factor (because of the $\chi_{(d\eta)_p}(\omega_p)$ factors). Due to the η_C insertion, each string crossing point on C gives an additional minus sign. The net effect will be a relative $(-1)^{V+S}$ sign between the contribution of $[\omega]$ to (10) and that of $[\tilde{\omega}]$ to (12), the moduli of these contributions, upon performing the Z_2 integrations, being equal. Notice that with an odd total number of vortices, $\prod_{p \in \Lambda} \chi_{(d\eta)_p}(\omega_p)$ is odd with respect to the global Z_2 symmetry $\omega \rightarrow -\omega$ of the rest of the integrand. This property ensures that $SO(3)$ configurations with an odd number of vortices give vanishing contribution to both the PF and the correlation function. Hence the number of vortices (mod 2) above and below C are the same and the above argument remains valid with V being the number of vortices below C . Notice also that $(-1)^{V+S}$ is not affected by any deformation of C . Whenever we lose or gain a vortex above C by a deformation of the path we also lose or gain an η string crossing point on C .

We have seen that the important $SO(3)$ configurations, the ones that "see" the two-point insertion in the partition function, are those with $V + S$ odd. There are two different types of configurations that can give an odd $V + S$:

–An η string running all the way through the lattice in the direction perpendicular to C (fig. 2a).

–A pair of vortices residing on different sides of C with their strings terminating on the edge of the lattice without piercing C (fig. 2b).

The asymptotic behaviour of the correlation function for large lattices depends upon the relative weight of these $SO(3)$ configurations. Pairs of vortices as in fig. 2b incur an energy cost proportional to their separation, because in (12) ω variables must be excited in the intervening gap to obtain a nonzero contribution; in the original variables, this reflects the necessary presence of a σ string. (Even in the $SO(3) \times SO(3)$ spin model, where in (1) the adjoint representation character is used and there is no σ string, the energy cost grows as the log of the separation.) Thus pairs of vortices tend to cluster together at large β . The crucial point, however, is that the configurations of fig. 2 contain long strings and the energetics of η strings is dominated by the $SO(3)$ part ($\bar{M}(U)$) of the action in (12). Simple semiclassical estimates indicate that if $d \leq 2$, due to flux spreading, the free energy

cost of these long η strings can remain finite in the large lattice limit, at large β . If this is the case, then the direct coupling of such η -strings to the correlation function noted above, can disorder it, even if the density of vortices the strings are attached to becomes exponentially small ($\sim \exp(-\text{const.}\beta)$) at large β . The resulting mass gap will itself be exponentially small.

The remarkable effectiveness of $SO(3)$ vortices in reducing the correlation length by many orders of magnitude was indeed noted in numerical simulations at large values of β , at which asymptotically free perturbation theory predicts enormous correlation lengths, while the vortices tend to cluster together at exponentially small densities [5]. The picture presented here provides an explanation for this phenomenon.

It is clearly worthwhile to substantiate this picture by rigorous estimates. This can be approached along the lines suggested in Ref. [4] for the closely analogous case of $SO(3)$ monopoles and Dirac sheets in 4D. We will report on such estimates elsewhere.

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