SHORT THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (PHD)

Diophantine Equations Related to Linear Recurrence Sequences

by

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The results described in the dissertation and this thesis have been published in the papers [**12, 13, 14, 15**] and accepted for publication in the papers [**16**, Mathematica Bohemica journal] and [**18**, Periodica Mathematica Hungarica journal]. Indeed, the main theme of our results is Diophantine equations involving linear recurrences sequences. We also note that we have a different result in [**17**] (that is accepted for publication in the journal "Rad HAZU, Matematičke znanosti") in which we use the frequency analysis technique to break a public key cryptosystem called ITRU, which is a variant of NTRU (N^{th} Degree Truncated Polynomial Ring) cryptosystem. However, to keep the presentation coherent, this result is not included in the dissertation and this thesis.

Introduction

This thesis has a detailed summary about the main areas that are touched by the three chapters of the dissertation. After a historical survey related to Diophantine equations, in the first chapter we mainly mention some types of Diophantine equations with their related results that appear throughout the dissertation. Then we recall some important concepts and notations related to linear recurrence sequences which we use with our main results. Furthermore, we recite some recent results related to the solutions of some Diophantine equations connected to linear recurrence sequences. Our main results are mainly described in the second and third chapters. In order to present our results, we start by recalling some standard notations, definitions and properties concerning linear recurrence sequences.

A sequence $\{G_n\}$ is called a linear recurrence relation of order k if the recurrence

$$G_{n+k} = a_1 G_{n+k-1} + a_2 G_{n+k-2} + \ldots + a_k G_n + f(n)$$

holds for all $n \geq 0$ with the coefficients $a_1, a_2, \ldots, (a_k \neq 0) \in \mathbb{C}$ and $f(n)$ a function depending on n only. If $f(n) = 0$ such a recurrence relation is called homogeneous, otherwise it is called nonhomogeneous.

For the homogeneous recurrence relation, the polynomial

$$F(X) = X^k - a_1 X^{k-1} - \ldots - a_k = \prod_{i=1}^{s} (X - \alpha_i)^{r_i} \in \mathbb{C}[X],$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $r_1, r_2, ..., r_s$ are respectively the distinct roots of $F(X)$ and their corresponding multiplicities, is called the characteristic polynomial of $\{G_n\}$. Thus, if $F(X) \in \mathbb{Z}[X]$ has k distinct roots, then there exist constants $c_1, c_2, \ldots, c_k \in \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that

$$G_n = \sum_{i=1}^{k} c_i \alpha_i^n$$

holds for all the nonnegative values of n. If $k = 3$, then the sequence is called a ternary linear recurrence sequence. Most of the well known ternary linear recurrence sequences are the Tribonacci sequence and Berstel's sequence, which are respectively defined by

$$T_0 = T_1 = 0, T_2 = 1, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n,$$
$$B_0 = B_1 = 0, B_2 = 1, \quad B_{n+3} = 2B_{n+2} - 4B_{n+1} + 4B_n,$$

for $n \geq 0$. On the other hand, if $k = 2$, then $\{G_n\}$ represents a binary recurrence sequence. In the following we recall some types of binary linear recurrence sequences with their properties. Let P and Q be nonzero relatively prime integers and $U_n = U_n(P, Q)$ and $V_n = V_n(P, Q)$ be defined by the following recurrence relations with their initials:

$$U_0 = 0, U_1 = 1, \quad U_n = PU_{n-1} - QU_{n-2} \qquad \text{for} \quad n \geq 2,$$
$$V_0 = 2, V_1 = P, \quad V_n = PV_{n-1} - QV_{n-2} \qquad \text{for} \quad n \geq 2.$$

The characteristic polynomial of the recurrences is given by

$$X^2 - PX + Q,$$

which has the roots

$$\alpha = \frac{P + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{D}}{2},$$

with $\alpha \neq \beta, \alpha + \beta = P, \alpha \cdot \beta = Q$ and $(\alpha - \beta)^2 = D$, where D is called the discriminant such that $D = P^2 - 4Q$. The sequences $\{U_n\}$ and $\{V_n\}$ are called the (first and second kind) Lucas sequences with the parameters (P, Q), respectively, and the terms of these sequences are the generalized Lucas numbers. The terms of Lucas sequences of the first and second kind satisfy the identity

$$V_n^2 = DU_n^2 + 4Q^n. \tag{1}$$

Moreover, the Lucas sequences of the first and second kind can be respectively written by the following formulas that are known as Binet's formulas:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad \text{for} \quad n \geq 0. \tag{2}$$

If the ratio $\zeta = \frac{\alpha}{\beta}$ is a root of unity, then the sequences $\{U_n\}$ and $\{V_n\}$ are said to be degenerate, and non-degenerate otherwise. Indeed, we mainly deal with non-degenerate linear recurrence sequences. Furthermore, the Lucas sequences for some values of P and Q have specific names such as the sequences of Fibonacci numbers, Pell numbers, Lucas numbers, Jacobsthal numbers and balancing numbers, which are given by

$$\begin{aligned}
&F_0 = 0, F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} && \text{for} \quad n \geq 2,\\
&P_0 = 0, P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} && \text{for} \quad n \geq 2,\\
&L_0 = 2, L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} && \text{for} \quad n \geq 2,\\
&J_0 = 0, J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} && \text{for} \quad n \geq 2,\\
&B_0 = 0, B_1 = 1, \quad B_n = 6B_{n-1} - B_{n-2} && \text{for} \quad n \geq 2,
\end{aligned}$$

respectively.

Diophantine equations related to reciprocals and repdigits with linear recurrence sequences

The aim of Chapter 2 is to study the solutions of some Diophantine equations involving reciprocals and repdigits with certain linear recurrence sequences, respectively. We first extend the result of Tengely [**37**] in which he determined all the integer solutions (n, x) with $x \geq 2$ of the equation

$$\frac{1}{U_n(P,Q)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k}.$$

In other words, we firstly determine the integral solutions (n, x) of the equation

$$\frac{1}{U_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k}, \tag{3}$$

for certain given pairs $(P_1, Q_1) \neq (P_2, Q_2)$. Here, we consider sequences with $1 \leq P \leq 3$ and $Q = \pm 1$. We also obtain the integral solutions (x, y) of the equation

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k}, \tag{4}$$

where the parameters of the Lucas sequence of the first kind represented by $1 \leq P \leq 3$ and $Q = \pm 1$, and the sequence $\{R_n\}$ is a ternary linear recurrence sequence represented by the Tribonacci sequence $\{T_n\}$ or Berstel's sequence $\{B_n\}$. Furthermore, we provide general results related to the integral solutions (x, y) of the equations

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2,Q_2)}{y^k}, \tag{5}$$

with arbitrary pairs $(P_1, Q_1) \neq (P_2, Q_2)$, and

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(a_2,a_1,a_0)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(b_2,b_1,b_0)}{y^k}, \tag{6}$$

where the triples $(a_2, a_1, a_0) \neq (b_2, b_1, b_0)$ and T_n denotes the general term of the generalized Tribonacci sequence that is given by

$$T_0(p,q,r) = T_1(p,q,r) = 0, T_2(p,q,r) = 1 \quad \text{and}$$

$$T_n(p,q,r) = pT_{n-1}(p,q,r) + qT_{n-2}(p,q,r) + rT_{n-3}(p,q,r),$$

for $n \geq 3$. Then we apply these results to completely resolve some concrete equations. Here, our main results also extend many former results obtained by e.g. Stancliff [**35**], Winans [**39**], Hudson and Winans [**19**], Long [**24**] and De Weger [**8**]. Before presenting our main results, we first define the following. Let the set S be defined as follows

$$S = \{u_1(n) = U_n(1,-1), u_2(n) = U_n(2,-1), u_3(n) = U_n(3,-1), u_4(n) = U_n(3,1)\}.$$

Moreover, in general we assume that the positive integers x, y in the investigated equations (3)–(6) satisfy the conditions of the following lemmas due to the results of Köhler [**23**]:

LEMMA. *Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $\{a_n\}$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula*

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{x^k} = \frac{a_0 x - Aa_0 + a_1}{x^2 - Ax - B}$$

holds for all complex x such that $|x|$ is larger than the absolute values of the zeros of $x^2 - Ax - B$.

LEMMA. *Let arbitrary complex numbers $A_0, A_1, \ldots, A_m$,$a_0, a_1, \ldots, a_m$ be given. Define the sequence $\{a_n\}$ by the recursion*

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \cdots + A_m a_{n-m}.$$

Then for all complex z such that $|z|$ is larger than the absolute values of all zeros of $q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m$, the formula

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{z^k} = \frac{p(z)}{q(z)}$$

holds with $p(z) = a_0 z^m + b_1 z^{m-1} + \cdots + b_m$, where $b_k = a_k - \sum_{i=0}^{k-1} A_i a_{k-1-i}$ for $1 \leq k \leq m$.

Then we prove the following theorems, that appear in the papers [**13, 14**].

THEOREM. *The equation*

$$\frac{1}{u_j(n)} = \sum_{k=1}^{\infty} \frac{u_i(k-1)}{x^k},$$

has the following solutions with $1 \leq i, j \leq 4, i \neq j$

$(i, j, n, x) \in \{(1,2,1,2), (1,2,3,3), (1,2,5,6), (1,3,1,2), (1,3,5,11), (1,3,7,35), (1,4,1,2), (1,4,5,8), (2,1,3,3), (2,1,9,7), (3,1,4,4), (3,1,14,21), (3,4,2,4), (3,4,7,21), (4,1,\{1,2\},3), (4,1,5,4), (4, 1,10,9), (4,1,11,11), (4,2,1,3), (4,2,3,4), (4,2,5,7), (4,3,1,3), (4,3,5,12), (4,3,7,36)\}$.

THEOREM. *Let $t \in \mathbb{N}$ such that $t \geq 2$. The complete list of solutions of the equation*

$$\sum_{k=1}^{\infty} \frac{u_j(k-1)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k},$$

with $u_n \in S, R_n \in \{B_n, T_n\}$ and positive integers x, y is as follows

u_n	R_n	(x,y)	u_n	R_n	(x,y)
u_1	B_n	$\{(25,9)\}$	u_1	T_n	$\{(2,2)\}$
u_2	B_n	$\{\}$	u_2	T_n	$\{(t(t^2-2)+1, t^2-1)\}$
u_3	B_n	$\{(6,3),(18,7)\}$	u_3	T_n	$\{\}$
u_4	B_n	$\{(26,9)\}$	u_4	T_n	$\{(3,2)\}$

THEOREM. *Let P_1, Q_1, P_2, Q_2 be non-zero integers such that $(P_1, Q_1) \neq (P_2, Q_2)$. If $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = d_1 d_2 \neq 0$ and $d_1 - d_2 \equiv -2P_1 \pmod 4, d_1 + d_2 \equiv -2P_2 \pmod 4$, then the positive integral solutions x, y of*

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2,Q_2)}{y^k}$$

satisfy

$$x = \frac{d_1 - d_2 + 2P_1}{4} > m(x^2 - P_1 x + Q_1),$$
$$y = \frac{d_1 + d_2 + 2P_2}{4} > m(x^2 - P_2 x + Q_2).$$

If $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = 0$ and $P_1 \equiv P_2 \pmod 2$, then the positive integral solutions x, y of

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2,Q_2)}{y^k}$$

satisfy

$$x > m(x^2 - P_1 x + Q_1), \quad y = \pm x + \frac{P_2 \mp P_1}{2} > m(x^2 - P_2 x + Q_2),$$

where $Q_2 = Q_1 + \frac{P_2^2 - P_1^2}{4}$, and $m(f) = \max\{|x| : f(x) = 0$, where $f(x)$ is a given polynomial over integers$\}$.

THEOREM. *If (x, y) is an integral solution of the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(a_2,a_1,a_0)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(b_2,b_1,b_0)}{y^k},$$

for given $(a_2, a_1, a_0) \neq (b_2, b_1, b_0)$, *then either*

$$9\left(a_2^2 - b_2^2 + 3\,a_1 - 3\,b_1\right)y + 2\,a_2^3 - 3\,a_2^2 b_2 + b_2^3 + 9\,a_1 a_2 - 9\,a_1 b_2 + 27\,a_0 - 27\,b_0 = 0$$

or in case of $|y| > B$ *we have*

$$|3x - 3y - a_2 + b_2| < C,$$

where B, C *are constants depending only on* $a_i, b_i, i = 0, 1, 2$.

As applications to the latter two theorems, we provide the following examples, that are described in [**14**].

EXAMPLE. *Let* $(P_1, Q_1) = (1, -1)$ *and* $(P_2, Q_2) = (18, 1)$, *then the solutions are as follows*

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{2^k} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{18^k} &&= 1,\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{7^k} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{20^k} &&= \frac{1}{41},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{10^k} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{22^k} &&= \frac{1}{89},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{15^k} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{26^k} &&= \frac{1}{209},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{26^k} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{36^k} &&= \frac{1}{649},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{79^k} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{88^k} &&= \frac{1}{6161}.
\end{aligned}$$

EXAMPLE. *In case of* $(P_1, Q_1) = (1, -1)$ *and* $(P_2, Q_2) = (2t + 1, t^2 + t - 1)$ *for some* $t \in \mathbb{Z}$, *we get that*

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(2t+1, t^2+t-1)}{(x+t)^k} = \frac{1}{x^2 - x - 1}$$

for $x \geq 2$.

EXAMPLE. *Consider the positive integral solutions x, y of the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-1,7,3)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(5,-5,-3)}{y^k}.$$

We obtain that the only integral solutions are given by

$$(x,y) \in \{(-1,1),(-3,3),(-2,4)\}.$$

Thus, we do not get positive integral solutions.

EXAMPLE. *Let us consider the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-4,-5,-6)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(1,8,18)}{y^k}.$$

Here, we get that the only positive solution is given by $(x,y) = (9,11)$, that is we have

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-4,-5,-6)}{9^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(1,8,18)}{11^k} = \frac{1}{1104}.$$

EXAMPLE. *Finally, we provide an example in which we obtain infinitely many solutions. Let $(a_2, a_1, a_0) = (1,6,5)$ and $(b_2, b_1, b_0) = (4,1,1)$. Indeed, the integral solutions are given by $(x,y) = (x, x+1)$ for all $x \geq 4$, that is we have*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(1,6,5)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(4,1,1)}{(x+1)^k} = \frac{1}{x^3 - x^2 - 6x - 5}, \quad x \geq 4.$$

In case of repdigits, we firstly use a direct approach to obtain a general finiteness result for the Diophantine equation

$$G_n = B \cdot \Big(\frac{g^{lm} - 1}{g^l - 1} \Big), \tag{7}$$

where n, m, g, l and B are positive integers such that $m > 1, g > 1$, l is even, $1 \leq B \leq g^l - 1$, and G_n denotes the general term of an integer linear recurrence sequence represented by $U_n(P,Q)$ and $V_n(P,Q)$, with $Q \in \{-1,1\}$. Indeed, the first finiteness result for equation (7), in case of $(G_n)_{n\geq 1}$ is an integer linear recurrence sequence and l is a positive integer, was given by Marques and Togbé [**29**] in which they used heavy computations followed by a result due to Matveev [**30**] on the lower bound on linear forms of logarithms of algebraic numbers to obtain bounds for n and m. As these bounds could be very high, they used a result due to Dujella and Pethő [**9**] on the Baker-Davenport reduction to reduce these bounds. Then they applied this result to determine all the solutions of the Diophantine equations

$$F_n = B \cdot \Big(\frac{10^{lm} - 1}{10^l - 1}\Big) \quad \text{and} \quad L_n = B \cdot \Big(\frac{10^{lm} - 1}{10^l - 1}\Big) \tag{8}$$

in positive integers m, n and l, with $m > 1, 1 \le l \le 10$ and $1 \le B \le 10^l - 1$, which are $(m, n, l) = (2, 10, 1)$ and $(m, n, l) = (2, 5, 1)$ in the Fibonacci and Lucas cases, respectively. It is clear that these equations have solutions only with $l = 1$. Here, one may ask the following natural questions:

- Is there another approach that is easier to apply to such concrete equations?
- Do the equations in (8) have solutions in any base g other than 10, say $g \ge 2$, in the case of $l = 1$?

In fact, here we answer the above questions positively. More precisely, our approach of obtaining a general finiteness result for equation (7) is mainly based on producing biquadratic elliptic curves of the following form (from combining equation (7) with identity (1)),

$$y^2 = ax^4 + bx^2 + c,$$

with integer coefficients a, b, c and discriminant

$$\Delta = 16ac(b^2 - 4ac)^2 \neq 0.$$

The finiteness of the number of the integral points on the latter curve is guaranteed by Baker's result [**4**] presented by the following theorem and its best improvement concerning the solutions of elliptic equations over $\mathbb{Q}$, that is due to Hajdu and Herendi [**11**].

THEOREM. *If the polynomial on the right of the Diophantine equation*

$$y^2 = a_0 x^n + a_1 x^{n-1} + \ldots + a_n,$$

where $n \ge 3$ *and* $a_0 \neq 0, a_1, \ldots, a_n \in \mathbb{Z}$*, possesses at least three simple zeros, then all of its solutions in integers* x, y *satisfy*

$$\max(|x|, |y|) < \exp\exp\exp\{(n^{10n} H)^{n^2}\},$$

where $H = \max_{0 \le i \le n} |a_i|$.

Also, the integral points of such curves can be determined using an algorithm implemented in Magma [**6**] as `SIntegralLjunggrenPoints()` (based on results obtained by Tzanakis [**38**]) or an algorithm described by Alekseyev and Tengely [**2**] in which they gave an algorithmic reduction of the search for integral points on such a curve by solving a finite number of Thue equations. As applications of our result, we apply our method on the sequences of Fibonacci numbers and Pell numbers that satisfy equation (7). Furthermore, with the first application we also generalize the result of Marques and Togbé in [**29**] in the case of Fibonacci numbers by determining all the solutions (n, m, g, B, l) of the equation

$$F_n = B \cdot \left(\frac{g^{lm} - 1}{g^l - 1} \right)$$

in case of $2 \leq g \leq 9$ and $l = 1$. Note that the case of Lucas numbers can be generalized similarly, therefore we omit the details of this case. More precisely, we use our approach in case where we have l is even, otherwise we follow the technique of Marques and Togbé in [**29**] of using the result of Matveev on linear forms in three logarithms and the result of Dujella and Pethő on the method of Baker-Davenport reduction. In fact, our main results here also extend other related results obtained by e.g. Luca [**25**] and Faye and Luca [**10**]. Before presenting our results, it is important to mention the following remark:

REMARK. *Since a finiteness result for equation* (7) *in case of* $G_n = U_n$ *or* $G_n = V_n$ *can be obtained in a similar way, we only present and prove this result in detail in the case of* $G_n = U_n$ *and omit the proof of the remaining case.*

Here, we prove the following theorems, that are obtained in [**16**].

THEOREM. *Let* P *and* Q *be nonzero relatively prime integers with* $Q \in \{-1, 1\}$ *and* t *be a positive integer. If* $G_n = U_n(P, Q)$ *is non-degenerate and* $l = 2t$*, then the Diophantine equation* (7) *has finitely many solutions of the form* (n, m, g, B, l)*, which can be effectively determined.*

THEOREM. *If* $G_n = F_n$*, then the Diophantine equation* (7) *has the following solutions with* $2 \leq g \leq 9, l \in \{1, 2, 4\}$ *and* $1 \leq B \leq$

$\min\{10, g^l - 1\}$.

$$(n, m, g, B, l) \in \{(4,2,2,1,1), (5,2,4,1,1), (6,2,3,2,1), (6,2,7,1,1), (7,3,3,1,1), (8,2,6,3,1), (8,3,4,1,1), (5,2,2,1,2), (8,3,2,1,2), (9,2,4,2,2), (9,2,2,2,4)\}.$$

Furthermore, suppose that $2 \le g \le 9, l = 2, 1 \le B \le \min\{5, g^l - 1\}$ *and* $G_n = P_n$*, then equation* (7) *has no more solutions other than* $(n, m, g, B, l) = (3, 2, 2, 1, 2)$.

Diophantine equations of the form $G(X,Y,Z) := AX^2 + BY^r + CZ^2$ involving linear recurrence sequences

The goal of Chapter 3 is to investigate the solutions of some Diophantine equations of the form $G(X,Y,Z) := AX^2+BY^r+CZ^2$ (that have infinitely many solutions in rational integers) from particular linear recurrence sequences for certain nonzero integers A, B, C and r. More precisely, we respectively study the solutions of such equations in case of $G(X,Y,Z) = 0$ and in case of $G(X,Y,Z) \neq 0$. First, let us consider the Diophantine equation

$$AX^2 + BY^r = C'Z^2, \tag{9}$$

where A, B, C' and r are nonzero integers such that $r > 1$. According to the following result of Beukers [**5**], equation (9) has either no solution or infinitely many relatively prime integer solutions (X, Y, Z).

THEOREM. *For any given integers* A_1, B_1, C_1, a, b, r *such that* $A_1B_1C_1 \neq 0$ *and* a, b *and* r *greater than* 1 *satisfying* $\frac{1}{a} + \frac{1}{b} + \frac{1}{r} > 1$, *equation*

$$A_1x^a + B_1y^b + C_1z^r = 0$$

has either no solution or infinitely many relatively prime integer solutions (x, y, z).

Moreover, if $a = b = 2, C_1 = -1$ and r is odd, then according to Mordell [**32**, page 111] one can obtain for the equation

$$A_1x^2 + B_1y^2 = z^r$$

the following parametrizations for its solutions by putting

$$z = A_1p^2 + B_1q^2,$$

where p and q arbitrary integers, and taking

$$x\sqrt{A_1} + y\sqrt{-B_1} = (p\sqrt{A_1} + q\sqrt{-B_1})^r,$$
$$x\sqrt{A_1} - y\sqrt{-B_1} = (p\sqrt{A_1} - q\sqrt{-B_1})^r.$$

Thus, equation (9) has infinitely many integer solutions if $B = 1$ and r is odd. Therefore, we here present a technique with which we can investigate the nontrivial integer solutions (X, Y, Z) of any equation of the form

$$AX^2 + Y^r = C'Z^2,$$

for certain nonzero integers A, C' and r with $r > 1$ being odd and $(X, Y) = (L_n, F_n)$ (or $(X, Y) = (F_n, L_n)$), where F_n and L_n denote the general terms of the sequences of Fibonacci numbers and Lucas numbers, respectively. We also remark that this technique can be applied on such equations for which they satisfy some conditions, that will be mentioned later in a procedure presented by Kedlaya in [**22**]. More precisely, we present the use of this technique for determining the solutions (X, Y, Z) of the Diophantine equation

$$7X^2 + Y^7 = Z^2, \tag{10}$$

where $(X, Y) = (L_n, F_n)$(or $(X, Y) = (F_n, L_n)$) and Z is a nonzero integer. From identity (1) (in case of $U_n(1, -1) = F_n$ and $V_n(1, -1) = L_n$), this technique shows that the solutions of equation (10) are equivalent to the solutions of the systems

$$x^2 - 5y^2 = \pm 4, \qquad 7x^2 + y^7 = z^2,$$
$$x^2 - 5y^2 = \pm 4, \qquad x^7 + 7y^2 = z^2,$$

where $x = L_n$, $y = F_n$ and $z = Z$ is a nonzero integer. More generally, a few techniques for investigating the integer solutions of certain systems of Diophantine equations of the form

$$x^2 - ay^2 = b, \qquad P(x, y) = z^2, \tag{11}$$

where a is a positive integer that is not a perfect square, b is a nonzero integer and $P(x, y)$ is a polynomial with integer coefficients, have been used by several authors such as Cohn [**7**] who considered the case where P is a linear polynomial. Cohn's method uses congruence arguments to eliminate some cases and a clever invocation of quadratic reciprocity to handle the remaining cases. The congruence arguments are very sufficient if there exists no solution in such a system, however they fail in the presence of a solution. This method was adapted by Mohanty and Ramasamy [**31**], Muriefah and Al Rashed [**1**], Peker and Cenberci [**33**] to study the solutions of particular systems. On the other hand, Kedlaya [**22**] gave a general procedure, based on the methods of Cohn and the theory of Pell equations, that solves many systems of the form (11). In fact, he applied this approach on several examples in which P is univariate with degree at most two. Moreover, in some cases this procedure fails to solve a system completely. In the following we summarize Kedlaya's procedure.

Kedlaya's procedure: Denote by (u_k, v_k) be the k^{th} solution of the Pell equation

$$u^2 - av^2 = 1.$$

For each base solution (x_0, y_0) of the equation $x^2 - ay^2 = b$, let S be the set of integers m such that (x_m, y_m) is in the given list of solutions. One can prove that $P(x_m, y_m)$ is a prefect square if and only if $m \in S$ as follows (without having to give up):

- For each $m \in S$, let $\alpha = P(-x_m, -y_m)$.
- If $|\alpha|$ is a perfect square, we give up; otherwise, let β be the product of all the primes that divide α an odd number of times.
- Let l be the period of $\{u_k (\text{mod } \beta)\}$ and d be the largest odd divisor of l.
- Let q be the largest integer such that $2^q | l$, unless 4 does not divide l, in which case let $q = 2$.
- Let s be the order of 2 in the group $(\mathbb{Z}/d\mathbb{Z})^{\times}$.
- Define the set $U = \{t \in \{0, \ldots, d-1\} : \left(\frac{u_{2^q t}}{\beta}\right) = -1\}$.
- If U is empty, we give up; otherwise find an odd number j such that for each $\epsilon = q, \ldots, q+s-1$, there exist $t \in U$ and $g \mid j$ with $2^{\epsilon - q} g \equiv t \pmod{\beta}$.
- Let $\gamma_m = 2^q j$ and γ be twice the least common multiple of γ_m for all $m \in S$.

- Find an integer δ with the following property: for every $k \in \{0, \dots, \delta\gamma - 1\}$, either $k \equiv m \pmod{2\gamma_m}$ for some $m \in S$; or there exists a prime number p such that $P(x_k, y_k)$ is a nonresidue $\pmod{p}$, with $\{x_i \pmod{p}\}$ and $\{y_i \pmod{p}\}$ have periods dividing $\delta\gamma$. The period condition can be guaranteed by having $p | v_\kappa$ for some κ, where $2\kappa | \delta\gamma$.
- Suppose that δ can be found satisfying the specified properties. To show that $P(x_m, y_m)$ is a prefect square if and only if $m \in S$, assume that there exists $k \notin S$ such that $P(x_k, y_k)$ is a perfect square. By the construction of δ, there exists m such that $k \equiv m \pmod{2\gamma_m}$, or else there exists a prime number p such that $P(x_k, y_k)$ is a nonresidue $\pmod{p}$. Since $k \notin S$, so $k \neq m$ and $k = m + 2^{\epsilon+1} jh$ for some h, ϵ with h odd and $\epsilon \geq q$. We have that

  $$x_k \equiv -x_m \pmod{u_{j2^\epsilon}}$$

  and

  $$y_k \equiv -y_m \pmod{u_{j2^\epsilon}}.$$

  Therefore,

  $$P(x_k, y_k) \equiv P(-x_m, -y_m) = \alpha \pmod{u_{j2^\epsilon}}.$$

  The construction gives that for some $t \in U$ and some $g \mid j$ with $2^{\epsilon - q} g \equiv t \pmod{\beta}$. It is clear that $\epsilon \geq q \geq 2$ and $\{u_k \pmod{8}\}$ has period dividing 4. Thus, the Jacobi symbols $\left(\frac{-1}{u_{2^\epsilon g}}\right)$ and $\left(\frac{2}{u_{2^\epsilon g}}\right)$ both equal 1. Since $|\alpha| / \beta$ is a perfect square and $u_{g2^\epsilon} | u_{j2^\epsilon}$, we have by quadratic reciprocity

  $$\begin{aligned} \left(\frac{P(x_k, y_k)}{u_{2^\epsilon g}}\right) = \left(\frac{\alpha}{u_{2^\epsilon g}}\right) = \left(\frac{\beta}{u_{2^\epsilon g}}\right) = \left(\frac{u_{2^\epsilon g}}{\beta}\right) \\ = \left(\frac{u_{2^q t}}{\beta}\right) = -1, \end{aligned}$$

  which contradicts the assumption that $P(x_k, y_k)$ is a perfect square.

Therefore, our technique mainly uses Kedlaya's procedure and similar techniques adapted by the methods of Mohanty and Ramasamy, Muriefah and Rashed, and Peker and Cenberci to prove the following theorems, that appear in [**12**].

THEOREM. *Suppose that* $X = L_n$ *and* $Y = F_n$*, then the Diophantine equation* (10) *has no more solutions other than* $(X, Y, Z) = (3, 1, \pm 8)$.

THEOREM. *The Diophantine equation* (10) *has no solutions in integers* X, Y *and* Z *if* $X = F_n$ *and* $Y = L_n$.

Next, we consider the following Diophantine equation, that is called Markoff equation,

$$x^2 + y^2 + z^2 = 3xyz$$

in positive integers $x \leq y \leq z$, which was deeply studied by Markoff [**27, 28**]. A triple (x, y, z) of positive integers that satisfies Markoff equation is called a Markoff triple, and the numbers x, y and z are called Markoff numbers. Indeed, Markoff showed that this equation has infinitely many solutions, which can be generated from the fundamental solution $(1, 1, 1)$ and the branching operation

$$\begin{array}{ccc} & (x, y, z) & \\ \swarrow & & \searrow \\ (x, z, 3xz - y) & & (y, z, 3yz - x). \end{array}$$

In these papers, Markoff numbers have been introduced to describe minimal values of indefinite quadratic forms with exceptionally large minima greater than $1/3$ of the square root of the discriminant. He showed that these forms are in one-to-one correspondence with the Markoff triples. This equation has been generalized by several authors. For instance, Rosenberger [**34**] considered the equation

$$ax^2 + by^2 + cz^2 = dxyz. \tag{12}$$

This equation is often called the Markoff-Rosenberger equation. Rosenberger proved that if $a, b, c, d \in \mathbb{N}$ are integers such that $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$ and $a, b, c | d$, then nontrivial solutions exist only if $(a, b, c, d) \in T$, where

$$T = \{(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,5,5), (1,2,3,6)\}.$$

The Markoff-Rosenberger equation was also generalized by Jin and Schmidt [**20**] in which they determined the positive integer solutions of the equation

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1. \tag{13}$$

Jin and Schmidt showed that equation (13) has a fundamental solution if and only if

$$(A, B, C, D) \in \{(2, 2, 3, 6), (2, 1, 2, 2), (7, 2, 14, 14), (3, 1, 6, 6), (6, 10, 15, 30), (5, 1, 5, 5), (1, t, t, 2t)\}, \text{with } t \in \mathbb{N}.$$

Respecting the authors of this generalization, we call equation (13) the Jin-Schmidt equation. As equations (12) and (13) have infinitely many solutions in integers, here we are interested in studying the solutions of these equations in some binary recurrence sequences. The idea of investigating the solutions of the Markoff equation in some binary linear recurrence sequences was initiated by Luca and Srinivasan [**26**] in which they proved that the only solution of Markoff equation with $x \le y \le z$ such that $(x, y, z) = (F_i, F_j, F_k)$ is given by the well-known identity related to the Fibonacci numbers

$$1 + F_{2n-1}^2 + F_{2n+1}^2 = 3F_{2n-1}F_{2n+1}.$$

Here, we extend the result of Luca and Srinivasan by simplifying their strategy with having upper bounds for the minimum of the indices to provide a direct approach for investigating such special solutions of the Markoff-Rosenberger equation (12) and the Jin-Schmidt equation (13). Indeed, we first determine the solutions $(X, Y, Z) = (F_I, F_J, F_K)$ in positive integers of the Jin-Schmidt equation (13), where F_I denotes the I^{th} Fibonacci number. In other words, we study the solutions of the following Diophantine equations in the sequence of Fibonacci numbers:

$$2X^2 + 2Y^2 + 3Z^2 = 6XYZ + 1, \tag{14}$$

$$2X^2 + Y^2 + 2Z^2 = 2XYZ + 1, \tag{15}$$

$$7X^2 + 2Y^2 + 14Z^2 = 14XYZ + 1, \tag{16}$$

$$3X^2 + Y^2 + 6Z^2 = 6XYZ + 1, \tag{17}$$

$$6X^2 + 10Y^2 + 15Z^2 = 30XYZ + 1, \tag{18}$$

$$5X^2 + Y^2 + 5Z^2 = 5XYZ + 1, \tag{19}$$

where $X = F_I, Y = F_J$ and $Z = F_K$. We also remark that the same technique can be applied in case of $(A, B, C, D) = (1, t, t, 2t)$ for given values of t. In the following we introduce the procedure of this technique which we use to study the existence and nonexistence of such special solutions of the Jin-Schmidt equation (particularly, equations (14)-(19)). Indeed, this technique can be adapted to study the solutions of any equation of the form $ax^2 + by^2 + cz^2 = dxyz + e$ (for certain nonzero integer coefficients) from certain binary linear recurrence sequences. Therefore, we call it the general investigative procedure.

General investigative procedure: To start the procedure off, we first have to obtain all the possible distinct equations

$$ax^2 + by^2 + cz^2 = dxyz + 1 \tag{20}$$

of equation (13) by permuting the coefficients A, B and C for

$$(A, B, C, D) \in S = \{(2, 2, 3, 6), (2, 1, 2, 2), (7, 2, 14, 14), (3, 1, 6, 6), (6, 10, 15, 30), (5, 1, 5, 5)\}.$$

The following steps summarize the technique of investigating all the solutions $(x, y, z) = (F_i, F_j, F_k)$ with $2 \le i \le j \le k$ for every equation of the form (20) for a given tuple (a, b, c, d); that is,

$$aF_i^2 + bF_j^2 + cF_k^2 = dF_iF_jF_k + 1, \tag{21}$$

where $2 \le i \le j \le k$. Note that we assumed that $i \ge 2$ since $F_1 = F_2 = 1$.

- Determining an upper bound for i in equation (21). We first rewrite the equation in the form

$$cF_k - dF_iF_j = -\frac{aF_i^2 + bF_j^2}{F_k} + \frac{1}{F_k}. \tag{22}$$

  Inserting the values of F_i, F_j and F_k in the left-hand side of equation (22) with using the Binet's Fibonacci numbers formula (that is obtained from the Binet's formula of the Lucas sequence of the first kind in (2) in case of $U_n(1, -1) =$

F_n): $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$, $n \geq 0$ and α is called the golden ratio with $\beta = \frac{-1}{\alpha}$. We obtain that

$$\begin{aligned}\frac{c}{\sqrt{5}}\alpha^k - \frac{d}{5}\alpha^{i+j} = -\frac{aF_i^2 + bF_j^2}{F_k} + \frac{1}{F_k} + \frac{c}{\sqrt{5}}\beta^k \\ - \frac{d}{5}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}).\end{aligned} \tag{23}$$

Based on the inequalities for the n^{th} Fibonacci number

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{for} \quad n \geq 1,$$

and $2 \leq i \leq j \leq k$ (that is $1 \leq F_i \leq F_j \leq F_k$), we have that

$$\begin{aligned}\frac{aF_i^2 + bF_j^2}{F_k} &\leq (a+b)\frac{F_j^2}{F_k} \leq (a+b)\alpha^{2j-k} \\ &\leq (a+b)\alpha^j,\end{aligned} \tag{24}$$

$$\frac{1}{F_k} \leq 1 < \alpha^j, \tag{25}$$

$$\begin{aligned}\left|\frac{c}{\sqrt{5}}\beta^k\right| = \left|-\frac{c}{\sqrt{5}}\alpha^{-k}\right| &\leq \frac{c}{\sqrt{5}}\alpha^{-j} \\ &\leq \frac{c}{5}\alpha^j,\end{aligned} \tag{26}$$

$$\left|\frac{d}{5}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j})\right| \leq \frac{d}{5}(2\alpha^j + 1) \leq \frac{3d}{5}\alpha^j. \tag{27}$$

Taking the absolute values to equation (23) and using the inequalities (24)-(27), we obtain that

$$\left|\frac{c}{\sqrt{5}}\alpha^k - \frac{d}{5}\alpha^{i+j}\right| < (1 + a + b + \frac{c+3d}{5})\alpha^j.$$

Multiplying across by $\frac{\sqrt{5}}{c\alpha^{i+j}}$, we get that

$$\left|\alpha^{k-i-j} - \frac{d}{c\sqrt{5}}\right| < \frac{h}{\alpha^i}, \tag{28}$$

where $h = \frac{\sqrt{5}}{c}(1 + a + b + \frac{c+3d}{5})$. Suppose that

$$\min_{n \in \mathbb{Z}} \left|\alpha^n - \frac{d}{c\sqrt{5}}\right| > g > 0,$$

so inequality (28) implies that

$$g < \frac{h}{\alpha^i},$$

which clearly gives

$$i \le \left\lfloor \frac{\ln(\frac{h}{g})}{\ln(\alpha)} \right\rfloor = l. \tag{29}$$

- Determining an upper bound for $k-j$ in equation (21). In fact, for a given i one can use inequality (28) to obtain an upper bound for $k-j$. Here, we provide such a bound using the upper bound for i (that is $i \le l$) and inequality (28). We have that $1 \le a, b, c \le 15$ and $2 \le d = D \le 30$, which imply that $h \le 52\sqrt{5} < 116.3$. Therefore, inequality (28) becomes

$$\begin{aligned} \left| \left|\alpha^{k-i-j}\right| - \left|\frac{d}{c\sqrt{5}}\right| \right| &\le \left|\alpha^{k-i-j} - \frac{d}{c\sqrt{5}}\right| \\ &< \frac{116.3}{\alpha^2} < 44.5 \end{aligned}$$

as $i \ge 2$, which leads to

$$\left|\alpha^{k-i-j}\right| < 44.5 + \left|\frac{d}{c\sqrt{5}}\right| < 44.5 + \frac{30}{\sqrt{5}} < 58$$

as $d \le 30$ and $c \ge 1$. Hence,

$$k - j < i + \frac{\ln(58)}{\ln(\alpha)} < l + 9 \quad \text{or} \quad k \le j + l + 8 \tag{30}$$

as $i \le l$.
- Eliminating the values of i for $i \in [2, l]$ in which equation (21) does not hold (and then equation (20) for which $(x, y, z) = (F_i, F_j, F_k)$ with $2 \le i \le j \le k$). For that, we solve the Diophantine equation

$$aF_i^2 + by^2 + cz^2 - dF_i yz - 1 = 0 \tag{31}$$

for y and z. This can be done by SageMath [**36**] using the function `solve_diophantine()`. If there exists no i for which equation (31) is satisfied, then equation (20) does not have any solution $(x, y, z) = (F_i, F_j, F_k)$ with $2 \le i \le j \le k$ at the tuple (a, b, c, d).

- Fixing i and k for an arbitrary $k \in \{j, j+1, \ldots, j+l+8\}$ in equation (21), we get that
$$bF_j^2 - sF_j + w = 0, \tag{32}$$
where $s = dF_iF_k$ and $w = aF_i^2 + cF_k^2 - 1$. We note that the equation above only depends on j for all $j \geq i \geq 2$.
- Determining whether there exists j for which equation (32) holds using any of the following arguments:
  (i) The technique of using the quadratic formula and the identity relationship between the Fibonacci numbers and Lucas numbers. This identity is obtained from (1) (in case of $U_n(1,-1) = F_n$ and $V_n(1,-1) = L_n$), and it is as follows
$$L_k^2 = 5F_k^2 \pm 4. \tag{33}$$
  Multiplying (32) by $4b$ and adding s^2 to both sides lead to
$$(2bF_j - s)^2 = s^2 - 4bw. \tag{34}$$
  Multiplying equations (33) and (34) together yields
$$Y_1^2 = (5X_1^2 \pm 4)(d^2F_i^2X_1^2 - 4b(aF_i^2 + cX_1^2 - 1)),$$
  where $X_1 = F_k$ and $Y_1 = L_k(2bF_j - dF_iF_k)$. Therefore, our problem is reduced to obtain integral points on these biquadratic genus 1 curves. This will be realized using an algorithm implemented in Magma [**6**] as `SIntegralLjunggrenPoints()` (based on results obtained by Tzanakis [**38**]) or an algorithm described by Alekseyev and Tengely [**2**] in which they gave an algorithmic reduction of the search for integral points on such a curve to solving a finite number of Thue equations.
  (ii) The Fibonacci identities substitution technique in which we use the Fibonacci sequence formula or some related identities to eliminate equation (32).
  (iii) The congruence argument technique in which we eliminate equation (32) modulo a prime number p.

- From every obtained solution $(x, y, z) = (F_i, F_j, F_k)$ of equation (20) at the tuple (a, b, c, d), we derive the corresponding solution $(X, Y, Z) = (F_I, F_J, F_K)$ of equation (13) at the tuple (A, B, C, D) by comparing the positions of the components of their tuples.

By using the above procedure, we prove the following theorem, that is obtained in [**15**].

THEOREM. *Let m be a positive integer greater than* 1. *If $(X, Y, Z) = (F_I, F_J, F_K)$ is a solution of equation* (13) *with $(A, B, C, D) \in S$, then the complete list of solutions is given by*

Eq.	*(A,B,C,D)*	$\{(X, Y, Z)\}$
(14)	$(2, 2, 3, 6)$	$\{(1,1,1), (1,2,1), (1,2,3), (2,1,1), (2,1,3), (F_{2m-1}, F_{2m+1}, 1), (F_{2m+1}, F_{2m-1}, 1)\}$
(15)	$(2, 1, 2, 2)$	$\{(2,3,2), (2,5,2), (2,5,8), (8,5,2)\}$
(16)	$(7, 2, 14, 14)$	$\{(1,2,1), (1,5,1), (3,2,1), (3,2,5)\}$
(17)	$(3, 1, 6, 6)$	$\{(1,2,1), (3,2,1), (3,2,5)\}$
(18)	$(6, 10, 15, 30)$	$\{(1,1,1), (1,2,1), (1,2,3)\}$
(19)	$(5, 1, 5, 5)$	$\{\}$

In case of the Markoff-Rosenberger equation (12), we generalize the strategy described in the general investigative procedure above to provide general results for the solutions $(x, y, z) = (R_i, R_j, R_k)$ of the Markoff-Rosenberger equation, where R_i denotes the i^{th} generalized Lucas number of first/second kind, i.e. $R_i = U_i$ or V_i. Then we apply the strategy of achieving these results to completely resolve concrete equations, e.g. we determine solutions containing only balancing numbers B_n and Jacobsthal numbers J_n, respectively. In other words, if $\mathfrak{T}$ is the set of all distinct tuples (a, b, c, d) derived from permuting the first three components of elements in T, then we prove the following results, that will appear in [**18**].

THEOREM. *Let $(a, b, c, d) \in \mathfrak{T}$, $P \geq 2, -P - 1 \leq Q \leq P - 1$ such that $Q \neq 0$, $D > 0$ and*

$$B_0 = \min_{I \in \mathbb{Z}} \left| \alpha^I - \frac{d}{c\sqrt{D}} \right|, \quad B_1 = \min_{I \in \mathbb{Z}} \left| \alpha^I - \frac{d}{c} \right|.$$

If $B_0 \neq 0$, then $B_0 \geq \alpha^{-4}$ and if $B_1 \neq 0$, then $B_1 \geq 0.17$. Furthermore, if $x = U_i, y = U_j$ and $z = U_k$ with $1 \leq i \leq j \leq k$ is a solution of (12) *and $B_0 \neq 0$, then $i \leq 12$. If $x = V_i, y = V_j$ and $z = V_k$ with $1 \leq i \leq j \leq k$ is a solution of* (12) *and $B_1 \neq 0$, then $i \leq 7$.*

Note that the cases where we have $B_1 = 0$ were completely studied in the proof of the above theorem. Thus, it remains to classify the cases satisfying $B_0 = 0$, the result is as follows.

PROPOSITION. *If $P \geq 2$, $-P-1 \leq Q \leq P-1$, $Q \neq 0$ and $D > 0$, then $B_0 \neq 0$ fulfills unless*

- $e = 1,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 0,$
- $e = 2,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 1,$
- $e = 2,\ P = 4, Q = 3,\ \alpha = 3,\ \sqrt{D} = 2,\ I = 0,$
- $e = 3,\ P = 5, Q = 4,\ \alpha = 4,\ \sqrt{D} = 3,\ I = 0,$
- $e = 4,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 2,$
- $e = 4,\ P = 6, Q = 5,\ \alpha = 5,\ \sqrt{D} = 4,\ I = 0,$
- $e = 4,\ P = 2, Q = -3,\ \alpha = 3,\ \sqrt{D} = 4,\ I = 0,$
- $e = 5,\ P = 7, Q = 6,\ \alpha = 6,\ \sqrt{D} = 5,\ I = 0,$
- $e = 5,\ P = 3, Q = -4,\ \alpha = 4,\ \sqrt{D} = 5,\ I = 0,$
- $e = 6,\ P = 4, Q = 3,\ \alpha = 3,\ \sqrt{D} = 2,\ I = 1,$
- $e = 6,\ P = 8, Q = 7,\ \alpha = 7,\ \sqrt{D} = 6,\ I = 0,$
- $e = 6,\ P = 4, Q = -5,\ \alpha = 5,\ \sqrt{D} = 6,\ I = 0,$

where $e = d/c$ such that $(a, b, c, d) \in \mathfrak{T}$.

Indeed, our results here also extend other related results obtained by e.g. Kafle, Srinivasan and Togbé [**21**] and Altassan and Luca [**3**]. As applications to the latter theorem, we also prove the following results, that will also appear in [**18**].

THEOREM. *If $(x, y, z) = (B_i, B_j, B_k)$ is a solution of the equation*

$$ax^2 + by^2 + cz^2 = dxyz$$

and $(a, b, c, d) \in \{(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,5,5), (1,2,3,6)\}$, then there is at most one solution given by $x = y = z = B_1 = 1$.

THEOREM. *If $(x, y, z) = (J_i, J_j, J_k)$ is a solution of the equation*

$$ax^2 + by^2 + cz^2 = dxyz$$

and $(a, b, c, d) \in \{(1,1,1,1),(1,1,1,3),(1,1,2,2),(1,1,2,4),(1,1,5,5),(1,2,3,6)\}$, *then the complete list of solutions is given by*

(a,b,c,d)	*solutions*
$(1,1,1,1)$	$\{(3,3,3)\}$
$(1,1,1,3)$	$\{(1,1,1)\}$
$(1,1,2,2)$	$\{\}$
$(1,1,2,4)$	$\{(1,1,1),(1,3,1),(1,3,5),(3,1,1),(3,1,5),(3,11,1),(11,3,1)\}$
$(1,1,5,5)$	$\{(1,3,1),(3,1,1)\}$
$(1,2,3,6)$	$\{(1,1,1),(5,1,1)\}$

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List of talks

(1) Solutions of generalizations of Markoff equation from linear recurrences, 64^{th} Annual Online Meeting of the Australian Mathematical Society, University of New England, Australia, December 8–11, 2020.
(2) Solutions of a generalized Markoff equation in Fibonacci numbers, 9^{th} Interdisciplinary Doctoral Conference, Doctoral Student Association of the University of Pécs, Hungary, November 27–28, 2020.
(3) Cryptanalysis of ITRU, 1^{st} Conference on Information Technology and Data Science, Faculty of Informatics, University of Debrecen, Hungary, November 6–8, 2020.
(4) Cryptanalysis of ITRU, 20^{th} Central European Conference on Cryptology, Department of Mathematics, Faculty of Science, University of Zagreb, Croatia, June 24–26, 2020.
(5) Diophantine equations related to reciprocals of linear recurrence sequences, 24^{th} Central European Number Theory Conference, J. Selye University, Komárno, Slovakia, September 1–6, 2019.
(6) Solutions of the Diophantine equation $7X^2 + Y^7 = Z^2$ from recurrence sequences. Institute of Mathematics, University of Debrecen, Hungary, April 12, 2019.
(7) Representations of reciprocals of Lucas sequences, CSM-The 5^{th} Conference of PhD Students in Mathematics, Bolyai Institute, University of Szeged, Hungary, June 25–27, 2018.

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Registry number: DEENK/59/2021.PL
Subject: PhD Publication List

Candidate: Hayder Raheem Hashim
Doctoral School: Doctoral School of Mathematical and Computational Sciences

List of publications related to the dissertation

Foreign language scientific articles in Hungarian journals (1)

1. **Hashim, H. R.**, Tengely, S.: Representations of reciprocals of Lucas sequences.
   Miskolc Math. Notes. 19 (2), 865-872, 2018. ISSN: 1787-2405.
   DOI: http://dx.doi.org/10.18514/MMN.2018.2520
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Foreign language scientific articles in international journals (5)

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   Math. Bohem. "Accepted by Publisher", 1-18, 2021. ISSN: 0862-7959.
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List of other publications

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9. **Hashim, H. R.**, Alkufi, M. A. H. J.: A proposed method for text encryption using symmetric and asymmetric cryptosystems.
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Tárgy: PhD Publikációs Lista

Jelölt: Hashim, Hayder Raheem
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A PhD értekezés alapjául szolgáló közlemények

Idegen nyelvű tudományos közlemények hazai folyóiratban (1)

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További közlemények

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