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LIE DERIVATIVES AND GEOMETRIC VECTOR FIELDS IN SPRAY AND FINSLER GEOMETRY

Egyetemi doktori (PhD) értekezés

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Lie derivatives and geometric vector fields in spray and Finsler geometry

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Soli Deo Gloria!

‘I will go before you
and will level the mountains;
I will break down gates of bronze
and cut through bars of iron.’
/Isaiah 45:2/

Contents

1 Introduction

The aim of this Thesis is threefold. First, to elaborate a (partly new) calculative background for Lie derivatives in the framework of Finsler bundles. Second, to apply the Finslerian Lie derivative, combining with other technical tools, for studying curvature collineations in spray manifolds. Third, to study projective and conformal (in particular, homothetic and Killing) vector fields on a Finsler manifold and describe some interrelations between them.

The theory of the above-mentioned 'geometric' vector fields has a vast literature. Let us quote here Mike Crampin. 'The transformation theory of sprays and Berwald connections was in vogue towards the middle of last century – Chapter VIII of Yano's book 'The Theory of Lie Derivatives and its Applications' [34] gives an excellent survey on the state of the art in 1957 – but went out of fashion; the subject has been taken up again very recently by Lovas [17]. The definition of an infinitesimal affine transformation of a Berwald connection is not entirely straightforward, because a Berwald connection is defined on a pull-back bundle (a pull back of a tangent bundle in fact). We feel that a concept of the Lie derivative of a section of such a pull-back bundle has not received the careful geometrical consideration that it deserves.' (See the Introduction of [8]; the numbering of items [17], [34] corresponds to the References in our Thesis.) In their just cited paper, Crampin and Saunders make an attempt to remedy the defect – and we continue their attempts here.

Going back to the historical roots, we mention that Gy. Soós important paper 'Über Gruppen von Affinitäten und Bewegungen in Finslerschen Räumen'([27]) has already been quoted in Yano's monograph. A good overview of the developments of the next two decades can be found in R. B. Misra's paper [23], written in 1981, revised and updated in 1993. In a two-part paper, M. Matsumoto clarified and improved some results of Yano in the framework of his theory of Finsler connections ([19], [20]). From the modern (and relatively modern, partly tensor calculus based) literature, beside the paper of R. L. Lovas, H. Akbar-Zadeh and J. Grifone works ([2], [3], [12], [13]) are worth mentioning. Grifone applies systematically the '$\tau_{TM}\colon TTM \to TM$ tangent bundle formalism', combining with the Frölicher–Nijenhuis calculus of vector-valued differential forms; Lovas formulates and proves his results in terms of the 'pull-back formalism'. This Thesis, is some sense, is a continuation of Grifone's and Lovas's work. The greater part of the theory is developed on a *pull-back* of a tangent bundle, however, the concepts and techniques of the tangent bundle geometry, including vertical calculus on TM, play an eminent role in our analysis. We use two types of Lie derivatives: the classical Lie derivative on the tensor algebra of a manifold and the Lie derivative of Finsler tensor fields with respect to *projectable* vector fields. (It turns out, as is expect, that the two types are closely related.) We also need the Lie derivative of a covariant derivative on a Finsler bundle as it has been introduced in [17]. In this Thesis, we define the concept of the Lie derivative of an Ehresmann connection $\mathcal{H}$; after that we can speak about $\mathcal{H}$-Killing vector fields.

We say, roughly speaking, that a vector field X on a manifold M is a curvature collineation of a curvature object $\mathbf{C}$ of a spray manifold if $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{C} = 0$, where X^{c} is the complete lift of X and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}$ is the Finslerian Lie derivative with respect to X^{c}. Curvature collineations play an important role in the study of geometry and physics of classical space-times; for an excellent account on the subject we refer to G. S. Hall's book [14], especially its last chapter. Similar investigations in the context of spray manifolds are new.

Most of our results are summarized in **18.2** (in English) and in **19.2** (in Hungarian), that is why we do not touch them here. To make the Thesis more readable, in Part I we briefly present the background material used throughout the other chapters.

Part I
Preliminary material

2 Manifolds and bundles

2.0 In general, we follow the notation and terminology of [29]. However, for convenience of the reader, we start here with some basic conventions which will be followed throughout this Thesis.

2.0.1 The identity transformation of a set S is denoted by 1_S. If $S \to T$ is a mapping and $A \subset S$, then $f \restriction A$ denotes the restriction of f to A. The (canonical) inclusion of A into S is $j_A := 1_S \restriction A$. Given two mappings $\varphi\colon M \to S$ and $\psi\colon M \to T$, (φ, ψ) denotes the mapping

$$M \to S \times T,\ p \mapsto (\varphi(p), \psi(p)). \tag{2.1}$$

The product $\varphi_1 \times \varphi_2$ of two mappings $\varphi_1\colon M_1 \to S_1$ and $\varphi_2\colon M_2 \to S_2$ is given by

$$\varphi_1 \times \varphi_2\,(s_1, s_2) := (\varphi_1(s_1), \varphi_2(s_2)); \tag{2.2}$$

it maps $M_1 \times M_2$ into $S_1 \times S_2$.

2.0.2 The set $\{0, 1, 2, \dots\}$ of natural numbers is denoted by $\mathbb{N}$. The symbols $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the integers, rationals and reals, respectively. If $A \subset \mathbb{R}$, we write $A^* := A \setminus \{0\}$ and $A_+ := \{a \in A|\, a \geqq 0\}$. Then $A^*_+ = \{a \in A|\, a > 0\}$. *Real-valued mappings will usually be mentioned as functions.*

2.0.3 For every $n \in \mathbb{N}^*$, we write $J_n := \{1, \dots, n\}$. The group of permutations of J_n is denoted by S_n, and $\epsilon(\sigma) \in \{-1, 1\}$ stands for the sign of $\sigma \in S_n$.

2.0.4 By a ring we mean a commutative ring with unit element 1. The zero element of a ring (and any additive group) will usually be denoted by the same symbol 0.

2.0.5 Let R be a ring and V a module over R (or an R-*module* for short). Then $V^* := L(V, \mathsf{R}) := \{f\colon V \to \mathsf{R}|\ f \text{ is } \mathsf{R}\text{-linear}\}$ is the dual of V, $\mathrm{End}_{\mathbb{R}}(V) := \mathrm{End}(V) := \{\varphi\colon V \to V|\ \varphi \text{ is } \mathsf{R}\text{-linear}\}$ is the ring of endomorphisms of V.

2.0.6 Let V be an R-module and $k \in \mathbb{N}^*$. The R-module of k-linear mappings $V^k \to \mathsf{R}$ (resp. $V^k \to V$) is denoted by $T_k(V)$ (resp. $T^1_k(V)$) and their elements are called *covariant tensors* (resp. *vector-valued tensors*) of degree k. Then $T_1(V) = V^*$, $T^1_1(V) = \mathrm{End}(V)$. We agree that $T_0(V) := \mathsf{R}$, $T^1_0(V) := V$. In this Thesis, *by a tensor we shall always mean a covariant tensor or vector-valued tensor*, so we use the term tensor in a restricted sense. The symbol $\otimes$ will stand for tensor product.

2.0.7 Let V be an R-module, and let $A \in T_k(V) \cup T^1_k(V)$, where $k \in \mathbb{N}^*$. Given a permutation $\sigma \in S_k$, we define a tensor σA by

$$\sigma A(v_1, \dots, v_k) := A(v_{\sigma(1)}, \dots, v_{\sigma(k)}). \tag{2.3}$$

The tensor A is called symmetric (resp. alternating) if $\sigma A = A$ (resp. $\sigma A = \epsilon(\sigma)A$). Both the symmetric and the alternating tensors form a submodule in $T_k(V)$ and $T^1_k(V)$. These submodules will be denoted by $S_k(V)$ and $S^1_k(V)$ in the symmetric case, $\mathcal{A}_k(V)$ and $\mathcal{A}^1_k(V)$ in the alternating case.

2.0.8 Let $V \neq \{0\}$ be an n-dimensional *real* vector space. A *volume form* on V is an element of $A_n(V) \setminus \{0\}$. Given a volume form $\mu \in A_n(V)$, for every linear transformation $\varphi \in \mathrm{End}(V)$ there exists a unique scalar $\mathrm{tr}\,\varphi \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \mu(v_1, \dots, \varphi(v_i), \dots, v_n) = \mathrm{tr}(\varphi)\mu(v_1, \dots, v_n), \tag{2.4}$$

where $v_1, \dots, v_n \in V$ (see [10], 4.23). This scalar is called the *trace* of φ. Obviously, $\mathrm{tr}\,\varphi$ depends linearly on φ. We define the *trace of a vector valued tensor* $A \in T^1_k(V)$ ($k \in \mathbb{N}^*$) as the covariant tensor $\mathrm{tr}A \in T_{k-1}(V)$ given by

$$(\mathrm{tr}A)(v_1, \dots, v_{k-1}) := \mathrm{tr}(v \in V \mapsto A(v, v_1, \dots, v_{k-1}) \in V). \tag{2.5}$$

2.0.9 We continue to assume that V is an n-dimensional, non-trivial real vector space. The nullspace of a tensor $b \in S_2(V)$ is the subspace

$$N_b := \{v \in V \,|\, b(u, v) = 0 \ \text{ for all } u \in V\}$$

of V. If $N_b = \{0\}$, then b is called *non-degenerate*, and we write

$$\mathrm{Met}(V) := \{b \in S_2(V) \,|\, N_b = \{0\}\}.$$

A tensor $b \in S_2(V)$ is *positive definite* if $b(v, v) > 0$ for all $v \in V \setminus \{0\}$;

$$\mathrm{Euc}(V) := \{b \in S_2(V) \,|\, b \text{ is positive definite }\}.$$

2.0.10 In coordinate terms, we shall use Einstein's summation convention in two forms. The weak form: *'The summation sign is not omitted, but summation is understood over all repeated indices. Frequently (but not always) the repeated index occurs exactly twice – once up, once down.'*(See [25], p. 10.) The standard form: *Whenever a term contains a repeated index, one as a superscript and the other as a subscript, summation is implied over this index.*

2.1 By a *manifold* we mean a second countable Hausdorff space endowed with a maximal smooth atlas. The letter M will be deserved for a manifold. The dimension of M is denoted by $\dim M$. All manifolds will be assumed at least 1-dimensional.

2.2 The set of k-times continuously differentiable mappings between manifolds M and N is denoted by $C^k(M,N)$. Here k is a natural number or $k=\infty$ with the convention that $C^0(M,N)$ stands for the set of continuous mappings of M into N. Elements of $C^\infty(M,N)$ are called *smooth mappings.* If $\varphi \in C^\infty(M,N)$ has a smooth inverse, we say that φ is a *diffeomorphism.* We write

$$\mathrm{Diff}(M,N) := \{\varphi \in C^\infty(M,N) \,|\, \varphi \text{ is a diffeomorphism}\}$$

and $\mathrm{Diff}(M) := \mathrm{Diff}(M,M)$.

2.3 The set of smooth real-valued functions on a manifold M is denoted by $C^\infty(M)$. If f, $g \in C^\infty(M)$, $\lambda \in \mathbb{R}$, and for any $p \in M$

$$(f+g)(p) := f(p)+g(p), \quad (\lambda f)(p) := \lambda f(g), \quad (fg)(p) := f(p)g(p),$$

then these operations make $C^\infty(M)$ into a ring and also an algebra over $\mathbb{R}$. The unit element of $C^\infty(M)$ is the constant function $\mathbf{1}\colon M \to \mathbb{R},\ p \mapsto \mathbf{1}(p) := 1$.

2.4 A triple (E,π,M) is a (smooth) *fibre bundle* with typical fibre F, briefly an *F-bundle*, if E, M, F are manifolds, π is a smooth mapping of E into M, and the following condition of *local triviality* is satisfied:

(LT) For every point $p \in M$ there exists a neighbourhood $\mathcal{U}$ of p in M together with a diffeomorphism $\psi\colon \mathcal{U} \times F \to \pi^{-1}(\mathcal{U})$ such that

$$\pi(\psi(q,v)) = q \text{ for all } (q,v) \in \mathcal{U} \times F.$$

Then E, M and π are called the *total manifold*, the *base manifold*, and the *projection* of the F-bundle (E, π, M), respectively. For each point $p \in M$, $E_p := \pi^{-1}(p)$ is the *fibre over* p (or *through* p). The diffeomorphism ψ in condition (LT) is a *trivializing map* for π (or for E). A family $(\mathcal{U}_i, \psi_i)_{i \in I}$ is called a *trivializing covering* for π (or for E by abuse of language), if $(\mathcal{U}_i)_{i \in I}$ is an open covering of M and $(\psi_i)_{i \in I}$ is a family of trivializing maps $\psi_i \colon \mathcal{U}_i \times F \to \pi^{-1}(\mathcal{U})$ for π. We shall frequently use the terms '$\pi \colon E \to M$ is a fibre bundle','π is a fibre bundle', or, less consequently, 'E is a fibre bundle'.

2.5 Let (E_i, π_i, M_i) be F_i-bundles, where $i \in \{1, 2\}$. A smooth mapping $\varphi \colon E_1 \to E_2$ is called *fibre preserving* if $\pi_1(z_1) = \pi_2(z_2)$ implies $\pi_2(\varphi(z_1)) = \pi_2(\varphi(z_2))$ $(z_1, z_2 \in E_1)$. Equivalently, φ is fibre preserving if there exists a smooth mapping $\varphi_B \colon M_1 \to M_2$ such that $\varphi_B \circ \pi_1 = \pi_2 \circ \varphi$. We say that φ_B is the mapping induced by the bundle map φ between the base manifolds.

2.6 A mapping $s \colon M \to E$ is a *section* of a fiber bundle $\pi \colon E \to M$ if $\pi \circ s = 1_M$. The set of *smooth* sections of π is denoted by $\Gamma(\pi)$ or (by abuse of notation) $\Gamma(E)$.

2.7 Let V be a finite-dimensional real vector space. A fibre bundle (E, π, M) with typical fibre V is said to be a *vector bundle* of rank $\dim V$ if every fibre E_p $(p \in M)$ is a real vector space, and there is a trivializing covering $(\mathcal{U}_i, \psi_i)_{i \in I}$ for π such that the mappings

$$(\psi_i)_p \colon V \to E_p,\ v \mapsto (\psi_i)_p(v) := \psi_i(p, v) \quad (i \in I, p \in \mathcal{U}_i)$$

are linear isomorphisms. A vector bundle $\pi' \colon E' \to M$ is a *subbundle* of a vector bundle $\pi \colon E \to M$ if, for every $p \in M$, E'_p is a linear subspace of E_p, and the induced inclusion mapping $j_{E'} \colon E' \to E, \quad j_{E'} \restriction E'_p := j_{E'_p}$ is smooth. (For the definition of $j_{E'_p}$ see **2.0.1**)

2.8 Let (E_1, π_1, M_1) and (E_2, π_2, M_2) be vector bundles. A smooth mapping $\varphi \colon E_1 \to E_2$ is called a *bundle map* if φ is fibre preserving and the restrictions $\varphi_p := \varphi \restriction (E_1)_p \colon (E_1)_p \to (E_2)_{\varphi_B(p)}$, $p \in M_1$, are linear mappings (φ_B is the mapping induced by φ, see **2.5**). If $M_1 = M_2 =: M$ and $\varphi_B = 1_M$, then we say that φ is a *strong bundle map.* A bundle map φ called an *isomorphism* if it is a diffeomorphism; this holds if, and only if, $\varphi_B \in \operatorname{Diff}(M_1, M_2)$ and the restrictions $\varphi_p \colon (E_1)_p \to (E_2)_{\varphi_B(p)}$ are linear isomorphisms.

2.9 The set $\Gamma(\pi)$ of smooth sections of a vector bundle $\pi\colon E \to M$ forms a $C^\infty(M)$-module under the pointwise operations

$$(s_1 + s_2)(p) := s_1(p) + s_2(p),\ (fs)(p) := f(p)s(p),\ p \in M,$$

where $s, s_1, s_2 \in \Gamma(\pi)$, $f \in C^\infty(M)$. The zero element of this module is the *zero section* o defined by $o(p) := 0_p :=$ the zero vector of the fibre E_p, $p \in M$.

2.10 The fundamental lemma of strong bundle maps. Let $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ be vector bundles over the same base manifold M. If $\varphi\colon E_1 \to E_2$ is a strong bundle map, then the mapping

$$\Phi\colon \Gamma(\pi_1) \to \Gamma(\pi_2),\ \ s \mapsto \Phi(s) := \varphi \circ s$$

is $C^\infty(M)$-linear. Conversely, let $\Phi\colon \Gamma(\pi_1) \to \Gamma(\pi_2)$ be a module homomorphism. Then there exists a strong bundle map $\varphi\colon E_1 \to E_2$ such that

$$\Phi(s) = \varphi \circ s \text{ for all } s \in \Gamma(\pi_1).$$

For a sketchy proof of this result, see [29], Proposition 2.2.31.

2.11 Let $\pi\colon E \to M$ be a vector bundle. A *scalar product* on π is a mapping

$$g\colon p \in M \mapsto g_p \in S_2(E_p)$$

such that the function

$$g(s_1, s_2)\colon M \to \mathbb{R},\ p \mapsto g(s_1, s_2)(p) := g_p(s_1(p), s_2(p))$$

is smooth for each $s_1, s_2 \in \Gamma(\pi)$. A vector bundle $\pi\colon E \to M$ with scalar product g is called *semi-Euclidean* if $g_p \in \mathrm{Met}(E_p)$ for all $p \in M$; *Euclidean* if $g_p \in \mathrm{Euc}(E_p)$ for all $p \in M$. *Every vector bundle admits a Euclidean scalar product.*

3 Tangent bundle and vector fields

3.1 A *tangent vector* to M at a point p of M is an $\mathbb{R}$-linear function $v\colon C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = v(f)g(p) + f(p)v(g) \text{ for all } f, g \in C^\infty(M).$$

Under the usual linear operations the tangent vectors form an n-dimensional real vector space $T_p(M)$, called the *tangent space* of M at p.

3.2 Let $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ be a chart of M at a point p of M. Here

$$u^i := e^i \circ u \colon \mathcal{U} \subset M \to u(\mathcal{U}) \subset \mathbb{R}^n \to \mathbb{R};$$

$(e^i)_{i=1}^n$ is the canonical coordinate system on $\mathbb{R}^n$, i.e., the dual of the canonical basis of $\mathbb{R}^n$. Then the functions $\left(\frac{\partial}{\partial u^i}\right)_p$ defined by

$$\left(\frac{\partial}{\partial u^i}\right)_p (f) = \left(\frac{\partial f}{\partial u^i}\right)(p) := D_i(f \circ u^{-1})(u(p)), \quad f \in C^\infty(M) \quad (3.1)$$

are the tangent vectors to M at p. The family $\left(\left(\frac{\partial}{\partial u^i}\right)_p\right)_{i=1}^n$ is a basis of T_pM. Using this basis, every tangent vector $v \in T_pM$ can uniquely be written in the form $v = \sum_{i=1}^n v(u^i)\left(\frac{\partial}{\partial u^i}\right)_p$.

3.3 Let $TM := \mathring{\bigcup}_{p \in M} T_pM$ (disjoint union) and define the projection $\tau \colon TM \to M$ by $\tau(v) := p$ if $v \in T_pM$. The topology and the smooth structure of M induce a unique (Hausdorff and second countable) topology and a smooth structure on TM such that for every chart $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ on M,

$$\begin{cases} (\tau^{-1}(\mathcal{U}), (x, y)) = (\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n)) \\ x^i := u^i \circ \tau, \quad y^i(v) := v(u^i) \end{cases} \quad (3.2)$$

is a chart on TM. We say that $(\tau^{-1}(\mathcal{U}), (x, y))$ is the *chart induced* by $(\mathcal{U}, u)$. The triple (TM, τ, M) is a vector bundle with typical fibre $\mathbb{R}^n$ whose fibre over a point $p \in M$ is the tangent space T_pM. The vector bundle obtained in this way is called the *tangent bundle* of M; its total manifold TM is the *tangent manifold* of M. The tangent bundle of TM will be denoted by $\tau_{TM} \colon TTM \to TM$, or simply by τ_{TM}, or less precisely by TTM.

3.4 Let $\varphi \colon M \to N$ be a smooth mapping between manifolds. Given any point $p \in M$, the mapping

$$\begin{cases} (\varphi_*)_p \colon T_pM \to T_{\varphi(p)}N, \quad v \mapsto (\varphi_*)_p(v), \\ (\varphi_*)_p(v)(h) := v(h \circ \varphi) \text{ for all } h \in C^\infty(N) \end{cases} \quad (3.3)$$

is a linear mapping, called the *derivative of* φ *at* p. The mapping

$$\varphi_* \colon TM \to TN, \quad \varphi_* \restriction T_pM := (\varphi_*)_p \quad (p \in M)$$

is a bundle map with induced mapping $(\varphi_*)_B = \varphi$ between the base manifolds. This bundle map is the *derivative of* φ.

3.5 A smooth section of the tangent bundle of M is called a *vector field* on M. The $C^\infty(M)$-module of *vector fields* on M is denoted by $\mathfrak{X}(M)$. Thus

$$\mathfrak{X}(M) := \Gamma(TM) = \{X \in C^\infty(M, TM) |\ \tau \circ X = 1_M\}.$$

If $\mathcal{U}$ is an open subset of M, then a *vector field on* $\mathcal{U}$ is a smooth mapping $X\colon \mathcal{U} \to TM$ such that $\tau \circ X = 1_{\mathcal{U}}$. They form a module over $C^\infty(\mathcal{U})$ denoted by $\mathfrak{X}(\mathcal{U})$. If, in particular, $\mathcal{U}$ is the domain of a local coordinate system $(u^i)_{i=1}^n$ of M, then the mappings

$$\frac{\partial}{\partial u^i}\colon \mathcal{U} \to TM, \quad p \mapsto \left(\frac{\partial}{\partial u^i}\right)_p \in T_pM \quad (i \in J_n),$$

where the right-hand side is defined by (3.1), are vector fields on $\mathcal{U}$, called the *coordinate vector fields* of the chart $(\mathcal{U}, (u^i)_{i=1}^n)$. A family $(X_i)_{i=1}^n$ of vector fields on $\mathcal{U}$ is a *frame field* on $\mathcal{U}$ if $((X_i)_p)_{i=1}^n$ is a basis of T_pM for all $p \in \mathcal{U}$. Thus the coordinate vector fields of a chart form a special frame field on their domain.

As an example consider the real line $\mathbb{R}$, endowed with its canonical smooth structure defined by the single chart $(\mathbb{R}, r) := (\mathbb{R}, 1_{\mathbb{R}})$. The coordinate vector field of this chart is the mapping

$$\frac{d}{dr}\colon t \in \mathbb{R} \mapsto \left(\frac{d}{dr}\right)_t \in T_t\mathbb{R},$$

where the tangent vectors $(\frac{d}{dr})_t$ $(t \in \mathbb{R})$ act as ordinary differentiations:

$$\left(\frac{d}{dr}\right)_t (h) := h'(t) \text{ for all } h \in C^\infty(\mathbb{R}). \tag{3.4}$$

3.6 Given a vector field $X \in \mathfrak{X}(M)$, the mapping

$$f \in C^\infty(M) \mapsto Xf \in C^\infty(M)$$

is a derivation of the $\mathbb{R}$-algebra $C^\infty(M)$: it is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$X(fh) = (Xf)(h) + f\,Xh; \quad f, h \in C^\infty(M).$$

Conversely, every derivation of $C^\infty(M)$ comes from a vector field. Thus *vector fields on M can be freely interpreted as derivations in the algebra*

$C^\infty(M)$. The *Lie bracket* $[X, Y]$ of two vector fields $X, Y \in \mathfrak{X}(M)$ is the unique vector field such that

$$[X, Y](f) = X(Yf) - Y(Xf) \text{ for all } f \in C^\infty(M). \tag{3.5}$$

This bracket operation is $\mathbb{R}$-bilinear, skew symmetric and satisfies the Jacobi identity, making $\mathfrak{X}(M)$ into a (real) Lie algebra. Moreover, for $f \in C^\infty(M)$ we have

$$[fX, Y] = f[X, Y] - (Yf)X \text{ and } [X, fY] = f[X, Y] + (Xf)Y. \tag{3.6 a-b}$$

3.7 Let $\varphi \colon M \to N$ be a smooth mapping between manifolds. Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called *φ-related* if

$$\varphi_* \circ X = Y \circ \varphi. \tag{3.7}$$

Then we write $X \underset{\varphi}{\sim} Y$. We say that a vector field X on M is *projectable* (by φ) if there exists a vector field Y on N such that $X \underset{\varphi}{\sim} Y$.

3.8 Let $X_1, X_2 \in \mathfrak{X}(M)$; $Y_1, Y_2 \in \mathfrak{X}(N)$. If $X_i \underset{\varphi}{\sim} Y_i$ $(i \in \{1, 2\})$, then $[X_1, X_2] \underset{\varphi}{\sim} [Y_1, Y_2]$ (*related vector field lemma*). Suppose, in particular, that $\varphi \in \mathrm{Diff}(M, N)$. The *push-forward* of a vector field $X \in \mathfrak{X}(M)$ by φ is

$$\varphi_\# X := \varphi_* \circ X \circ \varphi^{-1} \in \mathfrak{X}(N); \tag{3.8}$$

it is the unique vector field on N which is φ-related to X. The mapping

$$\varphi_\# \colon \mathfrak{X}(M) \to \mathfrak{X}(N), \quad X \mapsto \varphi_\# X$$

is a Lie algebra isomorphism, i.e., we have

$$\varphi_\#[X, Y] = [\varphi_\# X, \varphi_\# Y]; \quad X, Y \in \mathfrak{X}(M). \tag{3.9}$$

A vector field X on M is called *invariant under a diffeomorphism* ψ of M, if $\psi_\# X = X$, i.e, $\psi_* \circ X = X \circ \psi$.

4 Integral curves and flows

4.0 Throughout this section $I \subset \mathbb{R}$ is a nonempty open interval. To obtain coordinate expressions, we use a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M and the induced chart given by (3.2) on TM.

4.1 A smooth mapping $\alpha\colon I \to M$ is also called a *smooth curve* in M. The *tangent vector* (or *velocity vector*) $\dot{\alpha}(t) \in T_{\alpha(t)}M$ of α at $t \in I$ is defined by

$$\dot{\alpha}(t)(f) := (\alpha \circ f)'(t) = \lim_{s\to 0} \frac{f(\alpha(t+s)) - f(\alpha(t))}{s}, \quad f \in C^\infty(M).$$

Then we have

$$\dot{\alpha}(t) = (\alpha_*)_t \left(\frac{d}{dr}\right)_t, \quad t \in I. \tag{4.1}$$

The curve α is *regular* if $\dot{\alpha}(t) \neq 0$ for all $t \in I$. If $(\mathcal{U}, (u^i)_{i=1}^n)$ is a chart at $\alpha(t)$ and $\alpha^i := u^i \circ \alpha$, then

$$\dot{\alpha}(t) = \sum_{i=1}^n (\alpha^i)'(t) \left(\frac{\partial}{\partial u^i}\right)_{\alpha(t)}. \tag{4.2}$$

4.2 Let $\alpha\colon I \to M$ be a smooth curve. A *vector field along* α is a smooth mapping $X\colon I \to TM$ such that $X(t) \in T_{\alpha(t)}M$ for all $t \in I$, i.e., $\tau \circ X = \alpha$. The set of all vector fields along α forms the $C^\infty(I)$-module $\mathfrak{X}_\alpha(M) := \{X \in C^\infty(I, TM) |\ \tau \circ X = \alpha\}$. The *velocity vector field*

$$\dot{\alpha} := \alpha_* \circ \frac{d}{dr} \tag{4.3}$$

of α is an example of a vector field along α. By the *acceleration vector field* $\ddot{\alpha} \in \mathfrak{X}_{\dot{\alpha}}(TM)$ of α we mean the velocity vector field of the curve $\dot{\alpha}\colon I \to TM$, i.e.,

$$\ddot{\alpha} := (\dot{\alpha})^{\cdot} = (\dot{\alpha})_* \circ \frac{d}{dr}.$$

If $\alpha(t)$ is in the chart domain $\mathcal{U}$, then

$$\ddot{\alpha}(t) = \sum_{i=1}^n \left((\alpha^i)'(t) \left(\frac{\partial}{\partial x^i}\right)_{\dot{\alpha}(t)} + (\alpha^i)''(t) \left(\frac{\partial}{\partial y^i}\right)_{\dot{\alpha}(t)} \right). \tag{4.4}$$

4.3 Let V be an n-dimensional, non-trivial real vector space, endowed with the canonical smooth structure determined by a single chart (V, φ), where $\varphi\colon V \to \mathbb{R}^n$ is a linear isomorphism. Given any point p in V, V may be naturally identified with its tangent space T_pV via the mapping $\iota_p\colon V \to T_pV, v \mapsto \iota_p(v) := \dot{\alpha}_p(0)$, where $\alpha_p(t) := p + tv, \quad t \in \mathbb{R}$.

4.4 A curve $\alpha\colon I \to M$ is an *integral curve* of a vector field X on M if $\dot\alpha = X \circ \alpha$, i.e, $\dot\alpha(t) = X(\alpha(t))$ for all $t \in I$. If $\widetilde{I}$ is an open interval containing I, then an integral curve $\widetilde{\alpha}\colon \widetilde{I} \to M$ of X is an *extension* of α if $\widetilde{\alpha} \restriction I = \alpha$. An integral curve of X is *maximal* if it has no proper extension. A vector field on M is called *complete* if each of its maximal integral curves is defined on the entire real line.

4.5 Let X be a vector field on M and let a point $p \in M$ be given. *There exists a unique integral curve $\gamma_p\colon I_p \to M$ of X such that $\gamma_p(0) = p$.* We say that the integral curve γ_p *starts at* p. A function $f \in C^\infty(M)$ is called *a first integral* for $X \in \mathfrak{X}(M)$ if $Xf = 0$. This holds if, and only if, X is constant along the integral curves of f, i.e., the function $f \circ \alpha\colon I \to \mathbb{R}$ is constant for every integral curve $\alpha\colon I \to M$ of X.

4.6 Given a vector field X on M, there exists an open subset $\mathcal{D}(X)$ in $\mathbb{R} \times M$ and a smooth mapping $\varphi^X : \mathcal{D}(X) \to M$ satisfying the following conditions:

(a) For each $p \in M$, $\{t \in \mathbb{R} | (t,p) \in \mathcal{D}(X)\} = I_p$, and the mapping $I_p \to M$, $t \to \varphi^X(t,p)$ is the maximal integral curve of X starting at p. Thus, in particular, $\varphi^X(0,p) = p$.

(b) For each $t \in \mathbb{R}$, $\mathcal{D}_t(X) := \{p \in M |\ (t,p) \in \mathcal{D}(X)\}$ is an open subset of M and the mapping

$$\varphi_t^X : p \in \mathcal{D}_t(X) \mapsto \varphi_t^X(p) := \varphi^X(t,p) \in M$$

has the following properties:

(i) If (t,p) and $(s, \varphi_t^X(p))$ are elements of $\mathcal{D}(X)$, then $(s+t,p)$ is also an element of $\mathcal{D}(X)$, and we have

$$\varphi_s^X \circ \varphi_t^X = \varphi_{s+t}^X. \tag{4.5}$$

(ii) φ_t^X is a diffeomorphism of $\mathcal{D}_t(X)$ onto $\mathcal{D}_{-t}(X)$ with inverse φ_{-t}^X.

The mapping φ^X is called the *local flow* of X; it is uniquely determined by its (*infinitesimal*) *generator* X. In view of relation (4.5) we also say, less precisely, that (φ_t^X) is the *local one-parameter group generated by* X, whose tth *stage* is the (local) diffeomorphism φ_t^X from $\mathcal{D}_t(X)$ onto $\mathcal{D}_{-t}(X)$. When the vector field X is clear from the context, we simply write (φ_t). If X is complete, then $\mathcal{D}(X) = \mathbb{R} \times M$, and the smooth mapping φ^X is called the *global flow* of X. In this case we have:

$\varphi_0^X = 1_M; \varphi_s^X \circ \varphi_t^X = \varphi_{s+t}^X$ *for all* $s, t \in \mathbb{R}$*(so the stages of* φ^X *commute);* *every stage* φ_t^X *is a diffeomorphism of* M *with* $(\varphi_t^X)^{-1} = \varphi_{-t}^X$.

We also say that $(\varphi_t^X)_{t\in\mathbb{R}}$ (or $(\varphi_t)_{t\in\mathbb{R}}$) is the *(global) one-parameter group* generated by X. *Every vector field on compact manifold is complete.*

4.7 Let X be a vector field on M, and let $\varphi^X : \mathcal{D}(X) \to M$ its local flow. Suppose that $(t,p) \in \mathcal{D}(X)$. Then for every smooth function f on M

$$\begin{aligned}(Xf)(p) &= \lim_{t\to 0} \frac{1}{t}(f \circ \varphi_t^X(p) - f(p)) = \lim_{t\to 0} \frac{1}{t}(f \circ \alpha_p(t) - f(p)) \\ &= (f \circ \alpha_p)'(0) = \dot{\alpha}_p(0)(f)\end{aligned} \tag{4.6}$$

where α_p is the maximal integral curve of X starting at p. If Y is another vector field on M, then

$$\begin{aligned}[X,Y](p) &= \lim_{t\to 0} \frac{1}{t}\left(((\varphi_{-t}^X)_\# Y)_p - Y(p)\right) \\ &= \lim_{t\to 0} \frac{1}{t}\left((\varphi_{-t}^X)_* \circ Y \circ \varphi_t^X(p) - Y(p)\right).\end{aligned} \tag{4.7}$$

We abbreviate formulas (4.6) and (4.7) as

$$Xf = \lim_{t\to 0} \frac{1}{t}(f \circ \varphi_t^X - f) \tag{4.8}$$

and

$$[X,Y] = \lim_{t\to 0} \frac{1}{t}((\varphi_{-t}^X)_\# Y - Y), \tag{4.9}$$

respectively. Notice that relation (4.9) can also be written in the form

$$[X,Y] = \lim_{t\to 0} \frac{1}{t}(Y \circ \varphi_t^X - (\varphi_t^X)_* \circ Y). \tag{4.10}$$

To see this, note first that

$$\begin{aligned}&\lim_{t\to 0} \frac{1}{t}(Y \circ \varphi_t^X - (\varphi_t^X)_* \circ Y) = \lim_{t\to 0} (\varphi_t^X)_*((\varphi_{-t}^X)_* \circ Y \circ \varphi_t^X - Y) \\ &= \lim_{t\to 0} (\varphi_t^X)_*(\frac{1}{t}((\varphi_{-t}^X)_\# Y - Y)).\end{aligned}$$

Now let $p \in M$ be a fixed point and introduce the mappings

$$\eta\colon I_p \times T_pM \to M,\ (t,v) \mapsto \eta(t,v) := (\varphi_t^X)_*(v)$$

and

$$Z\colon I_p \to T_pM,\ t \to Z(t) := \frac{1}{t}(((\varphi_{-t}^X)_\# Y)_p - Y_p).$$

Then Z is continuous and $\lim\limits_{t\to 0} Z(t) \overset{(4.7)}{=} [X,Y]_p$, so we obtain

$$\lim_{t\to 0}\frac{1}{t}(Y_{\varphi_t^X(p)} - (\varphi_t^X)_*(Y_p)) = \lim_{t\to 0}\eta(t,Z(t)) = \eta(0,\lim_{t\to 0} Z(t))$$
$$= (\varphi_0^X)_*([X,Y]_p) = [X,Y]_p,$$

as we claimed.

4.8 Let X and Y be vector fields on M with local flows φ^X and φ^Y, respectively. Then following assertions are equivalent: The Lie bracket $[X,Y]$ vanishes; 'the vector field Y is invariant under the flow of X', i.e.,

$$(\varphi_{-t}^X)_\# Y = Y \restriction \mathcal{D}_t(X)$$

whenever $\mathcal{D}_t(X) \neq \emptyset$; 'the local flows of X and Y commute', i.e., $\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X$ whenever either side is defined.

5 Tensor fields and differential forms

5.1 Let M be a manifold. By a *tensor field* on M we mean a tensor in

$$T_k(\mathfrak{X}(M)) \cup T_k^1(\mathfrak{X}(M)),\ k \in \mathbb{N}.$$

Then we write

$$\mathcal{T}_k(M) := T_k(\mathfrak{X}(M)), \quad \mathcal{T}_k^1(M) := T_k^1(\mathfrak{X}(M))$$

In particular (see **2.0.6**)

$$\mathcal{T}_0(M) = C^\infty(M),\ \mathcal{T}_0^1(M) = \mathfrak{X}(M),\ \mathcal{T}_1(M) = \mathfrak{X}^*(M) := (\mathfrak{X}(M))^*,$$
$$\mathcal{T}_1^1(M) = \mathrm{End}_{C^\infty(M)}(\mathfrak{X}(M)).$$

Instead of 'tensor field on M' we also say simply that 'tensor on M'. The elements of $\mathfrak{X}^*(M)$ are called *1-forms* on M. If $f \in C^\infty(M)$, then

$$df\colon \mathfrak{X}(M) \to C^\infty(M),\ X \mapsto df(X) := Xf \tag{5.1}$$

is a 1-form on M, the *differential* of f.

Given a tensor field $A \in \mathcal{T}_k(M) \cup \mathcal{T}^1_k(M)$ and a point p of M, A has a well-defined value

$$A_p \in T_k(T_pM) \cup T^1_k(T_pM)$$

at p (see, e.g., [24], pp. 37-38). Using this fact, we define the *trace* of a tensor $A \in \mathcal{T}^1_1(M)$ as the smooth function

$$\mathrm{tr}A\colon M \to \mathbb{R},\ p \mapsto (\mathrm{tr}A)_p := \mathrm{tr}(A_p),$$

where the right-hand side is given by (2.5). This definition is extended to tensors $B \in \mathcal{T}^1_k(M)$, $k > 1$, as follows: $\mathrm{tr}B \in \mathcal{T}_{k-1}(M)$ such that

$$\mathrm{tr}B(X_1, \dots, X_k) := \mathrm{tr}(X \in \mathfrak{X}(M) \mapsto B(X, X_1, \dots, X_{k-1})) \in \mathfrak{X}(M). \tag{5.2}$$

5.2 The Grassmann algebra of a manifold The elements of

$$\mathcal{A}_k(M) := A_k(\mathfrak{X}(M)) \text{ and } \mathcal{A}^1_k(M) := A^1_k(\mathfrak{X}(M))$$

are called *k-forms* and *vector k-forms* on M, respectively. Notice that $\mathcal{A}_k(M) = \{0\}$ if $k > n = \dim M$; we agree that $\mathcal{A}_k(M) := \{0\}$, if k is a negative integer. We define the *wedge product* $\alpha \wedge \beta$ of a k-form $\alpha \in \mathcal{A}_k(M)$ and an l-form $\beta \in \mathcal{A}_l(M)$ by

$$\begin{aligned} &\alpha \wedge \beta(X_1, \dots, X_{k+l}) \\ &\quad := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \epsilon(\sigma)\alpha(X_{\sigma(1)}, \dots X_{\sigma(k)})\beta(X_{\sigma(k+1),\dots,X_{\sigma(k+l)}}), \end{aligned} \tag{5.3}$$

Then $\alpha \wedge \beta \in \mathcal{A}_{k+l}(M)$. The wedge product makes the direct sum $\mathcal{A}(M) := \oplus^n_{k=0}\mathcal{A}_k(M)$ into an algebra over the ring $C^\infty(M)$, called the *Grassmann algebra* of M. This algebra is

graded, i.e., $\alpha \wedge \beta \in \mathcal{A}_{k+l}(M)$ if $\alpha \in \mathcal{A}_k(M)$ and $\beta \in \mathcal{A}_l(M)$;

associative, i.e., $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ for all $\alpha, \beta, \gamma \in \mathcal{A}(M)$;

graded commutative, i.e., if $\alpha \in \mathcal{A}_k(M)$ and $\beta \in \mathcal{A}_l(M)$, then

$$\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha. \tag{5.4}$$

5.3 Derivations of the Grassmann algebra Let $r \in \mathbb{Z}$. An $\mathbb{R}$-linear mapping $D\colon \mathcal{A}(M) \to \mathcal{A}(M)$ is a *graded derivation of degree* r of $\mathcal{A}(M)$ if

(i) $D(\mathcal{A}_k(M)) \subset \mathcal{A}_{k+r}(M)$ for all $k \in \mathbb{Z}$;

(ii) for any $\alpha \in \mathcal{A}_k(M)$ and $\beta \in \mathcal{A}(M)$ we have

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{rk} \alpha \wedge D\beta. \tag{5.5}$$

Lemma 5.3.1. *Every graded derivation of the Grassmann algebra $\mathcal{A}(M)$ is uniquely determined by its action on the smooth functions on M and on their differentials.*

For a proof we refer to [29], Lemma 3.3.23.

5.3.2 If D_1 and D_2 are graded derivations of $\mathcal{A}(M)$ of degree r_1 and r_2, respectively, then their *graded commutator*

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1 \tag{5.6}$$

is a graded derivation of $\mathcal{A}(M)$ of degree $r_1 + r_2$. The graded commutator is *graded anticommutative* in the sense that

$$[D_1, D_2] = -(-1)^{r_1 r_2} [D_2, D_1], \tag{5.7}$$

and satisfies the *graded Jacobi identity*

$$\begin{aligned}(-1)^{r_1 r_3}[D_1, [D_2, D_3]] + (-1)^{r_2 r_1}[D_2, [D_3, D_1]]& \\ + (-1)^{r_3 r_2}[D_3, [D_1, D_2]] = 0,&\end{aligned} \tag{5.8}$$

where r_i is the degree of D_i, $i \in \{1, 2, 3\}$.

5.3.3 The classical graded derivations of $\mathcal{A}(M)$ are the *substitution operator* i_X, the *Lie derivative* $\mathcal{L}_X$ $(X \in \mathfrak{X}(M))$ and the *exterior derivative* d. Their degrees are -1,0 and 1, respectively, and they are defined by the following formulas:

$$(i_X \alpha)(X_2, \ldots, X_k) := \alpha(X, X_2, \ldots, X_k), \tag{5.9}$$

$$\begin{aligned}(\mathcal{L}_X \alpha)(X_1, \ldots, X_k) &:= X(\alpha(X_1, \ldots, X_k)) \\ &\quad - \sum_{i=1}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k),\end{aligned} \tag{5.10}$$

$$\begin{aligned}d\alpha(X_0, \ldots, X_k) &:= \sum_{i=0}^{k} (-1)^i X_i \, \alpha(X_0, \ldots, \breve{X}_i, \ldots, X_k) \\ &+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \breve{X}_i, \ldots, \breve{X}_j, \ldots, X_k);\end{aligned} \tag{5.11}$$

$$i_X f := 0,\ \mathcal{L}_X f := Xf,\ df(X) := Xf;\ f \in C^\infty(M). \qquad (5.12\,\text{a-c})$$

In formulas (5.9)-(5.11), $\alpha \in \mathcal{A}_k(M)$, $k > 1$. In (5.11) the notation $\breve{X}_i$ means that the argument X_i is deleted. These operators satisfy the identities

$$[i_X, i_Y] \overset{(5.6)}{=} i_X \circ i_Y + i_Y \circ i_X = 0, \qquad (5.13)$$
$$[\mathcal{L}_X, i_Y] = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X,Y]}, \qquad (5.14)$$
$$[i_X, d] = i_X \circ d + d \circ i_X = \mathcal{L}_X, \qquad (5.15)$$
$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]}, \qquad (5.16)$$
$$[\mathcal{L}_X, d] = \mathcal{L}_X \circ d - d \circ \mathcal{L}_X = 0, \qquad (5.17)$$
$$d^2 := d \circ d = 0. \qquad (5.18)$$

6 Covariant derivatives

6.1 A *covariant derivative* on a vector bundle $\pi\colon E \to M$ is a mapping

$$D\colon \mathfrak{X}(M) \times \Gamma(\pi) \to \Gamma(\pi),\ (X, s) \mapsto D_X s$$

which is tensorial in X and derivation in s, i.e, which satisfies the conditions $D_{fX}s = fD_X s$, $D_X fs = (Xf)s + fD_X s$ ($f \in C^\infty(M)$). The smooth section $D_X s$ is called the *covariant derivative of s with respect to X*. The *covariant differential* of a section $s \in \Gamma(\pi)$ is the mapping

$$Ds\colon \mathfrak{X}(M) \to \Gamma(\pi),\ X \mapsto Ds(X) := D_X s.$$

More generally, let $k \in \mathbb{N}^*$ and suppose that

$$A\colon (\Gamma(\pi))^k \to C^\infty(M) \text{ and } B\colon (\Gamma(\pi))^k \to \Gamma(\pi)$$

are $C^\infty(M)$-multilinear mappings. Then we say that A and B are *π-tensor fields* of type $(0, k)$ and $(1, k)$, respectively. We define their covariant differentials DA and DB by

$$\begin{aligned} DA(X, s_1, s_2, \dots, s_k) &:= (D_X A)(s_1, \dots, s_k) := X(A(s_1, \dots, s_k)) \\ &- \sum_{i=1}^{k} A(s_1, \dots, D_X s_i, \dots, s_k) \end{aligned} \qquad (6.1)$$

and

$$\begin{aligned} DB(X, s_1, \dots, s_k) &:= (D_X B)(s_1, \dots, s_k) := D_X(B(s_1, \dots, s_k)) \\ &- \sum_{i=1}^{k} B(s_1, \dots, D_X s_i, \dots, s_k), \end{aligned} \qquad (6.2)$$

respectively. Then $D_X A$ is also a type $(0,k)$, $D_X B$ is also a type $(1,k)$ π-tensor field.

If g is a scalar product on π (**2.11**), then a covariant derivative D on π is called *compatible* with g or a *metric derivative* if $Dg = 0$.

Lemma 6.1.1 (localization). *Let $\pi\colon E \to M$ be a vector bundle and D is a covariant derivative on π. Suppose that two sections $s_1, s_2 \in \Gamma(\pi)$ coincide in a neighbourhood of a point $p \in M$. Then*

$$(D_X s_1)(p) = (D_X s_2)(p) \text{ for all } X \in \mathfrak{X}(M).$$

For a proof, see, e.g., [5], Lemma 1.3.

Using this lemma we may define the covariant derivatives of (smooth) local sections of π.

6.2 The *curvature tensor field* (briefly the *curvature tensor*) of a covariant derivative D on $\pi\colon E \to M$ is the mapping

$$\begin{aligned} &R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(\pi) \to \Gamma(\pi), \\ &(X,Y,s) \mapsto R(X,Y)s := D_X D_Y s - D_Y D_X s - D_{[X,Y]} s. \end{aligned} \tag{6.3}$$

It can quickly be checked that R is $C^\infty(M)$-linear in all three arguments and skew-symmetric in the first two arguments.

6.3 By a *covariant derivative on a manifold M* we mean a covariant derivative

$$D\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), (X,Y) \mapsto D_X Y$$

on its tangent bundle. Then we define the *torsion tensor* $T \in \mathfrak{T}^1_2(M)$ of D by

$$T(X,Y) := D_X Y - D_Y X - [X,Y];\; X,Y \in \mathfrak{X}(M). \tag{6.4}$$

If $T = 0$ we say that D is *torsion-free* or *symmetric.*

Given a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ of M, and using the localization lemma 6.1.1, we define the *Christoffel symbols* $\Gamma^i_{jk} \in C^\infty(\mathcal{U})$ $(i,j,k \in J_n)$ of D with respect to the chosen chart by

$$D_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^k} = \sum \Gamma^i_{jk} \frac{\partial}{\partial u^i}. \tag{6.5}$$

6.4 A diffeomorphism $\varphi\colon U \to V$ between two open subsets of M is called a *local automorphism* of a covariant derivative D on M if

$$\varphi_\#(D_X Y \restriction U) = (D_{\varphi_\# X}\varphi_\# Y) \restriction V \quad \text{for all } X, Y \in \mathfrak{X}(M). \tag{6.6}$$

Then we also say that φ is a *(local) D-automorphism.* We define a vector field X on M to be *D-Killing* if the stages of its local one-parameter group are local automorphisms of D. We denote by $\mathrm{Kill}_D(M)$ the set of D-Killing vector fields on M. (Here we follow the terminology and notation of Serge Lang [16].)

Proposition 6.4.1. *Let M be a manifold together with a covariant derivative D on M. Then we have:*

$$X \in \mathrm{Kill}_D(M) \iff \mathcal{L}_X D = 0. \tag{6.7}$$

For a proof see, e.g., [26], 2.123 Proposition, (i).

6.5 Suppose that D is a covariant derivative on M. Given a vector field $X \in \mathfrak{X}(M)$, let

$$\begin{aligned}(\mathcal{L}_X D)(Y, Z) &:= \mathcal{L}_X(D_Y Z) - D_{\mathcal{L}_X Y} Z - D_Y(\mathcal{L}_X Z) \\ &= [X, D_Y Z] - D_{[X,Y]} Z - D_Y[X, Z]\end{aligned} \tag{6.8}$$

It is easily checked that $\mathcal{L}_X D$ is $C^\infty(M)$-linear in both of its arguments. So $\mathcal{L}_X D$ is a type $(1, 2)$ tensor field on M, called the *Lie derivative of D.*

Lemma 6.5.1 (see [26], 2.123 Proposition (ii))**.** *Let D be a covariant derivative on M with curvature R. If D is torsion-free, then*

$$(\mathcal{L}_X D)(Y, Z) = (R(X, Y) + D_Y(DX))(Z) \tag{6.9}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Thus

$$(\mathcal{L}_X D) = 0 \iff D_Y DX = R(Y, X) \text{ for all } Y \in \mathfrak{X}(M). \tag{6.10}$$

(where $R(X, Y)$ means the endomorphism

$$\mathfrak{X}(M) \to \mathfrak{X}(M),\ Z \mapsto R(X, Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.)$$

The proof is a straightforward calculation.

7 Constructions on the tangent bundle

7.1 Throughout this section, M is an n-dimensional manifold, $\tau\colon TM \to M$ is the tangent bundle of M, and $\tau_T M\colon TTM \to TM$ is the tangent bundle of TM. For local descriptions and calculations, we fix a chart $(\mathcal{U}, u) = (\mathcal{U}, (u^i)_{i=1}^n)$ on M, and consider the induced chart $(\tau^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ (see(3.2)) on TM. If f is a smooth function on M, then $f^{\mathsf{v}} := f \circ \tau$ and $f^{\mathsf{c}}\colon TM \to \mathbb{R}$, $v \mapsto f^{\mathsf{c}}(v) := v(f)$ are smooth functions on TM, called the *vertical lift* and the *complete lift* of f, respectively. Locally,

$$f^{\mathsf{c}} \underset{(\mathcal{U})}{=} \sum y^i \left(\frac{\partial f}{\partial u^i} \circ \tau \right). \tag{7.1}$$

7.2 The vertical bundle of TTM Given a vector $v \in TM$, the vector space

$$V_vTM := \operatorname{Ker}(\tau_*)_v := \{w \in T_vTM | (\tau_*)_v(w) = 0\} \subset T_vTM \tag{7.2}$$

is called the *vertical subspace* of T_vTM. The elements of V_vTM are mentioned as *vertical vectors* at v. Since $(\tau_*)_v\colon T_vTM \to T_{\tau(v)}M$ is a surjective linear mapping, it follows that $\dim V_vTM = \dim T_{\tau(v)}M = n$. A basis for V_vTM is the family $\left(\left(\frac{\partial}{\partial y^i} \right)_v \right)_{i=1}^n$. Let

$$VTM := \mathring{\bigcup}_{v \in TM} T_vTM, \quad \tau_{TM}^{\mathsf{v}} := \tau_{TM} \restriction VTM.$$

Then VTM has a unique smooth structure such that τ_{TM}^{v} becomes a subbundle of τ_{TM} (or TTM). This vector bundle is called the *vertical bundle* of τ_{TM}. We denote by $\mathfrak{X}^{\mathsf{v}}(TM)$ the $C^\infty(M)$-module of smooth sections of τ_{TM}^{v}, and we call the elements of $\mathfrak{X}^{\mathsf{v}}(TM)$ *vertical vector fields* on TM. It is easy to show that for a vector field $\xi \in \mathfrak{X}(TM)$ the following are equivalent:

(i) $\xi \in \mathfrak{X}^{\mathsf{v}}(TM)$;

(ii) $\xi \underset{\tau}{\sim} o$, where $o \in \mathfrak{X}(M)$ is the zero vector field;

(iii) $\xi(f^{\mathsf{v}}) = 0$ for all $f \in C^\infty(M)$

(see, e.g., [29], Lemma 4.1.28). From this and from the related vector field lemma 3.7.8 we conclude immediately that

$$\xi_1, \xi_2 \in \mathfrak{X}^{\mathsf{v}}(TM) \quad \text{implies} \quad [\xi_1, \xi_2] \in \mathfrak{X}^{\mathsf{v}}(TM),$$

and hence $\mathfrak{X}^{\mathsf{v}}(TM)$ *is a subalgebra of the Lie algebra* $\mathfrak{X}(TM)$.

7.3 Vertical lift Given a point $p \in M$ and two tangent vectors u, v at p, define a tangent vector $v^\uparrow(u) \in T_u TM$ by

$$v^\uparrow(u) := (i_p \circ \alpha)^\cdot(0), \tag{7.3}$$

where $i_p \colon T_pM \to TM$ is the canonical inclusion, and $\alpha \colon \mathbb{R} \to T_pM$ is a smooth curve given by $\alpha(t) := u + tv,\ t \in \mathbb{R}$. Applying (4.2), we find that

$$\begin{aligned}(i_p \circ \alpha)^\cdot(0) &= \sum ((x^i \circ i_p \circ \alpha)'(0) \left(\frac{\partial}{\partial x^i}\right)_u + (y^i \circ i_p \circ \alpha)'(0) \left(\frac{\partial}{\partial y^i}\right)_u \\ &= \sum y^i(v) \left(\frac{\partial}{\partial y^i}\right)_u ,\end{aligned}$$

i.e.,

$$v^\uparrow(u) = \sum y^i(v) \left(\frac{\partial}{\partial y^i}\right)_u . \tag{7.4}$$

Thus $v^\uparrow(u)$ is a vertical vector at u, called the *vertical lift of $v \in T_pM$ to $u \in T_pM$*. The *vertical lift of a vector field $X \in \mathfrak{X}(M)$* is the vertical vector field

$$X^{\mathsf{v}} \colon u \in TM \mapsto X^{\mathsf{v}}(u) := (X(\tau(u)))^\uparrow(u) \in V_v TM.$$

If $X \underset{(\mathcal{U})}{=} \sum X^i \dfrac{\partial}{\partial u^i}$, then

$$X^{\mathsf{v}} \underset{(\mathcal{U})}{=} \sum (X^i \circ \tau) \frac{\partial}{\partial y^i}. \tag{7.5}$$

This implies immediately that

$$(X + Y)^{\mathsf{v}} = X^{\mathsf{v}} + Y^{\mathsf{v}}, \quad (fX)^{\mathsf{v}} = f^{\mathsf{v}} X^{\mathsf{v}}, \tag{7.6}$$

for all $X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$.

Lemma 7.3.1. *Let X be a vector field on M.*

(i) *The vertical lift X^{v} of X is the unique vertical vector field on TM such that*

$$X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}} \quad \textit{for all} \quad f \in C^\infty(M). \tag{7.7}$$

(ii) *The vector field X^{v} is complete, and its flow is given by*

$$\varphi^{X^{\mathsf{v}}}(t, v) = v + tX(\tau(v)) \quad \textit{for all} \quad (t, v) \in \mathbb{R} \times TM. \tag{7.8}$$

7.4 The Liouville vector field We have a canonical vertical vector field on TM, the *Liouville vector field*

$$C\colon TM \to TTM,\ v \mapsto C(v) := v^{\uparrow}(v). \tag{7.9}$$

Locally,

$$C \underset{(\mathcal{U})}{=} \sum y^i \frac{\partial}{\partial y^i}. \tag{7.10}$$

Lemma 7.4.1. (i) *The Liouville vector field is the unique vertical vector field on TM such that*

$$Cf^{\mathsf{c}} = f^{\mathsf{c}} \textit{ for all } f \in C^\infty(M). \tag{7.11}$$

(ii) *Define the* dilatations μ_t $(t \in \mathbb{R}^*)$ *and* positive dilatations μ_t^+ *(t is a real number) of TM by*

$$\mu_t(v) := tv \quad \textit{and} \quad \mu_t^+(v) := e^t v, \tag{7.12}$$

respectively. Then

$$C \underset{\mu_t}{\sim} C, \quad \textit{i.e.,} \quad (\mu_t)_* \circ C = C \circ \mu_t \textit{ for all } t \in \mathbb{R}^*. \tag{7.13}$$

The Liouville vector field is complete, its flow is given by

$$\varphi^C(t,v) = e^t v \quad \textit{for all } (t,v) \in \mathbb{R} \times TM, \tag{7.14}$$

i.e., the one-parameter group generated by C is $(\mu_t^+)_{t\in\mathbb{R}}$.

7.5 The complete lift of a vector field Given a vector field X on M, there exists a unique vector field X^{c} on TM such that for every smooth function f on M,

$$X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}} \quad \text{and } X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}} \tag{7.15 a-b}$$

The vector field X^{c} is called the *complete lift* of X. Locally, if $X \underset{(\mathcal{U})}{=} \sum X^i \frac{\partial}{\partial u^i}$, then

$$\begin{aligned} X^{\mathsf{c}} &= \sum \left((X^i \circ \tau) \frac{\partial}{\partial x^i} + \sum y^j \left(\frac{\partial X^i}{\partial u^j} \circ \tau \right) \frac{\partial}{\partial y^i} \right) \\ &\overset{(7.1)}{=} \sum \left((X^i)^{\mathsf{v}} \frac{\partial}{\partial x^i} + (X^i)^{\mathsf{c}} \frac{\partial}{\partial y^i} \right). \end{aligned} \tag{7.16}$$

We have, in particular,

$$\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{c}} = \frac{\partial}{\partial x^i} \quad i \in J_n. \tag{7.17}$$

It can be seen immediately from (7.16), that X^{c} *is* τ*-related to* X, i.e.,

$$\tau_* \circ X^{\mathsf{c}} = X \circ \tau. \tag{7.18}$$

A further consequence of (7.16) is that

$$(X+Y)^{\mathsf{c}} = X^{\mathsf{c}} + Y^{\mathsf{c}}, \quad (fX)^{\mathsf{c}} = f^{\mathsf{v}} X^{\mathsf{c}} + f^{\mathsf{c}} X^{\mathsf{v}} \tag{7.19}$$

for all $X, Y \in \mathfrak{X}(M),\ f \in C^\infty(M)$. It can also easily be shown that

$$(\varphi_*)_\# X^{\mathsf{c}} = (\varphi_\# X)^{\mathsf{c}} \text{ for all } \varphi \in \mathrm{Diff}(M). \tag{7.20}$$

Lemma 7.5.1. *Let X be a vector field on M with local flow φ^X. Then the local flow $\varphi^{X^{\mathsf{c}}} \colon \mathcal{D}_{X^{\mathsf{c}}} \subset \mathbb{R} \times TM \to TM$ of X^{c} is given by*

$$\varphi^{X^{\mathsf{c}}}(t,v) = (\varphi_t^X)_*(v) \quad \textit{for all } (t,v) \in \mathcal{D}_{X^{\mathsf{c}}}. \tag{7.21}$$

Otherwise stated, if X generates the local one-parameter group (φ_t^X), then X^{c} generates the local one-parameter group $((\varphi_t^X)_*)$.

7.6 Formulas for Lie brackets For any vector fields X, Y on M we have

$$[X^{\mathsf{v}}, Y^{\mathsf{v}}] = 0,\ [X^{\mathsf{v}}, Y^{\mathsf{c}}] = [X,Y]^{\mathsf{v}},\ [X^{\mathsf{c}}, Y^{\mathsf{c}}] = [X,Y]^{\mathsf{c}} \tag{7.22 a-c}$$
$$[C, X^{\mathsf{v}}] = -X^{\mathsf{v}},\ [C, X^{\mathsf{c}}] = 0. \tag{7.23 a-b}$$

7.7 Homogeneity

7.7.1 Let $\widetilde{TM} \subset TM$ be an open subset, and let $\widetilde{\tau} := \tau \restriction \widetilde{TM}$. We say that $\widetilde{TM}$ is a *conic subset* of TM if $\widetilde{\tau}(\widetilde{TM}) = M$ and

$$\mu_t^+(v) \in \widetilde{TM} \ \text{ for all } v \in \widetilde{TM},\ t \in \mathbb{R}. \tag{7.24}$$

Obviously, TM is a conic subset of itself. A further important example is

$$\mathring{T}M := TM \setminus o(M),\ o \in \mathfrak{X}(M) \textit{ is the zero vector field.}$$

Then we write $\mathring{\tau} := \tau \restriction \mathring{T}M$, and call $\mathring{\tau} \colon \mathring{T}M \to M$ the *slit tangent bundle* of M.

If $(\mathcal{U}, (u^i)_{i=1}^n)$ is a chart on M, then we define the induced chart $(\widetilde{\tau}^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ on $\widetilde{TM}$ in the same way as in (3.2). For simplicity, the restriction of a vector field $\xi \in \mathfrak{X}(TM)$ to $\widetilde{TM}$ will usually be denoted also by ξ.

7.7.2 Let $\widetilde{TM}$ be a conic subset of TM, and let r be an integer. A function $F\colon \widetilde{TM} \to \mathbb{R}$ is called *positive-homogeneous of degree* r (or r^+*-homogeneous* for short) if $F \circ \mu_t^+ = e^{rt}F$ for all $t \in \mathbb{R}$.

The following basic results are well-known:
(i) A C^1-function $F\colon \widetilde{TM} \to \mathbb{R}$ is r^+-homogeneous if, and only if, $CF = rF$ (*Euler's theorem*).
(ii) If a function $F\colon TM \to \mathbb{R}$ is continuous on $o(M)$ and 0^+-homogeneous, then it is fibrewise constant.
(iii) If $F \in C^1(TM, \mathbb{R})$ and F is 1^+-homogeneous, then F is fibrewise linear, i.e., $F \restriction T_pM \in (T_pM)^*$ for all $p \in M$.
(iv) If $F \in C^2(TM, \mathbb{R})$ and F is 2^+-homogeneous, then $F \restriction T_pM$ is a quadratic form for all $p \in M$.
(v) Suppose that $r \in \mathbb{N}^*$, and let $\overset{\circ}{F}\colon \overset{\circ}{T}M \to \mathbb{R}$ be an r^+homogeneous continuous function. Then its extension

$$F\colon TM \to \mathbb{R},\ v \mapsto F(v) := \begin{cases} \overset{\circ}{F}(v) & \text{if } v \in \overset{\circ}{T}M \\ 0 & \text{if } v \in o(M) \end{cases}$$

is a continuous function on TM. If $r \geq 2$ and $\overset{\circ}{F}$ is of class C^1, then F is also of class C^1. For proofs see [29], 4.2.9 – 4.2.11.

7.7.3 We continue to assume that $\widetilde{TM} \subset TM$ is a conic subset and $r \in \mathbb{Z}$. A differential form $\omega \in \mathcal{A}(\widetilde{TM})$, resp. a vector k-form L in $\mathcal{A}_k^1(\widetilde{TM})$ is called r^+*-homogeneous* if

$$\mathcal{L}_C\omega = r\,\omega, \quad \text{resp.} \quad \mathcal{L}_C L = (r-1)L. \tag{7.25 a-b}$$

In the special case $L := \xi \in \mathfrak{X}(\widetilde{TM}) = \mathcal{A}_0^1(\widetilde{TM})$ condition (7.25 b) takes the form

$$[C, \xi] = (r-1)\xi. \tag{7.26}$$

From this and from (7.23 a-b) it follows that *the vertical lift of a vector field is* 0^+*-homogeneous, the complete lift of a vector field is* 1^+*-homogeneous.* Thus, in particular, for every $i \in J_n$,

$$\left[C, \frac{\partial}{\partial x^i}\right] = \left[C, \left(\frac{\partial}{\partial u^i}\right)^{\mathsf{c}}\right] = 0, \tag{7.27}$$

$$\left[C, \frac{\partial}{\partial y^i}\right] = \left[C, \left(\frac{\partial}{\partial u^i}\right)^{\mathsf{v}}\right] = -\frac{\partial}{\partial y^i}. \tag{7.28}$$

If $\xi \underset{(\mathcal{U})}{=} \sum \left(\xi^i \frac{\partial}{\partial x^i} + \xi^{n+i} \frac{\partial}{\partial y^i} \right)$, then

$$[C, \xi] \underset{(\mathcal{U})}{=} \sum \left((C\xi^i) \frac{\partial}{\partial x^i} + \xi^i \left[C, \frac{\partial}{\partial x^i} \right] + (C\xi^{n+i}) \frac{\partial}{\partial y^i} + \xi^{n+i} \left[C, \frac{\partial}{\partial y^i} \right] \right)$$
$$\overset{(7.27),\ (7.28)}{=} \sum \left((C\xi^i) \frac{\partial}{\partial x^i} + (C\xi^{n+i} - \xi^{n+i}) \frac{\partial}{\partial y^i} \right),$$

therefore ξ *is* r^+*-homogeneous if, and only if, the component functions* ξ^i *are* $(r-1)^+$*-homogeneous and the component functions* ξ^{n+i} *are* r^+*-homogeneous.*

Part II
Lie derivatives in Finslerian setting

8 Finsler bundles and canonical constructions

8.1 Let M be an n-dimensional manifold. Consider the tangent bundle $\tau\colon TM \to M$ of M, and let $\widetilde{\tau}\colon \widetilde{TM} \to M$ be a 'conic subbundle' of τ as described in **7.7.1**. Form the fibre product

$$\widetilde{TM} \times_M TM := \{(u,v) \in \widetilde{TM} \times TM \mid \widetilde{\tau}(u) = \tau(v)\},$$

and let $\widetilde{\pi} := \mathrm{pr}_1 \restriction \widetilde{TM} \times_M TM$. Then $\widetilde{\pi}\colon \widetilde{TM} \times_M TM \to \widetilde{TM}$ turns out to be a vector bundle of rank n over $\widetilde{TM}$ with fibres

$$\widetilde{\pi}^{-1}(u) = \{(u,v) \in \widetilde{TM} \times TM \mid v \in T_{\widetilde{\tau}(u)}M\} = \{u\} \times T_{\widetilde{\tau}(u)}M \cong T_{\tau(u)}M.$$

This vector bundle is called the *Finsler bundle* over $\widetilde{TM}$. The most important special cases are

$$\pi\colon TM \times_M TM \to TM - \text{ the Finsler bundle over } TM,$$
$$\mathring{\pi}\colon \mathring{T}M \times_M TM \to \mathring{T}M - \text{ the slit Finsler bundle.}$$

8.2 Finsler vector fields The smooth sections of $\widetilde{\pi}$ are of the form

$$\widetilde{X} = (1_{\widetilde{TM}}, \underline{X})\colon u \in \widetilde{TM} \mapsto (u, \underline{X}(u)) \in \widetilde{TM} \times_M TM, \tag{8.1}$$

where $\underline{X} \in C^\infty(\widetilde{TM}, TM)$ is such that $\tau \circ \underline{X} = \widetilde{\tau}$. We say that $\underline{X}$ is the principal part of $\widetilde{X}$. Elements of the $C^\infty(\widetilde{TM})$-module $\Gamma(\widetilde{\pi})$ are also called *Finsler vector fields* on $\widetilde{TM}$. *Finsler vector fields can be identified canonically with their principal parts.*

We have a canonical section $\widetilde{\delta}$ in $\Gamma(\widetilde{\pi})$ with principal part $1_{\widetilde{TM}}$. Thus

$$\widetilde{\delta}\colon \widetilde{TM} \to \widetilde{TM} \times_M TM,\ u \mapsto \widetilde{\delta}(u) := (u,u). \tag{8.2}$$

If $X \in \mathfrak{X}(M)$, then

$$\widehat{X} := (1_{\widetilde{TM}}, X \circ \widetilde{\tau}) : u \in \widetilde{TM} \mapsto (u, X(\widetilde{\tau}(u))) \tag{8.3}$$

is a Finsler vector field on $\widetilde{TM}$. Finsler vector fields of this type are called *basic*. *The* $C^\infty(\widetilde{TM})$*-module* $\Gamma(\widetilde{\pi})$ *is locally generated by the basic Finsler vector fields.* If $(\mathcal{U}, (u^i)_{i=1}^n)$ is a chart on M with induced chart $(\widetilde{\tau}^{-1}(\mathcal{U}), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ on $\widetilde{TM}$, then

$$\widetilde{\delta} \underset{(\mathcal{U})}{=} \sum y^i \widehat{\frac{\partial}{\partial u^i}}, \tag{8.4}$$

$$\widehat{X} = \sum (X^i \circ \tau) \widehat{\frac{\partial}{\partial u^i}} \quad \text{if} \quad X \in \mathfrak{X}(M), \; X \underset{(\mathcal{U})}{=} \sum X^i \frac{\partial}{\partial u^i} \tag{8.5}$$

Given a Finsler vector field $\widetilde{X} \in \Gamma(\pi)$ and a diffeomorphism φ of M, the mapping

$$\varphi_\# \widetilde{X} := (\varphi_* \times \varphi_*) \circ \widetilde{X} \circ (\varphi_*)^{-1} \tag{8.6}$$

is also a Finsler vector field, called the *push-forward* of $\widetilde{X}$ by φ (or, more precisely, by $(\varphi_* \times \varphi_*)$. Here the '$\times$' is defined by (2.2). If $\varphi_\# \widetilde{X} = \widetilde{X}$, then we say that $\widetilde{X}$ *is invariant under* φ (cf. **4.8**). We use the same terminology if φ is a diffeomorphism between two open subsets of M. The following equalities can easily be checked:

$$\varphi_\# \widehat{X} = \widehat{\varphi_\# X} \;\; (X \in \mathfrak{X}(M)), \quad \varphi_\# \widetilde{\delta} = \widetilde{\delta}. \tag{8.7 a-b}$$

8.3 Finsler tensor fields The elements of the $C^\infty(\widetilde{TM})$-modules $T_k(\Gamma(\widetilde{\pi}))$ and $T^1_k(\Gamma(\widetilde{\pi}))$ are called *Finsler tensor fields of type* $(0,k)$ *and* $(1,k)$, respectively. Then, for example, a Finsler tensor field $A \in T^1_k(\Gamma(\widetilde{\pi}))$ $(k \geqq 1)$ is a $C^\infty(\widetilde{TM})$-multilinear mapping from $(\Gamma(\widetilde{\pi}))^k$ to $\Gamma(\widetilde{\pi})$. As a tensor field on a manifold, a Finsler tensor field also has a well-defined value at each point of $\widetilde{TM}$. To illustrate this, we consider two examples.
(a) Let $A \in T^1_1(\Gamma(\widetilde{\pi}))$. Then, for every $v \in \widetilde{TM}$,

$$A_v \in T^1_1(\{v\} \times T_{\tau(v)}M) \cong T^1_1(T_{\tau(v)}M) = \operatorname{End}(T_{\tau(v)}M)$$

such that $(A(\widetilde{X}))(v) = A_v(\widetilde{X}(v))$ for all $\widetilde{X} \in \Gamma(\widetilde{\pi})$.
(b) If $g \in T_2(\Gamma(\widetilde{\pi}))$, then for every $v \in \widetilde{TM}$,

$$g_v \in T_2(\{v\} \times T_{\tau(v)}M) \cong T_2(T_{\tau(v)}M),$$

i.e., $g_v \colon T_{\tau(v)}M \times T_{\tau(v)}M \to \mathbb{R}$ is a bilinear function such that

$$g(\widetilde{X}, \widetilde{Y})(v) = g_v(\widetilde{X}(v), \widetilde{Y}(v)) \text{ for all } \widetilde{X}, \widetilde{Y} \in \Gamma(\widetilde{\pi}).$$

This interpretation makes it possible to define the crucial concept of the *trace* of a Finsler tensor field $A \in T^1_k(\Gamma(\widetilde{\pi}))$ $(k \geqq 1)$ on the analogy of **5.2**. If $k > 1$, then $\mathrm{tr}A \in T_{k-1}(\Gamma(\widetilde{\pi}))$ is given by

$$(\mathrm{tr}A)(\widetilde{X_1}, \dots, \widetilde{X_{l-1}}) := \mathrm{tr}(\widetilde{X} \mapsto A(\widetilde{X}, \widetilde{X_1}, \dots, \widetilde{X_{l-1}})) \in \Gamma(\widetilde{\pi}). \quad (8.8)$$

8.4 The bundle maps i, j and J In what follows, for simplicity, we consider the Finsler bundle $\pi \colon TM \times_M TM \to TM$. However, our constructions may be carried out without changes to the more general case $\widetilde{\pi} \colon \widetilde{TM} \times_M TM \to \widetilde{TM}$.

Definition and Lemma 8.4.1. (i) *The mapping*

$$\mathbf{i} \colon TM \times_M TM \to VTM, \ (u, v) \mapsto \mathbf{i}(u, v) := v^{\uparrow}(u) \quad (8.9)$$

is a strong bundle isomorphism of the Finsler bundle π *onto the vertical bundle* $\tau^{\mathsf{v}}_{TM} \colon VTM \to TM$.

(ii) *The mapping*

$$\mathbf{j} := (\tau_{TM}, \tau_*) \colon TTM \to TM \times_M TM, \ w \mapsto (\tau_{TM}(w), \tau_*(w)) \quad (8.10)$$

is a surjective strong bundle map from the tangent bundle of TM *onto the Finsler bundle over* TM. *Its kernel is the vertical bundle of* TTM.

(iii) *The composite mapping* $\mathbf{j} \circ \mathbf{i} : TM \times_M TM \to TM \times_M TM$ *is the zero mapping, i.e., for all* $(u, v) \in TM \times_M TM$ *we have* $\mathbf{j} \circ \mathbf{i}(u, v) = (\tau(u), 0)$, *where* $0 \in T_{\tau(u)}M$ *is the zero vector.*

(iv) *The composite mapping*

$$\mathbf{J} := \mathbf{i} \circ \mathbf{j} \colon TTM \to TM \times_M TM \to TTM \quad (8.11)$$

is a strong bundle map from TTM into itself, called the vertical endomorphism *of* TTM. *We have*

$$\mathrm{Im}(\mathbf{J}) = \mathrm{Ker}(\mathbf{J}) = VTM, \ \mathbf{J}^2 = 0. \quad (8.12 \text{ a-b})$$

(v) *The sequence*

$$0 \to TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \to 0 \quad (8.13)$$

is a short exact sequence of strong bundle maps in the sense that

$$\mathbf{i} \ \textit{is injective}, \ \mathbf{j} \ \textit{is surjective and} \ \mathrm{Im}(\mathbf{i}) = \mathrm{Ker}(\mathbf{j}). \quad (8.14)$$

(The zeros mean trivial vector bundles over TM, *whose typical fibre is the trivial* $\mathbb{R}$*-vector space* $\{0\}$.*)*

For a proof we refer to [29], Subsection 4.1.3.

Now, by **2.10**, it follows that we also have a short exact sequence

$$0 \to \Gamma(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}(TM) \xrightarrow{\mathbf{j}} \Gamma(\pi) \to 0 \tag{8.15}$$

of $C^\infty(TM)$-homomorphisms, where, for simplicity, we denote the module homomorphisms by the same symbols as the corresponding bundle maps in (8.13). In this interpretation,

$$\mathbf{J} := \mathbf{i} \circ \mathbf{j} \colon \mathfrak{X}(TM) \to \mathfrak{X}(TM)$$

is an endomorphism of the $C^\infty(TM)$-module of vector fields on TM such that

$$\operatorname{Im}(\mathbf{J}) = \operatorname{Ker}(\mathbf{J}) = \mathfrak{X}^{\mathsf{v}}(TM). \tag{8.16}$$

We say that a differential form $\alpha \in \mathcal{A}_k(TM)$ (resp. $A \in \mathcal{A}_k^1(TM)$) is *semibasic* if

$$i_{\mathbf{J}\xi}\alpha = 0 \text{ (resp. } i_{\mathbf{J}\xi}A = 0 \text{ and } \mathbf{J} \circ A = 0\text{) for all } \xi \in \mathfrak{X}(TM). \tag{8.17}$$

It can easily be seen that the mapping

$$\widetilde{A} \in T_1^1(\Gamma(\pi)) = \operatorname{End}(\Gamma(\pi)) \mapsto A := \mathbf{i} \circ \widetilde{A} \circ \mathbf{j} \in \operatorname{End}(\mathfrak{X}(TM)) \tag{8.18}$$

is a canonical isomorphism between the module of endomorphisms of $\Gamma(\pi)$ and the module of *semibasic* endomorphisms of $\mathfrak{X}(TM)$.

For a general discussion of such type of isomorphisms we refer to [28], **2.22.**

Lemma 8.4.2. *Concerning the $C^\infty(TM)$-homomorphisms* $\mathbf{i}$, $\mathbf{j}$ *and the endomorphism* $\mathbf{J} \in End(\mathfrak{X}(TM))$ *we have*

$$\mathbf{i}\widehat{X} = X^{\mathsf{v}}, \ \mathbf{j}X^{\mathsf{c}} = \widehat{X}, \tag{8.19 a-b}$$
$$\mathbf{J}X^{\mathsf{c}} = X^{\mathsf{v}}, \ \mathbf{J}X^{\mathsf{v}} = 0, \tag{8.20 a-b}$$
$$[\mathbf{J}, X^{\mathsf{c}}] = [\mathbf{J}, X^{\mathsf{v}}] = 0, \tag{8.21 a-b}$$

where $X \in \mathfrak{X}(M)$, and, furthermore

$$\mathbf{i}\widetilde{\delta} = C, \quad [\mathbf{J}, C] = \mathbf{J}. \tag{8.22 a-b}$$

The proof is routine.

Lemma 8.4.3. *If φ is a smooth transformation of M, then*

$$\varphi_{**} \circ \mathbf{i} = \mathbf{i} \circ (\varphi_* \times \varphi_*), \quad (\varphi_* \times \varphi_*) \circ \mathbf{j} = \mathbf{j} \circ \varphi_{**} \tag{8.23 a-b}$$

For a proof we refer to [29], Lemma 4.1.64.

9 Vertical calculus

9.1 The vertical endomorphism $\mathbf{J} \in \mathcal{A}_1^1(TM) = \mathrm{End}(\mathfrak{X}(TM))$ induces two graded derivations of the Grassmann algebra $\mathcal{A}(TM)$: a derivation $i_\mathbf{J}$ of degree 0 and a derivation $d_\mathbf{J}$ of degree 1. Referring to Lemma 5.3.1, we define the derivation $i_\mathbf{J}$ by its action on smooth functions and on their differentials:

$$i_\mathbf{J} F := 0 \quad \text{and} \quad i_\mathbf{J} dF := dF \circ \mathbf{J} \quad \text{for all } F \in C^\infty(TM). \tag{9.1}$$

Then the operator $d_\mathbf{J}$ is defined as the graded commutator

$$d_\mathbf{J} := [\mathbf{i}_\mathbf{J}, d] \overset{(5.6)}{=} i_\mathbf{J} \circ d - d \circ i_\mathbf{J}. \tag{9.2}$$

Then

$$d_\mathbf{J} F = i_\mathbf{J} dF = dF \circ \mathbf{J}, \quad d_\mathbf{J} dF = -d(dF \circ \mathbf{J}). \tag{9.3 a-b}$$

9.2 Given a Finsler vector field $\widetilde{X} \in \Gamma(\pi)$, we define a derivation $\nabla^\mathsf{v}_{\widetilde{X}}$ prescribing its action on smooth functions and Finsler vector fields as follows:

$$\nabla^\mathsf{v}_{\widetilde{X}} F := (\mathbf{i}\widetilde{X})F = (dF \circ \mathbf{i})(\widetilde{X}), \quad F \in C^\infty(TM); \tag{9.4}$$

$$\begin{cases} \nabla^\mathsf{v}_{\widetilde{X}} \widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \eta], \quad \widetilde{Y} \in \Gamma(\pi), \\ \eta \in \mathfrak{X}(TM) \ \text{ is such that } \ \mathbf{j}\eta = \widetilde{Y}. \end{cases} \tag{9.5}$$

Then $\nabla^\mathsf{v}_{\widetilde{X}} \widetilde{Y}$ is well-defined: does not depend on the choice of the vector field η satisfying $\mathbf{j}\eta = \widetilde{Y}$. The mapping

$$\nabla^\mathsf{v}_{\widetilde{X}} : \widetilde{Y} \in \Gamma(\pi) \mapsto \nabla^\mathsf{v}_{\widetilde{X}} \widetilde{Y} \in \Gamma(\pi)$$

is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$\nabla^\mathsf{v}_{\widetilde{X}} F\widetilde{Y} = (\nabla^\mathsf{v}_{\widetilde{X}} F)\widetilde{Y} + F \nabla^\mathsf{v}_{\widetilde{X}} \widetilde{Y}. \tag{9.6}$$

Now we extend the operator $\nabla^\mathsf{v}_{\widetilde{X}}$ to act on any Finsler tensor field in such a way that Leibniz's rule remains valid. For any Finsler tensor field $A \in T_k(\Gamma(\pi)) \cup T_k^1(\Gamma(\pi))$ $(k \geqq 1)$ we define the tensor $\nabla^\mathsf{v}_{\widetilde{X}} A$ of the same type by

$$\begin{aligned} (\nabla^\mathsf{v}_{\widetilde{X}} A)(\widetilde{X}_1, \dots, \widetilde{X}_k) := {} & \nabla^\mathsf{v}_{\widetilde{X}} (A(\widetilde{X}_1, \dots, \widetilde{X}_k)) \\ & - \sum_{i=1}^k A(\widetilde{X}_1, \dots \nabla^\mathsf{v}_{\widetilde{X}} \widetilde{\widetilde{X}}_i, \dots, \widetilde{X}_k). \end{aligned} \tag{9.7}$$

We say that $\nabla^{\mathsf{v}}_{\widetilde{X}} A$ is the canonical vertical covariant derivative, briefly the vertical derivative, of A with respect to the Finsler vector field $\widetilde{X}$. The *(canonical) vertical differential* of A is the Finsler tensor field $\nabla^{\mathsf{v}} A$ of type $(0, k+1)$ or $(1, k+1)$ given by

$$\nabla^{\mathsf{v}} A(\widetilde{X}, \widetilde{X}_1, \dots, \widetilde{X}_k) := (\nabla^{\mathsf{v}}_{\widetilde{X}} A)(\widetilde{X}_1, \dots, \widetilde{X}_k). \tag{9.8}$$

Examples. (a) If $F \in C^\infty(TM)$, then $\nabla^{\mathsf{v}} F = dF \circ \mathbf{i}$, therefore

$$\nabla^{\mathsf{v}} F(\widehat{X}) = X^{\mathsf{v}} F \quad \text{for all } X \in \mathfrak{X}(M). \tag{9.9}$$

The 1-form $d_{\mathbf{J}} F$ and the Finsler 1-form $\nabla^{\mathsf{v}} F$ are related by $d_{\mathbf{J}} F = \nabla^{\mathsf{v}} F \circ \mathbf{j}$.
(b) For every section $\widetilde{X} \in \Gamma(\pi)$ and vector field $Y \in \mathfrak{X}(M)$,

$$\nabla^{\mathsf{v}}_{\widetilde{X}} \widehat{Y} = 0. \tag{9.10}$$

Indeed, $\nabla^{\mathsf{v}}_{\widetilde{X}} \widehat{Y} = \mathbf{j}[\mathbf{i}\widetilde{X}, Y^{\mathsf{c}}]$. Since $\mathbf{i}\widetilde{X} \underset{\tau}{\sim} 0$ and $Y^{\mathsf{c}} \underset{\tau}{\sim} 0$, by the related vector field lemma (**3.7.8**) it follows that $[\mathbf{i}\widetilde{X}, Y^{\mathsf{c}}]$ is vertical, hence $\mathbf{j}[\mathbf{i}\widetilde{X}, Y^{\mathsf{c}}] = 0$.
(c) *The vertical differential of the canonical section is the identity transformation of* $\Gamma(\pi)$:

$$\nabla^{\mathsf{v}} \widetilde{\delta} = 1_{\Gamma(\pi)}. \tag{9.11}$$

This can be seen, for example, by an easy local calculation.

10 The classical Lie derivative

10.1 Let M be a manifold and let $X \in \mathfrak{X}(M)$. If $A \in \mathcal{T}_k(M)$, then we define the Lie derivative $\mathcal{L}_X A$ by (5.10) and (5.12 b). Thus, if $k \geqq 1$,

$$\begin{aligned}(\mathcal{L}_X A)(X_1, \dots, X_k) := {} & X(A(X_1, \dots, X_k)) \\ & - \sum_{i=1}^{k} A(X_1, \dots, [X, X_i], \dots, X_k)\end{aligned} \tag{10.1}$$

If $B \in \mathcal{T}^1_k(M)$ $(k \geqq 1)$, then we define

$$\begin{aligned}(\mathcal{L}_X B)(X_1, \dots, X_k) := {} & [X, B(X_1, \dots, X_k)] \\ & - \sum_{i=1}^{k} B(X_1, \dots, [X, X_i], \dots, X_k).\end{aligned} \tag{10.2}$$

If, in particular, $B \in \mathrm{End}(\mathfrak{X}(M))$, then we write

$$[B, Y] := -\mathcal{L}_Y B, \ (Y \in \mathfrak{X}(M)),$$

and we say that $[B, Y]$ is the *Frölicher-Nijenhuis bracket* of B and Y. From (10.2) we obtain

$$[B, Y]X = [BX, Y] - B([X, Y]); \ X \in \mathfrak{X}(M). \tag{10.3}$$

Lemma 10.1.2. *Let $A \in \mathcal{T}_l(M)$, $X \in \mathfrak{X}(M)$. Given a point $p \in M$, we have*

$$(\mathcal{L}_X A)_p = \lim_{t \to 0} \frac{1}{t}(((\varphi_t^X)^* A)_p - A_p), \tag{10.4}$$

where (φ_t^X) is the local one-parameter group generated by X.

Proof. (cf. [21], pp. 147-148 and [24], p. 250). First we note that, for small $t \neq 0$, the difference quotient at the right-hand side of (10.4) has meaning, because φ_t^X is defined in a neighbourhood of p, and $((\varphi_t^X)^* A)_p$ and A_p are elements of the finite-dimensional real vector space $T_l(T_p M)$.

Note further that, on the analogy of (4.8) and (4.9), relation (10.4) can be abbreviated as

$$\mathcal{L}_X A = \lim_{t \to 0} \frac{1}{t}((\varphi_t^X)^* A - A). \tag{10.5}$$

Now we turn to the actual proof. To simplify the writing, we assume that $l = 2$. Then we have to show that

$$(\mathcal{L}_X A)_p(Y_p, Z_p) = \lim_{t \to 0} \frac{1}{t}(((\varphi_t^X)^* A)_p(Y_p, Z_p) - A_p(Y_p, Z_p)), \tag{$*$}$$

for all $Y, Z \in \mathfrak{X}(M)$. By (10.1), the left-hand side of $(*)$ is equal to

$$X_p(A(Y, Z)) - A_p([X, Y]_p, Z_p) - A_p(Y_p, [X, Z]_p).$$

Adding and subtracting a suitable term, the right hand-side of $(*)$ can be manipulated as follows:

$$\begin{aligned}
&\lim_{t \to 0} \frac{1}{t}(((\varphi_t^X)^* A)_p(Y_p, Z_p) - A_p(Y_p, Z_p)) = \\
&\lim_{t \to 0} \frac{1}{t}(A_{\varphi_t^X(p)}((\varphi_t^X)_*(Y_p), (\varphi_t^X)_*(Z_p)) - A_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}, Z_{\varphi_t^X(p)})) \\
&+ \lim_{t \to 0} \frac{1}{t}(A_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}, Z_{\varphi_t^X(p)}) - A_p(Y_p, Z_p) =: L_1 + L_2.
\end{aligned}$$

Here

$$L_2 = \lim_{t\to 0} \frac{1}{t}(A(Y,Z) \circ \varphi_t^X(p) - A(Y,Z)(p)) \overset{(4.6)}{=} X_p(A(Y,Z)).$$

To manipulate expression L_1, we use the telescoping identity

$$A(u',v') - A(u,v) = A(u'-u,v') + A(u,v'-v).$$

Then we find that

$$\begin{aligned} L_1 &= \lim_{t\to 0} \frac{1}{t} A_{\varphi_t^X}(p)((\varphi_t^X)_*(Y_p) - Y_{\varphi_t^X(p)}, (\varphi_t^X)_*(Z_p)) \\ &+ \lim_{t\to 0} \frac{1}{t} A_{\varphi_t^X(p)}(Y_{\varphi_t^X(p)}, (\varphi_t^X)_*(Z_p) - Z_{\varphi_t^X(p)}) \\ &= A_p(\lim_{t\to 0} \frac{1}{t}(\varphi_t^X)_*(Y_p) - Y_{\varphi_t^X(p)}), Z_p) \\ &+ A_p(Y_p, \lim_{t\to 0} \frac{1}{t}(\varphi_t^X)_*(Z_p) - Z_{\varphi_t^X(p)}) \\ &\overset{(4.10)}{=} -A_p([X,Y]_p, Z_p) - A_p(Y_p, [X,Z]_p), \end{aligned}$$

Thus

$$L_1 + L_2 = L_2 + L_1 = X_p(A(Y,Z)) - A_p([X,Y]_p, Z_p) - A_p(Y_p, [X,Z]_p),$$

as was to be shown. □

Lemma 10.1.3. *Let* $B \in \mathrm{End}(\mathfrak{X}(\mathrm{M}))$ *and* $X \in \mathfrak{X}(M)$. *Then for every vector field* Y *on* M,

$$(\mathcal{L}_X B)(Y) = \lim_{t\to 0} \frac{1}{t}(\varphi_{-t}^X)_\# B(Y) - B((\varphi_{-t}^X)_\# Y)), \tag{10.6}$$

where (φ_t^X) is the local one-parameter group generated by X.

Proof. We have immediately that

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{t}(\varphi_{-t}^X)_\# B(Y) - B((\varphi_{-t}^X)_\# Y)) \\ &\lim_{t\to 0} \frac{1}{t}(\varphi_{-t}^X)_\# B(Y) - B(Y)) - B(\lim_{t\to 0} \frac{1}{t}(\varphi_{-t}^X)_\# Y - Y) \\ &= [X, BY] - B([X,Y]) \overset{(10.2)}{=} (\mathcal{L}_X B)(Y). \end{aligned}$$

□

Equality (10.6) can be abbreviated as follows:

$$\mathcal{L}_X B = \lim_{t\to 0} \frac{1}{t}(\varphi^X_{-t})_\# \circ B - B \circ (\varphi^X_{-t})_\#). \tag{10.7}$$

The next result belongs to the folklore, but we were unable to find a proof for it in the literature which was completely satisfactory for us. Due to its key importance, after the preparations above, we include here our own proof.

Proposition 10.1.4. *Let $B \in \mathrm{End}(\mathfrak{X}(\mathrm{M}))$, and let X be a vector field on M with local flow*

$$\varphi^X : \mathcal{D}(X) \subset \mathbb{R} \times M \to M.$$

Then $\mathcal{L}_X B = 0$ if, and only if, B commutes with the derivative of every stage of φ^X, i.e,

$$(\varphi^X_t)_* \circ B = B \circ (\varphi^X_t)_*,$$

where B is regarded as a smooth section of the vector bundle

$$\pi\colon \mathop{\dot{\bigcup}}_{p\in M} \mathrm{End}(T_pM) \to M,\ \ \pi(\psi) := m \textit{ if } \psi \in \mathrm{End}(T_mM).$$

Proof. Suppose first that for every stage φ^X_t of φ^X we have $(\varphi^X_t)_* \circ B = B \circ (\varphi^X_t)_*$. Then, for any $Y \in \mathfrak{X}(M)$,

$$((\varphi^X_{-t})_\# \circ B - B \circ (\varphi^X_{-t})_\#)(Y) = (\varphi^X_{-t})_* \circ B(Y) \circ \varphi^X_t$$
$$- B \circ (\varphi^X_{-t})_* \circ Y \circ \varphi^X_t = ((\varphi^X_{-t})_* \circ B(Y) - B \circ (\varphi^X_{-t})_* \circ Y) \circ \varphi^X_t = 0,$$

which implies by the previous lemma that $\mathcal{L}_X B = 0$.

Conversely, suppose that $\mathcal{L}_X B = 0$. Choose a point $p \in M$ and a tangent vector $v \in T_pM$. Note first that by our assumption, we have

$$[X, BY] = B[X, Y] \text{ for all } Y \in \mathfrak{X}(M).$$

Applying (4.10), we find that

$$\begin{aligned}[X, BY]_p &= \lim_{t\to 0} \frac{1}{t}((BY)_{\varphi^X_t(p)} - (\varphi^X_t)_*(BY)_p) \\ &= \lim_{t\to 0} \frac{1}{t}(B(Y_{\varphi^X_t(p)}) - (\varphi^X_t)_*(B(Y_p)));\end{aligned}$$

$$\begin{aligned}B[X, Y]_p &= B(\lim_{t\to 0} \frac{1}{t}(Y_{\varphi^X_t(p)}) - (\varphi^X_t)_*(Y_p)) \\ &= \lim_{t\to 0} \frac{1}{t}(B(Y_{\varphi^X_t(p)} - B((\varphi^X_t)_*(Y_p))),\end{aligned}$$

from which it follows that

$$\lim_{t\to 0}(B(\varphi_t^X)_*(Y_p)) - (\varphi_t^X)_*(B(Y_p)) = 0.$$

Thus $\mathcal{L}_X B = 0$ implies that

$$\lim_{t\to 0}\frac{1}{t}(B\circ(\varphi_t^X)_*(v) - (\varphi_t^X)_*\circ B(v)) = 0. \quad (*)$$

Now we define a mapping $h\colon I_p \to T_pM$ by

$$h(t) := (\varphi_{-t}^X)_*\circ B\circ(\varphi_t^X)_*(v).$$

Our next goal is to show that h is constant.

Let $t\in I_p$ be arbitrary, and let, for short, $w := (\varphi_t^X)_*(v)$. Then

$$\begin{aligned}
h'(t) &= \lim_{s\to 0}\frac{h(t+s)-h(t)}{s}\\
&= \lim_{s\to 0}\frac{(\varphi_{-t-s}^X)_*\circ B\circ(\varphi_{t+s}^X)_*(v) - (\varphi_{-t}^X)_*\circ B\circ(\varphi_t^X)_*(v)}{s}\\
&(\varphi_{-t}^X)_*\lim_{s\to 0}(\varphi_{-s}^X)_*\frac{B\circ(\varphi_s^X)_*(w) - (\varphi_s^X)_*\circ B(w)}{s}\\
&:= (\varphi_{-t}^X)_*(L(w)).
\end{aligned}$$

Continuing as in **4.7**, we define the mappings

$$\eta\colon I_p\times T_pM \to T_pM,\ (s,u)\mapsto \eta(s,u) := (\varphi_{-s}^X)_*(u)$$

and

$$Z\colon I_p\to T_pM,\ s\mapsto Z(s) := \frac{B\circ(\varphi_s^X)_*(w) - (\varphi_s^X)_*\circ B(w)}{s}.$$

Then $\eta(s,Z(s)) = (\varphi_{-s}^X)_*\frac{B\circ(\varphi_s^X)_*(w)-(\varphi_s^X)_*\circ B(w)}{s}$, so we obtain that

$$\begin{aligned}
L(w) &= \lim_{s\to 0}\eta(s,Z(s)) = \eta(0,\lim_{s\to 0}Z(s))\\
&= \lim_{s\to 0}\frac{B\circ(\varphi_s^X)_*(w) - (\varphi_s^X)_*\circ B(w)}{s} \overset{(*)}{=} 0.
\end{aligned}$$

Thus $h'(t) = 0$ for every $t\in I_p$, and hence h is constant. Since $h(0) = B(v)$, our assertion follows. □

11 The Finslerian Lie derivative

11.0 For simplicity, throughout this subsection we work on the Finsler bundle $\pi\colon TM \times_M TM \to TM$. What we say remains valid without any change in the more general context of Finsler bundles over $\widetilde{TM}$, where $\widetilde{TM}$ is a conic subbundle of TM (see **8.1**). The conventions fixed in **7.1** will be in force. We say that a vector field ξ on TM is *projectable* if it is τ-equivalent to a vector field on M, i.e., there exists a vector field $X \in \mathfrak{X}(M)$ such that $\tau_* \circ \xi = X \circ \tau$ (cf. **3.7**).

11.1 As a first step, we introduce the Lie derivative of a Finsler vector field with respect to a projectable vector field on TM. We note that our Lie derivative concept – suggested by M. Crampin and D. J. Saunders [8] – is a common generalization of the Lie derivatives with respect to the vertical, the complete and (in the presence of an Ehresmann connection) of the horizontal lift of a vector field on the base manifold; see [28], Section 2.39 and [18], §2.

Definition and Lemma 11.1.1. *Let ξ be a projectable vector field on TM, and $\widetilde{Y}$ be a section in $\Gamma(\pi)$.*
(i) *If*

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}], \tag{11.1}$$

then $\widetilde{\mathcal{L}}_\xi \widetilde{Y}$ is a well-defined section in $\Gamma(\pi)$, called the Lie-derivative *of $\widetilde{Y}$ with respect to ξ.*
(ii) *The mapping $\widetilde{\mathcal{L}}_\xi\colon \Gamma(\pi) \to \Gamma(\pi)$, $\widetilde{Y} \to \widetilde{\mathcal{L}}_\xi \widetilde{Y}$ satisfies the product rule*

$$\widetilde{\mathcal{L}}_\xi F\widetilde{Y} = (\xi F)\widetilde{Y} + F\widetilde{\mathcal{L}}_\xi \widetilde{Y}, \quad F \in C^\infty(TM). \tag{11.2}$$

(iii) *If η is another projectable vector field on TM, then*

$$[\widetilde{\mathcal{L}}_\xi, \widetilde{\mathcal{L}}_\eta] = \widetilde{\mathcal{L}}_{[\xi,\eta]}, \tag{11.3}$$

i.e.,

$$\widetilde{\mathcal{L}}_\xi \circ \widetilde{\mathcal{L}}_\eta(\widetilde{Z}) - \widetilde{\mathcal{L}}_\eta \circ \widetilde{\mathcal{L}}_\xi(\widetilde{Z}) = \widetilde{\mathcal{L}}_{[\xi,\eta]}\widetilde{Z} \text{ for all } \widetilde{Z} \in \Gamma(\pi). \tag{11.4}$$

(iv) *We have the following formulae:*

$$\widetilde{\mathcal{L}}_{X^c}\widetilde{\delta} = 0,\ \widetilde{\mathcal{L}}_{X^c}\widehat{Y} = \widehat{[X,Y]};\ X, Y \in \mathfrak{X}(M). \tag{11.5 a-b}$$

Proof. (i) Since the vector field ξ is a projectable, we have $\xi \underset{\tau}{\sim} X$, where $X \in \mathfrak{X}(M)$. On the other hand $\mathbf{i}\widetilde{Y} \underset{\tau}{\sim} 0$, because $\mathbf{i}\widetilde{Y} \in \mathfrak{X}^{\mathsf{v}}(TM)$. Thus, by the related vector field lemma (**3.7.8**) we conclude that $[\xi, \mathbf{i}\widetilde{Y}] \underset{\tau}{\sim} 0$, and hence $[\xi, \mathbf{i}\widetilde{Y}]$ is vertical. The injective $C^\infty(TM)$-linear mapping $\mathbf{i}\colon \Gamma(\pi) \to \mathfrak{X}(TM)$ is a bijection onto $\mathfrak{X}^{\mathsf{v}}(TM)$, so we have the inverse mapping $\mathbf{i}^{-1}\colon \Gamma(\pi) \to \mathfrak{X}(TM)$, and we can form the section $\mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}]$, as was to be shown.
(ii)

$$\begin{aligned}&\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} F\widetilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}(F\widetilde{Y})] = \mathbf{i}^{-1}[\xi, F(\mathbf{i}\widetilde{Y})] \overset{(3.6\,b)}{=} F\,\mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}]\\ &+ \mathbf{i}^{-1}(\xi F(\mathbf{i}\widetilde{Y})) = (\xi F)\widetilde{Y} + F\widetilde{\mathcal{L}}_\xi \widetilde{Y}.\end{aligned}$$

(iii) From the definition of $\widetilde{\mathcal{L}}_\xi$ and $\widetilde{\mathcal{L}}_\eta$,

$$\widetilde{\mathcal{L}}_\xi \circ \widetilde{\mathcal{L}}_\eta(\widetilde{Z}) = \widetilde{\mathcal{L}}_\xi(\mathbf{i}^{-1}[\eta, \mathbf{i}\widetilde{Z}]) = \mathbf{i}^{-1}[\xi, [\eta, \mathbf{i}\widetilde{Z}]].$$

Interchanging ξ and η and subtracting, we find that

$$\begin{aligned}&\widetilde{\mathcal{L}}_\xi \circ \widetilde{\mathcal{L}}_\eta(\widetilde{Z}) - \widetilde{\mathcal{L}}_\eta \circ \widetilde{\mathcal{L}}_\xi(\widetilde{Z}) = \mathbf{i}^{-1}([\xi, [\eta, \mathbf{i}\widetilde{Z}]] + [\eta, [\mathbf{i}\widetilde{Z}, \xi]])\\ &\overset{\text{Jacobi}}{=} \mathbf{i}^{-1}[[\xi, \eta], \mathbf{i}\widetilde{Z}] =: \widetilde{\mathcal{L}}_{[\xi,\eta]}\widetilde{Z},\end{aligned}$$

as wanted.
(iv) Since X^{c} is projectable (see (7.18)), formulae (11.5 a-b) have meaning. We obtain by an easy calculation that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{\delta} := \mathbf{i}^{-1}[X^{\mathsf{c}}, \mathbf{i}\widetilde{\delta}] \overset{(8.22\,a)}{=} \mathbf{i}^{-1}[X^{\mathsf{c}}, C] \overset{(7.23\,b)}{=} 0.$$

and

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y} := \mathbf{i}^{-1}[X^{\mathsf{c}}, \mathbf{i}\widehat{Y}] \overset{(8.19\,a)}{=} \mathbf{i}^{-1}[X^{\mathsf{c}}, Y^{\mathsf{v}}] \overset{(7.22\,b)}{=} \mathbf{i}^{-1}[X, Y]^{\mathsf{v}} = \widehat{[X, Y]},$$

which complete the proof. □

Proposition 11.1.2. *Let X and Y be vector fields on M. We have the following relations:*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{v}}}\widetilde{Y} = \nabla^{\mathsf{v}}_{\widehat{X}}\widetilde{Y} \textit{ for all } \widetilde{Y} \in \Gamma(\pi); \tag{11.6}$$

$$\mathbf{i} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \mathcal{L}_{X^{\mathsf{c}}} \circ \mathbf{i}; \tag{11.7}$$

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{j} = \mathbf{j} \circ \mathcal{L}_{X^{\mathsf{c}}}; \tag{11.8}$$

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{v}}_{\widehat{Y}} - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}} \tag{11.9}$$

Proof. (i) Let $\widetilde{Y} = \mathbf{j}\eta,\ \eta \in \mathfrak{X}(TM)$. Then, on the one hand,

$$\mathbf{i}\widetilde{\mathcal{L}}_{X^{\mathsf{v}}}\widetilde{Y} = \mathbf{i}\widetilde{\mathcal{L}}_{X^{\mathsf{v}}}\mathbf{j}\eta := [X^{\mathsf{c}}, \mathbf{J}\eta].$$

On the other hand,

$$\mathbf{i}\nabla^{\mathsf{v}}_{\widehat{X}}\widetilde{Y} = \mathbf{i}\nabla^{\mathsf{v}}_{\widehat{X}}\mathbf{j}\eta \overset{(9.5)}{:=} \mathbf{J}[X^{\mathsf{v}}, \eta].$$

Since $0 \overset{(8.21\,b)}{=} [\mathbf{J}, X^{\mathsf{v}}]\eta \overset{(10.3)}{=} [\mathbf{J}\eta, X^{\mathsf{v}}] - \mathbf{J}[\eta, X^{\mathsf{v}}]$, and hence

$$\mathbf{J}[X^{\mathsf{v}}, \eta] = [X^{\mathsf{v}}, \mathbf{J}\eta], \tag{11.10}$$

the equality (11.6) follows.
(ii) For every $\widetilde{Y} \in \Gamma(\pi)$,

$$\mathbf{i} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\widetilde{Y}) := [X^{\mathsf{c}}, \mathbf{i}\widetilde{Y}] = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{i}\widetilde{Y}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{i})(\widetilde{Y}).$$

(iii) Let $\eta \in \mathfrak{X}(TM)$. Observe that, as above, we have

$$0 \overset{(8.21\,a)}{=} [\mathbf{J}, X^{\mathsf{c}}]\eta = [\mathbf{J}\eta, X^{\mathsf{c}}] - \mathbf{J}[\eta, X^{\mathsf{c}}],$$

whence

$$\mathbf{J}[X^{\mathsf{c}}, \eta] = [X^{\mathsf{c}}, \mathbf{J}\eta]. \tag{11.11}$$

Taking this into account,

$$\mathbf{i}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}\eta := [X^{\mathsf{c}}, \mathbf{J}\eta] = \mathbf{J}[X^{\mathsf{c}}, \eta] = \mathbf{J}\mathcal{L}_{X^{\mathsf{c}}}\eta,$$

from which (11.8) follows.
(iv) Consider a section $\widetilde{Z} = \mathbf{j}\zeta \in \Gamma(\pi)$, where $\zeta \in \mathfrak{X}(TM)$. Then $\mathbf{i}\widetilde{Z} = \mathbf{J}\zeta$, and we obtain

$$\begin{aligned}
&\mathbf{i} \circ (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{v}}_{\widehat{Y}} - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}})(\widetilde{Z}) \overset{(9.5),\,(11.1)}{=} \mathbf{i}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}[Y^{\mathsf{v}}, \zeta] \\
&- \mathbf{i}\nabla^{\mathsf{v}}_{\widehat{Y}}(\mathbf{i}^{-1}[X^{\mathsf{c}}, \mathbf{J}\zeta]) \overset{(11.1),\,(11.11)}{=} [X^{\mathsf{c}}, \mathbf{J}[Y^{\mathsf{v}}, \zeta]] - \mathbf{i}\nabla^{\mathsf{v}}_{\widehat{Y}}\mathbf{j}[X^{\mathsf{c}}, \zeta] \\
&\overset{(11.10),\,(9.5)}{=} [X^{\mathsf{c}}, [Y^{\mathsf{v}}, \mathbf{J}\zeta]] - \mathbf{J}[Y^{\mathsf{v}}, [X^{\mathsf{c}}, \zeta]] \overset{(11.10),\,(11.11)}{=} [X^{\mathsf{c}}, [Y^{\mathsf{v}}, \mathbf{J}\zeta]] \\
&+ [Y^{\mathsf{v}}, [\mathbf{J}\zeta, X^{\mathsf{c}}]] \overset{\text{Jacobi}}{=} -[\mathbf{J}\zeta, [X^{\mathsf{c}}, Y^{\mathsf{v}}]] \overset{(7.22\,b)}{=} -[\mathbf{J}\zeta, [X, Y]^{\mathsf{v}}] \\
&= [[X, Y]^{\mathsf{v}}, \mathbf{i}\widetilde{Z}] = \mathbf{i}\widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}\widetilde{Z}.
\end{aligned}$$

This proves (11.9), and finishes the proof of the Proposition. □

Remark 11.1.3. In formula (11.9), $\nabla^{\mathsf{v}}_{\widehat{Y}}$ and $\widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}} \overset{(11.6)}{=} \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}$ annulate the basic sections, so it follows that

$$\nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Z} = 0 \quad \text{for all } X, Y, Z \in \mathfrak{X}(M). \tag{11.12}$$

Obviously, this relation can also be checked by an easy direct calculation.

Proposition 11.1.4. *Let X be a vector field on M, and let (φ_t) be the local one-parameter group of X. Then, for a Finsler vector field $\widetilde{Y} \in \Gamma(\pi)$, $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{Y} = 0$ if, and only if, $\widetilde{Y}$ is invariant under the stages of (φ_t), i.e., for every possible $t \in \mathbb{R}$ we have*

$$((\varphi_t)_* \times (\varphi_t)_*) \circ \widetilde{Y} = \widetilde{Y} \circ (\varphi_t)_*.$$

Proof. By (11.7), $\mathbf{i}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{Y} = \mathcal{L}_{X^{\mathsf{c}}}\mathbf{i}\widetilde{Y}$. Since $\mathbf{i}$ is injective, this implies that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{Y} = 0 \iff \mathcal{L}_{X^{\mathsf{c}}}\mathbf{i}\widetilde{Y} = 0.$$

Thus, taking into account **4.8** and Lemma 7.5.1, it follows that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{Y} = 0 \iff (\varphi_t)_{**} \circ (\mathbf{i}\widetilde{Y}) = (\mathbf{i}\widetilde{Y}) \circ (\varphi_t)_*.$$

Here $(\varphi_t)_{**} \circ \mathbf{i} \overset{(8.23\,a)}{=} \mathbf{i} \circ ((\varphi_t)_* \times (\varphi_t)_*)$, so our assertion follows. □

11.2 Let ξ again be a projectable vector field on TM. Now we extend the derivation

$$\widetilde{\mathcal{L}}_\xi \colon \Gamma(\pi) \to \Gamma(\pi),\ \widetilde{Y} \mapsto \widetilde{\mathcal{L}}_\xi \widetilde{Y}$$

to a derivation of Finsler tensor fields of type $(0,k)$ and $(1,k)$.
(a) We set $\widetilde{\mathcal{L}}_\xi F := \xi F$ if $F \in C^\infty(TM) =: T^0_0(\Gamma(\pi))$. We note that relations (11.3) and (11.9) remain valid over $C^\infty(TM)$. This is obvious in the first case, while in the second case it can be seen an easy calculation: for every smooth function F on TM we have

$$\begin{aligned}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{v}}_{\widehat{Y}} - \widetilde{\mathcal{L}}_{Y^{\mathsf{c}}} \circ \nabla^{\mathsf{v}}_{\widehat{X}})F &= X^{\mathsf{c}}(Y^{\mathsf{v}}F) - Y^{\mathsf{c}}(X^{\mathsf{v}}F) = [X^{\mathsf{c}}, Y^{\mathsf{v}}]F \\ &\overset{(7.22\ b)}{=} [X,Y]^{\mathsf{v}}F = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}F,\end{aligned}$$

as wanted.
(b) The Lie derivative of a Finsler tensor field

$$A \in T^0_l(\Gamma(\pi)) \cup T^1_l(\Gamma(\pi))\ (l \geqq 1)$$

with respect to ξ is defined by

$$(\widetilde{\mathcal{L}}_\xi A)(\widetilde{X}_1,\dots,\widetilde{X}_l) := \widetilde{\mathcal{L}}_\xi(A(\widetilde{X}_1,\dots,\widetilde{X}_l)) - \sum_{i=1}^{l} A(\widetilde{X}_1,\dots,\widetilde{\mathcal{L}}_\xi\widetilde{X}_i,\dots\widetilde{X}_l), \tag{11.13}$$

where $\widetilde{X}_1,\dots,\widetilde{X}_l \in \Gamma(\pi)$.

11.3 Now let $D\colon \mathfrak{X}(TM)\times\Gamma(\pi)\to\Gamma(\pi)$ be a covariant derivative on the Finsler bundle π (for the general definition see **6.1**). Given a projectable vector field ξ on TM, we define the Lie derivative $\widetilde{\mathcal{L}}_\xi D$ of D by

$$\begin{aligned}(\widetilde{\mathcal{L}}_\xi D)(\eta,\widetilde{Z}) &:= \widetilde{\mathcal{L}}_\xi(D_\eta\widetilde{Z}) - D_{\mathcal{L}_\xi\eta}\widetilde{Z} - D_\eta(\widetilde{\mathcal{L}}_\xi\widetilde{Z})\\ &= \widetilde{\mathcal{L}}_\xi(D_\eta\widetilde{Z}) - D_{[\xi,\eta]}\widetilde{Z} - D_\eta(\widetilde{\mathcal{L}}_\xi\widetilde{Z}),\end{aligned} \tag{11.14}$$

where $\eta\in\mathfrak{X}(TM)$, $\widetilde{Z}\in\Gamma(\pi)$. Then the mapping

$$\widetilde{\mathcal{L}}_\xi D\colon \mathfrak{X}(TM)\times\Gamma(\pi)\to\Gamma(\pi),\ (\eta,\widetilde{Z})\mapsto(\widetilde{\mathcal{L}}_\xi D)(\eta,\widetilde{Z})$$

is $C^\infty(TM)$-linear in both of its argument. Indeed, for example, if $F\in C^\infty(TM)$, then

$$\begin{aligned}(\widetilde{\mathcal{L}}_\xi D)(F\eta,\widetilde{Z}) &:= \widetilde{\mathcal{L}}_\xi(D_{F\eta}\widetilde{Z}) - D_{[\xi,F\eta]}\widetilde{Z} - D_{F\eta}(\widetilde{\mathcal{L}}_\xi\widetilde{Z})\\ &\overset{(3.6\,b),\,(6.1)}{=} \widetilde{\mathcal{L}}_\xi(FD_\eta\widetilde{Z}) - FD_{[\xi,\eta]}\widetilde{Z} - (\xi F)D_\eta\widetilde{Z} - FD_\eta(\widetilde{\mathcal{L}}_\xi\widetilde{Z})\\ &\overset{(11.2)}{=} F(\widetilde{\mathcal{L}}_\xi D)(\eta,\widetilde{Z}),\end{aligned}$$

as wanted.

11.4 We continue to assume that D is a covariant derivative on π. Consider a diffeomorphism $\varphi\colon\mathcal{U}\to\mathcal{V}$ between two open subsets of M. On the analogy of definition (6.6), if

$$\varphi_\#((D_\xi\widetilde{Y})\restriction\tau^{-1}(\mathcal{U})) = (D_{(\varphi_*)_\#\xi}\varphi_\#\widetilde{Y})\restriction\tau^{-1}(\mathcal{V}) \tag{11.15}$$

for all $\xi\in\mathfrak{X}(TM)$, $\widetilde{Y}\in\Gamma(\pi)$, then we say that φ is a *(local) D-automorphism*. (The push-forward of a Finsler vector field was defined in **8.2**). To continue the analogy, a vector field X on M is called *D-Killing*, if the stages of its local one-parameter group are D-automorphisms, and the set of D-Killing fields is denoted by $\mathrm{Kill}_D(\pi)$. Finally, the analogue of Proposition 6.4.1 is the following result:

$$X\in\mathrm{Kill}_D(\pi) \iff \widetilde{\mathcal{L}}_{X^c}D = 0. \tag{11.16}$$

Part III
Lie symmetries

12 Semisprays and sprays

We follow the conventions described in ***7.1***

12.1 A mapping $S\colon TM \to TTM$ is called a *semispray for* M (or *over* M) if it satisfies the following conditions:

(S1) $\tau_{TM} \circ S = 1_{TM}$, i.e., S is a section of the vector bundle $\tau_{TM}\colon TTM \to TM$.

(S2) S is of class C^1 on TM and smooth on $\overset{\circ}{T}M$.

(S3) $\tau_* \circ S = 1_{TM}$ or, equivalently, $\mathbf{J}S = C$.

If, in addition, we have

(S4) $[C, S] = S$, i.e., S is 2^+-homogeneous, then S is called a *spray.*

A spray is said to be *affine* or (*quadratic*) if it is of class C^2 (and hence smooth) on TM. A manifold together with a spray is called a *spray manifold.*

If $S\colon TM \to TTM$ is a semispray, then it can be expressed locally as

$$S \underset{(\mathcal{U})}{=} \sum \left(y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \right), \tag{12.1}$$

where the *semispray coefficients* $G^i\colon \tau^{-1}(\mathcal{U}) \to \mathbb{R}$ are of class C^1 and their restrictions to $\tau^{-1}(\mathcal{U}) \cap \overset{\circ}{T}M$ are smooth. In the special case when S is a spray, the *spray coefficients* $G^i\colon \tau^{-1}(\mathcal{U}) \to \mathbb{R}$ are 2^+-homogeneous and hence, by **7.7.2** (i), we have

$$\sum y^j \frac{\partial G^i}{\partial y^j} = 2G^i, \quad i \in J_n. \tag{12.2}$$

Suppose, finally, that S is an affine spray. Then the spray coefficients G^i are 2^+-homogeneous functions of class C^2, so, in view of **7.7.2** (iii) their restrictions $G^i \restriction T_pM$ ($p \in \mathcal{U}$) are quadratic functions. Thus there exist smooth functions

$$\Gamma^i_{jk}\colon \mathcal{U} \to \mathbb{R}; \quad i, j, k \in J_n$$

such that

$$G^i = \frac{1}{2}\sum y^j y^k(\Gamma^i_{jk}\circ\tau) \quad\text{and}\quad \Gamma^i_{jk} = \Gamma^i_{kj}. \tag{12.3}$$

Lemma 12.1.1. *Let $S\colon TM \to TTM$ be a semispray for M. Then*

(i) $Sf^{\mathsf{v}} = f^{\mathsf{c}}$ *for all* $f \in C^\infty(M)$*;*

(ii) $[X^{\mathsf{v}}, S] \underset{\tau}{\sim} X$ *for all* $X \in \mathfrak{X}(M)$*;*

(iii) $[X^{\mathsf{c}}, S] \in \mathfrak{X}^{\mathsf{v}}(\overset{\circ}{T}M)$ *for all* $X \in \mathfrak{X}(M)$*.*

Proof. For every $v \in TM$,

$$(S, f^{\mathsf{v}})(v) = S(v)(f\circ\tau) \overset{(3.3)}{=} \tau_*(S(v))(f) \overset{(\mathrm{S3})}{=} v(f) =: f^{\mathsf{c}}(v),$$

which proves (i). Given a smooth function f on M, we have

$$[X^{\mathsf{v}}, S](f\circ\tau) = X^{\mathsf{v}}(Sf^{\mathsf{v}}) - S(X^{\mathsf{v}}f^{\mathsf{v}}) \overset{(i),\mathbf{7.2}}{=} X^{\mathsf{v}}f^{\mathsf{c}} \overset{(7.7)}{=} (Xf)\circ\tau.$$

This implies that $[X^{\mathsf{v}}, S] \underset{\tau}{\sim} X$ (see, e.g., [24], p.14). Similarly, we find that

$$\begin{aligned}[X^{\mathsf{c}}, S]f^{\mathsf{v}} &= X^{\mathsf{c}}(Sf^{\mathsf{v}}) - S(X^{\mathsf{c}}f^{\mathsf{v}}) \overset{(i),(7.15\,a)}{=} X^{\mathsf{c}}f^{\mathsf{c}} - S(Xf)^{\mathsf{v}} \\ &\overset{(7.15\,b),(i)}{=} (Xf)^{\mathsf{c}} - (Xf)^{\mathsf{c}} = 0.\end{aligned}$$

This implies (see **7.2**) that $[X^{\mathsf{c}}, S]$ is vertical, and completes the proof. □

Lemma 12.1.2. *Let S be a semispray for M. Then, for every vector field ξ on TM,*

$$\mathbf{J}[\mathbf{J}\xi, S] = \mathbf{J}\xi \tag{12.4}$$

(Grifone's identity). *In particular,*

$$\mathbf{J}[X^{\mathsf{v}}, S] = X^{\mathsf{v}} \quad \textit{for all } X \in \mathfrak{X}(M). \tag{12.5}$$

For a proof, see [11], Proposition I.7 or [29] Lemma 5.1.9 and Corollary 5.1.10.

12.2 Automorphisms and symmetries

Lemma 12.2.1. *If $S\colon TM \to TTM$ is a semispray and $\varphi \in \mathrm{Diff}(M)$, then $(\varphi_*)_\# S = \varphi_{**}\circ S\circ\varphi_*^{-1}$ is also a semispray. This semispray is a spray whenever S is a spray.*

Proof. We show that $(\varphi_*)_\# S$ satisfies $(S3)$ if S is a semispray, and $[C, (\varphi_*)_\# S] = (\varphi_*)_\# S$ if S is a spray. Indeed, in the first case we find

$$\begin{aligned}\tau_* \circ (\varphi_*)_\# S &= \tau_* \circ \varphi_{**} \circ S \circ \varphi_*^{-1} = (\tau \circ \varphi_*)_* \circ S \circ \varphi_*^{-1} \\ &= (\varphi \circ \tau)_* \circ S \circ \varphi_*^{-1} = \varphi_* \circ \tau_* \circ S \circ \varphi_*^{-1} \overset{\text{(S2)}}{=} \varphi_* \circ \varphi_*^{-1} = 1_{TM},\end{aligned}$$

as desired. Now suppose that S is a spray. Since $(\varphi_*)_\# C = C$ (see [29], (4.1.112)), we get

$$[C, (\varphi_*)_\# S] = [(\varphi_*)_\# C, (\varphi_*)_\# S] \overset{(3.9)}{=} (\varphi_*)_\# [C, S] \overset{\text{(S4)}}{=} (\varphi_*)_\# S,$$

as was to be shown. □

Definition and Lemma 12.2.2. *Let S be a semispray for M.*

(i) *A diffeomorphism φ of M is called an* automorphism of S *if S is invariant under $\varphi_* \in \mathrm{Diff}(TM)$, i.e., $(\varphi_*)_\# S = S$. The automorphisms of S form a group under composition.*

(ii) *Let $\varphi\colon \mathcal{U} \to \mathcal{V}$ be a diffeomorphism between two open subsets of M. We say that φ is* local automorphism of S *if $S \restriction (\tau^{-1}(\mathcal{U}))$ is invariant under φ_*, i.e.,*

$$(\varphi_*)_\# (S \restriction \tau^{-1}(\mathcal{U})) = S \restriction \tau^{-1}(\mathcal{V}).$$

(iii) *A vector field X on M is called a* Lie symmetry *of S if the stages of the local one-parameter group (φ_t) generated by X are local automorphisms of S.*

(iv) *A vector field $X \in \mathfrak{X}(M)$ is a* Lie symmetry of S *if, and only if, $[X^{\mathsf{c}}, S] = 0$. The Lie symmetries of S form a subalgebra $Lie_S(M)$ of the Lie algebra $\mathfrak{X}(M)$.*

Proof. Only part (iv) requires some comments. If X generates the local one-parameter group (φ_t), then X^{c} generates the local one-parameter group $((\varphi_t)_*)$ by Lemma 7.5.1. So, as in **4.8**, $[X^{\mathsf{c}}, S] = 0$ if, and only if, the local one-parameter group of X consists of local automorphisms of S.

If $X, Y \in \mathrm{Lie}_S(M)$, then we obviously have

$$\lambda X + \mu Y \in \mathfrak{X}^S_{\mathrm{Lie}}(M);\ \ \lambda, \mu \in \mathbb{R}.$$

Since

$$[[X,Y]^{\mathsf{c}}, S] \overset{(7.22\ c)}{=} [X^{\mathsf{c}}, Y^{\mathsf{c}}], S] \overset{\text{Jacobi}}{=} -[[Y^{\mathsf{c}}, S], X^{\mathsf{c}}] - [[S, X^{\mathsf{c}}], Y^{\mathsf{c}}] = 0,$$

$[X, Y]$ also belongs to $\mathfrak{X}^S_{\mathrm{Lie}}(M)$, thus proving that $\mathrm{Lie}_S(M)$ is a subalgebra of $\mathfrak{X}(M)$. □

The last part of 12.2.2 can also be found in Lovas's paper [17] as a part of his Proposition 5.2 and his Corollary 5.3, in the framework of spray manifolds and with partly different proof.

Proposition 12.2.3 (cf. [7], Prop. 4.5.1)**.** *Let S be a semispray for M with semispray coefficients G^i ($i \in J_n$), and let X be a vector field on M with local expression $X \restriction \mathcal{U} = \sum X^i \frac{\partial}{\partial u^i}$.*

(i) *The vector field X is a Lie symmetry of S if, and only if, locally we have*

$$X^{\mathsf{c}} G^i = G^r \left(\frac{\partial X^i}{\partial u^r} \circ \tau\right) - \frac{1}{2} y^r y^s \left(\frac{\partial^2 X^i}{\partial u^r \partial u^s} \circ \tau\right) \ (i \in J_n) \tag{12.6}$$

(ii) *If, in addition, S is a spray, then $X \in \mathfrak{X}^S_{\mathrm{Lie}}(M)$ if, and only if,*

$$X^{\mathsf{c}} G^i_{jk} = -\frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau + \left(\frac{\partial X^i}{\partial u^r} \circ \tau\right) G^r_{jk} - \left(\frac{\partial X^r}{\partial u^j} \circ \tau\right) G^i_{rk} - \left(\frac{\partial X^r}{\partial u^k} \circ \tau\right) G^i_{jr} \tag{12.7}$$

(i, j, $k \in J_n$), where

$$G^i_j := \frac{\partial G^i}{\partial y^i}, \ G^i_{jk} := \frac{\partial G^i_j}{\partial y^k} = \frac{\partial G^i}{\partial y^j \partial y^k}. \tag{12.8}$$

Proof. **Step 1** We check assertion (i). This is just a calculation:

$$\begin{aligned}
[X^{\mathsf{c}}, S] \underset{(\mathcal{U})}{=} & \left[X^{\mathsf{c}}, y^r \frac{\partial}{\partial x^r} - 2G^r \frac{\partial}{\partial y^r}\right] \overset{(3.6)}{=} y^r \left[X^{\mathsf{c}}, \frac{\partial}{\partial x^r}\right] + (X^{\mathsf{c}} y^r) \frac{\partial}{\partial x^r} \\
& - 2G^r \left[X^{\mathsf{c}}, \frac{\partial}{\partial y^r}\right] - 2(X^{\mathsf{c}} G^i) \frac{\partial}{\partial y^i} \overset{(7.15\,b),\ (7.16)}{=} y^r \left[(X^i)^{\mathsf{v}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^r}\right] \\
& + y^r \left[(X^i)^{\mathsf{c}} \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^r}\right] + (X^i)^{\mathsf{c}} \frac{\partial}{\partial x^i} - 2G^r \left[(X^i)^{\mathsf{v}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^r}\right] \\
& - 2G^r \left[(X^i)^{\mathsf{c}} \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^r}\right] - 2(X^{\mathsf{c}} G^i) \frac{\partial}{\partial y^i} \overset{(7.1)}{=} -y^r \left(\frac{\partial X^i}{\partial u^r} \circ \tau\right) \frac{\partial}{\partial x^i} \\
& - y^r \left(\frac{\partial}{\partial x^r} \left(y^s \left(\frac{\partial X^i}{\partial u^s} \circ \tau\right)\right) \frac{\partial}{\partial y^i}\right) + y^r \left(\frac{\partial X^i}{\partial u^r} \circ \tau\right) \frac{\partial}{\partial x^i} \\
& + 2G^r \left(\frac{\partial}{\partial y^r} \left(y^s \left(\frac{\partial X^i}{\partial u^s} \circ \tau\right)\right) \frac{\partial}{\partial y^i}\right) = -y^r y^s \left(\frac{\partial^2 X^i}{\partial u^r \partial u^s} \circ \tau\right) \frac{\partial}{\partial y^i} \\
& = +2G^r \left(\frac{\partial X^i}{\partial u^r} \circ \tau\right) \frac{\partial}{\partial y^i} - 2\left(X^{\mathsf{c}} G^i\right) \frac{\partial}{\partial y^i}
\end{aligned}$$

whence

$$-\frac{1}{2}[X^{\mathsf{c}}, S] \underset{(\mathcal{U})}{=} \left(X^{\mathsf{c}} G^i - G^r \left(\frac{\partial X^i}{\partial u^r} \right) \circ \tau) + \frac{1}{2} y^r y^s \left(\frac{\partial^2 X^i}{\partial u^r \partial u^s} \circ \tau \right) \right) \frac{\partial}{\partial y^i}.$$

Thus $[X^{\mathsf{c}}, S] = 0$ if, and only if, we have (locally) relation (12.6).

Step 2. We show that (12.6) implies (12.7). To see this, we differentiate both side of (12.6) with respect to y^j and y^k. We find, on the one hand, that

$$\frac{\partial}{\partial y^j}(X^{\mathsf{c}} G^i) = \left[\frac{\partial}{\partial y^j}, X^{\mathsf{c}} \right] G^i + X^{\mathsf{c}} G^i_j \overset{(7.22\,b)}{=} \left[\frac{\partial}{\partial u^j}, X \right]^{\mathsf{v}} G^i + X^{\mathsf{c}} G^i_j$$
$$= \left(\frac{\partial X^r}{\partial u^j} \circ \tau \right) G^i_r + X^{\mathsf{c}} G^i_j,$$

therefore

$$\frac{\partial}{\partial y^k} \left(\frac{\partial}{\partial y^j} X^{\mathsf{c}} G^i \right) = \left(\frac{\partial X^r}{\partial u^j} \circ \tau \right) G^i_{rk} + \left[\frac{\partial}{\partial y^k}, X^{\mathsf{c}} \right] G^i_j + X^{\mathsf{c}} G^i_{jk}$$
$$= \left(\frac{\partial X^r}{\partial u^j} \circ \tau \right) G^i_{rk} + \left(\frac{\partial X^r}{\partial u^k} \circ \tau \right) G^i_{jr} + X^{\mathsf{c}} G^i_{jk}.$$

On the other hand,

$$\frac{\partial}{\partial y^k} \left(\frac{\partial}{\partial y^j} \left(G^r \left(\frac{\partial X^i}{\partial u^r} \circ \tau \right) \right) - \frac{1}{2} y^r y^s \left(\frac{\partial^2 X^i}{\partial u^r \partial u^s} \circ \tau \right) \right)$$
$$= \frac{\partial}{\partial y^k} \left(\left(\frac{\partial X^i}{\partial u^r} \circ \tau \right) G^r_j - y^s \left(\frac{\partial^2 X^i}{\partial u^i \partial u^s} \circ \tau \right) \right)$$
$$= \left(\frac{\partial X^i}{\partial u^r} \circ \tau \right) G^r_{jk} - \frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau,$$

so our claim follows.

Step 3. Now we assume that S is a spray, and we show that in this case (12.7) implies (12.6). Under our assumption the functions G^i and G^i_j are positive-homogeneous of degree 2 and 1, respectively, so we have

$$G^i_j y^j = 2G^i, \quad G^i_{jk} y^k = G^i_j. \tag{12.9}$$

Now we multiply both sides of (12.7) by $y^j y^k$. Then the left-hand side gives

$$(X^{\mathsf{c}} G^i_{jk}) y^j y^k = X^{\mathsf{c}} \left(G^i_{jk} y^j y^k \right) - G^i_{jk} (X^{\mathsf{c}} (u^j)^{\mathsf{c}}) y^k - G^i_{jk} y^j (X^{\mathsf{c}} (u^k)^{\mathsf{c}})$$
$$\overset{(12.9),\,(7.15\,b)}{=} 2X^{\mathsf{c}} G^i - G^i_j (X^j)^{\mathsf{c}} - G^i_k (X^k)^{\mathsf{c}} = 2 \left(X^{\mathsf{c}} G^i - G^i_r (X^r)^{\mathsf{c}} \right).$$

The right-hand side takes the form

$$-\left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau\right) y^j y^k + 2G^r \left(\frac{\partial X^i}{\partial u^r} \circ \tau\right) - 2G^i_r (X^r)^{\mathsf{c}},$$

so we obtain that

$$X^{\mathsf{c}} G^i = G^r \left(\frac{\partial X^i}{\partial u^r} \circ \tau\right) - \frac{1}{2}\left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau\right) y^i y^k.$$

This proves our claim, and completes the proof of the proposition. □

We note that in the book [7] of Bucataru and Miron, Lie symmetries of a semispray were defined by the condition 12.2.2 (iv), and were characterized locally, by (12.6).

13 $\mathcal{H}$-Killing vector fields

13.1 Ehresmann connections

13.1.1 Let M be a manifold and consider its slit tangent bundle $\overset{\circ}{\tau}\colon \overset{\circ}{T}M \to M$. By an *Ehresmann connection* in $\overset{\circ}{T}M$ we mean a $C^\infty(\overset{\circ}{T}M)$-linear mapping $\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ such that

$$\mathbf{j} \circ \mathcal{H} = 1_{\Gamma(\overset{\circ}{\pi})}. \tag{13.1}$$

The fundamental lemma of strong bundle maps (**2.10**) assures us that $\mathcal{H}$ can be equivalently be regarded as a strong bundle map

$$\mathcal{H}\colon \overset{\circ}{T}M \times_M TM \to T\overset{\circ}{T}M.$$

Then, for every $v \in \overset{\circ}{T}M$,

$$\mathcal{H}_v := \mathcal{H} \restriction \{v\} \times T_{\tau(v)}M\colon \{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M \to T_v\overset{\circ}{T}M$$

is an $\mathbb{R}$-linear mapping, and condition (13.1) reads as follows: for all $(v,w) \in \overset{\circ}{T}M \times_M TM$, $\mathbf{j} \circ \mathcal{H}_v(w) \overset{(8.10)}{=} (v, \tau_* \circ \mathcal{H}_v(w)) = (v,w)$, i.e.,

$$\mathbf{j} \circ \mathcal{H}_v = 1_{T_{\tau(v)}M} \text{ for all } v \in \overset{\circ}{T}M. \tag{13.2}$$

13.1.2 Let $\mathcal{H}\colon \Gamma(\mathring{\pi}) \to \mathfrak{X}(\mathring{T}M)$ be an Ehresmann connection in $\mathring{T}M$. Then $\mathfrak{X}^{\mathrm{h}}(\mathring{T}M) := \mathrm{Im}(\mathcal{H})$ is a submodule of $\mathfrak{X}(\mathring{T}M)$, and we have the direct sum decomposition

$$\mathfrak{X}(\mathring{T}M) = \mathfrak{X}^{\mathsf{v}}(\mathring{T}M) \oplus \mathfrak{X}^{\mathsf{h}}(\mathring{T}M). \tag{13.3}$$

Vector fields on $\mathring{T}M$ belonging to $\mathfrak{X}^{\mathsf{h}}(\mathring{T}M)$ are called *horizontal* (with respect to $\mathcal{H}$). Notice that horizontal vector fields do not form, in general, a subalgebra of the Lie algebra $\mathfrak{X}(\mathring{T}M)$.

The mappings

$$\mathbf{h} := \mathcal{H} \circ \mathbf{j}, \quad \mathbf{v} = 1_{\mathfrak{X}(\mathring{T}M)} - \mathbf{h}, \tag{13.4 a-b}$$

$$\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}\colon \mathfrak{X}(\mathring{T}M) \to \mathfrak{X}^{\mathsf{v}}(\mathring{T}M) \to \Gamma(\mathring{\pi}) \tag{13.5}$$

are the *horizontal projection*, the *vertical projection* and the *vertical mapping* associated to $\mathcal{H}$, respectively. Then $\mathbf{h}$ and $\mathbf{v}$ are indeed projections, i.e., we have $\mathbf{h}^2 = \mathbf{h}$ and $\mathbf{v}^2 = \mathbf{v}$. The vertical mapping $\mathcal{V}$ has properties

$$\mathcal{V} \circ \mathbf{i} = 1_{\Gamma(\mathring{\pi})}, \ \mathrm{Ker}(\mathcal{V}) = \mathrm{Im}(\mathcal{H}). \tag{13.6}$$

Since $C = \mathbf{i}(\widetilde{\delta})$ by $(8.22\,a)$, thus it follows that

$$\mathcal{V}(C) = \widetilde{\delta}. \tag{13.7}$$

Obviously, $\mathbf{h}, \mathbf{v}$ and $\mathcal{V}$ can be also regarded as strong bundle maps. If $\mathbf{h}_v := h \restriction T_v\mathring{T}M$, $\mathbf{v}_v := \mathbf{v} \restriction T_v\mathring{T}M$, then $\mathbf{h}_v,\ \mathbf{v}_v \in \mathrm{End}(T_v\mathring{T}M)$, and

$$\mathbf{h}_v(w) = \mathcal{H}(v, (\tau_*)_v(w)) \ \text{ for all } \ w \in T_v\mathring{T}M. \tag{13.8}$$

13.1.3 The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ with respect to an Ehresmann connection $\mathcal{H}\colon \Gamma(\mathring{\pi}) \to \mathfrak{X}(\mathring{T}M)$ is

$$X^h := \mathcal{H}(\widehat{X}) = \mathcal{H}(\mathbf{j}X^{\mathsf{c}}) \overset{(13.4)}{=} \mathbf{h}(X^{\mathsf{c}}). \tag{13.9}$$

(In this formula, $\widehat{X}$ and X^{c} are regarded as a section in $\Gamma(\mathring{\pi})$ and a vector field on $\mathring{T}M$, resp.; for simplicity, we make no notational distinction between them and the corresponding objects in $\Gamma(\pi)$ and $\mathfrak{X}(TM)$.) *The horizontal lift X^{h} of X is a projectable vector field,*

$$X^{\mathsf{h}} \underset{\tau}{\sim} X, \quad \text{i.e., } \tau_* \circ X^{\mathsf{h}} = X \circ \tau. \tag{13.10}$$

Indeed, for every $v \in \overset{\circ}{T}M$ we have

$$\tau_* \circ X^{\mathsf{h}}(v) := \tau_* \circ \mathcal{H} \circ \widehat{X}(v) \overset{(8.3)}{=} \tau_* \circ \mathcal{H}(v, X \circ \tau(v))$$
$$= \tau_* \circ \mathcal{H}_v(X \circ \tau(v)) \overset{(13.2)}{=} X \circ \tau(v).$$

13.1.4 An Ehresmann connection $\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ is called *homogeneous* if

$$[C, X^{\mathsf{h}}] = 0 \ \text{ for all } X \in \mathfrak{X}(M). \tag{13.11}$$

By **4.8** and Lemma 7.4.1 (ii), this hold if, and only if,

$$(\mu_t^+)_* \circ X^{\mathsf{h}} = X^{\mathsf{h}} \circ \mu_t^+ \ \text{ for all } t \in \mathbb{R}. \tag{13.12}$$

Then $\mathcal{H}$, as a strong bundle map of $\overset{\circ}{T}M \times TM$ into $T\overset{\circ}{T}M$, may be continuously extended to mapping from $TM \times_M TM$ into TTM such that

$$\mathcal{H}(0_p, w) = (\sigma_*)_p(w) \text{ for all } p \in M,\ w \in T_pM.$$

Thus, in what follows, we shall always assume that a *homogeneous Ehresmann connection is defined on the entire Finsler bundle* $TM \times_M TM$.

13.1.5 Given an Ehresmann connection $\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ and a Finsler vector field $\widetilde{X} \in \Gamma(\overset{\circ}{\pi})$, we define a differential operator $\nabla^{\mathsf{h}}_{\widetilde{X}}$, following the scheme of section **9.2**. First we prescribe its action

on *smooth functions* by $\nabla^{\mathsf{h}}_{\widetilde{X}} F := (\mathcal{H}\widetilde{X})F \quad (F \in C^\infty(\overset{\circ}{T}M))$; (13.13)

on *Finsler vector fields* by $\nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$ $(\widetilde{Y} \in \Gamma(\overset{\circ}{\pi}))$. (13.14)

Then the Leibniz rule $\nabla^{\mathsf{h}}_{\widetilde{X}} F\widetilde{Y} = (\nabla^{\mathsf{h}}_{\widetilde{X}} F)\widetilde{Y} + F\nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}$ is satisfied (cf. (9.6)). The mapping

$$\nabla^{\mathsf{h}}\colon \Gamma(\overset{\circ}{\pi}) \times \Gamma(\overset{\circ}{\pi}) \to \Gamma(\overset{\circ}{\pi}),\ (\widetilde{X}, \widetilde{Y}) \mapsto \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}$$

defined by (13.14) is called the *horizontal Berwald derivative* (or *h-Berwald derivative* for short) induced by $\mathcal{H}$. In the next step, we extend to the operators $\nabla^{\mathsf{h}}_{\widetilde{X}}$ $(\widetilde{X} \in \Gamma(\overset{\circ}{\pi}))$ to arbitrary Finsler tensor fields in such a way that derivation property be satisfied. Finally, we define the $\nabla^{\mathsf{h}}-$*differential* of Finsler tensor fields, formally in the same way as the ∇^{v}-differential in **9.2**. Thus, for example, if $\widetilde{Y} \in \Gamma(\mathring{\pi})$, $g \in T_2^0(\Gamma(\mathring{\pi}))$, $B \in T_k^1(\Gamma(\mathring{\pi}))$ $(k \geqq 1)$, then the Finsler tensor fields $\nabla^{\mathsf{h}}\widetilde{Y} \in T_1^1(\Gamma(\mathring{\pi})) \cong \mathrm{End}(\Gamma(\mathring{\pi}))$, $\nabla^{\mathsf{h}}g \in T_3^0((\Gamma(\mathring{\pi}))$, $\nabla^{\mathsf{h}}B \in T_{k+1}^1(\Gamma(\mathring{\pi}))$ are given by

$$(\nabla^{\mathsf{h}}\widetilde{Y})(\widetilde{X}) := \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}, \tag{13.15}$$

$$\begin{aligned}(\nabla^{\mathsf{h}}g)(\widetilde{X},\widetilde{Y},\widetilde{Z}) &:= (\nabla^{\mathsf{h}}_{\widetilde{X}}g)(\widetilde{Y},\widetilde{Z}) \\ &:= (\mathcal{H}\widetilde{X})g(\widetilde{Y},\widetilde{Z}) - g(\nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) - g(\widetilde{Y},\nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Z}),\end{aligned} \tag{13.16}$$

$$\begin{aligned}&(\nabla^{\mathsf{h}}B)(\widetilde{X},\widetilde{Y}_1,\widetilde{Y}_2,\dots,\widetilde{Y}_l) := (\nabla^{\mathsf{h}}_{\widetilde{X}}B)(\widetilde{Y}_1,\dots,\widetilde{Y}_l) \\ &:= \nabla^{\mathsf{h}}_{\widetilde{X}}(B(\widetilde{Y}_1,\dots,\widetilde{Y}_l)) - \sum_{i=1}^{k} B(\widetilde{Y}_1,\dots,\nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}_i,\dots,\widetilde{Y}_k).\end{aligned} \tag{13.17}$$

It is useful to note that

$$\mathbf{i}\nabla^{\mathsf{h}}_{\widehat{X}}\widehat{Y} = [X^{\mathsf{h}},Y^{\mathsf{v}}] \ \text{ for all } X,Y \in \mathfrak{X}(M). \tag{13.18}$$

13.1.6 Let an Ehresmann connection $\mathcal{H}$ be given in $\mathring{T}M$. Putting together the vertical derivative ∇^{v} and the h-Berwald derivative ∇^{h}, we obtain a particularly important covariant derivative on $\mathring{\pi}$, the *Berwald derivative* ∇ induced by $\mathcal{H}$. To be explicit,

$$\begin{aligned}&\nabla\colon \mathfrak{X}(\mathring{T}M) \times \Gamma(\mathring{\pi}) \mapsto \Gamma(\mathring{\pi}), \\ &(\xi,\widetilde{Y}) \mapsto \nabla_\xi\widetilde{Y} := \nabla^{\mathsf{v}}_{\mathcal{V}\xi}\widetilde{Y} + \nabla^{\mathsf{h}}_{\mathbf{j}\xi}\widetilde{Y} = \mathbf{j}[\mathbf{v}\xi,\mathcal{H}\widetilde{Y}] + \mathcal{V}[\mathbf{h}\xi,\mathbf{i}\widetilde{Y}].\end{aligned} \tag{13.19}$$

Then we have especially

$$\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}, \quad \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}. \tag{13.20 a-b}$$

With the help of the induced Berwald derivative ∇, we define the torsion $\mathbf{T}$ of an Ehresmann connection $\mathcal{H}$ by

$$\mathbf{T}(\widetilde{X},\widetilde{Y}) := \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}]; \quad \widetilde{X},\widetilde{Y} \in \Gamma(\mathring{\pi}). \tag{13.21}$$

Evaluating on basic sections, we obtain the more attractive formula

$$\mathbf{i}\mathbf{T}(\widehat{X},\widehat{Y}) = [X^{\mathsf{h}},Y^{\mathsf{v}}] - [Y^{\mathsf{h}},X^{\mathsf{v}}] - [X,Y]^{\mathsf{v}}; \ X,Y \in \mathfrak{X}(M). \tag{13.22}$$

13.1.7 Coordinate description Suppose that $\mathcal{H}$ is an Ehresmann connection in $\overset{\circ}{T}M$. Given a chart $(\mathcal{U},(u^i)_{i=1}^n)$ on M, consider the induced chart $(\overset{\circ}{\tau}^{-1}(\mathcal{U}),(x^i)_{i=1}^n,(y^i)_{i=1}^n)$ on $\overset{\circ}{T}M$.
(a) There exist unique smooth functions

$$N^i_j\colon \overset{\circ}{\tau}^{-1}(\mathcal{U})\to\mathbb{R};\ i,j\in J_n$$

such that

$$\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^j}}\right)=\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}}=\frac{\partial}{\partial x^j}-N^i_j\frac{\partial}{\partial y^i},\ j\in J_n. \tag{13.23}$$

We say that (N^i_j) is the family of *Christoffel symbols* $\mathcal{H}$ with respect to the chosen chart. If $X\in\mathfrak{X}(M)$ and $X\underset{(\mathcal{U})}{=}X^i\dfrac{\partial}{\partial u^i}$, then

$$X^{\mathsf{h}}\underset{(\mathcal{U})}{=}(X^i\circ\tau)\frac{\partial}{\partial x^i}-(X^j\circ\tau)N^i_j\frac{\partial}{\partial y^i}. \tag{13.24}$$

Thus

$$\begin{aligned}[C,X^{\mathsf{h}}]&\underset{(\mathcal{U})}{=}\left[C,(X^i\circ\tau)\frac{\partial}{\partial x^i}\right]-\left[C,(X^j\circ\tau)N^i_j\frac{\partial}{\partial y^i}\right]\\&=(X^i\circ\tau)\left[C,\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{c}}\right]-(X^j\circ\tau)(CN^i_j)\frac{\partial}{\partial y^i}\\&-(X^j\circ\tau)N^i_j\left[C,\left(\frac{\partial}{\partial u^i}\right)^{\mathsf{v}}\right]\overset{(7.23\,b)}{=}(X^j\circ\tau)(N^i_j-CN^i_j)\frac{\partial}{\partial y^i},\end{aligned}$$

from which we conclude, taking into account **7.7.2** (ii), that *an Ehresmann connection is homogeneous if, and only if, its Christoffel symbols are* 1^+*-homogeneous functions.*
(b) The *Christoffel symbols of the induced Berwald derivative* ∇ with respect to the chosen chart are the unique smooth function $N^i_{jk}\colon\overset{\circ}{\tau}^{-1}(\mathcal{U})\to\mathbb{R}$ such that

$$\nabla^{\mathsf{h}}_{\widehat{\frac{\partial}{\partial u^j}}}\widehat{\frac{\partial}{\partial u^k}}=N^i_{jk}\widehat{\frac{\partial}{\partial u^i}};\quad j,k\in J_n.$$

Since

$$\mathbf{i}\left(\nabla^{\mathsf{h}}_{\widehat{\frac{\partial}{\partial u^i}}}\widehat{\frac{\partial}{\partial u^k}}\right)=\left[\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}},\frac{\partial}{\partial y^k}\right]=\frac{\partial N^i_j}{\partial y^k}\frac{\partial}{\partial y^i},$$

it follows that

$$N^i_{jk} = \frac{\partial N^i_j}{\partial y^k};\ i,j,k \in J_n. \tag{13.25}$$

Now, taking into account (13.22), we find easily, that the components of the torsion of $\mathcal{H}$ are

$$T^i_{jk} = N^i_{jk} - N^i_{kj} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j}. \tag{13.26}$$

Then $\mathbf{T}(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}}) = T^i_{jk}\widehat{\frac{\partial}{\partial u^i}}$.
(c) Let $S_{\mathcal{H}} := \mathcal{H}(\widetilde{\delta})$. Since

$$\mathcal{H}(\widetilde{\delta}) \underset{(\mathcal{U})}{=} \mathcal{H}\left(y^i\widehat{\frac{\partial}{\partial u^i}}\right) = y^i\mathcal{H}\left(\widehat{\frac{\partial}{\partial u^i}}\right) \overset{(13.23)}{=} y^i\frac{\partial}{\partial x^i} - y^j N^i_j \frac{\partial}{\partial y^i},$$

it follows (see **12.1**) that $S_{\mathcal{H}}$ is a semispray with semispray coefficients

$$G^i := \frac{1}{2} y^j N^i_j. \tag{13.27}$$

We say that $S_{\mathcal{H}}$ is the *semispray associated to* $\mathcal{H}$. If $\mathcal{H}$ is homogeneous, then

$$CG^i = \frac{1}{2}(Cy^j)N^i_j + \frac{1}{2}y^j CN^i_j \overset{(7.11),(a)}{=} y^j N^i_j = 2G^i,$$

therefore $S_{\mathcal{H}}$ is a 2^+-homogeneous, so it is a spray.

13.1.8 Let an Ehresmann connection $\mathcal{H}$ be specified in $\overset{\circ}{T}M$, and let $\xi \in \mathfrak{X}(\overset{\circ}{T}M)$ be a projectable vector field. For every Finsler vector field $\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$, the vector field $[\xi, \mathbf{i}\widetilde{Y}]$ is vertical, as we have seen in **11.1.1**. Thus

$$[\xi, \mathbf{i}\widetilde{Y}] = \mathbf{v}[\xi, \mathbf{i}\widetilde{Y}] \overset{(13.5)}{=} \mathbf{i}\mathcal{V}[\xi, \mathbf{i}\widetilde{Y}],$$

so it follows that

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y} = \mathcal{V}[\xi, \mathbf{i}\widetilde{Y}]. \tag{13.28}$$

Since, as we have also seen above, the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ is projectable, the Lie derivative operator $\mathcal{L}_{X^{\mathsf{h}}}$ is defined. For every $\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$,

$$\mathcal{L}_{X^{\mathsf{h}}}\widetilde{Y} \overset{(13.28)}{=} \mathcal{V}[X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}] = \mathcal{V}[\mathcal{H}(\widehat{X}), \mathbf{i}\widetilde{Y}] \overset{(13.14)}{=} \nabla^{\mathsf{h}}_{\widehat{\widetilde{X}}}\widetilde{Y}.$$

As a conclusion, we find that

$$\nabla^{\mathsf{h}}_{\widehat{X}} = \widetilde{\mathcal{L}}_{X^{\mathsf{h}}} \ \text{ for all } \ X \in \mathfrak{X}(M). \tag{13.29}$$

Now we add to Proposition 11.1.2 the following result.

Proposition 13.1.1. *With the notation above, we have*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{h}}_{\widetilde{Y}} - \nabla^{\mathsf{h}}_{\widetilde{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]};\ X, Y \in \mathfrak{X}(M). \tag{13.30}$$

Proof. It is clear that the left-hand side and the right-hand side of (13.30) act in the same way on $C^\infty(\overset{\circ}{T}M)$. We show that $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{h}}_{\widetilde{Y}} - \nabla^{\mathsf{h}}_{\widetilde{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}) \restriction \Gamma(\overset{\circ}{\pi}) = \widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]} \restriction \Gamma(\overset{\circ}{\pi})$ also holds; then our claim follows.

For every Finsler vector field $\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$ we have

$$\begin{aligned}
&\mathbf{i} \circ (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{h}}_{\widetilde{Y}} - \nabla^{\mathsf{h}}_{\widetilde{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}})(\widetilde{Z}) \overset{(13.14),\ (11.1)}{=} \mathbf{i}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{V}[Y^{\mathsf{h}}, \mathbf{i}\widetilde{Z}]) \\
&- \mathbf{i}\nabla^{\mathsf{h}}_{\widetilde{Y}}\mathbf{i}^{-1}[X^{\mathsf{c}}, \mathbf{i}\widetilde{Z}] = [X^{\mathsf{c}}, \mathbf{i}\mathcal{V}[Y^{\mathsf{h}}, \mathbf{i}\widetilde{Z}]] - \mathbf{i}\mathcal{V}[Y^{\mathsf{h}}, [X^{\mathsf{c}}, \mathbf{i}\widetilde{Z}]] \\
&= [X^{\mathsf{c}}, \mathbf{v}[Y^{\mathsf{h}}, \mathbf{i}\widetilde{Z}]] - \mathbf{v}[Y^{\mathsf{h}}, [X^{\mathsf{c}}, \mathbf{i}\widetilde{Z}]] \overset{(*)}{=} [X^{\mathsf{c}}, [Y^{\mathsf{h}}, \mathbf{i}\widetilde{Z}]] + [Y^{\mathsf{h}}, [\mathbf{i}\widetilde{Z}, X^{\mathsf{c}}]] \\
&\overset{\text{Jacobi}}{=} -[\mathbf{i}\widetilde{Z}, [X^{\mathsf{c}}, Y^{\mathsf{h}}]] = [[X^{\mathsf{c}}, Y^{\mathsf{h}}], \mathbf{i}\widetilde{Z}] = \mathbf{i}\widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]\widetilde{Z}}\ ,
\end{aligned}$$

as was to be shown. In step $(*)$ we used the fact that the vector fields $[Y^{\mathsf{h}}, \mathbf{i}\widetilde{Z}]$ and $[\mathbf{i}\widetilde{Z}, X^{\mathsf{c}}]$ are vertical. □

13.2 The Lie derivative of an Ehresmann connection

13.2.1 Let $\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ be an Ehresmann connection, and let $\xi \in \mathfrak{X}(\overset{\circ}{T}M)$ be a projectable vector field. We define the Lie derivative $\widetilde{\mathcal{L}}_\xi\mathcal{H}$ of $\mathcal{H}$ by

$$(\widetilde{\mathcal{L}}_\xi\mathcal{H})(\widetilde{Y}) := \mathcal{L}_\xi(\mathcal{H}(\widetilde{Y})) - \mathcal{H}(\widetilde{\mathcal{L}}_\xi\widetilde{Y}) = [\xi, \mathcal{H}(\widetilde{Y})] - \mathcal{H}(\widetilde{\mathcal{L}}_\xi\widetilde{Y}), \tag{13.31}$$

where $\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$.

Proposition 13.2.2. *The Lie derivative*

$$\widetilde{\mathcal{L}}_\xi\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M),\ \widetilde{Y} \mapsto (\widetilde{\mathcal{L}}_\xi\mathcal{H})(\widetilde{Y})$$

has the following properties:

(i) It is $C^\infty(\overset{\circ}{T}M)$-linear.

(ii) For every vector field X on M,

$$\mathbf{j} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H} = 0, \tag{13.32}$$

and hence $\widetilde{\mathcal{L}}_\xi\mathcal{H}$ is not an Ehresmann connection.

(iii) If $\mathbf{h}$ is the horizontal projection associated to $\mathcal{H}$, then for every $X \in \mathfrak{X}(M)$,

$$\mathcal{L}_{X^c}\mathbf{h} = (\widetilde{\mathcal{L}}_{X^c}\mathcal{H}) \circ \mathbf{j}. \tag{13.33}$$

Proof. (i) The additivity of $\widetilde{\mathcal{L}}_\xi\mathcal{H}$ is clear. To see the $C^\infty(\overset{\circ}{T}M)$-homogeneity, let $\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$, $F \in C^\infty(\overset{\circ}{T}M)$. Then

$$\begin{aligned}(\widetilde{\mathcal{L}}_\xi\mathcal{H})(F\widetilde{Y}) &:= [\xi, \mathcal{H}(F\widetilde{Y})] - \mathcal{H}(\widetilde{\mathcal{L}}_\xi F\widetilde{Y}) \overset{(11.2)}{=} [\xi, F\mathcal{H}(\widetilde{Y})] \\ &- \mathcal{H}((\xi F))\widetilde{Y} + F\widetilde{\mathcal{L}}_\xi\widetilde{Y}) = (\xi F)\mathcal{H}(\widetilde{Y}) + F[\xi, \mathcal{H}(\widetilde{Y})] - (\xi F)\mathcal{H}(\widetilde{Y}) \\ &- F\mathcal{H}(\widetilde{\mathcal{L}}_\xi\widetilde{Y}) = F([\xi, \mathcal{H}(\widetilde{Y})] - \mathcal{H}(\widetilde{\mathcal{L}}_\xi\widetilde{Y})) = F(\widetilde{\mathcal{L}}_\xi\mathcal{H})(\widetilde{Y}),\end{aligned}$$

as wanted.
(ii) For any $X \in \mathfrak{X}(M)$, $\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$,

$$\begin{aligned}(\mathbf{j} \circ \widetilde{\mathcal{L}}_{X^c}\mathcal{H})(\widetilde{Y}) &\overset{(13.31)}{=} \mathbf{j} \circ \mathcal{L}_{X^c}(\mathcal{H}(\widetilde{Y})) - \mathbf{j} \circ \mathcal{H}(\widetilde{\mathcal{L}}_{X^c}\widetilde{Y}) \\ &\overset{(11.8),(13.1)}{=} \widetilde{\mathcal{L}}_{X^c}(\mathbf{j}(\mathcal{H}(\widetilde{Y}))) - \widetilde{\mathcal{L}}_{X^c}\widetilde{Y} = \widetilde{\mathcal{L}}_{X^c}\widetilde{Y} - \widetilde{\mathcal{L}}_{X^c}\widetilde{Y} = 0.\end{aligned}$$

(iii) For every vector field η on $\overset{\circ}{T}M$, we have

$$\begin{aligned}(\mathcal{L}_{X^c}\mathbf{h})(\eta) &\overset{(10.2)}{=} \mathcal{L}_{X^c}(\mathbf{h}\eta) - \mathbf{h}(\mathcal{L}_{X^c}\eta) = \mathcal{L}_{X^c}(\mathcal{H}(\mathbf{j}\eta)) - \mathcal{H}(\mathbf{j}\mathcal{L}_{X^c}\eta) \\ &\overset{(11.8)}{=} \mathcal{L}_{X^c}(\mathcal{H}(\mathbf{j}\eta)) - \mathcal{H}(\widetilde{\mathcal{L}}_{X^c}(\mathbf{j}\eta)) := (\widetilde{\mathcal{L}}_{X^c}\mathcal{H})(\mathbf{j}\eta) = (\widetilde{\mathcal{L}}_{X^c}\mathcal{H}) \circ \mathbf{j}(\eta),\end{aligned}$$

as was to be shown. □

Proposition 13.2.3. *With the notation as above, choose a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M, and consider the induced chart $(\overset{\circ}{\tau}^{-1}(\mathcal{U}),\ (x^i)_{i=1}^n,\ (y^i)_{i=1}^n)$ on $\overset{\circ}{T}M$. Then, for every vector field X on M,*

$$\begin{aligned}(\widetilde{\mathcal{L}}_{X^c}\mathcal{H})\left(\widehat{\frac{\partial}{\partial u^j}}\right) = \Bigg(&(N_j^k\left(\frac{\partial X^i}{\partial u^k} \circ \tau\right) - N_k^i\left(\frac{\partial X^k}{\partial u^j} \circ \tau\right) \\ &- X^c N_j^i - y^k\left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau\right)\Bigg)\frac{\partial}{\partial y^i},\end{aligned} \tag{13.34}$$

where (N_j^i) is the family of Christoffel symbols of $\mathcal{H}$ with respect to the chosen chart and $X \underset{(\mathcal{U})}{=} X^i \frac{\partial}{\partial u^i}$.

Proof. By (13.32) and (13.28)

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H})\left(\widehat{\frac{\partial}{\partial u^j}}\right) = \left[X^{\mathsf{c}}, \left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}}\right] - \mathcal{H}\circ\mathcal{V}\left[X^{\mathsf{c}}, \frac{\partial}{\partial y^j}\right].$$

Here

$$\left[X^{\mathsf{c}}, \left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}}\right] = \left[X^{\mathsf{c}}, \frac{\partial}{\partial x^j}\right] - \left[X^{\mathsf{c}}, N^i_j\frac{\partial}{\partial y^i}\right] = \left[X, \frac{\partial}{\partial u^j}\right]^{\mathsf{c}}$$
$$- (X^{\mathsf{c}}N^i_j)\frac{\partial}{\partial y^i} - N^i_j\left[X, \frac{\partial}{\partial u^i}\right]^{\mathsf{v}} = -\left(\frac{\partial X^i}{\partial u^j}\frac{\partial}{\partial u^i}\right)^{\mathsf{c}} - (X^{\mathsf{c}}N^i_j)\frac{\partial}{\partial y^i}$$
$$+ N^i_j\left(\frac{\partial X^k}{\partial u^i}\frac{\partial}{\partial u^k}\right)^{\mathsf{v}} = -\left(\frac{\partial X^k}{\partial u^j}\circ\tau\right)\frac{\partial}{\partial x^k} - y^k\left(\frac{\partial^2 X^i}{\partial u^i\partial u^k}\circ\tau\right)\frac{\partial}{\partial y^i}$$
$$- (X^{\mathsf{c}}N^i_j)\frac{\partial}{\partial y^i} + N^k_j\left(\frac{\partial X^i}{\partial u^k}\circ\tau\right)\frac{\partial}{\partial y^i};$$

$$\mathcal{H}\circ\mathcal{V}\left[X^{\mathsf{c}}, \frac{\partial}{\partial y^j}\right] = \mathcal{H}\circ\mathcal{V}\circ\mathbf{i}\,\widehat{\left[X, \frac{\partial}{\partial u^j}\right]} = \left[X^k\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^j}\right]^{\mathsf{h}}$$
$$= -\left(\frac{\partial X^k}{\partial u^j}\frac{\partial}{\partial u^k}\right)^{\mathsf{h}} = -\left(\frac{\partial X^k}{\partial u^j}\circ\tau\right)\left(\frac{\partial}{\partial u^k}\right)^{\mathsf{h}} = -\left(\frac{\partial X^k}{\partial u^j}\circ\tau\right)\frac{\partial}{\partial x^k}$$
$$+ N^i_k\left(\frac{\partial X^k}{\partial u^i}\circ\tau\right)\frac{\partial}{\partial y^i}.$$

Thus, taking the difference $\left[X^{\mathsf{c}}, (\frac{\partial}{\partial u^j})^{\mathsf{h}}\right] - \mathcal{H}\circ\mathcal{V}\left[X^{\mathsf{c}}, \frac{\partial}{\partial y^j}\right]$, we obtain the desired result. □

We note that in [7], by abuse of notation, the 'Lie derivative $\mathcal{L}_{X^{\mathsf{c}}}N^i_j$' was essentially *defined* by the right-hand side of (13.34); see loc.cit. (2.46).

13.3 $\mathcal{H}$-automorphisms and $\mathcal{H}$-Killing vector fields

Throughout, we assume that an Ehresmann connection $\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \mapsto \mathfrak{X}(\overset{\circ}{T}M)$ is given in $\overset{\circ}{T}M$. We recall that it can also be regarded as a strong bundle map $\mathcal{H}\colon \overset{\circ}{T}M \times_M TM \to T\overset{\circ}{T}M$.

A diffeomorphism $\varphi\colon U \to V$ between two open subsets of M is called a *(local) automorphism of* $\mathcal{H}$ (or simply an $\mathcal{H}$*-automorphism*) if over $\overset{\circ}{\tau}{}^{-1}(U) \times_U \tau^{-1}(U)$ we have

$$\varphi_{**} \circ \mathcal{H} = \mathcal{H} \circ (\varphi_* \times \varphi_*) \tag{13.35}$$

We say that a vector field X on M is an $\mathcal{H}$*-Killing vector field* if its local one-parameter group consists of $\mathcal{H}$-automorphisms. We denote by $\mathrm{Kill}_{\mathcal{H}}(M)$ the set of all $\mathcal{H}$-Killing vector fields on M.

Theorem 13.3.1. *Let X be a vector field on M, and let (φ_t) be the local one-parameter group generated by X. The following assertions are equivalent:*

(i) $X \in \mathrm{Kill}_{\mathcal{H}}(M)$.

(ii) *For every stage φ_t of the local flow of X we have*

$$(\varphi_t)_{**} \circ \mathbf{h} = \mathbf{h} \circ (\varphi_t)_{**}, \tag{13.36}$$

where $\mathbf{h}$ is the horizontal projection associated to $\mathcal{H}$.

(iii) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H} = 0$.

(iv) $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{h} = 0$.

If one (and hence all) of these conditions is satisfied, then locally we have

$$X^{\mathsf{c}} N^i_j = N^k_j \left(\frac{\partial X^i}{\partial u^k} \circ \tau\right) - N^i_k \left(\frac{\partial X^k}{\partial u^j} \circ \tau\right) - y^k \left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau\right); i, j \in J_n \tag{13.37}$$

where (N^i_j) is the family of Christoffel symbols of $\mathcal{H}$ with respect to a chart induced by a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ on M.

Proof. (i) $\Longleftrightarrow$ (ii) By definition, $X \in \mathrm{Kill}_{\mathcal{H}}(M)$ if, and only if, for every stage φ_t of the local flow of X we have

$$(\varphi_t)_{**} \circ \mathcal{H} = \mathcal{H} \circ ((\varphi_t)_* \times (\varphi_t)_*). \tag{$*$}$$

Since the strong bundle map $\mathbf{j}$ is surjective, this relation is equivalent to

$$(\varphi_t)_{**} \circ \mathcal{H} \circ \mathbf{j} = \mathcal{H} \circ ((\varphi_t)_* \times (\varphi_t)_*) \circ \mathbf{j}$$

Here $\mathcal{H} \circ \mathbf{j} =: \mathbf{h}$ and $((\varphi_t)_* \times (\varphi_t)_*) \circ \mathbf{j} \overset{(8.23\,b)}{=} \mathbf{j} \circ (\varphi_t)_{**}$, therefore $(*)$ is equivalent to (13.36), as we claimed.
(iii) $\Longleftrightarrow$ (iv) This is clear since $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{h} = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H}) \circ \mathbf{j}$ by Proposition 13.2.2 (iii), and $\mathbf{j}$ is surjective.
(iv) $\Longleftrightarrow$ (ii) In view of Lemma 7.5.1, the local one-parameter group of X^{c} is $((\varphi_t)_*)$. Thus, by Proposition 10.1.4, $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{h} = 0$ if, and only if $(\varphi_t)_{**} \circ \mathbf{h} = \mathbf{h} \circ (\varphi_t)_{**}$ for every stage $(\varphi_t)_*$ of the local flow of X^{c}.

The last statement of the theorem is immediate from Proposition 13.2.3. □

13.4 Lie symmetries of spray manifolds

13.4.1 Let M be a manifold, and suppose that $S\colon TM \to TTM$ is a semispray for M. Then there exists a unique Ehresmann connection $\mathcal{H}$ with vanishing torsion in $\overset{\circ}{T}M$ such that the horizontal lifts with respect to $\mathcal{H}$ are given by

$$X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) := \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]),\ X \in \mathfrak{X}(M). \tag{13.38}$$

The semispray associated to $\mathcal{H}$ is

$$S_{\mathcal{H}} := \mathcal{H}(\widetilde{\delta}) := \frac{1}{2}(S + [C, S]).$$

For a recent proof of this fundamental result we refer to [29], Proposition 7.3.4. We say that the connection $\mathcal{H}$ so defined is the *semispray connection* associated to S. If the semispray coefficients of S with respect to a chart are the functions $G^i \in C^\infty(\overset{\circ}{\tau}{}^{-1}(\mathcal{U}))$ as in **12.1**, then the Christoffel symbols of the associated semispray connection (with respect to the same chart) are

$$G^i_j := \frac{\partial G^i}{\partial y^j}; \quad i, j \in J_n. \tag{13.39}$$

Proposition 13.4.2. *Let $S\colon TM \to TTM$ be a semispray for M, and let $\mathcal{H}$ be the semispray connection associated to S. Then*

$$X \in \mathrm{Lie}_S(M) \Longrightarrow X \in \mathrm{Kill}_{\mathcal{H}}(M). \tag{13.40}$$

Proof. We calculate the Lie derivative $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H}$. Since $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H}$ is $C^\infty(\overset{\circ}{T}M)$-linear, it is sufficient to evaluate it on an arbitrary basic vector field

$\widehat{Y}$. Then we find

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H})(\widehat{Y}) \overset{(13.31)}{=} [X^{\mathsf{c}}, Y^{\mathsf{h}}] - \mathcal{H}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}) \overset{(11.5\,b)}{=} [X^{\mathsf{c}}, Y^{\mathsf{h}}] - \mathcal{H}(\widehat{[X,Y]})$$
$$\overset{(13.38)}{=} \frac{1}{2}([X^{\mathsf{c}}, Y^{\mathsf{c}} + [Y^{\mathsf{v}}, S]] - [X,Y]^{\mathsf{c}} - [[X,Y]^{\mathsf{v}}, S])$$
$$\overset{(7.22\,b-c)}{=} \frac{1}{2}([X^{\mathsf{c}}, [Y^{\mathsf{v}}, S]] - [[X^{\mathsf{c}}, Y^{\mathsf{v}}], S]) = \frac{1}{2}([X^{\mathsf{c}}, [Y^{\mathsf{v}}, S]] + [S, [X^{\mathsf{c}}, Y^{\mathsf{v}}]]).$$

Since, by condition, $[X^{\mathsf{c}}, S] = 0$ (see **12.2.2** (iv)), the Jacobi identity gives

$$0 = [X^{\mathsf{c}}, [Y^{\mathsf{v}}, S]] + [Y^{\mathsf{v}}, [S, X^{\mathsf{c}}]] + [S, [X^{\mathsf{c}}, Y^{\mathsf{v}}]]$$
$$= \frac{1}{2}[X^{\mathsf{c}}, [Y^{\mathsf{v}}, S]] + [S, [X^{\mathsf{c}}, Y^{\mathsf{v}}]],$$

thus concluding the proof. □

Remark 13.4.3. Applying the argument of R. L. Lovas in [17], Proposition 5.2, we show that the converse of implication (13.40) is also true when S is a *spray*. Indeed, if $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H} = 0$, then the calculation above leads to

$$[[X^{\mathsf{c}}, S], Y^{\mathsf{v}}] = 0 \quad \text{for all } Y \in \mathfrak{X}(M).$$

Since $[X^{\mathsf{c}}, S]$ is vertical by Lemma 12.1.1, this implies that $[X^{\mathsf{c}}, S]$ is a vertical lift, and hence

$$[C, [X^{\mathsf{c}}, S]] \overset{(7.23\,a)}{=} -[X^{\mathsf{c}}, S]. \qquad (*)$$

On the other hand, using the Jacobi identity, the 2^+-homogeneity of S and the 1^+-homogeneity of X^{c}, we find that

$$0 = [C, [X^{\mathsf{c}}, S]] + [X^{\mathsf{c}}, [S, C]] + [S, [C, X^{\mathsf{c}}]] = [C, [X^{\mathsf{c}}, S]] - [X^{\mathsf{c}}, S],$$

whence $[C, [X^{\mathsf{c}}, S]] = [X^{\mathsf{c}}, S]$. Comparing this with equality $(*)$, we conclude that $[X^{\mathsf{c}}, S] = 0$, and hence $X \in \mathrm{Lie}_S(M)$.

13.4.4 Now suppose that (M, S) is a spray manifold. Then the construction described in **13.4.1** leads to a *homogeneous* torsion-free Ehresmann connection $\mathcal{H}\colon TM \times_M TM \to TTM$ (see also the end of **13.1.4**). This *spray connection* will be called the *Berwald connection* of (M, S). Then the semispray associated to $\mathcal{H}$ (**13.1.7**(c)) is just the initial spray S. The Christoffel symbols $G^i_j = \frac{\partial G^i}{\partial y^j}$ are smooth on $\overset{\circ}{\tau}^{-1}(U)$ and continuous on $\tau^{-1}(U)$. The *Berwald derivative in a spray manifold* (M,S) is the covariant derivative ∇ on $\overset{\circ}{\pi}$ induced by the Berwald connection $\mathcal{H}$ according to (13.19). Its Christoffel symbols with respect to an induced chart on TM are the 0^+-homogeneous functions

$$G^i_{jk} \overset{(13.24)}{:=} \frac{\partial G^i_j}{\partial y^k} = \frac{\partial G^i}{\partial y^j \partial y^k} \in C^\infty(\overset{\circ}{\tau}^{-1}(U)).$$

Our next theorem is a supplement to Lovas's Proposition 5.2 in [17].

Theorem 13.4.5. *Let (M,S) be a spray manifold, equipped with the Berwald connection $\mathcal{H}$ associated to S and the Berwald derivative ∇ induced by $\mathcal{H}$. Let $\mathbf{h}$, $\mathbf{v}$ and $\mathcal{V}$ be the data defined by (13.4 a-b) and* (13.5)*. For a vector filed X on M, the following are equivalent:*

(i) $X \in \mathrm{Lie}_S(M)$*, i.e., X is a Lie symmetry of S;*

(ii) $[X^{\mathsf{c}}, S] = 0$*;*

(iii) $X \in \mathrm{Kill}_{\mathcal{H}}(M)$*;*

(iv) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H} = 0$*;*

(v) $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{h} = -[\mathbf{h}, X^{\mathsf{c}}] = 0$*;*

(vi) $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{v} = -[\mathbf{v}, X^{\mathsf{c}}] = 0$*;*

(vii) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla = 0$*;*

(viii) *For every vector field Y on M;*

$$[X^{\mathsf{c}}, Y^{\mathsf{h}}] = [X, Y]^{\mathsf{h}}; \tag{13.41}$$

(ix) *For every vector field Y on M,*

$$[\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}, \widetilde{\mathcal{L}}_{Y^{\mathsf{h}}}] = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{h}}}; \tag{13.42}$$

(x) *We have the commutation relation*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{L}_{X^{\mathsf{c}}}. \tag{13.43}$$

Proof. We begin with some remarks.

(1) The equivalence of conditions (i), (v) and (vii) has already been proven in Lovas's cited paper [17].

(2) We have shown for semisprays that (i) $\iff$ (ii), and for general Ehresmann connections that (iii) $\iff$ (iv) $\iff$ (v). We note that the equivalence of (*v*) and (*vi*) is evident, since $\mathbf{v} = 1_{\mathfrak{X}(\overset{\circ}{T}M)} - \mathbf{h}$, and $[1_{\mathfrak{X}(\overset{\circ}{T}M)}, \xi] = 0$ for every $\xi \in \mathfrak{X}(\overset{\circ}{T}M)$.

(3) By Proposition 13.4.2 and Remark 13.4.3 (*i*) $\Longrightarrow$ (*iii*) for every semispray, and (iii) $\Longrightarrow$(i) for sprays.

Thus, to complete the proof, it is enough to show the implications (v) $\iff$ (viii), (viii) $\iff$ (ix) and (vi) $\iff$ (x). The equivalence of conditions (i), (ii) and (iv) has already been proved in [17].

(v) $\iff$ (vii) For any vector field Y on M,

$$[\mathbf{h}, X^{\mathsf{c}}]Y^{\mathsf{c}} \overset{(10.3)}{=} [\mathbf{h}Y^{\mathsf{c}}, X^{\mathsf{c}}] - \mathbf{h}[Y^{\mathsf{c}}, X^{\mathsf{c}}] = [Y^{\mathsf{h}}, X^{\mathsf{c}}] - \mathbf{h}[Y, X]^{\mathsf{c}}$$
$$= [Y^{\mathsf{h}}, X^{\mathsf{c}}] - [Y, X]^{\mathsf{h}},$$

so $[\mathbf{h}, X^{\mathsf{c}}] = 0$ implies that $[X, Y]^{\mathsf{h}} = [X^{\mathsf{c}}, Y^{\mathsf{h}}]$. The converse is also true, since

$$[\mathbf{h}, X^{\mathsf{c}}]Y^{\mathsf{v}} = [\mathbf{h}Y^{\mathsf{v}}, X^{\mathsf{c}}] - \mathbf{h}[Y, X]^{\mathsf{v}} = 0,$$

and hence $[\mathbf{h}, X^{\mathsf{c}}] \restriction \mathfrak{X}^{\mathsf{v}}(\overset{\circ}{T}M) = 0$.

(vii) $\iff$ (ix) This is an immediate since $[\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}, \widetilde{\mathcal{L}}_{Y^{\mathsf{h}}}] \overset{(11.3)}{=} \widetilde{\mathcal{L}}_{[X^{\mathsf{c}}, Y^{\mathsf{h}}]}$.

(vi) $\iff$ (ix) For any vector field ξ on $\overset{\circ}{T}M$,

$$\mathbf{i}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathcal{V}\xi) \overset{(11.1)}{=} [X^{\mathsf{c}}, \mathbf{i}(\mathcal{V}\xi)] \overset{(13.5)}{=} [X^{\mathsf{c}}, \mathbf{v}\xi],$$
$$\mathbf{i}\mathcal{V}(\mathcal{L}_{X^{\mathsf{c}}}\xi) = \mathbf{v}[X^{\mathsf{c}}, \xi],$$

so $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathcal{V}\xi) = \mathcal{V}(\mathcal{L}_{X^{\mathsf{c}}}\xi)$ if, and only if,

$$0 = [\mathbf{v}\xi, X^{\mathsf{c}}] - \mathbf{v}[\xi, X^{\mathsf{c}}] = [\mathbf{v}, X^{\mathsf{c}}]\xi.$$

This concludes the proof. □

14 Curvature collineations in a spray manifold

Throughout this section, (M, S) is a spray manifold, $\mathcal{H}$ is the Berwald connection in (M, S) and ∇ is the Berwald derivative induced by $\mathcal{H}$. As always, we denote by **h**, **v** *and* $\mathcal{V}$ *the horizontal projection, the vertical projection and the vertical map associated to $\mathcal{H}$, respectively.*

14.1 In the language of classical tensor calculus, the basic curvature data of a spray manifold were introduced by Ludwig Berwald in his epoch-making, posthumously published paper [6], in an illuminating manner. Here we follow his approach, but we use an index-free formalism. In this spirit, we start with *Jacobi endomorphism* $\mathbf{K} \in T_1^1(\Gamma(\overset{\circ}{\pi}))$ (called *affine deviation* by Berwald) given by

$$\mathbf{K}(\widetilde{X}) := \mathcal{V}[S, \mathcal{H}(\widetilde{X})], \quad \widetilde{X} \in \Gamma(\overset{\circ}{\pi}). \tag{14.1}$$

Next, with the help of $\mathbf{K}$, we define the *fundamental affine curvature* $\mathbf{R} \in T_2^1(\Gamma(\overset{\circ}{\pi}))$ and the *affine curvature* $\mathbf{H} \in T_3^1(\Gamma(\overset{\circ}{\pi}))$ by the formulae

$$\mathbf{R}(\widetilde{X}, \widetilde{Y}) := \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{K}(\widetilde{X}, \widetilde{Y}) - \nabla^{\mathsf{v}}\mathbf{K}(\widetilde{Y}, \widetilde{X})) \tag{14.2}$$

and

$$\mathbf{H}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := -\nabla^{\mathsf{v}}\mathbf{R}(\widetilde{Z}, \widetilde{X}, \widetilde{Y}). \tag{14.3}$$

If $\mathbf{C} \in \{\mathbf{K}, \mathbf{R}, \mathbf{H}\}$, $X \in \mathfrak{X}(M)$ and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{C} = 0$, then we say X is a *curvature collineation* of $\mathbf{C}$. Notice that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0 \iff \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{K} = \mathbf{K} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}. \tag{14.4}$$

Indeed, for every $\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$ we have

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K})(\widetilde{Y}) \overset{(11.13)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widetilde{Y})) - \mathbf{K}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{Y}).$$

Proposition 14.1.1. (i) *Let* $\mathbf{K}^0$ *be the semibasic 1-form corresponding to the Jacobi endomorphism under the isomorphism given by* (8.18). *Then*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K}^0 = 0 \iff \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0 \quad (X \in \mathfrak{X}(M)).$$

(ii) *A vector field* X *on* M *is a curvature collineation of* $\mathbf{K}$ *if, and only if,* $\mathbf{K}$ *is invariant under the local flow of* X *in the sense that*

$$((\varphi_t)_* \times (\varphi_t)_*)) \circ \mathbf{K} = \mathbf{K} \circ ((\varphi_t)_* \times (\varphi_t)_*)$$

for every stage φ_t *of the local flow (here* $\mathbf{K}$ *is interpreted as a strong bundle endomorphism of* $\overset{\circ}{T}M \times_M TM$*).*

Proof. (i) Suppose that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0$. Then

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K}^0 \overset{(8.18)}{=} \mathcal{L}_{X^{\mathsf{c}}} \circ \mathbf{i} \circ \mathbf{K} \circ \mathbf{j} \overset{(11.7)}{=} \mathbf{i} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{K} \circ \mathbf{j} \overset{(14.4)}{=} \mathbf{i} \circ \mathbf{K} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{j}$$
$$\overset{(11.8)}{=} \mathbf{i} \circ \mathbf{K} \circ \mathbf{j} \circ \mathcal{L}_{X^{\mathsf{c}}} = \mathbf{K}^0 \circ \mathcal{L}_{X^{\mathsf{c}}},$$

which implies (as above) that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K}^0 = 0$.

Conversely, suppose that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K}^0 = 0$. Then $\mathcal{L}_{X^{\mathsf{c}}} \circ \mathbf{K}^0 = \mathbf{K}^0 \circ \mathcal{L}_{X^{\mathsf{c}}}$, and we obtain

$$\mathbf{i} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} \circ \mathbf{j} = \mathcal{L}_{X^{\mathsf{c}}}\mathbf{K}^0 = \mathcal{L}_{X^{\mathsf{c}}} \circ \mathbf{i} \circ \mathbf{K} \circ \mathbf{j} = \mathbf{i} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{K} \circ \mathbf{j}.$$

Since $\mathbf{i}$ is injective, $\mathbf{j}$ is surjective, from this we conclude that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{K} = \mathbf{K} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}$, and hence $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0$.
(ii) By part (i), Lemma 7.5.1 and Proposition 10.1.4,

$$\begin{aligned}
\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0 &\iff (\varphi_t)_{**} \circ \mathbf{K}^0 = \mathbf{K}^0 \circ (\varphi_t)_{**} \\
&\iff (\varphi_t)_{**} \circ \mathbf{i} \circ \mathbf{K} \circ \mathbf{j} = \mathbf{i} \circ \mathbf{K} \circ \mathbf{j} \circ (\varphi_t)_{**} \\
&\overset{(8.23\,a-b)}{\iff} \mathbf{i} \circ ((\varphi_t)_* \times (\varphi_t)_*) \circ \mathbf{K} \circ \mathbf{j} = \mathbf{i} \circ \mathbf{K} \circ ((\varphi_t)_* \times (\varphi_t)_*) \circ \mathbf{j} \\
&\iff ((\varphi_t)_* \times (\varphi_t)_*) \circ \mathbf{K} = \mathbf{K} \circ ((\varphi_t)_* \times (\varphi_t)_*),
\end{aligned}$$

where φ_t is any stage of the local flow of X. □

Theorem 14.1.2. *If a vector field X on M is a Lie symmetry of S, then it is a curvature collineation of the Jacobi endomorphism of (M, S).*

Proof. Suppose that $X \in \mathrm{Lie}_S(M)$. Then, for every vector field Y on M,

$$\begin{aligned}
&(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K})(\widehat{Y}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y})) - \mathbf{K}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}) \overset{(14.1)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathcal{V}[S, Y^{\mathsf{h}}]) \\
&-\mathcal{V}[S, \mathcal{H}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})] \overset{(11.1),\,(11.5\,b)}{=} \mathbf{i}^{-1}[X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] - \mathcal{V}[S, \mathcal{H}\widehat{[X,Y]}] \\
&\overset{(13.9)}{=} \mathbf{i}^{-1}[X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] - \mathcal{V}[S, [X,Y]^{\mathsf{h}}] = \mathbf{i}^{-1}([X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] \\
&-\mathbf{v}[S, [X,Y]^{\mathsf{h}}]) \overset{(13.41)}{=} \mathbf{i}^{-1}([X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] - \mathbf{v}[S, [X^{\mathsf{c}}, Y^{\mathsf{h}}]]) \\
&\overset{\text{Jacobi}}{=} \mathbf{i}^{-1}([X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] + \mathbf{v}[X^{\mathsf{c}}, [Y^{\mathsf{h}}, S]] \\
&+[Y^{\mathsf{h}}, [S, X^{\mathsf{c}}]]) \overset{X \text{ is a Lie symmetry}}{=} \mathbf{i}^{-1}([X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] - \mathbf{v}[X^{\mathsf{c}}, [S, Y^{\mathsf{h}}]] \\
&= -\mathbf{i}^{-1}([\mathbf{v}, X^{\mathsf{c}}][S, Y^{\mathsf{h}}]) \overset{(13.4.5),\,(\mathrm{vi})}{=} 0,
\end{aligned}$$

so we have the desired equality $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0$. □

Corollary 14.1.3. *If $X \in \mathrm{Lie}_S(M)$, then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R} = 0$, i.e., X is a curvature collineation of the fundamental affine curvature of (M, S).*

Proof. Since $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R}$ is $C^\infty(\overset{\circ}{T}M)$-linear in its both arguments, it is sufficient to show that $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R})(\widehat{Y}, \widehat{Z}) = 0$ for all $Y, Z \in \mathfrak{X}(M)$. By (11.13),

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R})(\widehat{Y}, \widehat{Z}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{R}(\widehat{Y}, \widehat{Z})) - \mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}, \widehat{Z}) - \mathbf{R}(\widehat{Y}, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}). \quad (*)$$

We calculate the three terms at the right-hand side of $(*)$:

$$\begin{aligned}
&(1)\quad \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{R}(\widehat{Y},\widehat{Z})) \overset{(14.2)}{=} \frac{1}{3}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{Z}) - \nabla^{\mathsf{v}}\mathbf{K}(\widehat{Z},\widehat{Y}))\\
&= \frac{1}{3}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla^{\mathsf{v}}_{\widehat{Y}}\mathbf{K})(\widehat{Z}) - (\nabla^{\mathsf{v}}_{\widehat{Z}}\mathbf{K})(\widehat{Y})) \overset{(9.10)}{=} \frac{1}{3}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}(\mathbf{K}(\widehat{Z})))\\
&- \frac{1}{3}\nabla^{\mathsf{v}}_{\widehat{Z}}(\mathbf{K}(\widehat{Y})) \overset{(11.9)}{=} \frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Z})) + \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}(\mathbf{K}(\widehat{Z})))\\
&- \frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y})) - \widetilde{\mathcal{L}}_{[X,Z]^{\mathsf{v}}}(\mathbf{K}(\widehat{Y})));
\end{aligned}$$

$$\begin{aligned}
&(2)\quad \mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) := \frac{1}{3}((\nabla^{\mathsf{v}}\mathbf{K})(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z})) - (\nabla^{\mathsf{v}}\mathbf{K})(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})\\
&= \frac{1}{3}(\nabla^{\mathsf{v}}_{\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}}(\mathbf{K}(\widehat{Z})) - \nabla^{\mathsf{v}}_{\widehat{Z}}(\mathbf{K}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})))\\
&\overset{(11.5\,b),\,(14.4)}{=} \frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{[X,Y]}}(\mathbf{K}(\widehat{Z})) - \nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y}))).
\end{aligned}$$

(3) Interchanging Y and Z in the above result,

$$\begin{aligned}
\mathbf{R}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}) &= -\mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{Y})\\
&= -\frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{[X,Z]}}(\mathbf{K}(\widehat{Y})) - (\nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Z})))).
\end{aligned}$$

Thus we obtain that

3 times the right-hand side of $(*)$

$$\begin{aligned}
&= \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widetilde{Z})) + \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}(\mathbf{K}(\widehat{Z})) - \nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y}))\\
&- \widetilde{\mathcal{L}}_{[X,Z]^{\mathsf{v}}}(\mathbf{K}(\widehat{Y})) - \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}(\mathbf{K}(\widehat{Z})) - \nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y}))\\
&+ \nabla^{\mathsf{v}}_{\widehat{[X,Z]}}(\mathbf{K}(\widehat{Y})) - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Z})) \overset{(11.6)}{=} \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}(\mathbf{K}(\widetilde{Z}))\\
&- \nabla^{\mathsf{v}}_{\widehat{[X,Z]}}(\mathbf{K}(\widetilde{Y})) - \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}(\mathbf{K}(\widetilde{Z})) + \nabla^{\mathsf{v}}_{\widehat{[X,Z]}}(\mathbf{K}(\widetilde{Y})) = 0,
\end{aligned}$$

as was to be shown. □

Notice that by (11.13) for all Y, $Z \in \mathfrak{X}(M)$ we have

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R} = 0 \iff \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{R}(\widehat{Y},\widehat{Z})) = \mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) + \mathbf{R}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}). \tag{14.5}$$

Corollary 14.1.4. *If $X \in \mathrm{Lie}_S(M)$, then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{H} = 0$, i.e., X is a curvature collineation of the affine curvature of (M,S).*

Proof. By the previous corollary, $X \in \mathrm{Lie}_S(M)$ implies that $\widetilde{\mathcal{L}}_{X^c}\mathbf{R} = 0$. Now we evaluate the Lie derivative $\widetilde{\mathcal{L}}_{X^c}\mathbf{H}$ on an arbitrary triple $(\widehat{Y}, \widehat{Z}, \widehat{U})$, where Y, Z, U are vector fields on M. Then we find that

$$
\begin{aligned}
&(\widetilde{\mathcal{L}}_{X^c}\mathbf{H})(\widehat{Y}, \widehat{Z}, \widehat{U}) \overset{(11.13)}{=} \widetilde{\mathcal{L}}_{X^c}(\mathbf{H}(\widehat{Y}, \widehat{Z}, \widehat{U}) - \mathbf{H}(\widetilde{\mathcal{L}}_{X^c}\widehat{Y}, \widehat{Z}, \widehat{U})\\
&-\mathbf{H}(\widehat{Y}, \widetilde{\mathcal{L}}_{X^c}\widehat{Z}, \widehat{U}) - \mathbf{H}(\widehat{Y}, \widehat{Z}, \widetilde{\mathcal{L}}_{X^c}\widehat{U}) \overset{(14.3)}{=} -\widetilde{\mathcal{L}}_{X^c}((\nabla^{\mathsf{v}}\mathbf{R})(\widehat{U}, \widehat{Y}, \widehat{Z}))\\
&+(\nabla^{\mathsf{v}}\mathbf{R})(\widehat{U}, \widetilde{\mathcal{L}}_{X^c}\widehat{Y}, \widehat{Z}) + (\nabla^{\mathsf{v}}\mathbf{R})(\widehat{U}, \widehat{Y}, \widetilde{\mathcal{L}}_{X^c}\widehat{Z}) + (\nabla^{\mathsf{v}}\mathbf{R})(\widetilde{\mathcal{L}}_{X^c}\widehat{U}, \widehat{Y}, \widehat{Z})\\
&\overset{(9.10),(11.5\,b)}{=} -\widetilde{\mathcal{L}}_{X^c}(\nabla^{\mathsf{v}}_{\widehat{U}}(\mathbf{R}(\widehat{Y}, \widehat{Z}))) + (\nabla^{\mathsf{v}}_{\widehat{U}}\mathbf{R})(\widetilde{\mathcal{L}}_{X^c}\widehat{Y}, \widehat{Z})\\
&+(\nabla^{\mathsf{v}}_{\widehat{U}}\mathbf{R})(\widehat{Y}, \widetilde{\mathcal{L}}_{X^c}\widehat{Z}) + (\nabla^{\mathsf{v}}_{\widehat{[X,U]}}\mathbf{R})(\widehat{Y}, \widehat{Z}) \overset{(11.9),(11.12)}{=}\\
&-\nabla^{\mathsf{v}}_{\widehat{U}} \circ \widetilde{\mathcal{L}}_{X^c}(\mathbf{R}(\widehat{Y}, \widehat{Z})) - \widetilde{\mathcal{L}}_{[X,U]^{\mathsf{v}}}(\mathbf{R}(\widehat{Y}, \widehat{Z})) + \nabla^{\mathsf{v}}_{\widehat{U}}(\mathbf{R}(\widetilde{\mathcal{L}}_{X^c}\widehat{Y}, \widehat{Z}))\\
&+\nabla^{\mathsf{v}}_{\widehat{U}}(\mathbf{R}(\widehat{Y}, \widetilde{\mathcal{L}}_{X^c}\widehat{Z})) + \nabla^{\mathsf{v}}_{\widehat{[X,U]}}(\mathbf{R}(\widehat{Y}, \widehat{Z})) \overset{(14.5)}{=} \nabla^{\mathsf{v}}_{\widehat{[X,U]}}\mathbf{R}(\widehat{Y}, \widehat{Z})\\
&-\widetilde{\mathcal{L}}_{[X,U]^{\mathsf{v}}}(\mathbf{R}(\widehat{Y}, \widehat{Z})) \overset{(11.6)}{=} 0,
\end{aligned}
$$

as was to be proved. □

14.2 Projective relatedness First we recall that a *geodesic* of S is a smooth curve $\gamma\colon I \to M$ whose velocity vector field is an integral curve of S, i.e., $S \circ \dot{\gamma} = \ddot{\gamma}$. If a smooth curve $\widetilde{\gamma}\colon \widetilde{I} \to M$ has a positive reparametrization as a geodesic, i.e., there exists a smooth function $\theta\colon I \to \widetilde{I}$ with positive derivative such that $\gamma := \widetilde{\gamma} \circ \theta\colon I \to M$ is a geodesic, then $\widetilde{\gamma}$ is called a *pregeodesic* of S. Two sprays over M are *projectively related* if they have the same pregeodesic. Projective relatedness of sprays is an equivalence relations, the *projective class* of S is denoted by $[S]$. Let $\overline{S}$ be another spray for M. By a classical result of the geometry of paths, $\overline{S} \in [S]$ if, and only if, there exists a function $P \in C^\infty(\overset{\circ}{T}M)$ such that

$$\overline{S} = S - 2PC. \tag{14.6}$$

Then the projective factor P is necessarily 1^+-homogeneous.

Let A be a Finsler tensor field constructed from S, and let $\overline{A}$ be a tensor constructed from $\overline{S} \in [S]$ by the same rule. If $\overline{A} = A$ for all $\overline{S} \in [S]$, then A is called a *projectively invariant* tensor of the spray manifold (M, S). The fundamental projectively invariant tensors of a spray manifold are the *Weyl tensors* $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ and the Douglas tensor $\mathbf{D}$. We recall here their definitions:

$$\mathbf{W_1} := \mathbf{K} - K\,\mathbf{1} - \frac{1}{n+1}(\mathrm{tr}\nabla^{\mathsf{v}}\mathbf{K} - \nabla^{\mathsf{v}}K) \otimes \widetilde{\delta} \tag{14.7}$$

where $\mathbf{K}$ is the Jacobi endomorphism defined above, $K := \frac{1}{n-1}\mathrm{tr}\mathbf{K}$, $n := \dim M \geqq 2$, $\mathbf{1} := 1_{\Gamma(\overset{\circ}{\pi})}$.

$$\mathbf{W_2}(\widetilde{X}, \widetilde{Y}) := \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{W_1}(\widetilde{X}, \widetilde{Y}) - \nabla^{\mathsf{v}}\,\mathbf{W_1}(\widetilde{Y}, \widetilde{X})), \tag{14.8}$$

$$\mathbf{W_3}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := \nabla^{\mathsf{v}}\mathbf{W_2}(\widetilde{Z}, \widetilde{X}, \widetilde{Y}); \quad \widetilde{X}, \widetilde{Y}\,\widetilde{Z} \in \Gamma(\overset{\circ}{\pi}). \tag{14.9}$$

The tensors $\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$ are called the *Weyl endomorphism*, the *fundamental projective curvature tensor* and the *projective curvature tensor* of (M, S), respectively.

To introduce the Douglas tensor, first we define the Berwald tensor $\mathbf{B} \in T^1_3(\Gamma(\overset{\circ}{\pi}))$ of (M, S) by

$$\begin{aligned}\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} &:= (\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{Z})(\widehat{X}, \widehat{Y}) \\ &= \nabla^{\mathsf{v}}_{\widehat{X}}(\nabla^{\mathsf{h}}\widehat{Z})(\widehat{Y}) \overset{(9.10)}{=} \nabla^{\mathsf{v}}_{\widehat{X}}(\nabla^{\mathsf{h}}_{\widehat{Y}}\widehat{Z}).\end{aligned} \tag{14.10}$$

Locally, with the notation of **13.4.4** (see also **13.1.7** (b)),

$$\mathbf{B}\left(\widehat{\frac{\partial}{\partial u^j}}, \widehat{\frac{\partial}{\partial u^k}}\right)\widehat{\frac{\partial}{\partial u^l}} = \nabla^{\mathsf{v}}_{\widehat{\frac{\partial}{\partial u^j}}}\left(G^i_{kl}\widehat{\frac{\partial}{\partial u^i}}\right) = \frac{\partial G^i_{kl}}{\partial y^z}\frac{\partial}{\partial y^i},$$

so the components of $\mathbf{B}$ with respect to an induced chart on TM are the $(-1)^+$-homogeneous functions

$$G^i_{jkl} := \frac{\partial G^i_{kl}}{\partial y^j} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} \in C^\infty(\overset{\circ}{\tau}^{-1}(U)),$$

where the functions G^i are the spray coefficients of S. After this preparatory step, we define the Douglas tensor of (M, S) by

$$\mathbf{D} := \mathbf{B} - \frac{1}{n-1}((\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B}) \otimes \widetilde{\delta} + (\mathrm{tr}\mathbf{B}) \odot \mathbf{1}), \tag{14.11}$$

where the symbol $\odot$ stands for the symmetric product without numerical factor. For a coordinate description we refer to [29], Remark 8.4.25.

Next we show that if $X \in \mathrm{Lie}_S(M)$, then X is a curvature collineation of the Weyl tensors, the Berwald tensor and the Douglas tensor in the same sense as above.

Theorem 14.2.1. *If $X \in \mathrm{Lie}_S(M)$, then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W_1} = 0$, i.e., X is a curvature collineation of the Weyl endomorphism of (M, S).*

Proof. If $X \in \mathrm{Lie}_S(M)$, then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0$ by Theorem 14.1.2. Taking into account that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{1} = 0$, $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{\delta} \overset{(11.5\,a)}{=} 0$ and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathrm{tr} = \mathrm{tr} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}$, we readily find that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(K\,\mathbf{1}) = \frac{1}{n-1}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathrm{tr}\mathbf{K})\mathbf{1} = \frac{1}{n-1}\mathrm{tr}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}K)\mathbf{1} = 0.$$

Next we show that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{v}}K = 0. \tag{14.12}$$

For every vector field Y on M,

$$\begin{aligned}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{v}}K)(\widehat{Y}) &= X^{\mathsf{c}}(Y^{\mathsf{v}}K) - \nabla^{\mathsf{v}}K(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}) = X^{\mathsf{c}}(Y^{\mathsf{v}}K) - \nabla^{\mathsf{v}}K\widehat{[X,Y]} \\ &= X^{\mathsf{c}}(Y^{\mathsf{v}}K) - [X,Y]^{\mathsf{v}}K = [X^{\mathsf{c}},Y^{\mathsf{v}}]K + Y^{\mathsf{v}}(X^{\mathsf{c}}K) - [X,Y]^{\mathsf{v}}K \\ &\overset{(7.22\,b)}{=} Y^{\mathsf{v}}(X^{\mathsf{c}}K) = \frac{1}{n-1}Y^{\mathsf{v}}(X^{\mathsf{c}}\,\mathrm{tr}\mathbf{K}) = \frac{1}{n-1}Y^{\mathsf{v}}(\mathrm{tr}\,\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K}) = 0,\end{aligned}$$

as we claimed. Taking these into account, we obtain

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W_1} = -\frac{1}{n-1}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\,\mathrm{tr}\nabla^{\mathsf{v}}\mathbf{K}) \otimes \widetilde{\delta}.$$

To finish the proof we show that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\,\mathrm{tr}\nabla^{\mathsf{v}}\mathbf{K} = 0$. From Corollary 8.2.8 in [29],

$$\mathrm{tr}\nabla^{\mathsf{v}}\mathbf{K} = 3\,\mathrm{tr}\mathbf{R} + \nabla^{\mathsf{v}}\mathrm{tr}\mathbf{K}.$$

By Corollary 14.1.3, $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R} = 0$. Thus

$$\begin{aligned}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\,\mathrm{tr}\,\nabla^{\mathsf{v}}\mathbf{K} &= 3\,\mathrm{tr}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R} + \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\,\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{K} = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\,\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{K} = (n-1)\,\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{v}}K \\ &\overset{(14.12)}{=} 0,\end{aligned}$$

which concludes the proof. □

Corollary 14.2.2. *If $X \in \mathrm{Lie}_S(M)$, then X is a curvature collineation both of the fundamental projective curvature and the projective curvature of (M,S).*

This can be shown in the same way as Corollaries 14.1.3 and 14.1.4, so we omit the analogous calculation.

Proposition 14.2.3. *If $X \in \mathrm{Lie}_S(M)$ and $\mathbf{B}$ is the Berwald tensor of (M,S), then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{B} = 0$.*

Proof. For any vector fields Y, Z, U on M,

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{B})(\widehat{Y},\widehat{Z},\widehat{U}) \overset{(11.13)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{B}(\widehat{Y},\widehat{Z})\,\widehat{U}) - \mathbf{B}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z})\,\widehat{U}$$
$$-\mathbf{B}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z})\widehat{U} - \mathbf{B}(\widehat{Y},\widehat{Z})\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U} \overset{(14.10),\,(11.5\,b)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{U})(\widehat{Y},\widehat{Z}))$$
$$-((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{U})(\widehat{[X,Y]},\widehat{Z})) - ((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{U})(\widehat{Y},\widehat{[X,Z]}))$$
$$-((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})(\widehat{Y},\widehat{Z})) \overset{(14.10)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U}) - \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U}$$
$$-\nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{[X,Z]})}\widehat{U} - \nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U} \overset{(11.9),\,(13.30)}{=} \nabla^{\mathsf{v}}_{\widehat{Y}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U})$$
$$+\widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U} - \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U} - \nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{[X,Z]}}\widehat{U} - \nabla^{\mathsf{v}}_{\widehat{Y}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U})$$
$$+\nabla^{\mathsf{v}}_{\widehat{Y}}\widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Z^{\mathsf{h}}]}\widehat{U} \overset{(11.6),\,(13.40)}{=} -\nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{[X,Z]}}\widehat{U} + \nabla^{\mathsf{v}}_{\widehat{Y}}\widetilde{\mathcal{L}}_{[X,Z]^{\mathsf{h}}}\widehat{U} \overset{(13.29)}{=} 0,$$

as was to be shown. □

Corollary 14.2.4. *If $X \in \mathrm{Lie}_S(M)$, then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{D} = 0$, i.e., X is a curvature collineation for the Douglas tensor.*

Proof. By the previous proposition, $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}$ kills the first and the third member of the right-hand side of (14.11), so it remains only to show that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B}) = 0$. Given any three vector fields Y, Z, U on M, we calculate:

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\,\mathrm{tr}\mathbf{B}))(\widehat{Y},\widehat{Z},\widehat{U}) = X^{\mathsf{c}}((\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B})(\widehat{Y},\widehat{Z},\widehat{U}))$$
$$-(\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B})(\widehat{[X,Y]},\widehat{Z},\widehat{U}) - (\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B})(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U})$$
$$-(\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B})(\widehat{Y},\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) = X^{\mathsf{c}}Y^{\mathsf{v}}(\mathrm{tr}\mathbf{B}(\widehat{Z},\widehat{U})) - [X,Y]^{\mathsf{v}}(\mathrm{tr}\mathbf{B}(\widehat{Z},\widehat{U}))$$
$$-Y^{\mathsf{v}}(\mathrm{tr}\mathbf{B}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U})) - Y^{\mathsf{v}}(\mathrm{tr}\mathbf{B}(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})) \overset{(7.22\,b)}{=} Y^{\mathsf{v}}(X^{\mathsf{c}}(\mathrm{tr}\mathbf{B})(\widehat{Z},\widehat{U})$$
$$-\mathrm{tr}\mathbf{B}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) - \mathrm{tr}\mathbf{B}(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})) = Y^{\mathsf{v}}((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\,\mathrm{tr}\mathbf{B})(\widehat{Z},\widehat{U}))$$
$$= Y^{\mathsf{v}}((\mathrm{tr}\,\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{B})(\widehat{Z},\widehat{U})) \overset{\text{Prop. 14.2.3}}{=} 0.$$

This proves our assertion. □

The main results of this chapter have been published in our paper [31].

Part IV
Geometric vector fields on Finsler manifolds

15 Basic objects of a Finsler manifold

Throughout this part, M is a manifold of dimension $n \geqq 2$.

15.1 We recall that a positive continuous function $F\colon TM \to \mathbb{R}$ is called a *Finsler function* for M if it is smooth on $\overset{\circ}{T}M$, 1^+-homogeneous, and the *fundamental tensor*

$$g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 =: \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E \in T_2^0(\Gamma(\overset{\circ}{\pi})) \tag{15.1}$$

is fibrewise non-degenerate. A *Finsler manifold* is a pair (M, F), where M is a manifold and F is a Finsler function for M. The function $E = \frac{1}{2}F^2$ is the *energy function* associated to F, or the *energy of* (M, F). Clearly, it is 2^+-homogeneous. An easy calculation shows that the energy function can be obtained from the fundamental tensor by

$$g(\widetilde{\delta}, \widetilde{\delta}) = 2\,E. \tag{15.2}$$

The *Hilbert 1-form* of (M, F) is

$$\text{\textit{in the pull-back formalism}} \quad \theta_g := \nabla^{\mathsf{v}}E = F\,\nabla^{\mathsf{v}}F, \tag{15.3}$$

$$\text{\textit{in the } } \tau_{TM} \text{ \textit{formalism}} \quad \theta_E := d_{\mathbf{J}}E. \tag{15.4}$$

The one-forms θ_g and θ_E are related by

$$\theta_E = \theta_g \circ \mathbf{j} \tag{15.5}$$

The two-form

$$\omega_E := d\theta_E = d\,d_{\mathbf{J}}E \in \mathcal{A}_2(\overset{\circ}{T}M) \tag{15.6}$$

is called the *fundamental 2-form* of (M, F). Its relation to the fundamental tensor is given by

$$\omega_E(\mathbf{J}\xi, \eta) = g(\mathbf{j}\xi, \mathbf{j}\eta);\ \ \xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M). \tag{15.7}$$

The non-degeneracy of g implies the non-degeneracy of ω_E, and vice versa.

Proposition 15.1.1. *Let (M, F) be a Finsler manifold and X a vector field on M. With the notation above,*

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_g) \circ \mathbf{j} = \mathcal{L}_{X^{\mathsf{c}}}\theta_E, \tag{15.8}$$

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{j}\xi, \mathbf{j}\eta) = (\mathcal{L}_{X^{\mathsf{c}}}\omega_E)(\mathbf{J}\xi, \eta). \tag{15.9}$$

Proof. For every vector field ξ on $\overset{\circ}{T}M$,

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_g)(\mathbf{j}\xi) \overset{(11.13)}{=} \mathcal{L}_{X^{\mathsf{c}}}(\theta_g(\mathbf{j}\xi)) - \theta_g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{j}\xi)) \overset{(15.5)}{=} \mathcal{L}_{X^{\mathsf{c}}}(\theta_E(\xi)) \\ - \theta_g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{j}(\xi)) \overset{(11.8),(15.5)}{=} \mathcal{L}_{X^{\mathsf{c}}}(\theta_E(\xi)) - \theta_E(\mathcal{L}_{X^{\mathsf{c}}}\xi) = (\mathcal{L}_{X^{\mathsf{c}}}\theta_E)(\xi),$$

whence (15.8). A little more calculation is necessary to prove (15.9). Starting with the definition of the classical Lie derivative, we find

$$(\mathcal{L}_{X^{\mathsf{c}}}\omega_E)(\mathbf{J}\xi, \eta) = X^{\mathsf{c}}\omega_E(\mathbf{J}\xi, \eta) - \omega_E(\mathcal{L}_{X^{\mathsf{c}}}\mathbf{J}\xi, \eta) - \omega_E(\mathbf{J}\xi, \mathcal{L}_{X^{\mathsf{c}}}\eta) \\ \overset{(11.8),(15.7)}{=} X^{\mathsf{c}}g(\mathbf{j}\xi, \mathbf{j}\eta) - \omega_E(\mathcal{L}_{X^{\mathsf{c}}}\mathbf{J}\xi, \eta) - g(\mathbf{j}\xi, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}\eta).$$

Observe now that the operators $\mathcal{L}_{X^{\mathsf{c}}}$ and $\mathbf{J}$ are interchangeable, i.e.,

$$\mathcal{L}_{X^{\mathsf{c}}} \circ \mathbf{J} = \mathbf{J} \circ \mathcal{L}_{X^{\mathsf{c}}} \text{ for all } X \in \mathfrak{X}(M). \tag{15.10}$$

Indeed, $\mathbf{i} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \mathcal{L}_{X^{\mathsf{c}}} \circ \mathbf{i}$ by (11.7). Composing both sides of this equality on the right with $\mathbf{j}$ and using (11.8), we obtain (15.10). Taking this into account,

$$\omega_E(\mathcal{L}_{X^{\mathsf{c}}}\mathbf{J}\xi, \eta) = \omega_E(\mathbf{J}\mathcal{L}_{X^{\mathsf{c}}}\xi, \eta) \overset{(15.7)}{=} g(\mathbf{j}\mathcal{L}_{X^{\mathsf{c}}}\xi, \mathbf{j}\eta) \overset{(11.8)}{=} g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{j}\xi), \mathbf{j}\eta).$$

Thus

$$(\mathcal{L}_{X^{\mathsf{c}}}\omega_E)(\mathbf{J}\xi, \eta) = X^{\mathsf{c}}g(\mathbf{j}\xi, \mathbf{j}\eta) - g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{j}\xi), \mathbf{j}\eta) - g(\mathbf{j}\xi, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{j}\eta)) \\ = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{j}\xi, \mathbf{j}\eta),$$

as was to be shown. □

15.1.2 Let $\omega_E^n = \omega_E \wedge \dots \wedge \omega_E$ (n factors). Then $w := \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\omega_E^n$ is a volume form on $\overset{\circ}{T}M$, called the *Dazord volume form* for (M, F). The *divergence* of a vector field ξ on $\overset{\circ}{T}M$ (with respect to w) is the unique smooth function $\operatorname{div}\xi \in C^\infty(\overset{\circ}{T}M)$ such that $\mathcal{L}_\xi\, w = (\operatorname{div}\xi)\, w$. It can easily be shown that

$$\operatorname{div} C = n; \tag{15.11}$$

see, e.g., [32], Corollary 1.

15.1.3 If (M,F) is a Finsler manifold, then there exists a unique spray S for M such that

$$i_S\,d\,d_{\mathbf{J}}E = -d\,E \quad \text{over } \overset{\circ}{T}M. \tag{15.12}$$

This spray is called the *canonical spray* of (M,F). Thus every Finsler manifold is a spray manifold at the same time. The Berwald connection of this spray manifold is called the *canonical connection* of (M,F). We denote it by $\mathcal{H}$; and $\mathbf{h}$, $\mathbf{v}, \mathcal{V}$ stand for the associated projection operators and the vertical mapping as in **13.1.2**. The canonical connection can be characterized as *the unique torsion-free Ehresmann connection for M which is compatible with the Finsler function* in the sense that $d\,F \circ H = 0$, or, equivalently,

$$\mathcal{H}(\widehat{X})F = X^{\mathsf{h}}F = 0 \qquad \text{for all } X \in \mathfrak{X}(M). \tag{15.13}$$

With the help of the canonical connection, we define the *Sasaki-Finsler metric* g^S by

$$g^S(\xi,\eta) := g(\mathbf{j}\xi,\mathbf{j}\eta) + g(\mathcal{V}\xi,\mathcal{V}\eta); \qquad \xi,\eta \in \mathfrak{X}(\overset{\circ}{T}M), \tag{15.14}$$

Then g^S is a Riemannian metric tensor on $\overset{\circ}{T}M$.

We shall need the following technical result.

Lemma 15.1.4. *If S is the canonical spray of the Finsler manifold (M,F), then*

$$\omega_E(C,S) = 2\,E, \quad \operatorname{div} S = 0. \tag{15.15 a-b}$$

Proof. Both equalities can be shown by a straightforward calculation:

$$\omega_E(C,S) \overset{(15.7)}{=} g(\mathbf{j}S,\mathbf{j}S) = g(\widetilde{\delta},\widetilde{\delta}) \overset{(15.2)}{=} 2E;$$
$$\mathcal{L}_S\omega_E = \mathcal{L}_S d\,d_{\mathbf{J}}E \overset{(5.15)}{=} i_S d\,d\,d_{\mathbf{J}}E + d\,i_S d\,d_{\mathbf{J}}E \overset{(5.18)}{=} d\,i_S d\,d_{\mathbf{J}}E$$
$$= -d\,dE = 0,$$

therefore $\mathcal{L}_S\omega_E = 0$, which implies (15.15 b). □

15.2 Covariant derivatives on a Finsler manifold

Since every Finsler manifold (M,F) is a spray manifold (M,S) with the canonical spray, we have the Berwald derivative ∇ of (M,S), induced by the canonical connection $\mathcal{H}$. In general, it is neither v-metric, nor h-metric, i.e., neither $\nabla^{\mathsf{v}} g$ nor $\nabla^{\mathsf{h}} g$ vanishes. We give names to these objects. The type $(0,3)$ Finsler tensor fields

$$\mathcal{C}_\flat := \nabla^{\mathsf{v}} g = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} E \quad \text{and} \quad \mathbf{L}_\flat := \nabla^{\mathsf{h}} g = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} E \tag{15.16 a-b}$$

are called the *Cartan-tensor* and *Landsberg-tensor* of (M,F), respectively. We give the same names to the metrically equivalent tensors $\mathcal{C}$ and $\mathbf{L}$, defined by

$$g(\mathcal{C}(\widetilde{X},\widetilde{Y}),\widetilde{Z}) := \mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z}), \text{ and } g(\mathbf{L}(\widetilde{X},\widetilde{Y}),\widetilde{Z}) = \mathbf{L}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z}). \tag{15.17 a-b}$$

We have three additional, important covariant derivatives on a Finsler manifold: the *Cartan derivative* D^C, the *Chern-Rund derivative* D^{Ch} and the *Hashiguchi derivative* D^{Hs}. They can be defined as follows:

$$D^C_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi,\widetilde{Y}) + \frac{1}{2}\mathbf{L}(\mathbf{j}\xi,\widetilde{Y}), \tag{15.18}$$

$$D^{Ch}_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathbf{L}(\mathbf{j}\xi,\widetilde{Y}), \tag{15.19}$$

$$D^{Hs}_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi,\widetilde{Y}). \tag{15.20}$$

Notice that the Cartan derivative is metric, the Chern-Rund derivative is h-metric, and the Hashiguchi-derivative is v-metric, i.e., we have

$$D^c g = 0; \ D^{Ch}_{\mathcal{H}\widetilde{X}} g = 0, \ D^{Hs}_{\mathbf{i}\widetilde{X}} g = 0 \quad (\widetilde{X} \in \Gamma(\overset{\circ}{\pi})). \tag{15.21 a-c}$$

Proposition 15.2.1. *With the notation above, for every vector field X on M we have*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat = \nabla^{\mathsf{v}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g). \tag{15.22}$$

Proof. For any vector fields $Y,\ Z,\ U$ on M

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathfrak{C}_\flat)(\widehat{Y},\widehat{Z},\widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}g))(\widehat{Y},\widehat{Z},\widehat{U}) \overset{(11.13)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla^{\mathsf{v}}g)(\widehat{Y},\widehat{Z},\widehat{U})) -(\nabla^{\mathsf{v}}g)(\widehat{[X,Y]},\widehat{Z},\widehat{U}) - (\nabla^{\mathsf{v}}g)(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) - (\nabla^{\mathsf{v}}g)(\widehat{Y},\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) \overset{(11.5\,b)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla_{Y^{\mathsf{v}}}g)(\widehat{Z},\widehat{U})) - (\nabla_{[X,Y]^{\mathsf{v}}}g)(\widehat{Z},\widehat{U}) - (\nabla_{Y^{\mathsf{v}}}g)(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) -(\nabla_{Y^{\mathsf{v}}}g)(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) \overset{(9.10)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla_{Y^{\mathsf{v}}}(g(\widehat{Z},\widehat{U}))) - \nabla_{[X,Y]^{\mathsf{v}}}(g(\widehat{Z},\widehat{U})) -\nabla_{Y^{\mathsf{v}}}(g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U})) - \nabla_{Y^{\mathsf{v}}}(g(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})) \overset{(11.9)}{=} \nabla_{Y^{\mathsf{v}}}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\widehat{Z},\widehat{U})) +\widetilde{\mathcal{L}}_{[X,V]^{\mathsf{v}}}(g(\widehat{Z},\widehat{U})) - \nabla_{[X,Y]^{\mathsf{v}}}(g(\widehat{Z},\widehat{U})) - \nabla_{Y^{\mathsf{v}}}(g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U})) -\nabla_{Y^{\mathsf{v}}}(g(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})) = \nabla_{Y^{\mathsf{v}}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\widehat{Z},\widehat{U})) - g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) -g(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})) \overset{(11.13)}{=} \nabla_{Y^{\mathsf{v}}}((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\widehat{Z},\widehat{U})) = (\nabla^{\mathsf{v}}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\widehat{Y},\widehat{Z},\widehat{U}).$$

This proves our assertion. □

Proposition 15.2.2. *If X, Y, Z, U are vector fields on M, then*

$$g((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathfrak{C})(\widehat{Y},\widehat{Z}),\widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathfrak{C}_\flat)(\widehat{Y},\widehat{Z},\widehat{U}) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widehat{U}), \tag{15.23}$$

$$g((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L})(\widehat{Y},\widehat{Z}),\widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat)(\widehat{Y},\widehat{Z},\widehat{U}) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{L}(\widehat{Y},\widehat{Z}),\widehat{U}). \tag{15.24}$$

Proof. From definition (15.17 a), $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathfrak{C}_\flat(\widehat{Y},\widehat{Z},\widehat{U})) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widehat{U}))$. By the product rule for derivations, we have

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathfrak{C}_\flat(\widehat{Y},\widehat{Z},\widehat{U})) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathfrak{C}_\flat)(\widehat{Y},\widehat{Z},\widehat{U}) + \mathfrak{C}_\flat(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z},\widehat{U})) + \mathfrak{C}_\flat(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) + \mathfrak{C}_\flat(\widehat{Y},\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U});$$

Similarly,

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widehat{U})) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widehat{U}) + g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathfrak{C}(\widehat{Y},\widehat{Z})),\widehat{U}) + g(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}).$$

So it follows that

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathfrak{C}_\flat)(\widehat{Y},\widehat{Z},\widehat{U}) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widehat{U}) = g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathfrak{C}(\widehat{Y},\widehat{Z})),\widehat{U}) + g(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) - \mathfrak{C}_\flat(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z},\widehat{U})) - \mathfrak{C}_\flat(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) - \mathfrak{C}_\flat(\widehat{Y},\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) \overset{(15.17\,a)}{=} g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathfrak{C}(\widehat{Y},\widehat{Z}))\,\widehat{U}) + g(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) - g(\mathfrak{C}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z})\,\widehat{U})) - g(\mathfrak{C}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z})\,\widehat{U}) - g(\mathfrak{C}(\widehat{Y},\widehat{Z}),\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) = g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathfrak{C}(\widehat{Y},\widehat{Z})) - \mathfrak{C}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) - \mathfrak{C}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z})\,\widehat{U}) = g((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathfrak{C})(\widehat{Y},\widehat{Z})\,\widehat{U}),$$

which finishes the proof of (15.23). Formula (15.24) can be shown in the same way, so we omit the essentially identical calculation. □

16 Killing vector fields on a Finsler manifold

16.1 To motivate our subsequent development, in this section we have a look at the semi-Riemannian metrics. We recall that if M is a manifold and $g \in \mathcal{T}_2^0(M)$ is a scalar product (resp. positive definite scalar product) on the tangent bundle of M (see **2.11**), then (M, g) is called a *semi-Riemannian* (resp. *Riemannian*) manifold. We also say in this case that g is a *metric tensor on* M. On a semi-Riemannian manifold (M, g) there exists a unique torsion-free metric derivative D, called the *Levi-Civita derivative on* M. It is characterized by the *Koszul formula*

$$\begin{aligned}2g(D_XY, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),\end{aligned} \tag{16.1}$$

where $X, Y, Z \in \mathfrak{X}(M)$.

16.2 On a semi-Riemannian manifold (M, g) one can conveniently define the well-known differential operators of classical vector analysis: *gradient, divergence* and *Laplacian.*
(i) The gradient of a function $f \in C^\infty(M)$ is the unique vector field $\mathrm{grad} f \in C^\infty(M)$ such that

$$g(\mathrm{grad} f, X) = df(X) = Xf \quad \text{for all} \quad X \in \mathfrak{X}(M). \tag{16.2}$$

(ii) The divergence of a vector field $X \in \mathfrak{X}(M)$ is the smooth function

$$\mathrm{div} X := \mathrm{tr} DX \overset{(2.5)}{=} \mathrm{tr}(Y \in \mathfrak{X}(M) \mapsto D_Y X \in \mathfrak{X}(M)), \tag{16.3}$$

where D is the Levi-Civita derivative on M.
(iii) The Laplacian of a function $f \in C^\infty(M)$ is

$$\Delta f := \mathrm{div}(\mathrm{grad} f). \tag{16.4}$$

Suppose for simplicity that g is a Riemannian metric, and let $(E_i)_{i=1}^n$ be a g-orthonormal frame field over an open subset $\mathcal{U}$ of M. ('g-orthonormal' means that $g(E_i, E_j) = \delta_{ij}; i, j \in J_n$.) Then we have

$$\mathrm{div} X \underset{(\mathcal{U})}{=} \sum g(D_{E_i} X, E_i). \tag{16.5}$$

A similar formula is valid also in the semi-Riemannian case.

16.3 Let (M, g) be a semi-Riemannian manifold. A diffeomorphism $\varphi\colon U \to V$ between two open subsets of M is called a *conformal transformation* if there exists a positive smooth function $f\colon U \to \mathbb{R}$ such that

$$g_{\varphi(p)}(\varphi_*(u), \varphi_*(v)) = f(p) g_p(u, v) \tag{16.6}$$

holds for all $p \in U$; $u, v \in T_pM$. Particular cases are *homotheties* (or *dilatations*) when f is a nonzero constant function, and *isometries* when $f(p) = 1$ for all point p in U. A vector field X on M is called *comformal*, *homothetic* and *Killing* if the stages of its local one-parameter group are conformal transformations, homotheties and isometries, respectively. A conformal vector field is *proper* if it is not homothetic. We use the following notation:

$\mathrm{Conf}_g(M)$ the set of conformal vector fields on M.

$\mathrm{Dil}_g(M)$ the set of homothetic vector fields on M.

$\mathrm{Kill}_g(M)$ the set of Killing vector fields on M.

The following results are well-known (see, e.g., [26]).

Proposition 16.3.1. *Let (M, g) be a semi-Riemannian manifold and X a vector field on M. Then*

$$X \in \mathrm{Conf}_g(M) \iff \mathcal{L}_X g = 2\sigma g \quad \textit{for some } \sigma \in C^\infty(M). \tag{16.7}$$

In particular,

$$X \in \mathrm{Dil}_g(M) \iff \mathcal{L}_X g = \alpha g \quad \textit{for some } \alpha \in \mathbb{R}^*; \tag{16.8}$$

$$X \in \mathrm{Kill}_g(M) \iff \mathcal{L}_X g = 0. \tag{16.9}$$

The function σ in equality (16.7) is called the *conformal function* of X.

Lemma 16.3.2. *Suppose (for simplicity) that (M, g) is a Riemannian manifold. If $X \in \mathrm{Conf}_g(M)$, then the conformal function of X is $\frac{1}{n}\mathrm{div}X$.*

Proof. Let $(E_i)_{i=1}^n$ be an orthonormal frame field over an open subset U of M. Then

$$\begin{aligned}
2\,\mathrm{div}X &\overset{(16.5)}{\underset{(\mathcal{U})}{=}} 2\sum g(D_{E_i}X, E_i) \\
&\overset{(16.1)}{=} \sum(E_i g(X, E_i) + X g(E_i, E_i) - E g(E_i, X)) \\
&+ \sum(-g(E_i, [X, E_i]) + g(X, [E_i, E_i]) + g(E_i, [E_i, X])) \\
&= 2\sum g(E_i, [E_i, X]).
\end{aligned}$$

On the other hand, we have

$$(\mathcal{L}_X g)(E_i, E_i) \overset{(10.1)}{=} Xg(E_i, E_i) - 2g([X, E_i], E_i) = 2g(E_i, [E_i, X]).$$

Thus

$$2\mathrm{div}X = \sum(\mathcal{L}_X g)(E_i, E_i) \overset{(16.7)}{=} 2\sum \sigma g(E_i, E_i) = 2\sigma n,$$

whence our claim. □

16.4 Let (M, F) be a Finsler manifold. A diffeomorphism $\varphi\colon \mathcal{U} \to \mathcal{V}$ between two open subsets of M is called a *(local) isometry* of (M, F) if its derivative preserves the Finslerian norms of the tangent vectors, i.e.,

$$F((\varphi_*)_p(v)) = F(v) \text{ for all } p \in \mathcal{U},\ v \in T_pM.$$

As in similar situations above, we say that a vector field on M is a *Killing vector field of* (M, F) if the stages of its local one-parameter group are ismotries. We denote by $\mathrm{Kill}_F(M)$ the set of all Killing vector field of (M, F).

The following result is partly known (the equivalence of (i) and (ii) is clearly folklore), and it will be generalized in the next section. However, because of its particular importance, we present it here together with a complete proof.

Lemma 16.4.1. *Let (M, F) be a Finsler manifold and X is a vector field on M. The following assertions are equivalent:*

(i) $X \in \mathrm{Kill}_F(M)$;	(iv) $\mathcal{L}_{X^c}\theta_E = 0$;
(ii) $\widetilde{\mathcal{L}}_{X^c} g = 0$;	(v) $\widetilde{\mathcal{L}}_{X^c}\theta_g = 0$;
(iii) $X^c F = 0$;	(vi) $\mathcal{L}_{X^c}\omega_E = 0$.

Proof. We organize our reasoning according to the following scheme:

$$\begin{array}{ccccc} \text{(ii)} & \Longrightarrow & \text{(iii)} & \Longleftrightarrow & \text{(i)} \\ \Downarrow & & \Downarrow & & \\ \text{(vi)} & \Longleftarrow & \text{(iv)} & \Longleftrightarrow & \text{(v)}. \end{array}$$

(ii)$\Longrightarrow$ (iii) $0 \overset{\text{(ii)}}{=} (\widetilde{\mathcal{L}}_{X^c} g)(\widetilde{\delta}, \widetilde{\delta}) \overset{(15.2)}{=} 2X^c E - 2g(\widetilde{\mathcal{L}}_{X^c}\widetilde{\delta}, \widetilde{\delta}) \overset{(11.5\,a)}{=} 2X^c E = F(X^c F)$. Since F is positive, this implies that $X^c F = 0$.

(iii)$\Longrightarrow$ (iv) For every vector field Y on M,

$$\begin{aligned}
&(\mathcal{L}_{X^{\mathsf{c}}}\theta_E)(Y^{\mathsf{c}}) = X^{\mathsf{c}}(\theta_E Y^{\mathsf{c}}) - \theta_E([X^{\mathsf{c}}, Y^{\mathsf{c}}]) \\
&= X^{\mathsf{c}}(d_{\mathbf{J}}E(Y^{\mathsf{c}})) - d_{\mathbf{J}}E([X,Y]^{\mathsf{c}}) = X^{\mathsf{c}}(Y^{\mathsf{v}}E) - [X,Y]^{\mathsf{v}}E \\
&= [X^{\mathsf{c}}, Y^{\mathsf{v}}]E + Y^{\mathsf{v}}(X^{\mathsf{c}}E) - [X,Y]^{\mathsf{v}}E \overset{(7.22\,b)}{=} Y^{\mathsf{v}}(X^{\mathsf{c}}E) = 0.
\end{aligned}$$

Since, as can easily be seen, $\mathcal{L}_{Z^{\mathsf{c}}}\theta_E \restriction \mathfrak{X}^{\mathsf{v}}(\overset{\circ}{T}M) = 0$ for every $Z \in \mathfrak{X}(M)$, our implication follows.

(iv)$\Longrightarrow$(vi) Indeed, $\mathcal{L}_{X^{\mathsf{c}}}\omega_E \overset{(15.6)}{=} \mathcal{L}_{X^{\mathsf{c}}}d\theta_E \overset{(5.17)}{=} d\mathcal{L}_{X^{\mathsf{c}}}\theta_E \overset{(iv)}{=} 0.$

(vi)$\Longrightarrow$ (ii) For any vector fields ξ, η on $\overset{\circ}{T}M$, $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{j}\xi, \mathbf{j}\eta) \overset{(15.9)}{=} (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\omega_E)(\mathbf{J}\xi, \eta) \overset{(vi)}{=} 0$

(iv) $\Longleftrightarrow$ (v) This is clear from (15.5).

(i)$\Longrightarrow$ (iii) Let (φ_t) be the local one-parameter group of X. Then, by Lemma 7.5.1, the local one-parameter group of X^{c} is $((\varphi_t)_*)$. Thus

$$X^{\mathsf{c}}F \overset{(4.6)}{=} \lim_{t\to 0}\frac{1}{t}(F\circ(\varphi_t)_* - F) \overset{(i)}{=} 0.$$

(iii)$\Longrightarrow$ (i) Let, as above, (φ_t) be the local one-parameter group of X. If $X^{\mathsf{c}}F = 0$, then

$$\lim_{t\to 0}\frac{F\circ(\varphi_t)_* - F}{t} = 0. \qquad (*)$$

Given a tangent vector $v \in \overset{\circ}{T}_pM$, define the function

$$f\colon I_p \to \mathbb{R},\ t \mapsto f(t) := F\circ(\varphi_t)_*(v),$$

where I_p is the domain of the maximal integral curve of X. At every $t_o \in I_p$,

$$f'(t_o) = \lim_{t\to t_o}\frac{f(t)-f(t_o)}{t-t_0} = \lim_{t\to t_0}\frac{F\circ(\varphi_t)_*(v) - F\circ(\varphi_{t_o})_*(v)}{t-t_0}.$$

Let, for a moment, $u := (\varphi_{t_o})_*(v)$. Then

$$v = (\varphi_{-t_o})_*(u) \quad (\varphi_t)_*(v) = (\varphi_{t-t_0})_*(u).$$

so we obtain that

$$f'(t_o) = \lim_{t-t_o\to 0}\frac{F((\varphi_{t-t_o})_*(u)) - F(u)}{t-t_o} \overset{(*)}{=} 0. \qquad (16.10)$$

Thus f is a constant function. Since

$$f(0) = F((\varphi_0)_*(v)) = F(v),$$

it follows that $F \circ (\varphi_t)_*(v) = F(v)$ for all $t \in I_p$. This finishes the proof. $\square$

Remark 16.4.2. Let (M, F) be a Finsler manifold, S its canonical spray, and let $\varphi \in \mathrm{Diff}(M)$. It was proved in [4] that if φ is an isometry of (M, F), then it is an automorphism of S, i.e., $\varphi_{**} \circ S = S \circ \varphi_*$. From this it follows immediately that *every Killing vector field of* (M, F) *is a Lie symmetry of the canonical spray of* (M, F), i.e.,

$$\mathrm{Kill}_F(M) \subset \mathrm{Lie}_S(M). \tag{16.11}$$

Proposition 16.4.3. *Let* (M, F) *be a Finsler manifold. If* X *is a Lie symmetry of the canonical spray of* (M, F)*, then*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \mathbf{L}_\flat = \nabla^{\mathsf{h}} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g. \tag{16.12}$$

Proof. We recall that by Theorem 13.4.5 ,

$$X \in \mathrm{Lie}_S(M) \iff [X^{\mathsf{c}}, Y^{\mathsf{h}}] = [X, Y]^{\mathsf{h}} \text{ for all } Y \in \mathfrak{X}(M).$$

We shall use this at step $(*)$ in our calculation below. To begin with, let Y, Z, U be vector fields on M. Next we calculate:

$$\begin{aligned}
&(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \mathbf{L}_\flat)(\widehat{Y}, \widehat{Z}, \widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{h}} g))(\widehat{Y}, \widehat{Z}, \widehat{U}) \overset{(11.13)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla^{\mathsf{h}} g)(\widehat{Y}, \widehat{Z}, \widehat{U}))\\
&- (\nabla^{\mathsf{h}} g)(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Y}, \widehat{Z}, \widehat{U}) - (\nabla^{\mathsf{h}} g)(\widehat{Y}, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Z}, \widehat{U}) - (\nabla^{\mathsf{h}} g)(\widehat{Y}, \widehat{Z}, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{U})\\
&\overset{(11.5\,b)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla_{Y^{\mathsf{h}}} g)(\widehat{Z}, \widehat{U})) - (\nabla_{[X,Y]^{\mathsf{h}}} g)(\widehat{Z}, \widehat{U}) - (\nabla_{Y^{\mathsf{h}}} g)(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Z}, \widehat{U})\\
&- (\nabla_{Y^{\mathsf{h}}} g)(\widehat{Z}, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{U}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla_{Y^{\mathsf{h}}}(g(\widehat{Z}, \widehat{U})) - \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\nabla_{Y^{\mathsf{h}}} \widehat{Z}, \widehat{U}))\\
&- \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\widehat{Z}, \nabla_{Y^{\mathsf{h}}} \widehat{U})) - \nabla_{[X,Y]^{\mathsf{h}}}(g(\widehat{Z}, \widehat{U})) + g(\nabla_{[X,Y]^{\mathsf{h}}} \widehat{Z}, \widehat{U})\\
&+ g(\widehat{Z}, \nabla_{[X,Y]^{\mathsf{h}}} \widehat{U}) - \nabla_{Y^{\mathsf{h}}}(g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Z}, \widehat{U})) + g(\nabla_{Y^{\mathsf{h}}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Z})), \widehat{U})\\
&+ g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Z}, \nabla_{Y^{\mathsf{h}}} \widehat{U}) - \nabla_{Y^{\mathsf{h}}}(g(\widehat{Z}, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{U})) + g(\nabla_{Y^{\mathsf{h}}} \widehat{Z}, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{U})\\
&+ g(\widehat{Z}, \nabla_{Y^{\mathsf{h}}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{U})) \overset{(13.30),(13.29)}{=} \nabla_{Y^{\mathsf{h}}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\widehat{Z}, \widehat{U}))) + \widetilde{\mathcal{L}}_{[X^{\mathsf{c}}, Y^{\mathsf{h}}]}(g(\widehat{Z}, \widehat{U}))\\
&- \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\nabla_{Y^{\mathsf{h}}} \widehat{Z}, \widehat{U})) - \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\widehat{Z}, \nabla_{Y^{\mathsf{h}}} \widehat{U})) - \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{h}}}(g(\widehat{Z}, \widehat{U}))
\end{aligned}$$

$$+ g(\nabla_{[X,Y]^{\mathsf{h}}}\widehat{Z},\widehat{U}) + g(\widehat{Z},\nabla_{[X,Y]^{\mathsf{h}}}\widehat{U}) - \nabla_{Y^{\mathsf{h}}}(g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}))$$
$$+ g(\nabla_{Y^{\mathsf{h}}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z})),\widehat{U}) + g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{U}) - \nabla_{Y^{\mathsf{h}}}(g(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}))$$
$$+ g(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) + g(\widehat{Z},\nabla_{Y^{\mathsf{h}}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})) \overset{(13.30);(*)}{=} \nabla_{Y^{\mathsf{h}}}((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\widehat{Z},\widehat{U}))$$
$$- \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widehat{U})) - \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(g(\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{U})) + g(\widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]}\widehat{Z},\widehat{U})$$
$$+ g(\widehat{Z},\widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]}\widehat{U}) + g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla_{Y^{\mathsf{h}}}\widehat{Z}),\widehat{U}) - g(\widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]}\widehat{Z},\widehat{U})$$
$$+ g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{U}) + g(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) + g(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla_{Y^{\mathsf{h}}}\widehat{U}))$$
$$- g(\widehat{Z},\widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]}\widehat{U})) = \nabla_{Y^{\mathsf{h}}}((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\widehat{Z},\widehat{U})) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widehat{U})$$
$$- (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{U}) =: (\nabla^{\mathsf{h}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g))(\widehat{Y},\widehat{Z},\widehat{U}).$$

This proves (16.12). □

Proposition 16.4.4. *If X is a Killing vector field of (M,F), then*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat = 0, \quad \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C} = 0; \tag{16.13 a-b}$$
$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat = 0, \quad \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L} = 0. \tag{16.14 a-b}$$

Proof. First we recall that $X \in \mathrm{Kill}_F(M) \iff \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g = 0$ by Lemma 16.4.1, and $X \in \mathrm{Kill}_F(M) \implies X \in \mathrm{Lie}_S M$ by Remark 16.4.2. Thus (16.13 a) follows from (15.22), (16.13 b) is a consequence of (16.13 a) and (15.23). Equality (16.14 a) can be seen from (16.12), and, finally (16.14 b) is an immediate consequence of (16.14 a) and (15.24). □

Theorem 16.4.5. *If X is a Killing vector field of a Finsler manifold, then X is also a D-Killing field, where $D \in \{\nabla, D^C, D^{Ch}, D^{Hs}\}$.*

Proof. Let $X \in \mathrm{Kill}_F(M)$. Then we also have $X \in \mathrm{Lie}_S(M)$ (Remark 16.4.2), so by Theorem 13.4.5, $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla = 0$. We show that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}D^C = 0$. For every $\xi \in \mathfrak{X}(\overset{\circ}{T}M)$ and $Y \in \mathfrak{X}(M)$,

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}D^C)(\xi,\widehat{Y}) \overset{(11.13)}{:=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(D^C_\xi\widehat{Y}) - D^C_{\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\xi}\widehat{Y} - D^C_\xi(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})$$
$$\overset{(15.18)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla_\xi\widehat{Y}) + \frac{1}{2}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathcal{C}(\mathcal{V}\xi,\widehat{Y})) + \frac{1}{2}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{L}(\mathbf{j}\xi,\widehat{Y})) - \nabla_{\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\xi}\widehat{Y}$$
$$- \frac{1}{2}\mathcal{C}(\mathcal{V}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\xi),\widehat{Y}) - \frac{1}{2}\mathbf{L}(\mathbf{j}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\xi),\widehat{Y}) - \nabla_\xi\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y} - \frac{1}{2}\mathcal{C}(\mathcal{V}\xi,\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})$$
$$- \frac{1}{2}\mathbf{L}(\mathbf{j}\xi,\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla)(\xi,\widehat{Y}) + \frac{1}{2}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C})(\mathcal{V}\xi,\widehat{Y}) + \frac{1}{2}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L})(\mathbf{j}\xi,\widehat{Y})$$
$$\overset{\text{Theo.13.4.5, Prop.16.4.4}}{=} 0.$$

We obtain in the same way that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}D^{Ch} = 0$ and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}D^{Hs} = 0$. □

17 Conformal and projective vector fields on a Finsler manifold

17.1 Motivated by (16.7), we say that $X \in \mathfrak{X}(M)$ is a *conformal vector field* of (M, F) if there exists a function $\sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M)$ such that

$$\widetilde{\mathcal{L}}_{X^c} g = \sigma g. \tag{17.1}$$

Here g is the fundamental tensor of (M, F); the function σ is called the *conformal function* of X (cf. **6.3**). We have chosen this definition for the sake of simplicity. Of course, the conformal property of X can also be expressed in terms of the local flow of X. If, in particular,

$$\widetilde{\mathcal{L}}_{X^c} g = \alpha g, \ \alpha \in \mathbb{R} \tag{17.2}$$

then X is called a *homothetic vector field of* (M, F). When $\alpha = 0$, X is a Killing vector field by Lemma 16.4.1. We denote by $\mathrm{Conf}_F(M)$ and $\mathrm{Dil}_F(M)$ the sets of conformal and homothetic vector fields of (M, F), respectively. Obviously,

$$\mathrm{Kill}_F(M) \subset \mathrm{Dil}_F M \subset \mathrm{Conf}_F(M).$$

A vector field X on M is called a *projective vector field* of (M, F) if

$$[X^c, S] = \varphi C \ \text{ for some } \varphi \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M), \tag{17.3}$$

where S is the canonical spray of (M, F). For the geometric meaning of this condition and some equivalent conditions we refer to section 7 of Lovas's paper [17]. The set of projective vector fields of (M, F) will be denoted by $\mathrm{Proj}_F(M)$. By Theorem 13.4.5, $\mathrm{Lie}_S(M) \subset \mathrm{Proj}_F(M)$.

Lemma 17.1.1. *Let X be a conformal vector field of a Finsler manifold with conformal function σ. Then*

$$X^c E = \sigma E, \tag{17.4}$$

and the conformal function is the vertical lift of a smooth function on M.

Proof. $2X^c E \overset{(15.2)}{=} X^c(g(\widetilde{\delta}, \widetilde{\delta})) = (\widetilde{\mathcal{L}}_{X^c} g)(\widetilde{\delta}, \widetilde{\delta}) + 2g(\widetilde{\mathcal{L}}_{X^c}\widetilde{\delta}, \widetilde{\delta}) \overset{(17.1),(11.5\,a)}{=} \sigma g(\widetilde{\delta}, \widetilde{\delta}) = 2\sigma E$, whence (17.4). Applying this observation, we find that $C(X^c E) = C(\sigma E) = (C\sigma)E + 2\sigma E$. On the other hand,

$$C(X^c E) = [C, X^c]E + X^c(CE) \overset{(7.23\,b)}{=} 2X^c E \overset{(17.4)}{=} 2\sigma E.$$

From these we conclude that $C\sigma = 0$, and hence σ is 0^+-homogeneous. This implies by **7.7.2** (ii) that there exists a smooth function f on M such that $\sigma = f \circ \tau$. □

The following result is the promised generalization of Lemma 16.4.1. Its proof is similar to the proof of the lemma, but the technical details are a little more complicated.

Theorem 17.1.2. *Let (M, F) be a Finsler manifold. For a vector field X on M, the following conditions are equivalent:*

(i) *X is a conformal vector field with conformal function σ;*

(ii) $X^{\mathsf{c}}E = \sigma\, E, \quad \sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M)$;

(iii) $\mathcal{L}_{X^{\mathsf{c}}}\theta_E = \sigma\, \theta_E, \quad \sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M)$;

(iv) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_g = \sigma\, \theta_g, \quad \sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M)$;

(v) $\mathcal{L}_{X^{\mathsf{c}}}\omega_E = f^{\mathsf{v}}\, \omega_E + df^{\mathsf{v}} \wedge d_{\mathbf{J}}E, \quad f \in C^\infty(M)$.

Proof. The arrangement of our argument is displayed by the following diagram:

$$\begin{array}{ccccc} \text{(i)} & \Longrightarrow & \text{(ii)} & & \\ \Uparrow & & \Downarrow & & \\ \text{(v)} & \Longleftarrow & \text{(iii)} & \Longleftrightarrow & \text{(iv)}. \end{array}$$

(i)$\Longrightarrow$ (ii) This has already been proved above.
(ii)$\Longrightarrow$ (iii) It can immediately be seen that

$$(\mathcal{L}_{X^{\mathsf{c}}}\theta_E - \sigma\theta_E) \restriction \mathfrak{X}^{\mathsf{v}}(\overset{\circ}{T}M) = 0. \tag{17.5}$$

On the other hand, for every vector field Y on M, $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_E)(Y^{\mathsf{c}}) = Y^{\mathsf{v}}(X^{\mathsf{c}}E)$ (see the proof of (iii)$\Longrightarrow$ (iv) in Lemma 16.4.1). In our case,

$$Y^{\mathsf{v}}(X^{\mathsf{c}}E) \overset{\text{(ii)}}{=} Y^{\mathsf{v}}(\sigma E) = (Y^{\mathsf{v}}\sigma)E + \sigma Y^{\mathsf{v}}E.$$

We saw in the proof of Lemma 17.1.1 that $X^{\mathsf{c}}E = \sigma E$ implies that σ is a vertical lift. So we have $Y^{\mathsf{v}}\sigma = 0$, therefore

$$(\mathcal{L}_{X^{\mathsf{c}}}\theta_E)(Y^{\mathsf{c}}) = \sigma(Y^{\mathsf{v}}E) = \sigma d_{\mathbf{J}}E(Y^{\mathsf{c}}).$$

This concludes the proof of the implication.

(iii)$\Longrightarrow$ (v) $\mathcal{L}_{X^c}\omega_E = \mathcal{L}_{X^c} d\,\theta_E \overset{(5.17)}{=} d\mathcal{L}_{X^c}\theta_E \overset{\text{(iii)}}{=} d(\sigma\,\theta_E) = d\,\sigma \wedge \theta_E + \sigma\, d\,\theta_E = \sigma\,\omega_E + d\sigma \wedge d_{\mathbf{J}} E$.

It remains to show that the function σ is a vertical lift. To do this, we evaluate both sides of (iii) at an arbitrary spray S. Then, one hand,

$$(\mathcal{L}_{X^c}\theta_E)(S) = X^c(\theta_E(S)) - \theta_E([X^c, S]) = X^c(dE(\mathbf{J}S)) - dE(\mathbf{J}[X^c, S]) \overset{\text{Lemma 12.1.1 (ii)}}{=} 2X^c E.$$

On the other hand $(\sigma\theta_E)(S) = \sigma d_J E(S) = 2\sigma E$, so it follows that $X^c E = \sigma E$. This implies (see above) that $\sigma = f^v$, $f \in C^\infty(M)$.

(v)$\Longrightarrow$ (i) For any vector fields ξ, η on $\overset{\circ}{T}M$,

$$\begin{aligned}(\widetilde{\mathcal{L}}_{X^c} g)(\mathbf{j}\xi, \mathbf{j}\eta) &\overset{(15.9)}{=} (\mathcal{L}_{X^c}\omega_E)(\mathbf{J}\xi, \eta) \overset{\text{(v)}}{=} (f^v\,\omega_E + (d\,f^v) \wedge d_{\mathbf{J}}E)(\mathbf{J}\xi, \eta)\\ &= f^v\omega_E(\mathbf{J}\xi, \eta) + df^v(\mathbf{J}\xi) d_{\mathbf{J}}E(\eta) - df^v(\eta) d_{\mathbf{J}}E(\mathbf{J}\xi) = f^v\omega_E(\mathbf{J}\xi, \eta)\\ &+ (\mathbf{J}\xi)(f^v) d_{\mathbf{J}}E(\eta) - df^v(\eta)(\mathbf{J}^2\xi)E = f^v g(\mathbf{j}\xi, \mathbf{j}\eta),\end{aligned}$$

because the vertical vector fields kill the vertical lifts of smooth functions on M and $\mathbf{J}^2 \overset{(8.12\,b)}{=} 0$. This proves what we wanted.

(iii) $\Longleftrightarrow$ (iv) If $\mathcal{L}_{X^c}\theta_E = \sigma\,\theta_E$, then for any vector field ξ on $\overset{\circ}{T}M$,

$$(\widetilde{\mathcal{L}}_{X^c}\theta_g)(\mathbf{j}\xi) \overset{(15.5)}{=} (\mathcal{L}_{X^c}\theta_E)(\xi) \overset{\text{(iii)}}{=} (\sigma\,\theta_E)(\xi) \overset{(15.5)}{=} \sigma\,\theta_g(\mathbf{j}\xi),$$

so we have $\widetilde{\mathcal{L}}_{X^c}\theta_g = \sigma\,\theta_g$. The reverse of the implication can be proved in the same way.

This concludes the proof of the theorem. □

We note that relation (v), as a characterization of conformal vector fields on a Finsler manifold, was first announced by J. Grifone [13]. In terms of the local flow (φ_t) of X, condition (ii) can be expressed as follows:

$$E \circ (\varphi_t)_* = (\exp \circ\, t f^v)\, E, \quad f \in C^\infty(M),$$

for every possible $t \in \mathbb{R}$; cf. the equivalence (i) $\Longleftrightarrow$ (iii) and its proof in Lemma 16.4.1.

The above theorem was obtained in 2011. Two years later, our condition (ii) was also be found by Libing Huang and Xiaohuan Mo [15]. From Theorem 17.1.2 we obtain immediately the next

Corollary 17.1.3. If (M, F) is a Finsler manifold and $X \in \mathfrak{X}(M)$, then the following conditions are equivalent:

(i) $X \in \mathrm{Dil}_F(M)$, i.e., $\widetilde{\mathcal{L}}_{X^c} g = \alpha\, g$, for some $\alpha \in \mathbb{R}$;

(ii) the energy function associated to F is an eigenfunction of X^c with eigenvalue α, i.e., $X^c E = \alpha\, E$;

(iii) $\mathcal{L}_{X^c}\theta_E = \alpha\, \theta_E$;

(iv) $\widetilde{\mathcal{L}}_{X^c}\theta_g = \alpha\, \theta_g$;

(v) $\mathcal{L}_{X^c}\omega_E = \alpha\, \omega_E$.

In conditions (iii)-(v), α is a real number. With the choice $\alpha := 0$ we re-obtain a part of Lemma 16.4.1.

17.2 In this concluding subsection we mainly deal with vector fields on M which have at least two of the properties 'Lie symmetry', 'conformal', 'projective', or one of them together with some additional property.

Theorem 17.2.1. *Let (M, F) be a Finsler manifold. If a vector field X on M is a conformal vector field of (M, F) and, at the same time, X is a Lie symmetry of the canonical spray of (M, F), then X^c is a conformal vector field on the Riemannian manifold $(\overset{\circ}{T}M, g^S)$, where g^S is the Sasaki-Finsler metric defined by* (15.14)*. Briefly,*

$$X \in \mathrm{Conf}_F(M) \cap \mathrm{Lie}_S(M) \Longrightarrow X^c \in \mathrm{Conf}_{g^S}(\overset{\circ}{T}M). \tag{17.6}$$

Conversely, if X^c is a conformal vector field of the Riemannian manifold $(\overset{\circ}{T}M, g^S)$, then X is a conformal vector field of (M, F):

$$X^c \in \mathrm{Conf}_{g^S}(\overset{\circ}{T}M) \Longrightarrow X \in \mathrm{Conf}_F(M). \tag{17.7}$$

Proof. Suppose first that $X \in \mathrm{Conf}_F(M) \cap \mathrm{Lie}_S(M)$. We calculate the Lie derivative $\mathcal{L}_{X^c} g^s$. For any vector fields ξ, η on $\overset{\circ}{T}M$,

$$\begin{aligned}
&(\mathcal{L}_{X^c} g^S)(\xi,\eta) = \mathcal{L}_{X^c}(g^S(\xi,\eta)) - g^S(\mathcal{L}_{X^c}\xi,\eta) - g^S(\xi,\mathcal{L}_{X^c}\eta)\\
&\overset{(15.14)}{=} \mathcal{L}_{X^c}(g(\mathbf{j}\xi,\mathbf{j}\eta)) + \mathcal{L}_{X^c}(g(\mathcal{V}\xi,\mathcal{V}\eta)) - g(\mathbf{j}\mathcal{L}_{X^c}\xi,\mathbf{j}\eta) - g(\mathcal{V}\mathcal{L}_{X^c}\xi,\mathcal{V}\eta)\\
&- g(\mathbf{j}\xi,\mathbf{j}\mathcal{L}_{X^c}\eta) - g(\mathcal{V}\xi,\mathcal{V}\mathcal{L}_{X^c}\eta) \overset{(11.8),(13.43)}{=} \widetilde{\mathcal{L}}_{X^c}(g(\mathbf{j}\xi,\mathbf{j}\eta))\\
&+ \widetilde{\mathcal{L}}_{X^c}(g(\mathcal{V}\xi,\mathcal{V}\eta)) - g(\widetilde{\mathcal{L}}_{X^c}(\mathbf{j}\xi),\mathbf{j}\eta) - g(\widetilde{\mathcal{L}}_{X^c}(\mathcal{V}\xi),\mathcal{V}\eta)\\
&- g(\mathbf{j}\xi,\widetilde{\mathcal{L}}_{X^c}(\mathbf{j}\eta)) - g(\mathcal{V}\xi,\widetilde{\mathcal{L}}_{X^c}(\mathcal{V}\eta)) = (\widetilde{\mathcal{L}}_{X^c} g)(\mathbf{j}\xi,\mathbf{j}\eta)\\
&+ (\widetilde{\mathcal{L}}_{X^c} g)(\mathcal{V}\xi,\mathcal{V}\eta) \overset{(17.1)}{=} \sigma g(\mathbf{j}\xi,\mathbf{j}\eta) + \sigma g(\mathcal{V}\xi,\mathcal{V}\eta) = \sigma g^S(\xi,\eta).
\end{aligned}$$

which proves that X^{c} is a conformal vector field of $(\overset{\circ}{T}M, g^S)$.

Conversely, suppose that $X^{\mathsf{c}} \in \mathrm{Conf}_{g^S}(\overset{\circ}{T}M)$. Then

$$\begin{aligned}&2\sigma\, E = \sigma\, g(\widetilde{\delta}, \widetilde{\delta}) = \sigma\, g(\mathcal{V}C, \mathcal{V}C) = \sigma\, g^S(C, C) \overset{\text{condition}}{=} (\mathcal{L}_{X^{\mathsf{c}}} g^S)(C, C)\\ &= X^{\mathsf{c}}(g^S(C, C)) - 2g^S([X^{\mathsf{c}}, C], C) = X^{\mathsf{c}}(g^S(C, C)) = X^{\mathsf{c}} g(\widetilde{\delta}, \widetilde{\delta})\\ &= 2X^{\mathsf{c}}E,\end{aligned}$$

so we have $X^{\mathsf{c}}E = \sigma E$. Thus, by Theorem (17.1.2), X is a conformal vector field of (M, F). □

Theorem 17.2.2. *Any homothetic vector field of a Finsler manifold is a Lie symmetry of the canonical spray of the Finsler manifold. Briefly,*

$$X \in \mathrm{Dil}_F(M) \Longrightarrow X \in \mathrm{Lie}_S(M.) \tag{17.8}$$

Proof. If $X \in \mathrm{Dil}_F(M)$, then by Corollary 17.1.3, $X^{\mathsf{c}}E = \alpha\, E$, or, equivalently, $\mathcal{L}_{X^{\mathsf{c}}}\, \omega_E = \alpha\, \omega_E$ for some real number α. Thus

$$\begin{aligned}&\mathcal{L}_{X^{\mathsf{c}}} dE = d(X^{\mathsf{c}}E) = \alpha\, dE \overset{(15.12)}{=} -\alpha\, i_S\, \omega_E = -i_S(\alpha\, \omega_E) = -i_S\, (\mathcal{L}_{X^{\mathsf{c}}}\omega_E)\\ &\overset{(5.14)}{=} -\mathcal{L}_{X^{\mathsf{c}}} i_S\, \omega_E + i_{[X^{\mathsf{c}}, S]}\omega_E = \mathcal{L}_{X^{\mathsf{c}}} dE + i_{[X^{\mathsf{c}}, S]}\omega_E,\end{aligned}$$

therefore $i_{[X^{\mathsf{c}}, S]}\omega_E = 0$. Since ω_E is non-degenerate, this implies that $[X^{\mathsf{c}}, S] = 0$, and hence $X \in \mathrm{Lie}_S(M)$. □

This result, published in 2011 in our paper [30], was rediscovered by Tian Huang-jia a few years later, see [33], Corollary 1.1.

Lemma 17.2.3. *If X is a conformal vector field of the Finsler manifold (M, F) with conformal function σ, then the divergence of X^{c} with respect to the Dazord volume form w **(15.1.2)** is*

$$\mathrm{div} X^{\mathsf{c}} = n\, \sigma. \tag{17.9}$$

Proof. Choose a frame $(X_i)_{i=1}^n$ on an open subset U of M. Then the family $(X_i^{\mathsf{v}}, X_i^{\mathsf{c}})_{i=1}^n$ is a frame on $\tau^{-1}(U) \subset TM$, and it can be shown by an inductive argument that

$$(\mathcal{L}_{X^{\mathsf{c}}}\omega_E)(X_1^{\mathsf{v}}, X_1^{\mathsf{c}}, ..., X_n^{\mathsf{v}}, X_n^{\mathsf{c}}) = n\, \sigma\, \omega_E(X_1^{\mathsf{v}}, X_1^{\mathsf{c}}, ..., X_n^{\mathsf{v}}, X_n^{\mathsf{c}}).$$

This implies our claim. □

Theorem 17.2.4. *Let (M, F) be a connected Finsler manifold. If a vector field X on M is both a projective and a conformal vector field of (M, F), then it is a homothetic vector field, i.e.,*

$$X \in \mathrm{Proj}_F(M) \cap \mathrm{Conf}_F(M) \Longrightarrow X \in \mathrm{Dil}_F(M). \tag{17.10}$$

Proof. Since $X \in \mathrm{Proj}_F(M)$,

$$[X^{\mathsf{c}}, S] = \psi\, C, \quad \text{for some } \ \psi \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M), \tag{$*$}$$

where S is the canonical spray of (M, F). On the other hand, by our condition $X \in \mathrm{Conf}_F(M)$, Theorem 17.1.2 and Lemma 17.1.1

$$X^{\mathsf{c}}E = f^{\mathsf{v}}E, \qquad f \in C^\infty(M). \tag{$**$}$$

Thus we find

$$\begin{aligned} &2\psi E = \psi(CE) \overset{(*)}{=} [X^{\mathsf{c}}, S]E = X^{\mathsf{c}}(SE) - S(X^{\mathsf{c}}E) \overset{(**)}{=} X^{\mathsf{c}}(SE) \\ &- (Sf^{\mathsf{v}})E - f^{\mathsf{v}}(SE) \overset{\text{Lemma 12.1.1}(i)}{=} -f^{\mathsf{c}}E + X^{\mathsf{c}}(SE) - f^{\mathsf{v}}(SE) \\ &= -f^{\mathsf{c}}E. \end{aligned}$$

In the last step we used the fact that S is horizontal with respect to the canonical connection of (M, F) (see, e.g., [29] Corollary 7.3.6), so we have $SE \overset{\mathbf{15.1.3}}{=} S(FS) = 0$. Our result $2\psi E = -f^{\mathsf{c}}E$ implies that $\psi = -\frac{1}{2}f^{\mathsf{c}}$. Hence equality $(*)$ takes the form

$$[X^{\mathsf{c}}, S] = -\frac{1}{2}f^{\mathsf{c}}C. \tag{$***$}$$

Now we calculate the divergence of both sides of $(***)$ with respect to the Dazord volume form w. Applying the formula can be found in [1], **6.5 F**,

$$\mathrm{div}[X^{\mathsf{c}}, S] = X^{\mathsf{c}}\,\mathrm{div}\, S - S\,\mathrm{div}\, X^{\mathsf{c}} \overset{(15.15\,b), (17.9)}{=} = -nf^{\mathsf{c}}.$$

As to the right-hand side, we have

$$\begin{aligned} &\mathrm{div}(-\frac{1}{2}f^{\mathsf{c}}C) \overset{\dagger}{=} \frac{1}{2}f^{\mathsf{c}}\,\mathrm{div}\, C - \frac{1}{2}Cf^{\mathsf{c}} \overset{(15.11), (7.11)}{=} -\frac{n}{2}f^{\mathsf{c}} - \frac{1}{2}f^{\mathsf{c}} \\ &= -\frac{1}{2}(n+1)f^{\mathsf{c}}. \end{aligned}$$

Here, at step $\dagger$, we used formula (8.4.28) in [29]. So it follows $(n-1)f^{\mathsf{c}} = 0$, whence $f^{\mathsf{c}} = 0$ (because $n \geqq 2$). This implies by the connectedness of M that f is a constant function. So the conformal function f^{v} of X is also constant, and hence $X \in \mathrm{Dil}_F(M)$. $\square$

We note that this result, which is an infinitesimal version of Theorem 2 in [32], was also rediscovered by Tian ([33], Corollary 1.2).

Theorem 17.2.5. *Let (M, F) be a Finsler manifold. Suppose that a vector field X on M preserves the Dazord volume form w of (M, F), i.e., $\mathcal{L}_{X^c} w = 0$. If, in addition,*

(i) *X is a projective vector field, then X is a Lie symmetry of the canonical spray of (M, F);*

(ii) *X is a conformal vector field, then X is a Killing vector field of (M, F).*

Proof. Note first that our condition $\mathcal{L}_{X^c} w = 0$ implies that $\operatorname{div} X^c = 0$.
(i) Let $X \in \operatorname{Proj}_F(M)$. Then

$$[X^c, S] = \psi\, C, \qquad \psi \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M), \tag{$*$}$$

where S is the canonical spray of (M, F). As a first step, we show that

$$C\,\psi = \psi \quad \text{ over } \overset{\circ}{T}M. \tag{$**$}$$

Using the Jacobi identity,

$$0 = [C, [X^c, S]] + [X^c, [S, C]] + [S, [C, X^c]] = [C, [X^c, S]] - [X^c, S],$$

hence

$$[X^c, S] = [C, [X^c, S]] \overset{(*)}{=} [C, \psi\, C] = (C\psi)C.$$

Comparing this to $(*)$, we obtain $(**)$.

Now, as in the proof of the preceding theorem, we calculate the divergence of both sides of $(*)$. Since in our case $\operatorname{div} X^c = \operatorname{div} S = 0$, we have on the one hand

$$\operatorname{div}[X^c, S] = X^c(\operatorname{div} S) - S \operatorname{div}(X^c) = 0.$$

On the other hand,

$$\operatorname{div}(\psi\, C) = \psi \operatorname{div} C + C\,\psi \overset{(15.11),\,(**)}{=} (n+1)\psi.$$

So it follows that $\psi = 0$, hence $[X^c, S] = 0$. Thus $X \in \operatorname{Lie}_S(M)$.
(ii) We suppose that $X \in \operatorname{Conf}_F(M)$. Then, by Theorem 17.1.2 and Lemma 17.1.1, $X^c E = f^v E$, where$f \in C^\infty(M)$. Since

$$n\, f^v \overset{(17.9)}{=} \operatorname{div} X^c \overset{\text{condition}}{=} 0,$$

it follows that $X^c E = 0$, and hence $X^c F = 0$. So, by Lemma 16.4.1, $X \in \operatorname{Kill}_F(M)$.

This concludes the proof of the theorem. □

Part V
Summaries

18 Summary

18.1 Notation and background

18.1.1 Let V be a module over a ring R and let $k \in \mathbb{N}$. The R-module of k-linear mappings $V^k \to \mathsf{R}$ (resp. $V^k \to V$) is denoted by $T_k(V)$ (resp. $T_k^1(V)$); $T_0(V) := \mathsf{R}$, $T_0^1(V) := V$. Then $T_1(V) =: V^*$ is the dual of V, $T_1^1(V) =: \mathrm{End}(V)$ is the ring of endomorphisms of V.

18.1.2 Throughout, M is an n-dimensional smooth manifold where $n \geqq 1$ or $n \geqq 2$. The symbols $C^\infty(M)$ and $\mathfrak{X}(M)$ stand for the ring of smooth functions on M and the $C^\infty(M)$-module of vector fields on M, respectively. We write

$$\mathcal{T}_k(M) := T_k(\mathfrak{X}(M)), \quad \mathcal{T}_k^1(M) := T_k^1(\mathfrak{X}(M)),$$
$$\mathcal{A}_k(V) := \{\alpha \in \mathcal{T}_k(M) |\ \alpha \text{ is alternating}\},$$
$$\mathcal{A}_k^1(M) := \{\beta \in \mathcal{T}_k^1(M) |\ \beta \text{ is alternating}\}.$$

Then $\mathcal{A}(M) := \oplus_{k=0}^n \mathcal{A}_k(M)$ is the Grassmann algebra of M. We agree that $\mathcal{A}_k(M) := \{0\}$ if k is a negative integer. An $\mathbb{R}$-linear transformation D is a graded derivation of $\mathcal{A}(M)$ of degree $r \in \mathbb{Z}$ if $D(\mathcal{A}_k(M)) \subset D(\mathcal{A}_{k+r}(M))$, and

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{kr} \alpha \wedge D\beta; \quad \alpha \in A_k(M),\ \beta \in \mathcal{A}(M),$$

where $\wedge$ denotes wedge product. The classical graded derivations of $\mathcal{A}(M)$ are the substitution operator i_X, the Lie derivative $\mathcal{L}_X$ ($X \in \mathfrak{X}(M)$) and the exterior derivative d of degree -1, 0 and 1, respectively.

18.1.3 The tangent bundle of M is $\tau\colon TM \to M$, the slit tangent bundle is $\mathring{\tau}\colon \mathring{T}M \to M$, where $\mathring{T}M \subset TM$ is the open set of nonzero tangent vectors to M and $\mathring{\tau} := \tau \upharpoonright \mathring{T}M$. The derivative of a smooth mapping $\varphi\colon M \to N$ is denoted by φ_*, it maps TM into TN. A vector field ξ on TM (or on $\mathring{T}M$) is projectable if there exists a vector field X on M such that $\tau_* \circ \xi = X \circ \tau$. If $\tau_* \circ \xi = o \circ \tau$, where $o \in \mathfrak{X}(M)$ is the zero vector field, then ξ is called vertical. We use the notation

$$\mathfrak{X}_{\mathrm{proj}}(TM) := \{\xi \in \mathfrak{X}(TM) |\ \xi \text{ is projectable}\},$$
$$\mathfrak{X}^{\mathsf{v}}(TM) := \{\xi \in \mathfrak{X}(TM) |\ \xi \text{ is vertical}\}.$$

18.1.4 Let $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$. Then $f^\mathsf{v} := f \circ \tau \in C^\infty(TM)$ is the vertical lift of f, the smooth function

$$f^\mathsf{c} \colon TM \to \mathbb{R},\ v \mapsto f^\mathsf{c}(v) := v(f) \in \mathbb{R}$$

is the complete lift of f. The vertical lift $X^\mathsf{v} \in \mathfrak{X}^\mathsf{v}(TM)$ and the complete lift $X^\mathsf{c} \in \mathfrak{X}(TM)$ are the unique vector fields on TM such that for every smooth function f on M,

$$X^\mathsf{v} f^\mathsf{c} = (Xf)^\mathsf{v},\ X^\mathsf{v} f^\mathsf{v} = 0;\ X^\mathsf{c} f^\mathsf{c} = (Xf)^\mathsf{c},\ X^\mathsf{c} f^\mathsf{v} = (Xf)^\mathsf{v}.$$

The Liouville vector field $C \in \mathfrak{X}^\mathsf{v}(TM)$ is the unique vertical vector field on TM such that $Cf^\mathsf{c} = f^\mathsf{c}$ for all $f \in C^\infty(M)$. A function $F \in C^\infty(\overset{\circ}{T}M)$ is k^+-homogeneous if $CF = kF$ $(k \in \mathbb{Z})$.

18.1.5 The vector bundles

$$\pi \colon TM \times_M TM \to TM \text{ and } \overset{\circ}{\pi} \colon \overset{\circ}{T}M \times_M TM \to \overset{\circ}{T}M$$

are the Finsler bundles over TM and $\overset{\circ}{T}M$, respectively. The fibre, e.g., of π over $v \in TM$ is the n-dimensional real vector space $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$. The modules of smooth sections of these vector bundles are denoted by $\Gamma(\pi)$ and $\Gamma(\overset{\circ}{\pi})$, respectively, and their elements are called Finsler vector fields. The elements of

$$T_k(\Gamma(\overset{\circ}{\pi})) \cup T_k^1(\Gamma(\overset{\circ}{\pi}))\ (k \in \mathbb{N})$$

are called Finsler tensor fields on $\overset{\circ}{T}M$. We use the following typography:

$X, Y, \dots$	–	vector fields on M,
$\xi, \eta, \dots$	–	vector fields on TM (or on $\overset{\circ}{T}M$),
$\widetilde{X}, \widetilde{Y}, \dots$	–	Finsler vector fields,
$\widehat{X}, \widehat{Y}, \dots$	–	basic Finsler vector fields,
$\widetilde{\delta}$	–	the canonical section in $\Gamma(\pi)$.

Here $\widehat{X}(v) := (v, X(\tau(v)))$, $\widetilde{\delta}(v) := (v, v)$ $(v \in TM)$.

18.1.6 We have the exact sequence of $C^\infty(TM)$-homomorphisms

$$0 \to \Gamma(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}(TM) \xrightarrow{\mathbf{j}} \Gamma(\pi) \to 0,$$

where $\mathbf{i}(\widehat{X}) = X^{\mathsf{v}}$; $\mathbf{j}(X^{\mathsf{v}}) = 0$, $\mathbf{j}(X^{\mathsf{c}}) = \widehat{X}$ $(X \in \mathfrak{X}(M))$, therefore

$$\mathrm{Im}(\mathbf{i}) = \mathrm{Ker}(\mathbf{j}) = \mathfrak{X}^{\mathsf{v}}(TM).$$

We have $C = \mathbf{i}(\widetilde{\delta})$. The vertical endomorphism of $\mathfrak{X}(TM)$ is $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$. It induces graded derivation $d_{\mathbf{J}}$ of degree 1 of $\mathcal{A}(M)$ specified by

$$d_{\mathbf{J}}F := dF \circ \mathbf{J}, \quad d_{\mathbf{J}}dF := dd_{\mathbf{J}}F \qquad (F \in C^{\infty}(TM)).$$

18.1.7 We use the operator ∇^{v} of the (canonical) vertical derivative. It is defined in the following steps:

$\nabla^{\mathsf{v}}_{\widetilde{X}}F := (\mathbf{i}\widetilde{X})F$ $(F \in C^{\infty}(TM))$;
$\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \eta]$, $\eta \in \mathfrak{X}(TM)$ is such that $\mathbf{j}\eta = \widetilde{Y}$;
$(\nabla^{\mathsf{v}}_{\widetilde{X}}A)(\widetilde{Y}_1, \dots \widetilde{Y}_k) := \nabla^{\mathsf{v}}_{\widetilde{X}}(A(\widetilde{Y}_1, \dots, Y_k)) - \sum_{i=1}^{k} A(\widetilde{Y}_1, \dots, \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}_i, \dots \widetilde{Y}_k)$,
$A \in T_k(\Gamma(\pi)) \cup T^1_k(\Gamma(\pi))$.

18.1.8 An Ehresmann connection in $\overset{\circ}{T}M$ is a $C^{\infty}(\overset{\circ}{T}M)$-linear mapping $\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ such that $\mathbf{j} \circ \mathcal{H} = 1_{\Gamma(\overset{\circ}{\pi})}$.
Data: $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ and $\mathbf{v} = 1_{\mathfrak{X}(\overset{\circ}{T}M)} - \mathbf{h}$ are the horizontal and vertical projection associated to $\mathcal{H}$, $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$ is the vertical mapping, $X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \mathbf{h}X^{\mathsf{c}}$ is the ($\mathcal{H}-$)horizontal lift of X. An Ehresmann connection $\mathcal{H}$ is homogeneous if $[C, X^{\mathsf{h}}] = 0$ for all $X \in \mathfrak{X}(M)$. The h-Berwald derivative ∇^{h} induced by $\mathcal{H}$ is defined in the following steps:

$\nabla^{\mathsf{h}}_{\widetilde{X}}F := (\mathcal{H}\widetilde{X})F$ $(F \in C^{\infty}(\overset{\circ}{T}M))$; $\quad \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$;
$(\nabla^{\mathsf{h}}_{\widetilde{X}}A)(\widetilde{Y}_1, \dots, \widetilde{Y}_k) := \nabla^{\mathsf{h}}_{\widetilde{X}}(A(\widetilde{Y}_1, \dots, \widetilde{Y}_k)) - \sum_{i=1}^{k} A(\widetilde{Y}_1, \dots, \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y}_i, \dots, \widetilde{Y}_k)$.

The mapping

$$\nabla\colon (\xi, \widetilde{Y}) \in \mathfrak{X}(\overset{\circ}{T}M) \times \Gamma(\overset{\circ}{\pi}) \mapsto \nabla_{\xi}\widetilde{Y} := \nabla^{\mathsf{v}}_{\mathcal{V}\xi}\widetilde{Y} + \nabla^{\mathsf{h}}_{\mathbf{j}\xi}\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$$

is a covariant derivative, the Berwald derivative on $\overset{\circ}{\pi}$.

18.1.9 A mapping $S\colon TM \to TTM$ is a semispray for M if it is of class C^1, smooth on $\overset{\circ}{T}M$ and satisfies the conditions $\tau_{TM} \circ S = 1_{TM}$,

$\mathbf{J}S = C$. If $[C, S] = S$, then S is called a spray. Every semispray induces an Ehresmann connection $\mathcal{H}$ such that

$$\mathcal{H}(\widehat{X}) = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]),\ X \in \mathfrak{X}(M).$$

This connection is torsion-free in the sense that

$$\nabla_{\mathcal{H}(\widetilde{X})}\widetilde{Y} - \nabla_{\mathcal{H}(\widetilde{Y})}\widetilde{X} = \mathbf{j}[\mathcal{H}(\widetilde{X}), \mathcal{H}(\widetilde{Y})];\ \widetilde{X}, \widetilde{Y} \in \Gamma(\overset{\circ}{\pi}).$$

If S is a spray, then $\mathcal{H}$ is a homogeneous and is called the Berwald connection of the spray manifold (M, S).

18.2 Results

18.2.1 Lie derivatives on a Finsler bundle Given a projectable vector field $\xi \in \mathfrak{X}_{\mathrm{proj}}(TM)$, we define the Lie derivatives of Finsler tensor fields with respect to ξ in the following steps:

$$\widetilde{\mathcal{L}}_\xi F := \mathcal{L}_\xi F = \xi F\ (F \in C^\infty(TM));\quad \widetilde{\mathcal{L}}_\xi \widetilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}];$$

$$(\widetilde{\mathcal{L}}_\xi A)(\widetilde{Y}_1, \dots, \widetilde{Y}_k) := \widetilde{\mathcal{L}}_\xi(A(\widetilde{Y}_1, \dots, \widetilde{Y}_k)) - \sum_{i=1}^{k} A(\widetilde{Y}_1, \dots, \widetilde{\mathcal{L}}_\xi \widetilde{Y}_i, \dots, \widetilde{Y}_k)$$

if $A \in T_k(\Gamma(\pi)) \cup T_k^1(\Gamma(\pi))$.

If $\mathcal{H}$ is an Ehresmann connection in $\overset{\circ}{T}M$, then we define its Lie derivative $\widetilde{\mathcal{L}}_\xi \mathcal{H}$ by $(\widetilde{\mathcal{L}}_\xi \mathcal{H})(\widetilde{Y}) := \mathcal{L}_\xi(\mathcal{H}(\widetilde{Y})) - \mathcal{H}(\widetilde{\mathcal{L}}_\xi \widetilde{Y})$. The Lie derivative of a covariant derivative $D\colon \mathfrak{X}(TM) \times \Gamma(\pi) \to \Gamma(\pi),\ (\eta, \widetilde{Z}) \mapsto D_\eta \widetilde{Z}$ with respect to ξ is the mapping

$$\begin{cases} \widetilde{\mathcal{L}}_\xi D\colon \mathfrak{X}(TM) \times \Gamma(\pi) \to \Gamma(\pi),\ (\eta, \widetilde{Z}) \mapsto (\widetilde{\mathcal{L}}_\xi D)(\eta, \widetilde{Z}), \\ (\widetilde{\mathcal{L}}_\xi D)(\eta, \widetilde{Z}) := \widetilde{\mathcal{L}}_\xi(D_\eta \widetilde{Z}) - D_{[\xi,\eta]}\widetilde{Z} - D_\eta(\widetilde{\mathcal{L}}_\xi \widetilde{Z}). \end{cases}$$

We derived the useful formulae:

(1) $[\widetilde{\mathcal{L}}_\xi, \widetilde{\mathcal{L}}_\eta] = \widetilde{\mathcal{L}}_{[\xi,\eta]}$;
(2) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y} = \widehat{[X, Y]}$;
(3)$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{\delta} = 0$;
(4)$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \restriction \Gamma(\pi) = \nabla^{\mathsf{v}}_{\widehat{X}} \restriction \Gamma(\pi)$;
(5) $\mathbf{i} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \mathcal{L}_{X^{\mathsf{c}}} \circ \mathbf{i}$;
(6) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{j} = \mathbf{j} \circ \mathcal{L}_{X^{\mathsf{c}}}$;
(7)$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{v}}_{\widehat{Y}} - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}$;
(8) $\widetilde{\mathcal{L}}_{X^{\mathsf{h}}} \restriction \Gamma(\overset{\circ}{\pi}) = \nabla^{\mathsf{h}}_{\widehat{X}} \restriction \Gamma(\overset{\circ}{\pi})$;

(9) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{h}}_{\widehat{Y}} - \nabla^{\mathsf{h}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \widetilde{\mathcal{L}}_{[X^{\mathsf{c}}, Y^{\mathsf{h}}]}$.

In the formulas above, $\xi, \eta \in \mathfrak{X}_{\mathrm{proj}}(TM)$; X and Y are vector fields on M. In (8) and (9) we assume that an Ehresmann connection is also specified in $\overset{\circ}{T}M$.

We showed that the vanishing of $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{Y}$ has the following *dynamical interpretation*:

Theorem 1. *Let (φ_t) be the local flow of X. Then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{Y} = 0$ if, and only if, $\widetilde{Y}$ is invariant under (φ_t), i.e.,*

$$((\varphi_t)_* \times (\varphi_t)_*) \circ \widetilde{Y} = \widetilde{Y} \circ (\varphi_t)_*,$$

for every stage φ_t of the flow.

18.2.2 $\mathcal{H}$-Killing vector fields Let an Ehresmann connection $\mathcal{H} \colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ be given. Note first that for every $\xi \in \mathfrak{X}_{\mathrm{proj}}(\overset{\circ}{T}M)$, the mapping $\widetilde{\mathcal{L}}_\xi \mathcal{H} \colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$, $\widetilde{Y} \mapsto (\widetilde{\mathcal{L}}_\xi \mathcal{H})(\widetilde{Y})$ is $C^\infty(\overset{\circ}{T}M)$-linear. If $X \in \mathfrak{X}(M)$, then $\mathbf{j} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = 0$, so the Lie derivative of an Ehresmann connection is definitely not an Ehresmann connection. We say that a vector field X on M is *$\mathcal{H}$-Killing* and we write that $X \in \mathrm{Kill}_{\mathcal{H}}(M)$, if $\mathcal{H}$ is invariant under the local flow of X in the sense that $(\varphi_t)_{**} \circ \mathcal{H} = \mathcal{H} \circ ((\varphi_t)_* \times (\varphi_t)_*)$, for every stage φ_t of the flow of X. (Here $\mathcal{H}$ is interpreted as a strong bundle map from $\overset{\circ}{T}M \times_M TM$ in $T\overset{\circ}{T}M$.) We have proved:

Theorem 2. *For a vector field X on M, the following are equivalent:*

1. *$X \in \mathrm{Kill}_{\mathcal{H}}(M)$, i.e., X is a $\mathcal{H}$-Killing vector field,*
2. *For every stage φ_t of the local flow of X,*

   $$(\varphi_t)_{**} \circ \mathbf{h} = \mathbf{h} \circ (\varphi_t)_{**},$$

   where $\mathbf{h}$ is the horizontal projection associated to $\mathcal{H}$,
3. *$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H} = 0$,*
4. *$\mathcal{L}_{X^{\mathsf{c}}}\mathbf{h} = 0$.*

If one (and hence all) of (1)-(4) is satisfied, then locally we have

$$X^{\mathsf{c}} N^i_j = N^k_j \left(\frac{\partial X^i}{\partial u^k} \circ \tau \right) - N^i_k \left(\frac{\partial X^k}{\partial u^j} \circ \tau \right) - y^k \left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau \right),$$

where $X^i \in C^\infty(\mathcal{U})$ are the components of X relative to a chart $(\mathcal{U}, (u^i)_{i=1}^n)$ of M, and (N^i_j) is the family of Christoffel symbols of $\mathcal{H}$ relative to the induced chart $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ on TM.

18.2.3 Lie symmetries Let S be a semispray for M. A vector field X on M is a Lie symmetry of S, if S is invariant under the local flow of X^{c}, i.e., $(\varphi_t)_{**} \circ S = S \circ (\varphi_t)_*$ for every stage φ_t of the flow of X. Then we write $X \in \mathrm{Lie}_S(M)$. It is clear from the dynamical interpretation of the classical Lie derivative that

$$X \in \mathrm{Lie}_S(M) \iff [X^{\mathsf{c}}, S] = 0.$$

We have: $\mathrm{Lie}_S(M) \subset \mathrm{Kill}_{\mathcal{H}}(M)$, where $\mathcal{H}$ is the Ehresmann connection induced by S.

Theorem 3. *Let (M, S) be a spray manifold, endowed with the Berwald connection $\mathcal{H}$ and the Berwald derivative ∇ induced by $\mathcal{H}$. For a vector field X on M, the following are equivalent:*

(1) $X \in \mathrm{Lie}_S(M)$,	(6) $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{v} = 0$,
(2) $[X^{\mathsf{c}}, S] = 0$,	(7) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla = 0$,
(3) $X \in \mathrm{Kill}_{\mathcal{H}}(M)$,	(8) $[X^{\mathsf{c}}, Y^{\mathsf{h}}] = [X, Y]^{\mathsf{h}}$,
(4) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{H} = 0$,	(9) $[\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}, \widetilde{\mathcal{L}}_{Y^{\mathsf{h}}}] = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{h}}}$,
(5) $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{h} = 0$,	(10) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{L}_{X^{\mathsf{c}}}$.

In conditions (8) and (9), Y is any vector field on M. We note that the equivalence of (1), (5) and (7) has already been proved by R. L. Lovas [17].

18.2.4 Curvature collineations Let (M, S) be a spray manifold. **(A)** The Finsler tensor fields $\mathbf{K}$, $\mathbf{R}$, $\mathbf{H}$ defined by

$$\begin{aligned}
\mathbf{K}(\widetilde{X}) &:= \mathcal{V}[S, \mathcal{H}(\widetilde{X})],\\
\mathbf{R}(\widetilde{X}, \widetilde{Y}) &:= \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{K}(\widetilde{X}, \widetilde{Y}) - \nabla^{\mathsf{v}}\mathbf{K}(\widetilde{Y}, \widetilde{X})),\\
\mathbf{H}(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= -\nabla^{\mathsf{v}}\mathbf{R}(\widetilde{Z}, \widetilde{X}, \widetilde{Y})
\end{aligned}$$

are the Jacobi endomorphism (or affine deviation), the fundamental affine curvature and the affine curvature of (M, S), respectively. If $\mathbf{C} \in \{\mathbf{K}, \mathbf{R}, \mathbf{H}\}$ and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{C} = 0$, then we say that X is a *curvature collineation* of $\mathbf{C}$.

Theorem 4. *A vector field X on M is a curvature collineation of the*

Jacobi endomorphism of (M, S) if, and only if, $\mathbf{K}$ *is invariant under the local flow of X in the sense that*

$$((\varphi_t)_* \times (\varphi_t)_*) \circ \mathbf{K} = \mathbf{K} \circ ((\varphi_t)_* \times (\varphi_t)_*)$$

for every stage φ_t of the local flow. (Here $\mathbf{K}$ *is interpreted as a strong bundle endomorphism of $\overset{\circ}{\pi}$.)*

Theorem 5. *If $X \in \mathrm{Lie}_S M$, then X is a curvature collineation of* $\mathbf{K}$*,* $\mathbf{R}$ *and* $\mathbf{H}$.

(B) A Finsler tensor field constructed from S is called projectively invariant if it remains invariant under the projective changes

$$S \rightsquigarrow S - 2PC, \quad P \in C^\infty(\overset{\circ}{T}M)$$

of S. The fundamental projectively invariant tensors of (M, S) are the Weyl tensors $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ and the Douglas tensor $\mathbf{D}$ defined as follows:

$$\mathbf{W}_1 := \mathbf{K} - K\,1_{\Gamma(\overset{\circ}{\pi})} - \frac{1}{n+1}(\mathrm{tr}\nabla^{\mathsf{v}}\mathbf{K} - \nabla^{\mathsf{v}}K) \otimes \widetilde{\delta} \quad \left(K := \frac{1}{n-1}\mathrm{tr}\mathbf{K}\right),$$

$$\mathbf{W}_2(\widetilde{X}, \widetilde{Y}) := \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{W}_1(\widetilde{X}, \widetilde{Y}) - \nabla^{\mathsf{v}}\,\mathbf{W}_1(\widetilde{Y}, \widetilde{X})),$$

$$\mathbf{W}_3(\widetilde{X}, \widetilde{Y})\widetilde{Z} := \nabla^{\mathsf{v}}\mathbf{W}_2(\widetilde{Z}, \widetilde{X}, \widetilde{Y}),$$

$$\mathbf{D} := \mathbf{B} - \frac{1}{n-1}((\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B}) \otimes \widetilde{\delta} + (\mathrm{tr}\mathbf{B}) \odot 1_{\Gamma(\overset{\circ}{\pi})}).$$

In the last formula, $\mathbf{B}$ is the Berwald tensor of (M, S) given by $\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} := (\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{Z})(\widehat{X}, \widehat{Y})$, and the symbol $\odot$ means symmetric product without numerical factor.

Theorem 6. *If $X \in \mathrm{Lie}_S(M)$, then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W}_i = 0, \quad i \in \{1, 2, 3\}$.*

Theorem 7. *If $X \in \mathrm{Lie}_S(M)$, then $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{B} = 0$, which implies that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{D} = 0$.*

18.2.5 Geometric vector fields on a Finsler manifold A positive continuous function $F\colon TM \to \mathbb{R}$ is a Finsler function for M if it is smooth on $\overset{\circ}{T}M$, 1^+-homogeneous and the fundamental tensor

$$g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 =: \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E$$

is fibrewise non-degenerate. A Finsler manifold is a pair (M, F) with M a manifold and F a Finsler function for M. First we recall some

basic data:
(1) $\theta_g := \nabla^{\mathsf{v}} E$ or $\theta_E := d_{\mathbf{J}} E = \theta_g \circ \mathbf{j}$ – the Hilbert 1-form of (M, F).
(2) $\omega_E := d\theta_E = dd_{\mathbf{J}} E$ – the fundamental 2-form of (M, F).
(3) $w := \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} \omega_E \wedge \cdots \wedge \omega_E$ (n factors) – the Dazord volume form of (M, F).
(4) The canonical spray of (M, F) is the unique spray S for M such that $i_S dd_{\mathbf{J}} E = -dE$ over $\overset{\circ}{T}M$. The canonical connection $\mathcal{H}$ of (M, F) is the Berwald connection of (M, S), ∇ stands for the Berwald derivative induced by $\mathcal{H}$.
(5) The Sasaki-Finsler metric g^S on $\overset{\circ}{T}M$ is given by

$$g^S(\xi, \eta) := g(\mathbf{j}\xi, \mathbf{j}\eta) + g(\mathcal{V}\xi, \mathcal{V}\eta).$$

(6) $\mathcal{C}_\flat := \nabla^{\mathsf{v}} g = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} E$ is the Cartan-tensor of (M, F); the type $(1, 2)$ Cartan-tensor $\mathcal{C}$ is given by $g(\mathcal{C}(\widetilde{X}, \widetilde{Y})\widetilde{Z}) = \mathcal{C}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$.
(7) $\mathbf{L}_\flat := \nabla^{\mathsf{h}} g = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} E$ is the Landsberg tensor of (M, F); the type $(1, 2)$ Landsberg tensor is given by $g(\mathbf{L}(\widetilde{X}, \widetilde{Y})\widetilde{Z}) = \mathbf{L}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$.
(8) D^C, D^{Ch} and D^{Hs} stand for the Cartan, the Chern-Rund and the Hashiguchi derivative on (M, F); they are given by

$$D^C_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi, \widetilde{Y}) + \frac{1}{2}\mathbf{L}(\mathbf{j}\xi, \widetilde{Y}),$$
$$D^{Ch}_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathbf{L}(\mathbf{j}\xi, \widetilde{Y}), \quad D^{Hs}_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi, \widetilde{Y}).$$

Definitions: A vector field X on M is a *Killing vector field* of (M, F) if the stages φ_t of its local flow preserve the Finslerian norms of the tangent vectors to M, i.e., $F \circ (\varphi_t)_* = F$ for every possible $t \in \mathbb{R}$. If

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g = \sigma g, \ \sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M),$$

then X is called a *conformal vector field* with conformal function σ. A conformal vector field is *homothetic* if its conformal function is constant. We say that X is a *projective vector field* if

$$[X^{\mathsf{c}}, S] = \varphi C, \quad \varphi \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M).$$

Notation: $\mathrm{Kill}_F(M)$, $\mathrm{Conf}_F(M)$, $\mathrm{Dil}_F(M)$ and $\mathrm{Proj}_F(M)$ are the sets of Killing, conformal, homothetic and projective vector fields of (M, F), respectively.

Theorem 8. (a) *For every vector field X on M,*

(i) $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_g)\circ \mathbf{j} = \mathcal{L}_{X^{\mathsf{c}}}\omega_E$;
(ii) $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{j}\xi, \mathbf{j}\eta) = (\mathcal{L}_{X^{\mathsf{c}}}\omega_E)(\mathbf{J}\xi, \eta)$;
(iii) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat = \nabla^{\mathsf{v}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)$;
(iv) $g((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C})(\widehat{Y}, \widehat{Z}), \widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat)(\widehat{Y}, \widehat{Z}, \widehat{U}) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathcal{C}(\widehat{Y}, \widehat{Z}), \widehat{U})$;
(v) $g((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L})(\widehat{Y}, \widehat{Z}), \widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat)(\widehat{Y}, \widehat{Z}, \widehat{U}) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{L}(\widehat{Y}, \widehat{Z}), \widehat{U})$.

(*b*) *If* $X \in \mathrm{Lie}_S(M)$, *then* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat = \nabla^{\mathsf{h}}(\mathcal{L}_{X^{\mathsf{c}}}g)$.
(*c*) *If* $X \in \mathrm{Kill}_F(M)$, *then*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat = 0,\ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C} = 0, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat = 0,\ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L} = 0.$$

Theorem 9. *If* $X \in \mathrm{Kill}_F(M)$ *and* $D \in \{\nabla, D^C, D^{Ch}, D^{Hs}\}$, *then* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}D = 0$.

Theorem 10. (a) *If* $X \in \mathrm{Conf}_F(M)$, *then its conformal function is a vertical lift.*
(b) *For a vector field* X *on* M, *the following are equivalent:*
(i) X *is a conformal vector field,*
(ii) $X^{\mathsf{c}}E = \sigma E$,
(iii) $\mathcal{L}_{X^{\mathsf{c}}}\theta_E = \sigma\theta_E$,
(iv) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_g = \sigma\theta_g$,
(v) $\mathcal{L}_{X^{\mathsf{c}}}\omega_E = f^{\mathsf{v}}\omega_E + df^{\mathsf{v}} \wedge d_{\mathbf{J}}E, \quad f \in C^\infty(M)$.
In conditions (ii)-(iv), $\sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M)$.

Theorem 11. $X \in \mathrm{Conf}_F(M) \cap \mathrm{Lie}_S(M) \Longrightarrow X^{\mathsf{c}} \in \mathrm{Conf}_{g^s}(\overset{\circ}{T}M)$,

$$X^{\mathsf{c}} \in \mathrm{Conf}_{g^S}(\overset{\circ}{T}M) \Longrightarrow X \in \mathrm{Conf}_F(M).$$

Theorem 12. $X \in \mathrm{Dil}_F(M) \Longrightarrow X \in \mathrm{Lie}_S(M)$.

Theorem 13. $X \in \mathrm{Proj}_F(M) \cap \mathrm{Conf}_F(M) \Longrightarrow X \in \mathrm{Dil}_F(M)$.

Theorem 14. $(X \in \mathrm{Proj}_F(M)$ *and* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\ w = 0) \Longrightarrow X \in \mathrm{Lie}_S(M)$.

Theorem 15. $(X \in \mathrm{Conf}_F(M)$ *and* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}w = 0) \Longrightarrow X \in \mathrm{Kill}_F(M)$.

19 Magyar nyelvű összefoglaló (Summary in Hungarian)

19.1 Jelölések és háttérismeretek

19.1.1 Legyen V egy R gyűrű fölötti modulus és legyen $k \in \mathbb{N}$. A $V^k \to \mathsf{R}$ (ill. $V^k \to V$) k-lineáris leképezések R-modulusára a $T_k(V)$ (ill. $T_k^1(V)$) jelölést használjuk; $T_0(V) := \mathsf{R}$, $T_0^1(V) := V$. Ekkor $T_1(V) =: V^*$ a V modulus duális modulusa, $T_1^1(V) =: \mathrm{End}(V)$ pedig V endomorfizmus gyűrűje.

19.1.2 M-mel mindvégig egy n-dimenziós sima sokaságot jelölünk, ahol $n \geqq 1$ vagy $n \geqq 2$. $C^\infty(M)$ az M sokaság sima függvényeinek gyűrűje, $\mathfrak{X}(M)$ az M fölötti vektormezők $C^\infty(M)$-modulusa. Alkalmazzuk a

$$\mathcal{T}_k(M) := T_k(\mathfrak{X}(M)), \quad \mathcal{T}_k^1(M) := T_k^1(\mathfrak{X}(M)),$$
$$\mathcal{A}_k(V) := \{\alpha \in \mathcal{T}_k(M) |\ \alpha \text{ alternáló}\},$$
$$\mathcal{A}_k^1(M) := \{\beta \in \mathcal{T}_k^1(M) |\ \beta \text{alternáló}\}$$

jelöléseket. Ekkor $\mathcal{A}(M) := \oplus_{k=0}^n \mathcal{A}_k(M)$ az M sokaság Grassmann algebrája. Megállapodunk abban, hogy $\mathcal{A}_k(M) := \{0\}$, ha k negatív egész. Egy D $\mathbb{R}$-lineáris transzformáció r-edfokú ($r \in \mathbb{Z}$) gradált derivációja $\mathcal{A}(M)$-nek, ha $D(\mathcal{A}_k(M)) \subset D(\mathcal{A}_{k+r}(M))$ és

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{kr} \alpha \wedge D\beta; \quad \alpha \in \mathcal{A}_k(M),\ \beta \in \mathcal{A}(M);$$

itt az $\wedge$ szimbólum ékszorzatot jelöl. A Grassmann algebra klasszikus gradált derivációi az i_X helyettesítési operátor, az $\mathcal{L}_X$ Lie-derivált ($X \in \mathfrak{X}(M)$) és a d külső derivált; ezek foka rendre -1, 0 és 1.

19.1.3 Az M sokaság érintőnyalábja $\tau \colon TM \to M$, a hasított érintőnyalábja $\overset{\circ}{\tau} \colon \overset{\circ}{T}M \to M$. Az utóbbinál $\overset{\circ}{T}M$ az M sokaság nemzérus érintővektorai alkotta nyílt részhalmaza TM-nek, $\overset{\circ}{\tau} := \tau \upharpoonright \overset{\circ}{T}M$. Egy $\varphi \colon M \to N$ sima leképezés deriváltját φ_* jelöli, ez TM-et TN-be képezi le. Egy TM-en (vagy $\overset{\circ}{T}M$-en) adott ξ vektormező vetíthető, ha van olyan X vektormező M-en, hogy $\tau_* \circ \xi = X \circ \tau$. Ha speciálisan $\tau_* \circ \xi = o \circ \tau$, ahol $o \in \mathfrak{X}(M)$ a zérus vektormező, akkor ξ-t vertikálisnak mondjuk. Használjuk az

$$\mathfrak{X}_{\mathrm{proj}}(TM) := \{\xi \in \mathfrak{X}(TM) |\ \xi \text{ vetíthető}\},$$
$$\mathfrak{X}^{\mathsf{v}}(TM) := \{\xi \in \mathfrak{X}(TM) |\ \xi \text{ vertikális}\}.$$

jelöléseket.

19.1.4 Legyen $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$. Ekkor $f^{\mathsf{v}} := f \circ \tau \in C^\infty(TM)$ f vertikális liftje, az $f^{\mathsf{c}}\colon TM \to \mathbb{R}$, $v \mapsto f^{\mathsf{c}}(v) := v(f) \in \mathbb{R}$ sima függvény pedig a teljes liftje. Az X vektormező $X^{\mathsf{v}} \in \mathfrak{X}^{\mathsf{v}}(TM)$ vertikális, ill. $X^{\mathsf{c}} \in \mathfrak{X}(TM)$ teljes liftje az az egyetlen vektormező TM-en, amelyre tetszőleges $f \in C^\infty(M)$ esetén

$$X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}},\ X^{\mathsf{v}} f^{\mathsf{v}} = 0;\ X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}},\ X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}}.$$

Létezik egy és csak egy olyan $C \in \mathfrak{X}^{\mathsf{v}}(TM)$ vertikális vektormező, hogy $Cf^{\mathsf{c}} = f^{\mathsf{c}}$ minden $f \in C^\infty(M)$ függvényre; ez a Liouville vektormező TM-en. Egy $F \in C^\infty(\overset{\circ}{T}M)$ függvény k^+-homogén, ha $CF = kF$ $(k \in \mathbb{Z})$.

19.1.5 A TM, ill. $\overset{\circ}{T}M$ fölötti Finsler-nyaláb a

$$\pi\colon TM \times_M TM \to TM \text{ és } \overset{\circ}{\pi}\colon \overset{\circ}{T}M \times_M TM \to \overset{\circ}{T}M$$

vektornyaláb. Itt például a π nyaláb $v \in TM$ fölötti fibruma a $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$ n-dimenziós valós vektortér. E vektornyalábok sima szeléseinek modulusát $\Gamma(\pi)$, ill. $\Gamma(\overset{\circ}{\pi})$ jelöli. $\Gamma(\pi)$ és $\Gamma(\overset{\circ}{\pi})$ elemeit Finsler vektormezőknek; a $T_k(\Gamma(\overset{\circ}{\pi})) \cup T^1_k(\Gamma(\overset{\circ}{\pi}))$ $(k \in \mathbb{N})$ modulusok elemeit $\overset{\circ}{T}M$-en adott Finsler vektormezőknek hívjuk. A következő tipográfiai megoldással élünk:

$X, Y, \dots$	–	vektormezők M-en,
$\xi, \eta, \dots$	–	vektormezők TM-en (vagy $\overset{\circ}{T}M$-en),
$\widetilde{X}, \widetilde{Y}, \dots$	–	Finsler vektormezők,
$\widehat{X}, \widehat{Y}, \dots$	–	bázikus Finsler vektormezők,
$\widetilde{\delta}$	–	$\Gamma(\pi)$ kanonikus szelése.

Itt $\widehat{X}(v) := (v, X(\tau(v)))$, $\widetilde{\delta}(v) := (v, v)$ $(v \in TM)$.

19.1.6 A $0 \to \Gamma(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}(TM) \xrightarrow{\mathbf{j}} \Gamma(\pi) \to 0$ sor, ahol $\mathbf{i}(\widehat{X}) = X^{\mathsf{v}}$; $\mathbf{j}(X^{\mathsf{v}}) = 0$, $\mathbf{j}(X^{\mathsf{c}}) = \widehat{X}$ $(X \in \mathfrak{X}(M))$ $C^\infty(TM)$-homomorfizmusok egzakt sora. Így $\operatorname{Im}(\mathbf{i}) = \operatorname{Ker}(\mathbf{j}) = \mathfrak{X}^{\mathsf{v}}(TM)$, s közvetlenül adódik, hogy $C = \mathbf{i}(\widetilde{\delta})$. A $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ endomorfizmus $\mathfrak{X}(TM)$ vertikális endomorfizmusa. Ez $\mathcal{A}(TM)$-nek egy $d_{\mathbf{J}}$ elsőfokú gradált derivációját indukálja, amely a

$$d_{\mathbf{J}}F := dF \circ \mathbf{J}, \quad d_{\mathbf{J}}dF := dd_{\mathbf{J}}F \qquad (F \in C^\infty(TM))$$

előírással értelmezhető.

19.1.7 Alkalmazzuk a (kanonikus) vertikális derivált ∇^{v} operátorát, melynek definíciója a következő lépésekben adható meg:

$$\nabla^{\mathsf{v}}_{\widetilde{X}} F := (\mathbf{i}\widetilde{X})F \ (F \in C^{\infty}(TM));$$

$$\nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \eta], \ \eta \in \mathfrak{X}(TM) \text{ olyan, hogy } \mathbf{j}\eta = \widetilde{Y};$$

$$(\nabla^{\mathsf{v}}_{\widetilde{X}} A)(\widetilde{Y}_1, \dots \widetilde{Y}_k) := \nabla^{\mathsf{v}}_{\widetilde{X}}(A(\widetilde{Y}_1, \dots, Y_k)) - \sum_{i=1}^{k} A(\widetilde{Y}_1, \dots, \nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y}_i, \dots \widetilde{Y}_k),$$

$$A \in T_k(\Gamma(\pi)) \cup T^1_k(\Gamma(\pi)).$$

19.1.8 Egy $\overset{\circ}{T}M$-beli Ehresmann-konnexió olyan $\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ $C^{\infty}(\overset{\circ}{T}M)$-lineáris leképezés, amelyre $\mathbf{j} \circ \mathcal{H} = 1_{\Gamma(\overset{\circ}{\pi})}$. Adatai: $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$, $\mathbf{v} = 1_{\mathfrak{X}(\overset{\circ}{T}M)} - \mathbf{h}$ és $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$ a $\mathcal{H}$-hoz csatolt vertikális és horizontális projekció, valamint vertikális leképezés; $X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \mathbf{h}X^{\mathsf{c}}$ az X vektormező ($\mathcal{H}-$)horizontális liftje. Az Ehresmann-konnexió homogén, ha $[C, X^{\mathsf{h}}] = 0$ minden $X \in \mathfrak{X}(M)$-re. A $\mathcal{H}$ által indukált ∇^{h} h-Berwald-derivált a következő lépésekben értelmezhető:

$$\nabla^{\mathsf{h}}_{\widetilde{X}} F := (\mathcal{H}\widetilde{X})F \ (F \in C^{\infty}(\overset{\circ}{T}M)); \quad \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}];$$

$$(\nabla^{\mathsf{h}}_{\widetilde{X}} A)(\widetilde{Y}_1, \dots, \widetilde{Y}_k) := \nabla^{\mathsf{h}}_{\widetilde{X}}(A(\widetilde{Y}_1, \dots, \widetilde{Y}_k)) - \sum_{i=1}^{k} A(\widetilde{Y}_1, \dots, \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y}_i, \dots, \widetilde{Y}_k).$$

A $\nabla\colon (\xi, \widetilde{Y}) \in \mathfrak{X}(\overset{\circ}{T}M) \times \Gamma(\overset{\circ}{\pi}) \mapsto \nabla_{\xi}\widetilde{Y} := \nabla^{\mathsf{v}}_{\mathcal{V}\xi}\widetilde{Y} + \nabla^{\mathsf{h}}_{\mathbf{j}\xi}\widetilde{Y} \in \Gamma(\overset{\circ}{\pi})$ leképezés kovariáns derivált a $\overset{\circ}{\pi}$ vektornyalábon, a Berwald-derivált.

19.1.9 Egy $S\colon TM \to TTM$ leképezés szemispray M fölött, ha C^1-osztályú, $\overset{\circ}{T}M$ fölött sima, és eleget tesz a $\tau_{TM} \circ S = 1_{TM}$, $\mathbf{J}S = C$ feltételeknek. Ha – ráadásul – $[C, S] = S$, akkor S spray M fölött. Minden szemispray indukál egy $\mathcal{H}$ Ehresmann-konnexiót, melyre

$$\mathcal{H}(\widehat{X}) = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]), \ \textit{bármely } X \in \mathfrak{X}(M) \textit{ esetén.}$$

Ez a konnexió torziómentes abban az értelemben, hogy tetszőleges $\widetilde{X}, \widetilde{Y}$ Finsler-vektormezőkre

$$\nabla_{\mathcal{H}(\widetilde{X})}\widetilde{Y} - \nabla_{\mathcal{H}(\widetilde{Y})}\widetilde{X} = \mathbf{j}[\mathcal{H}(\widetilde{X}), \mathcal{H}(\widetilde{Y})] \quad (\overset{\circ}{T}M \text{ fölött}).$$

Amennyiben S spray, úgy $\mathcal{H}$ homogén, és azt mondjuk, hogy $\mathcal{H}$ az (M, S) spray-sokaság Berwald-konnexiója.

19.2 Eredmények

19.2.1 Lie-derivatáltak Finsler-nyalábokon Megadva egy vetíthető $\xi \in \mathfrak{X}(TM)$ vektormezőt, a Finsler-tenzormezők ξ szerinti Lie-deriváltját a következő lépésekben definiáljuk:

$$\widetilde{\mathcal{L}}_\xi F := \mathcal{L}_\xi F = \xi F \ (F \in C^\infty(TM)); \quad \widetilde{\mathcal{L}}_\xi \widetilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}];$$

$$(\widetilde{\mathcal{L}}_\xi A)(\widetilde{Y}_1, \dots, \widetilde{Y}_k) := \widetilde{\mathcal{L}}_\xi(A(\widetilde{Y}_1, \dots, \widetilde{Y}_k)) - \sum_{i=1}^{k} A(\widetilde{Y}_1, \dots, \widetilde{\mathcal{L}}_\xi \widetilde{Y}_i, \dots, \widetilde{Y}_k),$$

itt $A \in T_k(\Gamma(\pi)) \cup T_k^1(\Gamma(\pi)$. Egy $\overset{\circ}{T}M$-beli $\mathcal{H}$ Ehresmann-konnexió $\widetilde{\mathcal{L}}_\xi \mathcal{H}$ Lie-deriváltját az

$$(\widetilde{\mathcal{L}}_\xi \mathcal{H})(\widetilde{Y}) := \mathcal{L}_\xi(\mathcal{H}(\widetilde{Y})) - \mathcal{H}(\widetilde{\mathcal{L}}_\xi \widetilde{Y}).$$

előírással értelmezzük; egy $D\colon \mathfrak{X}(TM) \times \Gamma(\pi) \to \Gamma(\pi),\ (\eta, \widetilde{Z}) \mapsto D_\eta \widetilde{Z}$ kovariáns derivált ξ szerinti Lie-deriváltja az

$$\begin{cases} \widetilde{\mathcal{L}}_\xi D\colon \mathfrak{X}(TM) \times \Gamma(\pi),\ (\eta, \widetilde{Z}) \mapsto (\widetilde{\mathcal{L}}_\xi D)(\eta, \widetilde{Z}), \\ (\widetilde{\mathcal{L}}_\xi D)(\eta, \widetilde{Z}) := \widetilde{\mathcal{L}}_\xi(D_\eta \widetilde{Z}) - D_{[\xi,\eta]}\widetilde{Z} - D_\eta(\widetilde{\mathcal{L}}_\xi \widetilde{Z}). \end{cases}$$

leképezés. Levezettük a következő hasznos formulákat:

(1) $[\widetilde{\mathcal{L}}_\xi, \widetilde{\mathcal{L}}_\eta] = \widetilde{\mathcal{L}}_{[\xi,\eta]}$;
(2) $\widetilde{\mathcal{L}}_{X^c}\widehat{Y} = \widehat{[X,Y]}$;
(3) $\widetilde{\mathcal{L}}_{X^c}\widetilde{\delta} = 0$;
(4) $\widetilde{\mathcal{L}}_{X^c} \restriction \Gamma(\pi) = \nabla^{\mathsf{v}}_{\widehat{X}} \restriction \Gamma(\pi)$;
(5) $\mathbf{i} \circ \widetilde{\mathcal{L}}_{X^c} = \mathcal{L}_{X^c} \circ \mathbf{i}$;
(6) $\widetilde{\mathcal{L}}_{X^c} \circ \mathbf{j} = \mathbf{j} \circ \mathcal{L}_{X^c}$;
(7) $\widetilde{\mathcal{L}}_{X^c} \circ \nabla^{\mathsf{v}}_{\widehat{Y}} - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^c} = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}$;
(8) $\widetilde{\mathcal{L}}_{X^{\mathsf{h}}} \restriction \Gamma(\overset{\circ}{\pi}) = \nabla^{\mathsf{h}}_{\widehat{X}} \restriction \Gamma(\overset{\circ}{\pi})$;
(9) $\widetilde{\mathcal{L}}_{X^c} \circ \nabla^{\mathsf{h}}_{\widehat{Y}} - \nabla^{\mathsf{h}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^c} = \widetilde{\mathcal{L}}_{[X^c,Y^{\mathsf{h}}]}$.

A fenti formulákban $\xi, \eta \in \mathfrak{X}_{\mathrm{proj}}(TM)$; $X, Y \in \mathfrak{X}(M)$. (8)-ban és (9)-ben föltesszük, hogy egy Ehresmann-konnexió is adva van $\overset{\circ}{T}M$-ben.

Megmutattuk, hogy $\widetilde{\mathcal{L}}_{X^c}\widetilde{Y}$ eltűnésére a következő dinamikai interpretáció lehetséges

1. Tétel *Legyen (φ_t) az X vektormező lokális folyama. $\widetilde{\mathcal{L}}_{X^c}\widetilde{Y} = 0$ pontosan akkor teljesül, ha $\widetilde{Y}$ invariáns a folyammal szemben, azaz*

$$((\varphi_t)_* \times (\varphi_t)_*) \circ \widetilde{Y} = \widetilde{Y} \circ (\varphi_t)_*,$$

minden szóbajövő $t \in \mathbb{R}$ esetén.

19.2.2 $\mathcal{H}$-Killing vektormezők Legyen adva egy $\mathcal{H}$ Ehresmann-konnexió $\overset{\circ}{T}M$-en. Jegyezzük meg először is, hogy tetszőleges $\xi \in \mathfrak{X}_{\mathrm{proj}}(\overset{\circ}{T}M)$ esetén az $\widetilde{\mathcal{L}}_\xi\mathcal{H}\colon \Gamma(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$, $\widetilde{Y} \mapsto (\widetilde{\mathcal{L}}_\xi\mathcal{H})(\widetilde{Y})$ leképezés $C^\infty(\overset{\circ}{T}M)$-lineáris. Tetszőleges $X \in \mathfrak{X}(M)$ vektormező esetén $\mathbf{j} \circ \widetilde{\mathcal{L}}_{X^c} = 0$, ami mutatja, hogy egy Ehresmann-konnexió Lie-deriváltja már nem Ehresmann-konnexió.

Egy $X \in \mathfrak{X}(M)$ vektormezőt *$\mathcal{H}$-Killing vektormezőnek* nevezünk és azt írjuk, hogy $X \in \mathrm{Kill}_{\mathcal{H}}(M)$, ha $\mathcal{H}$ invariáns X lokális folyamával szemben, abban az értelemben, hogy minden szóbajövő valós t-re

$$(\varphi_t)_{**} \circ \mathcal{H} = \mathcal{H} \circ ((\varphi_t)_* \times (\varphi_t)_*).$$

(Itt $\mathcal{H}$-t $\overset{\circ}{T}M \times_M TM \to T\overset{\circ}{T}M$ erős nyalábleképezésként interpretáljuk.) Megmutattuk a következőt:

2. Tétel *Egy $X \in \mathfrak{X}(M)$ vektormezőre az alábbi feltételek ekvivalensek:*

(1) *X $\mathcal{H}$-Killing vektormező,*

(2) *Ha X lokális folyama φ_t, akkor minden szóbajövő t-re teljesül, hogy $(\varphi_t)_{**} \circ \mathbf{h} = \mathbf{h} \circ (\varphi_t)_{**}$,*

(3) $\widetilde{\mathcal{L}}_{X^c}\mathcal{H} = 0$,

(4) $\mathcal{L}_{X^c}\mathbf{h} = 0$.

Amennyiben (1)-(4) valamelyike - és így bármelyike fennáll, úgy

$$X^c N^i_j = N^k_j \left(\frac{\partial X^i}{\partial u^k} \circ \tau \right) - N^i_k \left(\frac{\partial X^k}{\partial u^j} \circ \tau \right) - y^k \left(\frac{\partial^2 X^i}{\partial u^j \partial u^k} \circ \tau \right),$$

ahol az $X^i \in C^\infty(\mathcal{U})$ függvények X komponensei M egy $(\mathcal{U}, (u^i)_{i=1}^n)$ térképére vonatkozóan, (N^i_j) pedig $\mathcal{H}$ Christoffel-szimbólumainak családja a TM-en indukált $(\tau^{-1}(\mathcal{U}), ((x^i)_{i=1}^n, (y^i)_{i=1}^n))$ térképre vonatkozóan.

19.2.3 Lie-szimmetriák Legyen S az M sokaság fölötti szemispray. Egy $X \in \mathfrak{X}(M)$ vektormező Lie-szimmetriája S-nek, ha S invariáns X^c lokális folyamával szemben, azaz, $(\varphi_t)_{**} \circ S = S \circ (\varphi_t)_*$ minden szóbajövő t-re, ahol (φ_t) X lokális folyama. Ekkor azt írjuk, hogy $X \in \mathrm{Lie}_S(M)$. A klasszikus Lie-derivált dinamikai interpretációjából azonnal látható, hogy

$$X \in \mathrm{Lie}_S(M) \iff [X^c, S] = 0.$$

Amennyiben $\mathcal{H}$ az S által indukált Ehresmann-konnexió, úgy $\mathrm{Lie}_S(M) \subset \mathrm{Kill}_{\mathcal{H}}(M)$.

3. Tétel *Legyen (M, S) spray-sokaság, ellátva a $\mathcal{H}$ Berwald-konnexióval és a $\mathcal{H}$ által indukált ∇ Berwald-deriválttal. Az M sokaság egy X vektormezőjére a következők ekvivalensek:*

(1) $X \in \mathrm{Lie}_S(M)$,	(6) $\mathcal{L}_{X^c}\mathbf{v} = 0$,
(2) $[X^c, S] = 0$,	(7) $\widetilde{\mathcal{L}}_{X^c}\nabla = 0$,
(3) $X \in \mathrm{Kill}_{\mathcal{H}}(M)$,	(8) $[X^c, Y^h] = [X, Y]^h$,
(4) $\widetilde{\mathcal{L}}_{X^c}\mathcal{H} = 0$,	(9) $[\widetilde{\mathcal{L}}_{X^c}, \widetilde{\mathcal{L}}_{Y^h}] = \widetilde{\mathcal{L}}_{[X,Y]^h}$,
(5) $\mathcal{L}_{X^c}\mathbf{h} = 0$,	(10) $\widetilde{\mathcal{L}}_{X^c} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{L}_{X^c}$.

Itt (8)-ban és (9)-ben $Y \in \mathfrak{X}(M)$ tetszőleges. Megjegyezzük, hogy (1), (5) és (7) ekvivalenciáját korábban Lovas Rezső már igazolta, ld. [17].

19.2.4 Görbületi kollineációk **(A)** Egy (M, S) spray-sokaság Jacobi endomorfizmusa (vagy affin elhajlási tenzora), fundamentális affin görbülete és affin görbülete rendre az a $\mathbf{K}$, $\mathbf{R}$, és $\mathbf{H}$ Finsler tenzormező, amelyet a

$$\mathbf{K}(\widetilde{X}) := \mathcal{V}[S, \mathcal{H}(\widetilde{X})], \quad \mathbf{R}(\widetilde{X}, \widetilde{Y}) := \frac{1}{3}(\nabla^v\mathbf{K}(\widetilde{X}, \widetilde{Y}) - \nabla^v\mathbf{K}(\widetilde{Y}, \widetilde{X})),$$
$$\mathbf{H}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := -\nabla^v\mathbf{R}(\widetilde{Z}, \widetilde{X}, \widetilde{Y})$$

előírás értelmez. Ha $\mathbf{C} \in \{\mathbf{K}, \mathbf{R}, \mathbf{H}\}$ és $\widetilde{\mathcal{L}}_{X^c}\mathbf{C} = 0$, akkor azt mondjuk, hogy X *görbületi kollineációja* $\mathbf{C}$-nek.

4. Tétel *Egy $X \in \mathfrak{X}(M)$ vektormező pontosan akkor görbületi kollineációja az (M, S) spray-sokaság Jacobi endomorfizmusának, ha invariáns X (φ_t) lokális folyamával szemben, abban az értelemben, hogy*

$$((\varphi_t)_* \times (\varphi_t)_*) \circ \mathbf{K} = \mathbf{K} \circ ((\varphi_t)_* \times (\varphi_t)_*),$$

minden lehetséges valós t-re. (Itt $\mathbf{K}$-t a $\mathring{\pi}$ vektornyaláb erős nyalábendomorfizmusaként interpretáljuk.)

5. Tétel *Ha $X \in \mathrm{Lie}_S M$, akkor X görbületi kollineációja a $\mathbf{K}$, $\mathbf{R}$ és $\mathbf{H}$ tenzoroknak.*

(B) Egy, az S sprayből konstruált Finsler tenzormező projektíven invariáns, ha nem változik az S spray $S \rightsquigarrow S - 2PC$, $\quad P \in C^\infty(\mathring{T}M)$

projektív változtatásai során. Egy (M, S) spray-sokaság alapvető projektíven invariáns tenzorai a $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ Weyl-tenzorok és a $\mathbf{D}$ Douglas tenzor. Ezek definíciói rendre a következők:

$$\mathbf{W_1} := \mathbf{K} - K\, 1_{\Gamma(\overset{\circ}{\pi})} - \frac{1}{n+1}(\mathrm{tr}\nabla^{\mathsf{v}}\mathbf{K} - \nabla^{\mathsf{v}}K) \otimes \widetilde{\delta} \quad (K := \frac{1}{n-1}\mathrm{tr}\mathbf{K}),$$

$$\mathbf{W_2}(\widetilde{X}, \widetilde{Y}) := \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{W_1}(\widetilde{X}, \widetilde{Y}) - \nabla^{\mathsf{v}}\, \mathbf{W_1}(\widetilde{Y}, \widetilde{X})),$$

$$\mathbf{W_3}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := \nabla^{\mathsf{v}}\mathbf{W_2}(\widetilde{Z}, \widetilde{X}, \widetilde{Y}),$$

$$\mathbf{D} := \mathbf{B} - \frac{1}{n-1}((\nabla^{\mathsf{v}}\mathrm{tr}\mathbf{B}) \otimes \widetilde{\delta} + (\mathrm{tr}\mathbf{B}) \odot 1_{\Gamma(\overset{\circ}{\pi})}).$$

Az utolsó formulában $\mathbf{B}$ a spray-sokaság Berwald-tenzora, amely megadható a $\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} := (\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{Z})(\widehat{X}, \widehat{Y})$ előírással, a $\odot$ szimbólum pedig numerikus faktor nélküli szimmetrikus szorzatot jelöl.

6. Tétel *Ha* $X \in \mathrm{Lie}_S(M)$, *akkor* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W}_i = 0, \quad i \in \{1, 2, 3\}$.

7. Tétel *Ha* $X \in \mathrm{Lie}_S(M)$, *akkor* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{B} = 0$, *és ebből következően* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{D} = 0$.

19.2.5 Geometriai vektormezők Finsler-sokaságokon Egy $F\colon TM \to \mathbb{R}$ pozitív, folytonos függvény M fölötti Finsler-függvény, ha $\overset{\circ}{T}M$-en sima, 1^+-homogén és a

$$g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 =: \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E$$

alaptenzor (fibrumonként) nemelfajuló. Egy Finsler-sokaság olyan (M, F) pár, amelyet egy M sokaság és egy M fölötti Finsler-függvény alkot. Néhány fontosabb adata:

(1) $\theta_g := \nabla^{\mathsf{v}}E$ vagy $\theta_E := d_{\mathbf{J}}E = \theta_g \circ \mathbf{j}$ – (M, F) Hilbert 1-formája.

(2) $\omega_E := d\theta_E = dd_{\mathbf{J}}E$ – (M, F) fundamentális 2-formája.

(3) $w := \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\omega_E \wedge \cdots \wedge \omega_E$ (n tényező) – a Dazord-féle térfogati forma $\overset{\circ}{T}M$-en.

(4) (M, F) kanonikus spray-je az az S spray, amelyet $\overset{\circ}{T}M$ fölött az $i_S dd_{\mathbf{J}}E = -dE$ feltétel határoz meg. (M, F) $\mathcal{H}$-val jelölt kanonikus konnexiója az (M, S) spray-sokaság Berwald-konnexiója; ∇ a kanonikus konnexió által indukált Berwald-derivált.

(5) $\overset{\circ}{T}M$-en a $g^S(\xi, \eta) := g(\mathbf{j}\xi, \mathbf{j}\eta) + g(\mathcal{V}\xi, \mathcal{V}\eta)$ előírással értelmezett Riemann-metrika a Sasaki-Finsler metrika.

(6) $\mathcal{C}_\flat := \nabla^{\mathsf{v}}g = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E$ a Finsler-sokaság Cartan-tenzora; $\mathcal{C}$ a vele

metrikusan ekvivalens $(1,2)$-típusú tenzor, amelyet a $g(\mathcal{C}(\widetilde{X},\widetilde{Y})\widetilde{Z}) = \mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z})$ formula értelmez.
(7) $\mathbf{L}_\flat := \nabla^{\mathsf{h}} g = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} E$ a Landsberg-tenzor; $\mathbf{L}$ a vele metrikusan ekvivalens $(1,2)$-típusú tenzor, amelyet a $g(\mathbf{L}(\widetilde{X},\widetilde{Y})\widetilde{Z}) := \mathbf{L}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z})$ formula ad meg.
(8) D^C, D^{Ch} és D^{Hs} rendre a Cartan, a Chern-Rund és a Hashiguchi-derivált (M,F)-en. Értelmezésük:

$$D^C_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi,\widetilde{Y}) + \frac{1}{2}\mathbf{L}(\mathbf{j}\xi,\widetilde{Y}),$$
$$D^{Ch}_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathbf{L}(\mathbf{j}\xi,\widetilde{Y}), \quad D^{Hs}_\xi \widetilde{Y} := \nabla_\xi \widetilde{Y} + \frac{1}{2}\mathcal{C}(\mathcal{V}\xi,\widetilde{Y}).$$

Definíciók: Legyen $X \in \mathfrak{X}(M)$, és legyen (φ_t) X lokális folyama. Az X vektormező *Killing vektormezője* (M,F)-nek, ha a φ_t transzformációk megőrzik az érintőnyalábok Finsler normáját, azaz $F \circ (\varphi_t)_* = F$ minden szóbajövő t-re. Ha

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g = \sigma g, \text{ ahol } \sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M),$$

akkor azt mondjuk, hogy X *konform vektormező*, amelynek a *konform függvénye* σ. Ha a konform függvény konstans, *homotetikus vektormezőről* beszélünk. Az X vektormező *projektív vektormezője* (M,F)-nek, ha

$$[X^{\mathsf{c}}, S] = \varphi C, \quad \varphi \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M).$$

Jelölés: $\mathrm{Kill}_F(M)$, $\mathrm{Conf}_F(M)$, $\mathrm{Dil}_F(M)$ és $\mathrm{Proj}_F(M)$ rendre (M,F) Killing-, konform, homotetikus és projektív vektormezőinek halmaza.

8. Tétel (a) *Tetszőleges $X \in \mathfrak{X}(M)$ vektormező esetén*
(i) $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_g) \circ \mathbf{j} = \mathcal{L}_{X^{\mathsf{c}}}\omega_E$;
(ii) $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g)(\mathbf{j}\xi,\mathbf{j}\eta) = (\mathcal{L}_{X^{\mathsf{c}}}\omega_E)(\mathbf{J}\xi,\eta)$;
(iii) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat = \nabla^{\mathsf{v}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g)$;
(iv) $g((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C})(\widehat{Y},\widehat{Z}),\widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat)(\widehat{Y},\widehat{Z},\widehat{U}) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g)(\mathcal{C}(\widehat{Y},\widehat{Z}),\widehat{U})$;
(v) $g((\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L})(\widehat{Y},\widehat{Z}),\widehat{U}) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat)(\widehat{Y},\widehat{Z},\widehat{U}) - (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g)(\mathbf{L}(\widehat{Y},\widehat{Z}),\widehat{U})$.

(b) *Ha $X \in \mathrm{Lie}_S(M)$, akkor $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat = \nabla^{\mathsf{h}}(\mathcal{L}_{X^{\mathsf{c}}} g)$.*
(c) *Ha $X \in \mathrm{Kill}_F(M)$, akkor*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C}_\flat = 0, \ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{C} = 0, \quad \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L}_\flat = 0, \ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{L} = 0.$$

9. Tétel *Ha $X \in \mathrm{Kill}_F(M)$ és $D \in \{\nabla, D^C, D^{Ch}, D^{Hs}\}$, akkor $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} D = 0$.*

10. Tétel (a) *Ha* $X \in \mathrm{Conf}_F(M)$, *akkor* X *konform függvénye vertikális lift.* (*b*) *Egy* $X \in \mathfrak{X}(M)$ *vektormezőre a következők ekvivalensek:*
(i) $X \in \mathrm{Conf}_F(M)$,
(ii) $X^{\mathsf{c}}E = \sigma E$,
(iii) $\mathcal{L}_{X^{\mathsf{c}}}\theta_E = \sigma\theta_E$,
(iv) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\theta_g = \sigma\theta_g$,
(v) $\mathcal{L}_{X^{\mathsf{c}}}\omega_E = f^{\mathsf{v}}\omega_E + df^{\mathsf{v}} \wedge d_{\mathbf{J}}E, \quad f \in C^\infty(M)$.
Az (ii)-(iv) feltételekben $\sigma \in C^0(TM) \cap C^\infty(\overset{\circ}{T}M)$.

11. Tétel $X \in \mathrm{Conf}_F(M) \cap \mathrm{Lie}_S(M) \Longrightarrow X^{\mathsf{c}} \in \mathrm{Conf}_{g^s}(\overset{\circ}{T}M)$,

$$X^{\mathsf{c}} \in \mathrm{Conf}_{g^S}(\overset{\circ}{T}M) \Longrightarrow X \in \mathrm{Conf}_F(M).$$

12. Tétel $X \in \mathrm{Dil}_F(M) \Rightarrow X \in \mathrm{Lie}_S(M)$.

13. Tétel $X \in \mathrm{Proj}_F(M) \cap \mathrm{Conf}_F(M) \Rightarrow X \in \mathrm{Dil}_F(M)$.

14. Tétel $(X \in \mathrm{Proj}_F(M)$ *és* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\, w = 0) \Rightarrow X \in \mathrm{Lie}_S(M)$.

15. Tétel $(X \in \mathrm{Conf}_F(M)$ *és* $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} w = 0) \Rightarrow X \in \mathrm{Kill}_F(M)$.

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