1949

On stretch Finsler metrics and six-dimensional filiform nilmanifolds

Dissertation for the Degree of Doctor of Philosophy (PhD)

by
Sameer Annon Abbas

Supervisor: Dr. László Kozma, co-supervisor: Dr. Ágota Figula

University of Debrecen

Doctoral Council of Natural Sciences and Information Technology

Doctoral School of Mathematical and Computational Sciences

Debrecen, 2024

Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences of the University of Debrecen in order to obtain a PhD Degree in Natural Sciences from the University of Debrecen.

I declare that the results published in this thesis are not reported in any other PhD theses.

Debrecen, 2024.

. . . .
Sameer Annon Abbas
doctoral candidate

Hereby I confirm that the candidate Sameer Annon Abbas conducted his studies with my supervision within the Differential Geometry and its Applications Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen between 2020 and 2024. The independent studies and research work of the candidate significantly contributed to the results published in this thesis. I also declare that the results published in the thesis are not reported in any other PhD theses.

I support the acceptance of the dissertation.
Debrecen, 2024.

László Kozma
supervisor

Ágota Figula
co-supervisor

On stretch Finsler metrics and six-dimensional filiform nilmanifolds

Dissertation submitted in partial fulfilment of the requirements for the doctoral (PhD) degree in mathematics and computing.

Written by: Sameer Annon Abbas certified mathematician

Prepared in the framework of the Differential Geometry and its Applications program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen.

Dissertation supervisor: Dr. László Kozma (sup.), Dr. Ágota Figula (co-sup.)

The official opponents of the dissertation:

Dr. ....
Dr. ....

The evaluation committee:

chairperson:	Dr.	....
members:	Dr.	....
	Dr.	....
	Dr.	....
	Dr.	....

The date of the dissertation defence: .... 2024.

Acknowledgments

I am deeply grateful to my supervisors, Dr. László Kozma and Dr. Ágota Figula. I could not have accomplished this work without their mentoring and support. Thank you very much.

I would like to express my gratitude to the whole academic staff of the Institute of Mathematics at the University of Debrecen. Particularly the academic staff of the Department of Geometry, which had served as a second home for the last four years. Thank you very much.

I would like to thank my mom, my brothers, and my sisters for their continuous support during the years. Thank you very much.

Contents

Introduction

This thesis addresses some investigations of Finsler geometry and homogeneous Riemannian nilmanifolds.

Finsler geometry is a relatively recent area of geometric study when compared to its basis in differential geometry. Riemann hinted in [40] a remark at generalized case of Riemannian metrics, which later labeled Finsler metric and denoted by F. Along the time lots mathematicians aimed to adjust mathematical tools which were effective in Riemannian geometry such as the theory of connections, Jacobi vector fields, sectional curvature to a more general one. In 1918, Finsler devoted his Ph.D. dissertation to clear the way to start that approach in the field of Finsler Geometry (see [23]). In Finsler geometry, there exists many geometric objects that can not exist for a Riemannian manifold such as the Cartan torsion **C**, the Berwald curvature **B**, and the Landsberg curvature **L**, etc. These are said to be non-Riemannian quantities because they vanish when F is Riemannian. In Riemannian geometry, one of the most fundamental concepts is the Riemannian curvature tensor. It defines how the manifold is curved, or how it differs from a Euclidean space. In Finsler geometry, it can be expressed as a $(1, 1)$-tensor over the pullback bundle π^*TM.

In [8], L. Berwald presented the concept of the stretch curvature tensor and showed that it vanishes if and only if a vector's length stays constant under the parallel displacement along an infinitesimal parallelogram. Then, the stretch curvature was studied by C. Shibata in 1978 [42], and M. Matsumoto in 1997 and 2004 (see [33, 34]). Shibata proved that a Kropina metric is a Berwald metric if and only if the stretch curvature tensor vanishes under the condition that the Kropina metric is positive-definite. Matsumoto aimed to provide a fresh understanding of the stretch curvature within the framework of modern Finsler connection theory. S. Bácsó and M. Matsumoto proved that a Douglas metric with vanishing stretch curvature is R-quadratic if and only if its **E**-curvature vanishes (see [7]). It is interesting to find conditions under which Douglas metrics with vanishing stretch curvature reduce to Berwald metrics. A. Tayebi and T. Tabatabaeifar in 2015 [49] proved that every Douglas-Randers metric with vanishing stretch curvature is a Berwald metric. B. Najafi and A. Tayebi in 2017 [37] introduced a new non-Riemannian quantity named *mean stretch curvature*. A Finsler metric with vanishing mean stretch curvature is called *weakly stretch metric*. This class of Finsler metrics contains the class of stretch metrics. In 2018, they showed that every complete P-reducible weakly stretch metric with bounded Cartan torsion is a Landsberg metric. Furthermore, they classified complete weakly stretch surfaces and showed that every complete weakly stretch surface is Riemannian or Landsbergian [38].

In [46] A. Tayebi and B. Najafi showed that every homogeneous (α, β)-metric is a stretch metric if and only if it is a Berwald metric. A. Tayebi and H. Sadeghi characterized the stretch (α, β)-metrics with vanishing S-curvature, more precisely they proved a regular non-Randers type (α, β)-metric with vanishing S-curvature is stretchian if and only if it is Berwaldian (see [48]). In [45] A. Tayebi N. Izadian proved that every Douglas-square metric is a Berwald metric if and only if it is a weakly stretch metric. In [44] with the authors M. Faghfuri and N. Jazer proved that every compact Finsler manifold with posi-tive (or negative) relatively isotropic mean stretch curvature is a weakly Landsberg metric. In [47] A. Tayebi with B. Najafi classified the almost regular weakly stretch non-Randers-type (α, β)-metrics with vanishing S-curvature. In [43] with M. Bahadori and H. Sadeghi proved that a spherically symmetric Finsler metric is a stretch metric if and only if it is R-quadratic. In [27] with F. Kamelaei and B. Najafi proved that every homogeneous Finsler metric has relatively isotropic stretch curvature if and only if it is a Landsberg metric. Recently, some properties of the weakly stretch Finsler metric of the special Finsler metric were investigated (see[13, 50]).

The contribution of the first chapter of the dissertation is to use of the Berwald curvature instead of the Cartan torsion, and the investigation of the relationships among the classes obtained analogously to the Landsberg and the stretch curvatures. This will enhance the understanding of the role of the relevant tensors in characterizing the new classes of Finsler metrics.

Z. Shen [41, page 139] introduced a non-Riemannian quantity $\widetilde{\mathbf{B}}$ which is obtained from the Berwald curvature $\mathbf{B}$ by the covariant horizontal differentiation along Finslerian geodesics. For a vector $y \in \mathcal{T}_pM$, define $\widetilde{\mathbf{B}}_y : T_pM \times T_pM \times T_pM \to T_pM$ by $\widetilde{\mathbf{B}}_y(u,v,w) := \widetilde{B}^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i} \mid_x$, where

$$\widetilde{B}^i_{jkl} := B^i_{jkl|m} y^m.$$

The Finsler metric F is called $\widetilde{\boldsymbol{B}}$*-metric* if and only if $\widetilde{\mathbf{B}} = 0$.

The goal of section 1.2 is to use the quantity $\widetilde{\mathbf{B}}$ to study a class of Finsler metrics containing the class of Berwald metric. The quantity resulted from $\widetilde{\mathbf{B}}$ has the following form

$$\mathcal{K}^i_{jklm} := 2\left(\widetilde{B}^i_{jkl|m} - \widetilde{B}^i_{jkm|l}\right).$$

The family $\boldsymbol{K} := \{\boldsymbol{K}_y : y \in \mathcal{T}_pM\}$ is called the *stretch* $\widetilde{\boldsymbol{B}}$*-curvature.* A Finsler metric F is said to be $\widetilde{\boldsymbol{B}}$*-stretch metric* if and only if $\boldsymbol{K} = 0$, where $\boldsymbol{K}_y(u,v,w,z) := \mathcal{K}^i_{jklm}(y)u^j v^k w^l z^m \frac{\partial}{\partial x^i} \mid_x$. It is interesting to find some curvature properties conditions under which a $\widetilde{\mathbf{B}}$-stretch metric reduces to a $\widetilde{\mathbf{B}}$-metric (see Theorems 1.4, 1.5, 1.7, 1.8).

The aim of section 1.3 is to study a class of Finsler metrics that includes the class of Douglas metrics. Finsler metrics in this class are known as generalized Douglas metrics. We prove that every generalized Douglas metric with vanishing $\widetilde{\mathbf{B}}$-stretch tensor is a Douglas metric under the condition that the mean Berwald curvature is horizontally constant along geodesics of F. Then, we show that if (M, F) is a Douglas Finsler manifold then the Finsler metric F is an $\mathbf{H}$-stretch if and only if it is a $\widetilde{\mathbf{B}}$-metric. The results of this section are introduced in Theorems 1.20, 1.22, 1.23.

The second chapter of the dissertation is devoted to investigate homogeneous Riemannian nilmanifolds. Denote by M a connected Riemannian manifold and by $\mathcal{I}(M)$ the Lie group of its isometries. If $\mathcal{I}(M)$ contains a nilpotent Lie subgroup acting transitively on M, then M is called a *homogeneous Riemannian nilmanifold.* E. N. Wilson proved that there exists a unique nilpotent simply transitive normal subgroup N of $\mathcal{I}(M)$ (see [52, Theorem 2]). Therefore the nilmanifold M can be treated as the nilpotent Lie group N and the metric on M induces a left invariant metric $\langle .,. \rangle_N$ of N. Let $\mathfrak{n}$ be the Lie algebra of N. The metric on N determines a Euclidean inner product $\langle ., . \rangle$ on $\mathfrak{n}$ and conversely. A metric Lie algebra is a Lie algebra endowed with a Euclidean inner product. We denote by $\mathcal{OA}(\mathfrak{n})$ the group of automorphisms of the Lie algebra $\mathfrak{n}$ preserving the inner product $\langle ., . \rangle$ on $\mathfrak{n}$ and we call it the group of orthogonal automorphisms. The isometry group $\mathcal{I}(N)$ of $(N, \langle ., . \rangle_N)$ is the semi-direct product $N \rtimes \mathcal{OA}(\mathfrak{n})$ of the group N and the group $\mathcal{OA}(\mathfrak{n})$. As a result, there exists a one-to-one correspondence between isometry equivalence classes of homogeneous connected and simply connected Riemannian nilmanifolds, and the classes of isometrically isomorphic metric nilpotent Lie algebras (cf. [52, Theorem 3]). Hence to find the isometry equivalence classes of nilmanifolds is useful to work on nilpotent metric Lie algebras. A nilpotent Lie group is called two-step nilpotent if the lower central series of its Lie algebra has one non-trivial subalgebra. In this sense two-step nilpotent Lie algebras are non-abelian Lie algebras which are one step away from being abelian. The connected and simply connected two-step Riemannian nilmanifolds of dimension at most 6 were intensively studied applying two-step nilpotent metric Lie algebras. Their isometry equivalence classes and isometry groups are determined in [14, 26, 31]. In [22, 31] the classification of the at most 5-dimensional Riemannian nilmanifolds of higher nilpotency class and the determination of their isometry groups carried out successfully by working on metric nilpotent Lie algebras of nilpotency class greater than 2.

Among nilpotent Lie groups with higher nilpotency class the filiform Lie groups play an essential role. An n-dimensional filiform Lie algebra has the maximal possible nilpotency class $n-1$. That is its lower central series has $n-2$ non-trivial subalgebra. The 3-dimensional non-abelian nilpotent Lie algebra, the Heisenberg Lie algebra, is two-step nilpotent. At the same time it is the unique filiform Lie algebra of dimension 3. One way to generalize the Heisenberg Lie algebra leads to the notion of the Heisenberg type Lie algebras (cf. [28]). These are two-step nilpotent Lie algebras but the dimension of the top step is enlarged. The n-dimensional filiform Lie algebras are another generalization of the Heisenberg Lie algebra. In this case the number of steps grows up as the dimension increases (cf. [29], p. 2). In [22] the isometry equivalence classes and the isometry groups of connected and simply connected filiform Riemannian nilmanifolds of arbitrary dimension were thoroughly studied. It turns out that every filiform metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ has a decomposition into orthogonal direct sum of 1-dimensional subspaces which is preserved by all orthogonal automorphisms of $(\mathfrak{n}, \langle ., . \rangle)$. Hence the isometry groups of the filiform nilmanifolds have the same dimension as the nilmanifolds (cf. [22, Corollary 5]). In particular the detailed description for the isometrically isomorphic equivalence classes of standard filiform metric Lie algebras and the isometry groups of the corresponding nilmanifolds are given in [22, Theorem 7, Corollary 8]. Using this approach in section 2.2 we make analogous consideration for the 6-dimensional connected simply connected filiform nilmanifolds. Applying the classification procedure given by [22, pp. 371-372], we find in Theorems 2.10, 2.13, 2.16, and 2.19 the classes of isometrically isomorphic 6-dimensional filiform metric Lie algebras and the group of their orthogonal automorphisms. To receive these results we proceed as follows. Throughout this section we denote by $\mathbb{E}^6$ a 6-dimensional Euclidean vector space equipped with a distinguished orthonormal basis $\mathcal{E} = \{E_1, E_2, \cdots, E_6\}$.

We use the list of W. A. de Graaf in [16, pp. 646-647] to fix a basis $(G_1, \cdots, G_6)$ of the non-isomorphic 6-dimensional filiform Lie algebras $\mathfrak{l}_{6,k}$, $k = 14, \ldots, 18$ such that their Lie brackets have the simple form (2.3).

Applying the Gram-Schmidt process to the ordered basis $(G_6, G_5, \cdots, G_1)$ in the metric Lie algebra $(\mathfrak{l}, \langle ., . \rangle)$ we receive an orthonormal basis $\mathcal{F} = \{F_1, F_2, \cdots, F_6\}$ such that $F_i = \sum_{k=i}^{6} a_{ik} G_k$, where $a_{ik} \in \mathbb{R}$, $a_{ii} > 0$. Conversely, each basis $\mathcal{F} = \{F_1, F_2, \cdots, F_6\}$ of $\mathfrak{l}$ with the shape $F_i = \sum_{k=i}^{6} a_{ik} G_k$, $a_{ik} \in \mathbb{R}$, $a_{ii} > 0$ becomes an orthonormal basis for an inner product on the Lie algebra $\mathfrak{l}$. The inner products on Lie algebra $\mathfrak{l}$ are determined by these bases.

We compute the structure coefficients of the Lie brackets of the metric Lie algebra $(\mathfrak{l}, \langle ., . \rangle)$ with respect to its basis $\mathcal{F}$ and we define a Lie bracket on $\mathbb{E}^6$ with the same structure coefficients with respect to its distinguished basis $\mathcal{E}$. The received metric Lie algebra on $\mathbb{E}^6$ depends on real parameters $\alpha_i > 0$, β_j and it is isometrically isomorphic to $(\mathfrak{l}, \langle ., . \rangle)$.

Finally, we find conditions on the real parameters $\alpha_i > 0$, β_j, of metric Lie algebras on $\mathbb{E}^6$ to obtain a one-to-one correspondence between the equiva-

lence classes of isometrically isomorphic metric Lie algebras and a family of metric Lie algebras on $\mathbb{E}^6$.
Using the above steps the 6-dimensional filiform metric Lie algebras are isometrically isomorphic to one of the metric Lie algebras given by the following non-trivial Lie brackets with respect to an orthonormal basis $\mathcal{E}$ on $\mathbb{E}^6$: (see [21]):

The Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$

$$\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_2, E_3] = \alpha_3 E_5 + \beta_6 E_6,\\
&[E_1, E_3] = \alpha_2 E_4 - \Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\Big) E_5 + \beta_4 E_6, && [E_2, E_4] = \beta_7 E_6,\\
&[E_1, E_4] = \frac{\alpha_1\alpha_5}{\alpha_4} E_5 + \beta_5 E_6, && [E_2, E_5] = \alpha_4 E_6,\\
& && [E_4, E_3] = \alpha_5 E_6,
\end{aligned} \tag{1}$$

such that $\alpha_i > 0, i = 1, \dots, 5$, $\beta_j \in \mathbb{R}, j = 1, \dots, 7$, and if the set $J = \{j \in \{1, 4, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$,

The Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$

$$\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_2, E_3] = \frac{\alpha_2\alpha_5}{\alpha_4} E_5 + \beta_7 E_6,\\
&[E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, && [E_2, E_4] = \alpha_5 E_6.\\
&[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, && \\
&[E_1, E_5] = \alpha_4 E_6, &&
\end{aligned} \tag{2}$$

such that $\alpha_i > 0, i = 1, \dots, 5$, $\beta_j \in \mathbb{R}, j = 1, \dots, 7$, and if the set $J = \{j \in \{1, 3, 4, 6, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$,

The Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$

$$\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_2, E_3] = \beta_7 E_6,\\
&[E_1, E_3] = \alpha_2 E_4 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) E_5 + \beta_4 E_6, && [E_2, E_4] = \beta_8 E_6,\\
&[E_1, E_4] = \alpha_3 E_5 + \beta_5 E_6, && [E_2, E_5] = \alpha_4 E_6,\\
&[E_1, E_5] = \beta_6 E_6, && [E_4, E_3] = \frac{\alpha_3\alpha_4}{\alpha_1} E_6.
\end{aligned} \tag{3}$$

such that $\alpha_i > 0, i = 1, \ldots, 4$, $\beta_j \in \mathbb{R}, j = 1, \ldots, 8$, and one of the following cases is satisfied:

1. $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_8 = 0$,
2. $\beta_3 > 0$ or $\beta_5 > 0$, $\beta_1 = \beta_4 = \beta_6 = \beta_8 = 0$,
3. $\beta_6 > 0$ or $\beta_4 > 0$, $\beta_1 = \beta_3 = \beta_5 = \beta_8 = 0$,
4. $\beta_1 > 0$ or $\beta_8 > 0$, $\beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$,
5. at least two elements of the set $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8\}$ are positive with the exceptions $(\beta_1 > 0, \beta_8 > 0)$, $(\beta_3 > 0, \beta_5 > 0)$, $(\beta_4 > 0, \beta_6 > 0)$,

The Lie algebra $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$

$$\begin{aligned} &[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, \\ &[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, && [E_1, E_5] = \alpha_4 E_6, \\ &[E_2, E_3] = \alpha_5 E_6, && \end{aligned} \tag{4}$$

such that $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}, j = 1, \ldots, 6$ and if the set $J = \{j \in \{1, 3, 4, 6\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. Moreover, the Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ is defined by the same Lie brackets such that the parameter α_5 is missing.

The group of all isometries of the corresponding connected and simply connected filiform nilmanifolds are given in Corollaries 2.11, 2.14, 2.17, 2.20, 2.21.

The purpose of sections 2.3, 2.4 is to study the totally geodesic subgroups of connected simply connected 6-dimensional filiform Riemannian nilmanifolds. The Levi-Civita connection of the metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is defined by

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle)$$

for left invariant vector fields $X, Y, Z \in \mathfrak{n}$. A subalgebra $\mathfrak{h}$ of $(\mathfrak{n}, \langle ., . \rangle)$ is totally geodesic if for all $X, Y \in \mathfrak{h}$ one has $\nabla_X Y \in \mathfrak{h}$. A subgroup H of $(N, \langle ., . \rangle_N)$ which passes through the identity $e \in N$ is totally geodesic precisely if the corresponding subalgebra $\mathfrak{h}$ is totally geodesic. The left cosets

xH, $x \in N$, with respect to a totally geodesic subgroup H give a totally geodesic foliation on N. The investigation of the geometry of two-step nilmanifolds began with the works [18, 19] of P. Eberlein. He studies curvatures and totally geodesic subgroups in non-singular two-step nilmanifolds. His results were generalized to the case where N does not have the non-singularity condition in [17]. It is also discussed in [17, 19] criteria under which a subalgebra of a two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is totally geodesic. The work [36] is devoted to investigate totally geodesic subalgebras in two-step nilpotent metric Lie algebras. P. T. Nagy and S. Homolya proved there that for isomorphic Lie algebras $\mathfrak{n}$ and $\mathfrak{n}^*$ there exists a bijective linear map $\mathfrak{n} \to \mathfrak{n}^*$ preserving the flat totally geodesic property of subalgebras. Although the totally geodesic property of subalgebras is very sensitive with respect to the change of the inner product of a metric Lie algebra their outstanding result shows that the linear structure of totally geodesic subalgebras of two-step nilpotent metric Lie algebras depends only on the isomorphism class of the Lie algebra. The generating vector of a one-dimensional totally geodesic subalgebra is called geodesic. They also determined geodesic vectors and flat totally geodesic subalgebras in two-step nilpotent metric Lie algebras of dimension less than or equal to 6. In [29] M. M. Kerr and T. L. Payne extended the study of the geometry of two-step nilmanifolds for two infinite families of nilmanifolds belonging to filiform nilpotent Lie algebras with special inner products. Their paper is an initial source to study curvatures and totally geodesic subgroups of filiform nilmanifolds.

The continuation of their studies was presented by G. Cairns, A. Hinić Galić and Y. Nikolayevsky in [11, 12]. There the authors gave several results on the possible dimensions of totally geodesic subalgebras of nilpotent metric Lie algebras. They also found examples, where the obtained bounds on the dimensions of totally geodesic subalgebras are attained. Furthermore, they gave an example of a 6-dimensional filiform nilpotent Lie algebra which does not allow any totally geodesic subalgebra of dimension greater than 2. Using the classification of the non two-step nilpotent metric Lie algebras of dimension at most 5 given in [36], A. Al-Abayechi and Á. Figula determined the geodesic vectors and flat totally geodesic subalgebras in these metric Lie algebras. They found that the flat totally geodesic subalgebras and geodesic vectors in non-filiform metric Lie algebras with one-dimensional center are independent of the choice of the inner product (see [5]).

The aim in section 2.3 is to determine the sets of the geodesic vectors and hence the one-dimensional totally geodesic subalgebras in the 6-dimensional filiform metric Lie algebras. Applying the results (1), (2), (3), (4) describing the classes of isometrically isomorphic 6-dimensional filiform metric Lie algebras and the claim that a non-zero vector $Y \in (\mathfrak{n}, \langle ., . \rangle)$ is geodesic precisely if for all $X \in (\mathfrak{n}, \langle ., . \rangle)$ one has $\langle [X, Y], Y \rangle = 0$, in Theorems 2.22, 2.23, 2.24, 2.25 the set of the geodesic vectors is decomposed as the disjoint union of subsets depending on the parameters α_i, β_j. In section 2.4 we deal with the question under which conditions on the parameters α_i, β_j exists a flat totally geodesic subalgebra of dimension greater than 1 in the class $\mathcal{C}$ of the six-dimensional filiform metric Lie algebras. It follows from [12, Proposition 1.13] that for filiform nilpotent metric Lie algebras there does not exist any totally geodesic subalgebra of codimension one. From [12, Theorems 2.17, 2.18] it turns out that only metric Lie algebras corresponding to the standard filiform Lie algebra can allow a flat totally geodesic subalgebra of codimension 2. Some filiform Lie algebras possess only low dimensional totally geodesic subalgebras independently of the choice of inner product.

Our investigation shows that in the class $\mathcal{C}$ with the exception of the metric Lie algebras corresponding to the standard filiform Lie algebra the flat totally geodesic subalgebras of all metric Lie algebras have dimension at most two (see Theorem 2.26). A metric Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ defined on $\mathbb{E}^6$ by the Lie brackets (1) allow a 2-dimensional flat totally geodesic subalgebra if and only if either the parameters β_3, β_4 and β_5 are 0 or the parameters β_1 and β_7 are 0. In the former case the flat totally geodesic subalgebra is spanned by the orthonormal basis vectors E_1 and E_6, whereas in the later case the orthonormal basis vectors E_2 and E_4 generate the 2-dimensional flat totally geodesic subalgebra (see Theorem 2.27). According to Theorem 2.28 a metric Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ defined on $\mathbb{E}^6$ by the Lie brackets (2) possesses a 2-dimensional flat totally geodesic subalgebra precisely if one has $\beta_5 = \beta_7 = 0$. In this case the flat totally geodesic subalgebra is the subspace span(E_3, E_6). For a metric Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$, respectively $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ defined on $\mathbb{E}^6$ by the Lie brackets (3), respectively (4) the situation is more complex, it is discussed in Theorems 2.29, 2.30. A metric Lie algebra corresponding to the 6-dimensional standard filiform Lie algebra has a 4-dimensional flat totally geodesic subalgebra if and only if the conditions $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1\alpha_3}{\alpha_2}$ are satisfied. In this case the subalgebra has either the form

$$\text{span}(E_2 - \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}E_6, \quad E_3, \quad E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5),$$

or

$$\text{span}(E_2, \quad E_3 - \frac{\alpha_2}{\alpha_3}E_5, \quad E_4, \quad E_6)$$

(see Theorem 2.36). The possibilities for the existence of 3- and 2-dimensional flat totally geodesic subalgebras in a metric Lie algebra corresponding to the 6-dimensional standard filiform Lie algebra are described in Theorem 2.36 cases 2 and 3.

Chapter 1

Stretch Finsler metrics

Let (M, F) be a Finsler manifold. The third-order derivative of $\left(\frac{1}{2}F_p^2\right)$ at $y \in \mathcal{T}_pM$ is called *Cartan torsion.* The Landsberg curvature $\mathbf{L}$ is a description of how Cartan torsion $\mathbf{C}$ along geodesics changes. A Landsberg metric is a Finsler metric that fulfills $\mathbf{L} = 0$. In 1926, L. Berwald [8] introduced a Finslerian quantity during the investigation on the generalization of Landsberg curvature. This quantity is called a *stretch curvature* and is denoted by Σ_y. Geometrically, a Finsler metric is stretch-type if and only if a vector's length stays constant under parallel displacement along an infinitesimal parallelogram.

The structure of this chapter is as follows:

In section 1.1, we listed the basic notations and definitions of Finsler geometry that will be needed in the later sections of this chapter.

In section 1.2, we study a class of Finsler metrics containing the class of Berwald (weakly Berwald) metric (respectively). A Finsler metric in this class is called $\widetilde{\mathbf{B}}$-stretch ($\mathbf{H}$-stretch) metric (respectively). First, we show that every complete $\widetilde{\mathbf{B}}$-stretch metric ($\mathbf{H}$-stretch metric) is a $\widetilde{\mathbf{B}}$-metric ($\mathbf{H}$-metric). Then we prove that every compact Finsler manifold with positive (negative) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature (stretch $\mathbf{H}$-curvature) is $\widetilde{\mathbf{B}}$-metric ($\mathbf{H}$-metric).

The aim of section 1.3 is to study a class of Finsler metrics that includes the class of Douglas metrics. Finsler metrics in this class are known as generalized Douglas metrics. We prove that every generalized Douglas metric with vanishing $\widetilde{\mathbf{B}}$-stretch tensor is a Douglas metric under the condition that the mean Berwald curvature is horizontally constant along geodesics of F. Then, we show that if (M, F) is a Douglas Finsler manifold then the Finsler metric F is an $\mathbf{H}$-stretch if and only if it is a $\widetilde{\mathbf{B}}$-metric.

1.1 Basic concepts and tools

This section reviews some important definitions and concepts of Finsler manifolds that are extensively used in next sections.

Throughout this chapter, M denotes a C^∞-smooth n-dimensional manifolds, (TM, π, M) denotes the tangent bundle, and by $\mathcal{T}M = TM \setminus \{0\}$ the tangent manifold with zero section removed.

Let $(\mathcal{U}, \varphi = (x^i))$ be a local coordinate system at $p \in M$ from a fixed atlas of C^∞-class. We denote by $(\pi^{-1}(\mathcal{U}), \bar{\varphi} = (x^i, y^i))$ the induced local coordinate system at $\xi \in \pi^{-1}(p) \subset TM$. The linear map $\hat{\pi}_\xi := (d\pi_\xi) : T_\xi TM \to T_{\pi(\xi)}M$ induced by the canonical submersion π is an epimorphism of linear spaces for each $\xi \in TM$. Therefore, its kernel determines a regular, n-dimensional, integrable distribution

$$\mathcal{V} : \xi \in TM \mapsto \mathcal{V}_\xi TM \subset T_\xi TM, \tag{1.1}$$

where

$$\mathcal{V}_\xi TM := \text{Ker}(\hat{\pi})_\xi = \text{span}\left\{ \frac{\partial}{\partial y^i} \mid_\xi \right\},$$

which is called the *vertical distribution*. A non-linear connection on TM is a vector bundle morphism

$$\mathcal{N} : TTM \to \mathcal{V}TM.$$

The kernel of the morphism $\mathcal{N}$ is a vector subbundle of the tangent $(TTM, \hat{\pi}, TM)$, denoted by $\mathcal{H}TM$ and called the *horizontal subbundle*. Its fibers $\mathcal{H}_\xi TM$ determine a regular n-dimensional distribution

$$\mathcal{H} : \xi \in TM \mapsto \mathcal{H}_\xi TM \subset T_\xi TM,$$

which is supplementary to the vertical distribution (1.1). This means that for every $\xi \in TM$ we have the direct decomposition:

$$T_\xi TM = \mathcal{V}_\xi TM \oplus \mathcal{H}_\xi TM. \tag{1.2}$$

Finsler spaces are an important field in modern differential geometry of the tangent bundle (TTM, π, TM) of a manifold M. Contributions to the geometry of these spaces have been made by by D. Bao, S.S. Chern and Z. Shen [15], L. Berwald [9], M. Matsumoto [32] and H. Busemann [10].

Let $x^i = x^i(t), a \leq t \leq b$ be the equations of a segment of a curve γ in a coordinate neighborhood $\mathcal{U}$. The length $\mathcal{L}$ of the segment is given by the integral

$$\mathcal{L} = \int_a^b F\left(x(t), \dot{x}(t)\right) dt.$$

The manifold M equipped with such a notion of length is called an n-dimensional *Finsler manifold* with fundamental function $F(x, y)$ and denoted by (M, F) if F satisfies the following conditions:

1. **Regularity:** F is C^∞ on the slit tangent bundle $\mathcal{T}M$, and C^1 on TM.
2. **Positive homogeneity:** for any point $p \in M$ the restriction $F_p := F\mid_{T_pM}$ is positively homogeneous of degree one, i.e.,

$$F(p, \lambda y) = \lambda F(p, y), \quad \lambda > 0, y \in \mathcal{T}M.$$

3. **Strong convexity:** the $n \times n$ Hessian matrix

$$\mathbf{g}_{ij} := \left(\left[\frac{1}{2}F^2\right]_{y^i y^j}\right)$$

is positive-definite at every point of $\mathcal{T}M$.

Let F be a Finsler metric on an n-dimensional manifold M. It induces a *spray* G on TM. In local coordinates in TM, it is expressed by

$$G(y) := y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}.$$

The local functions $G^i(x, y)$ satisfy $G^i(x, ky) = k^2 G^i(x, y), \forall k > 0$. For a non-zero vector $y \in \mathcal{T}_pM$, define the Cartan torsion of F as follows:

$$\mathbf{C}_y : T_pM \times T_pM \times T_pM \to \mathbb{R},$$

where

$$\mathbf{C}_y(v_1, v_2, v_3) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tv_3}(v_1, v_2)\Big]_{t=0}, \quad v_1, v_2, v_3 \in \mathcal{T}_pM.$$

For a vector $y \in \mathcal{T}M$, $\mathbf{I}_y : T_pM \to \mathbb{R}$ is defined by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} \mathbf{C}_y(e_i, e_j, u)g^{ij}(y),$$

where $\{e_i\}_{i=1}^n$ is basis vectors for T_pM at $p \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in\mathcal{T}M}$ is called the *mean Cartan torsion*. Assume the following conventions:

$$G^i_{\ j} := \frac{\partial G^i}{\partial y^j}, \quad G^i_{\ jk} := \frac{\partial G^i_{\ j}}{\partial y^k}.$$

The local functions $G^i_{\ j}$ are coefficients of a connection in the pullback tangent bundle π^*TM which is called the Berwald connection. The derivatives of a vector field V and a 2-covariant tensor $T = T_{ij}dx^i \otimes dx^j$ is given by

$$V^i_{|m} = \frac{\delta V^i}{\delta x^m} + V^s G^i_{sm}, \quad V_{i|m} = \frac{\delta V^i}{\delta x^m} - V_s G^s_{im},$$

$$T_{ij|m} = \frac{\delta T_{ij}}{\delta x^m} - T_{sj}G^s_{im} - T_{is}G^s_{mj},$$

where $\frac{\delta}{\delta x^m} := \frac{\partial}{\partial x^m} - G^i_{\ m}\frac{\partial}{\partial y^i}$. A curve $\gamma = \gamma(t)$ is *a geodesic* if and only if its coordinates $(\gamma^i(t))$ satisfy

$$\ddot{\gamma}^i + 2G^i \circ \dot{\gamma} = 0,$$

where $\dot{\gamma} = \dot{\gamma}^i\frac{\partial}{\partial x^i}$.

The vector field $Y(t) = Y^i(t)\frac{\partial}{\partial y^i}\ |_{\gamma(t)}$ along $\gamma(t), 0 \le t \le 1$ is called *parallel* if its covariant derivative is identically zero, i.e., it is a solution of the differential equation

$$D_{\dot{\gamma}}Y(t) := \left(\frac{dY^i(t)}{dt} + G^i_j(\gamma(t), Y(t))\dot{\gamma}^j(t)\right)\frac{\partial}{\partial y^i} = 0.$$

The *parallel translation* $P_\gamma : T_pM \to T_qM$ along $\gamma(t)$ can be introduced as follows: for any $v \in \mathcal{T}_pM$ let Y be the parallel vector field along γ for which $Y(0) = v$. Then $P_\gamma(v) := Y(1)$.

For a non-zero vector $y \in \mathcal{T}_pM$, let us define $\mathbf{B}_y : T_pM \times T_pM \times T_pM \to T_pM$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i} |_x$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

We have a $(0, 2)$-tensor, which is $\mathbf{E}_y : T_pM \times T_pM \to \mathbb{R}$ by $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$, where

$$E_{jk} := \frac{1}{2} B^m_{jkm}.$$

The quantities **B** and **E** are non-Riemannian quantities called the *Berwald curvature* and *mean Berwald curvature*, respectively. A Finsler metric F is said to be *Berwald metric* if $\mathbf{B} = 0$, while if $\mathbf{E} = 0$, it called *weakly Berwald metric*.

There are some important classes of Finsler metrics containing the class of Berwald metrics. For $y \in \mathcal{T}_pM$, let us define the *Landsberg curvature* $\mathbf{L}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$ and *mean Landsberg curvature* $\mathbf{J}_y : T_pM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y(\mathbf{B}_y(u, v, w), y), \quad \mathbf{J}_y(u) := \sum_{i,j=1}^{n} \mathbf{L}_y(e_i, e_j, u) g^{ij}(y).$$

In the local coordinates (x^i, y^i)

$$\mathbf{L}_y(u, v, w) := L_{ijk} u^i v^j w^k, \quad \mathbf{J}_y(u) := J_i(y) u^i,$$

where

$$L_{ijk} := -\frac{1}{2} y^m g_{ml}(y) B^l_{ijk}, \quad J_i := g^{jk} C_{ijk}.$$

Note that $\mathbf{L}_y(u, v, w)$ is symmetric in u, v and w and $\mathbf{L}_y(y, v, w) = 0$. A Finsler metric F is called a *Landsberg metric* (*weakly Landsberg metric*) if $\mathbf{L}_y = 0$ ($\mathbf{J}_y = 0$), respectively. It is easy that every Berwald metric is a Landsberg metric.

For $y \in \mathcal{T}_pM$, the *stretch curvature* $\mathbf{\Sigma}_y : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$ is given by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl} u^i v^j w^k z^l$, where

$$\Sigma_{ijkl} := 2\left(L_{ijk|l} - L_{ijl|k}\right).$$

A Finsler metric is said to be a *stretch metric* if and only if $\mathbf{\Sigma} = 0$.

We have the following relation

$$\{\text{Berwald metrics}\} \subseteq \{\text{Landsberg metrics}\} \subseteq \{\text{stretch metrics}\}.$$

The Douglas curvature is a non Riemannian projective invariant constructed from the Berwald curvature. A Finsler metric is said to be a *Douglas metric* if there exists positively 1-homogeneous function $P(x, y)$ on M where the spray coefficients $G^i := G^i(x, y)$ of a Finsler metric F can be expressed in the following form

$$G^i(y) = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k + P(x, y) y^i.$$

Douglas metrics are also characterized by vanishing Douglas tensor. Let

$$D^i_{jkl} := B^i_{jkl} - \frac{2}{n+1}\left[\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(\frac{\partial G^s}{\partial y^s} y^i\right)\right]. \tag{1.3}$$

The tensor $\mathbf{D} := D^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}\ |_x$ is called Douglas tensor. A Finsler metric is a Douglas metric if and only if $\mathbf{D} = 0$ (see [6]). According to (1.3), the Douglas tensor can be written as follows

$$D^i_{jkl} = B^i_{jkl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk;l} y^i\Big\},$$

where $E_{jk;l} = \frac{\partial E_{jk}}{\partial y^l}$.

1.2 $\widetilde{\mathbf{B}}$-, and H-stretch metric

Z. Shen [41, page 139] introduced a non-Riemannian quantity $\widetilde{\mathbf{B}}$ which is obtained from the Berwald curvature $\mathbf{B}$ by the covariant horizontal differentiation along Finslerian geodesics. For a vector $y \in \mathcal{T}_pM$, define $\widetilde{\mathbf{B}}_y$:

$T_pM \times T_pM \times T_pM \to T_pM$ by $\widetilde{\mathbf{B}}_y(u,v,w) := \widetilde{B}^i_{jkl}(y)u^jv^kw^l \frac{\partial}{\partial x^i}\mid_x$, where

$$\widetilde{B}^i_{jkl} := B^i_{jkl|m}y^m. \tag{1.4}$$

The Finsler metric F is called $\widetilde{\mathbf{B}}$*-metric* if and only if $\widetilde{\mathbf{B}} = 0$. In this case, (M,F) is called a $\widetilde{\mathbf{B}}$*- Finsler manifold.*

Definition 1.1. *For a vector $y \in T_pM$, we define $\boldsymbol{K}_y : T_pM \times T_pM \times T_pM \times T_pM \to T_pM$ by*

$$\boldsymbol{K}_y(u,v,w,z) := \mathcal{K}^i_{jklm}(y)u^jv^kw^lz^m \frac{\partial}{\partial x^i}\mid_x,$$

where

$$\mathcal{K}^i_{jklm} := 2\left(\widetilde{B}^i_{jkl|m} - \widetilde{B}^i_{jkm|l}\right)$$

and " $|$ " is the horizontal derivation with respect to the Berwald connection of F.

The family $\boldsymbol{K} := \{\boldsymbol{K}_y : y \in T_pM\}$ is called the *stretch* $\widetilde{\mathbf{B}}$*-curvature.* A Finsler metric F is said to be $\widetilde{\mathbf{B}}$*-stretch metric* if and only if $\boldsymbol{K} = 0$. Especially, every $\widetilde{\mathbf{B}}$-metric is a $\widetilde{\mathbf{B}}$-stretch metric. Therefore, on the contrary, it is interesting to find some topological condition on the manifold M such that every $\widetilde{\mathbf{B}}$-stretch metric on M reduces to a $\widetilde{\mathbf{B}}$-metric.

Let us introduce a non-trivial example, where $|.|$ and $\langle,\rangle$ denote the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively.

Example 1.2. *The Finsler function*

$$F(x,y) = \frac{\left(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \varepsilon\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}$$

on the unit ball $\mathbb{B}^n$ is a $\widetilde{\mathbf{B}}$-stretch metric when $n=2$ and $n=3$. This can be shown using the Finsler package and Maple program [53]. We guess it should work in the general dimension but the calculation is very tedious and a bit complicated.

We have the following inclusions

$$\{\text{Berwald metric}\} \subset \left\{\widetilde{\mathbf{B}}\text{-metric}\right\} \subset \left\{\widetilde{\mathbf{B}}\text{-stretch metric}\right\}.$$

The Finslerian quantity $\mathbf{H}$ was introduced by H. Akbar-Zadeh in 1988 [4] to characterization of Finsler metrics of constant flag curvature which is obtained from the mean Berwald curvature $\mathbf{E}$ by the covariant horizontal differentiation along geodesics. For a vector $y \in \mathcal{T}_pM$, $\mathbf{H}_y : T_pM \times T_pM \to \mathbb{R}$ is given by $\mathbf{H}_y(u,v) := H_{jk}(y)u^j v^k$, where

$$H_{jk} := E_{jk|l}y^l.$$

The Finsler metric F is called ***H**-metric* if and only if $\mathbf{H} = 0$. Moreover, we say that the Finsler manifold (M, F) is a ***H**-Finsler manifold.*

Definition 1.3. *For a vector $y \in \mathcal{T}_pM$, we define $\boldsymbol{\kappa}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$, by*

$$\boldsymbol{\kappa}_y(u,v,w) := \kappa_{jkl}(y)u^j v^k w^l,$$

where

$$\kappa_{jkl} := 2\left(H_{jk|l} - H_{jl|k}\right).$$

The Finsler metric F is called ***H**-stretch metric* if and only if $\boldsymbol{\kappa} = 0$. We have the following inclusion relations

$$\{\text{weakly Berwald metric}\} \subset \{\mathbf{H}\text{-metric}\} \subset \{\mathbf{H}\text{-stretch metric}\}.$$

In this section, we introduce the following results:

Theorem 1.4. *Suppose that F is a positively complete $\widetilde{\mathbf{B}}$-stretch metric with bounded Berwald curvature. Then F must be a $\widetilde{\mathbf{B}}$-metric and the Berwald curvature is constant along any geodesic.*

Theorem 1.5. *Every complete **H**-stretch metric with bounded mean Berwald curvature is a **H**-metric.*

Definition 1.6. *Let (M, F) be a Finsler manifold. The Finsler metric F is called a relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature if its stretch $\widetilde{\mathbf{B}}$-curvature is given by*

$$\mathcal{K}^i_{jklm} := \lambda F \left(B^i_{jkl|m} - B^i_{jkm|l} \right),$$

*where $\lambda := \lambda(x, y)$ is scalar function on TM. In this case, (M, F) is called a relatively isotropic $\widetilde{\mathbf{B}}$-stretch manifold. If $\lambda > 0\,(\lambda < 0, \lambda = \text{constant})$, then F is said to be negative (positive or constant) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature (respectively). Furthermore, if the stretch **H**-curvature is given by*

$$\kappa_{jkl} := \lambda F \left(E_{jk|l} - E_{jl|k} \right),$$

*then F is said to be negative (positive, constant) relatively isotropic stretch **H**-curvature if we have $\lambda > 0\,(\lambda < 0, \lambda = \text{constant})$ (respectively).*

In view of Theorem 1.4, every complete $\widetilde{\mathbf{B}}$-stretch metric is a $\widetilde{\mathbf{B}}$-metric. Thus, a compact $\widetilde{\mathbf{B}}$-stretch metric reduces to a $\widetilde{\mathbf{B}}$-metric. We generalize this result as follows.

Theorem 1.7. *A compact Finsler manifold (M, F) with negative (positive) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature is a $\widetilde{\mathbf{B}}$-Finsler manifold. More precisely, a complete Finsler metric with constant relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature and bounded $\widetilde{\mathbf{B}}$-curvature is a $\widetilde{\mathbf{B}}$-metric.*

According to Theorem 1.5, every complete **H**-stretch metric is a **H**-metric. Thus, a compact **H**-stretch metric reduces to a **H**-metric. We generalize this result as follows.

Theorem 1.8. *A compact Finsler manifold (M, F) with positive (negative) relatively isotropic stretch **H**-curvature is a **H**-Finsler manifold. More precisely, a complete Finsler metric with constant relatively isotropic stretch **H**-curvature and bounded **H**-curvature is a **H**-metric.*

Proof of Theorems 1.4, 1.5, 1.7, 1.8

In what follows we are going to prove Theorem 1.4. We need the following

Proposition 1.9. *Let (M, F) be a Finsler manifold. Suppose that F is $\widetilde{\boldsymbol{B}}$-stretch metric and $\gamma = \gamma(t)$ is a geodesic. Put $\boldsymbol{B}(t) := \boldsymbol{B}_{\dot\gamma}(U(t), V(t), W(t))$, where $U(t), V(t)$ and $W(t)$ are the parallel vector fields along γ. Then, the following equation holds:*

$$\boldsymbol{B}(t) = \widetilde{\boldsymbol{B}}(0)t + \boldsymbol{B}(0). \tag{1.5}$$

Proof. Let p be an arbitrary point of M, $y, u, v, w \in T_pM$ and $\gamma : (-\infty, \infty) \to M$ be the unit speed geodesic passing from p and $\frac{d\gamma}{dt}(0) = y$. For $U(t), V(t)$ and $W(t)$ are the parallel vector fields along γ with $U(0) = u, V(0) = v, W(0) = w$ we put

$$\widetilde{\mathbf{B}}(t) := \widetilde{\mathbf{B}}_{\dot\gamma}(U(t), V(t), W(t)).$$

By definition of $\widetilde{\mathbf{B}}$-curvature, we have

$$\widetilde{\mathbf{B}}(t) = \mathbf{B}'(t). \tag{1.6}$$

Let

$$\widetilde{\mathbf{B}}'(t) := \widetilde{\mathbf{B}}'_{\dot\gamma}(U(t), V(t), W(t)).$$

Since F is $\widetilde{\mathbf{B}}$-stretch metric, then we have

$$\widetilde{\mathbf{B}}'(t) = 0,$$

which implies that $\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0)$. By (1.6), the proof is complete. □

Lemma 1.10 ([41, Lemma 7.3.2])**.** *Let (M, F) be a Finsler manifold. Assume that $\gamma : [a, b] \to M$ is a geodesic from p to q. Then the parallel translation P_γ preserves the inner product $\mathbf{g}_{\dot\gamma}$, i.e.*

$$\mathbf{g}_{\dot\gamma(b)}\left(P(u), P(v)\right) = \mathbf{g}_{\dot\gamma(a)}\left(u, v\right), \quad u, v \in T_pM.$$

Now let us begin to prove Theorem 1.4.

Proof of Theorem 1.4. For an arbitrary unit vector $y \in T_pM$ and an arbitrary vector $v \in T_pM$, let $\gamma = \gamma(t)$ be the geodesic with $\dot\gamma(0) = y$, and $V(t)$ be the parallel vector field along γ with $V(0) = v$.

Then by Proposition 1.9 we get

$$\mathbf{B}(t) = \widetilde{\mathbf{B}}(0)t + \mathbf{B}(0).$$

Suppose that Berwald curvature is bounded at $p \in M$, i.e.

$$\|\mathbf{B}\|_x := \sup_{y\in \mathcal{T}M}\left[\sup_{v\in T_pM}\frac{\mathbf{B}_y(v)}{\left[\mathbf{g}_y(v,v)\right]^{\frac{3}{2}}}\right] < \infty.$$

Using Lemma 1.10, we get that

$$W := \mathbf{g}_{\dot\gamma(t)}(V(t),V(t))$$

is a positive constant. Thus

$$|\mathbf{B}(t)| \le W^{\frac{3}{2}}\|\mathbf{B}\| < \infty.$$

Let us put $t \to +\infty$. Then, we get

$$\widetilde{\mathbf{B}}_y(v) = \widetilde{\mathbf{B}}(0) = 0.$$

Therefore $\widetilde{\mathbf{B}} = 0$. This completes the proof. □

It is clear that every Finsler metric with vanishing $\widetilde{\mathbf{B}}$-curvature has vanishing $\mathbf{H}$-curvature, that means, every $\widetilde{\mathbf{B}}$-metric is a $\mathbf{H}$-metric. By Theorem 1.4 a $\widetilde{\mathbf{B}}$-stretch Finsler metric reduces to a $\widetilde{\mathbf{B}}$-metric. Then, we get the following

Corollary 1.11. *Let (M,F) be a Finsler manifold. Then, every $\widetilde{\mathbf{B}}$-stretch metric is a $\mathbf{H}$-metric.*

Proposition 1.12. *Let (M,F) be a Landsberg manifold. If the Riemannian curvature $\mathbf{R} = 0$, then the Finsler metric F is a $\widetilde{\mathbf{B}}$-stretch metric.*

Proof. Since (M,F) is Landsberg space then the horizontal covariant derivatives of Berwald and Cartan connections coincide, i.e.

$$G^h_{ij} = \Gamma^h_{ij}.$$

We have

$$g_{jk|h} = 0. \tag{1.7}$$

Differentiating (1.7) with respect to y^l, we get

$$2C_{jkl|h} - \left(B^r_{hkl}g_{rj} + B^r_{hjl}g_{rk} + B^r_{jhk}g_{rl}\right) + B_{hjkl},$$

where $B^h_{ijk} = \frac{\partial G^h_{ij}}{\partial y^k}$.

We have the property that the tensor $B^r_{hjl}g_{rk} = B_{khjl}$ is totally symmetric. Then we get

$$C_{jkl|h} = B_{jkhl}.$$

Equivalently

$$C^h_{jk|l} = B^h_{jkl}.$$

Now, we have

$$\tilde{B}^h_{jkl} = B^h_{jkl|m}y^m = C^h_{jk|l|m}y^m.$$

Since $R = 0$, $C^h_{jk|l|m} = C^h_{jk|m|l}$ holds, so

$$\tilde{B}^h_{jkl} = C^h_{jk|m|l}y^m = 0.$$

Taking into account $C^h_{jk|m}y^m = 0$. Therefore, we obtain $\boldsymbol{K} = 0$. □

To prove Theorem 1.5, we need the following

Proposition 1.13. *Let (M, F) be a Finsler manifold. Suppose that F is **H**-stretch metric. Then, for any geodesic $\gamma = \gamma(t)$ and any parallel vector field $V = V(t)$ along γ, the function $\boldsymbol{E}(t) := \boldsymbol{E}_{\dot{\gamma}}(V(t))$ must be in the following form:*

$$\boldsymbol{E}(t) = \boldsymbol{H}(0)t + \boldsymbol{E}(0).$$

Proof. Let $\gamma : [0, +\infty] \to M$ be the geodesic parameterized by the arc length on M with the start point $\gamma(0) = p$ and the tangent vector $\dot{\gamma}(0) = y$. Suppose that $U = U(t), V = V(t)$ are two parallel vector fields along $\gamma = \gamma(t)$ with $U(0) = u, V(0) = v$.

Since F is **H**-stretch metric, then we get $\boldsymbol{\kappa} = 0$, that means

$$H_{jk|l} = H_{jl|k}. \tag{1.8}$$

Contracting (1.8) with y^l, we have

$$H_{jk|l}y^l = 0.$$

Let

$$\mathbf{H}(t) := \mathbf{H}_{\dot\gamma}(U(t), V(t)) = H_{jk}(\gamma(t), \dot\gamma(t))U^j(t)V^k(t). \tag{1.9}$$

We have $\mathbf{H}(t) = \mathbf{E}'(t)$, by (1.9)

$$\mathbf{E}''(t) = \mathbf{H}'(t) = H_{jk|l}\dot\gamma^l(t)(\gamma(t), \dot\gamma(t))U^j(t)V^k(t) = 0.$$

Thus yields $\mathbf{E}(t) = \mathbf{H}(0)t + \mathbf{E}(0)$. □

Let us start to prove Theorem 1.5.

Proof of Theorem 1.5. Let (M, F) be complete Finsler manifold. Suppose that F is **H**-stretch metric. Take an arbitrary unit vector $y \in T_pM$ and an arbitrary vector $v \in T_pM$. Let $\gamma = \gamma(t)$ be the geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = y$, and $W(t)$ be the parallel vector field along γ with $W(0) = w$. Then by Proposition 1.13, we get

$$\mathbf{E}(t) = \mathbf{H}(0)t + \mathbf{E}(0). \tag{1.10}$$

Suppose that $\mathbf{E}_y$ is bounded, i.e. there is a constant $A < \infty$ such that

$$||\mathbf{E}||_x := \sup_{y \in TpM}\left[\sup_{v\in TpM}\frac{\mathbf{E}_y(w)}{\left[\mathbf{g}_y(w,w)\right]^{\frac{3}{2}}}\right] \leq A.$$

According to Lemma 1.10, one has

$$|\mathbf{E}(t)| \leq AQ^{\frac{3}{2}} < \infty$$

for some constant Q. Therefore $\mathbf{E}(t)$ is a bounded function on $(-\infty, \infty)$. Letting $t \to \infty$ in (1.10), it implies that $\mathbf{H}_y(v) = \mathbf{H}(0) = 0$. □

In view of Theorem 1.5, a **H**-stretch metric reduces to a **H**-metric. Tayebi et al. in [44] proved that any **H**-metric is a $\widetilde{\mathbf{B}}$-metric for a Finsler surface (M, F). Then, we get the following corollary

Corollary 1.14. *Let (M, F) be a Finsler surface. Then F is a* **H***-stretch metric if and only if it is a* $\widetilde{\mathbf{B}}$*-metric.*

In the sequel the following results are used:

Theorem 1.15 ([30]). *Suppose M is a compact, oriented manifold with a volume element ω . Then for every vector field X over M, we have $\int_M (divX)_\omega = 0$.*

Theorem 1.16 ([30]). *Suppose M is an oriented manifold with the volume form ω and ∇ is a torsion-free connection where $\nabla_\omega = 0$. Then for every vector field X over M, $y \in T_pM$ with $x \in M$ we have*

$$(divX)_x = -trace(Y \to \nabla_Y X) = \nabla_i X^i.$$

Proof of Theorem 1.7. Let $p \in M$, and $y, u, v, w \in T_pM$, and $\gamma : (-\infty, \infty) \to M$ is the geodesic with $\gamma(0) = p$ and $\frac{d\gamma}{dt}(0) = y$ and $U(t), V(t)$ and $W(t)$ are parallel vector fields along γ such that $U(0) = u, V(0) = v, W(0) = w$.

We put

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}_{\dot\gamma}((U(t), V(t), W(t)).$$
$$\widetilde{\mathbf{B}}'(t) = \widetilde{\mathbf{B}}'_{\dot\gamma}((U(t), V(t), W(t)).$$

However, the Finsler manifold (M, F) has non-negative (non-positive, respectively) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature or is constant. By the definition and multiplying by y^l, it is simple to get

$$\widetilde{B}^i_{jkm|l} y^l = \lambda F \widetilde{B}^i_{jkm}, \tag{1.11}$$

where $\lambda := \lambda(x, y)$ is a negative (positive, respectively) or constant homogeneous function over $\mathcal{T}M$.

First suppose $\lambda := \lambda(x, y)$ is a negative (positive, respectively) function over $\mathcal{T}M$.

By putting

$$\Gamma(x, y) := \widetilde{B}^r_{szmq} \widetilde{B}^{szmq}_r$$

we have

$$\dot\Gamma(x, y) = \Gamma_{|n} y^n$$

$$
\begin{aligned}
&= \lambda F \widetilde{B}^r_{szmq}\widetilde{B}^{szmq}_r + \widetilde{B}^r_{szmq}\lambda\widetilde{B}^{szmq}_r \\
&= 2\lambda F\Gamma. \qquad (1.12)
\end{aligned}
$$

As F and Γ have positive values, if λ is negative (positive), then $\dot{\Gamma}$ is negative (positive). In view of Theorem 1.16, one gets

$$\dot{\Gamma}(x,y) = \Gamma_{|n}y^n = \xi(\Gamma) = \overline{\text{div}}(\Gamma\xi).$$

Using Theorem 1.16, one obtains $\overline{\text{div}}\,\xi = 0$ for every geodesic vector field $\xi = y^i\frac{\delta}{\delta x^i}$ on unit tangent sphere bundle SM. Since M, SM are compact. The volume form ω_{SM} over SM is obtained from the volume form ω on M. According to Theorem 1.15, we obtain

$$\int_{SM}\dot{\Gamma}\omega_{SM} = 0.$$

Since $\dot{\Gamma}$ is homogeneous function and its non-negative (non-positive) sign, then $\dot{\Gamma} = 0$. By equation (1.12), we get $\Gamma = 0$. Hence $\widetilde{\mathbf{B}} = 0$. In this case we obtain that

$$\widetilde{\mathbf{B}}' = \widetilde{B}^i_{jkm|l}y^l = 0.$$

Thus

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0).$$

So Berwald curvature is equal to

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0)t + \mathbf{B}(0).$$

Letting $t \to \mp\infty$ and using $\|\mathbf{B}\| < \infty$, we get $\widetilde{\mathbf{B}}(0) = 0$. Thus

$$\widetilde{\mathbf{B}}(t) = 0.$$

Now, if $\lambda =$ constant, the general answer to equation (1.11) is as follows:

$$\widetilde{\mathbf{B}}(t) = e^{t\lambda}\widetilde{\mathbf{B}}(0).$$

Using $\|\mathbf{B}\| < \infty$ and letting $t \to \mp\infty$, this implies that

$$\widetilde{\mathbf{B}}(t) = \widetilde{\mathbf{B}}(0) = 0.$$

Thus, the second part of Theorem 1.7 is also proved. □

Corollary 1.17. *Let (M, F) be Finsler manifold. If F is an negative (positive) relatively isotropic stretch $\widetilde{\mathbf{B}}$-curvature, then it is an $\mathbf{H}$-metric.*

Proof of Theorem 1.8. Let $p \in M$ and $y, u, v \in T_pM$. Let $\gamma : (-\infty, \infty) \to M$ be the unit speed geodesic such that $\gamma(0) = p$, $\dot{\gamma}(0) = y$. Suppose $U = U(t), V = V(t)$ are the parallel vector fields along γ with $U(0) = u, V(0) = v$. Put

$$\mathbf{H}(t) = \mathbf{H}(U(t), V(t)),$$
$$\mathbf{H}'(t) = \mathbf{H}'(U(t), V(t)).$$

By assumption, F has non-positive (non-negative) relatively isotropic stretch $\mathbf{H}$-curvature. Then

$$H_{jk|l} - H_{jl|k} = \lambda F \left(E_{jk|l} - E_{jl|k} \right), \tag{1.13}$$

where $\lambda := \lambda(x, y)$ is a positive (negative) or constant function on TM.

Contraction with y^k implies that

$$H_{jl|k} y^k = \lambda F E_{jl|k} y^k,$$

since $H_{jl} = E_{jl|k} y^k$. Thus we get

$$H_{jl|k} y^k = \lambda F H_{jl}.$$

First let $\lambda := \lambda(x, y)$ be a negative scalar function on TM. Put

$$\phi := H^{zn} H_{zn}.$$

Then we have

$$\phi' = 2\lambda F \phi.$$

By definition, F and ϕ have positive value. If λ is negative (positive), then ϕ' is negative (positive). By Theorem 1.16, we get

$$\phi' = \phi_{|m} y^m = \zeta(\phi) = \overline{\text{div}}(\phi\zeta),$$

where $\zeta = y^i \frac{\delta}{\delta x^i}$ is a geodesic vector field on the unit sphere tangent bundle SM and $\overline{\text{div}}(\zeta) = 0$. By Theorem 1.15, we get

$$\int_{SM} \phi' \omega_{SM} = 0.$$

Thus the volume form ω_{SM} on SM is obtained from volume form ω on M. Since ϕ' is homogeneous function, and its sign is negative (positive), then $\phi' = 0$, and we have $\phi = 0$. Hence $H = 0$. In this case

$$\mathbf{H}' = E_{jl|k} y^k = 0.$$

Thus $\mathbf{H}(t) = \mathbf{H}(0)$, which implies that

$$\mathbf{H}(t) = t\mathbf{H}(t) + \mathbf{E}(0).$$

Letting $t \to \mp\infty$, and using $\|\mathbf{E}\| < \infty$, we get $\mathbf{H}(0) = 0$. Thus

$$\mathbf{H}(t) = 0.$$

Now, suppose $\lambda =$ constant, then the general answer of (1.13) is as follows

$$\mathbf{H}(t) = \mathbf{H}(0)\exp(t\lambda).$$

Using $\|\mathbf{E}\| < \infty$ and letting $t \to \pm\infty$ this implies that $\mathbf{H}(0) = 0$, thus $\mathbf{H}(t) = 0$. □

Corollary 1.18. *Let (M, F) be Finsler surface. If F has a negative (positive) relatively isotropic stretch **H**-curvature, then it is a $\widetilde{\mathbf{B}}$-metric.*

1.3 Generalized Douglas curvature

One of the important quantities in Finsler geometry is Douglas' curvature which is an invariant tensor by a projective change $\phi : F \to \bar{F}$. S. Bácsó and M. Matsumoto in [6] established the concept of Douglas space as an extension of Berwald spaces by studying the geodesics curves in a Finsler space. Additionally, they investigate the relationship between Douglas spaces and other particular Finsler spaces, including Wagner spaces and Landsberg spaces. Douglas metrics can be characterized by

$$\mathcal{G}^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i,$$

where $\Gamma^i_{jk}(x)$ are local functions on M and $P(x,y)$ is a local positively homogeneous function of degree one. The class of Douglas metrics is much larger than that of Berwald metrics. The aim of this section is to study a class of Finsler metrics that includes the classes of Douglas metrics.

Definition 1.19. *We call the Finsler metric F a generalized Douglas metric if and only if the quantity $\widetilde{\boldsymbol{B}}$-curvature in* (1.4) *is given by*

$$\widetilde{B}^i_{jkl} := B^i_{jkl} + \omega_{jk}\delta^i_l + \omega_{jl}\delta^i_k + \omega_{kl}\delta^i_j + E_{jk;l}y^i, \tag{1.14}$$

where ω is a smooth map $M \to \wedge^2\mathcal{T}_pM$ given by $\omega(p) := \omega_{ij}(p)dx^i \wedge dx^j$ at any point $p \in M$.

We have the following

$$\{\text{ Berwald metric}\} \subset \{\text{Douglas metric}\} \subset \{\text{generalized Douglas metric}\}.$$

In the next result, we show that every generalized Douglas metric with vanishing $\widetilde{\mathbf{B}}$-stretch tensor is a Douglas metric under the condition that the mean Berwald curvature is horizontally constant along geodesics of F.

Theorem 1.20. *Let (M,F) be a generalized Douglas Finsler manifold. Suppose that F is a $\widetilde{\boldsymbol{B}}$-stretch metric. Then F is a Douglas metric.*

The converse of Theorem 1.20 is not true. Let us introduce a famous example of Finsler metrics introduced by a physicist G. Randers in [39].

Example 1.21. *Let $|.|$ and $\langle .,. \rangle$ be the Euclidean norm and the inner product in $\mathbb{R}^n$ respectively. Let $F = \alpha + \beta$ be a Finsler metric with $\|\beta_x\|_\alpha < 1$ where*

$$\alpha = \frac{\sqrt{|y|^2 + \varepsilon\left(|x|^2|y|^2 - \langle x,y\rangle^2\right)}}{1+\varepsilon|x|^2}, \quad \beta = \frac{\sqrt{-\varepsilon}\langle x,y\rangle}{1+\varepsilon|x|^2}, \qquad \varepsilon < 0,$$

and $x \in \mathbb{B}^n(\delta) \subset \mathbb{R}^n$, $\delta = \frac{1}{\sqrt{-\varepsilon}}$, $y \in T_x\mathbb{B}^n(\delta)$. Putting $\varepsilon = -1$, we get a metric called *Funk metric* which satisfies $F_{x^s} - FF_{y^s} = 0$. A Funk metric F is a Douglas metric but it is not $\widetilde{\mathbf{B}}$-stretch.

Every Finsler metric with vanishing $\widetilde{\mathbf{B}}$-curvature has vanishing $\mathbf{H}$-curvature. Thus, every $\widetilde{\mathbf{B}}$-metric is an $\mathbf{H}$-metric, and hence it is $\mathbf{H}$-stretch metric. But the converse might not hold. A. Tayebi et al. in [44] showed that it is true on Finsler surfaces. We generalized this result as follows

Theorem 1.22. *Let (M, F) be a Douglas Finsler manifold with $n \geq 3$. Then every $\mathbf{H}$-stretch metric is a $\widetilde{\mathbf{B}}$-metric.*

Exploiting the above statements, the next result gives a formula involving the relation between the quantities stretch $\widetilde{\mathbf{B}}$-curvature and stretch $\mathbf{H}$-curvature. Namely, we have the following

Theorem 1.23. *Let (M, F) be a Douglas Finsler manifold. Then, the stretch $\widetilde{\mathbf{B}}$-curvature of F is given by*

$$\mathcal{K}^i_{jklm} := \frac{2}{n+1}\left\{\kappa_{jlm}h^i_k + \kappa_{klm}h^i_j\right\}. \tag{1.15}$$

Now we can proceed to prove the main result of this section:

Proof of Theorems 1.20, 1.22, 1.23

Proof of Theorem 1.20. Let F be a generalized Douglas metric,

$$\widetilde{B}^i_{jkl} = B^i_{jkl} + \omega_{jk}\delta^i_l + \omega_{jl}\delta^i_k + \omega_{kl}\delta^i_j + E_{jk;l}y^i. \tag{1.16}$$

Differentiating (1.16) along the direction $y^s\frac{\delta}{\delta x^s}$ yields,

$$\widetilde{B}^i_{jkl,s}y^s = \widetilde{B}^i_{jkl} + \omega'_{jk}\delta^i_l + \omega'_{jl}\delta^i_k + \omega'_{kl}\delta^i_j, \tag{1.17}$$

where $\omega'_{ij} := \frac{\delta\omega_{ij}}{\delta x^s}y^s$.

Plugging (1.16) into (1.17), we obtain

$$\widetilde{B}^i_{jkl,s}y^s = B^i_{jkl} + (\omega_{jk}+\omega'_{jk})\delta^i_l + (\omega_{jl}+\omega'_{jl})\delta^i_k + (\omega_{kl}+\omega'_{kl})\delta^i_j + E_{jk;l}y^i. \tag{1.18}$$

Since F is $\widetilde{\mathbf{B}}$-stretch, that is

$$\widetilde{B}^i_{jkl,s} = \widetilde{B}^i_{jks,l}, \tag{1.19}$$

contracting (1.19) with y^s yields

$$\widetilde{B}^i_{jkl,s} y^s = 0. \tag{1.20}$$

By (1.18) and (1.20), we have

$$B^i_{jkl} = -1 \left[(\omega_{jk} + \omega^{'}_{jk})\delta^i_l + (\omega_{jl} + \omega^{'}_{jl})\delta^i_k + (\omega_{kl} + \omega^{'}_{kl})\delta^i_j + E_{jk;l} y^i \right]. \tag{1.21}$$

Contracting (1.21) with h^m_i, one has

$$B^m_{jkl} = -1 \left[(\omega_{jk} + \omega^{'}_{jk})h^m_l + (\omega_{jl} + \omega^{'}_{jl})h^m_k + (\omega_{kl} + \omega^{'}_{kl})h^m_j \right]. \tag{1.22}$$

Multiplying (1.22) with $\frac{1}{2} g_{ms} g^{ls}$ and using the following relations

$$g^{ls} h_{ls} = n - 1 \quad and \quad g_{ms} g^{ls}(\omega_{jl} h^m_k) = g_{ms} g^{ls}(\omega_{kl} h^m_j) = \omega_{jk},$$

it implies that

$$E_{jk} = \frac{-(n+1)}{2}(\omega_{jk} + \omega^{'}_{jk}). \tag{1.23}$$

Alternatively, we can write (1.23) as

$$\omega_{jk} + \omega^{'}_{jk} = \frac{-2}{n+1} E_{jk}. \tag{1.24}$$

Plugging (1.24) in (1.21), we get

$$B^i_{jkl} = \frac{2}{n+1} \left\{ E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk;l} y^i \right\}.$$

We clearly have that F is a Douglas metric, which concludes our proof. □

Now, we are going to prove Theorem 1.22.

Let us begin with the auxiliary lemma which turns out to be the key tool in the proof of our theorem.

Lemma 1.24. *Let (M,F) be a Douglas manifold. Suppose that F is weakly $\boldsymbol{H}$-stretch metric. Then for any geodesic $\gamma(t)$ and any parallel vector field $V(t)$ along γ, the function $\boldsymbol{B} := \boldsymbol{B}_{\dot\gamma}(V(t))$ must be in the following form*

$$\boldsymbol{B}(s) = s\widetilde{\boldsymbol{B}}(0) + \boldsymbol{B}(0). \tag{1.25}$$

Proof. By definition, we have

$$\kappa_{jkl} := H_{jk,l} - H_{jl,k}.$$

By assumption F is **H**-stretch metric, we obtain

$$H_{jk,l} = H_{jl,k}. \tag{1.26}$$

Contracting (1.26) with y^l gives rise the constancy of the rate of change of the **H**-curvature along geodesic of F, i.e.

$$H_{jk,l}y^l = 0. \tag{1.27}$$

Since F is Douglas metric, we have

$$B^i_{jkl} = \frac{2}{n+1}\left\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk;l}y^i\right\}. \tag{1.28}$$

Then

$$\widetilde{B}^i_{jkl} = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{jl}\delta^i_k + H_{kl}\delta^i_j + \mathcal{E}y^i\right\}, \tag{1.29}$$

where $\mathcal{E} = \frac{\delta E_{jk;l}}{\delta x^s}y^s$.

Taking a horizontal derivative of (1.29) along Finslerian geodesics implies that

$$\widetilde{B}^i_{jkl,n}y^n = \frac{2}{n+1}\left\{H_{jk,n}y^n\delta^i_l + H_{jl,n}y^n\delta^i_k + H_{kl,n}y^n\delta^i_j + \mathcal{E}'y^i\right\}, \tag{1.30}$$

where $\mathcal{E}' = \frac{\delta\mathcal{E}}{\delta x^n}y^n$.

Contracting (1.30) with h^m_i, one obtains

$$\widetilde{B}^m_{jkl,n}y^n = \frac{2}{n+1}\left\{H_{jk,n}y^n h^m_l + H_{jl,n}y^n h^m_k + H_{kl,n}y^n h^m_j\right\}, \tag{1.31}$$

By (1.27) and (1.31), we get

$$\widetilde{B}^m_{jkl,n}y^n = 0. \tag{1.32}$$

From our definition of $\widetilde{\mathbf{B}}_y$, we have $\widetilde{\mathbf{B}}(s) = \mathbf{B}'(s)$. We obtain

$$\mathbf{B}''(s) = \widetilde{\mathbf{B}}'(s) = 0.$$

Thus (1.25) follows. This completes the proof. □

In the following, we are going to prove Theorem 1.22. The proof involves applying Lemma 1.24 as follows:

Proof of Theorem 1.22. Let (M, F) be a Finsler manifold. Take an arbitrary unit vector $y \in T_pM$ and an arbitrary vector $v \in T_pM$. Let $\gamma(t)$ be the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = y$ and $V(t)$ be the parallel vector along γ with $V(0) = v$. Then by Lemma 1.24, we get

$$\mathbf{B}(s) = s\widetilde{\mathbf{B}}(0) + \mathbf{B}(0). \tag{1.33}$$

Suppose that $\mathbf{B}_y$ is bounded. i.e., there is a constant $D < 1$ such that

$$\|\mathbf{B}\|_p := \sup_{y\in T_pM_\star}\ \sup_{v\in T_pM}\left[\frac{\mathbf{B}_y(v)}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}\right] \leq D. \tag{1.34}$$

In view of Lemma 1.10, which yields that $|\mathbf{B}(s)| \leq DA^{\frac{3}{2}} < \infty$, for some constant A. Therefore, $\mathbf{B}(s)$ is a bounded function on $[0, \infty)$. (1.33) implies that $\widetilde{\mathbf{B}}_y(v) = \widetilde{\mathbf{B}}(0) = 0$. Hence, F is a $\widetilde{\mathbf{B}}$-metric. Thus, the proof is done. □

Bácsó and Matsumoto [6] showed that if the 1-form β is a closed, then any Randers metric $F = \alpha + \beta$ is a Douglas metric. Then by Theorem 1.22, we get the following.

Corollary 1.25. *Let $F = \alpha + \beta$ is a Randers metric on a manifold M with closed 1-form β. Then F is a $\widetilde{\mathbf{B}}$-metric if and only if it is a $\mathbf{H}$-stretch metric.*

Finally we prove Theorem 1.23 as follows:

Proof of Theorem 1.23. Since F is a Douglas metric, we have

$$B^i_{jkl} = \frac{2}{n+1}\left\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk;l}y^i\right\}. \tag{1.35}$$

By taking horizontal derivatives of (1.35) and contracting with h^m_i then replacing indices, one has

$$\widetilde{B}^m_{jkl,s} = \frac{2}{n+1}\{H_{jk,s}h^m_l + H_{jl,s}h^m_k + H_{kl,s}h^m_j\}, \tag{1.36}$$

and

$$\widetilde{B}^m_{jks,l} = \frac{2}{n+1}\{H_{jk,l}h^m_s + H_{js,l}h^m_k + H_{ks,l}h^m_j\}. \tag{1.37}$$

Subtracting (1.36) from (1.37), we get

$$\left(\widetilde{B}^m_{jkl,s} - \widetilde{B}^m_{jks,l}\right) = \frac{2}{n+1}\left\{(H_{jl,s} - H_{js,l})h^m_k + (H_{kl,s} - H_{ks,l})h^m_j\right\}.$$

By definition of $\widetilde{\mathbf{B}}$-stretch and $\mathbf{H}$-stretch metrics, we get (1.15). □

Chapter 2

6-dimensional filiform nilmanifolds and the corresponding metric Lie algebras

The main contribution of this chapter is to study the 6-dimensional filiform Riemannian nilmanifolds and the corresponding metric Lie algebras. We organize it as follows:

In section 2.1, we collect the necessary notions, tools and results that will be used in later sections.

In section 2.2, we use the Lie brackets of the non-isomorphic 6-dimensional filiform Lie algebras given in [16]. We apply systematically the method developed in [22] to obtain the equivalence classes of isometrically isomorphic filiform metric Lie algebras. We determine the isometry equivalence classes of connected simply connected nilmanifolds on 6-dimensional filiform Lie groups and compute the group of their isometries.

Our aim in the section 2.3 is to determine the sets of geodesic vectors in 6-dimensional filiform metric Lie algebras. This yields the one-dimensional totally geodesic subalgebras of the 6-dimensional filiform metric Lie algebras. To obtain this result we apply the description of the isometrically isomorphic equivalence classes of filiform metric Lie algebras resulted in section 2.2 of this chapter.

In section 2.4, we compute the flat totally geodesic subalgebras of dimension greater than 1 of the six-dimensional filiform metric Lie algebras. With the exception of the metric Lie algebras corresponding to the standard filiform Lie algebra the dimension of the flat totally geodesic subalgebras of every metric Lie algebra is at least one and at most two. Only some metric Lie algebras corresponding to the standard filiform Lie algebra allow a flat totally geodesic subalgebra of dimension four.

2.1 Basic concepts and tools

For any Lie algebra $\mathfrak{n}$ we define the lower central series of $\mathfrak{n}$ by

$$\mathfrak{n} = \ell^0\mathfrak{n} \supset \ell^1\mathfrak{n} \supset \ldots \supset \ell^j\mathfrak{n} \supset \ell^{j+1}\mathfrak{n} \supset \ldots \supset \{0\},$$

where $\ell^{j+1}\mathfrak{n} = [\mathfrak{n}, \ell^j\mathfrak{n}]$, $j \in \mathbb{N}$. We say that a Lie algebra $\mathfrak{n}$ is k-step nilpotent if $\ell^k\mathfrak{n} = \{0\}$ but $\ell^{k-1}\mathfrak{n} \neq \{0\}$ for some $k \in \mathbb{N}$. If an n-dimensional Lie algebra $\mathfrak{n}$ is $(n-1)$-step nilpotent, then it is called *filiform*. A filiform Lie algebra is standard filiform, if it contains a basis $\{B_1, ..., B_n\}$ such that the non-trivial Lie bracket relations are given by $[B_1, B_j] = B_{j+1}, j = 2, ..., n-1$. A Lie algebra $\mathfrak{n}$ is abelian if its Lie bracket is trivial $[X, Y] = 0$ for all $X, Y \in \mathfrak{n}$ (see [24, 25, 51] for detailed study of these terminologies).

A metric nilpotent Lie algebra is a nilpotent Lie algebra endowed with a Euclidean inner product $\langle ., . \rangle$. Let N be the connected and simply connected nilpotent Lie group having $\mathfrak{n}$ as its Lie algebra. The inner product on $\mathfrak{n}$ determines a left invariant Riemannian metric on the Lie group N and conversely given a left invariant Riemannian metric $\langle ., . \rangle_N$ on the Lie group N this determines an Euclidean inner product $\langle ., . \rangle$ on $\mathfrak{n}$. An automorphism of $(\mathfrak{n}, \langle ., . \rangle)$ is called orthogonal automorphism, if it preserves the inner product on $\mathfrak{n}$. We denote by $\mathcal{OA}(\mathfrak{n})$ the group of orthogonal automorphisms of $(\mathfrak{n}, \langle ., . \rangle)$. A connected Riemannian manifold M which admits a transitive nilpotent Lie group of isometries is said to be a Riemannian nilmanifold. For every Riemannian nilmanifold M there exists a unique nilpotent Lie subgroup N of the group $\mathcal{I}(M)$ of isometries of M which acts simply transitively on M (see [52], Theorem 2). Therefore the Riemannian nilmanifold M can be treated as the nilpotent Lie group N equipped with a left invariant metric $\langle ., . \rangle_N$. Furthermore, the group $\mathcal{I}(N)$ of all isometries of $(N, \langle ., . \rangle_N)$ is the semi-direct product $N \rtimes \mathcal{OA}(\mathfrak{n})$ of the group N and the group $\mathcal{OA}(\mathfrak{n})$ such that $(\mathfrak{n}, \langle ., . \rangle)$ is the metric Lie algebra corresponding to $(N, \langle ., . \rangle_N)$. Therefore the classification of the connected and simply connected Riemannian nilmanifolds up to isometry is equivalent to the determination of the classes of isometrically isomorphic metric nilpotent Lie algebras. Hence the determination of the isometry equivalence classes of nilmanifolds can be successfully carried out by working in their metric Lie algebras ([52, Theorem 3]). Each filiform metric Lie algebra has a decomposition into the direct sum of one-dimensional orthogonal subspaces such that this decomposition is preserved under all orthogonal automorphisms of the metric Lie algebra. This motivates the following definition.

Definition 2.1 ([22, Definition 2]). *An orthogonal direct sum decomposition $\mathfrak{n} = V_1 \oplus \ldots \oplus V_n$ on one-dimensional subspaces $V_1, \ldots, V_n$ of a metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is called a framing, if any orthogonal automorphism of $(\mathfrak{n}, \langle .,. \rangle)$ preserves this decomposition. An orthonormal basis $\{G_1, G_2, \ldots, G_n\}$ of a metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is adapted to the framing $\mathfrak{n} = V_1 \oplus \ldots \oplus V_n$ if $V_i = \mathbb{R}\, G_i$ for $i \in \{1, \ldots, n\}$. The metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is said to be framed, if it has a framing.*

Theorem 2.2 ([22, Theorem 4]). *Any filiform metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ of dimension at least 4 has a framing.*

In particular, any filiform metric Lie algebra possesses a descending series of ideals such that it is invariant under all automorphisms of $\mathfrak{n}$ and the dimension of the consecutive members of the series decreases by one. In this case a framing can be found easily using this series. Hence every filiform metric Lie algebra has a framing determined by ideals (see [20]).

The structure constants for isometrically isomorphic framed metric Lie algebras are characterized as follows.

Lemma 2.3 ([22, Lemma 1]). *Let $(\mathfrak{n}, \langle .,. \rangle)$ and $(\mathfrak{n}^*, \langle .,. \rangle^*)$ be isometrically isomorphic framed metric Lie algebras of dimension n with framings $\mathfrak{n} = \mathbb{R}\, G_1 \oplus \ldots \oplus \mathbb{R}\, G_n$ and $\mathfrak{n}^* = \mathbb{R}\, G_1^* \oplus \ldots \oplus \mathbb{R}\, G_n^*$, where $\{G_1, \ldots, G_n\}$, respectively $\{G_1^*, \ldots, G_n^*\}$ are orthonormal bases. If the commutators $[.,.]$ of $\mathfrak{n}$ and $[.,.]^*$ of $\mathfrak{n}^*$ are of the form*

$$[G_i, G_j] = \sum_{k=1}^{n} a_{i,j}^k G_k \quad \textit{and} \quad [G_i^*, G_j^*]^* = \sum_{k=1}^{n} {a^*}_{i,j}^k G_k^*,$$

then $a_{i,j}^k = \pm {a^*}_{i,j}^k$ for all $i, j, k \in \{1, \ldots, n\}$. In particular, if $a_{i,j}^k, {a^*}_{i,j}^k \geq 0$, then we obtain $a_{i,j}^k = {a^*}_{i,j}^k$.

Taking into account Lemma 2.3 the group $\mathcal{OA}(\mathfrak{n})$ of a framed metric nilpotent Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is a subgroup of the group $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ such that the number of the factors is less than or equal to the dimension of $\mathfrak{n}$. In this case the connected component of the group $\mathcal{I}(N)$ of isometries of the corresponding nilmanifold $(N, \langle ., . \rangle_N)$ is isomorphic to the Lie group N.

We say that a submanifold M^* of a Riemannian manifold M is totally geodesic, if each geodesic of M^* is a geodesic of M, too. A Riemannian manifold M is called flat, if the Riemannian curvature tensor of M is everywhere zero.

The exponential map $\exp : \mathfrak{n} \to N$ is a diffeomorphism for a simply connected nilpotent Lie group N. As the tangent space $T_e N$ can be identify by $\mathfrak{n}$ we can identify an element of $\mathfrak{n}$ by a left invariant vector field on N. Denote by $\nabla_\xi \eta$ the covariant derivative on $\mathfrak{n}$, where ξ, η are left invariant vector fields on N. So $\nabla_\xi \eta$ is also left invariant. The Levi-Civita connection ∇ on the metric Lie algebra $\mathfrak{n}$ is defined by

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle)$$

for all left invariant vector fields $X, Y, Z \in \mathfrak{n}$ (cf. [35, (5.3), pp. 310]). A subalgebra $\mathfrak{h}$ of $(\mathfrak{n}, \langle ., . \rangle)$ is called *totally geodesic*, respectively *flat*, if $\nabla_X Y \in \mathfrak{h}$, respectively $\nabla_X Y = 0$ for all $X, Y \in \mathfrak{h}$. A totally geodesic, respectively flat subgroup H of a Lie group N with left invariant Riemannian metric $\langle ., . \rangle_N$ is a Lie subgroup which is also totally geodesic, respectively flat as a submanifold. A subalgebra $\mathfrak{h}$ is totally geodesic, respectively flat if and only if its exponential image H is totally geodesic, respectively flat submanifold in the induced left invariant Riemannian manifold on the group N. We say that a non-zero vector $X \in \mathfrak{n}$ is *geodesic* if the generated subalgebra $\{tX; t \in \mathbb{R}\}$ is totally geodesic. Throughout Chapter 2 we use the following criteria for a subalgebra to be totally geodesic.

Lemma 2.4 ([12, Lemma 1.2, Remark 1.4]). *A subalgebra $\mathfrak{h}$ of a metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is totally geodesic if and only if one has*

$$\langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle = 0 \tag{2.1}$$

for all $Y, Z \in \mathfrak{h}$ *and* X *in the orthogonal complement* $\mathfrak{h}^\perp$ *of* $\mathfrak{h}$ *in* $(\mathfrak{n}, \langle ., . \rangle)$. *Furthermore, a non-zero vector* $Y \in (\mathfrak{n}, \langle ., . \rangle)$ *is geodesic precisely if for all* $X \in (\mathfrak{n}, \langle ., . \rangle)$ *we have*

$$\langle [X, Y], Y \rangle = 0. \tag{2.2}$$

P. T. Nagy and S. Homolya proved the following useful relations between abelian subalgebras and flat subalgebras as well as between flat totally geodesic subalgebras and geodesic vectors.

Lemma 2.5 ([36, Lemmas 1, 2])**.** *Let* $\mathfrak{h}$ *be a subalgebra in the nilpotent metric Lie algebra* $(\mathfrak{n}, \langle ., . \rangle)$. *The subalgebra* $\mathfrak{h}$ *is flat if and only if it is abelian. The subalgebra* $\mathfrak{h}$ *is flat totally geodesic if and only if all non-zero elements of* $\mathfrak{h}$ *are geodesic.*

Let $(\mathfrak{n}, \langle ., . \rangle)$ be a k-step nilpotent metric Lie algebra. The center ζ of $\mathfrak{n}$ is a proper subalgebra in the ideal $\ell^{k-2}\mathfrak{n}$. Let $\mathfrak{a}_{k-1}$ denote the orthogonal complement of the center ζ in $\ell^{k-2}\mathfrak{n}$. We obtain $\ell^{k-2}\mathfrak{n} = \mathfrak{a}_{k-1} \oplus \zeta$ and $\ell^{k-1}\mathfrak{n} = [\ell^{k-2}\mathfrak{n}, \mathfrak{n}] = [\mathfrak{a}_{k-1}, \mathfrak{n}] \subseteq \zeta$. Denote by $\mathfrak{a}_j$ the orthogonal complement of $\ell^j\mathfrak{n}$ in $\ell^{j-1}\mathfrak{n}$, $j \in \{1, 2, \dots, k-2\}$. One gets $\mathfrak{n} = \oplus_{j=1}^{k-1}\mathfrak{a}_j \oplus \zeta$, $[\mathfrak{a}_j, \mathfrak{n}] \subseteq \ell^j\mathfrak{n}$ and for $i < j$ we have $[\mathfrak{a}_i, \mathfrak{a}_j] \subseteq \ell^j\mathfrak{n}$ (see [5, pp. 662]).

Let $(\mathfrak{n}, \langle ., . \rangle)$ be a 6-dimensional filiform metric Lie algebra and $\{E_1, E_2, \cdots, E_6\}$ be its orthonormal basis on the Euclidean vector space $\mathbb{E}^6$. The lower central series of $(\mathfrak{n}, \langle ., . \rangle)$ is given by

$$\ell^1\mathfrak{n} = \text{span}(E_3, E_4, E_5, E_6),$$
$$\ell^2\mathfrak{n} = \text{span}(E_4, E_5, E_6),$$
$$\ell^3\mathfrak{n} = \text{span}(E_5, E_6) \text{ and}$$
$$\ell^4\mathfrak{n} = \text{span}(E_6) \text{ which is the center } \zeta \text{ of } \mathfrak{n}.$$

Hence we have

$$\mathfrak{a}_1 = \text{span}(E_1, E_2),$$
$$\mathfrak{a}_2 = \text{span}(E_3),$$
$$\mathfrak{a}_3 = \text{span}(E_4),$$
$$\mathfrak{a}_4 = \text{span}(E_5) \text{ and } \mathfrak{n} = \oplus_{i=1}^{4}\mathfrak{a}_i \oplus \zeta.$$

The following Lemma, which generalizes the results of [36, Lemma 4, Corollary 5] was proved in [5, Corollary 2.4].

Lemma 2.6. *Let $(\mathfrak{n}, \langle ., . \rangle)$ be a k-step nilpotent metric Lie algebra, $k \geq 2$. Each non-zero vector in $\cup_{i=1}^{k-1} \mathfrak{a}_i \cup \zeta$ is geodesic. Any subalgebra contained in $\cup_{i=1}^{k-1} \mathfrak{a}_i \cup \zeta$ is flat totally geodesic.*

G. Cairns, A. Hinić Galić, and Y. A. Nikolayevsky proved the following claims about the dimension of totally geodesic subalgebras of filiform metric Lie algebras (see [12, Proposition 1.13, Theorems 1.17, 1.18]).

Proposition 2.7. *a) Filiform nilpotent metric Lie algebras of dimension n do not have any totally geodesic subalgebra of dimension $n-1$.*

b) An n-dimensional filiform nilpotent metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ possesses a totally geodesic subalgebra of dimension $n-2$ if and only if the Lie algebra $\mathfrak{n}$ is isomorphic to the standard filiform Lie algebra of dimension $n \geq 3$.

The following result shows that there exists filiform Lie algebra which allows only low dimensional totally geodesic subalgebras.

Theorem 2.8 ([12, Theorems 1.19]). *Let $\mathfrak{n}$ be the 6-dimensional filiform Lie algebra defined by*

$$[E_1, E_i] = E_{i+1}, \quad i = 2, \ldots, 5, \quad [E_2, E_3] = -E_6.$$

For no choice of inner product, does $\mathfrak{n}$ possess a totally geodesic subalgebra of dimension greater than two.

2.2 Isometry equivalence classes and isometry group of 6-dimensional filiform nilmanifolds

In this section we consider the classification given by W. A. de Graaf in [16]. The 6-dimensional filiform Lie algebras are defined by the following non-vanishing commutators (see [16, pp. 646-647]),

$$
\begin{aligned}
\mathfrak{l}_{6,14} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_3] = G_5,\\
&[G_2, G_5] = G_6,\ [G_4, G_3] = G_6,\\
\mathfrak{l}_{6,15} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_3] = G_5,\\
&[G_1, G_5] = G_6,\ [G_2, G_4] = G_6,\\
\mathfrak{l}_{6,16} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_5] = G_6,\\
&[G_4, G_3] = G_6,\\
\mathfrak{l}_{6,17} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_1, G_5] = G_6,\\
&[G_2, G_3] = G_6,\\
\mathfrak{l}_{6,18} :&[G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_1, G_5] = G_6,
\end{aligned} \tag{2.3}
$$

with respect to a distinguished basis $\{G_1, G_2, G_3, G_4, G_5, G_6\}$ which is called the canonical basis of the corresponding Lie algebra.

For the classification of 6-dimensional filiform metric Lie algebras up to isometric isomorphisms, we take into account the following organized approach (see [22, pp. 371-372]):

[T1] Firstly, we consider the Gram-Schmidt process on the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ in the metric Lie algebra $(\mathfrak{l}, \langle ., .\rangle)$ to obtain an orthonormal basis $\mathcal{F} = \{F_1, F_2, \cdots, F_6\}$ expressed by $F_i = \sum_{k=i}^{6} a_{ik}G_k, a_{ik} \in \mathbb{R}$, such that $a_{ii} \geq 0$.

[T2] We define a Lie bracket on the Euclidean vector space $\mathbb{E}^6$ with the same structure coefficients with respect to its distinguished basis $\mathcal{E}$ as the metric Lie algebra $(\mathfrak{l}, \langle ., .\rangle)$ has with respect to its basis F. The obtained metric Lie algebra $(\mathfrak{n}, \langle ., .\rangle)$ on $\mathbb{E}^6$ determined by the structure coefficients is isometrically isomorphic to $(\mathfrak{l}, \langle ., .\rangle)$.

[T3] Finally, we have to find conditions on the real parameters of metric Lie algebras on $\mathbb{E}^6$ to get a one-to-one correspondence between the equivalence classes of isometrically isomorphic metric Lie algebras and a family of metric Lie algebras on $\mathbb{E}^6$.

Our first aim is to study the 6-dimensional filiform Lie algebra $\mathfrak{l}_{6,14}$.

Definition 2.9. *Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$, $\alpha_i, \beta_j \in \mathbb{R}, i \in \{1, \ldots, 5\}, j \in \{1, \ldots, 7\}$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators*

$$
\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_2, E_3] = \alpha_3 E_5 + \beta_6 E_6,\\
&[E_1, E_3] = \alpha_2 E_4 - \Big(\frac{\alpha_5 \beta_1 + \alpha_2 \beta_7}{\alpha_4}\Big) E_5 + \beta_4 E_6, && [E_2, E_4] = \beta_7 E_6,\\
& && [E_2, E_5] = \alpha_4 E_6,\\
&[E_1, E_4] = \frac{\alpha_1 \alpha_5}{\alpha_4} E_5 + \beta_5 E_6, && [E_4, E_3] = \alpha_5 E_6.
\end{aligned}
\tag{2.4}
$$

The bracket operation (2.4) *satisfies the Jacobi identity.*

In what follows, let us present the following result, which states:

Theorem 2.10. *Let $\langle ., . \rangle$ be an inner product on the 6-dimensional filiform Lie algebra $\mathfrak{l}_{6,14}$.*

1. *There is a unique metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ with $\alpha_i, \beta_j \in \mathbb{R}$ such that $\alpha_i > 0$ satisfying:*

   (a) $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{6,14}, \langle ., . \rangle)$,

   (b) if the set $J = \{j \in \{1, 4, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$.

2. *The group $\mathcal{OA}(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ is the group:*

   (a) if the set $J = \emptyset$, then we receive
   $\mathcal{OA}(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle) = \{TE_i = \varepsilon E_i, i \in \{1, 3, 5, 6\}, \varepsilon = \pm 1,\ TE_j = E_j, j \in \{2, 4\}\} \simeq \mathbb{Z}_2$,

   (b) if the set $J \neq \emptyset$, then we obtain that $\mathcal{OA}(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ is trivial.

Proof. Firstly we apply the Gram-Schmidt process on the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of the metric Lie algebra $(\mathfrak{l}_{6,14}, \langle .,. \rangle)$ such that the vector F_i is a positive multiple of G_i modulo the subspace span$(G_j; j > i)$ and orthogonal to span$(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \ldots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,14}, \langle .,. \rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$ we get

$$\begin{aligned}
&[F_1, F_2] = \alpha_1 F_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, && [F_2, F_3] = \alpha_3 F_5 + \beta_6 F_6,\\
&[F_1, F_3] = \alpha_2 F_4 - \Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\Big) F_5 + \beta_4 F_6, && [F_2, F_4] = \beta_7 F_6,\\
& && [F_2, F_5] = \alpha_4 F_6,\\
&[F_1, F_4] = \frac{\alpha_1\alpha_5}{\alpha_4} F_5 + \beta_5 F_6, && [F_4, F_3] = \alpha_5 F_6,
\end{aligned} \tag{2.5}$$

where $\alpha_i > 0, i \in \{1, \ldots, 5\}$ and $\beta_j \in \mathbb{R}, j \in \{1, \ldots, 7\}$. Changing the orthonormal basis as follow $\tilde{F_1} = -F_1,\ \tilde{F_2} = F_2,\ \tilde{F_3} = -F_3,\ \tilde{F_4} = F_4,\ \tilde{F_5} = -F_5,\ \tilde{F_6} = -F_6$ we obtain

$$\begin{aligned}
&[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_3} - \beta_1 \tilde{F_4} + \beta_2 \tilde{F_5} + \beta_3 \tilde{F_6}, && [\tilde{F_2}, \tilde{F_3}] = \alpha_3 \tilde{F_5} + \beta_6 \tilde{F_6},\\
&[\tilde{F_1}, \tilde{F_3}] = \alpha_2 \tilde{F_4} + \Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\Big) \tilde{F_5} - \beta_4 \tilde{F_6}, && [\tilde{F_2}, \tilde{F_4}] = -\beta_7 \tilde{F_6},\\
& && [\tilde{F_2}, \tilde{F_5}] = \alpha_4 \tilde{F_6},\\
&[\tilde{F_1}, \tilde{F_4}] = \frac{\alpha_1\alpha_5}{\alpha_4} \tilde{F_5} + \beta_5 \tilde{F_6}, && [\tilde{F_4}, \tilde{F_3}] = \alpha_5 \tilde{F_6}.
\end{aligned}$$

Thereupon, there is an orthonormal basis satisfying (2.5) such that if the set $J = \{j \in \{1, 4, 7\}, \beta_j \neq 0\} \neq \emptyset$, then we may assume that $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. Therefore, the existence of metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle .,. \rangle)$ satisfying Theorem 2.10 (1) is proved. To prove the uniquence, let $T : (\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle .,. \rangle) \to (\mathfrak{n}_{6,14}(\alpha'_i, \beta'_j), \langle .,. \rangle)$ be an isometric isomorphism. The decomposition $\mathbb{R}E_1 \oplus \ldots \oplus \mathbb{R}E_6$ is a framing of both metric Lie algebras, where $\alpha_i, \alpha'_i > 0, i \in \{1, \ldots, 5\}$. In view of Lemma 2.3, we have that $\alpha_i = \alpha'_i, i \in \{1, \ldots, 5\}$ and $|\beta'_j| = \beta_j,\ j \in \{1, \ldots, 7\}$. Let us consider $T(E_i) = \varepsilon_i E_i,\ \varepsilon_i = \pm 1,\ i \in \{1, \ldots, 6\}$. Thus we obtain from $[TE_i, TE_j]' = T[E_i, E_j],\ i, j \in \{1, \ldots, 6\}$, using the commutation relations

(2.5), the equations

$$\begin{aligned}
&\varepsilon_1\varepsilon_2\left(\alpha_1E_3+\beta_1'E_4+\beta_2'E_5+\beta_3'E_6\right)=\\
&\qquad\qquad\alpha_1\varepsilon_3E_3+\beta_1\varepsilon_4E_4+\beta_2\varepsilon_5E_5+\beta_3\varepsilon_6E_6,\\
&\varepsilon_1\varepsilon_3\Big(\alpha_2E_4-(\frac{\alpha_5\beta_1'+\alpha_2\beta_7'}{\alpha_4})E_5+\beta_4'E_6\Big)=\\
&\qquad\qquad\alpha_2\varepsilon_4E_4-(\frac{\alpha_5\beta_1+\alpha_2\beta_7}{\alpha_4})\varepsilon_5E_5+\beta_4\varepsilon_6E_6,\\
&\varepsilon_1\varepsilon_4\Big(\frac{\alpha_1\alpha_5}{\alpha_4}E_5+\beta_5'E_6\Big)=\frac{\alpha_1\alpha_5}{\alpha_4}\varepsilon_5E_5+\beta_5\varepsilon_6E_6,\\
&\varepsilon_2\varepsilon_3\left(\alpha_3E_5+\beta_6'E_6\right)=\alpha_3\varepsilon_5E_5+\beta_6\varepsilon_6E_6,\\
&\varepsilon_2\varepsilon_4\left(\beta_7'E_6\right)=\beta_7\varepsilon_6E_6,\\
&\varepsilon_2\varepsilon_5\left(\alpha_4E_6\right)=\alpha_4\varepsilon_6E_6,\\
&\varepsilon_4\varepsilon_3\left(\alpha_5E_6\right)=\alpha_5\varepsilon_6E_6.
\end{aligned}\tag{2.6}$$

From (2.6) it follows that $\varepsilon_1\varepsilon_2=\varepsilon_3,\varepsilon_1\varepsilon_3=\varepsilon_4,\varepsilon_1\varepsilon_4=\varepsilon_2\varepsilon_3=\varepsilon_5,\varepsilon_2\varepsilon_5=\varepsilon_4\varepsilon_3=\varepsilon_6$, which implies that $\varepsilon_2=\varepsilon_4=1$, $\varepsilon_1=\varepsilon_3=\varepsilon_5=\varepsilon_6$. Hence we get $\varepsilon_1\varepsilon_2=\varepsilon_1=\varepsilon_5$, $\varepsilon_1\varepsilon_2=\varepsilon_1=\varepsilon_6$, $\varepsilon_1\varepsilon_4=\varepsilon_1=\varepsilon_6$, $\varepsilon_2\varepsilon_3=\varepsilon_3=\varepsilon_6$. Therefore we obtain that $\beta_2'=\beta_2$, $\beta_3'=\beta_3$, $\beta_5'=\beta_5$, $\beta_6'=\beta_6$. Assume that $J\neq\emptyset$. If $\beta_1=\beta_1'>0$, then one has the additional condition $\varepsilon_1\varepsilon_2=\varepsilon_4$. If $\beta_4=\beta_4'>0$, then we get that $\varepsilon_1\varepsilon_3=\varepsilon_6$. If $\beta_7=\beta_7'>0$, then we have that $\varepsilon_2\varepsilon_4=\varepsilon_6$. In all these cases we obtain that $\varepsilon_i=1,i\in\{1,\dots,6\}$. This proves that the metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle.,.\rangle)$ is uniquely determined.

Thus the only remaining part to be proved is that (2) holds. To this end, let us consider the map $E_i\longmapsto\varepsilon_iE_i$, which is an orthogonal automorphism of $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle.,.\rangle)$. In consequence, the system of equations given by (2.6) is satisfied with $\alpha_i>0,i\in\{1,\dots,5\}$, $\beta_j'=\beta_j,j\in\{1,\dots,7\}$. Therefore, in the case that $J=\emptyset$, which implies that $\varepsilon_2=\varepsilon_4=1$, $\varepsilon_1=\varepsilon_3=\varepsilon_5=\varepsilon_6$, the group of orthogonal automorphisms of $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle.,.\rangle)$ is isomorphic to the group given by (2a). On the assumption $J\neq\emptyset$, we obtain that $\varepsilon_i=1,i\in\{1,\dots,6\}$, which implies that the group of orthogonal automorphisms of $(\mathfrak{n}_{6,14}(\alpha_i,\beta_j),\langle.,.\rangle)$ is trivial. This is emphasized that the part (2) is valid. Hence Theorem 2.10 is proved. □

As an immediate consequence of the above theorem. We have the following:

Corollary 2.11. *Let* $(N_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the connected and simply connected Riemannian nilmanifold corresponding to* $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$. *The isometry group of* $(N_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ *is*

$$\mathcal{I}(N_{6,14}(\alpha_i, \beta_j)) = \begin{cases} \mathbb{Z}_2 \ltimes N_{6,14}(\alpha_i, \beta_j) & if\, J = \emptyset, \\ N_{6,14}(\alpha_i, \beta_j) & if\, J \neq \emptyset. \end{cases}$$

Now, we treat the 6-dimensional filiform Lie algebra $\mathfrak{l}_{6,15}$. Our method yields the following:

Definition 2.12. *Let* $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ *be an orthonormal basis in the Euclidean vector space* $\mathbb{E}^6$. *Denote by* $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle), \alpha_i, \beta_j \in \mathbb{R}, i \in \{1, \dots, 5\}, j \in \{1, \dots, 7\}$ *with* $\alpha_i \neq 0$ *the metric Lie algebra defined on* $\mathbb{E}^6$ *given by the non-vanishing commutators*

$$\begin{aligned} &[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_2, E_3] = \frac{\alpha_2 \alpha_5}{\alpha_4} E_5 + \beta_7 E_6, \\ &[E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, && [E_2, E_4] = \alpha_5 E_6. \\ &[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, \\ &[E_1, E_5] = \alpha_4 E_6, \end{aligned} \tag{2.7}$$

The bracket operation (2.7) *satisfies the Jacobi identity.*

Theorem 2.13. *Let* $\langle ., . \rangle$ *be an inner product on the* 6*-dimensional filiform Lie algebra* $\mathfrak{l}_{6,15}$.

1. *There is a unique metric Lie algebra* $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i, \beta_j \in \mathbb{R}$ *such that* $\alpha_i > 0$ *satisfying:*
   (a) $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ *is isometrically isomorphic to the metric Lie algebra* $(\mathfrak{l}_{6,15}, \langle ., . \rangle)$,
   (b) *if the set* $J = \{j \in \{1, 3, 4, 6, 7\} : \beta_j \neq 0\} \neq \emptyset$, *then* $\beta_{j_\circ} > 0$ *for the minimal element* $j_\circ \in J$.

2. *The group* $\mathcal{OA}\left(\mathfrak{n}_{6,15}(\alpha_i,\beta_j),\langle .,.\rangle\right)$ *of orthogonal automorphisms of the metric Lie algebra* $\left(\mathfrak{n}_{6,15}(\alpha_i,\beta_j),\langle .,.\rangle\right)$ *is the group:*

   (a) *if the set* $J=\emptyset$, *then we obtain* $\mathcal{OA}\left(\mathfrak{n}_{6,15}(\alpha_i,\beta_j),\langle .,.\rangle\right) = \{TE_i = \varepsilon E_i, i\in\{1,3,5\}, \varepsilon=\pm1, TE_j=E_j, j\in\{2,4,6\}\}\simeq\mathbb{Z}_2$,

   (b) *if the set* $J\neq\emptyset$, *then* $\mathcal{OA}\left(\mathfrak{n}_{6,15}(\alpha_i,\beta_j),\langle .,.\rangle\right)$ *is trivial.*

Proof. First off, our aim is to prove the existence property. To this end, we apply the Gram-Schmidt process on the ordered basis $\{G_6,G_5,G_4,G_3,G_2,G_1\}$, which yields the orthonormal basis $\{F_6,F_5,F_4,F_3,F_2,F_1\}$ of $\mathfrak{l}_{6,15}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span$(G_j; j>i)$ and orthogonal to span$(G_j; j>i)$. The orthogonal direct sum $\mathbb{R}F_1\oplus\ldots\oplus\mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,15},\langle .,.\rangle)$ and the vectors $F_i, i\in\{1,\ldots,6\}$ of the new basis can be written into the form $F_i=\sum_{k=1}^{6}a_{ik}G_k$ with $a_{ii}>0$. Hence we receive

$$\begin{aligned}
&[F_1,F_2]=\alpha_1F_3+\beta_1F_4+\beta_2F_5+\beta_3F_6, && [F_2,F_3]=\frac{\alpha_2\alpha_5}{\alpha_4}F_5+\beta_7F_6,\\
&[F_1,F_3]=\alpha_2F_4+\beta_4F_5+\beta_5F_6, && [F_2,F_4]=\alpha_5F_6,\\
&[F_1,F_4]=\alpha_3F_5+\beta_6F_6, &&\\
&[F_1,F_5]=\alpha_4F_6, &&
\end{aligned}\tag{2.8}$$

with $\alpha_i>0, i\in\{1,\ldots,5\}$ and $\beta_j\in\mathbb{R}, j\in\{1,\ldots,7\}$. Replacing the orthonormal basis $\{F_1,F_2,F_3,F_4,F_5,F_6\}$ to $\{-F_1,F_2,-F_3,F_4,-F_5,F_6\}$ we get

$$\begin{aligned}
&[F_1,F_2]=\alpha_1F_3-\beta_1F_4+\beta_2F_5-\beta_3F_6, && [F_2,F_3]=\frac{\alpha_2\alpha_5}{\alpha_4}F_5-\beta_7F_6,\\
&[F_1,F_3]=\alpha_2F_4-\beta_4F_5+\beta_5F_6, && [F_2,F_4]=\alpha_5F_6.\\
&[F_1,F_4]=\alpha_3F_5-\beta_6F_6, &&\\
&[F_1,F_5]=\alpha_4F_6, &&
\end{aligned}$$

Thus, there is an orthonormal basis satisfying (2.8) such that if the set $J=\{j\in\{1,3,4,6,7\},\beta_j\neq0\}\neq\emptyset$, then we may assume that $\beta_{j_\circ}>0$ for the minimal element $j_\circ \in J$. Hence the existence of the metric Lie algebra $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ satisfying Theorem 2.13 (1) is proved. To verify that $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is uniquely determined, let us consider an isometric isomorphism $T : (\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle) \to (\mathfrak{n}_{6,15}(\alpha_i', \beta_j'), \langle ., . \rangle)$. The decomposition $\mathbb{R}E_1 \oplus \ldots \oplus \mathbb{R}E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha_i' > 0, i \in \{1, \ldots, 5\}$. Therefore, according to Lemma 2.3, we have that $\alpha_i = \alpha_i', i \in \{1, \ldots, 5\}$ and $|\beta_j'| = \beta_j$, $j \in \{1, \ldots, 7\}$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i \in \{1, \ldots, 6\}$. Using the commutation relations (2.8) we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j \in \{1, \ldots, 6\}$, the equations

$$
\begin{aligned}
&\varepsilon_1\varepsilon_2 \left(\alpha_1 E_3 + \beta_1' E_4 + \beta_2' E_5 + \beta_3' E_6\right) = \\
&\qquad\qquad\qquad \alpha_1\varepsilon_3 E_3 + \beta_1\varepsilon_4 E_4 + \beta_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6, \\
&\varepsilon_1\varepsilon_3 \left(\alpha_2 E_4 + \beta_4' E_5 + \beta_5' E_6\right) = \alpha_2\varepsilon_4 E_4 + \beta_4\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6, \\
&\varepsilon_1\varepsilon_4 \left(\alpha_3 E_5 + \beta_6' E_6\right) = \alpha_3\varepsilon_5 E_5 + \beta_6\varepsilon_6 E_6, \\
&\varepsilon_2\varepsilon_3 \left(\frac{\alpha_2\alpha_5}{\alpha_4} E_5 + \beta_7' E_6\right) = \frac{\alpha_2\alpha_5}{\alpha_4}\varepsilon_5 E_5 + \beta_7\varepsilon_6 E_6, \\
&\varepsilon_1\varepsilon_5 \left(\alpha_4 E_6\right) = \alpha_4\varepsilon_6 E_6, \\
&\varepsilon_2\varepsilon_4 \left(\alpha_5 E_6\right) = \alpha_5\varepsilon_6 E_6.
\end{aligned}
\tag{2.9}
$$

It follows that $\varepsilon_1\varepsilon_2 = \varepsilon_3, \varepsilon_1\varepsilon_3 = \varepsilon_4, \varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_3 = \varepsilon_5, \varepsilon_1\varepsilon_5 = \varepsilon_2\varepsilon_4 = \varepsilon_6$, which implies that $\varepsilon_2 = \varepsilon_4 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$. Adopting these relations one obtains $\varepsilon_1\varepsilon_2 = \varepsilon_1 = \varepsilon_5$, $\varepsilon_1\varepsilon_3 = \varepsilon_6 = 1$. Therefore, we get that $\beta_2' = \beta_2$, $\beta_5' = \beta_5$. Assume that $J \neq \emptyset$. In case that $\beta_1 = \beta_1' > 0$ one has the addition condition $\varepsilon_1\varepsilon_2 = \varepsilon_4$. On the assumption that $\beta_3 = \beta_3' > 0$ we obtain that $\varepsilon_1\varepsilon_2 = \varepsilon_6$. Supposing that $\beta_4 = \beta_4' > 0$ we get $\varepsilon_1\varepsilon_3 = \varepsilon_5$. The condition $\beta_6 = \beta_6' > 0$ implies that $\varepsilon_1\varepsilon_4 = \varepsilon_6$. If $\beta_7 = \beta_7' > 0$, then we have $\varepsilon_2\varepsilon_3 = \varepsilon_6$. In this setting, all these cases yield that $\varepsilon_i = 1, i \in \{1, \ldots, 6\}$. Hence we obtain that the metric Lie algebra $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is uniquely determined, as it was desired, and with this proof of (1) ends.

We now discuss the second assertion, if the map $E_i \longmapsto \varepsilon_i E_i, \varepsilon_i = \pm 1, i \in \{1, \ldots, 6\}$ is an orthogonal automorphism of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$, then the system of equations given by (2.9) is satisfied with $\alpha_i > 0, i \in \{1, \ldots, 5\}$ and $\beta_j' = \beta_j, j \in \{1, \ldots, 7\}$. Suppose that $J = \emptyset$, which yields that $\varepsilon_2 = \varepsilon_4 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$. It follows that the group of orthogonal automorphisms of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is isomorphic to the group given by (2a). On the assumption that $J \neq \emptyset$, we obtain that $\varepsilon_i = 1, i \in \{1, \dots, 6\}$. Therefore, the group of orthogonal automorphisms of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is trivial, which proves the inclusion (2b). Hence assertion (2) is proved. Thus, the proof is complete. □

Exploiting the above statement, we provide the following corollary:

Corollary 2.14. *Let $(N_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$. The isometry group of $(N_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is*

$$\mathcal{I}(N_{6,15}(\alpha_i, \beta_j) = \begin{cases} \mathbb{Z}_2 \ltimes N_{6,15}(\alpha_i, \beta_j) & if J = \emptyset, \\ N_{6,15}(\alpha_i, \beta_j) & if J \neq \emptyset. \end{cases}$$

Now we consider the 6-dimensional filiform Lie algebra $\mathfrak{l}_{6,16}$. We describe the isometrically isomorphic equivalence classes of the metric Lie algebras $(\mathfrak{l}_{6,16}, \langle ., . \rangle)$ and the group of orthogonal automorphisms of the representatives of these classes.

Definition 2.15. *Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$, $\alpha_i, \beta_j \in \mathbb{R}, i \in \{1, \dots, 4\}, j \in \{1, \dots, 8\}$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators*

$$\begin{aligned} &[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_2, E_3] = \beta_7 E_6, \\ &[E_1, E_3] = \alpha_2 E_4 - \Big(\frac{\alpha_3 \beta_1}{\alpha_1} + \frac{\alpha_2 \beta_8}{\alpha_4}\Big) E_5 + \beta_4 E_6, && [E_2, E_4] = \beta_8 E_6, \\ & && [E_2, E_5] = \alpha_4 E_6, \\ &[E_1, E_4] = \alpha_3 E_5 + \beta_5 E_6, && [E_4, E_3] = \frac{\alpha_3 \alpha_4}{\alpha_1} E_6. \\ &[E_1, E_5] = \beta_6 E_6, && \end{aligned} \tag{2.10}$$

The bracket operation (2.10) *satisfies the Jacobi identity.*

Theorem 2.16. *Let $\langle ., . \rangle$ be an inner product on the 6-dimensional filiform Lie algebra $\mathfrak{l}_{6,16}$.*

1. *There is a unique metric Lie algebra* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ *which is isometrically isomorphic to the metric Lie algebra* $(\mathfrak{l}_{6,16}, \langle .,. \rangle)$ *with the properties that* $\alpha_i > 0, i \in \{1, \dots, 4\}$ *and one of the following cases is satisfied:*

   (a) $\beta_j = 0, j \in \{1, 3, 4, 5, 6, 8\}$,

   (b) $\beta_3 > 0$ *or* $\beta_5 > 0,\ \beta_j = 0, j \in \{1, 4, 6, 8\}$,

   (c) $\beta_6 > 0$ *or* $\beta_4 > 0,\ \beta_j = 0, j \in \{1, 3, 5, 8\}$,

   (d) $\beta_1 > 0$ *or* $\beta_8 > 0,\ \beta_j = 0, j \in \{3, 4, 5, 6\}$,

   (e) *at least two of the elements of the set* $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8\}$ *are positive with exception of the pairs* $\{\beta_1 > 0, \beta_8 > 0\}$, $\{\beta_3 > 0, \beta_5 > 0\}$, $\{\beta_4 > 0, \beta_6 > 0\}$.

2. *The group* $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ *of orthogonal automorphisms of the metric Lie algebra* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ *is the following group:*

   (a) *in case* (1a), *one has* $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle) = \{TE_i = \varepsilon_1 E_i, i \in \{1, 6\},\ TE_i = \varepsilon_2 E_i, i \in \{2, 4\},\ TE_i = \varepsilon_1\varepsilon_2 E_i, i \in \{3, 5\},\ \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,

   (b) *in case* (1b), *one has* $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle) = \{TE_j = E_j, j \in \{2, 4\},\ TE_i = \varepsilon E_i, i \in \{1, 3, 5, 6\},\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,

   (c) *in case* (1c), *we receive* $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle) = \{TE_j = E_j, j \in \{3, 5\},\ TE_i = \varepsilon E_i, i \in \{1, 2, 4, 6\},\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,

   (d) *in case* (1d), *we obtain* $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle) = \{TE_j = E_j, j \in \{1, 6\},\ TE_i = \varepsilon E_i, i \in \{2, 3, 4, 5\},\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,

   (e) *in case* (1e), *the group* $\mathcal{OA}(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ *is trivial.*

Proof. Firstly, we are going to prove the existence of the metric Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ which satisfies the inclusion (1). Applying the Gram-Schmidt process on the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$, we get an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of the Lie algebra $\mathfrak{l}_{6,16}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span$(G_j; j > i)$ and orthogonal to span$(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \ldots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,16}, \langle .,. \rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$ we get

$$\begin{aligned}
&[F_1, F_2] = \alpha_1 F_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, && [F_2, F_3] = \beta_7 F_6,\\
&[F_1, F_3] = \alpha_2 F_4 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) F_5 + \beta_4 F_6, && [F_2, F_4] = \beta_8 F_6,\\
& && [F_2, F_5] = \alpha_4 F_6,\\
&[F_1, F_4] = \alpha_3 F_5 + \beta_5 F_6, && [F_4, F_3] = \frac{\alpha_3\alpha_4}{\alpha_1} F_6,\\
&[F_1, F_5] = \beta_6 F_6,
\end{aligned} \tag{2.11}$$

with $\alpha_i > 0, i \in \{1, \dots 4\}$ and $\beta_j \in \mathbb{R}, j \in \{1, \dots, 8\}$. The change of the orthonormal basis $\tilde{F}_1 = -F_1,\ \tilde{F}_2 = F_2,\ \tilde{F}_3 = -F_3,\ \tilde{F}_4 = F_4,\ \tilde{F}_5 = -F_5,\ \tilde{F}_6 = -F_6$ gives

$$\begin{aligned}
&[\tilde{F}_1, \tilde{F}_2] = \alpha_1 \tilde{F}_3 - \beta_1 \tilde{F}_4 + \beta_2 \tilde{F}_5 + \beta_3 \tilde{F}_6, && [\tilde{F}_2, \tilde{F}_3] = \beta_7 \tilde{F}_6,\\
&[\tilde{F}_1, \tilde{F}_3] = \alpha_2 \tilde{F}_4 + \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) \tilde{F}_5 - \beta_4 \tilde{F}_6, && [\tilde{F}_2, \tilde{F}_4] = -\beta_8 \tilde{F}_6,\\
& && [\tilde{F}_2, \tilde{F}_5] = \alpha_4 \tilde{F}_6,\\
&[\tilde{F}_1, \tilde{F}_4] = \alpha_3 \tilde{F}_5 + \beta_5 \tilde{F}_6, && [\tilde{F}_4, \tilde{F}_3] = \frac{\alpha_3\alpha_4}{\alpha_1} \tilde{F}_6.\\
&[\tilde{F}_1, \tilde{F}_5] = -\beta_6 \tilde{F}_6,
\end{aligned}$$

Applying another change as follows $\tilde{F}_1 = F_1,\ \tilde{F}_2 = -F_2,\ \tilde{F}_3 = -F_3,\ \tilde{F}_4 = -F_4,\ \tilde{F}_5 = -F_5,\ \tilde{F}_6 = F_6$ we obtain

$$\begin{aligned}
&[\tilde{F}_1, \tilde{F}_2] = \alpha_1 \tilde{F}_3 + \beta_1 \tilde{F}_4 + \beta_2 \tilde{F}_5 - \beta_3 \tilde{F}_6, && [\tilde{F}_2, \tilde{F}_3] = \beta_7 \tilde{F}_6,\\
&[\tilde{F}_1, \tilde{F}_3] = \alpha_2 \tilde{F}_4 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) \tilde{F}_5 - \beta_4 \tilde{F}_6, && [\tilde{F}_2, \tilde{F}_4] = \beta_8 \tilde{F}_6,\\
& && [\tilde{F}_2, \tilde{F}_5] = \alpha_4 \tilde{F}_6,\\
&[\tilde{F}_1, \tilde{F}_4] = \alpha_3 \tilde{F}_5 - \beta_5 \tilde{F}_6, && [\tilde{F}_4, \tilde{F}_3] = \frac{\alpha_3\alpha_4}{\alpha_1} \tilde{F}_6.\\
&[\tilde{F}_1, \tilde{F}_5] = -\beta_6 \tilde{F}_6,
\end{aligned}$$

Accordingly, there is an orthonormal basis such that in commutators (2.11) one has $\alpha_i > 0, i = 1, \dots, 4$, and one of the cases in assertion 1. is satisfied. This proves the existence of $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ with the properties in assertion 1. To prove the uniqueness of metric the Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ let

$T : (\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle) \to (\mathfrak{n}_{6,16}(\alpha'_i, \beta'_j), \langle ., . \rangle)$ be an isometric isomorphism. The decomposition $\mathbb{R}E_1 \oplus \ldots \oplus \mathbb{R}E_6$ is a framing of both metric Lie algebras, where $\alpha_1, \alpha'_1, \alpha_2, \alpha'_2, \alpha_3, \alpha'_3, \alpha_4, \alpha'_4 > 0$. In view of Lemma 2.3, we obtain that $\alpha_i = \alpha'_i > 0, i \in \{1,2,3,4\}$ and $|\beta'_j| = \beta_j$, $j \in \{1, \ldots, 8\}$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1, i \in \{1, \ldots, 6\}$. Using the commutation relations (2.11) and the form $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j \in \{1, \ldots, 6\}$, we obtain straightforwardly the following equations

$$\begin{aligned}
&\varepsilon_1\varepsilon_2 \left(\alpha_1 E_3 + \beta'_1 E_4 + \beta'_2 E_5 + \beta'_3 E_6\right) = \\
&\qquad\qquad \alpha_1\varepsilon_3 E_3 + \beta_1\varepsilon_4 E_4 + \beta_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6, \\
&\varepsilon_1\varepsilon_3 \Big(\alpha_2 E_4 - (\frac{\alpha_3\beta'_1}{\alpha_1} + \frac{\alpha_2\beta'_8}{\alpha_4})E_5 + \beta'_4 E_6\Big) = \\
&\qquad\qquad \alpha_2\varepsilon_4 E_4 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})\varepsilon_5 E_5 + \beta_4\varepsilon_6 E_6, \\
&\varepsilon_1\varepsilon_4 \left(\alpha_3 E_5 + \beta'_5 E_6\right) = \alpha_3\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6, \\
&\varepsilon_1\varepsilon_5 \left(\beta'_6 E_6\right) = \beta_6\varepsilon_6 E_6, \\
&\varepsilon_2\varepsilon_3 \left(\beta'_7 E_6\right) = \beta_7\varepsilon_6 E_6, \\
&\varepsilon_2\varepsilon_4 \left(\beta'_8 E_6\right) = \beta_8\varepsilon_6 E_6, \\
&\varepsilon_2\varepsilon_5 \left(\alpha_4 E_6\right) = \alpha_4\varepsilon_6 E_6, \\
&\varepsilon_4\varepsilon_3 \Big(\frac{\alpha_3\alpha_4}{\alpha_1} E_6\Big) = \frac{\alpha_3\alpha_4}{\alpha_1}\varepsilon_6 E_6.
\end{aligned} \tag{2.12}$$

From (2.12) it follows that $\varepsilon_1\varepsilon_2 = \varepsilon_3, \varepsilon_1\varepsilon_3 = \varepsilon_4, \varepsilon_1\varepsilon_4 = \varepsilon_5, \varepsilon_2\varepsilon_5 = \varepsilon_4\varepsilon_3 = \varepsilon_6$, which implies that $\varepsilon_4 = \varepsilon_2, \varepsilon_1\varepsilon_2 = \varepsilon_3 = \varepsilon_5, \varepsilon_6 = \varepsilon_1$. Using these relations we obtain that $\varepsilon_1\varepsilon_2 = \varepsilon_5$, $\varepsilon_2\varepsilon_3 = \varepsilon_1 = \varepsilon_6$. Therefore one has $\beta'_2 = \beta_2$, $\beta'_7 = \beta_7$. If $\beta_1 = \beta'_1 > 0$ or $\beta_8 = \beta'_8 > 0$, then we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_4$ or $\varepsilon_2\varepsilon_4 = \varepsilon_6$. In both cases we obtain $\varepsilon_1 = \varepsilon_6 = 1$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5$. If $\beta_3 = \beta'_3 > 0$ or $\beta_5 = \beta'_5 > 0$, then we obtain in addition $\varepsilon_1\varepsilon_2 = \varepsilon_6$ or $= \varepsilon_1\varepsilon_4 = \varepsilon_6$. Therefore in both cases we receive $\varepsilon_2 = \varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6$. If $\beta_4 = \beta'_4 = 0$ or $\beta_6 = \beta'_6 > 0$, then we have in addition $\varepsilon_1\varepsilon_3 = \varepsilon_6$ or $\varepsilon_1\varepsilon_5 = \varepsilon_6$ and hence in both cases $\varepsilon_3 = \varepsilon_5 = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_6$.
Using these relations in assertion 1. of the Theorem
in case (1a) we get $\varepsilon_2 = \varepsilon_4$, $\varepsilon_1\varepsilon_2 = \varepsilon_3 = \varepsilon_5$, $\varepsilon_2\varepsilon_3 = \varepsilon_6 = \varepsilon_1$,

in case (1b) we obtain $\varepsilon_2 = \varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6$,
in case (1c) we have $\varepsilon_3 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_6$,
in case (1d) we receive $\varepsilon_1 = \varepsilon_6 = 1$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5$,
in case (1e) we get $\varepsilon_i = 1, i = 1, \ldots, 6$.
Hence the system of equations (2.12) holds with $\beta'_j = \beta_j, j = 1, \ldots, 7$ in cases (1a)- (1e) of the Theorem. Therefore the uniqueness of the Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ in cases (1a)- (1e) is proved. This yields assertion 1.

At this stage, (2) is the only remaining part to be proved. To this end, we consider the map $E_i \longmapsto \varepsilon_i E_i, \varepsilon_i = \pm 1, i \in \{1, \ldots, 6\}$ which is an orthogonal automorphism of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$. Therefore, the system of equations given by (2.12) is satisfied with $\alpha_i > 0, i \in \{1, \ldots, 4\}$ and $\beta'_j = \beta_j, j \in \{1, \ldots, 8\}$. Therefore in cases (1a)- (1e) the conditions for $\varepsilon_i, i = 1, \ldots, 6$, are given above. Taking this into consideration the group of orthogonal automorphisms of $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ in case $1e$ is trivial, in cases (1a)- (1d) is isomorphic to the group given by (2a) - (2d). This proves assertion (2). This completes the proof of the theorem 2.16. □

Due to the above discussion, we can conclude the following:

Corollary 2.17. *Let $(N_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{6,16}((\alpha_i, \beta_j), \langle ., . \rangle)$. The isometry group of $(N_{6,16}((\alpha_i, \beta_j), \langle ., . \rangle)$ is*

$$\mathcal{I}(\aleph_{6,16}(\alpha_i, \beta_j) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes N_{6,16}(\alpha_i, \beta_j) & \textit{if } (1a) \textit{ holds}, \\ \mathbb{Z}_2 \ltimes N_{6,16}(\alpha_i, \beta_j) & \textit{if } (1b), (1c), (1d) \textit{ are satisfied}, \\ N_{6,16}(\alpha_i, \beta_j) & \textit{if } (1e) \textit{ holds}. \end{cases}$$

Finally, we consider the 6-dimensional filiform Lie algebras $\mathfrak{l}_{6,17}$ and $\mathfrak{l}_{6,18}$ with their canonical basis $\{G_1, G_2, G_3, G_4, G_5, G_6\}$.

Definition 2.18. *Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. We denote by $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle), \alpha_i, \beta_j \in \mathbb{R}, i \in \{1, \ldots, 4\}, j \in \{1, \ldots, 6\}$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ by* *the non-vanishing commutators*

$$\begin{aligned} &[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, \quad &&[E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, \\ &[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, &&[E_1, E_5] = \alpha_4 E_6. \end{aligned} \tag{2.13}$$

Denote by $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle), \alpha_i, \beta_j \in \mathbb{R}, i \in \{1, \ldots, 5\}, j \in \{1, \ldots, 6\}$ *with* $\alpha_i \neq 0$ *the metric Lie algebra defined on* $\mathbb{E}^6$ *given by* (2.13) *and by the additional commutator*

$$[E_2, E_3] = \alpha_5 E_6. \tag{2.14}$$

The bracket operations (2.13) *as well as* (2.14) *satisfy the Jacobi identity.*

Theorem 2.19. *Let* $\langle ., . \rangle$ *be an inner product on the* 6*-dimensional filiform Lie algebra* $\mathfrak{l}_{6,k}, k \in \{17, 18\}$.

1. *There is a unique metric Lie algebra* $(\mathfrak{n}_{6,k}(\alpha_i, \beta_j), \langle ., . \rangle), k \in \{17, 18\}$, *with* $\alpha_i, \beta_j \in \mathbb{R}$ *such that* $\alpha_i > 0$ *satisfying:*
   (a) $(\mathfrak{n}_{6,k}(\alpha_i, \beta_j), \langle ., . \rangle), k \in \{17, 18\}$*, is isometrically isomorphic to the metric Lie algebra* $(\mathfrak{l}_{6,k}, \langle ., . \rangle), k \in \{17, 18\}$,
   (b) *if the set* $J = \{j \in \{1, 3, 4, 6\} : \beta_j \neq 0\} \neq \emptyset$*, then* $\beta_{j_\circ} > 0$ *for the minimal element* $j_\circ \in J$.
2. *The group* $\mathcal{OA}(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *of orthogonal automorphisms of the metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *is the group:*
   (a) *if the set* $J = \emptyset$*, then we obtain* $\mathcal{OA}(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle) = \{TE_i = \varepsilon E_i, i \in \{1, 2, 4, 6\}, TE_3 = E_3, TE_5 = E_5, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
   (b) *if the set* $J \neq \emptyset$*, then* $\mathcal{OA}(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *is trivial.*
3. *The group* $\mathcal{OA}(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ *of orthogonal automorphisms of the metric Lie algebra* $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ *is the group:*
   (a) *if* $J = \emptyset$*, then we obtain* $\mathcal{OA}(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle) = \{TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_2 E_i, i \in \{2, 4, 6\}, TE_j = \varepsilon_1 \varepsilon_2 E_j, j \in \{3, 5\}, \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,

(b) *if $J \neq \emptyset$, then we receive*
$\mathcal{OA}\left(\mathfrak{n}_{6,18}(\alpha_i,\beta_j),\langle .,.\rangle\right) = \{TE_1 = E_1, TE_i = \varepsilon E_i, i \in \{2,\ldots,6\}, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.

Proof. Depending on the method considered, we obtain an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\mathfrak{l}_{6,17}$, respectively $\mathfrak{l}_{6,18}$ after applying the Gram-Schmidt process on the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ such that the vector F_i is a positive multiple of G_i modulo the subspace $\text{span}(G_j; j > i)$ and orthogonal to $\text{span}(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \ldots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,17},\langle .,.\rangle)$, respectively $(\mathfrak{l}_{6,18},\langle .,.\rangle)$. The vectors of the new basis have the form $F_i = \sum_{k=i}^{6} a_{ik}G_k$ with $a_{ii} > 0$. We get the metric Lie algebras $(\mathfrak{l}_{6,17},\langle .,.\rangle)$ and $(\mathfrak{l}_{6,18},\langle .,.\rangle)$

$$\begin{aligned} &[F_1,F_2] = \alpha_1 F_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, && [F_1,F_3] = \alpha_2 F_4 + \beta_4 F_5 + \beta_5 F_6,\\ &[F_1,F_4] = \alpha_3 F_5 + \beta_6 F_6, && [F_1,F_5] = \alpha_4 F_6, \end{aligned} \tag{2.15}$$

and for $(\mathfrak{l}_{6,17},\langle .,.\rangle)$ in addition

$$[F_2,F_3] = \alpha_5 F_6, \tag{2.16}$$

where $\alpha_i > 0$ $i \in \{1,\ldots,5\}$, and $\beta_j \in \mathbb{R}, j \in \{1,\ldots,6\}$. Changing the orthonormal basis as follow $F_1 \mapsto -F_1,\ F_2 \mapsto -F_2,\ F_3 \mapsto F_3,\ F_4 \mapsto -F_4,\ F_5 \mapsto F_5,\ F_6 \mapsto -F_6$ we obtain

$$\begin{aligned} &[F_1,F_2] = \alpha_1 F_3 - \beta_1 F_4 + \beta_2 F_5 - \beta_3 F_6, && [F_1,F_3] = \alpha_2 F_4 - \beta_4 F_5 + \beta_5 F_6,\\ &[F_1,F_4] = \alpha_3 F_5 - \beta_6 F_6, && [F_1,F_5] = \alpha_4 F_6,\\ &[F_2,F_3] = \alpha_5 F_6. \end{aligned}$$

Hence there is an orthonormal basis satisfying (2.15) such that if the set $J = \{j \in \{1,3,4,6\}, \beta_j \neq 0\} \neq \emptyset$, then we may assume that $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. Consequently, the existence of the metric Lie algebras $(\mathfrak{n}_{6,17}(\alpha_i,\beta_j),\langle .,.\rangle)$, respectively $(\mathfrak{n}_{6,18}(\alpha_i,\beta_j),\langle .,.\rangle)$ satisfying Theorem 2.19 (1) is proved. Let $T : (\mathfrak{n}_{6,k}(\alpha_i,\beta_j),\langle .,.\rangle) \to \left(\mathfrak{n}_{6,k}(\alpha'_i,\beta'_j),\langle .,.\rangle\right),\ k \in \{17,18\}$, be an isometric isomorphism. The decomposition $\mathbb{R}E_1 \oplus \ldots \oplus \mathbb{R}E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha'_i > 0, i \in \{1,\ldots,5\}$. Hence, in view of Lemma 2.3, we have $\alpha_i = \alpha_i', i \in \{1, \ldots, 5\}$ and $|\beta_j'| = \beta_j$ for all $j \in \{1, \ldots, 6\}$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i \in \{1, \ldots, 6\}$. Using the commutation relations (2.15) and (2.16) we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j \in \{1, \ldots, 6\}$, for $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ and $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ the equations

$$\begin{aligned} \varepsilon_1\varepsilon_2(\alpha_1 E_3 + \beta_1' E_4 + \beta_2' E_5 + \beta_3' E_6) &= \alpha_1 \varepsilon_3 E_3 + \beta_1 \varepsilon_4 E_4 + \beta_2 \varepsilon_5 E_5 + \beta_3 \varepsilon_6 E_6, \\ \varepsilon_1\varepsilon_3(\alpha_2 E_4 + \beta_4' E_5 + \beta_5' E_6) &= \alpha_2 \varepsilon_4 E_4 + \beta_4 \varepsilon_5 E_5 + \beta_5 \varepsilon_6 E_6, \\ \varepsilon_1\varepsilon_4(\alpha_3 E_5 + \beta_6' E_6) &= \alpha_3 \varepsilon_5 E_5 + \beta_6 \varepsilon_6 E_6, \\ \varepsilon_1\varepsilon_5(\alpha_4 E_6) &= \alpha_4 \varepsilon_6 E_6, \end{aligned} \tag{2.17}$$

and in addition for $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ the equation

$$\varepsilon_2\varepsilon_3(\alpha_5 E_6) = \alpha_5 \varepsilon_6 E_6. \tag{2.18}$$

From (2.17) and (2.18) for the metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ we receive that $\varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_1\varepsilon_3 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_5$, $\varepsilon_1\varepsilon_5 = \varepsilon_2\varepsilon_3 = \varepsilon_6$. Therefore one obtains that $\varepsilon_3 = \varepsilon_5 = 1$ $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_6$. Using these relations we have $\varepsilon_1\varepsilon_2 = \varepsilon_5 = 1$, $\varepsilon_1\varepsilon_3 = \varepsilon_1 = \varepsilon_6$ hence $\beta_2' = \beta_2$, $\beta_5' = \beta_5$. Assume that $J \neq \emptyset$. If $\beta_1 = \beta_1' > 0$, then one has the additional condition $\varepsilon_1\varepsilon_2 = \varepsilon_4$. If $\beta_3 = \beta_3' > 0$, then we obtain $\varepsilon_1\varepsilon_2 = \varepsilon_6$. If $\beta_4 = \beta_4' > 0$, then we get $\varepsilon_1\varepsilon_3 = \varepsilon_5$. If $\beta_6 = \beta_6' > 0$, then one has $\varepsilon_1\varepsilon_4 = \varepsilon_6$. In all these cases we receive that $\varepsilon_i = 1, i \in \{1, \ldots, 6\}$. Analogously to the previous case, one gets $\varepsilon_i = 1, i \in \{1, \ldots, 6\}$ from (2.17) for metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$. This proves the uniqueness of the Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$, respectively $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$. Hence assertion (1) is proved.

If the map $T(E_i) = \varepsilon_i E_i, \varepsilon_i = \pm 1, i \in \{1, \ldots, 6\}$, is an orthogonal automorphism of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$, respectively $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$, then the system of equations given by (2.17), respectively (2.18) is satisfied with $\beta_j' = \beta_j, j \in \{1, \ldots, 6\}$. In the case $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$, if $J = \emptyset$, then we obtain $\varepsilon_3 = \varepsilon_5 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_6$. It follows that the group of orthogonal automorphisms of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ is isomorphic to the group given by (2a). If $J \neq \emptyset$, then $\varepsilon_i = 1, i \in \{1, \ldots, 6\}$. Hence the group of orthogonal automorphisms of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ is trivial. This proves assertion

(2). In the case $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., .\rangle)$, if $J = \emptyset$, then we obtain $\varepsilon_1\varepsilon_2 = \varepsilon_5 = 1$, $\varepsilon_1\varepsilon_3 = \varepsilon_1 = \varepsilon_6$. It follows that the group of orthogonal automorphisms of $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., .\rangle)$ is isomorphic to the group given by (3a). If $J \neq \emptyset$, then $\varepsilon_i = 1, i \in \{1, \dots, 6\}$. Hence the group of orthogonal automorphisms of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., .\rangle)$ is trivial. This proves assertion (3).Thus, the prove of Theorem 2.19 is complete. □

Corollary 2.20. *Let $(N_{6,17}(\alpha_i, \beta_j), \langle ., .\rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., .\rangle)$. The isometry group of $(N_{6,17}(\alpha_i, \beta_j), \langle ., .\rangle)$ is*

$$\mathcal{I}(N_{6,17}(\alpha_i, \beta_j) = \begin{cases} \mathbb{Z}_2 \ltimes N_{6,17}(\alpha_i, \beta_j) & if J = \emptyset, \\ N_{6,17}(\alpha_i, \beta_j) & if J \neq \emptyset. \end{cases}$$

Corollary 2.21. *Let $(N_{6,18}(\alpha_i, \beta_j), \langle ., .\rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., .\rangle)$. The isometry group of $(N_{6,18}(\alpha_i, \beta_j), \langle ., .\rangle)$ is*

$$\mathcal{I}(N_{6,18}(\alpha_i, \beta_j) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes N_{6,18}(\alpha_i, \beta_j) & if J = \emptyset, \\ \mathbb{Z}_2 \ltimes N_{6,18}(\alpha_i, \beta_j) & if J \neq \emptyset. \end{cases}$$

2.3 Geodesic vectors of 6-dimensional filiform metric Lie algebras

Our aim in this section is to determine the sets of geodesic vectors of the six-dimensional filiform metric Lie algebras. According to Lemma 2.6, any element belonging to $\cup_{i=1}^{4}\mathfrak{a}_i \cup \zeta$ is geodesic. Therefore, we want to find the sets of geodesic vectors not belonging to $\cup_{i=1}^{4}\mathfrak{a}_i \cup \zeta$. Let us introduce the sets:

$C_1 := \{bE_2 + cE_3 + dE_4 : b\,(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0 : b, c, d \in \mathbb{R}\}$,

such that at least two of the numbers b, c, d are non-zero

with exception of the cases:

1. $b = 0$,
2. $d = 0$,
3. $c = 0$ with $\beta_1 \neq 0$,

$$
\begin{aligned}
C_2 :=\Big\{ & a(E_1 - \frac{\alpha_2}{\alpha_5}E_6) + cE_3 + dE_4 + eE_5 : a \neq 0, a, c, d, e \in \mathbb{R}, \\
& \alpha_5\beta_1 + \alpha_2\beta_7 = 0 = \beta_4, \\
& e = (\beta_5 a - \alpha_5 c)\frac{\alpha_2\alpha_4}{\alpha_1\alpha_5^2}, \\
& a(\alpha_1 c + \beta_1 d + \beta_2 e - \frac{\alpha_2\beta_3}{\alpha_5}) - c(\alpha_3 e - \frac{\alpha_2\beta_6}{\alpha_5}a) + \frac{a\alpha_2}{\alpha_5}(\beta_7 d + \alpha_4 e) = 0\Big\},
\end{aligned}
$$

$$
\begin{aligned}
C_3 :=\Big\{ & a\Big(E_1 - \frac{\alpha_2}{\alpha_5}E_6 + (\frac{\beta_5}{\alpha_5} + \frac{\alpha_1\beta_4}{\alpha_5\beta_1 + \alpha_2\beta_7})E_3 - \frac{\alpha_2\alpha_4\beta_4}{\alpha_5(\alpha_5\beta_1 + \alpha_2\beta_7)}E_5\Big) + \\
& dE_4 : a \neq 0,\ a, d \in \mathbb{R}, \\
& a(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c\,(\alpha_3 e + \beta_6 f) - f(\beta_7 d + \alpha_4 e) = 0\Big\},
\end{aligned}
$$

$$
\begin{aligned}
C_4 :=\Big\{ & aE_1 + cE_3 + dE_4 + eE_5 + fE_6 : a \neq 0,\ f \neq -a\frac{\alpha_2}{\alpha_5},\ f \neq 0, \\
& a, c, d, e, f \in \mathbb{R}, \quad e = (c\alpha_5 - a\beta_5)\frac{f\alpha_4}{a\alpha_1\alpha_5}, \\
& d = \frac{a}{\alpha_5 f + \alpha_2 a}\Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e - \beta_4 f\Big), \\
& a\,(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c\,(\alpha_3 e + \beta_6 f) - f(\beta_7 d + \alpha_4 e) = 0\Big\},
\end{aligned}
$$

$$
\begin{aligned}
C_5 :=\Big\{ & cE_3 + dE_4 + eE_5 + fE_6 : f \neq 0,\ c \neq 0,\ c, d, e, f \in \mathbb{R}, \\
& d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big), \\
& \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big)(\alpha_3 e + \beta_6 f + c\alpha_2) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0\Big\},
\end{aligned}
$$

$C_6 := \Big\{ bE_2 + cE_3 + dE_4 + eE_5 : b(\alpha_1 c + \beta_1 d + \beta_2 e) + c(\alpha_2 d + \gamma e) +$

$d\alpha_3 e = 0,\ b, c, d, e \in \mathbb{R} \Big\}$, where $\gamma = -\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big)$ in the case of the metric Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$, whereas $\gamma = \beta_4$ in the case of the metric Lie algebra $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ such that at least two of the numbers b, c, d, e are non-zero with exception of the cases:

1. $b = c = 0$,
2. $b = e = 0$,
3. $d = e = 0$,
4. $c = d = 0,\ \beta_2 \neq 0$,
5. $c = e = 0,\ \beta_1 \neq 0$,
6. $b = d = 0$ with $\alpha_3\alpha_4\beta_1 + \alpha_1\alpha_2\beta_8 \neq 0$ in the case of the metric Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$,
7. $b = d = 0$ with $\beta_4 \neq 0$ in the case of the metric Lie algebra $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$,

$$C_7 := \Big\{ a\Big(E_1 - \frac{\beta_6}{\alpha_4}E_2\Big) + cE_3 + dE_4 + eE_5 + fE_6 : af \neq 0,\ a, c, d, e, f \in \mathbb{R},$$

$$ae = \frac{f}{\alpha_3}\Big(\frac{a\beta_6\beta_8}{\alpha_4} + c\frac{\alpha_3\alpha_4}{\alpha_1} - a\beta_5\Big),$$

$$ac = \frac{f}{\alpha_1}(c\beta_7 + d\beta_8 + e\alpha_4) - \frac{a}{\alpha_1}\left(\beta_1 d + \beta_2 e + \beta_3 f\right),$$

$$ad = \frac{a}{\alpha_2}\Big(\big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\big)e - \beta_4 f\Big) + \frac{f}{\alpha_2}\Big(\frac{a\beta_6\beta_7}{\alpha_4} - d\frac{\alpha_3\alpha_4}{\alpha_1}\Big),$$

$$d(\alpha_2 c + \alpha_3 e) = a\frac{\beta_6}{\alpha_4}\left(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f\right) + ce\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big)$$

$$- f\left(c\beta_4 + d\beta_5 + e\beta_6\right)\Big\},$$

$C_8 := \{dE_4 + eE_5 + fE_6 : d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0,\ d, e, f \in \mathbb{R}\}$,
such that at least two of the numbers d, e, f are non-zero
with exception of the cases:
1. $f = 0$,
2. $d = 0$,
3. $e = 0$ with $\beta_6 \neq 0$,

$C_9 := \{bE_2 + cE_3 + dE_4 + eE_5 + fE_6 : b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) +$
$c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0, b, c, d, e, f \in \mathbb{R}\}$,
such that at least two of the numbers b, c, d, e, f are non-zero
with exception of the cases:
1. $b = c = d = 0$,
2. $b = c = f = 0$,
3. $d = e = f = 0$,
4. $b = c = e = 0$ with $\beta_6 \neq 0$,
5. $c = d = f = 0$ with $\beta_2 \neq 0$,
6. $c = d = e = 0$ with $\beta_3 \neq 0$.

Using the above sets we obtain:

Theorem 2.22. *Let $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by the non-vanishing commutators* (2.4) *with $\alpha_i > 0$, $\beta_j \in \mathbb{R},\ i \in \{1, \dots, 5\}$, $j \in \{1, \dots 7\}$. The geodesic vectors of $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ not belonging to $\cup_{i=1}^{4} \mathfrak{a}_i \cup \zeta$ are the non-zero elements of the set $C_1 \cup C_2 \cup C_4$ in the case $\alpha_5\beta_1 + \alpha_2\beta_7 = 0 = \beta_4$, for $\alpha_5\beta_1 + \alpha_2\beta_7 \neq 0$ these are the non-zero elements of the set $C_1 \cup C_3 \cup C_4$.*

Theorem 2.23. *Let $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by the non-vanishing commutators* (2.7) *with $\alpha_i > 0, \beta_j \in \mathbb{R},\ i \in \{1, \dots, 5\}$, $j \in \{1, \dots 7\}$. The geodesic vectors in $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ not belonging to $\cup_{i=1}^{4} \mathfrak{a}_i \cup \zeta$ are the non-zero elements of the set $C_1 \cup C_5$.*

Theorem 2.24. *Let $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the metric Lie algebra defined on $\mathbb{E}^6$ by the non-vanishing commutators* (2.10) *with $\alpha_i > 0, \beta_j \in \mathbb{R},\ i \in \{1, \dots, 4\}$,*

$j \in \{1, \dots 8\}$. *The geodesic vectors of* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ *not belonging to* $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$ *are the non-zero elements of the set* $C_6 \cup C_7$.

Theorem 2.25. *Let* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the metric Lie algebra defined on* $\mathbb{E}^6$ *by the non-vanishing commutators* (2.13) *and* (2.14) *with* $\alpha_i > 0, \beta_j \in \mathbb{R}, i \in \{1, \dots, 5\}, j \in \{1, \dots 6\}$. *The geodesic vectors of* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *not belonging to* $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$ *are the non-zero elements of the set* $C_6 \cup C_8$. *The geodesic vectors of metric Lie algebra* $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators* (2.13) *with* $\alpha_i > 0, \beta_j \in \mathbb{R}, i \in \{1, \dots, 4\}, j \in \{1, \dots 6\}$ *not belonging to* $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$ *are the non-zero elements of the set* C_9.

Proof of Theorems 2.22, 2.23, 2.24, 2.25

Proof of Theorem 2.22. According to (2.2) a non-zero vector $Y = aE_1 + bE_2 + cE_3 + dE_4 + eE_5 + fE_6 \in \mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ is geodesic if and only if one has $\langle [X, Y], Y \rangle = 0$ for all $X = \sum_{i=1}^5 x_i E_i \in \mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ or equivalently if and only if the system of equations

$$\begin{cases} b\alpha_4 f = 0, \\ a(\frac{\alpha_1\alpha_5}{\alpha_4} e + \beta_5 f) - c\alpha_5 f = 0, \\ a(\alpha_2 d - \frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4} e + \beta_4 f) + b\alpha_3 e + d\alpha_5 f = 0, \\ a\left(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f\right) - c\left(\alpha_3 e + \beta_6 f\right) - f(\beta_7 d + \alpha_4 e) = 0, \\ b\left(\alpha_1 c + \beta_1 d + \beta_2 e\right) + c(\alpha_2 d - \frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4} e + \beta_4 f) \\ \qquad + d(\frac{\alpha_1\alpha_5}{\alpha_4} e + \beta_5 f) = 0 \end{cases} \tag{2.19}$$

is satisfied. Taking into account that $\alpha_i \neq 0$, $i = \{1, \dots, 5\}$ and assume that $f = 0$ the system of equations (2.19) reduces to

$$\begin{cases} ae = 0, \\ a\alpha_2 d + b\alpha_3 e = 0, \\ a(\alpha_1 c + \beta_1 d) - c\alpha_3 e = 0, \\ b\left(\alpha_1 c + \beta_1 d + \beta_2 e\right) + c(\alpha_2 d - \frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4} e) + d\frac{\alpha_1\alpha_5}{\alpha_4} e = 0. \end{cases} \tag{2.20}$$

If $a = 0$, then from (2.20) we obtain $be = 0 = ce$, $b(\alpha_1 c + \beta_1 d) + c\alpha_2 d + d\frac{\alpha_1\alpha_5}{\alpha_4} e = 0$. In the case $e = 0 = a = f$ the geodesic vectors of metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ are the non-zero elements of the set

$$\overline{C_1} := \{bE_2 + cE_3 + dE_4 : b(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0,\ b, c, d \in \mathbb{R}\}.$$

For $b = 0$ the element $Y = cE_3 + dE_4 \in \overline{C_1}$ satisfies the condition $\alpha_2 cd = 0$. Since $\alpha_2 \neq 0$ we receive that either $c = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$ or $d = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. If $d = 0$, then for the element $Y = bE_2 + cE_3 \in \overline{C_1}$ the condition $\alpha_1 bc = 0$ holds. As $\alpha_1 \neq 0$ we obtain that either $c = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$ or $b = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. If $c = 0$ and $\beta_1 \neq 0$, then for the element $Y = bE_2 + dE_4 \in \overline{C_1}$ the condition $bd = 0$ is satisfied. Hence we get either $b = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$ or $d = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. Therefore the conditions for the set C_1 are proved. In the case $b = c = 0 = a = f$ the vector $Y = dE_4 + eE_5$ is geodesic if and only if either $d = 0$ or $e = 0$ and hence it lies either in $\mathfrak{a}_4$ or in $\mathfrak{a}_3$. If $e = 0$, then from (2.20) one has $ad = 0 = ac,\ b(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0$. The case $a = 0 = e = f$ is discussed above. In the case $d = c = 0 = e = f$ we receive that any vector $Y = aE_1 + bE_2$ is geodesic since it lies in the set $\mathfrak{a}_1$.

Now, we suppose that $b = 0$. In this case the system (2.19) of equations reduces to the following

$$\begin{cases} a\frac{\alpha_1\alpha_5}{\alpha_4}e = f(c\alpha_5 - a\beta_5), \\ d\alpha_5 f = -a(\alpha_2 d - \frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e + \beta_4 f), \\ a(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c(\alpha_3 e + \beta_6 f) - f(\beta_7 d + e\alpha_4) = 0, \\ c(\alpha_2 d - \frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e + \beta_4 f) + d(\frac{\alpha_1\alpha_5}{\alpha_4}e + \beta_5 f) = 0. \end{cases} \tag{2.21}$$

Assume that $a = 0$. The first and second equations of (2.21) give $cf = 0 = df$. The case $f = 0 = a$ is already discussed. In the case $c = d = 0 = a$ the third equation of (2.21) gives $fe = 0$. In this case the geodesic vector Y has either the form $eE_5 \in \mathfrak{a}_4$ or the form $fE_6 \in \zeta$. If $a \neq 0$, then the system (2.21) of equations is equivalent to the following system:

$$\begin{cases} \frac{\alpha_1\alpha_5}{\alpha_4}e = \frac{f}{a}(c\alpha_5 - a\beta_5), \\ d(\alpha_5 f + \alpha_2 a) = a(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e - \beta_4 f), \\ a(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c(\alpha_3 e + \beta_6 f) - f(\beta_7 d + e\alpha_4) = 0. \end{cases} \tag{2.22}$$

Let $f = -a\frac{\alpha_2}{\alpha_5}$. Putting this expression into the first and second equations of (2.22) we receive

$$(a\beta_5 - c\alpha_5)\frac{\alpha_2\alpha_4}{\alpha_1(\alpha_5)^2} = e, \tag{2.23}$$

$$\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e + \frac{\alpha_2}{\alpha_5}a\beta_4 = 0. \tag{2.24}$$

The substitution of (2.23) into (2.24) yields that

$$c(\alpha_5\beta_1 + \alpha_2\beta_7) = \frac{a}{\alpha_5}\Big((\alpha_5\beta_1 + \alpha_2\beta_7)\beta_5 + \alpha_1\alpha_5\beta_4\Big). \tag{2.25}$$

If $\alpha_5\beta_1 + \alpha_2\beta_7 = 0$, then the equation (2.25) gives $\beta_4 = 0$. In this case the geodesic vectors of $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ are the elements of the set C_2. If $\alpha_5\beta_1 + \alpha_2\beta_7 \neq 0$, then from equation (2.25) we obtain:

$$c = a\Big(\frac{\beta_5}{\alpha_5} + \frac{\alpha_1\beta_4}{\alpha_5\beta_1 + \alpha_2\beta_7}\Big). \tag{2.26}$$

Putting (2.26) into (2.23) we receive

$$e = -\frac{\alpha_2\alpha_4\beta_4}{\alpha_5(\alpha_5\beta_1 + \alpha_2\beta_7)}a.$$

Hence in this case the geodesic vectors of $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ are in the set C_3. If $f \neq -a\frac{\alpha_2}{\alpha_5}$, then from the first and second equations of (2.22) we have

$$e = \Big(c\alpha_5 - a\beta_5\Big)\frac{f\alpha_4}{a\alpha_1\alpha_5}, \quad d = \frac{a}{\alpha_5 f + \alpha_2 a}\Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e - \beta_4 f\Big).$$

In this case the geodesic vectors of $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ are the vectors of the set C_4 such that $f \neq -a\frac{\alpha_2}{\alpha_5}$ is a real coefficient. For $f = 0$ the elements in C_4 have the shape $Y = aE_1 + cE_3$ with $a \neq 0$ and satisfy the condition $\alpha_1 ac = 0$. Therefore we have $c = 0$ and hence the element $Y = aE_1$ lies in $\mathfrak{a}_1$. The intersection $C_1 \cap C_2 \cap C_4$ is empty because for the elements of C_1 one has $a = 0$ but for the elements in $C_2 \cup C_4$ we have $a \neq 0$. Moreover, for the elements in C_2 one gets $f = -a\frac{\alpha_2}{\alpha_5}$, in contrast to this for the elements in C_4 we have $f \neq -a\frac{\alpha_2}{\alpha_5}$. Hence the sets C_2 and C_4 are disjoint. Similarly we can see that the sets C_1, C_3 and C_4 are disjoint. This completes the proof. □

Proof of Theorem 2.23. Applying the commutators (2.7) the equation (2.2) gives that the non-zero element $Y = aE_1 + bE_2 + cE_3 + dE_4 + eE_5 + fE_6 \in \mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ is geodesic if and only if for the real numbers a, b, c, d, e, f with respect to a basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ the system of equations

$$\begin{cases} af = 0, \\ a\alpha_3 e + b\alpha_5 f = 0, \\ a(\alpha_2 d + \beta_4 e) + b(\frac{\alpha_2\alpha_5}{\alpha_4} e + \beta_7 f) = 0, \\ a(\alpha_1 c + \beta_1 d + \beta_2 e) - c(\frac{\alpha_2\alpha_5}{\alpha_4} e + \beta_7 f) - d\alpha_5 f = 0, \\ b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + c(\alpha_2 d + \beta_4 e + \beta_5 f) \\ \qquad\qquad +d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0 \end{cases} \tag{2.27}$$

is satisfied. If $a = 0$, then the second and the third equations of (2.27) gives $bf = 0 = be$. In the case $f = e = 0 = a$ the vector $Y = bE_2 + cE_3 + dE_4$ is geodesic of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ if and only if it lies in the set C_1. The validity of the condition for the coefficients b, c, d in the case of the set C_1 was shown in the proof of the Theorem 2.22. In the case $b = 0 = a$ the system (2.27) reduces to:

$$\begin{cases} c(\frac{\alpha_2\alpha_5}{\alpha_4} e + \beta_7 f) + d\alpha_5 f = 0, \\ c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0. \end{cases} \tag{2.28}$$

Suppose that $f = 0$. From the first equation of (2.28) we receive that either $c = 0$ or $e = 0$. In the case $c = 0 = f = b = a$ the vector $Y = dE_4 + eE_5$ is geodesic of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ if and only if it lies either in $\mathfrak{a}_4$ or in $\mathfrak{a}_3$, whereas if $e = 0 = f = b = a$ the vector $Y = cE_3 + dE_4$ is geodesic of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ precisely if it lies either in $\mathfrak{a}_2$ or in $\mathfrak{a}_3$. Assume that $f \neq 0$. From the first equation of (2.28) we obtain

$$d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4} e + \beta_7 f\Big). \tag{2.29}$$

Substituting (2.29) into the second equation of (2.28) we receive that

$$\frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4} e + \beta_7 f\Big)(c\alpha_2 + \alpha_3 e + \beta_6 f) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0.$$

In this case the vector $Y = cE_3 + dE_4 + eE_5 + fE_6 \in \mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ with $f \neq 0$ is geodesic if and only if it lies in the set

$$\begin{aligned}\overline{C_5} :=& \{cE_3 + dE_4 + eE_5 + fE_6 : f \neq 0,\ d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big),\\ &\frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big)(\alpha_3 e + \beta_6 f + c\alpha_2) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0\}.\end{aligned}$$

If $c = 0$, then the set $\overline{C_5}$ consists of the vectors $Y = fE_6$, $f \neq 0$, which are in ζ. Therefore we obtain that $c \neq 0$ and we get the set C_5. Now, if $f = 0$, the system (2.27) of equations reduces to

$$\begin{cases} ae = 0, \\ ad + \frac{\alpha_5}{\alpha_4}be = 0, \\ a(\alpha_1 c + \beta_1 d) - \frac{\alpha_2\alpha_5}{\alpha_4}ce = 0, \\ b(\alpha_1 c + \beta_1 d + \beta_2 e) + c(\alpha_2 d + \beta_4 e) + d\alpha_3 e = 0. \end{cases} \tag{2.30}$$

We discussed the case $a = 0 = f$ above. In the case $e = 0 = f$, from the second and the third equations of (2.30) one has $ad = 0 = ac$. The case $e = 0 = a = f$ is discussed above. In the case $e = 0 = d = c = f$ any vector $Y = aE_1 + bE_2$ is geodesic because it lies in $\mathfrak{a}_1$. The intersection $C_1 \cap C_5$ is empty since for the elements of C_1 one has $f = 0$ but for the elements in C_5 we have $f \neq 0$. This proves Theorem 2.23. □

Proof of Theorem 2.24. According to (2.2) the vector $Y = aE_1 + bE_2 + cE_3 + dE_4 + eE_5 + fE_6$ is geodesic of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ if and only if the following system of equations

$$\begin{cases} f\,(a\beta_6 + b\alpha_4) = 0, \\ a\,(\alpha_3 e + \beta_5 f) + f(b\beta_8 - c\frac{\alpha_3\alpha_4}{\alpha_1}) = 0, \\ a\Big(\alpha_2 d - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e + \beta_4 f\Big) + f(b\beta_7 + d\frac{\alpha_3\alpha_4}{\alpha_1}) = 0, \\ a\,(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - f\,(c\beta_7 + d\beta_8 + e\alpha_4) = 0, \\ b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + c\Big(\alpha_2 d - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e + \beta_4 f\Big) \\ \qquad + d(\alpha_3 e + \beta_5 f) + e\beta_6 f = 0 \end{cases} \tag{2.31}$$

is satisfied for $a, b, c, d, e, f \in \mathbb{R}$. Firstly, we suppose that $f = 0$. Take into account that $\alpha_i > 0$, $i = 1, \dots, 4$, the system (2.31) of equations gives $ae = 0 = ad = ac$. For $a = 0 = f$ the vector $Y = bE_2 + cE_3 + dE_4 + eE_5$ is geodesic of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), .\langle ., .\rangle)$ if and only if it lies in set

$$\overline{C_6} := \Big\{ bE_2 + cE_3 + dE_4 + eE_5 : b(\alpha_1 c + \beta_1 d + \beta_2 e) + c\Big(\alpha_2 d - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e\Big) + d\alpha_3 e = 0,\ b, c, d, e \in \mathbb{R} \Big\},$$

If $b = c = 0$, then for the element $Y = dE_4 + eE_5 \in \overline{C_6}$ one has $d\alpha_3 e = 0$. As $\alpha_3 \neq 0$ we obtain that Y lies either in $\mathfrak{a}_4$ or in $\mathfrak{a}_3$. If $b = e = 0$, then for the element $Y = cE_3 + dE_4 \in \overline{C_6}$ we have $c\alpha_2 d = 0$. Since $\alpha_2 \neq 0$ we obtain that Y lies either in $\mathfrak{a}_3$ or in $\mathfrak{a}_2$. In the case $d = e = 0$ for the element $Y = bE_2 + cE_3 \in \overline{C_6}$ one gets $b\alpha_1 c = 0$. As $\alpha_1 \neq 0$ we receive that Y lies either in $\mathfrak{a}_1$ or in $\mathfrak{a}_2$. In the case $c = d = 0$ for the element $Y = bE_2 + eE_5 \in \overline{C_6}$ we receive $b\beta_2 e = 0$. If $\beta_2 \neq 0$, then the element Y lies either in $\mathfrak{a}_4$ or in $\mathfrak{a}_1$. In the case $c = e = 0$ for the element $Y = bE_2 + dE_4 \in \overline{C_6}$ we receive that $b\beta_1 d = 0$. If $\beta_1 \neq 0$, then the element Y lies either in $\mathfrak{a}_3$ or in $\mathfrak{a}_1$. In the case $b = d = 0$ for the element $Y = cE_3 + eE_5 \in \overline{C_6}$ one obtains $ce(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}) = 0$. If $\alpha_3\alpha_4\beta_1 + \alpha_1\alpha_2\beta_8 \neq 0$, then we have $ce = 0$ or equivalently the element Y is either in $\mathfrak{a}_4$ or in $\mathfrak{a}_2$. This proves the conditions for the set C_6. For $e = d = c = 0 = f$ any vector $Y = aE_1 + bE_2$ is geodesic of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), .\langle ., .\rangle)$ because it lies in $\mathfrak{a}_1$.

Secondly, assume that $a\beta_6 + b\alpha_4 = 0$. Hence we receive $b = -\frac{a\beta_6}{\alpha_4}$. Putting this expression into (2.31) one obtains that

$$\left\{ \begin{array}{l} ae = \frac{f}{\alpha_3}(\frac{a\beta_6\beta_8}{\alpha_4} + c\frac{\alpha_3\alpha_4}{\alpha_1} - a\beta_5), \\ ad = \frac{a}{\alpha_2}\Big((\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e - \beta_4 f\Big) + \frac{f}{\alpha_2}(\frac{a\beta_6\beta_7}{\alpha_4} - d\frac{\alpha_3\alpha_4}{\alpha_1}), \\ ac = \frac{f}{\alpha_1}(c\beta_7 + d\beta_8 + e\alpha_4) - \frac{a}{\alpha_1}(\beta_1 d + \beta_2 e + \beta_3 f), \\ a\frac{\beta_6}{\alpha_4}(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + ce(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}) \\ \qquad\qquad - f(c\beta_4 + d\beta_5 + e\beta_6) = d(\alpha_2 c + \alpha_3 e). \end{array} \right. \tag{2.32}$$

Suppose that $a = 0$. Hence one has $b = 0$. From system (2.32) of equations we receive $fc = 0 = fd = fe$. The case $f = 0 = a = b$ is discussed above. If

$c = d = e = 0 = a = b$, then any vector $Y = fE_6$ is geodesic since it lies in ζ. Assume that $a \neq 0$ the geodesic vectors of $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ are in the set C_7 such that f is a real coefficient. If $f = 0$, then from (2.32) it follows that $e = d = c = 0$. Hence any element $Y = a(E_1 - \frac{\beta_6}{\alpha_4} E_2)$, $a \neq 0$, lies in $\mathfrak{a}_1$. Therefore we may assume that $f \neq 0$. The intersection $C_6 \cap C_7$ is empty since for the elements of C_6 one has $a = 0$ and $f = 0$ but for the elements in C_7 we have $af \neq 0$. This completes the proof. □

Proof of Theorem 2.25. Using the commutators (2.13) and (2.14) the claim (2.2) yields that the non-zero element $Y = aE_1 + bE_2 + cE_3 + dE_4 + eE_5 + fE_6 \in \mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ is geodesic if and only if for the real numbers a, b, c, d, e, f with respect to a basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ the system of equations

$$\begin{cases} af = 0, \ ae = 0, \\ a\alpha_2 d + b\alpha_5 f = 0, \\ a(\alpha_1 c + \beta_1 d) - c\alpha_5 f = 0, \\ b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) \\ \qquad + e\alpha_4 f = 0 \end{cases} \tag{2.33}$$

is satisfied. If $a = 0$, then the third and the fourth equations of (2.33) give $bf = 0 = cf$. In the case $b = c = 0 = a$, the vector $Y = dE_4 + eE_5 + fE_6$ is geodesic of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ if and only if it lies in the set

$$\overline{C_8} := \{dE_4 + eE_5 + fE_6 : d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0, \ d, e, f \in \mathbb{R}\}.$$

For $f = 0$ the element $Y = dE_4 + eE_5 \in \overline{C_8}$ satisfies the condition $\alpha_3 de = 0$. Since $\alpha_3 \neq 0$ we obtain that either $d = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$ or $e = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. If $d = 0$, then for the element $Y = eE_5 + fE_6 \in \overline{C_8}$ the condition $\alpha_4 ef = 0$ holds. Since $\alpha_4 \neq 0$ we receive that either $e = 0$ and the element $Y = fE_6$ is in ζ or $f = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$. If $e = 0$ with $\beta_6 \neq 0$, then for the element $Y = dE_4 + fE_6 \in \overline{C_8}$ the condition $df = 0$ is satisfied. Hence we get either $d = 0$ and the element $Y = fE_6$ is in ζ or $f = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. Therefore the conditions for the set C_8 are proved. In the case $f = 0 = a$ the non-zero vector $Y = bE_2 + cE_3 + dE_4 + eE_5$ is geodesic of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ if and only if it lies in the set

$$\begin{aligned}\overline{C_6} :=& \{bE_2 + cE_3 + dE_4 + eE_5 : b(\alpha_1 c + \beta_1 d + \beta_2 e) + c(\alpha_2 d + \beta_4 e) \\ &+ d\alpha_3 e = 0,\ b, c, d, e \in \mathbb{R}\}.\end{aligned}$$

If $b = c = 0$, then the element $Y = dE_4 + eE_5 \in \overline{C_6}$ satisfies the condition $\alpha_3 de = 0$. Since $\alpha_3 \neq 0$ we obtain that either $d = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$ or $e = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. If $b = e = 0$, then for the element $Y = cE_3 + dE_4 \in \overline{C_6}$ the condition $\alpha_2 cd = 0$ holds. Since $\alpha_2 \neq 0$ we receive that either $c = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$ or $d = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. If $d = e = 0$, then for the element $Y = bE_2 + cE_3 \in \overline{C_6}$ the condition $\alpha_1 bc = 0$ is satisfied. As $\alpha_1 \neq 0$ we get either $b = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$ or $c = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. If $c = d = 0$ with $\beta_2 \neq 0$, then for the element $Y = bE_2 + eE_5 \in \overline{C_6}$ the condition $be = 0$ holds. Therefore we get either $b = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$ or $e = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. If $c = e = 0$ with $\beta_1 \neq 0$, then for the element $Y = bE_2 + dE_4 \in \overline{C_6}$ the condition $bd = 0$ holds. It follows that either $b = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$ or $d = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. Finally, if $b = d = 0$ with $\beta_4 \neq 0$, then for the element $Y = cE_3 + eE_5 \in \overline{C_6}$ the condition $ce = 0$ holds. Hence we have either $c = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$ or $e = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$. This proves the conditions for the set C_6. If $f = e = 0$, then from the third and fourth equations of (2.33) we obtain that $ad = 0 = ac$. The case $a = 0 = f = e$ is discussed above. In the case $d = c = 0 = f = e$ any vector $Y = aE_1 + bE_2$ is geodesic because it lies in $\mathfrak{a}_1$. The intersection $C_8 \cap C_6$ is empty, because for any element of C_8 one has $b = c = 0$ and any element of C_6 one gets $f = 0$. Hence the set of the geodesic vectors of $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ is shown.

In the case of the metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ the system (2.33) of equations is satisfied with $\alpha_5 = 0$. Therefore we obtain that $af = 0 = ae = ad = ac$. The case $f = e = d = c = 0$ is discussed above. In the case $a = 0$ the vector $Y = bE_2 + cE_3 + dE_4 + eE_5 + fE_6$ is geodesic of $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ precisely if it lies in the set

$$\overline{C_9} := \{bE_2 + cE_3 + dE_4 + eE_5 + fE_6 : b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f)$$

$$+ c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0,\ b, c, d, e, f \in \mathbb{R}\}.$$

If $b = c = d = 0$, then the element $Y = eE_5 + fE_6 \in \overline{C_9}$ satisfies the condition $\alpha_4 ef = 0$. Since $\alpha_4 \neq 0$ we obtain that either $e = 0$ and the element $Y = fE_6$ is in ζ, or $f = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$. If $b = c = f = 0$, then for the element $Y = dE_4 + eE_5 \in \overline{C_9}$ the condition $\alpha_3 de = 0$ holds. Since $\alpha_3 \neq 0$ we receive that either $d = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$, or $e = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. If $d = e = f = 0$, then for the element $Y = bE_2 + cE_3 \in \overline{C_9}$ the condition $\alpha_1 bc = 0$ is satisfied. As $\alpha_1 \neq 0$ we get either $b = 0$ and the element $Y = cE_3$ is in $\mathfrak{a}_2$, or $c = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. In the case $b = c = e = 0$ with $\beta_6 \neq 0$, for the element $Y = dE_4 + fE_6 \in \overline{C_9}$ the condition $df = 0$ holds. Hence we have either $d = 0$ and the element $Y = fE_6$ is in ζ, or $f = 0$ and the element $Y = dE_4$ is in $\mathfrak{a}_3$. If $c = d = f = 0$ with $\beta_2 \neq 0$, then for the element $Y = bE_2 + eE_5 \in \overline{C_9}$ the condition $be = 0$ holds. Therefore we get either $b = 0$ and the element $Y = eE_5$ is in $\mathfrak{a}_4$, or $e = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. Finally, if $c = d = e = 0$ with $\beta_3 \neq 0$, then for the element $Y = bE_2 + fE_6 \in \overline{C}_9$ the condition $bf = 0$ holds. It follows that either $b = 0$ and the element $Y = fE_6$ is in ζ, or $f = 0$ and the element $Y = bE_2$ is in $\mathfrak{a}_1$. This shows the conditions for the set C_9. Hence Theorem 2.25 is proved. □

2.4 Flat totally geodesic subalgebras of 6-dimensional filiform metric Lie algebras

In this section, we describe the flat totally geodesic subalgebras of the 6-dimensional filiform metric Lie algebras. As any non-zero vector in $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$ is geodesic, we focus on the study of subalgebras not belonging to $\cup_{i=1}^4 \mathfrak{a}_i \cup \zeta$. Since any non-zero geodesic vector Y generates a 1-dimensional flat totally geodesic subalgebra $\{tY, t \in \mathbb{R}\}$, here we wish to find totally geodesic subalgebras of dimension greater than 1.

2.4.1 Metric Lie algebras corresponding to non-standard filiform Lie algebras

In this subsection, we deal with the filiform metric Lie algebras $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$, $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ and $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$.
We introduce the following theorem.

Theorem 2.26. *The flat totally geodesic subalgebras of the non-standard filiform metric Lie algebras* $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17$ *have dimension at most* 2.

To prove Theorem 2.26 we characterize the flat totally geodesic subalgebras of the filiform metric Lie algebras $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17$, as follows:

Theorem 2.27. *Let* $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the metric Lie algebra defined on* $\mathbb{E}^6$ *by the commutators (2.4). The flat totally geodesic subalgebras of dimension* > 1 *in the metric Lie algebra* $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ *are the* 2*-dimensional subalgebras:*

1. $\mathfrak{h}_2 = span(E_1, E_6)$ *in the case* $\beta_3 = \beta_4 = \beta_5 = 0$,
2. $\mathfrak{h}_3 = span(E_2, E_4)$ *in the case* $\beta_1 = \beta_7 = 0$.

Theorem 2.28. *Let* $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the metric Lie algebra defined on* $\mathbb{E}^6$ *by the commutators (2.7). Then the unique flat totally geodesic subalgebra of dimension* > 1 *in* $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ *is* $\mathfrak{h}_6 = span(E_3, E_6)$ *in the case* $\beta_5 = \beta_7 = 0$.

Theorem 2.29. *Let* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the metric Lie algebra defined on* $\mathbb{E}^6$ *by the commutators (2.10). The flat totally geodesic subalgebras of dimension* > 1 *in* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ *are the* 2*-dimensional subalgebras:*

1. $\mathfrak{h}_2 = span(E_1 - \frac{\beta_6}{\alpha_4} E_2, E_6)$ *in the case* $\beta_3 = 0$, $\beta_4 = \frac{\beta_6 \beta_7}{\alpha_4}$ *and* $\beta_5 = \frac{\beta_6 \beta_8}{\alpha_4}$,
2. $\mathfrak{h}_3 = span(E_2 + \frac{\beta_8 \alpha_1}{\alpha_3 \alpha_4} E_3 - \frac{1}{\alpha_3}(\beta_1 + \frac{\beta_8 \alpha_1 \alpha_2}{\alpha_3 \alpha_4}) E_5, \quad E_4)$ *if and only if for* α_i, β_j, $i = 1, 2, 3, 4$, $j = 1, 2, 8$ *the equation*
$$\frac{(\alpha_1)^2 \beta_8}{\alpha_3 \alpha_4} + \Big(\beta_1 + \frac{\beta_8 \alpha_1}{\alpha_3 \alpha_4}\Big)\Big(-\frac{\beta_2}{\alpha_3} + \frac{\beta_8 \alpha_1}{(\alpha_3)^2 \alpha_4}(\frac{\alpha_3 \beta_1}{\alpha_1} + \frac{\alpha_2 \beta_8}{\alpha_4})\Big) = 0 \quad (2.34)$$
holds,

3. $\mathfrak{h}_5 = span(E_3, E_5)$ *in the case* $\alpha_3\alpha_4\beta_1 + \alpha_2\alpha_1\beta_8 = 0$,

4. $\mathfrak{h}_{20} = span(E_2 + k_1E_4 + k_2E_5, \quad E_3 + l_1E_4 + l_2E_5)$ *if and only if the equations*

$$\begin{aligned}
&\beta_7 + l_1\beta_8 + l_2\alpha_4 + k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,\\
&\beta_1k_1 + \beta_2k_2 + k_1\alpha_3k_2 = 0,\\
&\alpha_2l_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})l_2 + l_1\alpha_3l_2 = 0,\\
&\alpha_1 + \beta_1l_1 + \beta_2l_2 + \alpha_2k_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})k_2 + \alpha_3(k_1l_2 + l_1k_2) = 0
\end{aligned}$$

are satisfied.

Theorem 2.30. *Let* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the metric Lie algebra defined on* $\mathbb{E}^6$ *by the commutators* (2.13) *and* (2.14). *The flat totally geodesic subalgebras of dimension* > 1 *in* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *are the* 2*-dimensional subalgebras:*

1. $\mathfrak{h}_3 = span(E_2 - \frac{\beta_1+\alpha_3k_2}{\alpha_2}E_3 + k_2E_5, \quad E_4)$, *where* k_2 *is a solution of the equation*

$$\alpha_3\beta_4k_2^2 + (\alpha_1\alpha_3 + \beta_1\beta_4 - \alpha_2\beta_2)k_2 + \alpha_1\beta_1 = 0, \tag{2.35}$$

2. $\mathfrak{h}_5 = span\,(E_3, \quad E_5)$ *in the case* $\beta_4 = 0$,

3. $\mathfrak{h}_7 = span(E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5)$ *in the case* $\beta_6 = 0$,

4. $\mathfrak{h}_8 = span\,(E_4, \quad E_6)$ *in the case* $\beta_6 = 0$,

5. $\mathfrak{h}_{14} = span(E_2 + k_1E_3 - \frac{\beta_2+\beta_4k_1}{\alpha_3}E_4, \quad E_5)$, *where* k_1 *is a solution of the equation*

$$\alpha_2\beta_4k_1^2 + (\beta_1\beta_4 + \alpha_2\beta_2 - \alpha_1\alpha_3)k_1 + \beta_1\beta_2 = 0, \tag{2.36}$$

6. $\mathfrak{h}_{15} = span(E_3 - \frac{\alpha_2}{\alpha_3}E_5, \quad E_4)$ *in the case* $\beta_4 = 0$.

Proof of Theorems 2.27, 2.28, 2.29, 2.30

We need the following Lemmas.

Lemma 2.31. *There does not exist any flat totally geodesic subalgebra of the* 6*-dimensional filiform metric Lie algebras containing the element* E_5+lE_6 *with* $l \neq 0$.

Proof. The vector $Y = E_5 + lE_6$, $l \in \mathbb{R}$, is geodesic in a metric Lie algebra $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, \cdots, 18$, if and only if for all $X \in \mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, \cdots, 18$, one has $\langle [X, Y], Y\rangle = 0$ (cf. (2.2)). Applying the commutators (2.4) in the case of the metric Lie algebras $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, respectively (2.10) for the metric Lie algebras $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ we obtain that $\langle [E_2, Y], Y\rangle = \langle \alpha_4 E_6, Y\rangle = \alpha_4 l = 0$. Using the commutators (2.7) in the case of the metric Lie algebras $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$, respectively (2.13) for the metric Lie algebras $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ and $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ we receive that $\langle [E_1, Y], Y\rangle = \langle \alpha_4 E_6, Y\rangle = \alpha_4 l = 0$. Since $\alpha_4 \neq 0$ these imply that $l = 0$. Hence from Lemma 2.5 follows the assertion. □

Lemma 2.32. *There does not exist any flat totally geodesic subalgebra of the metric Lie algebras* $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 16, 17$, *which contains the element* $Y = E_3 + k_1E_4 + k_2E_5 + k_3E_6$, $k_1, k_2, k_3 \in \mathbb{R}$, *with* $k_3 \neq 0$.

Proof. According to (2.2) the vector $Y = E_3 + k_1E_4 + k_2E_5 + k_3E_6$, $k_i \in \mathbb{R}$, $i = 1, 2, 3$, is geodesic in a metric Lie algebra $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 16, 17$, precisely if for all $X \in \mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 16, 17$, we get $\langle [X, Y], Y\rangle = 0$. In $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, respectively in $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ one has $\langle [E_4, Y], Y\rangle = \alpha_5 k_3$, respectively $\langle [E_2, Y], Y\rangle = \alpha_5 k_3$. Moreover one gets $\langle [E_4, Y], Y\rangle = \frac{\alpha_3\alpha_4}{\alpha_1} k_3$ in $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$. As $\alpha_3\alpha_4\alpha_5 \neq 0$ we obtain that $k_3 = 0$. Using Lemma 2.5 the assertion follows. □

Lemma 2.33. *The element* $E_4 + mE_5 + lE_6$, $m, l \in \mathbb{R}$, *is geodesic in the metric Lie algebras* $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16$, *if and only if one has* $m = l = 0$.

Proof. According to (2.2) the non-zero vector $Y = E_4 + mE_5 + lE_6$, $m, l \in \mathbb{R}$, is geodesic in a metric Lie algebra $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16$, precisely if for all $X \in \mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16$, one has $\langle [X, Y], Y\rangle = 0$. Applying the commutators (2.4) in the case of the metric Lie algebras $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ we have $\langle [E_3, Y], Y\rangle = \langle -\alpha_5 E_6, Y\rangle = -\alpha_5 l$. Using the commutators (2.7) in the case of the metric Lie algebras $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ we receive $\langle [E_2, Y], Y\rangle = \langle \alpha_5 E_6, Y\rangle = \alpha_5 l$. Taking into account the commutators (2.10) of the metric Lie algebras $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ we obtain $\langle [E_3, Y], Y\rangle = \langle -\frac{\alpha_3\alpha_4}{\alpha_1} E_6, Y\rangle = -\frac{\alpha_3\alpha_4}{\alpha_1} l$. Since $\alpha_3\alpha_4\alpha_5 \neq 0$ we get that $l = 0$. Taking this into account we obtain $\langle [E_1, E_4 + mE_5], E_4 + mE_5\rangle = \frac{\alpha_1\alpha_5}{\alpha_4} m$ in $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, $\langle [E_1, E_4 + mE_5], E_4 + mE_5\rangle = \alpha_3 m$ in $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 15, 16$. As $\alpha_1\alpha_3\alpha_5 \neq 0$ we obtain $m = 0$. Since $E_4 \in \mathfrak{a}_3$ it is a geodesic vector which proves the assertion. □

Lemma 2.34. *There does not exist any flat totally geodesic subalgebra of the 6-dimensional filiform metric Lie algebras $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17$, which contains the vector $Y = E_2 + b_1E_3 + b_2E_4 + b_3E_5 + b_4E_6$ with $b_4 \neq 0$.*

Proof. Using (2.2) the vector $Y = E_2 + b_1E_3 + b_2E_4 + b_3E_5 + b_4E_6$, $b_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, is geodesic in a metric Lie algebra $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17$, if and only if for all $X \in \mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17$, we obtain $\langle [X, Y], Y\rangle = 0$. Since one has $\langle [E_5, Y], Y\rangle = -\alpha_4 b_4 \in \mathfrak{n}_{6,k}(\alpha_i, \beta_j), k = 14, 16$, and $\langle [E_4, Y], Y\rangle = -\alpha_5 b_4 \in \mathfrak{n}_{6,15}(\alpha_i, \beta_j)$, as well as $\langle [E_3, Y], Y\rangle = -\alpha_5 b_4 \in \mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ we receive that $b_4 = 0$. Hence from Lemma 2.5 follows the assertion. □

Lemma 2.35. *The vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5$, $k_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, is geodesic in a 6-dimensional filiform metric Lie algebra $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17, 18$, if and only if we have $k_2 = k_3 = k_4 = 0$. Moreover, the vector $E_1 + k_1E_2 + E_6$, $k_1 \in \mathbb{R}$, is not a geodesic vector in the metric Lie algebras $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 15, 17, 18$.*

Proof. Applying (2.2) the vector $Y = E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5$ is geodesic in a metric Lie algebra $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17, 18$, if and only if for all $X \in \mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17, 18$, we obtain $\langle [X, Y], Y\rangle = 0$. We have $\langle [E_4, Y], Y\rangle = -\frac{\alpha_1\alpha_5}{\alpha_4} k_4$ in $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ as well as $\langle [E_4, Y], Y\rangle = -\alpha_3 k_4$ in $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 15, 16, 17, 18$. Since $\alpha_1\alpha_5\alpha_3 \neq 0$ the claim (2.2) gives that $k_4 = 0$. Using this we obtain $\langle [E_3, Y], Y\rangle = -\alpha_2 k_3$ in $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 14, 15, 16, 17, 18$. As $\alpha_2 \neq 0$ we obtain from (2.2) that $k_3 = 0$. Taking this into account we receive $\langle [E_2, Y], Y\rangle = -\alpha_1 k_2$ in $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k =$

$14, 15, 16, 17, 18$. Because of $\alpha_1 \neq 0$ it follows from (2.2) that $k_2 = 0$. This proves the first assertion. As $\langle [E_5, E_1 + k_1 E_2 + E_6], E_1 + k_1 E_2 + E_6 \rangle = -\alpha_4 \neq 0$ in $\mathfrak{n}_{6,k}(\alpha_i, \beta_j)$, $k = 15, 17, 18$, we obtain a contradiction to (2.2) and the second assertion is proved. □

Now we proof the following:

Proof of Theorem 2.27. In view of Proposition 2.7 any flat totally geodesic subalgebra of metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle .,. \rangle)$ has dimension at most 3. Hence firstly we determine the 2- and 3-dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$.

The 2-dimensional abelian subalgebras of $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ are:

$$\begin{aligned}
\mathfrak{h}_1 &= \operatorname{span}(E_1 + k_2 E_3 + k_3 E_4 + k_4 E_6, \quad E_5 + l_1 E_6), \\
\mathfrak{h}_2 &= \operatorname{span}(E_1 + k_1 E_2 + k_2 E_3 + k_3 E_4 + k_4 E_5, \quad E_6), \\
\mathfrak{h}_3 &= \operatorname{span}(E_2 + k_1 E_3 + k_2 E_5 + k_3 E_6, \quad E_4 + l_1 E_5 + l_2 E_6) \\
&\qquad \text{with } \beta_7 + l_1 \alpha_4 - k_1 \alpha_5 = 0, \\
\mathfrak{h}_4 &= \operatorname{span}(E_2 + k_1 E_3 + k_2 E_4 + k_3 E_5, \quad E_6), \\
\mathfrak{h}_5 &= \operatorname{span}(E_3 + k_1 E_4 + k_2 E_6, \quad E_5 + l_1 E_6), \\
\mathfrak{h}_6 &= \operatorname{span}(E_3 + k_1 E_4 + k_2 E_5, \quad E_6), \\
\mathfrak{h}_7 &= \operatorname{span}(E_4 + s_1 E_6, \quad E_5 + l_1 E_6), \\
\mathfrak{h}_8 &= \operatorname{span}(E_4 + l_1 E_5, \quad E_6), \\
\mathfrak{h}_9 &= \operatorname{span}(E_5, \quad E_6),
\end{aligned} \tag{2.37}$$

whereas the 3-dimensional abelian subalgebras of $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ are:

$$\begin{aligned}
\mathfrak{h}_{10} &= \operatorname{span}(E_1 + k_2 E_3 + k_3 E_4, \quad E_5, \quad E_6), \\
\mathfrak{h}_{11} &= \operatorname{span}(E_3 + k_1 E_4, \quad E_5, \quad E_6), \\
\mathfrak{h}_{12} &= \operatorname{span}(E_4, \quad E_5, \quad E_6), \\
\mathfrak{h}_{13} &= \operatorname{span}(E_2 + k_1 E_3 + k_2 E_5, \quad E_4 + l_1 E_5, \quad E_6) \\
&\qquad \text{with } \beta_7 + l_1 \alpha_4 - k_1 \alpha_5 = 0,
\end{aligned} \tag{2.38}$$

where $k_1, k_2, k_3, k_4, l_1, l_2, s_1 \in \mathbb{R}$. According to Lemma 2.31 the subalgebras $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, and $\mathfrak{h}_{12}$ are not flat totally geodesic because they contain the vector E_5+E_6. Therefore the subalgebras $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$ are excluded. Because the vector $Y = E_4 + l_1E_5 + E_6 \in \mathfrak{h}_8 \cap \mathfrak{h}_{13}$ is not geodesic (see Lemma 2.33) the subalgebras $\mathfrak{h}_8$ and $\mathfrak{h}_{13}$ are not flat totally geodesic (cf. Lemma 2.4). Hence the subalgebras $\mathfrak{h}_8$, $\mathfrak{h}_{13}$ are excluded, too. Since the subalgebra $\mathfrak{h}_4$ contains the vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6$ by Lemma 2.34 it is not flat totally geodesic. Hence $\mathfrak{h}_4$ is excluded. Since the element $Y = E_3 + k_1E_4 + k_2E_5 + E_6 \in \mathfrak{h}_6$ is not geodesic (cf. Lemma 2.32) the subalgebra $\mathfrak{h}_6$ is not flat totally geodesic and hence it is excluded.

By the proof of Lemma 2.31 the vector $Y = E_5 + l_1E_6 \in \mathfrak{h}_1 \cap \mathfrak{h}_5 \cap \mathfrak{h}_7$ is geodesic if and only if $l_1 = 0$.

We treat the subalgebra $\mathfrak{h}_1$. The vector $E_1 + k_2E_3 + k_3E_4 + k_4E_6 \in \mathfrak{h}_1$ is geodesic if and only if for $b = e = 0, a = 1, c = k_2, d = k_3, f = k_4$ the system (2.19) of equations is satisfied. From the second equation of (2.19) we receive

$$k_4(\beta_5 - \alpha_5k_2) = 0. \tag{2.39}$$

Furthermore, the element $E_1 + k_2E_3 + k_3E_4 + k_4E_6 + E_5 \in \mathfrak{h}_1$ is geodesic if and only if for $b = 0, a = e = 1, c = k_2, d = k_3, f = k_4$ the system (2.19) of equations holds. The second equation of (2.19) gives

$$\frac{\alpha_1\alpha_5}{\alpha_4} + k_4(\beta_5 - \alpha_5k_2) = 0. \tag{2.40}$$

Taking into account (2.39), equation (2.40) yields the contradiction $\frac{\alpha_1\alpha_5}{\alpha_4} = 0$. Hence the subalgebra $\mathfrak{h}_1$ is not flat totally geodesic and therefore it is excluded.

Next we consider the subalgebra $\mathfrak{h}_5$. The vector $E_3+k_1E_4+k_2E_6 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = e = 0, c = 1, d = k_1, f = k_2$, the system (2.19) of equations holds. From Lemma 2.32 one obtains $k_2 = f = 0$. Using this the fifth equation of (2.19) gives $\alpha_2k_1 = 0$. Since $\alpha_2 \neq 0$ we receive $k_1 = 0$. But the vector $Y = E_3 + E_5 \in \mathfrak{h}_5$ is not geodesic because one has $\langle [E_2, Y], Y\rangle = \alpha_3 \neq 0$. Therefore the subalgebra $\mathfrak{h}_5$ is excluded, too.

We deal with the subalgebra $\mathfrak{h}_7$. By Lemma 2.33 the element $E_4 + s_1E_6 \in \mathfrak{h}_7$ is geodesic if and only if $s_1 = 0$. But in this case the vector $Y = E_4+E_5 \in \mathfrak{h}_7$ is not geodesic (see Lemma 2.33). Hence the subalgebra $\mathfrak{h}_7$ is excluded.

Now we treat the subalgebra $\mathfrak{h}_2$. The vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 \in \mathfrak{h}_2$ is geodesic if and only if $k_2 = k_3 = k_4 = 0$ (see Lemma 2.35). The element $E_1 + k_1E_2 + E_6 \in \mathfrak{h}_2$ is geodesic if and only if for $e = c = d = 0, a = f = 1, b = k_1$ the system (2.19) of equations holds. From the first equation of (2.19) one has $\alpha_4 k_1 = 0$. This implies that $k_1 = 0$ since $\alpha_4 \neq 0$. Additionally, the vector $E_1 + E_6 \in \mathfrak{h}_2$ is geodesic if and only if for $b = c = d = e = 0, a = f = 1$ the system (2.19) of equations is satisfied. From the second, third, and fourth equations we get $\beta_5 = 0$, $\beta_4 = 0$, and $\beta_3 = 0$. Hence the assertion (1) of Theorem 2.27 follows.

Finally we consider the subalgebra $\mathfrak{h}_3$. The vector $E_2 + k_1E_3 + k_2E_5 + k_3E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $a = d = 0, b = 1, c = k_1, e = k_2, f = k_3$ the system (2.19) of equations holds. In view of Lemma 2.34, one has $k_3 = f = 0$. Using this from the third equation of (2.19) we obtain $\alpha_3 k_2 = 0$. As $\alpha_3 \neq 0$ we receive $k_2 = e = 0$. Applying this the fifth equation of (2.19) gives $\alpha_1 k_1 = 0$. Since $\alpha_1 \neq 0$ this yields $k_1 = c = 0$. The element $E_4 + l_1E_5 + l_2E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $a = b = c = 0, d = 1, e = l_1, f = l_2$ the system (2.19) of equations is satisfied. According to Lemma 2.33 we receive that $l_1 = l_2 = 0$. In addition, the vector $E_2 + E_4 \in \mathfrak{h}_3$ is geodesic if and only if for $a = c = f = 0, b = d = 1$ the system (2.19) of equations holds. The fifth equation of (2.19) gives $\beta_1 = 0$. Taking into account the condition for the subalgebra $\mathfrak{h}_3$ to be abelian, when $l_1 = k_1 = 0$ we receive that $\beta_7 = 0$. Therefore the case (2) of Theorem 2.27 is shown. This proves Theorem 2.27. □

Proof of Theorem 2.28. In view of Proposition 2.7, the dimension of the flat totally geodesic subalgebras of $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is at most 3. Hence firstly we determine the 2- and 3-dimensional abelian subalgebras in the Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$.

The 2-dimensional and the 3-dimensional abelian subalgebras of Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ are:

$$\begin{aligned}
&\mathfrak{h}_2,\ \mathfrak{h}_4,\ \mathfrak{h}_5,\ \mathfrak{h}_6,\ \mathfrak{h}_7,\ \mathfrak{h}_8,\ \mathfrak{h}_9 \text{ in } (2.37), \mathfrak{h}_{11} \text{ and } \mathfrak{h}_{12} \text{ in } (2.38),\\
&\mathfrak{h}_{14} = \mathrm{span}(E_2 + k_1E_3 + k_2E_4 + k_3E_6, \quad E_5 + l_1E_6),\\
&\mathfrak{h}_{15} = \mathrm{span}(E_3 + k_1E_5 + k_2E_6, \quad E_4 + l_1E_5 + l_2E_6),\\
&\mathfrak{h}_{16} = \mathrm{span}(E_3 + k_1E_5, \quad E_4 + l_1E_5, \quad E_6),\\
&\mathfrak{h}_{17} = \mathrm{span}(E_2 + k_1E_3 + k_2E_4, \quad E_5, \quad E_6),\\
&\mathfrak{h}_{18} = \mathrm{span}(E_3 + k_1E_6, \quad E_4 + s_1E_6, \quad E_5 + l_1E_6),
\end{aligned} \tag{2.41}$$

where $k_1, k_2, k_3, l_1, l_2, s_1 \in \mathbb{R}$. According to Lemmas 2.31, 2.33, the subalgebras $\mathfrak{h}_8$, $\mathfrak{h}_9$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$, $\mathfrak{h}_{16}$ and $\mathfrak{h}_{17}$ are excluded. The vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 \in \mathfrak{h}_2$ is geodesic if and only if $k_2 = k_3 = k_4 = 0$ (see Lemma 2.35). But the vector $Y = E_1 + k_1E_2 + E_6 \in \mathfrak{h}_2$ is not geodesic (see Lemma 2.35). Therefore the subalgebra $\mathfrak{h}_2$ is not flat totally geodesic (cf. Lemma 2.5) and hence it is excluded. The subalgebra $\mathfrak{h}_4$ is not flat totally geodesic since it contains the vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_4$ (see Lemma 2.34). Hence the subalgebra $\mathfrak{h}_4$ is excluded.

Using the proof of Lemma 2.31 the vector $E_5 + l_1E_6 \in \mathfrak{h}_5 \cap \mathfrak{h}_7 \cap \mathfrak{h}_{14} \cap \mathfrak{h}_{18}$ is geodesic if and only if $l_1 = 0$. The vector $E_4 + s_1E_6 \in \mathfrak{h}_7 \cap \mathfrak{h}_{18}$ is geodesic precisely if one has $s_1 = 0$ (see the proof of Lemma 2.33). Using this it follows from the proof of Lemma 2.33 that the vector $E_4 + E_5 \in \mathfrak{h}_7 \cap \mathfrak{h}_{18}$ is not geodesic. Hence the subalgebras $\mathfrak{h}_7$, $\mathfrak{h}_{18}$ are excluded, too.

Next we consider the subalgebra $\mathfrak{h}_5$. The element $E_3 + k_1E_4 + k_2E_6 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = e = 0$, $c = 1$, $d = k_1$, $f = k_2$ the system (2.27) of equations is satisfied. The fourth equation gives

$$\beta_7k_2 + \alpha_5k_1k_2 = 0. \tag{2.42}$$

In addition, the vector $E_3 + k_1E_4 + k_2E_6 + E_5 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = 0$, $c = e = 1$, $d = k_1$, $f = k_2$ the system (2.27) of equations holds. The fourth equation of (2.27) yields

$$\frac{\alpha_2\alpha_5}{\alpha_4} + \beta_7k_2 + \alpha_5k_1k_2 = 0. \tag{2.43}$$

Comparing (2.42) with (2.43) we obtain the contradiction $\frac{\alpha_2\alpha_5}{\alpha_4} = 0$. Hence the vector $E_3 + k_1E_4 + k_2E_6 + E_5 \in \mathfrak{h}_5$ is not geodesic and the subalgebra $\mathfrak{h}_5$ is excluded.

Now we treat the subalgebra $\mathfrak{h}_{14}$. The element $Y = E_2 + k_1E_3 + k_2E_4 + k_3E_6 + E_5 \in \mathfrak{h}_{14}$ is geodesic if and only if for $a = 0, b = e = 1, c = k_1, d = k_2$, $f = k_3$ the system (2.27) of equations is satisfied. From Lemma 2.34 we get $k_3 = 0 = f$. Using this the third equation of (2.27) yields the contradiction $\frac{\alpha_2\alpha_5}{\alpha_4} = 0$. Hence the vector $E_2 + k_1E_3 + k_2E_4 + k_3E_6 + E_5 \in \mathfrak{h}_{14}$ is not geodesic, which excludes the subalgebra $\mathfrak{h}_{14}$ (cf. Lemma 2.5).

Next we treat the subalgebra $\mathfrak{h}_{15}$. The vector $E_4 + l_1E_5 + l_2E_6 \in \mathfrak{h}_{15}$ is geodesic if and only if $l_1 = l_2 = 0$ (cf. Lemma 2.33). The element $E_3 + k_1E_5 + k_2E_6 \in \mathfrak{h}_{15}$ is geodesic if for $a = b = d = 0, c = 1, e = k_1, f = k_2$ the system (2.27) of equations is satisfied. From the fourth and fifth equations of (2.27) we get

$$\frac{\alpha_2\alpha_5}{\alpha_4}k_1 + \beta_7k_2 = 0,\ \beta_4k_1 + \beta_5k_2 + \alpha_4k_1k_2 = 0. \tag{2.44}$$

The vector $E_3 + E_4 + k_1E_5 + k_2E_6 \in \mathfrak{h}_{15}$ is geodesic precisely if for $a = b = 0$, $c = d = 1$, $e = k_1$, $f = k_2$ the system (2.27) of equations is valid. From the fourth equation of (2.27) one has

$$\frac{\alpha_2\alpha_5}{\alpha_4}k_1 + \beta_7k_2 + \alpha_5k_2 = 0. \tag{2.45}$$

Comparing it with the first equation of (2.44) we obtain $\alpha_5k_2 = 0$. Since $\alpha_5 \neq 0$ one gets $k_2 = 0$. Using this from (2.45) we receive that also $k_1 = 0$. But the vector $Y = E_3 + E_4 \in \mathfrak{h}_{15}$ is not geodesic because $\langle [E_1, Y], Y\rangle = \alpha_2 \neq 0$ which is a contradiction to (2.2). Therefore the subalgebra $\mathfrak{h}_{15}$ is excluded.

Finally we treat the subalgebra $\mathfrak{h}_6$. The element $E_3 + k_1E_4 + k_2E_5 \in \mathfrak{h}_6$ is geodesic if and only if the system (2.27) of equations are satisfied for $a = b = f = 0$, $c = 1$, $d = k_1$, $e = k_2$. From the fourth and fifth equations of (2.27) we obtain

$$\frac{\alpha_2\alpha_5}{\alpha_4}k_2 = 0, \quad \alpha_2k_1 + \beta_4k_2 + k_1\alpha_3k_2 = 0.$$

As $\alpha_2\alpha_5 \neq 0$ we get $k_2 = k_1 = 0$. Using this the vector $E_3 + E_6$ lies in $\mathfrak{h}_6$. Moreover, it is geodesic precisely if for $a = b = d = e = 0$, $c = f = 1$ the system (2.27) is valid. From the fourth and fifth equations of (2.27) we receive $\beta_7 = \beta_5 = 0$. Therefore the subalgebra $\mathfrak{h}_6 = \text{span}(E_3, E_6)$ is flat totally geodesic in the case $\beta_5 = \beta_7 = 0$. Hence Theorem 2.28 is proved. □

Proof of Theorem 2.29. According to Proposition 2.7 any totally geodesic subalgebra $\mathfrak{h}$ of metric Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle .,. \rangle)$ has dimension less than or equal to 3. Firstly we determine the 2- and the 3-dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$.

The 2-dimensional and the 3-dimensional abelian subalgebras of Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ are:

$$\begin{aligned}
&\mathfrak{h}_2,\ \mathfrak{h}_3 \text{ with } \beta_8 + l_1\alpha_4 - k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,\ \mathfrak{h}_4,\ \mathfrak{h}_5,\ \mathfrak{h}_6,\ \mathfrak{h}_7,\ \mathfrak{h}_8\ , \mathfrak{h}_9 \text{ in (2.37)},\\
&\mathfrak{h}_{11},\ \mathfrak{h}_{12},\ \mathfrak{h}_{13} \text{ with } \beta_8 + l_1\alpha_4 - k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0, \ \text{ in } (2.38),\\
&\mathfrak{h}_{19} = \text{span}(E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_6, \quad E_5 + l_1E_6)\\
&\qquad \text{with } k_1 = -\frac{\beta_6}{\alpha_4},\\
&\mathfrak{h}_{20} = \text{span}(E_2 + k_1E_4 + k_2E_5 + k_3E_6, \quad E_3 + l_1E_4 + l_2E_5 + l_3E_6)\\
&\qquad \text{with } \beta_7 + l_1\beta_8 + l_2\alpha_4 + k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,\\
&\mathfrak{h}_{21} = \text{span}(E_1 + k_1E_2 + k_2E_3 + k_3E_4, \quad E_5, \quad E_6) \text{ with } \beta_6 + k_1\alpha_4 = 0,\\
&\mathfrak{h}_{22} = \text{span}(E_2 + h_1E_4 + h_2E_5, \quad E_3 + k_1E_4 + k_2E_5, \quad E_6)\\
&\qquad \text{with } \beta_7 + k_1\beta_8 + k_2\alpha_4 + h_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,
\end{aligned} \tag{2.46}$$

where $h_1, h_2, k_1, k_2, k_3, k_4, l_1, l_2, l_3 \in \mathbb{R}$. The subalgebras $\mathfrak{h}_8$, $\mathfrak{h}_9$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$, $\mathfrak{h}_{21}$ and $\mathfrak{h}_{13}$ are excluded (see Lemmas 2.31, 2.33). Since the vector $Y = E_3 + k_1E_4 + k_2E_5 + E_6 \in \mathfrak{h}_6 \cap \mathfrak{h}_{22}$ is not geodesic (cf. Lemma 2.32) the subalgebras $\mathfrak{h}_6$ and $\mathfrak{h}_{22}$ are not flat totally geodesic. The subalgebra $\mathfrak{h}_4$ is not flat totally geodesic because the vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_4$ is not geodesic (see Lemma 2.34). The subalgebra $\mathfrak{h}_{13}$ is not flat totally geodesic because the vector $E_4 + l_1E_4 + E_6 \in \mathfrak{h}_{13}$ is not geodesic (see Lemma 2.33).

We treat the subalgebra $\mathfrak{h}_2$. The vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 \in \mathfrak{h}_2$ is geodesic if and only if $k_2 = k_3 = k_4 = 0$ (cf. Lemma 2.35). Using this the vector $E_1 + k_1E_2 + E_6 \in \mathfrak{h}_2$ is geodesic if and only if for $a = 1, b = k_1, c = d = e = 0, f = 1$ the system (2.31) is valid. From the first equation of (2.31) we receive that $k_1 = -\frac{\beta_6}{\alpha_4}$. Using this it follows from the second equation of (2.31) that $\beta_5 = \frac{\beta_6\beta_8}{\alpha_4}$, whereas from the third equation of (2.31) that $\beta_4 = \frac{\beta_6\beta_7}{\alpha_4}$. From the fourth equation of (2.31) we obtain that $\beta_3 = 0$. This proves case (1) of the Theorem 2.29.

Here we treat the subalgebra $\mathfrak{h}_3$. The vector $E_4 + l_1E_5 + l_2E_6 \in \mathfrak{h}_3$ is geodesic precisely if $l_1 = l_2 = 0$ (see Lemma 2.33). Using this, from the condition $\beta_8 + l_1\alpha_4 - k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0$ to be abelian the subalgebra $\mathfrak{h}_3$ we receive that $k_1 = \frac{\beta_8\alpha_1}{\alpha_3\alpha_4}$. The element $E_2 + k_1E_3 + k_2E_5 + k_3E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $a = d = 0$, $b = 1$, $c = k_1$, $e = k_2$, $f = k_3$ the system (2.31) of equations holds. From the first equation of (2.31) we have $k_3\alpha_4 = 0$. As $\alpha_4 \neq 0$ it follows that $f = k_3 = 0$. Using this the fifth equation of (2.31) yields

$$\alpha_1 k_1 + \beta_2 k_2 - k_1 k_2 (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}) = 0. \tag{2.47}$$

The element $E_2 + k_1E_3 + k_2E_5 + E_4 \in \mathfrak{h}_3$ is geodesic precisely if for $a = f = 0$, $b = d = 1$, $c = k_1$, $e = k_2$ the system (2.31) of equations is valid. From the fifth equation of (2.31) we obtain

$$\alpha_1 k_1 + \beta_1 + \beta_2 k_2 + k_1\Big(\alpha_2 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})k_2\Big) + \alpha_3 k_2 = 0. \tag{2.48}$$

Taking into account equation (2.47), equation (2.48) reduces to

$$\beta_1 + k_1\alpha_2 + \alpha_3 k_2 = 0. \tag{2.49}$$

Putting the expression for k_1 into (2.49) we obtain that $k_2 = -\frac{1}{\alpha_3}(\beta_1 + \frac{\beta_8\alpha_1\alpha_2}{\alpha_3\alpha_4})$. Substituting the expressions of k_1 and k_2 into (2.47) for the parameters α_i, β_j, $i = 1, 2, 3, 4$, $j = 1, 2, 8$ of the filiform metric Lie algebra $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ we obtain equation (2.34). This proves case (2) of Theorem 2.29.

The vector $E_5 + l_1E_6 \in \mathfrak{h}_5 \cap \mathfrak{h}_7 \cap \mathfrak{h}_{19}$ is geodesic if and only if $l_1 = 0$ (see proof of Lemma 2.31). We deal with the subalgebra $\mathfrak{h}_5$. The vector

$E_3 + k_1E_4 + k_2E_6 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = e = 0$, $c = 1$, $d = k_1$, $f = k_2$ the system (2.31) of equations holds. The second, fourth and fifth equations yield

$$k_2\frac{\alpha_3\alpha_4}{\alpha_1} = 0, \quad k_2(\beta_7 + k_1\beta_8) = 0, \quad \alpha_2k_1 + \beta_4k_2 + k_1\beta_5k_2 = 0.$$

As $\frac{\alpha_3\alpha_4}{\alpha_1} \neq 0, \alpha_2 \neq 0$ we receive that $k_2 = k_1 = 0$. The vector $E_3 + E_5 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = d = f = 0$, $c = e = 1$, the system (2.31) of equations is satisfied. The fifth equation gives $\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4} = 0$. Hence the subalgebra $\mathfrak{h}_5 = \text{span}(E_3, E_5)$ is flat totally geodesic if and only if $\alpha_3\alpha_4\beta_1 + \alpha_2\alpha_1\beta_8 = 0$. This proves the case (3) in Theorem 2.29.

Next we deal with the subalgebra $\mathfrak{h}_7$. By Lemma 2.33 the vector $E_4 + s_1E_6 \in \mathfrak{h}_7$ is geodesic if and only if one has $s_1 = 0$. But in this case the vector $E_4 + E_5 \in \mathfrak{h}_7$ is not geodesic (see Lemma 2.33). Therefore the subalgebra $\mathfrak{h}_7$ is not flat totally geodesic.

In what follows, we consider the subalgebra $\mathfrak{h}_{19}$. The vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_6 \in \mathfrak{h}_{19}$ is geodesic if and only if for $a = 1, e = 0$, $b = k_1 = -\frac{\beta_6}{\alpha_4}, c = k_2, d = k_3, f = k_4$ the equations (2.31) holds. The second equation of (2.31) gives

$$k_4(\beta_5 + \beta_8k_1 - \frac{\alpha_3\alpha_4}{\alpha_1}k_2) = 0. \tag{2.50}$$

Furthermore, the element $E_1 + E_5 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_6 \in \mathfrak{h}_{19}$ is geodesic if and only if for $a = 1, e = 1, b = k_1 = -\frac{\beta_6}{\alpha_4}, c = k_2, d = k_3, f = k_4$ the system (2.31) of equations is valid. From the second equation of (2.31) one has

$$\alpha_3 + k_4(\beta_5 + \beta_8k_1 - \frac{\alpha_3\alpha_4}{\alpha_1}k_2) = 0. \tag{2.51}$$

Comparing (2.51) with (2.50) we obtain the contradiction $\alpha_3 = 0$, which excludes the subalgebra $\mathfrak{h}_{19}$.

Finally, we consider the subalgebra $\mathfrak{h}_{20}$. The element $E_2 + k_1E_4 + k_2E_5 + k_3E_6 \in \mathfrak{h}_{20}$ is geodesic if and only if for $a = c = 0, b = 1, d = k_1, e = k_2, f = k_3$ the system (2.31) of equations holds. By Lemma 2.34 one gets $f = k_3 = 0$. Applying this the fifth equation of (2.31) gives

$$\beta_1k_1 + \beta_2k_2 + k_1\alpha_3k_2 = 0. \tag{2.52}$$

The vector $E_3 + l_1E_4 + l_2E_5 + l_3E_6 \in \mathfrak{h}_{20}$ is geodesic if and only if for $a = b = 0$, $c = 1, d = l_1, e = l_2, f = l_3$ the system (2.31) of equations is satisfied. It follows from the second equation of (2.31) that $\frac{\alpha_3\alpha_4}{\alpha_1}l_3 = 0$. Since $\frac{\alpha_3\alpha_4}{\alpha_1} \neq 0$ we receive $f = l_3 = 0$. Using this the fifth equation of (2.31) gives

$$\alpha_2 l_1 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) l_2 + l_1\alpha_3 l_2 = 0. \tag{2.53}$$

Let us consider the vector $E_2 + E_3 + (k_1 + l_1)E_4 + (k_2 + l_2)E_5 \in \mathfrak{h}_{20}$. It is geodesic if and only for $a = f = 0, b = c = 1, d = k_1 + l_1, e = k_2 + l_2$ the system (2.31) of equations is valid. From the fifth equation of (2.31) one has

$$\begin{aligned} &\alpha_1 + \beta_1(k_1 + l_1) + \beta_2(k_2 + l_2) + \alpha_2(k_1 + l_1) \\ &-\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big)(k_2 + l_2) + (k_1 + l_1)\alpha_3(k_2 + l_2) = 0. \end{aligned} \tag{2.54}$$

Taking into account (2.52) and (2.53), equation (2.54) reduces to

$$\alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \alpha_2 k_1 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big)k_2 + \alpha_3(k_1 l_2 + l_1 k_2) = 0.$$

Therefore the subalgebra $\mathfrak{h}_{20}$ is flat totally geodesic if and only if it satisfies the conditions of the case (4) in Theorem 2.29. Thus Theorem 2.29 is shown. □

Proof of Theorem 2.30. The Lie algebra $\mathfrak{n}$ in Theorem 2.8 is isomorphic to the Lie algebra $\mathfrak{l}_{6,17}$ in (2.3) since the map $E_2 \mapsto E_2$, $E_i \mapsto -E_i, i = 1, 3, 4, 5, 6$, is an isomorphism $\mathfrak{n} \to \mathfrak{l}_{6,17}$. Hence the metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., .\rangle)$ does not have a totally geodesic subalgebra of dimension greater than 2 (cf. Theorem 2.8). Therefore one needs to compute only the 2-dimensional abelian subalgebras in the Lie algebra $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$. These subalgebras are:

$$\mathfrak{h}_2,\ \mathfrak{h}_3,\ \mathfrak{h}_4,\ \mathfrak{h}_5,\ \mathfrak{h}_6,\ \mathfrak{h}_7\ , \mathfrak{h}_8, \mathfrak{h}_9 \text{ in (2.37), and } \mathfrak{h}_{14}, \mathfrak{h}_{15} \text{ in (2.41),}$$

According to Lemma 2.31, the subalgebra $\mathfrak{h}_9$ is not flat totally geodesic because it contains the vector $Y = E_5 + E_6$. Hence the subalgebra $\mathfrak{h}_9$ is excluded. By Lemma 2.35 the vector $Y = E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 \in \mathfrak{h}_2$ is geodesic precisely if $k_2 = k_3 = k_4 = 0$. But the vector $Y = E_1 + k_1E_2 + E_6 \in$

$\mathfrak{h}_2$ is not geodesic (cf. Lemma 2.35). By Lemma 2.5, the subalgebra $\mathfrak{h}_2$ is not flat totally geodesic and it is excluded, too. According to Lemma 2.34 the vector $Y = E_2 + k_1 E_3 + k_2 E_4 + k_3 E_5 + E_6 \in \mathfrak{h}_4$ is not geodesic, which excludes the subalgebra $\mathfrak{h}_4$ (cf. Lemma 2.5). Since the vector $Y = E_3 + k_1 E_4 + k_2 E_5 + E_6 \in \mathfrak{h}_6$ is not geodesic the subalgebra $\mathfrak{h}_6$ is not flat totally geodesic (see Lemma 2.32).

Next we deal with the subalgebra $\mathfrak{h}_3$. The vector $Y = E_4 + l_1 E_5 + l_2 E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $a = b = c = 0, d = 1, e = l_1, f = l_2$ the system (2.33) of equations is satisfied. The fifth equation of (2.33) yields

$$\alpha_3 l_1 + \beta_6 l_2 + l_1 \alpha_4 l_2 = 0. \tag{2.55}$$

The element $E_2 + k_1 E_3 + k_2 E_5 + k_3 E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $a = d = 0, b = 1, c = k_1, e = k_2, f = k_3$ the system (2.33) of equation is valid. From the third equation of (2.33) one has $\alpha_5 k_3 = 0$. As $\alpha_5 \neq 0$ we receive $k_3 = f = 0$. Using this the fifth equation of (2.33) gives

$$\alpha_1 k_1 + \beta_2 k_2 + k_1 \beta_4 k_2 = 0. \tag{2.56}$$

Additionally, the vector $E_2 + E_4 + k_1 E_3 + (k_2 + l_1) E_5 + l_2 E_6 \in \mathfrak{h}_3$ is geodesic precisely if for $a = 0, b = d = 1, c = k_1, e = k_2 + l_1, f = l_2$ the system (2.33) of equations holds. The third equation of (2.33) gives $\alpha_5 l_2 = 0$. Since $\alpha_5 \neq 0$ we get $l_2 = f = 0$. Using this from the equation (2.55) we get $\alpha_3 l_1 = 0$. As $\alpha_3 \neq 0$ we obtain $l_1 = e = 0$, and from the fifth equation of (2.33) we obtain

$$\alpha_1 k_1 + \beta_1 + \beta_2 k_2 + k_1 \alpha_2 + k_1 \beta_4 k_2 + \alpha_3 k_2 = 0. \tag{2.57}$$

Applying (2.56), equation (2.57) reduces to

$$\beta_1 + \alpha_2 k_1 + \alpha_3 k_2 = 0.$$

The above equation gives $k_1 = -\frac{\beta_1 + \alpha_3 k_2}{\alpha_2}$. Putting this expression into (2.56) we receive the second order equation (2.35). Hence the case (1) of Theorem 2.30 is proved.

The vector $E_5 + l_1 E_6 \in \mathfrak{h}_5 \cap \mathfrak{h}_7 \cap \mathfrak{h}_{14}$ is geodesic if and only if $l_1 = 0$ (see proof of Lemma 2.31).

In what follows we treat the subalgebra $\mathfrak{h}_5$. The element $E_3 + k_1 E_4 + k_2 E_6 \in \mathfrak{h}_5$ is geodesic if and only if for $a = b = e = 0, c = 1, d = k_1, f = k_2$ the system (2.33) of equations is satisfied. From Lemma 2.32 we get $k_2 = f = 0$. Using this the fifth equation of (2.33) gives $\alpha_2 k_1 = 0$. Since $\alpha_2 \neq 0$ we obtain $k_1 = 0$. The vector $E_3 + E_5 \in \mathfrak{h}_5$ is geodesic precisely if for $a = b = d = f = 0$, $c = e = 1$, the system (2.33) of equations holds. From the fifth equation of (2.33) we get $\beta_4 = 0$. This gives the case (2) of Theorem 2.30.

Now, we consider the subalgebra $\mathfrak{h}_7$. The vector $Y = E_4 + s_1 E_6 \in \mathfrak{h}_7$ is geodesic precisely if for $a = b = c = e = 0$, $d = 1$, $f = s_1$ the equations (2.33) are satisfied. From the fifth equation we obtain

$$\beta_6 s_1 = 0. \tag{2.58}$$

Furthermore, the element $E_4 + s_1 E_6 + E_5 \in \mathfrak{h}_7$ is geodesic if and only if for $a = b = c = 0$, $d = e = 1, f = s_1$ the system (2.33) of equations is valid. It follows from the fifth equation of (2.33) that

$$\alpha_3 + \beta_6 s_1 + \alpha_4 s_1 = 0. \tag{2.59}$$

Taking into account (2.58), equation (2.59) reduces to

$$\alpha_3 + \alpha_4 s_1 = 0. \tag{2.60}$$

The equation (2.60) gives $s_1 = -\frac{\alpha_3}{\alpha_4}$. Putting this expression into (2.58) one has $\beta_6 = 0$. Therefore the case (3) of Theorem 2.30 is proved.

We treat the subalgebra $\mathfrak{h}_8$. The vector $E_4 + l_1 E_5 \in \mathfrak{h}_8$ is geodesic if and only if for $a = b = c = f = 0$, $d = 1$, $e = l_1$ the system (2.33) of equations is valid. It follows from the fifth equation of (2.33) that $\alpha_3 l_1 = 0$. As $\alpha_3 \neq 0$ we get $l_1 = 0$. The vector $E_4 + E_6 \in \mathfrak{h}_8$ is geodesic precisely if for $a = b = c = e = 0, d = f = 1$ the system (2.33) of equations holds. The fifth equation gives $\beta_6 = 0$. Therefore the case (4) of Theorem 2.30 is shown.

Here we consider the subalgebra $\mathfrak{h}_{14}$. The element $E_2 + k_1 E_3 + k_2 E_4 + k_3 E_6 \in \mathfrak{h}_{14}$ is geodesic precisely if for $a = e = 0, b = 1, c = k_1, d = k_2, f = k_3$ the system (2.33) of equations holds. From Lemma 2.34 it follows that $k_3 = 0 = f$. Using this the fifth equation of (2.33) gives

$$\alpha_1 k_1 + \beta_1 k_2 + k_1 \alpha_2 k_2 = 0. \tag{2.61}$$

Moreover, the element $E_2 + k_1E_3 + k_2E_4 + E_5 \in \mathfrak{h}_{14}$ is geodesic if and only if for $a = f = 0, b = e = 1, c = k_1, d = k_2$ the system (2.33) of equations is valid. From the fifth equation of (2.33) we receive

$$\alpha_1k_1 + \beta_1k_2 + \beta_2 + k_1\alpha_2k_2 + k_1\beta_4 + k_2\alpha_3 = 0. \tag{2.62}$$

Taking into account (2.61), equation (2.62) reduces to

$$\beta_2 + \beta_4k_1 + \alpha_3k_2 = 0. \tag{2.63}$$

From (2.63) one has $k_2 = -\frac{\beta_2+\beta_4k_1}{\alpha_3}$. Putting this expression into (2.61) we receive the second order equation (2.36). Therefore the case (5) in Theorem 2.30 is proved.

Finally, we deal with the subalgebra $\mathfrak{h}_{15}$. The vector $E_3 + k_1E_5 + k_2E_6 \in \mathfrak{h}_{15}$ is geodesic precisely if for $a = b = d = 0, c = 1, e = k_1, f = k_2$ the system (2.33) of equations holds. From Lemma 2.32 one obtains $\alpha_5k_2 = 0$. As $\alpha_5 \neq 0$ we get $k_2 = f = 0$. Using this the fifth equation of (2.33) gives

$$\beta_4k_1 = 0. \tag{2.64}$$

In addition, the element $E_3 + E_4 + (k_1 + l_1)E_5 + l_2E_6 \in \mathfrak{h}_{15}$ is geodesic if and only if for $a = b = 0, c = d = 1, e = k_1 + l_1, f = l_2$ the system (2.33) of equations is valid. acoording to Lemma 2.32, we obtain $l_2 = f = 0$. Using this from the equation (2.55) we get $\alpha_3l_1 = 0$. As $\alpha_3 \neq 0$ we receive $l_1 = 0$, and from the fifth equation of (2.33) yields

$$\alpha_2 + \beta_4k_1 + \alpha_3k_1 = 0. \tag{2.65}$$

Applying (2.64), equation (2.65) reduces to

$$\alpha_2 + \alpha_3k_1 = 0.$$

From the above equation one has $k_1 = -\frac{\alpha_2}{\alpha_3}$. Putting this expression into (2.64) we receive that $\beta_4 = 0$. This proves case (6) of Theorem 2.30. Hence Theorem 2.30 is proved. □

2.4.2 Metric Lie algebras corresponding to the standard filiform Lie algebras

In this subsection, we consider the filiform metric Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$. We prove the following result.

Theorem 2.36. *Let* $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the metric Lie algebra defined on* $\mathbb{E}^6$ *by the commutators* (2.13). *The flat totally geodesic subalgebras of dimension* > 1 *in* $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ *are:*

1. *the* 4*-dimensional subalgebras*

   (a) $\mathfrak{h}_{23} = span(E_2 - \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}E_6, \quad E_3, \quad E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5)$ *in the case* $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1\alpha_3}{\alpha_2}$,

   (b) $\mathfrak{h}_{24} = span(E_2, \quad E_3 - \frac{\alpha_2}{\alpha_3}E_5, \quad E_4, \quad E_6)$ *in the case* $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1\alpha_3}{\alpha_2}$,

2. *the* 3*-dimensional subalgebras*

   (a) $\mathfrak{h}_{13} = span(E_2 + k_1E_3 + k_2E_5, \quad E_4, \quad E_6)$ *such that one of the following cases is satisfied:*

      i. $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_2 = -\frac{\beta_3 + k_1\beta_5}{\alpha_4}$, *and* k_1 *is a solution of the equation*

      $$\beta_4\beta_5(k_1)^2 + k_1(\beta_2\beta_5 + \beta_3\beta_4 - \alpha_1\alpha_4) + \beta_2\beta_3 = 0, \quad (2.66)$$

      ii. $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_1 = \frac{\alpha_3\beta_3 - \alpha_4\beta_1}{\alpha_2\alpha_4 - \alpha_3\beta_5}$, $k_2 = \frac{\beta_3(\alpha_2\alpha_4 - \alpha_3\beta_5) + \beta_5(\beta_3\alpha_3 - \beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4 - \alpha_3\beta_5)}$, *and the equation*

      $$(\alpha_2\alpha_4 - \alpha_3\beta_5)\Big((\alpha_1\alpha_4 - \beta_3\beta_4 - \beta_2\beta_5)(\beta_3\alpha_3 - \beta_1\alpha_4) - \beta_2\beta_3(\alpha_2\alpha_4 - \alpha_3\beta_5) - \beta_4\beta_5(\beta_3\alpha_3 - \beta_1\alpha_4)^2\Big) = 0 \quad (2.67)$$

      holds,

   (b) $\mathfrak{h}_{16} = span(E_3 - \frac{\alpha_2}{\alpha_3}E_5, \quad E_4, \quad E_6)$ *in the case* $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

(c) $\mathfrak{h}_{18} = span(E_3, \quad E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5)$ *in the case* $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

(d) $\mathfrak{h}_{22} = span\,(E_2 + l_1E_4 + l_2E_5, \quad E_3 + k_1E_4 + k_2E_5, \quad E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_1 l_1 + \beta_2 l_2 + l_1\alpha_3 l_2 = 0,\\
&\alpha_2 k_1 + \beta_4 k_2 + k_1 k_2 \alpha_3 = 0,\\
&\alpha_1 + \beta_1 k_1 + \beta_2 k_2 + \alpha_2 l_1 + \beta_4 l_2 + l_1 k_2\alpha_3 + k_1 l_2 \alpha_3 = 0,\\
&\beta_3 + \beta_6 l_1 + \alpha_4 l_2 = 0,\\
&\beta_5 + k_1\beta_6 + k_2\alpha_4 = 0
\end{aligned}$$

hold,

(e) $\mathfrak{h}_{28} = span(E_2+k_1E_5+k_2E_6,\ E_3+l_1E_5+l_2E_6,\ E_4+s_1E_5+s_2E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_2 k_1 + \beta_3 k_2 + k_1\alpha_4 k_2 = 0,\\
&\beta_4 l_1 + \beta_5 l_2 + l_1\alpha_4 l_2 = 0,\\
&\alpha_3 s_1 + \beta_6 s_2 + s_1\alpha_4 s_2 = 0,\\
&\alpha_1 + \beta_2 l_1 + \beta_3 l_2 + \beta_4 k_1 + \beta_5 k_2 + k_1 l_2\alpha_4 + l_1 k_2 \alpha_4 = 0,\\
&\beta_1 + \beta_2 s_1 + \beta_3 s_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 s_2\alpha_4 + s_1 k_2\alpha_4 = 0,\\
&\alpha_2 + \beta_4 s_1 + \beta_5 s_2 + \alpha_3 l_1 + \beta_6 l_2 + l_1 s_2\alpha_4 + s_1 l_2\alpha_4 = 0,\\
&\beta_2 k_1 + \beta_3 k_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 s_1 + \beta_6 s_2 + k_1 s_2 + k_1 k_2\alpha_4\\
&\qquad\qquad + l_1 l_2\alpha_4 + s_1 s_2\alpha_4 = 0
\end{aligned}$$

are satisfied,

(f) $\mathfrak{h}_{29} = span\,(E_2 + k_1E_4 + k_2E_6, \quad E_3 + l_1E_4 + l_2E_6, \quad E_5)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_1 k_1 + \beta_3 k_2 + k_1\beta_6 k_2 = 0,\\
&\alpha_2 l_1 + \beta_5 l_2 + l_1\beta_6 l_2 = 0,\\
&\alpha_1 + \beta_1 l_1 + \beta_3 l_2 + \alpha_2 k_1 + \beta_5 k_2 + k_1 l_2\beta_6 + l_1 k_2\beta_6 = 0,\\
&\beta_2 + \alpha_3 k_1 + \alpha_4 k_2 = 0,
\end{aligned}$$

$$\beta_4 + \alpha_3 l_1 + \alpha_4 l_2 = 0$$

are satisfied,

(g) $\mathfrak{h}_{30} = span(E_2 + k_1 E_3 + k_2 E_6, \quad E_4 + s_1 E_6, \quad E_5)$ *such that one of the following cases is satisfied:*

i. $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_2 = -\frac{\beta_2+k_1\beta_4}{\alpha_4}$, $s_1 = -\frac{\alpha_3}{\alpha_4}$ *and* k_1 *is a solution of the equation* (2.66)

ii. $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}, k_1 = \frac{\alpha_3\beta_3-\alpha_4\beta_1}{\alpha_2\alpha_4-\alpha_3\beta_5}, k_2 = -\frac{\beta_2(\alpha_2\alpha_4-\alpha_3\beta_5)+\beta_4(\beta_3\alpha_3-\beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4-\alpha_3\beta_5)}$, $s_1 = -\frac{\alpha_3}{\alpha_4}$, *and the equation* (2.67) *holds,*

3. *the* 2*-dimensional subalgebras*

(a) $\mathfrak{h}_3 = span\,(E_2 + k_1 E_3 + k_2 E_5 + k_3 E_6, \quad E_4 + l_1 E_5 + l_2 E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\alpha_3 l_1 + \beta_6 l_2 + l_1\alpha_4 l_1 = 0,\\
&\alpha_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + k_1\beta_4 k_2 + k_1\beta_5 k_3 + k_2\alpha_4 k_3 = 0,\\
&\beta_1 + \beta_2 l_1 + \beta_3 l_2 + \alpha_2 k_1 + k_1\beta_4 l_1 + k_1\beta_5 l_2 + \alpha_3 k_2\\
&\qquad\qquad + \beta_6 k_3 + \alpha_4 k_2 l_2 + \alpha_4 l_1 k_3 = 0
\end{aligned}$$

hold,

(b) $\mathfrak{h}_4 = span(E_2 + k_1 E_3 + k_2 E_4 - \frac{\beta_3+k_1\beta_5+k_2\beta_6}{\alpha_4}E_5, \quad E_6)$ *such that the equation*

$$\begin{aligned}
&(\alpha_3 k_2 + \beta_2 + \beta_4 k_1)(k_1\beta_5 + k_2\beta_6 + \beta_3)\\
&\quad - \alpha_2\alpha_4 k_1 k_2 - \alpha_1\alpha_4 k_1 - \beta_1\alpha_4 k_2 = 0
\end{aligned} \tag{2.68}$$

is satisfied,

(c) $\mathfrak{h}_5 = span(E_3 + k_1 E_4 - \frac{\beta_4+\alpha_3 k_1}{\alpha_4}E_6, \quad E_5)$, *where* k_1 *is a solution of the equation*

$$\alpha_3\beta_6(k_1)^2 + k_1(\alpha_3\beta_5 + \beta_4\beta_6 - \alpha_2\alpha_4) + \beta_4\beta_5 = 0, \tag{2.69}$$

(d) $\mathfrak{h}_6 = span(E_3 + k_1 E_4 - \frac{\beta_5+\beta_6 k_1}{\alpha_4}E_5, \quad E_6)$, *where* k_1 *is a solution of the equation* (2.69),

(e) $\mathfrak{h}_7 = span(E_4 - \frac{\alpha_3}{\alpha_4}E_6, \quad E_5)$ *in the case* $\beta_6 = 0$,

(f) $\mathfrak{h}_8 = span\,(E_4, \quad E_6)$ *in the case* $\beta_6 = 0$,

(g) $\mathfrak{h}_{14} = span(E_2 + k_1E_3 + k_2E_4 - \frac{\beta_2+k_1\beta_4+k_2\alpha_3}{\alpha_4}E_6, \quad E_5)$ *such that the equation* (2.68) *holds,*

(h) $\mathfrak{h}_{15} = span(E_3 + k_1E_5 + k_2E_6, \quad E_4 + l_1E_5 + l_2E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_4k_1 + \beta_5k_2 + k_1\alpha_4k_2 = 0,\\
&\alpha_3l_1 + \beta_6l_2 + l_1\alpha_4l_2 = 0,\\
&\alpha_2 + \beta_4l_1 + \beta_5l_2 + \alpha_3k_1 + \beta_6k_2 + k_1l_2\alpha_4 + l_1k_2\alpha_4 = 0
\end{aligned}$$

are satisfied,

(i) $\mathfrak{h}_{20} = span(E_2 + k_1E_4 + k_2E_5 + k_3E_6, \quad E_3 + l_1E_4 + l_2E_5 + l_3E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_1k_1 + \beta_2k_2 + \beta_3k_3 + k_1\alpha_3k_2 + k_1\beta_6k_3 + k_2\alpha_4k_3 = 0,\\
&\alpha_2l_1 + \beta_4l_2 + \beta_5l_3 + l_1\alpha_3l_2 + l_1\beta_6l_3 + l_2\alpha_4l_3 = 0,\\
&\alpha_1 + \beta_1l_1 + \beta_2l_2 + \beta_3l_3 + \alpha_2k_1 + \beta_4k_2 + \beta_5k_3 + k_1l_2\alpha_3\\
&\qquad + l_1k_2\alpha_3 + k_1l_3\beta_6 + l_1k_3\beta_6 + k_2l_3\alpha_4 + l_2l_3\alpha_4 = 0
\end{aligned} \tag{2.70}$$

are satisfied.

Proof. According to Proposition 1.13 b) the dimension of the flat totally geodesic subalgebras of $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ is at most 4. Firstly we list the 4-dimensional, the 3-dimensional and the 2-dimensional abelian subalgebras of the Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ defined by the commutators (2.13).

The 2-, 3- and 4-dimensional subalgebras have one of the following forms:

$\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6, \mathfrak{h}_7, \mathfrak{h}_8, \mathfrak{h}_9$ in (2.37), $\mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}$ in (2.38), $\mathfrak{h}_{14}, \mathfrak{h}_{15}, \mathfrak{h}_{16}, \mathfrak{h}_{17}$, $\mathfrak{h}_{18}$ in (2.41), $\mathfrak{h}_{20}, \mathfrak{h}_{22}$ in (2.46),

$\mathfrak{h}_{23} = \text{span}\,(E_2 + k_2E_6, \quad E_3 + k_1E_6, \quad E_4 + s_1E_6, \quad E_5 + l_1E_6)\,,$

$\mathfrak{h}_{24} = \text{span}\,(E_2 + s_1E_5, \quad E_3 + k_1E_5, \quad E_4 + l_1E_5, \quad E_6)\,,$

$\mathfrak{h}_{25} = \text{span}\,(E_2 + k_1E_4, \quad E_3 + k_2E_4, \quad E_5, \quad E_6)\,,$

$\mathfrak{h}_{26} = \text{span}(E_2, E_4, E_5, E_6)$,
$\mathfrak{h}_{27} = \text{span}(E_3, E_4, E_5, E_6$,
$\mathfrak{h}_{28} = \text{span}\left(E_2 + k_1E_5 + k_2E_6, \quad E_3 + l_1E_5 + l_2E_6, \quad E_4 + s_1E_5 + s_2E_6\right)$,
$\mathfrak{h}_{29} = \text{span}\left(E_2 + k_1E_4 + k_2E_6, \quad E_3 + l_1E_4 + l_2E_6, \quad E_5 + l_1E_6\right)$,
$\mathfrak{h}_{30} = \text{span}\left(E_2 + k_1E_3 + k_2E_6, \quad E_4 + s_1E_6, \quad E_5 + l_1E_6\right)$,

where $k_1, k_2, l_1, l_2, s_1, s_2 \in \mathbb{R}$. By Lemma 2.35 the vector $E_1 + k_1E_2 + k_2E_3 + k_3E_4 + k_4E_5 \in \mathfrak{h}_2$ is geodesic precisely if $k_2 = k_3 = k_4 = 0$. But the vector $Y = E_1 + k_1E_2 + E_6 \in \mathfrak{h}_2$ is not geodesic (cf. Lemma 2.35). According to Lemma 2.5 the subalgebra $\mathfrak{h}_2$ is not flat totally geodesic hence it is excluded.

For all elements of the remaining subalgebras we have $a = 0$. Since α_5 is missing for the metric Lie algebra $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$, the system (2.33) of equations reduces to the equation

$$\begin{gathered} b(\alpha_1c + \beta_1d + \beta_2e + \beta_3f) + c(\alpha_2d + \beta_4e + \beta_5f) + \\ d(\alpha_3e + \beta_6f) + e\alpha_4f = 0. \end{gathered} \tag{2.71}$$

Acoording to Lemma 2.31, the subalgebras $\mathfrak{h}_{25}, \mathfrak{h}_{26}, \mathfrak{h}_{27}, \mathfrak{h}_{17}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_9$ are not flat totally geodesic because they contain the vector $E_5 + E_6$. Hence they are excluded. According to the proof of Lemma 2.31, the vector $E_5 + l_1E_6 \in \mathfrak{h}_{14} \cap \mathfrak{h}_5$ is geodesic if and only if one has $l_1 = 0$.

The subalgebra $\mathfrak{h}_{23}$ is totally geodesic precisely if for all $Y, Z \in \mathfrak{h}_{23}$ and $X \in \mathfrak{h}_{23}^{\perp} = \text{span}(E_1, E_6 - k_2E_2 - k_1E_3 - s_1E_4 - l_1E_5)$ equation (2.1) holds. The subalgebra $\mathfrak{h}_{24}$ is totally geodesic if and only if for all $Y, Z \in \mathfrak{h}_{24}$ and $X \in \mathfrak{h}_{24}^{\perp} = \text{span}(E_1, E_5 - s_1E_2 - k_1E_3 - l_1E_4)$ equation (2.1) is satisfied. Since the commutation relations of the element $E_6 - k_2E_2 - k_1E_3 - s_1E_4 - l_1E_5$ and the elements of $\mathfrak{h}_{23}$ as well as the element $E_5 - s_1E_2 - k_1E_3 - l_1E_4$ and the elements of $\mathfrak{h}_{24}$ are zero, we may assume that $X = E_1$. In addition, the element $X = E_1$ lies in the orthogonal complement $\mathfrak{h}_i^{\perp}$ for all subalgebras $\mathfrak{h}_i$, $i = 3, \cdots, 30$. Using the equation (2.1) we receive the following:

1. For $Y = Z = E_5 + l_1E_6 \in \mathfrak{h}_{23} \cap \mathfrak{h}_{29} \cap \mathfrak{h}_{30} \cap \mathfrak{h}_{18}$ we have $2\alpha_4l_1 = 0$ and hence $l_1 = 0$.

2. Taking the elements $Y = E_4 + s_1E_6,\ Z = E_5$ in $\mathfrak{h}_{23} \cap \mathfrak{h}_{30} \cap \mathfrak{h}_{18}$ we get $\alpha_3 + \alpha_4s_1 = 0$ and hence $s_1 = -\frac{\alpha_3}{\alpha_4} < 0$.

3. For $Y = Z = E_4 + s_1E_6 \in \mathfrak{h}_{23} \cap \mathfrak{h}_{30} \cap \mathfrak{h}_{18}$ one has $2\beta_6 s_1 = 0$ and hence $\beta_6 = 0$.

4. The elements $Y = E_3 + k_1E_6$, $Z = E_4 + s_1E_6$ in $\mathfrak{h}_{23} \cap \mathfrak{h}_{18}$ yield that $\alpha_2 + \beta_5 s_1 + \beta_6 k_1 = \alpha_2 - \beta_5 \frac{\alpha_3}{\alpha_4} = 0$ and hence $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3} > 0$.

5. For $Y = E_3 + k_1E_6$, $Z = E_5$ in $\mathfrak{h}_{23} \cap \mathfrak{h}_{18}$ we obtain $\beta_4 + \alpha_4 k_1 = 0$ and hence $k_1 = -\frac{\beta_4}{\alpha_4}$.

6. For $Y = Z = E_3 + k_1E_6 \in \mathfrak{h}_{23} \cap \mathfrak{h}_{18}$, one has $2\beta_5 k_1 = 0$ and hence $k_1 = 0$ and $\beta_4 = 0$.

7. For the elements $Y = E_2 + k_2E_6$, $Z = E_5$ in $\mathfrak{h}_{23}$ we get $\beta_2 + \alpha_4 k_2 = 0$ and therefore $k_2 = -\frac{\beta_2}{\alpha_4}$.

8. For the elements $Y = E_2 + k_2E_6$, $Z = E_3$ in $\mathfrak{h}_{23}$ we receive that $\alpha_1 + \beta_5 k_2 = \alpha_1 - \frac{\alpha_2}{\alpha_3}\beta_2 = 0$ and hence $\beta_2 = \frac{\alpha_1\alpha_3}{\alpha_2} > 0$, moreover $k_2 = -\frac{\alpha_1\alpha_3}{\alpha_4\alpha_2} < 0$.

9. Taking the elements $Y = Z = E_2 + k_2E_6 \in \mathfrak{h}_{23}$ one has $2\beta_3 k_2 = 0$ and hence $\beta_3 = 0$.

10. For $Y = E_2 + k_2E_6$, $Z = E_4 + s_1E_6$ of $\mathfrak{h}_{23}$ we obtain $\beta_1 + \beta_3 s_1 + \beta_6 k_2 = \beta_1 = 0$.

Taking into account (1)–(10) the subalgebra $\mathfrak{h}_{23}$ is flat totally geodesic if and only if $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0,\ \beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3},\ \beta_2 = \frac{\alpha_1\alpha_3}{\alpha_2}$. Hence the case (1a) is proved.

11. For $Y = Z = E_4 + l_1E_5 \in \mathfrak{h}_{24} \cap \mathfrak{h}_{13} \cap \mathfrak{h}_{16}$ we obtain that $\alpha_3 l_1 = 0$ and hence $l_1 = 0$.

12. For the elements $Y = E_4,\ Z = E_6$ of $\mathfrak{h}_{24} \cap \mathfrak{h}_{13} \cap \mathfrak{h}_{16}$ we receive that $\beta_6 = 0$.

13. Taking the elements $Y = E_3 + k_1E_5,\ Z = E_4$ in $\mathfrak{h}_{24} \cap \mathfrak{h}_{16}$ one gets $\alpha_2 + \alpha_3 k_1 = 0$ which implies that $k_1 = -\frac{\alpha_2}{\alpha_3} < 0$.

14. For $Y = Z = E_3 + k_1 E_5 \in \mathfrak{h}_{24} \cap \mathfrak{h}_{16}$ we have $\beta_4 k_1 = 0$ and hence $\beta_4 = 0$.

15. For the elements $Y = E_3 + k_1 E_5,\ Z = E_6$ of $\mathfrak{h}_{24} \cap \mathfrak{h}_{16}$ one obtains $\beta_5 + \alpha_4 k_1 = 0$ which yields $\beta_5 = \frac{\alpha_4 \alpha_2}{\alpha_3}$.

16. For $Y = E_2 + s_1 E_5,\ Z = E_3 + k_1 E_5$ in $\mathfrak{h}_{24}$ we receive $\alpha_1 + \beta_2 k_1 + \beta_4 s_1 = 0$ and hence $\beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2} > 0$.

17. For $Y = Z = E_2 + s_1 E_5 \in \mathfrak{h}_{24}$ one has $s_1 \beta_2 = 0$ and hence $s_1 = 0$.

18. Taking the elements $Y = E_2,\ Z = E_4$ of $\mathfrak{h}_{24}$ we get that $\beta_1 = 0$.

19. For the elements $Y = E_2,\ Z = E_6$ in $\mathfrak{h}_{24}$ we receive that $\beta_3 = 0$.

According to (11)–(19) the subalgebra $\mathfrak{h}_{24}$ is flat totally geodesic precisely if $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0,\ \beta_5 = \frac{\alpha_2 \alpha_4}{\alpha_3},\ \beta_2 = \frac{\alpha_1 \alpha_3}{\alpha_2}$. Hence the assertion (1b) follows.

Now we treat the subalgebra $\mathfrak{h}_{13}$. According to (11)–(12) one has $l_1 = \beta_6 = 0$. The element $E_2 + k_1 E_3 + k_2 E_5 \in \mathfrak{h}_{13}$ is geodesic if and only if for $b = 1$, $c = k_1$, $e = k_2$, $d = f = 0$ the equation (2.71) is satisfied. This gives the equation

$$\alpha_1 k_1 + \beta_3 k_2 + k_1 \beta_4 k_2 = 0. \tag{2.72}$$

The element $E_2 + k_1 E_3 + k_2 E_5 + E_4 \in \mathfrak{h}_{13}$ is geodesic if and only if for $b = d = 1$, $c = k_1$, $e = k_2$, $f = 0$ the equation (2.71) is valid. Using equations (2.71) and (2.72) we receive

$$\beta_1 + k_1 \alpha_2 + \alpha_3 k_2 = 0. \tag{2.73}$$

Furthermore, the element $E_2 + k_1 E_3 + k_2 E_5 + E_6 \in \mathfrak{h}_{13}$ is geodesic precisely if for $b = f = 1$, $c = k_1$, $e = k_2$, $d = 0$ the equation (2.71) holds. Taking into account (2.72) from equation (2.71) we get

$$\beta_3 + k_1 \beta_5 + k_2 \alpha_4 = 0. \tag{2.74}$$

Hence we obtain $k_2 = -\frac{\beta_3 + k_1 \beta_5}{\alpha_4}$. Putting this expression of k_2 into (2.73) we receive

$$k_1(\alpha_2 \alpha_4 - \alpha_3 \beta_5) = \alpha_3 \beta_3 - \alpha_4 \beta_1. \tag{2.75}$$

If $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, then from (2.75) we obtain $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$ and from (2.72) we get the second order equation (2.66). This proves case 2(a)i. If $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, then from (2.75) we have $k_1 = \frac{\beta_3\alpha_3-\beta_1\alpha_4}{\alpha_2\alpha_4-\alpha_3\beta_5}$. Putting this into (2.74) we obtain $k_2 = -\frac{\beta_3(\alpha_2\alpha_4-\alpha_3\beta_5)+\beta_5(\beta_3\alpha_3-\beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4-\alpha_3\beta_5)}$. Substituting the expression of k_1 and k_2 into (2.72) we receive equation (2.67). Hence case 2(a)ii is proved.

In what follows we address the subalgebras $\mathfrak{h}_{16}$ and $\mathfrak{h}_{18}$. Taking into account (1)–(6) for the subalgebra $\mathfrak{h}_{18}$ and (11)–(15) for the subalgebra $\mathfrak{h}_{16}$ it follows from equation (2.71) that the subalgebra $\mathfrak{h}_{18}$ in (2c) as well as the subalgebra $\mathfrak{h}_{16}$ in (2b) with the conditions $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$ are flat totally geodesic.

Now we consider the subalgebra $\mathfrak{h}_{22}$. The vector $E_2 + l_1E_4 + l_2E_5 \in \mathfrak{h}_{22}$ is geodesic if and only if for $b = 1$, $d = l_1$, $e = l_2$, $c = f = 0$ the equation (2.71) is satisfied. This gives the equation

$$\beta_1 l_1 + \beta_2 l_2 + l_1\alpha_3 l_2 = 0. \tag{2.76}$$

The element $E_3 + k_1E_4 + k_2E_5 \in \mathfrak{h}_{22}$ is geodesic if and only if for $c = 1$, $d = k_1$, $e = k_2$, $b = f = 0$ the equation (2.71) holds. Hence we receive

$$\alpha_2 k_1 + \beta_4 k_2 + k_1 k_2 \alpha_3 = 0. \tag{2.77}$$

The vector $E_2 + E_3 + (l_1 + k_1)E_4 + (l_2 + k_2)E_5 \in \mathfrak{h}_{22}$ is geodesic if and only if for $b = c = 1$, $d = l_1 + k_1$, $e = l_2 + k_2$, $f = 0$ the equation (2.71) valid. Using (2.76), (2.77) one has the following equation

$$\alpha_1 + \beta_1 k_1 + \beta_2 k_2 + \alpha_2 l_1 + \beta_4 l_2 + l_1 k_2 \alpha_3 + k_1 l_2 \alpha_3 = 0. \tag{2.78}$$

The element $E_2 + E_6 + l_1E_4 + l_2E_5 \in \mathfrak{h}_{22}$ is geodesic if and only if for $b = f = 1$, $d = l_1$, $e = l_2$, $c = 0$ the equation (2.71) is satisfied. Applying (2.76) one gets

$$\beta_3 + \beta_6 l_1 + \alpha_4 l_2 = 0. \tag{2.79}$$

The element $E_3 + k_1E_4 + k_2E_5 + E_6 \in \mathfrak{h}_{22}$ is geodesic if and only if for $c = f = 1$, $d = k_1$, $e = k_2$, $b = 0$ the equation (2.71) holds. Using (2.77) we receive

$$\beta_5 + k_1\beta_6 + k_2\alpha_4 = 0. \tag{2.80}$$

The vector $E_2 + E_3 + (l_1 + k_1)E_4 + (l_2 + k_2)E_5 + E_6 \in \mathfrak{h}_{22}$ is geodesic if and only if for $c = b = f = 1$, $d = l_1 + k_1$, $e = l_2 + k_2$ the equation (2.71) is satisfied. This gives

$$\begin{aligned} &\alpha_1 + \beta_1 l_1 + \beta_1 k_1 + \beta_2 l_2 + \beta_2 k_2 + \beta_3 + \alpha_2 l_1 + \alpha_2 k_1 \\ &\quad + \beta_4 l_2 + \beta_4 k_2 + \beta_5 + l_1 l_2 \alpha_3 + l_1 k_2 \alpha_3 + k_1 l_2 \alpha_3 \\ &\quad + k_1 k_2 \alpha_3 + l_1 \beta_6 + k_1 \beta_6 + l_2 \alpha_4 + k_2 \alpha_4 = 0. \end{aligned} \tag{2.81}$$

Using the equations (2.76), (2.77), (2.78), (2.79), and (2.80), the equation (2.81) holds. Therefore, conditions in case (2d) for the subalgebra $\mathfrak{h}_{22}$ are proved.

Here we deal with the subalgebra $\mathfrak{h}_{28}$. The element $E_2 + k_1 E_5 + k_2 E_6 \in \mathfrak{h}_{28}$ is geodesic if and only if for $b = 1$, $e = k_1$, $f = k_2$, $c = d = 0$ the equation (2.71) holds. From this we obtain

$$\beta_2 k_1 + \beta_3 k_2 + k_1 \alpha_4 k_2 = 0. \tag{2.82}$$

The vector $E_3 + l_1 E_5 + l_2 E_6 \in \mathfrak{h}_{28}$ is geodesic if and only if for $c = 1$, $e = l_1$, $f = l_2$, $b = d = 0$ the equation (2.71) is satisfied. This gives

$$\beta_4 l_1 + \beta_5 l_2 + l_1 \alpha_4 l_2 = 0. \tag{2.83}$$

The element $E_4 + s_1 E_5 + s_2 E_6 \in \mathfrak{h}_{28}$ is geodesic if and only if for $d = 1$, $e = s_1$, $f = s_2$, $b = c = 0$ the equation (2.71) holds. This gives

$$\alpha_3 s_1 + \beta_6 s_2 + s_1 \alpha_4 s_2 = 0. \tag{2.84}$$

The element $E_2 + E_3 + (k_1 + l_1)E_5 + (k_2 + l_2)E_6 \in \mathfrak{h}_{28}$ is geodesic if and only if for $b = c = 1$, $e = k_1 + l_1$, $f = k_2 + l_2$, $d = 0$ the equation (2.71) is satisfied. Using (2.82), (2.83) we receive

$$\alpha_1 + \beta_2 l_1 + \beta_3 l_2 + \beta_4 k_1 + \beta_5 k_2 + k_1 l_2 \alpha_4 + l_1 k_2 \alpha_4 = 0. \tag{2.85}$$

The element $E_2 + E_4 + (k_1 + s_1)E_5 + (k_2 + s_2)E_6 \in \mathfrak{h}_{28}$ is geodesic if and only if for $d = b = 1$, $e = k_1 + s_1$, $f = k_2 + s_2$, $c = 0$ the equation (2.71) is satisfied. Using (2.82), (2.84) we get

$$\beta_1 + \beta_2 s_1 + \beta_3 s_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 s_2 \alpha_4 + s_1 k_2 \alpha_4 = 0. \tag{2.86}$$

The element $E_3 + E_4 + (l_1 + s_1)E_5 + (l_2 + s_2)E_6 \in \mathfrak{h}_{28}$ is geodesic if and only if for $d = c = 1$, $e = l_1 + s_1$, $f = l_2 + s_2$, $b = 0$ the equation (2.71) is satisfied. Using (2.83), (2.84) we obtain

$$\alpha_2 + \beta_4 s_1 + \beta_5 s_2 + \alpha_3 l_1 + \beta_6 l_2 + l_1 s_2 \alpha_4 + s_1 l_2 \alpha_4 = 0. \tag{2.87}$$

The element $E_2 + E_3 + E_4 + (k_1 + l_1 + s_1)E_5 + (k_2 + l_2 + s_2)E_6 \in \mathfrak{h}_{28}$ is geodesic if and only if for $b = d = c = 1$, $e = k_1 + l_1 + s_1$, $f = k_2 + l_2 + s_2$, $b = 0$ the equation (2.71) is satisfied. Using (2.85), (2.86), and (2.87) we receive

$$\beta_2 k_1 + \beta_3 k_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 s_1 + \beta_6 s_2 + k_1 s_2 + k_1 k_2 \alpha_4 + l_1 l_2 \alpha_4 + s_1 s_2 \alpha_4 = 0. \tag{2.88}$$

This gives the case (2e).

Now we deal with the subalgebra $\mathfrak{h}_{29}$. Taking into account (1) we get $l_1 = 0$. The element $E_2 + k_1 E_4 + k_2 E_6 \in \mathfrak{h}_{29}$ is geodesic if and only if for $b = 1$, $d = k_1$, $f = k_2$, $c = e = 0$ the equation (2.71) holds. This gives the equation

$$\beta_1 k_1 + \beta_3 k_2 + k_1 \beta_6 k_2 = 0. \tag{2.89}$$

The element $E_3 + l_1 E_4 + l_2 E_6 \in \mathfrak{h}_{29}$ is geodesic precisely if for $c = 1$, $d = l_1$, $f = l_2$, $b = e = 0$ the equation (2.71) holds. Hence we obtain the equation

$$\alpha_2 l_1 + \beta_5 l_2 + l_1 \beta_6 l_2 = 0. \tag{2.90}$$

The element $E_2 + E_3 + (k_1 + l_1)E_4 + (k_2 + l_2)E_6 \in \mathfrak{h}_{29}$ is geodesic if and only if for $b = c = 1$, $d = k_1 + l_1$, $f = k_2 + l_2$, $e = 0$ the equation (2.71) is valid. Using (2.89) and (2.90) we receive the equation

$$\alpha_1 + \beta_1 l_1 + \beta_3 l_2 + \alpha_2 k_1 + \beta_5 k_2 + k_1 l_2 \beta_6 + l_1 k_2 \beta_6 = 0. \tag{2.91}$$

The element $E_2 + k_1 E_4 + k_2 E_6 + E_5 \in \mathfrak{h}_{29}$ is geodesic precisely if for $b = e = 1$, $d = k_1$, $f = k_2$, $c = 0$ the equation (2.71) holds. Using (2.89) one gets the equation

$$\beta_2 + \alpha_3 k_1 + \alpha_4 k_2 = 0. \tag{2.92}$$

The element $E_3 + l_1 E_4 + l_2 E_6 + E_5 \in \mathfrak{h}_{29}$ is geodesic precisely if for $c = e = 1$, $d = l_1$, $f = l_2$, $b = 0$ the equation (2.71) is satisfied. Applying (2.90) one obtains the equation

$$\beta_4 + \alpha_3 l_1 + \alpha_4 l_2 = 0. \tag{2.93}$$

The equations (2.89), (2.90), (2.91), (2.92), (2.93) yield the case (2f).

Now we consider the subalgebra $\mathfrak{h}_{30}$. Taking into account (1)–(3) we have $l_1 = \beta_6 = 0$, $s_1 = -\frac{\alpha_3}{\alpha_4}$. The element $E_2 + k_1 E_3 + k_2 E_6 \in \mathfrak{h}_{30}$ is geodesic if and only if for $b = 1$, $c = k_1$, $f = k_2$, $d = e = 0$ the equation (2.71) is satisfied. This gives the equation

$$\alpha_1 k_1 + \beta_3 k_2 + k_1 \beta_5 k_2 = 0. \tag{2.94}$$

The element $E_2 + k_1 E_3 + k_2 E_6 + E_5 \in \mathfrak{h}_{30}$ is geodesic if and only if for $b = e = 1, c = k_1, f = k_2, d = 0$ the equation (2.71) is valid. Using equations (2.71) and (2.94) we obtain

$$\beta_2 + k_1 \beta_4 + \alpha_4 k_2 = 0. \tag{2.95}$$

Since $\alpha_4 \neq 0$ from equation (2.95) we receive

$$k_2 = -\frac{\beta_2 + k_1 \beta_4}{\alpha_4}. \tag{2.96}$$

The element $E_2 + E_4 + k_1 E_3 + (k_2 - \frac{\alpha_3}{\alpha_4}) E_6 \in \mathfrak{h}_{30}$ is geodesic precisely if for $b = d = 1, c = k_1, f = k_2 - \frac{\alpha_3}{\alpha_4}, e = 0$ the equation (2.71) holds. Taking into account (2.94) from equation (2.71) we get

$$\beta_1 - \frac{\alpha_3}{\alpha_4}\beta_3 + k_1(\alpha_2 - \beta_5 \frac{\alpha_3}{\alpha_4}). \tag{2.97}$$

If $\beta_5 = \frac{\alpha_2 \alpha_4}{\alpha_3}$, then from (2.97) we obtain $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$ and from (2.94) we get the second order equation (2.66). This proves case 2(g)i. If $\beta_5 \neq \frac{\alpha_2 \alpha_4}{\alpha_3}$, then from (2.97) we have $k_1 = \frac{\beta_3 \alpha_3 - \beta_1 \alpha_4}{\alpha_2 \alpha_4 - \alpha_3 \beta_5}$. Putting this into (2.96) we obtain $k_2 = -\frac{\beta_2(\alpha_2\alpha_4 - \alpha_3\beta_5) + \beta_4(\beta_3\alpha_3 - \beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4 - \alpha_3\beta_5)}$. Substituting the expression of k_1 and k_2 into (2.94) we receive equation (2.67). Hence case 2(g)ii is shown.

The vector $E_4 + l_1 E_5 + l_2 E_6 \in \mathfrak{h}_3 \cap \mathfrak{h}_{15}$ coincides with the filiform metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$. It is geodesic if the equation

$$\alpha_3 l_1 + \beta_6 l_2 + l_1 \alpha_4 l_2 = 0 \tag{2.98}$$

holds.

Now, we deal with the subalgebra $\mathfrak{h}_3$. The element $E_2 + k_1E_3 + k_2E_5 + k_3E_6 \in \mathfrak{h}_3$ is geodesic if and only if for $d = 0, b = 1, c = k_1, e = k_2, f = k_3$ the equation (2.71) is valid. Hence we get

$$\alpha_1k_1 + \beta_2k_2 + \beta_3k_3 + k_1\beta_4k_2 + k_1\beta_5k_3 + k_2\alpha_4k_3 = 0. \tag{2.99}$$

In addition, the vector $E_2 + E_4 + k_1E_3 + (k_2 + l_1)E_5 + (k_3 + l_2)E_6 \in \mathfrak{h}_3$ is geodesic precisely if for $b = d = 1, c = k_1, e = k_2 + l_1, f = k_3 + l_2$ the system (2.71) is satisfied. Therefore we obtain

$$\begin{aligned}\alpha_1k_1 &+ \beta_1 + \beta_2k_2 + \beta_2l_1 + \beta_3k_3 + \beta_3l_2 + k_1\alpha_2\\ &+ k_1\beta_4k_2 + k_1\beta_4l_1 + k_1\beta_5k_3 + k_1\beta_5l_2 +\\ &\alpha_3k_2 + \alpha_3l_1 + \beta_6k_3 + \beta_6l_2 + k_2k_3\alpha_4\\ &+ k_2l_2\alpha_4 + l_1k_3\alpha_4 + l_1l_2\alpha_4 = 0.\end{aligned} \tag{2.100}$$

Due to the equations (2.98) and (2.99), equation (2.100) becomes

$$\begin{aligned}\beta_1 + \beta_2l_1 + \beta_3l_2 + \alpha_2k_1 + k_1\beta_4l_1 + k_1\beta_5l_2\\ + \alpha_3k_2 + \beta_6k_3 + \alpha_4k_2l_2 + \alpha_4l_1k_3 = 0.\end{aligned}$$

This proves case (3a).

Next we treat the subalgebra $\mathfrak{h}_4$. The element $E_2 + k_1E_3 + k_2E_4 + k_3E_5 \in \mathfrak{h}_4$ is geodesic precisely if for $f = 0, b = 1, c = k_1, d = k_2, e = k_3$ the equation (2.71) is satisfied. From this we get

$$\alpha_1k_1 + \beta_1k_2 + \beta_2k_3 + k_1\alpha_2k_2 + k_1\beta_4k_3 + k_2\alpha_3k_3 = 0. \tag{2.101}$$

Additionally, the vector $E_2 + k_1E_3 + k_2E_4 + k_3E_5 + E_6 \in \mathfrak{h}_4$ is geodesic if and only if for $b = f = 1, c = k_1, d = k_2, e = k_3$, the equation (2.71) holds. Hence one obtains

$$\alpha_1k_1+\beta_1k_2+\beta_2k_3+\beta_3+k_1\alpha_2k_2+k_1\beta_4k_3+k_1\beta_5+k_2\alpha_3k_3+k_2\beta_6+k_3\alpha_4 = 0. \tag{2.102}$$

Applying (2.101), equation (2.102) reduces to

$$\beta_3 + k_1\beta_5 + k_2\beta_6 + \alpha_4k_3 = 0. \tag{2.103}$$

From (2.103) we obtain $k_3 = -\frac{\beta_3+k_1\beta_5+k_2\beta_6}{\alpha_4}$. Putting this expression into (2.101) we obtain equation (2.68). Hence the case (3b) is shown.

Below we consider the subalgebra $\mathfrak{h}_5$. The vector $E_3 + k_1E_4 + k_2E_6 \in \mathfrak{h}_5$ is geodesic precisely if for $b = e = 0, c = 1, d = k_1, f = k_2$ the equation (2.71) holds. This gives

$$\alpha_2k_1 + \beta_5k_2 + k_1\beta_6k_2 = 0. \tag{2.104}$$

Additionally, the element $E_3 + k_1E_4 + k_2E_6 + E_5 \in \mathfrak{h}_5$ is geodesic if and only if for $b = 0$, $c = e = 1, d = k_1, f = k_2$ the equation (2.71) is satisfied. From this it follows that

$$\alpha_2k_1 + \beta_4 + \beta_5k_2 + k_1\alpha_3 + k_1\beta_6k_2 + \alpha_4k_2 = 0. \tag{2.105}$$

Comparing with equation (2.104), equation (2.105) reduces to

$$\beta_4 + \alpha_3k_1 + \alpha_4k_2 = 0. \tag{2.106}$$

From (2.106) one has $k_2 = -\frac{\beta_4+\alpha_3k_1}{\alpha_4}$. Substituting this expression into (2.104) we have the second order equation (2.69) for k_1. This gives the case (3c).

Now we consider the subalgebra $\mathfrak{h}_6$. The element $E_3 + k_1E_4 + k_2E_5 \in \mathfrak{h}_6$ is geodesic precisely if for $b = f = 0, c = 1, d = k_1, e = k_2$ the system (2.33) of equation is satisfied. Hence we receive

$$\alpha_2k_1 + \beta_4k_2 + k_1\alpha_3k_2 = 0. \tag{2.107}$$

Moreover, the vector $E_3 + k_1E_4 + k_2E_5 + E_6 \in \mathfrak{h}_6$ is geodesic if and only if for $b = 0$, $c = f = 1, d = k_1, e = k_2$ the equation (2.71) holds. This gives

$$\alpha_2k_1 + \beta_4k_2 + \beta_5 + k_1\alpha_3k_2 + k_1\beta_6 + \alpha_4k_2 = 0. \tag{2.108}$$

Using equation (2.107) equation (2.108) reduces to

$$\beta_5 + \beta_6k_1 + \alpha_4k_2 = 0. \tag{2.109}$$

From (2.109) we obtain $k_2 = -\frac{\beta_5+\beta_6k_1}{\alpha_4}$. Putting this expression into (2.107) we have the second order equation (2.69) for k_1. Thus, the case (3d) is proved.

The proof of subalgebras $\mathfrak{h}_7$ and $\mathfrak{h}_8$ is coincides with the proof in Theorem 2.30 of metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$. Hence the cases (3e) and (3f) are valid.

In what follows we deal with the subalgebra $\mathfrak{h}_{14}$. The element $E_2 + k_1 E_3 + k_2 E_4 + k_3 E_6 \in \mathfrak{h}_{14}$ is geodesic precisely if for $a = e = 0, b = 1, c = k_1, d = k_2, f = k_3$ the equation (2.71) is satisfied. This yields

$$\alpha_1 k_1 + \beta_1 k_2 + \beta_3 k_3 + k_1 \alpha_2 k_2 + k_1 \beta_5 k_3 + k_2 \beta_6 k_3 = 0. \tag{2.110}$$

Furthermore, the vector $E_2 + k_1 E_3 + k_2 E_4 + k_3 E_6 + E_5 \in \mathfrak{h}_{14}$ is geodesic if and only if for $b = e = 1, c = k_1, d = k_2, f = k_3$ the equation (2.71) holds. This gives

$$\alpha_1 k_1 + \beta_1 k_2 + \beta_2 + \beta_3 k_3 + k_1 \alpha_2 k_2 + k_1 \beta_4 + k_1 \beta_5 k_3 + k_2 \alpha_3 + k_2 \beta_6 k_3 + \alpha_4 k_3 = 0. \tag{2.111}$$

Taking into account (2.110), equation (2.111) can be written as follows

$$\beta_2 + k_1 \beta_4 + k_2 \alpha_3 + \alpha_4 k_3 = 0. \tag{2.112}$$

From equation (2.112) we obtain that $k_3 = -\frac{\beta_2 + k_1 \beta_4 + k_2 \alpha_3}{\alpha_4}$. Putting this expression into equation (2.110) we receive the equation (2.68). This proves the case (3g).

Here we consider the case $\mathfrak{h}_{15}$. The element $E_3 + k_1 E_5 + k_2 E_6 \in \mathfrak{h}_{15}$ is geodesic if and only if for $b = d = 0, c = 1, e = k_1, f = k_2$ the equation (2.71) is satisfied. From this we obtain

$$\beta_4 k_1 + \beta_5 k_2 + k_1 \alpha_4 k_2 = 0. \tag{2.113}$$

Moreover, the vector $E_3 + E_4 + (k_1 + l_1) E_5 + (k_2 + l_2) E_6 \in \mathfrak{h}_{15}$ is geodesic precisely if for $b = 0, c = 1, d = 1, e = k_1 + l_1, f = k_2 + l_2$ the equation (2.71) is valid. This gives

$$\begin{aligned} \alpha_2 + \beta_4 k_1 + \beta_4 l_1 + \beta_5 k_2 + \beta_5 l_2 + \alpha_3 k_1 + \alpha_3 l_1 + \beta_6 k_2 + \beta_6 l_2 \\ + k_1 k_2 \alpha_4 + k_1 l_2 \alpha_4 + l_1 k_2 \alpha_4 + l_1 l_2 \alpha_4 = 0. \end{aligned} \tag{2.114}$$

Exploiting equations (2.98) and (2.113), equation (2.114) reduces to

$$\alpha_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 l_2 \alpha_4 + l_1 k_2 \alpha_4 = 0.$$

This proves the case (3h).

Finally, we consider the subalgebra $\mathfrak{h}_{20}$. The vector $E_2 + k_1 E_4 + k_2 E_5 + k_3 E_6 \in \mathfrak{h}_{20}$ is geodesic precisely if for $c = 0, b = 1, d = k_1, e = k_2, f = k_3$ the equation (2.71) is valid. Hence we get

$$\beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + k_1 \alpha_3 k_2 + k_1 \beta_6 k_3 + k_2 \alpha_4 k_3 = 0. \tag{2.115}$$

The element $E_3 + l_1 E_4 + l_2 E_5 + l_3 E_6 \in \mathfrak{h}_{20}$ is geodesic if and only if for $b = 0$, $c = 1, d = l_1, e = l_2, f = l_3$ the equation (2.71) holds. From this we receive

$$\alpha_2 l_1 + \beta_4 l_2 + \beta_5 l_3 + l_1 \alpha_3 l_2 + l_1 \beta_6 l_3 + l_2 \alpha_4 l_3 = 0. \tag{2.116}$$

Furthermore, the vector $E_2 + E_3 + (k_1 + l_1) E_4 + (k_2 + l_2) E_5 + (k_3 + l_3) E_6 \in \mathfrak{h}_{20}$ is geodesic precisely if for $b = c = 1$, $d = k_1 + l_1, e = k_2 + l_2, f = k_3 + l_3$ the equation (2.71) is satisfied. Therefore one obtains

$$\begin{aligned} &\alpha_1 + \beta_1 k_1 + \beta_1 l_1 + \beta_2 k_2 + \beta_2 l_2 + \beta_3 k_3 + \beta_3 l_3 + \alpha_2 k_1 \\ &\quad + \alpha_2 l_1 + \beta_4 k_2 + \beta_4 l_2 + \beta_5 k_3 + \beta_5 l_3 + k_1 k_2 \alpha_3 + k_1 l_2 \alpha_3 \\ &\quad + l_1 k_2 \alpha_3 + l_1 l_2 \alpha_3 + k_1 k_3 \beta_6 + k_1 l_3 \beta_6 + l_1 k_3 \beta_6 \\ &\quad + l_1 l_3 \beta_6 + k_2 k_3 \alpha_4 + k_2 l_3 \alpha_4 + l_2 k_3 \alpha_4 + l_2 l_3 \alpha_4 = 0. \end{aligned} \tag{2.117}$$

Taking into account (2.115) and (2.116), equation (2.117) reduces to

$$\begin{aligned} \alpha_1 + \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \alpha_2 k_1 + \beta_4 k_2 + \beta_5 k_3 + k_1 l_2 \alpha_3 + l_1 k_2 \alpha_3 \\ + k_1 l_3 \beta_6 + l_1 k_3 \beta_6 + k_2 l_3 \alpha_4 + l_2 l_3 \alpha_4 = 0. \end{aligned} \tag{2.118}$$

This gives (3i). Hence Theorem 2.36 is proved. □

Chapter 3

Summary

The main contribution of this dissertation is some investigations on stretch Finsler metrics and six-dimensional filiform nilmanifolds. The results of the dissertation have been published in papers [1, 2, 3, 21]. The dissertation contains two chapters. The first section of each chapter summarizes the concepts, tools and methods necessary to understand the later sections (see Sections 1.1, 2.1).

Below we present a brief survey of our results as follows:

3.1 Stretch Finsler metrics

In this chapter of the dissertation, we investigate some properties of stretch and Douglas curvature.

3.1.1 $\widetilde{\mathbf{B}}$-, and H-stretch metric

In this section, we deal with two classes of stretch metrics. By the covariant horizontal differentiation along Finslerian geodesics of the Berwald curvature **B** (mean Berwald curvature **E**), respectively. Z. Shen (H. Akbar-Zadeh) proposed a non-Riemannian quantity $\widetilde{\mathbf{B}}$-curvature (**H**-curvature), respectively (see [41, page 139]) and [4].
Extending this concepts,we can introduce the following

Definition 3.1. *For a vector* $y \in \mathcal{T}_pM$, *we define* $\boldsymbol{K}_y : T_pM \times T_pM \times T_pM \times T_pM \to T_pM$ *by*

$$\boldsymbol{K}_y(u, v, w, z) := \mathcal{K}^i_{jklm}(y)u^j v^k w^l z^m \frac{\partial}{\partial x^i} |_x,$$

where

$$\mathcal{K}^i_{jklm} := 2\left(\widetilde{B}^i_{jkl|m} - \widetilde{B}^i_{jkm|l}\right).$$

Definition 3.2. *For a vector* $y \in \mathcal{T}_pM$*, we define* $\boldsymbol{\kappa}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$*, by*

$$\boldsymbol{\kappa}_y(u, v, w) := \kappa_{jkl}(y)u^j v^k w^l,$$

where

$$\kappa_{jkl} := 2\left(H_{jk|l} - H_{jl|k}\right).$$

A Finsler metric F is said to be $\widetilde{\boldsymbol{B}}$*-stretch metric* ($\boldsymbol{H}$*-stretch metric*) if and only if $\boldsymbol{K} = 0$ ($\boldsymbol{\kappa} = 0$), respectively.

Definition 3.3. *Let* (M, F) *be a Finsler manifold. The Finsler metric* F *is called a relatively isotropic stretch* $\widetilde{\boldsymbol{B}}$*-curvature if its stretch* $\widetilde{\boldsymbol{B}}$*-curvature is given by*

$$\mathcal{K}^i_{jklm} := \lambda F\left(B^i_{jkl|m} - B^i_{jkm|l}\right),$$

where $\lambda := \lambda(x, y)$ *is scalar function on* TM*. In this case,* (M, F) *is called a relatively isotropic* $\widetilde{\boldsymbol{B}}$*-stretch manifold. If* $\lambda > 0\,(\lambda < 0, \lambda =$ *constant), then* F *is said to be negative (positive or constant) relatively isotropic stretch* $\widetilde{\boldsymbol{B}}$*-curvature (respectively). Furthermore, if the stretch* $\boldsymbol{H}$*-curvature is given by*

$$\kappa_{jkl} := \lambda F\left(E_{jk|l} - E_{jl|k}\right),$$

then F *is said to be negative (positive, constant) relatively isotropic stretch* $\boldsymbol{H}$*-curvature if we have* $\lambda > 0\,(\lambda < 0, \lambda =$ *constant) (respectively).*

The following results are based on the paper [2].

Theorem 3.4. *Suppose that* F *is a positively complete* $\widetilde{\boldsymbol{B}}$*-stretch metric with bounded Berwald curvature. Then* F *must be a* $\widetilde{\boldsymbol{B}}$*-metric and the Berwald curvature is constant along any geodesic.*

Theorem 3.5. *Every complete* $\boldsymbol{H}$*-stretch metric with bounded mean Berwald curvature is a* $\boldsymbol{H}$*-metric.*

Theorem 3.6. *A compact Finsler manifold with negative (positive) relatively isotropic stretch* $\widetilde{\boldsymbol{B}}$*-curvature is* $\widetilde{\boldsymbol{B}}$*-Finsler manifold. More precisely, a complete Finsler manifold with constant relatively isotropic stretch* $\widetilde{\boldsymbol{B}}$*-curvature and bounded* $\widetilde{\boldsymbol{B}}$*-curvature is a* $\widetilde{\boldsymbol{B}}$*-metric.*

Theorem 3.7. *Every compact Finsler manifold with positive (negative) relatively isotropic stretch **H**-curvature is a **H**-Finsler manifold. More precisely, a complete Finsler metric with constant relatively isotropic stretch **H**-curvature and bounded **H**-curvature is a **H**-metric.*

3.1.2 Generalized Douglas curvature

Douglas' curvature is an invariant tensor by a projective change $\phi : F \to \bar{F}$ established in 1997 by S. Bácsó and M. Matsumoto (see [6]). In this section, we study a class of Finsler metrics that includes the class of Douglas metrics. We introduce the following:

Definition 3.8. *We call the Finsler metric F is a generalized Douglas metric if and only if the quantity $\widetilde{\boldsymbol{B}}$-curvature in* (1.4) *has the form*

$$\widetilde{B}^i_{jkl} := B^i_{jkl} + \omega_{jk}\delta^i_l + \omega_{jl}\delta^i_k + \omega_{kl}\delta^i_j + E_{jk;l}y^i,$$

where ω is a smooth map $M \to \wedge^2 \mathcal{T}_p M$ given by $\omega(p) := \omega_{ij}(p) dx^i \wedge dx^j$ at any point $p \in M$.

The following results are introduced in [3].

Theorem 3.9. *Let (M, F) be a generalized Douglas Finsler manifold. Suppose that F is a $\widetilde{\boldsymbol{B}}$-stretch metric. Then F is a Douglas metric.*

Theorem 3.10. *Let (M, F) be a Douglas Finsler manifold with $n \geq 3$. Then every **H**-stretch metric is a $\widetilde{\boldsymbol{B}}$-metric.*

Theorem 3.11. *Let (M, F) be a Douglas Finsler manifold. Then, the stretch $\widetilde{\boldsymbol{B}}$-curvature of F is given by*

$$\mathcal{K}^i_{jklm} := \frac{2}{n+1}\left\{\kappa_{jlm}h^i_k + \kappa_{klm}h^i_j\right\}.$$

3.2 6-dimensional filiform nilmanifolds and the corresponding metric Lie algebras

In this chapter of the dissertation we determine the isometry equivalence classes of nilmanifolds on 6-dimensional filiform Lie groups. The representatives of these classes are 6-dimensional connected simply connected filiform Lie groups N equipped with a left invariant metric $\langle .,. \rangle_N$. We study the isometry groups of the obtained filiform Lie groups N having left invariant metric $\langle .,. \rangle_N$. Moreover, we investigate the geodesic vectors and the flat totally geodesic subalgebras of the metric Lie algebras corresponding to the received filiform Lie groups N with left invariant metric $\langle .,. \rangle_N$.

3.2.1 Isometry equivalence classes and isometry group of 6-dimensional filiform nilmanifolds

Our aim in this section is to classify the isometrically isomorphic equivalence classes of 6-dimensional filiform metric Lie algebras $(\mathfrak{n}, \langle .,. \rangle)$ and to determine the group $\mathcal{OA}(\mathfrak{n})$ of all orthogonal automorphisms of the representatives of the obtained classes. The semi-direct product $N \rtimes \mathcal{OA}(\mathfrak{n})$ of the connected simply connected Lie group N of $\mathfrak{n}$ and the group $\mathcal{OA}(\mathfrak{n})$ gives the isometry group of the corresponding connected simply connected Lie group N with left invariant metric $\langle .,. \rangle_N$ induced by the Euclidean inner product $\langle .,. \rangle$ of $\mathfrak{n}$. To proceed this classification we use the list of W. de Graaf in [16] to fix a basis $\{G_1, G_2, \cdots, G_6\}$ of the non-isomorphic 6-dimensional filiform Lie algebras $\mathfrak{l}_{6,k}, k = 14, \ldots, 18$ and apply the classification procedure given in [22], pp. 371-372. This procedure describes the representative of each isometrically isomorphic equivalence classes of the 6-dimensional filiform metric Lie algebras $(\mathfrak{l}, \langle .,. \rangle)$ as a filiform Lie algebra $\mathfrak{n}$ isomorphic to $\mathfrak{l}$ such that the non-trivial Lie brackets of $\mathfrak{n}$ are defined on the Euclidean vector space $\mathbb{E}^6$ with a distinguished orthonormal basis $\{E_1, E_2, \cdots, E_6\}$ as follows:

Theorem 3.12. *A metric Lie algebra $(\mathfrak{l}_{6,14}, \langle .,. \rangle)$ is isometrically isomorphic to a unique metric Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}$,*

$j = 1, \ldots, 7$, *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators*

$$[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, \qquad [E_2, E_3] = \alpha_3 E_5 + \beta_6 E_6,$$
$$[E_1, E_3] = \alpha_2 E_4 - \Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\Big) E_5 + \beta_4 E_6, \qquad [E_2, E_4] = \beta_7 E_6,$$
$$[E_2, E_5] = \alpha_4 E_6,$$
$$[E_1, E_4] = \frac{\alpha_1\alpha_5}{\alpha_4} E_5 + \beta_5 E_6, \qquad [E_4, E_3] = \alpha_5 E_6$$

such that if the set $J = \{j \in \{1, 4, 7\} : \beta_j \neq 0\} \neq \emptyset$, *then* $\beta_{j_\circ} > 0$ *for the minimal element* $j_\circ \in J$.

Theorem 3.13. *A metric Lie algebra* $(\mathfrak{l}_{6,15}, \langle ., . \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$, $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 7$, *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators*

$$[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, \qquad [E_2, E_3] = \frac{\alpha_2\alpha_5}{\alpha_4} E_5 + \beta_7 E_6,$$
$$[E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6, \qquad [E_2, E_4] = \alpha_5 E_6.$$
$$[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6,$$
$$[E_1, E_5] = \alpha_4 E_6,$$

such that if the set $J = \{j \in \{1, 3, 4, 6, 7\} : \beta_j \neq 0\} \neq \emptyset$, *then* $\beta_{j_\circ} > 0$ *for the minimal element* $j_\circ \in J$.

Theorem 3.14. *A metric Lie algebra* $(\mathfrak{l}_{6,16}, \langle ., . \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$, $\alpha_i > 0$, $i = 1, \ldots, 4$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 8$, *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators*

$$[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, \qquad [E_2, E_3] = \beta_7 E_6,$$
$$[E_1, E_3] = \alpha_2 E_4 - \Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) E_5 + \beta_4 E_6, \qquad [E_2, E_4] = \beta_8 E_6,$$
$$[E_2, E_5] = \alpha_4 E_6,$$
$$[E_1, E_4] = \alpha_3 E_5 + \beta_5 E_6, \qquad [E_4, E_3] = \frac{\alpha_3\alpha_4}{\alpha_1} E_6.$$
$$[E_1, E_5] = \beta_6 E_6,$$

such that one of the following cases is satisfied:

1. $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_8 = 0$,

2. $\beta_3 > 0$ *or* $\beta_5 > 0$, $\beta_1 = \beta_4 = \beta_6 = \beta_8 = 0$,

3. $\beta_6 > 0$ *or* $\beta_4 > 0$, $\beta_1 = \beta_3 = \beta_5 = \beta_8 = 0$,

4. $\beta_1 > 0$ *or* $\beta_8 > 0$, $\beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$,

5. *at least two elements of the set* $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_8\}$ *are positive with the exceptions* $(\beta_1 > 0, \beta_8 > 0)$, $(\beta_3 > 0, \beta_5 > 0)$, $(\beta_4 > 0, \beta_6 > 0)$.

Theorem 3.15. *A metric Lie algebra* $(\mathfrak{l}_{6,17}, \langle ., . \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$, $\alpha_i > 0, i = 1, \dots, 5$, $\beta_j \in \mathbb{R}$, $j = 1, \dots, 6$, *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators*

$$\begin{aligned}
&[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6,\\
&[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, && [E_1, E_5] = \alpha_4 E_6,\\
&[E_2, E_3] = \alpha_5 E_6, &&
\end{aligned} \tag{3.1}$$

such that if the set $J = \{j \in \{1, 3, 4, 6\} : \beta_j \neq 0\} \neq \emptyset$, *then* $\beta_{j_\circ} > 0$ *for the minimal element* $j_\circ \in J$. *Moreover, a metric Lie algebra* $(\mathfrak{l}_{6,18}, \langle ., . \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *defined on* $\mathbb{E}^6$ *by the non-vanishing commutators (3.1) such that the constant* α_5 *is missing.*

The groups of all isometries of the corresponding connected and simply connected filiform nilmanifolds are given in Corollaries 2.11, 2.14, 2.17, 2.20, 2.21.

3.2.2 Geodesic vectors and flat totally geodesic subalgebras of 6-dimensional filiform metric Lie algebras

This section is devoted to study the totally geodesic subgroups of connected simply connected 6-dimensional filiform Riemannian nilmanifolds $(N, \langle ., . \rangle_N)$. A subgroup H of $(N, \langle ., . \rangle_N)$ is totally geodesic if and only if the corresponding subalgebra $\mathfrak{h}$ is totally geodesic. The left cosets xH, $x \in N$, give a totally geodesic foliation on N. A subalgebra $\mathfrak{h}$ of $(\mathfrak{n}, \langle ., . \rangle)$ is totally geodesic if for all $X, Y \in \mathfrak{h}$ one has $\nabla_X Y \in \mathfrak{h}$, where ∇ denotes the Levi-Civita connection of $(\mathfrak{n}, \langle ., . \rangle)$. Taking the representatives of the isometrically isomorphic equivalence classes of the 6-dimensional filiform metric Lie algebras given in Theorems 3.12, 3.13, 3.14 and 3.15 first we determine the sets of the geodesic vectors and hence the one-dimensional totally geodesic subalgebras in the six-dimensional filiform metric Lie algebras. To do this we systematically use the following claim: a non-zero vector $Y \in (\mathfrak{n}, \langle ., . \rangle)$ is geodesic if and only if for all $X \in (\mathfrak{n}, \langle ., . \rangle)$ we have $\langle [X, Y], Y \rangle = 0$. According to Lemma 2.6, any non-zero element of the union $C_0 = \langle E_1, E_2 \rangle \cup \langle E_3 \rangle \cup \langle E_4 \rangle \cup \langle E_5 \rangle \cup \langle E_6 \rangle$ is geodesic.

Theorem 3.16. *Let the sets C_i, $i = 1, \cdots, 9$, be defined as follows:*

$$C_1 := \{bE_2 + cE_3 + dE_4 : b(\alpha_1 c + \beta_1 d) + c\alpha_2 d = 0 : b, c, d \in \mathbb{R}\},$$

such that at least two of the numbers b, c, d are non-zero

with exception of the cases:

1. $b = 0$,
2. $d = 0$,
3. $c = 0$ *with* $\beta_1 \neq 0$,

$$\begin{aligned} C_2 := \Big\{ & a(E_1 - \frac{\alpha_2}{\alpha_5} E_6) + cE_3 + dE_4 + eE_5 : a \neq 0, a, c, d, e \in \mathbb{R}, \\ & \alpha_5 \beta_1 + \alpha_2 \beta_7 = 0 = \beta_4, \\ & e = (\beta_5 a - \alpha_5 c) \frac{\alpha_2 \alpha_4}{\alpha_1 \alpha_5^2}, \\ & a(\alpha_1 c + \beta_1 d + \beta_2 e - \frac{\alpha_2 \beta_3}{\alpha_5}) - c(\alpha_3 e - \frac{\alpha_2 \beta_6}{\alpha_5} a) + \frac{a\alpha_2}{\alpha_5}(\beta_7 d + \alpha_4 e) = 0 \Big\}, \end{aligned}$$

$$\begin{aligned} C_3 := \Big\{ & a\Big(E_1 - \frac{\alpha_2}{\alpha_5} E_6 + (\frac{\beta_5}{\alpha_5} + \frac{\alpha_1 \beta_4}{\alpha_5 \beta_1 + \alpha_2 \beta_7}) E_3 - \frac{\alpha_2 \alpha_4 \beta_4}{\alpha_5(\alpha_5 \beta_1 + \alpha_2 \beta_7)} E_5\Big) + \\ & dE_4 : a \neq 0,\ a, d \in \mathbb{R}, \\ & a(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c(\alpha_3 e + \beta_6 f) - f(\beta_7 d + \alpha_4 e) = 0 \Big\}, \end{aligned}$$

$$C_4 := \Big\{ aE_1 + cE_3 + dE_4 + eE_5 + fE_6 : a \neq 0,\ f \neq -a\frac{\alpha_2}{\alpha_5},\ f \neq 0,$$
$$a, c, d, e, f \in \mathbb{R}, \quad e = (c\alpha_5 - a\beta_5)\frac{f\alpha_4}{a\alpha_1\alpha_5},$$
$$d = \frac{a}{\alpha_5 f + \alpha_2 a}\Big(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}e - \beta_4 f\Big),$$
$$a\,(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) - c\,(\alpha_3 e + \beta_6 f) - f(\beta_7 d + \alpha_4 e) = 0\Big\},$$

$$C_5 := \Big\{ cE_3 + dE_4 + eE_5 + fE_6 : f \neq 0,\ c \neq 0,\ c, d, e, f \in \mathbb{R},$$
$$d = \frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big),$$
$$\frac{-c}{\alpha_5 f}\Big(\frac{\alpha_2\alpha_5}{\alpha_4}e + \beta_7 f\Big)(\alpha_3 e + \beta_6 f + c\alpha_2) + c(\beta_4 e + \beta_5 f) + e\alpha_4 f = 0\Big\},$$

$$C_6 := \Big\{ bE_2 + cE_3 + dE_4 + eE_5 : b(\alpha_1 c + \beta_1 d + \beta_2 e) + c(\alpha_2 d + \gamma e) +$$
$$d\alpha_3 e = 0,\ b, c, d, e \in \mathbb{R}\Big\},$$ *where* $\gamma = -\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big)$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$, *whereas* $\gamma = \beta_4$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ *such that at least two of the numbers* b, c, d, e *are non-zero with exception of the cases:*

1. $b = c = 0$,
2. $b = e = 0$,
3. $d = e = 0$,
4. $c = d = 0,\ \beta_2 \neq 0$,
5. $c = e = 0,\ \beta_1 \neq 0$,
6. $b = d = 0$ *with* $\alpha_3\alpha_4\beta_1 + \alpha_1\alpha_2\beta_8 \neq 0$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$,
7. $b = d = 0$ *with* $\beta_4 \neq 0$ *in the case of the metric Lie algebra* $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$,

$C_8 :=\{dE_4 + eE_5 + fE_6 : d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0,\ d, e, f \in \mathbb{R}\},$

such that at least two of the numbers d, e, f *are non-zero*

with exception of the cases:

1. $f = 0,$
2. $d = 0,$
3. $e = 0$ *with* $\beta_6 \neq 0,$

$$C_7 :=\Big\{a\Big(E_1 - \frac{\beta_6}{\alpha_4}E_2\Big) + cE_3 + dE_4 + eE_5 + fE_6 : af \neq 0,\ a, c, d, e, f \in \mathbb{R},$$

$$ae = \frac{f}{\alpha_3}\Big(\frac{a\beta_6\beta_8}{\alpha_4} + c\frac{\alpha_3\alpha_4}{\alpha_1} - a\beta_5\Big),$$

$$ac = \frac{f}{\alpha_1}(c\beta_7 + d\beta_8 + e\alpha_4) - \frac{a}{\alpha_1}(\beta_1 d + \beta_2 e + \beta_3 f),$$

$$ad = \frac{a}{\alpha_2}\Big((\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})e - \beta_4 f\Big) + \frac{f}{\alpha_2}\Big(\frac{a\beta_6\beta_7}{\alpha_4} - d\frac{\alpha_3\alpha_4}{\alpha_1}\Big),$$

$$d(\alpha_2 c + \alpha_3 e) = a\frac{\beta_6}{\alpha_4}(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f) + ce\Big(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4}\Big) -$$

$$f(c\beta_4 + d\beta_5 + e\beta_6)\Big\},$$

$C_9 :=\{bE_2 + cE_3 + dE_4 + eE_5 + fE_6 : b(\alpha_1 c + \beta_1 d + \beta_2 e + \beta_3 f)+$
$c(\alpha_2 d + \beta_4 e + \beta_5 f) + d(\alpha_3 e + \beta_6 f) + e\alpha_4 f = 0, b, c, d, e, f \in \mathbb{R}\},$

such that at least two of the numbers b, c, d, e, f *are non-zero*

with exception of the cases:

1. $b = c = d = 0,$
2. $b = c = f = 0,$
3. $d = e = f = 0,$
4. $b = c = e = 0$ *with* $\beta_6 \neq 0,$
5. $c = d = f = 0$ *with* $\beta_2 \neq 0,$
6. $c = d = e = 0$ *with* $\beta_3 \neq 0.$

1. *The geodesic vectors of the metric Lie algebra* $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$*, with* $\alpha_i > 0$*,* $\beta_j \in \mathbb{R}$*,* $i \in \{1, \ldots, 5\}$*,* $j \in \{1, \ldots 7\}$ *not belonging to* C_0 *are*

the non-zero elements of the set $C_1 \cup C_2 \cup C_4$ *in the case* $\alpha_5\beta_1 + \alpha_2\beta_7 = 0 = \beta_4$*, for* $\alpha_5\beta_1 + \alpha_2\beta_7 \neq 0$ *these are the non-zero elements of the set* $C_1 \cup C_3 \cup C_4$.

2. *The geodesic vectors in the metric Lie algebra* $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0, i = 1, \ldots, 5$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 7$, *not belonging to* C_0 *are the non-zero elements of the set* $C_1 \cup C_5$.

3. *The geodesic vectors of the metric Lie algebra* $(\mathfrak{n}_{6,16}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0$, $i = 1, \ldots, 4$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 8$, *not belonging to* C_0 *are the non-zero elements of the set* $C_6 \cup C_7$.

4. *The geodesic vectors in the metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0$, $\beta_j \in \mathbb{R}$, $i \in \{1, \ldots, 5\}$, $j \in \{1, \ldots 6\}$ *not belonging to* C_0 *are the non-zero elements of the set* $C_6 \cup C_8$.

5. *The geodesic vectors of the metric Lie algebra* $(\mathfrak{n}_{6,18}(\alpha_i, \beta_j), \langle ., . \rangle)$ *with* $\alpha_i > 0$, $\beta_j \in \mathbb{R}$, $i \in \{1, \ldots, 4\}$, $j \in \{1, \ldots 6\}$ *not belonging to* C_0 *are the non-zero elements of the set* C_9.

Hence every 6-dimensional filiform metric Lie algebra has a flat totally geodesic subalgebra of dimension at least 1. The criterion that a 6-dimensional filiform metric Lie algebra possesses a flat totally geodesic subalgebra of dimension greater than 1 gives strong restrictions on the structure constants α_i, β_j. Our investigation shows that the 6-dimensional filiform Lie algebras not belonging to the standard filiform Lie algebra possess flat totally geodesic subalgebras of dimension 1 or 2.

Theorem 3.17. *Let* $\mathcal{C}$ *be the class of the* 6*-dimensional filiform metric Lie algebras not corresponding to the standard filiform Lie algebra. The maximal dimension of the flat totally geodesic subalgebras of a metric Lie algebra in* $\mathcal{C}$ *is* 2. *A metric Lie algebra* $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ *has the subalgebra* $\mathfrak{h} = span(E_1, E_6)$ *as a flat totally geodesic subalgebra if and only if* $\beta_3 = \beta_4 = \beta_5 = 0$. *A metric Lie algebra* $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2, E_4)$ *precisely if* $\beta_1 = \beta_7 = 0$.

A metric Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ *possesses the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_3, E_6)$ *if and only if* $\beta_5 = \beta_7 = 0$.

A metric Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *has the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_1 - \frac{\beta_6}{\alpha_4}E_2, E_6)$ *precisely if* $\beta_3 = 0$, $\beta_4 = \frac{\beta_6\beta_7}{\alpha_4}$ *and* $\beta_5 = \frac{\beta_6\beta_8}{\alpha_4}$. *A metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_3, E_5)$ *if and only if* $\alpha_3\alpha_4\beta_1 + \alpha_2\alpha_1\beta_8 = 0$. *A metric Lie algebra* $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *has the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 + \frac{\beta_8\alpha_1}{\alpha_3\alpha_4}E_3 - \frac{1}{\alpha_3}(\beta_1 + \frac{\beta_8\alpha_1\alpha_2}{\alpha_3\alpha_4})E_5,\ E_4)$ *precisely if for* α_i, β_j, $i = 1,2,3,4$, $j = 1,2,8$ *the equation*

$$\frac{(\alpha_1)^2\beta_8}{\alpha_3\alpha_4} + \Big(\beta_1 + \frac{\beta_8\alpha_1\alpha_2}{\alpha_3\alpha_4}\Big)\Big(-\frac{\beta_2}{\alpha_3} + \frac{\beta_8\alpha_1}{(\alpha_3)^2\alpha_4}(\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})\Big) = 0$$

holds. A metric Lie algebra $\mathfrak{n}_{6,16}(\alpha_i, \beta_j)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 + k_1E_4 + k_2E_5,\ E_3 + l_1E_4 + l_2E_5)$ *if and only if the equations*

$$\begin{aligned}
&\beta_7 + l_1\beta_8 + l_2\alpha_4 + k_1\frac{\alpha_3\alpha_4}{\alpha_1} = 0,\\
&\beta_1k_1 + \beta_2k_2 + k_1\alpha_3k_2 = 0,\\
&\alpha_2l_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})l_2 + l_1\alpha_3l_2 = 0,\\
&\alpha_1 + \beta_1l_1 + \beta_2l_2 + \alpha_2k_1 - (\frac{\alpha_3\beta_1}{\alpha_1} + \frac{\alpha_2\beta_8}{\alpha_4})k_2 + \alpha_3(k_1l_2 + l_1k_2) = 0
\end{aligned}$$

are satisfied.

A metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$ *processes the flat totally geodesic subalgebras* $\mathfrak{h} = span(E_4 - \frac{\alpha_3}{\alpha_4}E_6,\ E_5)$ *and* $\mathfrak{h} = span\,(E_4, E_6)$ *if and only if* $\beta_6 = 0$. *A metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$ *has the flat totally geodesic subalgebras* $\mathfrak{h} = span\,(E_3, E_5)$ *and* $\mathfrak{h} = span(E_3 - \frac{\alpha_2}{\alpha_3}E_5,\ E_4)$ *precisely if* $\beta_4 = 0$. *A metric Lie algebra* $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$ *allows the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 + k_1E_3 - \frac{\beta_2+\beta_4k_1}{\alpha_3}E_4,\ E_5)$ *if and only if* k_1 *is a solution of the equation*

$$\alpha_2\beta_4k_1^2 + (\beta_1\beta_4 + \alpha_2\beta_2 - \alpha_1\alpha_3)k_1 + \beta_1\beta_2 = 0.$$

A metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle .,. \rangle)$ *possesses the flat totally geodesic subalgebra* $\mathfrak{h} = span(E_2 - \frac{\beta_1+\alpha_3k_2}{\alpha_2}E_3 + k_2E_5,\ E_4)$ *precisely if* k_2 *is a solution of* *the equation*

$$\alpha_3\beta_4 k_2^2 + (\alpha_1\alpha_3 + \beta_1\beta_4 - \alpha_2\beta_2)k_2 + \alpha_1\beta_1 = 0.$$

Now we describe the flat totally geodesic subalgebras of dimension > 1 of the metric Lie algebras belonging to the 6-dimensional standard filiform Lie algebra.

Theorem 3.18. *The maximal dimension of the flat totally geodesic subalgebras of a metric Lie algebra* $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *is* 4.

A metric Lie algebra $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *has the* 4*-dimensional subalgebras* $\mathfrak{h} = span(E_2 - \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}E_6,\ E_3,\ E_4 - \frac{\alpha_3}{\alpha_4}E_6,\ E_5)$ *and* $\mathfrak{h} = span(E_2,\ E_3 - \frac{\alpha_2}{\alpha_3}E_5,\ E_4,\ E_6)$ *as flat totally geodesic subalgebras if and only if* $\beta_1 = \beta_3 = \beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $\beta_2 = \frac{\alpha_1\alpha_3}{\alpha_2}$.

The 3*-dimensional flat totally geodesic subalgebras of* $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *are:*

$\mathfrak{h} = span(E_3 - \frac{\alpha_2}{\alpha_3}E_5,\ E_4,\ E_6)$ *and* $\mathfrak{h} = span(E_3,\ E_4 - \frac{\alpha_3}{\alpha_4}E_6,\ E_5)$ *precisely if* $\beta_4 = \beta_6 = 0$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$,

$\mathfrak{h} = span(E_2 + k_1E_3 + k_2E_6,\ E_4 + s_1E_6,\ E_5)$ *such that one of the following cases is satisfied:*

1. $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_2 = -\frac{\beta_2 + k_1\beta_4}{\alpha_4}$, $s_1 = -\frac{\alpha_3}{\alpha_4}$ *and* k_1 *is a solution of the equation*

$$\beta_4\beta_5 k_1^2 + k_1(\beta_2\beta_5 + \beta_3\beta_4 - \alpha_1\alpha_4) + \beta_2\beta_3 = 0, \tag{3.2}$$

2. $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_1 = \frac{\alpha_3\beta_3 - \alpha_4\beta_1}{\alpha_2\alpha_4 - \alpha_3\beta_5}$, $k_2 = -\frac{\beta_2(\alpha_2\alpha_4 - \alpha_3\beta_5) + \beta_4(\beta_3\alpha_3 - \beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4 - \alpha_3\beta_5)}$, $s_1 = -\frac{\alpha_3}{\alpha_4}$ *and the equation*

$$\begin{aligned}&(\alpha_2\alpha_4 - \alpha_3\beta_5)\Big((\alpha_1\alpha_4 - \beta_3\beta_4 - \beta_2\beta_5)(\beta_3\alpha_3 - \beta_1\alpha_4) - \\ &\beta_2\beta_3(\alpha_2\alpha_4 - \alpha_3\beta_5) - \beta_4\beta_5(\beta_3\alpha_3 - \beta_1\alpha_4)^2\Big) = 0\end{aligned} \tag{3.3}$$

holds,

3. $\mathfrak{h} = span(E_2 + k_1E_3 + k_2E_5,\ E_4,\ E_6)$ *such that one of the following cases is satisfied:*

(a) $\beta_1 = \frac{\alpha_3}{\alpha_4}\beta_3$, $\beta_5 = \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_2 = -\frac{\beta_3+k_1\beta_5}{\alpha_4}$ *and* k_1 *is a solution of the equation* (3.2),

(b) $\beta_5 \neq \frac{\alpha_2\alpha_4}{\alpha_3}$, $k_1 = \frac{\alpha_3\beta_3-\alpha_4\beta_1}{\alpha_2\alpha_4-\alpha_3\beta_5}$, $k_2 = \frac{\beta_3(\alpha_2\alpha_4-\alpha_3\beta_5)+\beta_5(\beta_3\alpha_3-\beta_1\alpha_4)}{\alpha_4(\alpha_2\alpha_4-\alpha_3\beta_5)}$ *and the equation* (3.3) *holds,*

4. $\mathfrak{h} =$ *span* $(E_2 + l_1E_4 + l_2E_5,\ E_3 + k_1E_4 + k_2E_5,\ E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_1 l_1 + \beta_2 l_2 + l_1\alpha_3 l_2 = 0,\\
&\alpha_2 k_1 + \beta_4 k_2 + k_1 k_2 \alpha_3 = 0,\\
&\alpha_1 + \beta_1 k_1 + \beta_2 k_2 + \alpha_2 l_1 + \beta_4 l_2 + l_1 k_2 \alpha_3 + k_1 l_2 \alpha_3 = 0,\\
&\beta_3 + \beta_6 l_1 + \alpha_4 l_2 = 0,\\
&\beta_5 + k_1\beta_6 + k_2\alpha_4 = 0
\end{aligned}$$

are satisfied,

5. $\mathfrak{h} =$ *span* $(E_2 + k_1E_4 + k_2E_6,\ E_3 + l_1E_4 + l_2E_6,\ E_5)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_1 k_1 + \beta_3 k_2 + k_1\beta_6 k_2 = 0,\\
&\alpha_2 l_1 + \beta_5 l_2 + l_1\beta_6 l_2 = 0,\\
&\alpha_1 + \beta_1 l_1 + \beta_3 l_2 + \alpha_2 k_1 + \beta_5 k_2 + k_1 l_2 \beta_6 + l_1 k_2 \beta_6 = 0,\\
&\beta_2 + \alpha_3 k_1 + \alpha_4 k_2 = 0,\\
&\beta_4 + \alpha_3 l_1 + \alpha_4 l_2 = 0
\end{aligned}$$

hold,

6. $\mathfrak{h} =$ *span*$(E_2 + k_1E_5 + k_2E_6,\ E_3 + l_1E_5 + l_2E_6,\ E_4 + s_1E_5 + s_2E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_2 k_1 + \beta_3 k_2 + k_1\alpha_4 k_2 = 0,\\
&\beta_4 l_1 + \beta_5 l_2 + l_1\alpha_4 l_2 = 0,\\
&\alpha_3 s_1 + \beta_6 s_2 + s_1\alpha_4 s_2 = 0,\\
&\alpha_1 + \beta_2 l_1 + \beta_3 l_2 + \beta_4 k_1 + \beta_5 k_2 + k_1 l_2 \alpha_4 + l_1 k_2 \alpha_4 = 0,
\end{aligned}$$

$$\beta_1 + \beta_2 s_1 + \beta_3 s_2 + \alpha_3 k_1 + \beta_6 k_2 + k_1 s_2 \alpha_4 + s_1 k_2 \alpha_4 = 0,$$
$$\alpha_2 + \beta_4 s_1 + \beta_5 s_2 + \alpha_3 l_1 + \beta_6 l_2 + l_1 s_2 \alpha_4 + s_1 l_2 \alpha_4 = 0,$$
$$\beta_2 k_1 + \beta_3 k_2 + \beta_4 l_1 + \beta_5 l_2 + \alpha_3 s_1 + \beta_6 s_2 + k_1 s_2 + k_1 k_2 \alpha_4 + l_1 l_2 \alpha_4 + s_1 s_2 \alpha_4 = 0$$

are satisfied.

The 2-dimensional flat totally geodesic subalgebras of $\mathfrak{n}_{6,18}(\alpha_i, \beta_j)$ *are:*

(a) $\mathfrak{h} = span(E_4 - \frac{\alpha_3}{\alpha_4} E_6,\ E_5)$ *and* $\mathfrak{h} = span\,(E_4,\ E_6)$ *if and only if* $\beta_6 = 0$,

(b) $\mathfrak{h} = span(E_2 + k_1 E_3 + k_2 E_4 - \frac{\beta_3 + k_1\beta_5 + k_2\beta_6}{\alpha_4} E_5,\ E_6)$ *and* $\mathfrak{h} = span(E_2 + k_1 E_3 + k_2 E_4 - \frac{\beta_2 + k_1\beta_4 + k_2\alpha_3}{\alpha_4} E_6,\ E_5)$ *if and only if the equation*

$$(\alpha_3 k_2 + \beta_2 + \beta_4 k_1)(k_1\beta_5 + k_2\beta_6 + \beta_3) - \alpha_2\alpha_4 k_1 k_2 - \alpha_1\alpha_4 k_1 - \beta_1\alpha_4 k_2 = 0$$

is satisfied,

(c) $\mathfrak{h} = span(E_3 + k_1 E_4 - \frac{\beta_4 + \alpha_3 k_1}{\alpha_4} E_6,\ E_5)$ *and* $\mathfrak{h} = span(E_3 + k_1 E_4 - \frac{\beta_5 + \beta_6 k_1}{\alpha_4} E_5,\ E_6)$, *precisely if* k_1 *is a solution of the equation*

$$\alpha_3\beta_6 k_1^2 + k_1(\alpha_3\beta_5 + \beta_4\beta_6 - \alpha_2\alpha_4) + \beta_4\beta_5 = 0,$$

(d) $\mathfrak{h} = span\,(E_2 + k_1 E_3 + k_2 E_5 + k_3 E_6,\ E_4 + l_1 E_5 + l_2 E_6)$ *if and only if the following equations*

$$\alpha_3 l_1 + \beta_6 l_2 + l_1 \alpha_4 l_1 = 0,$$
$$\alpha_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + k_1\beta_4 k_2 + k_1\beta_5 k_3 + k_2\alpha_4 k_3 = 0,$$
$$\beta_1 + \beta_2 l_1 + \beta_3 l_2 + \alpha_2 k_1 + k_1\beta_4 l_1 + k_1\beta_5 l_2 + \alpha_3 k_2 + \beta_6 k_3 + \alpha_4 k_2 l_2 + \alpha_4 l_1 k_3 = 0$$

hold,

(e) $\mathfrak{h} =$ *span*$(E_3 + k_1E_5 + k_2E_6,\ E_4 + l_1E_5 + l_2E_6)$ *if and only if the following equations*

$$\begin{aligned}
&\beta_4k_1 + \beta_5k_2 + k_1\alpha_4k_2 = 0,\\
&\alpha_3l_1 + \beta_6l_2 + l_1\alpha_4l_2 = 0,\\
&\alpha_2 + \beta_4l_1 + \beta_5l_2 + \alpha_3k_1 + \beta_6k_2 + k_1l_2\alpha_4 + l_1k_2\alpha_4 = 0
\end{aligned}$$

are satisfied,

(f) $\mathfrak{h} =$ *span*$(E_2 + k_1E_4 + k_2E_5 + k_3E_6, \quad E_3 + l_1E_4 + l_2E_5 + l_3E_6)$ *precisely if the following equations*

$$\begin{aligned}
&\beta_1k_1 + \beta_2k_2 + \beta_3k_3 + k_1\alpha_3k_2 + k_1\beta_6k_3 + k_2\alpha_4k_3 = 0,\\
&\alpha_2l_1 + \beta_4l_2 + \beta_5l_3 + l_1\alpha_3l_2 + l_1\beta_6l_3 + l_2\alpha_4l_3 = 0,\\
&\alpha_1 + \beta_1l_1 + \beta_2l_2 + \beta_3l_3 + \alpha_2k_1 + \beta_4k_2 + \beta_5k_3 + k_1l_2\alpha_3\\
&\qquad + l_1k_2\alpha_3 + k_1l_3\beta_6 + l_1k_3\beta_6 + k_2l_3\alpha_4 + l_2l_3\alpha_4 = 0
\end{aligned}$$

hold.

Bibliography

[1] S. A. Abbas and Á. Figula. Geodesic vectors and flat totally geodesic subalgebras of six-dimensional filiform metric Lie algebras. *J. Geom.*, **115**(5):1–39, 2024.

[2] S. A. Abbas and L. Kozma. On new classes of stretch Finsler metrics. *J. Finsler Geom. Appl.*, **3**(1):86–99, 2022.

[3] S. A. Abbas and L. Kozma. On generalized Douglas curvature of Finsler metrics. *Glob. J. Adv. Res. Class. Mod. Geom.*, **12**(2):229–236, 2023.

[4] H. Akbar-Zadeh. Sur les espaces de Finsler à courbures sectionnelles constantes [On Finsler spaces with constant sectional curvature]. *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, **74**(10):281–322, 1988.

[5] A. Al-Abayechi and Á. Figula. Geodesic vectors and flat totally geodesic subalgebras in nilpotent metric Lie algebras. *J. Math Sci.*, **275**:660–673, 2023.

[6] S. Bácsó and M. Matsumoto. On Finsler spaces of Douglas type—a generalization of the notion of Berwald space. *Publ. Math. Debrecen*, **51**(3-4):385–406, 1997.

[7] S. Bácsó and M. Matsumoto. Finsler spaces with the h-curvature tensor dependent on position alone. *Publ. Math. Debrecen*, **55**(1-2):199–210, 1999.

[8] L. Berwald. Über Parallelübertragung in Räumen mit allgemeiner Maßbestimmung. *Jber. Deutsch. Math.-Ver.*, **34**:213–220, 1926.

[9] L. Berwald. On Finsler and Cartan geometries. III: Two-dimensional Finsler spaces with rectilinear extremals. *Ann. of Math. (2)*, **42**:84–112, 1941.

[10] H. Busemann. *The Geometry of Geodesics.* Academic Press, Inc., New York, 1955.

[11] G. Cairns, A. Hinić Galić, and Y. A. Nikolayevsky. Totally geodesic subalgebras of filiform nilpotent Lie algebras. *J. Lie Theory*, **23**(4):1051–1074, 2013.

[12] G. Cairns, A. Hinić Galić, and Y. A. Nikolayevsky. Totally geodesic subalgebras of nilpotent Lie algebras. *J. Lie Theory*, **23**(4):1023–1049, 2013.

[13] G. Chen and L. Liu. On weakly stretch Randers metrics. *J. Math. Anal. Appl.*, **514**(1):1–11, 2022.

[14] S. Console, A. Fino, and E. Samiou. The moduli space of six-dimensional two-step nilpotent Lie algebras. *Ann. Global Anal. Geom.*, **27**(1):17–32, 2005.

[15] S. S. Chern D. Bao and Z. Shen. *An Introduction to Riemann-Finsler Geometry*. Graduate Texts in Mathematics, Springer-Verlag, 2000.

[16] W. A. de Graaf. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. *J. Algebra*, **309**(2):640–653, 2007.

[17] R. C. Decoste and L. Demeyer. Totally geodesic subalgebras in 2-step nilpotent Lie algebras. *Rocky Mountain J. Math.*, **45**(5):1425–1444, 2015.

[18] P. Eberlein. Geometry of 2-step nilpotent groups with a left invariant metric. *Ann. Sci. École Norm. Sup. (4)*, **27**(5):611–660, 1994.

[19] P. Eberlein. Geometry of 2-step nilpotent groups with a left invariant metric. II. *Trans. Amer. Math. Soc.*, **343**(2):805–828, 1994.

[20] K. Ficzere and Á. Figula. Isometry groups of six-dimensional nilmanifolds. *Aequationes Math.*, **97**(4):725–752, 2023.

[21] Á. Figula and S. A. Abbas. Isometry groups of six-dimensional filiform nilmanifolds. *Int. J. Group Theory*, **12**(2):67–80, 2023.

[22] Á. Figula and P. T. Nagy. Isometry classes of simply connected nilmanifolds. *J. Geom. Phys.*, **132**:370–381, 2018.

[23] P. Finsler. Über Kurven und Flächen in allgemeinen Räumen. *Birkhäuser Verlag, Basel*, 1951, (Dissertation, Göttingen,1918).

[24] M. Goze and Y. Khakimdjanov. *Nilpotent Lie Algebras*. Mathematics and its Applications, **361**. Kluwer Academic Publishers Group, Dordrecht, 1996.

[25] M. Goze and Y. Khakimdjanov. Nilpotent and solvable Lie algebras. In *Handbook of Algebra*, volume **2**, pages 615–663. North-Holland, , Amsterdam, 2000.

[26] S. Homolya and O. Kowalski. Simply connected two-step homogeneous nilmanifolds of dimension 5. *Note Mat.*, **26**(1):69–77, 2006.

[27] F. Kamelaei, A. Tayebi, and B. Najafi. On the strech curvature of homogeneous Finsler metrics. *Facta Univ. Ser. Math. Inform.*, **36**(5):1143–1153, 2021.

[28] A. Kaplan. On the geometry of groups of Heisenberg type. *Bull. London Math. Soc.*, **15**(1):35–42, 1983.

[29] M. M. Kerr and T. L. Payne. The geometry of filiform nilpotent Lie groups. *Rocky Mountain J. Math.*, **40**(5):1587–1610, 2010.

[30] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1963.

[31] J. Lauret. Homogeneous nilmanifolds of dimensions 3 and 4. *Geom. Dedic.*, **68**(2):145–155, 1997.

[32] M. Matsumoto. *Foundations of Finsler Geometry and Special Finsler Spaces.* Kaiseisha Press, Shigaken, 1986.

[33] M. Matsumoto. On the stretch curvature of a Finsler space and certain open problems. *J. Nat. Acad. Math. India*, **11**:22–32, 1997.

[34] M. Matsumoto. An improved proof of Numata and Shibata's theorems on Finsler spaces of scalar curvature. *Publ. Math. Debrecen*, **64**(3-4):489–500, 2004.

[35] J. Milnor. Curvatures of left invariant metrics on Lie groups. *Advances in Math.*, **21**(3):293–329, 1976.

[36] P. T. Nagy and S. Homolya. Geodesic vectors and subalgebras in two-step nilpotent metric Lie algebras. *Adv. Geom.*, **15**(1):121–126, 2015.

[37] B. Najafi and A. Tayebi. Weakly stretch Finsler metrics. *Publ. Math. Debrecen*, **91**(3-4):441–454, 2017.

[38] B. Najafi and A. Tayebi. A rigidity result on Finsler surfaces. *Balkan J. Geom. Appl.*, **23**(2):34–40, 2018.

[39] G. Randers. On an asymmetrical metric in the four-space of general relativity. *Phys. Rev. (2)*, **59**:195–199, 1941.

[40] B. Riemann. Über die Hypothesen, welche der Geometrie zu Grunde liegen. (Mitgetheilt durch R. Dedekind). *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen*, **13**:133–150, 1854.

[41] Z. Shen. *Differential Geometry of Spray and Finsler Spaces.* Kluwer Academic Publishers, Dordrecht, 2001.

[42] C. Shibata. On Finsler spaces with Kropina metric. *Rep. Mathematical Phys.*, **13**(1):117–128, 1978.

[43] A. Tayebi, M. Bahadori, and H. Sadeghi. On spherically symmetric Finsler metrics with some non-Riemannian curvature properties. *J. Geom. Phys.*, **163**:1–13, 2021.

[44] A. Tayebi, M. Faghfuri, and N. Jazer. On mean stretch curvatures of Finsler metrics. *Acta Math. Univ. Comenian. (N.S.)*, **89**(2):335–342, 2020.

[45] A. Tayebi and N. Izadian. Douglas-square metrics with vanishing mean stretch curvature. *Int. Electron. J. Geom.*, **12**(2):188–201, 2019.

[46] A. Tayebi and B. Najafi. A class of homogeneous Finsler metrics. *J. Geom. Phys.*, **140**:265–270, 2019.

[47] A. Tayebi and B. Najafi. The weakly generalized unicorns in Finsler geometry. *Sci. China Math.*, **64**(9):2065–2076, 2021.

[48] A. Tayebi and H. Sadeghi. On a class of stretch metrics in Finsler geometry. *Arab. J. Math.*, **8**(2):153–160, 2019.

[49] A. Tayebi and T. Tabatabeifar. Douglas-Randers manifolds with vanishing stretch tensor. *Publ. Math. Debrecen*, **86**(3-4):423–432, 2015.

[50] H.T. Vishkaei, M. Toomanian, R. C. Katamy, and M. Nadjafikhah. On homogeneous weakly stretch Finsler metrics. *Bull. Iranian Math. Soc.*, **48**:19–30, 2022.

[51] F. W. Warner. *Foundations of Differentiable Manifolds and Lie groups.* Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1983.

[52] E. N. Wilson. Isometry groups on homogeneous nilmanifolds. *Geom. Dedic.*, **12**(3):337–346, 1982.

[53] N. L. Youssef and S. G. Elgendi. New Finsler package. *Comput. Phys. Commun.*, **185**(3):986–997, 2014.

UNIVERSITY AND NATIONAL LIBRARY
UNIVERSITY OF DEBRECEN
H-4002 Egyetem tér 1, Debrecen
Phone: +3652/410-443, email: publikaciok@lib.unideb.hu

Registry number: DEENK/81/2024.PL
Subject: PhD Publication List

Candidate: Sameer Annon Abbas Al-Janabi
Doctoral School: Doctoral School of Mathematical and Computational Sciences
MTMT ID: 10087428

List of publications related to the dissertation

Foreign language scientific articles in international journals (4)

1. **Al-Janabi, S. A. A.**, Figula, Á.: Geodesic vectors and flat totally geodesic subalgebras of six-dimensional filiform metric Lie algebras.
   J. Geom. 115 (1), 1-39, 2023. ISSN: 0047-2468.
   DOI: http://dx.doi.org/10.1007/s00022-023-00703-4
   IF: 0.6 (2022)
2. Figula, Á., **Al-Janabi, S. A. A.**: Isometry groups of six-dimensional filiform nilmanifolds.
   Int. J. Group Theory. 12 (2), 67-80, 2023. ISSN: 2251-7650.
   DOI: http://dx.doi.org/10.22108/IJGT.2022.131891.1767
   IF: 0.2 (2022)
3. **Al-Janabi, S. A. A.**, Kozma, L.: On generalized Douglas curvature of Finsler metrics.
   Global Journal of Advanced Research on Classical and Modern Geometries. 12 (2), 229-236, 2023. ISSN: 2284-5569.
4. **Al-Janabi, S. A. A.**, Kozma, L.: On New Classes of Stretch Finsler Metrics.
   Journal of Finsler Geometry and its Applications 3 (1), 86-99, 2022. EISSN: 2783-0500.
   DOI: http://dx.doi.org/10.22098/jfga.2022.10115.1058

List of other publications

Foreign language scientific articles in international journals (4)

5. Al-khafaji, S. N., Hussain, A. H., **Al-Janabi, S. A. A.**: Third Hankel Determinant for Certain Class of Bazilević Functions Associated with Linear Differential Operato.
   Discontinuity, Nonlinearity, and Complexity. 10 (2), 323-331, 2021. ISSN: 2164-6376.
   DOI: http://dx.doi.org/10.5890/DNC.2021.06.011

UNIVERSITY of DEBRECEN

UNIVERSITY AND NATIONAL LIBRARY
UNIVERSITY OF DEBRECEN
H-4002 Egyetem tér 1, Debrecen
Phone: +3652/410-443, email: publikaciok@lib.unideb.hu

6. Salman, A. M., Mouajeeb, E. K., Al-Janabi, A. N., Hussain, A. H., **Al-Janabi, S. A. A.**: Weakly w-compact spaces via w-open sets.
J. Interdiscip. Math. 24 (3), 713-719, 2021. ISSN: 0972-0502.
DOI: http://dx.doi.org/10.1080/09720502.2020.1861784

7. Hussain, A. H., **Al-Janabi, S. A. A.**, Salman, A. M., Hussein, N. A.: Semi soft local function which generated a new topology in soft ideal spaces.
Journal of Interdisciplinary Mathematics. 22 (8), 1509-1517, 2019. ISSN: 0972-0502.
DOI: http://dx.doi.org/10.1080/09720502.2019.1706848

8. **Al-Janabi, S. A. A.**, Hussein, N. A., Salman, A. M., Hussain, A. H.: Soft-local function via soft point.
J. Interdiscip. Math. 22 (8), 1549-1558, 2019. ISSN: 0972-0502.
DOI: http://dx.doi.org/10.1080/09720502.2019.1706856

Foreign language conference proceedings (2)

9. **Al-Janabi, S. A. A.**, Hussain, A. H., Salman, A. M., Hussein, N. A.: On Soft Closure Function in Soft Ideal Spaces.
J. Phys. Conf. Ser. 1804 (1), 1-9, 2021. ISSN: 1742-6588.
DOI: http://dx.doi.org/10.1088/1742-6596/1804/1/012115

10. Al-khafaji, S. N., Al-Fayadh, A., Hussain, A. H., **Al-Janabi, S. A. A.**: Toeplitz Determinant whose Its Entries are the Coefficients for Class of Non-Bazilevi'c Functions.
J. Phys. Conf. Ser. 1660 (1), 1-7, 2020. ISSN: 1742-6588.
DOI: http://dx.doi.org/10.1088/1742-6596/1660/1/012091

Foreign language abstracts (1)

11. **Al-Janabi, S. A. A.**, Figula, Á.: Totally geodesic subalgebras of 6-dimensional nilmanifolds having nilpotency classes 3 and 4.
In: Book of Abstracts, RIGA 2023 /ed. Ion Mihai, Adela Mihai, Bucharest University Press, Bucharest, 2, 2023.

Total IF of journals (all publications): 0,8
Total IF of journals (publications related to the dissertation): 0,8

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

11 March, 2024

List of Talks

1. **DGA, 17-23 July, 2022**: Conference Differential Geometry and its Applications- Hradec Kralove, Czech Republic. Title of the talk: *On New classes of stretch Finsler metrics.*

2. **CFGA, 12-16 June, 2023**: Colloquium on Finsler Geometry and its Applications-Debrecen, Hungary. Title of the talk: *Isometry groups and totally geodesic subalgebras of* 6*-dimensional filiform nilmanifolds.*

3. **RIGA, 22-24 September, 2023**: The International Conference Riemannian Geometry and Applications-Bucharest, Romania. Title of the talk: *Totally geodesic subalgebras of* 6*-dimensional nilmanifolds having nilpotency classes* 3 *and* 4.

4. **RIGA, 12-15 January, 2021**: The International Conference Riemannian Geometry and Applications-Bucharest, Romania. *Attendance.*

5. **DGDS, 1-4 September, 2022** : The XVI-th International Conference of Differential Geometry and Dynamical Systems-Bucharest, Romania. *Attendance.*