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Systems of first order ordinary differential equations allowing a given 3-dimensional Lie group as a subgroup of their symmetry group

Kornélia Ficzere and Ágota Figula

Abstract. We determine systems of the first order ordinary differential equations such that their group of symmetries contains a three-dimensional Lie subgroup G. We represent the basis vectors of the Lie algebra $\mathfrak{g}$ of G by vector fields in the three-dimensional real space. Two cases are distinguished according to whether the infinitesimal generators of $\mathfrak{g}$ do not contain any component or contain component with respect to the independent variable of the system.

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1. Introduction

Many research are done to find Lie symmetries for existing systems in physical and biological sciences (see [14–17, 22]). Applications of symmetry groups to understand and solve differential equations go back to S. Lie's work [11]. He observed that the knowledge of an appropriate group of symmetries of a system of first order ordinary differential equations is useful to obtain its general solution. Namely, if one knows a one-parameter symmetry group, then one can find the solution by quadratures (indefinite integrals) from the solution to a system of first order ordinary differential equations with one fewer equation in it. By the knowledge of an n-dimensional solvable group of symmetries we can reduce the number of equations by n ([9], p. 100, [18], p. 154). Also many books have been dedicated to Lie's method and its generalizations (e.g., [1, 5, 7, 8, 19–21]).

Kornélia Ficzere and Ágota Figula have contributed equally to this work.

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The determination of Lie algebras of vector fields, up to local diffeomorphisms, played an important role in the research of S. Lie. He classified the Lie groups given by the Lie algebras of their infinitesimal generators which act either on the complex line or on the complex plane (cf. [12], pp. 767–773, [13], pp. 1–94, [4], p. 1164). Furthermore, he improved a method in [12, Sect. X, p. 243–248] to receive the ordinary differential equations which admit a given Lie group as a group of their symmetries. Using his classifications of Lie algebras of vector fields he demonstrated the power of this method in §1–4. of [12, Sect. X, p. 249–273], where he classified the Lie groups, according to the number of invariant first order differential equations. The possible cases are: infinitely many, precisely 2, precisely 1 or none. In particular, N. H. Ibragimov in [9, Chap. III] presented Lie's method for the second order differential equations such that the Lie algebra admitted by a given equation is defined by vector fields in the complex $\mathbb{C}^2$-plane. G. Czichowski in [2] described three different actions of the group $SL(2)$ on the real $\mathbb{R}^2$-plane, gave the second order differential equations which are invariant under these actions and discussed the connection between the problem to solve these differential equations (using the symmetries) and to determine the type for the $SL(2)$-action in the solution space.

When Lie symmetry method is applied for a given system in physics or biology the main task is to use the infinitesimal symmetry condition (see (2.76) in [18, p. 131] and Theorem 5 in [6, p. 59]) for the determination of a sufficiently large group of symmetries of the given system and with the help of this group to integrate the system. This procedure was carried out successfully for example in the Lie symmetry analysis of the Kepler problem and the Anderson's HIV model (see [14,22]). The Kepler problem, which describes the interaction of two point particles with an inverse square law of attraction, can be given by a system of three second order ordinary differential equations. The Lie algebra of its symmetry group contains the direct sum of the 2-dimensional non-abelian Lie algebra $\mathfrak{g}_2$ and the simple Lie algebra $so_3(\mathbb{R})$ as a proper subalgebra (see [21]). In [22] the Anderson's HIV model was translated into a system of first order non-linear ordinary differential equations. After making a condition on the form of the infinitesimal generators of the symmetries it turns out that the symmetry group of this system admits a solvable Lie subgroup such that its Lie algebra is isomorphic to the direct sum $\mathbb{R} \oplus \mathfrak{g}_2$.

There is an alternative approach for study Lie symmetry group. It is the classification of all ordinary differential equations allowing a given Lie group as a subgroup of their symmetry group. The purpose of this paper is to determine the systems of first order ordinary differential equations which admit a given three-dimensional Lie group as their symmetry group such that the infinitesimal generators of its Lie algebra are represented by vector fields in the space $\mathbb{R}^3$. For the representation of the basis vectors of the solvable three-dimensional Lie algebras we used the results in [3, pp. 161–162], whereas to represent the basis vectors of the simple Lie algebras $sl_2(\mathbb{R})$, $so_3(\mathbb{R})$ we applied the results of [10, p. 385]. These representations are given by (11) in Sect. 3.

Analogously to the Lie's method, in Sect. 2 we formulate the appropriate necessary condition for a system of first order ordinary differential equations allowing a given Lie group as a group of their symmetries. Here we represent the infinitesimal generators of the Lie algebra of this Lie group in the n-dimensional real space with respect to the coordinates $y_1, y_2, \ldots, y_n$. First, we discuss the case when one of the coordinates $y_1, y_2, \ldots, y_n$ is the independent variable x of the system and the generators of the Lie algebra are considered as time-dependent symmetries. This means they can contain component with respect to the direction $\frac{\partial}{\partial x}$. Next, we investigate the case when t is the independent variable, $y_1(t)$, $y_2(t)$, $\ldots$, $y_n(t)$ are the dependent variables of the system and all generators of the Lie algebra are time-preserving symmetries. That is, they do not contain any component in the direction $\frac{\partial}{\partial t}$. In Sect. 3 we apply it for the three-dimensional Lie algebras acting on the three-dimensional real x, y, z-space. In Sect. 3.1 we obtain for each three-dimensional Lie algebra $\mathfrak{g}$ the system of first order ordinary differential equations which is invariant under the action of $\mathfrak{g}$ such that the infinitesimal generators of the Lie algebras are time-dependent symmetries. According to whether $y_1 = x$, $y_2 = y$, or $y_3 = z$ is the independent variable we have three different tasks (see Theorem 1). It turns out that for some systems of first order ordinary differential equations the equivalent second order systems describe motions of a particle under the action of a viscous force (see Remark 2). Section 3.2 is devoted to find the systems of first order ordinary differential equations which are invariant under the action of a given three-dimensional Lie algebra, the infinitesimal generators of which are time-preserving symmetries (see Theorem 2). In Sect. 3.3 we explain how one can find the solutions of the systems of first order ordinary differential equations given in Theorems 1 and 2 using the given Lie algebras of the infinitesimal generators of their symmetries. The obtained solutions are given in Proposition 1.

2. Preliminaries and methods

Symmetries of a differential equation are transformations that move continuously a solution of the equation into another solution. Thus, for each symmetry there exists a corresponding vector field (the infinitesimal generator of the symmetry). The symmetries of a differential equation form a group, which is called the symmetry group. We say that a symmetry is time-preserving if its action on the time coordinate t (it is the independent variable of the differential equation) is trivial. In this case the infinitesimal generator of the symmetry has no component in the direction $\frac{\partial}{\partial t}$ (see [6], pp. 60–61). If a symmetry contains component in the direction of the time coordinate, then we call it time-dependent symmetry.

We represent the basis vectors of the Lie algebra $\mathfrak{g}$ of the symmetry group G in the n-dimensional space $\mathbb{R}^n$ with respect to the coordinates $y_1, y_2, \ldots, y_n$. First, we treat these infinitesimal generators as time-dependent symmetries of systems of the first order ordinary differential equations. Since

every coordinate can be chosen as the independent variable (the time coordinate) we have n different issues for finding those systems which are invariant under the action of the Lie algebra $\mathfrak{g}$.
Let $y_i = x$ be the independent variable, and $(y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n) = (y_1, y_2, \dots, y_{n-1})$ be the dependent variables of the following time-dependent system of first order ordinary differential equations:

$$f_k(x, y_1, \dots, y_{n-1}, y'_1, \dots, y'_{n-1}) = 0, \quad k = 1, 2, \dots, n-1, \tag{1}$$

with the notation $y'_j = \frac{dy_j}{dx}$, $j \in \{1, \dots, n-1\}$.

The infinitesimal generators of the Lie algebra $\mathfrak{g}$ are the vector fields

$$X_l(x, y_1, \dots, y_{n-1}) = \phi_l(x, y_1, \dots, y_{n-1}) \frac{\partial}{\partial x} + \sum_{j=1}^{n-1} \alpha_l^j(x, y_1, \dots, y_{n-1}) \frac{\partial}{\partial y_j},$$

$l = 1, \dots, r$, where r is the dimension of $\mathfrak{g}$.

If x denotes the independent variable, then the first prolonged vector field of X_l with respect to x can be written as follows:

$$\begin{aligned} &X_l^{(1)}(x, y_1, \dots, y_{n-1}, y'_1, \dots, y'_{n-1}) \\ &\quad = X_l + \sum_{j=1}^{n-1} \alpha_l^{j(1)}(x, y_1, \dots, y_{n-1}, y'_1, \dots, y'_{n-1}) \frac{\partial}{\partial y'_j}, \end{aligned}$$

where

$$\alpha_l^{j(1)} = \frac{\partial \alpha_l^j}{\partial x} + \sum_{k=1}^{n-1} \frac{\partial \alpha_l^j}{\partial y_k}(y_k)' - (y_j)' \left(\frac{\partial \phi_l}{\partial x} + \sum_{k=1}^{n-1} \frac{\partial \phi_l}{\partial y_k}(y_k)' \right) \tag{2}$$

is the first prolongation of the function α_l^j, $l = 1, \dots, r$, $j = 1, \dots, n-1$.

The first order system (1) could admit the given group G with Lie algebra $\mathfrak{g}$ as its symmetries if and only if the functions f_k, $k = 1, 2, \dots, n-1$, satisfy the following system of partial differential equations

$$\phi_l \frac{\partial f_k}{\partial x} + \sum_{j=1}^{n-1} \alpha_l^j \frac{\partial f_k}{\partial y_j} + \sum_{j=1}^{n-1} \alpha_l^{j(1)} \frac{\partial f_k}{\partial y'_j} = 0, \tag{3}$$

where $l = 1, \dots, r$.

Using the $(2n-1) \times r$-matrix

$$M = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_r \\ \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \dots & \alpha_r^1 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \alpha_3^{n-1} & \dots & \alpha_r^{n-1} \\ \alpha_1^{1(1)} & \alpha_2^{1(1)} & \alpha_3^{1(1)} & \dots & \alpha_r^{1(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^{n-1(1)} & \alpha_2^{n-1(1)} & \alpha_3^{n-1(1)} & \dots & \alpha_r^{n-1(1)} \end{pmatrix}, \tag{4}$$

the system (3) of partial differential equations can be considered as the system

$$\left(\frac{\partial f_k}{\partial x}\ \frac{\partial f_k}{\partial y_1}\ \cdots\ \frac{\partial f_k}{\partial y_{n-1}}\ \frac{\partial f_k}{\partial y_1'}\ \cdots\ \frac{\partial f_k}{\partial y_{n-1}'}\right)\cdot M = \left(0\ \ldots\ 0\right)^T$$

of homogeneous linear equations in the variables $\frac{\partial f_k}{\partial x}$, $\frac{\partial f_k}{\partial y_1}$, $\ldots$, $\frac{\partial f_k}{\partial y_{n-1}}$, $\frac{\partial f_k}{\partial y_1'}$, $\ldots$, $\frac{\partial f_k}{\partial y_{n-1}'}$. In order to get non-trivial solutions f_k, $k = 1, 2, \ldots, n-1$, of the system of equations provided by (3) the rank of the matrix M must be less than or equal to $2n-2$. If $\operatorname{rank} M \geq n+1$, then the obtained solutions for the vectors $\left(\frac{\partial f_k}{\partial x}\ \frac{\partial f_k}{\partial y_1}\ \cdots\ \frac{\partial f_k}{\partial y_{n-1}}\ \frac{\partial f_k}{\partial y_1'}\ \cdots\ \frac{\partial f_k}{\partial y_{n-1}'}\right)$, $k = 1, \ldots, n-1$, are linearly dependent. This means that the received systems of differential equations contain less than or equal to $n-2$ equations instead of $n-1$ equations for the $n-1$ dependent variables $y_1, \ldots, y_{n-1}$. Hence, we require that

$$\operatorname{rank} M < n+1. \tag{5}$$

Now we investigate the case when the infinitesimal generators of the Lie algebra $\mathfrak{g}$ of the given real Lie group G are time-preserving symmetries. Representing the basis vectors of $\mathfrak{g}$ in $\mathbb{R}^n$ we deal with the following system of first order ordinary differential equations:

$$f_k(x, y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x)) = 0, \tag{6}$$

where x is the independent variable, $y_1(x), \ldots, y_n(x)$ are the dependent variables, and $y_j'(x) = \frac{dy_j(x)}{dx}$, $j, k = 1, \ldots, n$. Let $\dim(\mathfrak{g}) = r$. The basis elements of $\mathfrak{g}$ can be written as the vector fields

$$X_l(y_1(x), \ldots, y_n(x)) = \sum_{j=1}^{n} \alpha_l^j(y_1(x), \ldots, y_n(x))\frac{\partial}{\partial y_j},\ \ l = 1, \ldots, r.$$

The formula

$$\begin{aligned} &X_l^{(1)}(y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x)) \\ &\quad = X_l + \sum_{j=1}^{n} \alpha_l^{j(1)}(y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x))\frac{\partial}{\partial y_j'} \end{aligned}$$

with

$$\alpha_l^{j(1)} = \sum_{k=1}^{n} \frac{\partial \alpha_l^j}{\partial y_k}(y_k(x))' \tag{7}$$

defines the first prolonged vector field of $X_l(y_1(x), \ldots, y_n(x))$, $l = 1, \ldots, r$, with respect to the variable x. The system (6) admits the given group G as its symmetry group precisely if the functions f_k, $k = 1, \ldots, n$, fulfill the following system of partial differential equations

$$\sum_{j=1}^{n} \alpha_l^j \frac{\partial f_k}{\partial y_j} + \sum_{j=1}^{n} \alpha_l^{j(1)} \frac{\partial f_k}{\partial y_j'} = 0. \tag{8}$$

The system (8) is equivalent to the following system of homogeneous linear equations of the variables $\frac{\partial f_k}{\partial y_1}$, $\ldots$, $\frac{\partial f_k}{\partial y_n}$, $\frac{\partial f_k}{\partial y_1'}$, $\ldots$, $\frac{\partial f_k}{\partial y_n'}$:

$$\left(\frac{\partial f_k}{\partial y_1}\ \cdots\ \frac{\partial f_k}{\partial y_n}\ \frac{\partial f_k}{\partial y_1'}\ \cdots\ \frac{\partial f_k}{\partial y_n'}\right)\cdot M = \left(0\ \ldots\ 0\right)^T,$$

where M is the $(2n) \times r$-matrix

$$M = \begin{pmatrix} \alpha_1^1 & \dots & \alpha_r^1 \\ \vdots & & \vdots \\ \alpha_1^n & \dots & \alpha_r^n \\ \alpha_1^{1(1)} & \dots & \alpha_r^{1(1)} \\ \vdots & & \vdots \\ \alpha_1^{n(1)} & \dots & \alpha_r^{n(1)} \end{pmatrix}. \tag{9}$$

To receive non-trivial solutions f_k, $k = 1, \dots, n$, of the system defined by (8) it is required that for the rank of the coefficient matrix M one should have $\text{rank}M < 2n$. The vectors $\left(\frac{\partial f_k}{\partial y_1} \dots \frac{\partial f_k}{\partial y_n} \frac{\partial f_k}{\partial y_1'} \dots \frac{\partial f_k}{\partial y_n'}\right)$, $k = 1, \dots, n$, are linearly independent if

$$\text{rank}M \leq n. \tag{10}$$

3. Results

In this paper we will consider the 3-dimensional Lie algebras as Lie algebras of vector fields over 3-dimensional space $\mathbb{R}^3$.

The 3-dimensional indecomposable solvable Lie algebras $\mathfrak{g}_{3.i}$, $i = 1, 2, 3, 4, 5a, 6, 7a$, and the Lie algebra $\mathbb{R} \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2$ is the 2-dimensional non-abelian Lie algebra, and their basis vectors can be found in [3, pp. 161–162]. The linear representations of the non-isomorphic Lie algebras $\mathfrak{g}_{3.3}$, $\mathfrak{g}_{3.4}$, $\mathfrak{g}_{3.5a}$, are given with the representation of the Lie algebra $\mathfrak{g}_{3.5a}$ such that $a = 1$ for the Lie algebra $\mathfrak{g}_{3.3}$, $a = -1$ for the Lie algebra $\mathfrak{g}_{3.4}$, and $a = \mathbb{R} \setminus \{1, -1\}$ for the Lie algebra $\mathfrak{g}_{3.5a}$ (see [3], pp. 161-162). The linear representations of the non-isomorphic Lie algebras $\mathfrak{g}_{3.6}$, $\mathfrak{g}_{3.7a}$ are given with the representation of the Lie algebra $\mathfrak{g}_{3.7a}$ such that $a = 0$ for the Lie algebra $\mathfrak{g}_{3.6}$, and $a = \mathbb{R} \setminus \{0\}$ for the Lie algebra $\mathfrak{g}_{3.7a}$ (see [3], p. 162). Therefore, we consider the linear representations of the Lie algebras $\mathfrak{g}_{3.5a}$ and $\mathfrak{g}_{3.7a}$ with $a \in \mathbb{R}$.

There are two non-isomorphic simple Lie algebras: $sl_2(\mathbb{R}) = \mathfrak{g}_{3.8}$ and $so_3(\mathbb{R}) = \mathfrak{g}_{3.9}$. Their basis vectors can be found in [10, p. 385].
The non-trivial Lie brackets of the above Lie algebras are

$$\begin{array}{llll}
\mathbb{R} \oplus \mathfrak{g}_2 : & [X_2, X_3] = X_2, & & \\
\mathfrak{g}_{3.1} : & [X_2, X_3] = X_1, & & \\
\mathfrak{g}_{3.2} : & [X_1, X_3] = X_1, & [X_2, X_3] = X_1 + X_2, & \\
\mathfrak{g}_{3.5a} : & [X_1, X_3] = X_1, & [X_2, X_3] = aX_2, & \\
\mathfrak{g}_{3.7a} : & [X_1, X_3] = aX_1 - X_2, & [X_2, X_3] = X_1 + aX_2, & \\
\mathfrak{g}_{3.8} : & [X_1, X_2] = X_1, & [X_1, X_3] = -X_2, & [X_2, X_3] = X_3, \\
\mathfrak{g}_{3.9} : & [X_1, X_2] = X_3, & [X_1, X_3] = -X_2, & [X_2, X_3] = X_1.
\end{array}$$

Their infinitesimal generators are

$$\begin{array}{llll}
\mathbb{R}\oplus\mathfrak{g}_2: & X_1=\dfrac{\partial}{\partial x}, & X_2=\dfrac{\partial}{\partial y}, & X_3=y\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z},\\
\mathfrak{g}_{3.1}: & X_1=\dfrac{\partial}{\partial x}, & X_2=\dfrac{\partial}{\partial y}, & X_3=y\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial z},\\
\mathfrak{g}_{3.2}: & X_1=\dfrac{\partial}{\partial x}, & X_2=\dfrac{\partial}{\partial y}, & X_3=\dfrac{\partial}{\partial z}+(x+y)\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y},\\
\mathfrak{g}_{3.5a}: & X_1=\dfrac{\partial}{\partial x}, & X_2=\dfrac{\partial}{\partial y}, & X_3=\dfrac{\partial}{\partial z}+x\dfrac{\partial}{\partial x}+ay\dfrac{\partial}{\partial y},\\
\mathfrak{g}_{3.7a}: & X_1=\dfrac{\partial}{\partial x}, & X_2=\dfrac{\partial}{\partial y}, & X_3=\dfrac{\partial}{\partial z}+(ax+y)\dfrac{\partial}{\partial x}+(ay-x)\dfrac{\partial}{\partial y},\\
\mathfrak{g}_{3.8}: & X_1=z\dfrac{\partial}{\partial x}+x\dfrac{\partial}{\partial y}, & X_2=y\dfrac{\partial}{\partial y}-z\dfrac{\partial}{\partial z}, & X_3=x\dfrac{\partial}{\partial z}+y\dfrac{\partial}{\partial x},\\
\mathfrak{g}_{3.9}: & X_1=-y\dfrac{\partial}{\partial x}+x\dfrac{\partial}{\partial y}, & X_2=-z\dfrac{\partial}{\partial y}+y\dfrac{\partial}{\partial z}, & X_3=z\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial z}.
\end{array} \tag{11}$$

3.1. Time-dependent symmetries

Using the infinitesimal generators in (11) the matrix M in (4) reduces to a 5×3 matrix. The condition rank$M \leq 3$ is always true. Hence, the rank condition (5) of the matrix M is satisfied. Therefore, we have to solve (3) in $\frac{\partial f_k}{\partial x}$, $\frac{\partial f_k}{\partial y_1}$, $\frac{\partial f_k}{\partial y_2}$, $\frac{\partial f_k}{\partial y_1'}$, $\frac{\partial f_k}{\partial y_2'}$, and check if any solution yields a nontrivial system of differential equations f_1, f_2.

We will suppose that the functions f_1, f_2 have the following reduced explicit form when x is the independent variable

$$\begin{aligned} f_1(x,y_1,y_2,y_1',y_2') &= y_1' - g_1(x,y_1,y_2)=0,\\ f_2(x,y_1,y_2,y_1',y_2') &= y_2' - g_2(x,y_1,y_2)=0. \end{aligned} \tag{12}$$

Remark 1. In (12) the function f_1 is independent of y_2' and f_2 does not depend on y_1'.

Theorem 1. *Systems of the first order ordinary differential equations invariant under the action of the infinitesimal generators (11) are given by the following systems of differential equations for the Lie algebra $\mathbb{R}\oplus\mathfrak{g}_2$, and $\mathfrak{g}_{3.i}$, $i=1,2,5a,7a,8,9$:*

1. *if x is the independent variable, then*
   (a) $f_1(z,y') = y' - C_1e^z = 0,\quad f_2(z') = z' - C_2 = 0,\quad C_1, C_2\in\mathbb{R},$
   (b) $f_1(y') = y' = 0,\quad f_2(z') = z' - C_2 = 0,\quad C_2\in\mathbb{R},$
   $f_1(z,y') = y' - (z+C_1)^{-1} = 0,$
   $f_2(z,z') = z' - C_2(z+C_1)^{-1} = 0,\quad C_1, C_2\in\mathbb{R},$
   (c) $f_1(y') = y' = 0,\quad f_2(z,z') = z' - C_2e^z = 0,\quad C_2\in\mathbb{R},$
   $f_1(z,y') = y' - (z+C_1)^{-1} = 0,$
   $f_2(z,z') = z' - C_2e^{-z}(z+C_1)^{-1} = 0,\ C_1, C_2\in\mathbb{R},$
   (d) $f_1(z,y') = y' - C_1e^{(a-1)z} = 0,$
   $f_2(z,z') = z' - C_2e^{-z} = 0,\quad C_1, C_2\in\mathbb{R},$
   (e) $f_1(z,y') = y' + tan(z+C_1) = 0,$
   $f_2(z,z') = z' - C_2e^{-az}(cos(z+C_1))^{-1} = 0,\quad C_1, C_2\in\mathbb{R},$
   (f) $f_1(x,y,y') = y' - \frac{y}{x} = 0,\quad f_2(x,z,z') = z' - \frac{z}{x} = 0,\quad x\neq 0,$
   $f_1(x,y,z,y') = y' - \frac{x\pm\sqrt{x^2-2yz}}{2z} = 0,$
   $f_2(x,y,z,z') = z' - \frac{x\mp\sqrt{x^2-2yz}}{2y} = 0,$
   $y\neq 0,\quad z\neq 0$ *and* $x^2 > 2yz,$

(g) $f_1(x,y,y') = y' - \frac{y}{x} = 0, \quad f_2(x,z,z') = z' - \frac{z}{x} = 0, \quad x \neq 0,$

2. *if y is the independent variable, then*
   (a) $f_1(z,x') = x' - C_1e^{-z} = 0,$
   $f_2(z,z') = z' - C_2e^{-z} = 0, \quad C_1, C_2 \in \mathbb{R},$
   (b) $f_1(z,x') = x' - z - C_1 = 0,$
   $f_2(z') = z' - C_2 = 0, \quad C_1, C_2 \in \mathbb{R},$
   (c) $f_1(z,x') = x' - z - C_1 = 0,$
   $f_2(z,z') = z' - C_2e^{-z} = 0, \quad C_1, C_2 \in \mathbb{R},$
   (d) $f_1(z,x') = x' - C_1e^{(1-a)z} = 0,$
   $f_2(z,z') = z' - C_2e^{-az} = 0, \quad C_1, C_2 \in \mathbb{R},$
   (e) $f_1(z,x') = x' - tan(z + C_1) = 0,$
   $f_2(z,z') = z' - C_2e^{-az}(cos(z + C_1))^{-1} = 0, \quad C_1, C_2 \in \mathbb{R},$
   (f) $f_1(y,x,z,x') = x' - \frac{x \pm \sqrt{x^2-2yz}}{y} = 0,$
   $f_2(y,x,z,z') = z' - \frac{x^2-yz \pm x\sqrt{x^2-2yz}}{y^2} = 0, \quad y \neq 0 \;\; and \;\; x^2 \geq 2yz,$
   (g) $f_1(y,x,x') = x' - \frac{x}{y} = 0, \quad f_2(y,z,z') = z' - \frac{z}{y} = 0, \quad y \neq 0,$
3. *if z is the independent variable, then*
   (a) $f_1(x') = x' - C_1 = 0, \quad f_2(z,y') = y' - C_2e^{z} = 0, \quad C_1, C_2 \in \mathbb{R},$
   (b) $f_1(z,x') = x' - C_2z - C_1 = 0, \quad f_2(y') = y' - C_2 = 0, \quad C_1, C_2 \in \mathbb{R},$
   (c) $f_1(z,x') = x' - C_1e^{z} - C_2e^{z}z = 0,$
   $f_2(z,y') = y' - C_2e^{z} = 0, \quad C_1, C_2 \in \mathbb{R},$
   (d) $f_1(z,x') = x' - C_1e^{z} = 0, \quad f_2(z,y') = y' - C_2e^{az} = 0, \quad C_1, C_2 \in \mathbb{R},$
   (e) $f_1(z,x') = x' - C_1e^{az}sin(z) - C_2e^{az}cos(z) = 0,$
   $f_2(z,y') = y' - C_1e^{az}cos(z) + C_2e^{az}sin(z) = 0, \quad C_1, C_2 \in \mathbb{R},$
   (f) $f_1(z,x,y,x') = x' - \frac{x \pm \sqrt{x^2-2yz}}{z} = 0,$
   $f_2(z,x,y,y') = y' - \frac{x^2-yz \pm x\sqrt{x^2-2yz}}{z^2} = 0, \quad z \neq 0 \;\; and \;\; x^2 \geq 2yz,$
   (g) $f_1(z,x,x') = x' - \frac{x}{z} = 0, \quad f_2(z,y,y') = y' - \frac{y}{z} = 0, \quad z \neq 0.$

Proof. 1) Let x be the independent variable, $y_1 = y$, $y_2 = z$ be the dependent variables, and $y_1' = y' = \frac{dy}{dx}$, $y_2' = z' = \frac{dz}{dx}$. Using the infinitesimal generators given by (11) the functions ϕ_l, α_l^j, $j = 1, 2$, $l = 1, 2, 3$, are

$$\begin{array}{lll}
\mathbb{R} \oplus \mathfrak{g}_2 : \phi = (1,0,0), & \alpha^1 = (0,1,y), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.1} : \phi = (1,0,y), & \alpha^1 = (0,1,0), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.2} : \phi = (1,0,x+y), & \alpha^1 = (0,1,y), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.5a} : \phi = (1,0,x), & \alpha^1 = (0,1,ay), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.7a} : \phi = (1,0,ax+y), & \alpha^1 = (0,1,ay-x), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.8} : \phi = (z,0,y), & \alpha^1 = (x,y,0), & \alpha^2 = (0,-z,x), \\
\mathfrak{g}_{3.9} : \phi = (-y,0,z), & \alpha^1 = (x,-z,0), & \alpha^2 = (0,y,-x),
\end{array} \tag{13}$$

where $\phi = (\phi_1, \phi_2, \phi_3)$, $\quad \alpha^1 = \left(\alpha_1^1, \alpha_2^1, \alpha_3^1\right)$, $\quad \alpha^2 = \left(\alpha_1^2, \alpha_2^2, \alpha_3^2\right)$.
Applying the formula (2) for (13) the first prolongations of the functions α_l^j, $j = 1, 2$, $l = 1, 2, 3$, are

$$\begin{array}{lll}
\mathbb{R} \oplus \mathfrak{g}_2 : & \alpha^{1(1)} = (0,0,y'), & \alpha^{2(1)} = (0,0,0), \\
\mathfrak{g}_{3.1} : & \alpha^{1(1)} = (0,0,-(y')^2), & \alpha^{2(1)} = (0,0,-y'z'), \\
\mathfrak{g}_{3.2} : & \alpha^{1(1)} = (0,0,-(y')^2), & \alpha^{2(1)} = (0,0,-(1+y')z'),
\end{array}$$

$$\begin{aligned}
&\mathfrak{g}_{3.5a}: && \alpha^{1(1)} = (0,0,(a-1)y'), && \alpha^{2(1)} = (0,0,-z'),\\
&\mathfrak{g}_{3.7a}: && \alpha^{1(1)} = (0,0,-(1+(y')^2)), && \alpha^{2(1)} = (0,0,-(a+y')z'),\\
&\mathfrak{g}_{3.8}: && \alpha^{1(1)} = (1-y'z', y', -(y')^2), && \alpha^{2(1)} = (-(z')^2, -z', 1-y'z'),\\
&\mathfrak{g}_{3.9}: && \alpha^{1(1)} = (1+(y')^2, -z', -y'z'), && \alpha^{2(1)} = (y'z', y', -(1+(z')^2)),
\end{aligned}$$

where $\alpha^{1(1)} = \left(\alpha_1^{1(1)}, \alpha_2^{1(1)}, \alpha_3^{1(1)}\right)$, $\alpha^{2(1)} = \left(\alpha_1^{2(1)}, \alpha_2^{2(1)}, \alpha_3^{2(1)}\right)$.

Hence, the functions f_1 and f_2 fulfill the system (3) precisely if for f_k, $k = 1, 2$, the following systems of partial differential equations hold:

Case	System of partial differential equations
$\mathbb{R} \oplus \mathfrak{g}_2$	$\frac{\partial f_k}{\partial x} = 0, \quad \frac{\partial f_k}{\partial y} = 0, \quad y\frac{\partial f_k}{\partial y} + \frac{\partial f_k}{\partial z} + y'\frac{\partial f_k}{\partial y'} = 0$
$\mathfrak{g}_{3.1}$	$\frac{\partial f_k}{\partial x} = 0, \quad \frac{\partial f_k}{\partial y} = 0, \quad y\frac{\partial f_k}{\partial x} + \frac{\partial f_k}{\partial z} - (y')^2\frac{\partial f_k}{\partial y'} - y'z'\frac{\partial f_k}{\partial z'} = 0$
$\mathfrak{g}_{3.2}$	$\frac{\partial f_k}{\partial x} = 0, \quad \frac{\partial f_k}{\partial y} = 0, \quad (x+y)\frac{\partial f_k}{\partial x} + y\frac{\partial f_k}{\partial y} + \frac{\partial f_k}{\partial z} - (y')^2\frac{\partial f_k}{\partial y'} - (1+y')z'\frac{\partial f_k}{\partial z'} = 0$
$\mathfrak{g}_{3.5a}$	$\frac{\partial f_k}{\partial x} = 0, \quad \frac{\partial f_k}{\partial y} = 0, \quad x\frac{\partial f_k}{\partial x} + ay\frac{\partial f_k}{\partial y} + \frac{\partial f_k}{\partial z} + (a-1)y'\frac{\partial f_k}{\partial y'} - z'\frac{\partial f_k}{\partial z'} = 0$
$\mathfrak{g}_{3.7a}$	$\frac{\partial f_k}{\partial x} = 0, \quad \frac{\partial f_k}{\partial y} = 0,$ $(ax+y)\frac{\partial f_k}{\partial x} + (ay-x)\frac{\partial f_k}{\partial y} + \frac{\partial f_k}{\partial z} - (1+(y')^2)\frac{\partial f_k}{\partial y'} - (a+y')z'\frac{\partial f_k}{\partial z'} = 0$
$\mathfrak{g}_{3.8}$	$z\frac{\partial f_k}{\partial x} + x\frac{\partial f_k}{\partial y} + (1-y'z')\frac{\partial f_k}{\partial y'} - (z')^2\frac{\partial f_k}{\partial z'} = 0,$ $y\frac{\partial f_k}{\partial y} - z\frac{\partial f_k}{\partial z} + y'\frac{\partial f_k}{\partial y'} - z'\frac{\partial f_k}{\partial z'} = 0,$ $y\frac{\partial f_k}{\partial x} + x\frac{\partial f_k}{\partial z} - (y')^2\frac{\partial f_k}{\partial y'} + (1-y'z')\frac{\partial f_k}{\partial z'} = 0$
$\mathfrak{g}_{3.9}$	$-y\frac{\partial f_k}{\partial x} + x\frac{\partial f_k}{\partial y} + (1+(y')^2)\frac{\partial f_k}{\partial y'} + y'z'\frac{\partial f_k}{\partial z'} = 0,$ $-z\frac{\partial f_k}{\partial y} + y\frac{\partial f_k}{\partial z} - z'\frac{\partial f_k}{\partial y'} + y'\frac{\partial f_k}{\partial z'} = 0,$ $z\frac{\partial f_k}{\partial x} - x\frac{\partial f_k}{\partial z} - y'z'\frac{\partial f_k}{\partial y'} - (1+(z')^2)\frac{\partial f_k}{\partial z'} = 0$

Taking into account the first and the second equations in the cases $\mathbb{R} \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i = 1, 2, 5a, 7a$, the functions f_k, $k = 1, 2$, don't depend on the variables x, y. Using (12) the functions f_k, $k = 1, 2$, have the shape

$$f_1(z, y') = y' - g_1(z) = 0, \quad f_2(z, z') = z' - g_2(z) = 0. \tag{14}$$

Applying (14) the third equations in the cases $\mathbb{R}\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i = 1, 2, 5a, 7a$, yield the following.

For $\mathbb{R}\oplus\mathfrak{g}_2$ we obtain $-g_1'(z) + y' = 0$ and $-g_2'(z) = 0$. Hence, we get $g_1(z) = C_1 e^z$, $C_1 \in \mathbb{R}$ and $g_2(z) = C_2$, $C_2 \in \mathbb{R}$. This proves the assertion 1a.

For $\mathfrak{g}_{3.1}$ one has $-g_1'(z) - (y')^2 = 0$ and $-g_2'(z) - y'z' = 0$. The first equation gives either $g_1(z) = 0$, or $g_1(z) = (z+C_1)^{-1}$, $C_1 \in \mathbb{R}$. Since from (14) we have $y' = g_1(z)$, $z' = g_2(z)$, the second equation yields either $-g_2'(z) = 0$, or $-g_2'(z) - (z + C_1)^{-1} g_2(z) = 0$. Therefore, we receive either $g_2(z) = C_2$, or $g_2(z) = C_2(z + C_1)^{-1}$, $C_1, C_2 \in \mathbb{R}$. This shows the assertion 1b.

For $\mathfrak{g}_{3.2}$ we have $-g_1'(z) - (y')^2 = 0$ and $-g_2'(z) - (1+y')z' = 0$. Solving the first equation we obtain either $g_1(z) = 0$, or $g_1(z) = (z + C_1)^{-1}$, $C_1 \in \mathbb{R}$. Since from (14) we get $y' = g_1(z)$, $z' = g_2(z)$, the second equation yields either $-g_2'(z) - g_2(z) = 0$, or $-g_2'(z) - (1 + (z + C_1)^{-1})g_2(z) = 0$. Hence, we obtain either $g_2(z) = C_2 e^{-z}$, or $g_2(z) = C_2 e^{-z}(z + C_1)^{-1}$, $C_1, C_2 \in \mathbb{R}$, which proves the assertion 1c.

For $\mathfrak{g}_{3.5a}$ we receive $-g_1'(z) + (a-1)y' = 0$ and $-g_2'(z) - z' = 0$. Hence, we have $g_1(z) = C_1 e^{(a-1)z}$, $C_1 \in \mathbb{R}$ and $g_2(z) = C_2 e^{-z}$, $C_2 \in \mathbb{R}$. Therefore, the assertion 1d is shown.

For $\mathfrak{g}_{3.7a}$ we get $-g_1'(z) - (1 + (y')^2) = 0$. Solving this Ricatti differential equation we obtain $g_1(z) = -tan(z + C_1)$, $C_1 \in \mathbb{R}$. Using this the third equation gives $-g_2'(z) - (a + y')z' = -g_2'(z) - (a - tan(z + C_1))z' = 0$ for the function f_2. Hence, one has $g_2(z) = C_2 e^{-az}(cos(z + C_1))^{-1}$, $C_1, C_2 \in \mathbb{R}$, and this proves the assertion 1e.

We multiply in the case $\mathfrak{g}_{3.8}$, respectively in $\mathfrak{g}_{3.9}$, the first, the second and the third equations with $(-y)$, x, z, respectively, with z, x, y, and add the obtained new equations. Moreover, it follows from (12) that the functions f_1, f_2 have the form

$$f_1(x, y, z, y') = y' - g_1(x, y, z) = 0, \quad f_2(x, y, z, z') = z' - g_2(x, y, z) = 0. \tag{15}$$

According to (15) we get for f_1 and f_2 in the case $\mathfrak{g}_{3.8}$, respectively in $\mathfrak{g}_{3.9}$, the system of differential equations

$$\left.\begin{array}{l} xy' - z(y')^2 - y + yy'z' = 0 \\ y(z')^2 - xz' + z - zy'z' = 0 \end{array}\right\}, \quad \text{respectively,} \quad \left.\begin{array}{l} z + z(y')^2 - xz' - yy'z' = 0 \\ zy'z' + xy' - y - y(z')^2 = 0 \end{array}\right\}. \tag{16}$$

Putting $z' = \frac{z(y')^2 - xy' + y}{yy'}$, respectively, $z' = \frac{z + z(y')^2}{x + yy'}$ from the first equation into the second equation of (16), and simplifying the obtained equations we receive

$$2xz(y')^3 - (2x^2 + 2yz)(y')^2 + 3xyy' - y^2 = 0, \text{ respectively,} \tag{17}$$

$$(xy^2 + xz^2)(y')^3 + (2x^2y - yz^2 - y^3)(y')^2 + (x^3 + xz^2 - 2xy^2)y' \\ -(x^2y + yz^2) = 0. \tag{18}$$

Both Eqs. (17) and (18) have the solution $y_1' = \frac{y}{x}$, $x \neq 0$, and, hence, $z_1' = \frac{z}{x}$, $x \neq 0$. The Eq. (17) has additionally the solutions $y_{2,3}' = \frac{x \pm \sqrt{x^2 - 2yz}}{2z}$, $z \neq$

0, $x^2 > 2yz$, and, hence, $z'_{2,3} = \frac{x \mp \sqrt{x^2-2yz}}{2y}$, $y \neq 0$, $x^2 > 2yz$, which satisfy the system of partial differential equations in the case $\mathfrak{g}_{3.8}$. Finally, if $z = 0$, then the Eq. (18) has the solution $y'_2 = -\frac{x}{y}$, too. But the second equation of the second system of (16) gives the contradiction $(z'_2)^2 = -\left(\frac{x^2}{y^2} + 1\right)$. This proves the assertions 1f and 1g.

2) Let y be the independent variable, $y_1 = x$, $y_2 = z$ be the dependent variables, $y'_1 = x' = \frac{dx}{dy}$ and $y'_2 = z' = \frac{dz}{dy}$. Using the infinitesimal generators given by (11) the functions ϕ_l, α^j_l, $j = 1, 2$, $l = 1, 2, 3$, are

$$\begin{array}{llll}
\mathbb{R} \oplus \mathfrak{g}_2: & \phi = (0,1,y), & \alpha^1 = (1,0,0), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.1}: & \phi = (0,1,0), & \alpha^1 = (1,0,y), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.2}: & \phi = (0,1,y), & \alpha^1 = (1,0,x+y), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.5a}: & \phi = (0,1,ay), & \alpha^1 = (1,0,x), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.7a}: & \phi = (0,1,ay-x), & \alpha^1 = (1,0,ax+y), & \alpha^2 = (0,0,1), \\
\mathfrak{g}_{3.8}: & \phi = (x,y,0), & \alpha^1 = (z,0,y), & \alpha^2 = (0,-z,x), \\
\mathfrak{g}_{3.9}: & \phi = (x,-z,0), & \alpha^1 = (-y,0,z), & \alpha^2 = (0,y,-x),
\end{array} \tag{19}$$

where $\phi = (\phi_1, \phi_2, \phi_3)$, $\alpha^1 = \left(\alpha^1_1, \alpha^1_2, \alpha^1_3\right)$, $\alpha^2 = \left(\alpha^2_1, \alpha^2_2, \alpha^2_3\right)$.

Applying the formula (2) for (19) the first prolongations of the functions α^j_l, $j = 1, 2$, $l = 1, 2, 3$, are

$$\begin{array}{lll}
\mathbb{R} \oplus \mathfrak{g}_2: & \alpha^{1(1)} = (0,0,-x'), & \alpha^{2(1)} = (0,0,-z'), \\
\mathfrak{g}_{3.1}: & \alpha^{1(1)} = (0,0,1), & \alpha^{2(1)} = (0,0,0), \\
\mathfrak{g}_{3.2}: & \alpha^{1(1)} = (0,0,1), & \alpha^{2(1)} = (0,0,-z'), \\
\mathfrak{g}_{3.5a}: & \alpha^{1(1)} = (0,0,(1-a)x'), & \alpha^{2(1)} = (0,0,-az'), \\
\mathfrak{g}_{3.7a}: & \alpha^{1(1)} = (0,0,1+(x')^2), & \alpha^{2(1)} = (0,0,(x'-a)z'), \\
\mathfrak{g}_{3.8}: & \alpha^{1(1)} = (z'-(x')^2,-x',1), & \alpha^{2(1)} = (-x'z',-2z',x'), \\
\mathfrak{g}_{3.9}: & \alpha^{1(1)} = (-(1+(x')^2),x'z',z'), & \alpha^{2(1)} = (-x'z',1+(z')^2,-x'),
\end{array}$$

where $\alpha^{1(1)} = \left(\alpha^{1(1)}_1, \alpha^{1(1)}_2, \alpha^{1(1)}_3\right)$, $\alpha^{2(1)} = \left(\alpha^{2(1)}_1, \alpha^{2(1)}_2, \alpha^{2(1)}_3\right)$.

Hence, the functions f_1 and f_2 satisfy the system (3) precisely if for f_k, $k = 1, 2$, the following systems of partial differential equations are valid:

Case	System of partial differential equations
$\mathbb{R}\oplus\mathfrak{g}_2$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad y\frac{\partial f_k}{\partial y}+\frac{\partial f_k}{\partial z}-x'\frac{\partial f_k}{\partial x'}-z'\frac{\partial f_k}{\partial z'}=0$
$\mathfrak{g}_{3.1}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad y\frac{\partial f_k}{\partial x}+\frac{\partial f_k}{\partial z}+\frac{\partial f_k}{\partial x'}=0$
$\mathfrak{g}_{3.2}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad y\frac{\partial f_k}{\partial y}+(x+y)\frac{\partial f_k}{\partial x}+\frac{\partial f_k}{\partial z}+\frac{\partial f_k}{\partial x'}-z'\frac{\partial f_k}{\partial z'}=0$
$\mathfrak{g}_{3.5a}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad ay\frac{\partial f_k}{\partial y}+x\frac{\partial f_k}{\partial x}+\frac{\partial f_k}{\partial z}+(1-a)x'\frac{\partial f_k}{\partial x'}-az'\frac{\partial f_k}{\partial z'}=0$
$\mathfrak{g}_{3.7a}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,$ $(ay-x)\frac{\partial f_k}{\partial y}+(ax+y)\frac{\partial f_k}{\partial x}+\frac{\partial f_k}{\partial z}+(1+(x')^2)\frac{\partial f_k}{\partial x'}$ $+(x'-a)z'\frac{\partial f_k}{\partial z'}=0$
$\mathfrak{g}_{3.8}$	$x\frac{\partial f_k}{\partial y}+z\frac{\partial f_k}{\partial x}+(z'-(x')^2)\frac{\partial f_k}{\partial x'}-x'z'\frac{\partial f_k}{\partial z'}=0,$ $y\frac{\partial f_k}{\partial y}-z\frac{\partial f_k}{\partial z}-x'\frac{\partial f_k}{\partial x'}-2z'\frac{\partial f_k}{\partial z'}=0,$ $y\frac{\partial f_k}{\partial x}+x\frac{\partial f_k}{\partial z}+\frac{\partial f_k}{\partial x'}+x'\frac{\partial f_k}{\partial z'}=0$
$\mathfrak{g}_{3.9}$	$x\frac{\partial f_k}{\partial y}-y\frac{\partial f_k}{\partial x}-(1+(x')^2)\frac{\partial f_k}{\partial x'}-x'z'\frac{\partial f_k}{\partial z'}=0,$ $-z\frac{\partial f_k}{\partial y}+y\frac{\partial f_k}{\partial z}+x'z'\frac{\partial f_k}{\partial x'}+(1+(z')^2)\frac{\partial f_k}{\partial z'}=0,$ $z\frac{\partial f_k}{\partial x}-x\frac{\partial f_k}{\partial z}+z'\frac{\partial f_k}{\partial x'}-x'\frac{\partial f_k}{\partial z'}=0$

From the first and the second equations in the cases $\mathbb{R}\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i=1,2,5a,7a$, it follows that the functions f_k, $k=1,2$, don't depend on the variables x, y. From (12) it follows

$$f_1(z,x')=x'-g_1(z)=0,\quad f_2(z,z')=z'-g_2(z)=0. \tag{20}$$

Using (20) the third equations in the cases $\mathbb{R}\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i=1,2,5a,7a$, give the following.

In the case $\mathbb{R}\oplus\mathfrak{g}_2$ we receive $-g_1'(z)-x'=0$ and $-g_2'(z)-z'=0$. Hence, we get $g_1(z)=C_1e^{-z}$, $C_1\in\mathbb{R}$, $g_2(z)=C_2e^{-z}$, $C_2\in\mathbb{R}$, and this proves the assertion 2a.

In the case $\mathfrak{g}_{3.1}$ one has $-g_1'(z)+1=0$ and $-g_2'(z)=0$. Therefore, we obtain $g_1(z)=z+C_1$, $C_1\in\mathbb{R}$, $g_2(z)=C_2$, $C_2\in\mathbb{R}$, and this shows the assertion 2b.

In the case $\mathfrak{g}_{3.2}$ we have $-g_1'(z)+1=0$ and $-g_2'(z)-z'=0$. Hence, we receive $g_1(z)=z+C_1$, $C_1\in\mathbb{R}$ and $g_2(z)=C_2e^{-z}$, $C_2\in\mathbb{R}$, which proves the assertion 2c.

In the case $\mathfrak{g}_{3.5a}$ we obtain $-g_1'(z)+(1-a)x'=0$ and $-g_2'(z)-az'=0$. Therefore, one has $g_1(z)=C_1e^{(1-a)z}$, $C_1\in\mathbb{R}$ and $g_2(z)=C_2e^{-az}$, $C_2\in\mathbb{R}$. This gives the assertion 2d.

In the case $\mathfrak{g}_{3.7a}$ we have $-g_1'(z)+(1+(x')^2)=0$. The solution of this Ricatti differential equation is $g_1(z)=tan(z+C_1)$, $C_1\in\mathbb{R}$. Using this the third equation yields $-g_2'(z)+(x'-a)z'=-g_2'(z)+(tan(z+C_1)-a)z'=0$ for the function f_2. Hence, we receive $g_2(z)=C_2e^{-az}(cos(z+C_1))^{-1}$, $C_1,C_2\in\mathbb{R}$, and this proves the assertion 2e.

In the case $\mathfrak{g}_{3.8}$, respectively in $\mathfrak{g}_{3.9}$, we multiply the first, the second and the third equations with $(-y)$, x, z, respectively, with z, x, y, and add the obtained new equations. It follows from (12) that the functions f_k, $k = 1, 2$, have the shape

$$f_1(x, y, z, x') = x' - g_1(x, y, z) = 0, \quad f_2(x, y, z, z') = z' - g_2(x, y, z) = 0. \tag{21}$$

Applying (21) we obtain for f_1 and f_2 in the case $\mathfrak{g}_{3.8}$, respectively in $\mathfrak{g}_{3.9}$, the system of differential equations

$$\left.\begin{array}{l} y(x')^2 - xx' - yz' + z = 0 \\ zx' - 2xz' + yx'z' = 0 \end{array}\right\}, \quad \text{respectively,} \quad \left.\begin{array}{l} yz' + xx'z' - z - z(x')^2 = 0 \\ x + x(z')^2 - yx' - zx'z' = 0 \end{array}\right\}. \tag{22}$$

Substituting $z' = \frac{y(x')^2 - xx' + z}{y}$, respectively $z' = \frac{z + z(x')^2}{xx' + y}$, from the first equation into the second equation of (22), and simplifying the obtained equations we receive

$$y^2(x')^3 - 3xy(x')^2 + (2x^2 + 2yz)x' - 2xz = 0, \text{ respectively,} \tag{23}$$

$$(x^2y + yz^2)(x')^3 + (2xy^2 - x^3 - xz^2)(x')^2 + (y^3 + yz^2 - 2x^2y)x' \\ -(xy^2 + xz^2) = 0. \tag{24}$$

In both cases we obtain as solution of the above equations $x_1' = \frac{x}{y}$, $y \neq 0$, and, hence, $z_1' = \frac{z}{y}$, $y \neq 0$. The equation (23) has in addition two solutions $x_{2,3}' = \frac{x \pm \sqrt{x^2 - 2yz}}{y}$, $y \neq 0$, $x^2 > 2yz$, and, hence, $z_{2,3}' = \frac{x^2 - yz \pm x\sqrt{x^2 - 2yz}}{y^2}$, $y \neq 0$, $x^2 > 2yz$, which satisfy the system of partial differential equations in the case $\mathfrak{g}_{3.8}$. Finally, if $z = 0$, then the Eq. (24) has the solution $x_2' = -\frac{y}{x}$, too. Using this the second equation of the second system of (22) gives the contradiction $(z_2')^2 = -\left(\frac{y^2}{x^2} + 1\right)$. This proves the assertions 2f and 2g.

3) Let z be the independent variable, $y_1 = x$, $y_2 = y$ be the dependent variables, $y_1' = x' = \frac{dx}{dz}$ and $y_2' = y' = \frac{dy}{dz}$. Using the infinitesimal generators given by (11) the functions ϕ_l, α_l^j, $j = 1, 2$, $l = 1, 2, 3$, are

$$\begin{array}{llll} \mathbb{R} \oplus \mathfrak{g}_2 : & \phi = (0, 0, 1), & \alpha^1 = (1, 0, 0), & \alpha^2 = (0, 1, y), \\ \mathfrak{g}_{3.1} : & \phi = (0, 0, 1), & \alpha^1 = (1, 0, y), & \alpha^2 = (0, 1, 0), \\ \mathfrak{g}_{3.2} : & \phi = (0, 0, 1), & \alpha^1 = (1, 0, x + y), & \alpha^2 = (0, 1, y), \\ \mathfrak{g}_{3.5a} : & \phi = (0, 0, 1), & \alpha^1 = (1, 0, x), & \alpha^2 = (0, 1, ay), \\ \mathfrak{g}_{3.7a} : & \phi = (0, 0, 1), & \alpha^1 = (1, 0, ax + y), & \alpha^2 = (0, 1, ay - x), \\ \mathfrak{g}_{3.8} : & \phi = (0, -z, x), & \alpha^1 = (z, 0, y), & \alpha^2 = (x, y, 0), \\ \mathfrak{g}_{3.9} : & \phi = (0, y, -x), & \alpha^1 = (-y, 0, z), & \alpha^2 = (x, -z, 0), \end{array} \tag{25}$$

where $\phi = (\phi_1, \phi_2, \phi_3)$, $\alpha^1 = \left(\alpha_1^1, \alpha_2^1, \alpha_3^1\right)$, $\alpha^2 = \left(\alpha_1^2, \alpha_2^2, \alpha_3^2\right)$.

Applying the formula (2) for (25) the first prolongations of the functions α_l^j, $j=1,2$, $l=1,2,3$, are

$$\begin{array}{lll}
\mathbb{R}\oplus\mathfrak{g}_2: & \alpha^{1(1)}=(0,0,0), & \alpha^{2(1)}=(0,0,y'),\\
\mathfrak{g}_{3.1}: & \alpha^{1(1)}=(0,0,y'), & \alpha^{2(1)}=(0,0,0),\\
\mathfrak{g}_{3.2}: & \alpha^{1(1)}=(0,0,x'+y'), & \alpha^{2(1)}=(0,0,y'),\\
\mathfrak{g}_{3.5a}: & \alpha^{1(1)}=(0,0,x'), & \alpha^{2(1)}=(0,0,ay'),\\
\mathfrak{g}_{3.7a}: & \alpha^{1(1)}=(0,0,ax'+y'), & \alpha^{2(1)}=(0,0,ay'-x'),\\
\mathfrak{g}_{3.8}: & \alpha^{1(1)}=(1,x',y'-(x')^2), & \alpha^{2(1)}=(x',2y',-x'y'),\\
\mathfrak{g}_{3.9}: & \alpha^{1(1)}=(-y',-x'y',1+(x')^2), & \alpha^{2(1)}=(x',-(1+(y')^2),x'y'),
\end{array}$$

where $\alpha^{1(1)}=\left(\alpha_1^{1(1)},\alpha_2^{1(1)},\alpha_3^{1(1)}\right)$, $\alpha^{2(1)}=\left(\alpha_1^{2(1)},\alpha_2^{2(1)},\alpha_3^{2(1)}\right)$.

Using this the functions f_1, f_2 satisfy the system (3) precisely if for f_k, $k=1,2$, the following systems of partial differential equations are fulfilled:

Case	System of partial differential equations
$\mathbb{R}\oplus\mathfrak{g}_2$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad \frac{\partial f_k}{\partial z}+y\frac{\partial f_k}{\partial y}+y'\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.1}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad \frac{\partial f_k}{\partial z}+y\frac{\partial f_k}{\partial x}+y'\frac{\partial f_k}{\partial x'}=0$
$\mathfrak{g}_{3.2}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad \frac{\partial f_k}{\partial z}+(x+y)\frac{\partial f_k}{\partial x}+y\frac{\partial f_k}{\partial y}$ $+(x'+y')\frac{\partial f_k}{\partial x'}+y'\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.5a}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,\quad \frac{\partial f_k}{\partial z}+x\frac{\partial f_k}{\partial x}+ay\frac{\partial f_k}{\partial y}+x'\frac{\partial f_k}{\partial x'}+$ $ay'\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.7a}$	$\frac{\partial f_k}{\partial x}=0,\quad \frac{\partial f_k}{\partial y}=0,$ $\frac{\partial f_k}{\partial z}+(ax+y)\frac{\partial f_k}{\partial x}+(ay-x)\frac{\partial f_k}{\partial y}+(ax'+y')\frac{\partial f_k}{\partial x'}$ $+(ay'-x')\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.8}$	$z\frac{\partial f_k}{\partial x}+x\frac{\partial f_k}{\partial y}+\frac{\partial f_k}{\partial x'}+x'\frac{\partial f_k}{\partial y'}=0,$ $-z\frac{\partial f_k}{\partial z}+y\frac{\partial f_k}{\partial y}+x'\frac{\partial f_k}{\partial x'}+2y'\frac{\partial f_k}{\partial y'}=0,$ $x\frac{\partial f_k}{\partial z}+y\frac{\partial f_k}{\partial x}+(y'-(x')^2)\frac{\partial f_k}{\partial x'}-x'y'\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.9}$	$-y\frac{\partial f_k}{\partial x}+x\frac{\partial f_k}{\partial y}-y'\frac{\partial f_k}{\partial x'}+x'\frac{\partial f_k}{\partial y'}=0,$ $y\frac{\partial f_k}{\partial z}-z\frac{\partial f_k}{\partial y}-x'y'\frac{\partial f_k}{\partial x'}-(1+(y')^2)\frac{\partial f_k}{\partial y'}=0,$ $-x\frac{\partial f_k}{\partial z}+z\frac{\partial f_k}{\partial x}+(1+(x')^2)\frac{\partial f_k}{\partial x'}+x'y'\frac{\partial f_k}{\partial y'}=0$

Taking into account the first and the second equations in the cases $\mathbb{R}\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i=1,\ 2,\ 5a, 7a$, the functions f_k, $k=1,2$, are independent of the variables x, y. Hence, from (12) we get

$$f_1(z,x')=x'-g_1(z)=0,\quad f_2(z,y')=y'-g_2(z)=0. \tag{26}$$

Applying (26) the third equations in the cases $\mathbb{R}\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i=1,\ 2,$ $5a, 7a$, give the following.

For $\mathbb{R} \oplus \mathfrak{g}_2$ we obtain $-g_1'(z) = 0$ and $-g_2'(z) + y' = 0$. Hence, one has $g_1(z) = C_1$, $C_1 \in \mathbb{R}$ and $g_2(z) = C_2 e^z$, $C_2 \in \mathbb{R}$, which proves the assertion 3a.

For $\mathfrak{g}_{3.1}$ one has $-g_2'(z) = 0$ and, hence, $g_2(z) = C_2$, $C_2 \in \mathbb{R}$. Putting this into (26) we get $y' = C_2$. Therefore, the third equation yields $-g_1'(z) + y' = -g_1'(z) + C_2 = 0$ for f_1, and, hence, we get $g_1(z) = C_2 z + C_1$, $C_1, C_2 \in \mathbb{R}$. This shows the assertion 3b.

For $\mathfrak{g}_{3.2}$ we have $-g_2'(z) + y' = 0$ and, hence, $g_2(z) = C_2 e^z$, $C_2 \in \mathbb{R}$. Substituting this into (26) we receive $y' = C_2 e^z$. Hence, the third equation gives $-g_1'(z) + (x' + y') = -g_1'(z) + x' + C_2 e^z = 0$. This leads to the linear differential equation $g_1'(z) - g_1(z) = C_2 e^z$. Solving the homogeneous part we get $g_1(z) = C_1 e^z$, $C_1 \in \mathbb{R}$, and, therefore, we obtain $g_1(z) = C_1 e^z + C_2 e^z z$, $C_1, C_2 \in \mathbb{R}$. This proves the assertion 3c.

For $\mathfrak{g}_{3.5a}$ we receive $-g_1'(z) + x' = 0$ and $-g_2'(z) + ay' = 0$. Therefore, we get $g_1(z) = C_1 e^z$, $C_1 \in \mathbb{R}$ and $g_2(z) = C_2 e^{az}$, $C_2 \in \mathbb{R}$. Hence, the assertion 3d is proved.

For $\mathfrak{g}_{3.7a}$ one has $-g_1'(z) + ax' + y' = 0$ and $-g_2'(z) + ay' - x' = 0$. From (26) we get $x' = g_1(z)$ and $y' = g_2(z)$. Putting these into the first equation we obtain $g_2(z) = g_1'(z) - ag_1(z)$. Hence, one has $g_2'(z) = g_1''(z) - ag_1'(z)$. Substituting the forms of x', $g_2(z)$ and $g_2'(z)$ into the second equation and solving the obtained second order linear differential equation we get $g_1(z) = C_1 e^{az} sin(z) + C_2 e^{az} cos(z)$, $C_1, C_2 \in \mathbb{R}$. Therefore, we receive $g_2(z) = C_1 e^{az} cos(z) - C_2 e^{az} sin(z)$, $C_1, C_2 \in \mathbb{R}$, and this proves the assertion 3e.

In the case $\mathfrak{g}_{3.8}$, respectively in $\mathfrak{g}_{3.9}$, we multiply the first, the second and the third equations with $(-y)$, x, z, respectively, with z, x, y, and add the obtained new equations. From (12) we obtain that the functions f_k, $k = 1, 2$, have the form

$$f_1(x, y, z, x') = x' - g_1(x, y, z) = 0, \quad f_2(x, y, z, y') = y' - g_2(x, y, z) = 0. \tag{27}$$

Applying (27) for f_1 and f_2 in the case $\mathfrak{g}_{3.8}$, respectively in $\mathfrak{g}_{3.9}$, we receive the system of differential equations

$$\left.\begin{array}{l} xx' - z(x')^2 + zy' - y = 0 \\ 2xy' - yx' - zx'y' = 0 \end{array}\right\}, \quad \text{respectively,} \quad \left.\begin{array}{l} y(x')^2 - zy' - xx'y' + y = 0 \\ zx' - x(y')^2 + yx'y' - x = 0 \end{array}\right\}. \tag{28}$$

Putting $y' = \frac{z(x')^2 + y - xx'}{z}$, respectively $y' = \frac{y + y(x')^2}{z + xx'}$, from the first equation into the second equation of (28), and simplifying the obtained equations we receive

$$z^2(x')^3 - 3xz(x')^2 + (2x^2 + 2yz)x' - 2xy = 0, \text{ respectively,} \tag{29}$$

$$(x^2 z + y^2 z)(x')^3 + (2xz^2 - x^3 - xy^2)(x')^2 + (z^3 + y^2 z - 2x^2 z)x' - (xy^2 + xz^2) = 0. \tag{30}$$

The Eqs. (29) and (30) have the common solution $x_1' = \frac{x}{z}$, $z \neq 0$, and, hence, $y_1' = \frac{y}{z}$, $z \neq 0$. Moreover, the Eq. (29) has two further solutions $x_{2,3}' = \frac{x \pm \sqrt{x^2 - 2yz}}{z}$, $z \neq 0$, $x^2 > 2yz$, and, hence, $y_{2,3}' = \frac{x^2 - yz \pm x\sqrt{x^2 - 2yz}}{z^2}$,

$z \neq 0$, $x^2 > 2yz$, which satisfy the system of partial differential equations in the case $\mathfrak{g}_{3.8}$. Furthermore, if $y = 0$, then the Eq. (30) has the solution $x_2' = -\frac{z}{x}$, too. It is excluded because the second equation of the second system of (28) yields the contradiction $(y')^2 = -\left(\frac{z^2}{x^2} + 1\right)$. This proves the assertions 3f and 3g. □

Corollary 1. *Since the Lie algebra* $\mathfrak{g}_2$ *is an ideal of the Lie algebra* $\mathbb{R} \oplus \mathfrak{g}_2$ *from Theorem 1 we obtain the following first order ordinary differential equations which are invariant under the action of the infinitesimal generators* $X_1 = \frac{\partial}{\partial y}$, $X_2 = y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$:

- *if* y *is the independent variable, then* $f_1(z, z') = z' - Ce^{-z} = 0, \quad C \in \mathbb{R}$,
- *if* z *is the independent variable, then* $f_1(z, y') = y' - Ce^{z} = 0, \quad C \in \mathbb{R}$.

Remark 2. The equation of geodesics in an arbitrary coordinate frame of a Riemannian space is a second order ordinary differential equation of the form

$$(x^i)'' + \Gamma^i_{jk}(x^j)'(x^k)' + F(x^i, (x^j)') = 0,$$

where $F(x^i, (x^j)')$ is an arbitrary function of its arguments and the functions Γ^i_{jk} are the coefficients of the connection of the space. It is also an equation of motion with the action of a velocity dependent force (see [23]). We consider the systems 1a–1e, 2a–2e in Theorem 1. The equivalent second order ordinary differential equations can be obtained by expressing z from the equation f_1 and substituting it into the equation f_2. They have the form

$$u'' = F(u'). \tag{31}$$

In Eq. (31) the force depends on the velocity, i.e. it is a viscous force that a fluid exerts on a particle. In particular, the second order system corresponding to the system 1a is $u'' = C_2 u'$, $C_2 \in \mathbb{R}$, if we identify y with u. For $a = 0$ the second order system corresponding to the system 1d is $u'' = -\frac{C_2}{C_1}(u')^2$, $C_2 \in \mathbb{R}$, $C_1 \neq 0$. The first equation with $C_2 < 0$ describes horizontal motion with linear resistance, whereas the second equation with $-\frac{C_2}{C_1} < 0$ describes that with quadratic resistance.

3.2. Time-preserving symmetries

Using the notations in Sect. 2, let $x = t$ be the independent variable, $y_1(x) = x(t)$, $y_2(x) = y(t)$, $y_3(x) = z(t)$ be the dependent variables, $y_1'(x) = x'(t)$, $y_2'(x) = y'(t)$ and $y_3'(x) = z'(t)$. Applying the infinitesimal generators in (11) the matrix M in (9) reduces to a 6×3 matrix. In this case rank$M \leq 3$ is true. Hence, the rank condition (10) is satisfied. Therefore, we have to solve (8) in $\frac{\partial f_k}{\partial x}$, $\frac{\partial f_k}{\partial y}$, $\frac{\partial f_k}{\partial z}$, $\frac{\partial f_k}{\partial x'}$, $\frac{\partial f_k}{\partial y'}$, $\frac{\partial f_k}{\partial z'}$, and check if any solution yields a non-trivial system of differential equations f_k, $k = 1, 2, 3$.

We deal with systems having the reduced explicit form

$$\begin{aligned} f_1(t, x(t), y(t), z(t), x'(t)) &= x'(t) - g_1(t, x(t), y(t), z(t)) = 0, \\ f_2(t, x(t), y(t), z(t), y'(t)) &= y'(t) - g_2(t, x(t), y(t), z(t)) = 0, \\ f_3(t, x(t), y(t), z(t), z'(t)) &= z'(t) - g_3(t, x(t), y(t), z(t)) = 0, \end{aligned} \tag{32}$$

that is in formula (6) the function f_1 is independent of the variables $y'(t)$, $z'(t)$, the function f_2 is independent of $x'(t)$, $z'(t)$, and the function f_3 is independent of $x'(t)$, $y'(t)$.

Theorem 2. *If t is the independent variable, then systems of the first order ordinary differential equations which are invariant under the action of the infinitesimal generators (11) are*

1. *for* $\mathbb{R} \oplus \mathfrak{g}_2$

$$f_1(t, x'(t)) = x'(t) - C_1(t) = 0,$$
$$f_2(t, z(t), y'(t)) = y'(t) - C_2(t)e^{z(t)} = 0, \; f_3(t, z'(t)) = z'(t) - C_3(t) = 0,$$

2. *for* $\mathfrak{g}_{3.1}$

$$f_1(t, z(t), x'(t)) = x'(t) - C_2(t)z(t) - C_1(t) = 0,$$
$$f_2(t, y'(t)) = y'(t) - C_2(t) = 0, \; f_3(t, z'(t)) = z'(t) - C_3(t) = 0,$$

3. *for* $\mathfrak{g}_{3.2}$

$$f_1(t, z(t), x'(t)) = x'(t) - C_1(t)e^{z(t)} - C_2(t)e^{z(t)}z(t) = 0,$$
$$f_2(t, z(t), y'(t)) = y'(t) - C_2(t)e^{z(t)} = 0, \; f_3(t, z'(t)) = z'(t) - C_3(t) = 0,$$

4. *for* $\mathfrak{g}_{3.5a}$

$$f_1(t, z(t), x'(t)) = x'(t) - C_1(t)e^{z(t)} = 0,$$
$$f_2(t, z(t), y'(t)) = y'(t) - C_2(t)e^{az(t)} = 0, \; f_3(t, z'(t)) = z'(t) - C_3(t) = 0,$$

5. *for* $\mathfrak{g}_{3.7a}$

$$f_1(t, z(t), x'(t)) = x'(t) - C_1(t)e^{az(t)}sin(z(t)) - C_2(t)e^{az(t)}cos(z(t)) = 0,$$
$$f_2(t, z(t), y'(t)) = y'(t) - C_1(t)e^{az(t)}cos(z(t)) + C_2(t)e^{az(t)}sin(z(t)) = 0,$$
$$f_3(t, z'(t)) = z'(t) - C_3(t) = 0,$$

6. *for* $\mathfrak{g}_{3.8}$ *and* $\mathfrak{g}_{3.9}$

$$f_1(t, x(t), x'(t)) = x'(t) - C(t)x(t) = 0,$$
$$f_2(t, y(t), y'(t)) = y'(t) - C(t)y(t) = 0,$$
$$f_3(t, z(t), z'(t)) = z'(t) - C(t)z(t) = 0,$$

where $C_1(t)$, $C_2(t)$, $C_3(t)$, $C(t)$ are continuous real functions.

Proof. In the following we write shortly $x(t) = x$, $y(t) = y$, $z(t) = z$, $x'(t) = x' = \frac{dx}{dt}$, $y'(t) = y' = \frac{dy}{dt}$ and $z'(t) = z' = \frac{dz}{dt}$. Applying the infinitesimal generators given by (11) the functions α_l^j, $j = 1, 2, 3$, $l = 1, 2, 3$, are

$$\begin{array}{llll} \mathbb{R}\oplus\mathfrak{g}_2: & \alpha^1 = (1,0,0), & \alpha^2 = (0,1,y), & \alpha^3 = (0,0,1), \\ \mathfrak{g}_{3.1}: & \alpha^1 = (1,0,y), & \alpha^2 = (0,1,0), & \alpha^3 = (0,0,1), \\ \mathfrak{g}_{3.2}: & \alpha^1 = (1,0,x+y), & \alpha^2 = (0,1,y), & \alpha^3 = (0,0,1), \\ \mathfrak{g}_{3.5a}: & \alpha^1 = (1,0,x), & \alpha^2 = (0,1,ay), & \alpha^3 = (0,0,1), \\ \mathfrak{g}_{3.7a}: & \alpha^1 = (1,0,ax+y), & \alpha^2 = (0,1,ay-x), & \alpha^3 = (0,0,1), \\ \mathfrak{g}_{3.8}: & \alpha^1 = (z,0,y), & \alpha^2 = (x,y,0), & \alpha^3 = (0,-z,x), \\ \mathfrak{g}_{3.9}: & \alpha^1 = (-y,0,z), & \alpha^2 = (x,-z,0), & \alpha^3 = (0,y,-x), \end{array} \tag{33}$$

where $\alpha^j = \left(\alpha_1^j, \alpha_2^j, \alpha_3^j\right)$, $j = 1, 2, 3$.

Using the formula (7) for (33) the first prolongations of the functions α_l^j, $j = 1, 2, 3$, $l = 1, 2, 3$, are

$$\begin{array}{llll} \mathbb{R}\oplus\mathfrak{g}_2: & \alpha^{1(1)} = (0,0,0), & \alpha^{2(1)} = (0,0,y'), & \alpha^{3(1)} = (0,0,0), \\ \mathfrak{g}_{3.1}: & \alpha^{1(1)} = (0,0,y'), & \alpha^{2(1)} = (0,0,0), & \alpha^{3(1)} = (0,0,0), \\ \mathfrak{g}_{3.2}: & \alpha^{1(1)} = (0,0,x'+y'), & \alpha^{2(1)} = (0,0,y'), & \alpha^{3(1)} = (0,0,0), \\ \mathfrak{g}_{3.5a}: & \alpha^{1(1)} = (0,0,x'), & \alpha^{2(1)} = (0,0,ay'), & \alpha^{3(1)} = (0,0,0), \\ \mathfrak{g}_{3.7a}: & \alpha^{1(1)} = (0,0,ax'+y'), & \alpha^{2(1)} = (0,0,ay'-x'), & \alpha^{3(1)} = (0,0,0), \\ \mathfrak{g}_{3.8}: & \alpha^{1(1)} = (z',0,y'), & \alpha^{2(1)} = (x',y',0), & \alpha^{3(1)} = (0,-z',x'), \\ \mathfrak{g}_{3.9}: & \alpha^{1(1)} = (-y',0,z'), & \alpha^{2(1)} = (x',-z',0), & \alpha^{3(1)} = (0,y',-x'), \end{array}$$

where $\alpha^{j(1)} = \left(\alpha_1^{j(1)}, \alpha_2^{j(1)}, \alpha_3^{j(1)}\right)$, $j = 1, 2, 3$.

The functions f_1, f_2 and f_3 given by (6) fulfill the system (8) precisely if for $f_k, k = 1, 2, 3$, the systems of partial differential equations in Table 1 are satisfied.

Taking into account the first and the second equations in the cases $\mathbb{R}\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i = 1,\ 2,\ 5a, 7a$, the functions f_k, $k = 1, 2, 3$, don't depend on the variables x, y. Using (32) the functions f_k, $k = 1, 2, 3$, have the form

$$\begin{aligned} f_1(t, z, x') &= x' - g_1(t, z) = 0, \\ f_2(t, z, y') &= y' - g_2(t, z) = 0, \\ f_3(t, z, z') &= z' - g_3(t, z) = 0. \end{aligned} \tag{34}$$

Applying (34) the third equations in the cases $\mathbb{R}\oplus\mathfrak{g}_2$ and $\mathfrak{g}_{3.i}$, $i = 1,\ 2,$ $5a, 7a$, yield the following.

In the case $\mathbb{R}\oplus\mathfrak{g}_2$ we have $-\frac{\partial g_1(t,z)}{\partial z} = 0$, $-\frac{\partial g_2(t,z)}{\partial z} + y' = 0$, and $-\frac{\partial g_3(t,z)}{\partial z} = 0$. Hence, we receive $g_1(t) = C_1(t)$, $g_2(t,z) = C_2(t)e^z$, $g_3(t) = C_3(t)$, which proves assertion 1.

In the case $\mathfrak{g}_{3.1}$ one has $-\frac{\partial g_2(t,z)}{\partial z} = 0$ and $-\frac{\partial g_3(t,z)}{\partial z} = 0$. Therefore, we get $g_2(t) = C_2(t)$ and $g_3(t) = C_3(t)$. Using this it follows from (34) that $y' =$

Table 1. Systems of partial differential equations when t is the independent variable.

Case	System of partial differential equations
$\mathbb{R}\oplus\mathfrak{g}_2$	$\frac{\partial f_k}{\partial x}=0,\ \ \frac{\partial f_k}{\partial y}=0,\ \ y\frac{\partial f_k}{\partial y}+\frac{\partial f_k}{\partial z}+y'\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.1}$	$\frac{\partial f_k}{\partial x}=0,\ \ \frac{\partial f_k}{\partial y}=0,\ \ y\frac{\partial f_k}{\partial x}+\frac{\partial f_k}{\partial z}+y'\frac{\partial f_k}{\partial x'}=0$
$\mathfrak{g}_{3.2}$	$\frac{\partial f_k}{\partial x}=0,\ \ \frac{\partial f_k}{\partial y}=0,\ \ (x+y)\frac{\partial f_k}{\partial x}+y\frac{\partial f_k}{\partial y}+\frac{\partial f_k}{\partial z}$ $+(x'+y')\frac{\partial f_k}{\partial x'}+y'\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.5a}$	$\frac{\partial f_k}{\partial x}=0,\ \ \frac{\partial f_k}{\partial y}=0,\ \ x\frac{\partial f_k}{\partial x}+ay\frac{\partial f_k}{\partial y}+\frac{\partial f_k}{\partial z}+x'\frac{\partial f_k}{\partial x'}+$ $ay'\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.7a}$	$\frac{\partial f_k}{\partial x}=0,\ \ \frac{\partial f_k}{\partial y}=0,$ $(ax+y)\frac{\partial f_k}{\partial x}+(ay-x)\frac{\partial f_k}{\partial y}+\frac{\partial f_k}{\partial z}+(ax'+y')\frac{\partial f_k}{\partial x'}$ $+(ay'-x')\frac{\partial f_k}{\partial y'}=0$
$\mathfrak{g}_{3.8}$	$z\frac{\partial f_k}{\partial x}+x\frac{\partial f_k}{\partial y}+z'\frac{\partial f_k}{\partial x'}+x'\frac{\partial f_k}{\partial y'}=0,$ $y\frac{\partial f_k}{\partial y}-z\frac{\partial f_k}{\partial z}+y'\frac{\partial f_k}{\partial y'}-z'\frac{\partial f_k}{\partial z'}=0,$ $y\frac{\partial f_k}{\partial x}+x\frac{\partial f_k}{\partial z}+y'\frac{\partial f_k}{\partial x'}+x'\frac{\partial f_k}{\partial z'}=0$
$\mathfrak{g}_{3.9}$	$-y\frac{\partial f_k}{\partial x}+x\frac{\partial f_k}{\partial y}-y'\frac{\partial f_k}{\partial x'}+x'\frac{\partial f_k}{\partial y'}=0,$ $-z\frac{\partial f_k}{\partial y}+y\frac{\partial f_k}{\partial z}-z'\frac{\partial f_k}{\partial y'}+y'\frac{\partial f_k}{\partial z'}=0,$ $z\frac{\partial f_k}{\partial x}-x\frac{\partial f_k}{\partial z}+z'\frac{\partial f_k}{\partial x'}-x'\frac{\partial f_k}{\partial z'}=0$

$C_2(t)$. Hence, the third equation yields $-\frac{\partial g_1(t,z)}{\partial z}+y'=-\frac{\partial g_1(t,z)}{\partial z}+C_2(t)=0$ for f_1. Its solution is $g_1(t,z)=C_2(t)z+C_1(t)$. This proves the assertion 2.

In the case $\mathfrak{g}_{3.2}$ we receive $-\frac{\partial g_2(t,z)}{\partial z}+y'=0$, and, hence, one has $g_2(t,z)=C_2(t)e^z$. Using this we obtain from (34) that $y'=C_2(t)e^z$. Hence, the third equation gives $-\frac{\partial g_1(t,z)}{\partial z}+x'+y'=-\frac{\partial g_1(t,z)}{\partial z}+x'+C_2(t)e^z=0$ for f_1. This leads to the partial differential equation $\frac{\partial g_1(t,z)}{\partial z}-g_1(t,z)=C_2(t)e^z$, the solution of which is $g_1(t,z)=C_1(t)e^z+C_2(t)e^z z$. Finally, for the function f_3 we have $-\frac{\partial g_3(t,z)}{\partial z}=0$, and, $g_3(t)=C_3(t)$. This proves the assertion 3.

In the case $\mathfrak{g}_{3.5a}$ we obtain $-\frac{\partial g_1(t,z)}{\partial z}+x'=0$, $-\frac{\partial g_2(t,z)}{\partial z}+ay'=0$, and $-\frac{\partial g_3(t,z)}{\partial z}=0$. Therefore, one has $g_1(t,z)=C_1(t)e^z$, $g_2(t,z)=C_2(t)e^{az}$, and $g_3(t)=C_3(t)$. This shows the assertion 4.

In the case $\mathfrak{g}_{3.7a}$ one has $-\frac{\partial g_1(t,z)}{\partial z}+ax'+y'=0$, $-\frac{\partial g_2(t,z)}{\partial z}+ay'-x'=0$, and $-\frac{\partial g_3(t,z)}{\partial z}=0$. From (34) it follows $x'=g_1(t,z)$ and $y'=g_2(t,z)$. Putting these into the first equation we obtain $g_2(t,z)=\frac{\partial g_1(t,z)}{\partial z}-ag_1(t,z)$. Hence, one has $\frac{\partial g_2(t,z)}{\partial z}=\frac{\partial^2 g_1(t,z)}{\partial z^2}-a\frac{\partial g_1(t,z)}{\partial z}$. Substituting the forms of x', $g_2(t,z)$ and $\frac{\partial g_2(t,z)}{\partial z}$ into the second equation and solving the obtained second order partial differential equation we get $g_1(t,z)=C_1(t)e^{az}sin(z)+C_2(t)e^{az}cos(z)$. Moreover, we receive $g_2(t,z)=C_1(t)e^{az}cos(z)-C_2(t)e^{az}sin(z)$. Finally, the solution of the third equation is $g_3(t) = C_3(t)$, and this proves the assertion 5.

In the case $\mathfrak{g}_{3.8}$, respectively in $\mathfrak{g}_{3.9}$, we multiply the first, the second and the third equations with $(-y)$, x, z, respectively, with z, x, y, and add the obtained new equations. Taking into account (32) and arranging the received new equations for f_1, f_2 and f_3 we get

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z}.$$

Denote these quotients by the function $C(t)$, the assertion 6 is proved. □

Corollary 2. *The Lie algebra $\mathfrak{g}_2$ is an ideal of the Lie algebra $\mathbb{R} \oplus \mathfrak{g}_2$. Hence, if t is the independent variable, then it follows from Theorem 2 case 1 that system of the first order ordinary differential equations which is invariant under the action of the infinitesimal generators $X_1 = \frac{\partial}{\partial y}$, $X_2 = y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ is given by*

$$f_1(t, z(t), y'(t)) = y'(t) - C_1(t)e^{z(t)} = 0, \quad f_2(t, z'(t)) = z'(t) - C_2(t) = 0.$$

3.3. Solutions of the systems of first order ordinary differential equations

Now we solve the systems of first order ordinary differential equations given by Theorems 1 and 2 with the aid of the given Lie subgroups of their symmetry groups. To do this we use the following Theorem (see [18], Theorems 2.66 and 2.68).

Theorem 3. *Let*

$$\frac{dy_k}{dx} = g_k(x, y_1, \ldots, y_n), \quad k = 1, \ldots, n,$$

be a system of n first order ordinary differential equations. Suppose G is a one-parameter group of symmetries of the system. Then there is a change of variables $(\tau, w) = \varphi(x, y)$ under which the system takes the form

$$\frac{dw_k}{d\tau} = H_k(\tau, w_1, \ldots, w_{n-1}), \quad k = 1, \ldots, n.$$

Thus the system reduces to a system of $n-1$ first order ordinary differential equations for $w_1, \ldots, w_{n-1}$ together with the quadrature

$$w_n(\tau) = \int H_n(\tau, w_1(\tau), \ldots, w_{n-1}(\tau))d\tau + c.$$

In particular, if the original system consists of q first order ordinary differential equations and is invariant under a q-dimensional solvable group, then its general solution can be found by quadratures alone.

Proposition 1. *The solutions of the systems of first order ordinary differential equations given by Theorems 1 and 2 are the following:*

if x is the independent variable, then

1a	$y(x) = \frac{C_1}{C_2} e^{C_2 x + C_3} + C_4, \quad z(x) = C_2 x + C_3, \quad \text{if} \;\; C_2 \neq 0,$ $y(x) = C_1 e^{C_3} x + C_4, \quad z(x) = C_3, \quad \text{if} \;\; C_2 = 0$
1b	$y(x) = C_3, \quad z(x) = C_2 x + C_4$ $y(x) = \pm \frac{\sqrt{C_1^2 + 2C_2 x + 2C_3}}{C_2} + C_4,$ $z(x) = -C_1 \pm \sqrt{C_1^2 + 2C_2 x + 2C_3} \;\text{if}\; C_2 \neq 0,$ $y(x) = \frac{1}{C_1 + C_3} x + C_4, \quad z(x) = C_3, \quad C_3 \in \mathbb{R} \setminus \{-C_1\}, \quad \text{if} \;\; C_2 = 0$
1c	$y(x) = C_3, \quad z(x) = \ln \lvert C_2 x + C_4 \rvert$ $C_2 x + C_3 = e^{z(x)} \left(z(x) + C_1 - 1 \right), \quad C_2 y(x) + C_4 = e^{z(x)}, \quad \text{if} \;\; C_2 \neq 0,$ $y(x) = \frac{1}{C_1 + C_3} x + C_4, \quad z(x) = C_3, \quad C_3 \in \mathbb{R} \setminus \{-C_1\}, \quad \text{if} \;\; C_2 = 0$
1d	$y(x) = \frac{C_1}{C_2 a} (C_2 x + C_3)^a + C_4, \quad z(x) = \ln \lvert C_2 x + C_3 \rvert, \quad \text{if} \;\; a \neq 0, \quad C_2 \neq 0$ $y(x) = \frac{C_1}{C_2} \ln \lvert C_2 x + C_3 \rvert + C_4, \quad z(x) = \ln \lvert C_2 x + C_3 \rvert, \quad \text{if} \;\; a = 0, \quad C_2 \neq 0,$ $y(x) = C_1 e^{(a-1) C_3} x + C_4, \quad z(x) = C_3, \quad \text{if} \;\; C_2 = 0$
1e	$C_2 x + C_3 = e^{a z(x)} \left(sin(z(x) + C_1) + a cos(z(x) + C_1) \right) (a^2 + 1)^{-1},$ $-C_2 y + C_4 = e^{a z(x)} \left(a sin(z(x) + C_1) - cos(z(x) + C_1) \right) (a^2 + 1)^{-1}, \text{if}\; C_2 \neq 0,$ $y(x) = -tan(C_1 + C_3) x + C_4, \;\; z(x) = C_3, \;\; C_3 \neq \frac{(2k+1)\pi}{2} - C_1, \;\; k \in \mathbb{Z}, \;\; \text{if}\; C_2 = 0$
1f	$y(x) = C_1 x, \quad z(x) = C_2 x, \quad x \neq 0$ $y(x) = \frac{C_2}{2} x \mp \frac{C_2 \sqrt{C_1}}{2}, \quad z(x) = \frac{1}{C_2} x \pm \frac{\sqrt{C_1}}{C_2}, \quad C_1 > 0, \quad C_2 \neq 0, \quad y \neq 0, \quad z \neq 0$
1g	$y(x) = C_1 x, \quad z(x) = C_2 x, \quad x \neq 0$

if y is the independent variable, then

2a	$x(y) = \frac{C_1}{C_2}\ln\lvert C_2y+C_3\rvert + C_4,\quad z(y)=\ln\lvert C_2y+C_3\rvert\quad \textit{if}\;\; C_2\neq 0,$ $x(y)=C_1e^{-C_3}y+C_4,\quad z(y)=C_3,\quad \textit{if}\;\; C_2=0$
2b	$x(y)=\frac{1}{2}C_2y^2+(C_1+C_3)y+C_4,\quad z(y)=C_2y+C_3$
2c	$x(y)=\frac{(C_2y+C_3)\ln\lvert C_2y+C_3\rvert}{C_2}+(C_1-1)y+C_4,$ $z(y)=\ln\lvert C_2y+C_3\rvert,\;\; \textit{if}\; C_2\neq 0,$ $x(y)=(C_1+C_3)y+C_4,\quad z(y)=C_3,\quad \textit{if}\;\; C_2=0$
2d	$x(y)=\frac{C_1}{C_2}\left(a(C_2y+C_3)\right)^{\frac{1}{a}}+C_4,$ $z(y)=\frac{1}{a}\ln\lvert a(C_2y+C_3)\rvert,\;\; \textit{if}\; a\neq 0,\;\; C_2\neq 0$ $x(y)=\frac{C_1}{C_2}e^{C_2y+C_3}+C_4,\quad z(y)=C_2y+C_3,\quad \textit{if}\;\; a=0,\quad C_2\neq 0$ $x(y)=C_1e^{(1-a)C_3}y+C_4,\quad z(y)=C_3,\quad \textit{if}\;\; C_2=0$
2e	$C_2y+C_3=e^{az(y)}\left(sin(z(y)+C_1)+acos(z(y)+C_1)\right)(a^2+1)^{-1},$ $-C_2x+C_4=e^{az(y)}\left(asin(z(y)+C_1)-cos(z(y)+C_1)\right)(a^2+1)^{-1}, \textit{if}\; C_2\neq 0,$ $x(y)=tan(C_1+C_3)y+C_4,\;\; z(y)=C_3,\;\; C_3\neq\frac{(2k+1)\pi}{2}-C_1,\;\; k\in\mathbb{Z},\;\; \textit{if}\; C_2=0$
2f	$x(y)=C_1y,\quad z(y)=C_2y,\quad y\neq 0$ $x(y)=\frac{2}{C_2}y\mp\sqrt{C_1},\;\; z(y)=2\left(\frac{1}{(C_2)^2}y\mp\frac{\sqrt{C_1}}{C_2}\right),\;\; C_1>0,\;\; C_2\neq 0,\;\; y\neq 0,\;\; z\neq 0$
2g	$x(y)=C_1y,\quad z(y)=C_2y,\quad y\neq 0$

if z is the independent variable, then

3a	$x(z)=C_1z+C_3,\quad y(z)=C_2e^z+C_4$
3b	$x(z)=\frac{1}{2}C_2z^2+C_1z+C_3,\quad y(z)=C_2z+C_4$
3c	$x(z)=C_1e^z+C_2e^z(z-1)+C_3,\quad y(z)=C_2e^z+C_4$
3d	$x(z)=C_1e^z+C_3,\quad y(z)=\frac{1}{a}C_2e^{az}+C_4,\quad \textit{if}\;\; a\neq 0$ $x(z)=C_1e^z+C_3,\quad y(z)=C_2z+C_4,\quad \textit{if}\;\; a=0$
3e	$x(z)=\left(C_1e^{az}\left(asin(z)-cos(z)\right)+C_2e^{az}\left(sin(z)+acos(z)\right)\right)(a^2+1)^{-1},$ $y(z)=\left(C_1e^{az}\left(sin(z)+acos(z)\right)-C_2e^{az}\left(asin(z)-cos(z)\right)\right)(a^2+1)^{-1}$
3f	$x(z)=C_1z,\quad y(z)=C_2z,\quad z\neq 0$ $x(z)=C_2z\mp\sqrt{C_1},\quad y(z)=\frac{(C_2)^2}{2}z\mp C_2\sqrt{C_1},\quad C_1>0,\quad C_2\in\mathbb{R},\quad z\neq 0$
3g	$x(z)=C_1z,\quad y(z)=C_2z,\quad z\neq 0$

if t is the independent variable, then

1	$x(t)=\int C_1(t)dt,\quad y(t)=\int C_2(t)e^{\int C_3(t)dt}dt,\quad z(t)=\int C_3(t)dt$
2	$x(t)=\int C_2(t)\left(\int C_3(t)dt\right)dt+\int C_1(t)dt,\quad y(t)=\int C_2(t)dt,\quad z(t)=\int C_3(t)dt$
3	$x(t)=\int C_1(t)e^{\int C_3(t)dt}dt+\int C_2(t)e^{\int C_3(t)dt}\left(\int C_3(t)dt\right)dt,$ $y(t)=\int C_2(t)e^{\int C_3(t)dt}dt,\quad z(t)=\int C_3(t)dt$
4	$x(t)=\int C_1(t)e^{\int C_3(t)dt}dt,\quad y(t)=\int C_2(t)e^{a\int C_3(t)dt}dt,\quad z(t)=\int C_3(t)dt$
5	$x(t)=\int C_1(t)e^{a\int C_3(t)dt}sin\left(\int C_3(t)dt\right)dt$ $+\int C_2(t)e^{a\int C_3(t)dt}cos\left(\int C_3(t)dt\right)dt,$ $y(t)=\int C_1(t)e^{a\int C_3(t)dt}cos\left(\int C_3(t)dt\right)dt$ $-\int C_2(t)e^{a\int C_3(t)dt}sin\left(\int C_3(t)dt\right)dt,$ $z(t)=\int C_3(t)dt$
6	$x(t)=e^{\int C(t)dt}+C_1,\quad y(t)=e^{\int C(t)dt}+C_2,\quad z(t)=e^{\int C(t)dt}+C_3$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$, and $C_1(t), C_2(t), C_3(t), C(t)$ are continuous real functions.

Proof. The systems 1a–1e, 2a–2e, 3a–3e of 2 first order ordinary differential equations in Theorem 1 and the systems 1–5 of 3 first order ordinary differential equations in Theorem 2 allow a 3-dimensional solvable Lie group as a subgroup of their symmetry group. By Theorem 3 they can be solved using quadratures. If x or y is the independent variable of the system, then one can start to solve the separable differential equation f_2. Putting the obtained solution z into the differential equation f_1 one can find the solution for y or x using indefinite integral. If z is the independent variable of the system, then the solutions $x(z)$ and $y(z)$ can be found directly by indefinite integrals. If t is the independent variable, then we find first the solution z from the differential equation f_3 by indefinite integral and after the substitution of z into the differential equations f_1, f_2 we obtain the solutions x, y applying two further indefinite integrals.

Since the systems 1g, 2g, 3g of Theorem 1 and the system 6 in Theorem 2 consist of separable differential equations their solutions can be obtained by elementary integration method.

Now we explain how the knowledge of a suitable symmetry group helps in finding the solution of the system when the system allows the Lie group $SL(2)$ as a subgroup of its symmetry group and does not consist of separable differential equations. These systems are listed in cases 1f, 2f, 3f of Theorem 1. They consist of 2 first order ordinary differential equations and admit a 2-dimensional solvable Lie group G as a subgroup of their symmetry group. Hence, their solutions can be found by quadratures. The Lie algebra of G has the basis vectors $X_1 = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, $X_2 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$ with the Lie bracket $[X_1, X_2] = X_1$. Therefore, we perform a reduction on the systems using the infinitesimal generator X_1 (see [18], p. 156). First, we find new coordinates by a suitable change of variables

$$\tau = \varphi_1(x, y, z), \quad w_1 = \varphi_2(x, y, z), \quad w_2 = \varphi_3(x, y, z),$$

such that the functions φ_1 and φ_2 are functionally independent invariants of the one-parameter group of symmetries belonging to X_1, i.e. $X_1(\varphi_1) = X_1(\varphi_2) = 0$, while the function φ_3 satisfies $X_1(\varphi_3) = 1$ (see [18], p. 155). To obtain invariants φ_1 and φ_2 (see [18], pp. 87–90) we solve the corresponding characteristic system of ordinary differential equations

$$\frac{dz}{0} = \frac{dx}{z} = \frac{dy}{x}. \tag{35}$$

The general solution of (35) is $z = k_1$, $x^2 - 2yz = k_2$, where k_1, k_2 are real constants. Furthermore, one can see that $\varphi_3 = \frac{x \pm \sqrt{x^2-2yz}}{z}$ satisfies $X_1(\varphi_3) = 1$. In the new coordinates

$$\tau = z, \quad w_1 = x^2 - 2yz, \quad w_2 = \frac{x \pm \sqrt{x^2 - 2yz}}{z}$$

the systems 1f, 2f, 3f, which do not consist of separable differential equations, are equivalent to the systems

$$\frac{dw_1}{d\tau} = \frac{dw_1/dx}{d\tau/dx} = 0, \quad \frac{dw_2}{d\tau} = \frac{dw_2/dx}{d\tau/dx} = 0, \tag{36}$$

$$\frac{dw_1}{d\tau} = \frac{dw_1/dy}{d\tau/dy} = 0, \quad \frac{dw_2}{d\tau} = \frac{dw_2/dy}{d\tau/dy} = \frac{\mp 2\sqrt{w_1}}{\tau^2}, \tag{37}$$

$$\frac{dw_1}{d\tau} = \frac{dw_1/dz}{d\tau/dz} = 0, \quad \frac{dw_2}{d\tau} = \frac{dw_2/dz}{d\tau/dz} = 0, \tag{38}$$

respectively. The solution of (36) and (38), respectively (37), is

$$w_1(\tau) = C_1, \quad w_2(\tau) = C_2, \quad C_1, C_2 \in \mathbb{R}, \quad \text{respectively,}$$
$$w_1(\tau) = C_1, \quad w_2(\tau) = \frac{\pm 2\sqrt{C_1}}{\tau} + C_2, \quad C_1 \geq 0,\ C_2 \in \mathbb{R}.$$

Finally, we need to express the obtained solutions in the original coordinates x, y, z. The solutions of the systems 1f, 2f, 3f, which do not consist of separable differential equations, are

$$y(x) = \frac{C_2}{2}x \mp \frac{C_2\sqrt{C_1}}{2}, \quad z(x) = \frac{1}{C_2}x \pm \frac{\sqrt{C_1}}{C_2}, \quad C_2 \neq 0,\ y \neq 0,\ z \neq 0,$$
$$x(y) = \frac{2}{C_2}y \mp \sqrt{C_1}, \quad z(y) = \frac{2}{(C_2)^2}y \mp \frac{2\sqrt{C_1}}{C_2}, \quad C_2 \neq 0,\ y \neq 0,\ z \neq 0,$$
$$x(z) = C_2 z \mp \sqrt{C_1}, \quad y(z) = \frac{(C_2)^2}{2}z \mp C_2\sqrt{C_1}, \quad C_2 \in \mathbb{R},\ z \neq 0,$$

where $C_1 \geq 0$. The solutions of the systems listed in Theorems 1 and 2 are found in the assertion. Hence, the proof is complete. □

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Kornélia Ficzere
Doctoral School of Mathematical and Computational Sciences
University of Debrecen
Egyetem Square 1
Debrecen 4032
Hungary
e-mail: `ficzere.kornelia@science.unideb.hu`

Ágota Figula (✉)
Institute of Mathematics
University of Debrecen
Egyetem Square 1
Debrecen 4032
Hungary
e-mail: `figula@science.unideb.hu`

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