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AN ALTERNATIVE EQUATION FOR GENERALIZED POLYNOMIALS INVOLVING MEASURE AND CATEGORY CONSTRAINTS

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Abstract. In this paper we consider a generalized polynomial $f\colon \mathbb{R}^N \to \mathbb{R}$ that satisfies the additional equation $f(x)f(y) = 0$ for the pairs $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ has a positive Lebesgue measure or it is a second category Baire set. We prove that $f(x) = 0$ for all $x \in \mathbb{R}^N$. In fact, the first statement is established in a considerably more general setting. Then we formulate statements concerning the signs of generalized monomials $g\colon \mathbb{R} \to \mathbb{R}$ of even degree that satisfy the inequality $g(x)g(y) \geq 0$ for the pairs $(x, y) \in E$, where $E \subseteq \mathbb{R}^2$ has a positive planar Lebesgue measure or it is a second category Baire set.

1. Introduction

Let $(G, +)$ and $(H, +)$ denote Abelian groups and let $\mathbb{R}$ denote the set of real numbers. We call a function $f\colon G \to H$ *additive* if $f(x + y) = f(x) + f(y)$ holds for all $x, y \in G$. As it is well-known, the additivity of f implies the identity

$$f(rx) = rf(x) \tag{1}$$

for every $x \in G$ and for every integer r. Let X and Y denote linear spaces over the field of rationals. If $f\colon X \to Y$ is additive, then (1) is valid for every $x \in X$ and for every rational number r as well (cf. the proof of [7, Theorem 5.2.1]).

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Let m denote a positive integer. A function $F\colon G^m \to H$ is called *m-additive* if F is additive in each of its variables. Clearly, if $F\colon X^m \to Y$ is m-additive, then

$$F(x_1, \dots, x_{j-1}, rx_j, x_{j+1}, \dots, x_m) = rF(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots x_m)$$

holds for every $j \in \{1, 2, \dots, m\}$, for every rational number r and for all $x_i \in X$ $(i = 1, 2, \dots, m)$. If $f\colon G \to H$ is defined as a diagonalization (or trace) of an m-additive mapping $F\colon G^m \to H$ as

$$f(x) = F(x, \dots, x) \tag{2}$$

for every $x \in G$, we say that f is a *generalized monomial of degree m*. As a corollary, every generalized monomial $f\colon X \to Y$ of degree m fulfills the identity

$$f(rx) = r^m f(x) \tag{3}$$

for every $x \in X$ and for every rational number r. Replacing F with its symmetric part, we obtain that for every generalized monomial $f\colon G \to Y$ of degree m there exists a symmetric m-additive function $F\colon G^m \to Y$ such that (2) holds for every $x \in G$.

We may also call constant functions as (generalized) monomials of degree zero.

If f is a finite sum of generalized monomials, then f is called a *generalized polynomial*.

Z. Kominek, L. Reich and J. Schwaiger [6] investigated additive real functions that satisfy the additional equation

$$f(x)f(y) = 0 \tag{4}$$

for every $(x, y) \in D$, considering various subsets D of $\mathbb{R}^2$. In several cases, in particular, when D is a Lebesgue measurable set with a positive planar Lebesgue measure, they obtained $f(x) = 0$ for every $x \in \mathbb{R}$. This particular result has been extended by the present authors [2] to the case when $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial (or an almost monomial function) of degree $m \geq 1$.

The purpose of the present paper is to extend these results to generalized polynomials (or almost polynomial functions) on more general domains. We also establish such results to the case when D is a second category Baire set. Finally, using our results for real monomials [2, Lemma 2.2, Theorems 4.3 and 5.1], as well as the category analogies, we shall also investigate real generalized monomials (respectively, almost monomial functions) f fulfilling the inequality

$$f(x)f(y) \geq 0 \tag{5}$$

for every $(x, y) \in D$.

Investigations involving subsets of Cartesian products are usually based on an analysis of the cross sections of the given set. It is therefore reasonable to fix our related notations, that will be used in the subsequent sections several times. For an arbitrary subset $D \subseteq X \times Y$, for any $x_0 \in X$ and for any $y_0 \in Y$, let us write

$$D_{x_0} = \big\{ y \in Y \mid (x_0, y) \in D \big\} \quad \text{and} \quad D^*_{y_0} = \big\{ x \in X \mid (x, y_0) \in D \big\}.$$

Clearly, if we consider D as a relation, we have $D_{x_0} = D(x_0)$ and $D^*_{y_0} = D^{-1}(y_0)$.

2. Basic properties of generalized polynomials

Let us suppose that f is a diagonalization of a symmetric m-additive function $F\colon G^m \to Y$. It is well known [7, Lemma 15.9.2] that we have

$$\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_m} f(x) = m!\, F(h_1, h_2, \dots, h_m) \tag{6}$$

for all $x, h_1, h_2, \dots, h_m \in G$, where $\Delta_y f(x) = f(x+y) - f(x)$ $(x, y \in G)$. This shows the uniqueness of F. Moreover (as it is also stated in [7, Lemma 15.9.2]), equation (6) implies

$$\Delta_{h_1}\Delta_{h_2}\dots\Delta_{h_n} f(x) = 0 \tag{7}$$

for all $x, h_1, h_2, \dots, h_n \in G$ if n is an integer such that $n > m$.

It is a consequence of the identity (6) that any generalized monomial $f\colon G \to Y$ of degree m satisfies the m-monomial functional equation

$$\Delta_y^m f(x) = m! f(y) \quad (x, y \in G), \tag{8}$$

where Δ_y^m denotes the m-th iterate of the operation Δ_y. In fact ([7, Chapter 15], [13, Chapter 1]), generalized monomials of degree m are characterized as the solutions of the m-monomial functional equation (8) if G is uniquely divisible by $(m+1)!$.

Let p denote a positive integer, $F_0 \in Y$, $f_0(x) = F_0$ for all $x \in G$, $F_k\colon G^k \to Y$ be a symmetric, k-additive function and

$$f_k(x) = F_k(\overbrace{x, x, \dots, x}^{k}) \quad (x \in G),$$

for every $k \in \big\{1, 2, \dots, p\big\}$. Moreover, let

$$f(x) = \sum_{k=0}^{p} f_k(x) \quad (x \in G). \tag{9}$$

Functions with representation (9) (defined for all $x \in G$) are called *generalized polynomials* (or *polynomial functions*) *of degree at most* p. As it follows from the above equation (7), if $f\colon G \to Y$ is a generalized polynomial of degree at most p, we have

$$\Delta_y^{p+1} f(x) = 0 \quad (x, y \in G). \tag{10}$$

In fact ([7, Chapter 15], [13, Chapter 1]), generalized polynomials of degree at most p are characterized as solutions of the functional equation (10) if G is uniquely divisible by $(p+1)!$.

The existence of discontinuous generalized monomials among real functions is a well known consequence of the axiom of choice. However, we have established the following limit property [2, Lemma 2.2]:

LEMMA 2.1. *If* $f\colon \mathbb{R} \to \mathbb{R}$ *is a generalized monomial of degree* m *and* $x, y \in \mathbb{R}$, *then*

$$\lim_{n\to\infty} f\left(x + \frac{y}{n}\right) = f(x).$$

3. An alternative equation

3.1. An alternative equation with measure constraints. In this subsection we consider functions defined on particular locally compact Abelian groups. It is well known [4, § 58. Theorem B (p. 254)] that there exists a Haar measure on such a group (which is unique on Borel sets up to a constant factor [4, § 60, Theorem C (p. 263)]). In our statements and arguments we refer to such a measure.

We shall make use of Székelyhidi's result on the zeros of generalized polynomials ([14, Theorem 2], [13, Theorem 3.3]):

THEOREM 3.1. *Let* $\mathcal{G}$ *be a locally compact Abelian group which is generated by any neighborhood of zero and let* Z *be a complex linear space. If a generalized polynomial* $p\colon \mathcal{G} \to Z$ *vanishes on a Haar measurable set of a positive Haar measure, then it vanishes everywhere.*

Now we can formulate the main theorem of this section.

THEOREM 3.2. *Let* $\mathcal{G}$ *be a locally compact Abelian group which is generated by any neighborhood of zero. Let* μ *denote the Haar measure on* $\mathcal{G}$, *and let us assume that* μ *is* σ*-finite. Let* $f\colon \mathcal{G} \to \mathbb{C}$ *be a generalized polynomial fulfilling* (4) *for all* $(x, y) \in D$, *where* $D \subseteq \mathcal{G}^2$ *is a* $\mu \times \mu$*-measurable subset with a positive measure* $\mu \times \mu$. *Then* $f(x) = 0$ *for every* $x \in \mathcal{G}$.

PROOF. Let us consider the set

$$P = \left\{ x \in \mathcal{G} \mid f(x) = 0 \right\}.$$

We wish to prove the existence of $A \subseteq P$ such that A is μ-measurable and $\mu(A) > 0$.

Clearly, we have

$$D \subseteq (P \times \mathcal{G}) \cup (\mathcal{G} \times P)\,.$$

According to the properties of the product measure [4, § 34. Theorem A (p. 141), § 35. Theorems A and B (pp. 143–144)], D_x is μ-measurable for every $x \in \mathcal{G}$, the mapping $x \mapsto \mu(D_x)$ is μ-measurable and

$$(\mu \times \mu)(D) = \int_{\mathcal{G}} \mu(D_x)\, d\mu(x)\,.$$

Due to our assumption on the set D and the definition of the integral there exists a real number $r > 0$ and a μ-measurable set $B \subseteq \mathbb{R}$ such that $\mu(B) > 0$ and, for every $x \in B$, $\mu(D_x) \geq r$. If $D_x \subseteq P$ for some $x \in B$, we can verify the above statement with $A = D_x$. Otherwise we have $B \subseteq P$ as $y \in D_x \setminus P$ implies $(x, y) \in D \setminus (\mathcal{G} \times P)$ and thus $(x, y) \in P \times \mathcal{G}$. Hence we can verify our statement with $A = B$.

Then we may apply Theorem 3.1 to complete the proof. □

3.2. An alternative equation with category constraints.

3.2.1. Second category Baire sets. In this subsection we elaborate an analogy of the previous results when sets of positive measure are replaced with second category Baire sets in Euclidean spaces. We recall that $B \subseteq \mathbb{R}^N$ has the Baire property (or shortly, B is a Baire set) if there exist an open set $G \subseteq \mathbb{R}^N$ and a first category set $T \subseteq \mathbb{R}^N$ such that $B = G \triangle T$ (where the set operation $\triangle$ denotes the symmetric difference, as usual [10], [7, Chapter 2]).

In our arguments we shall use the notation $d(x, y)$ for the standard distance of $x, y \in \mathbb{R}^N$ and we denote the open ball with center $x \in \mathbb{R}^N$ and radius $r > 0$ by $B_r(x)$.

The following lemma is motivated by Steinhaus' celebrated theorem [12, Thèoréme VIII] for sets with a positive Lebesgue measure.

LEMMA 3.3. *Let $m \in \mathbb{N}$ and $A \subseteq \mathbb{R}^N$ such that A is a second category Baire set. Then there exists a neighborhood U of zero such that, for every $y \in U$, there exists $x \in \mathbb{R}^N$ fulfilling*

$$x + ky \in A \quad (k = 0, 1, \ldots, m).$$

PROOF. Let $A = G \triangle T$, where $G \subseteq \mathbb{R}^N$ is a non-void open set and $T \subseteq \mathbb{R}^N$ is of the first category. Hence $G \subseteq A \cup T$.

Let $r > 0$ and $x_0 \in G$ such that $B_{(m+1)r}(x_0) \subseteq G$, and let $U = B_r(0)$. Note that, for all $x \in B_r(x_0)$, $y \in U$ and $k \in \{0, 1, \dots, m\}$ we have

$$d(x + ky, x_0) \leq d(x, x_0) + \sum_{j=1}^{k} d(x + jy, x + (j-1)y) < (k+1)r\,,$$

which implies $x + ky \in G$. Then we have $x \in G - ky$, and thus $x \in A \cup T - ky$. Hence, for all $x \in B_r(x_0)$ and $y \in U$, we have

$$x \in \bigcap_{k=0}^{m} [(A \cup T) - ky] = \bigcap_{k=0}^{m} [(A - ky) \cup (T - ky)]\,,$$

which yields

$$x \in \bigcap_{k=0}^{m} (A - ky) \cup \bigcup_{k=0}^{m} (T - ky) = \bigcap_{k=0}^{m} (A - ky) \cup \tilde{T}[y]\,,$$

where $\tilde{T}[y] = \bigcup_{k=0}^{m} (T - ky)$ is of the first category for every $y \in U$. Then $B_r(x_0)$ cannot be a subset of $\tilde{T}[y]$, hence, for every $y \in U$, there exists $x \in B_r(x_0)$ such that

$$x \in \bigcap_{k=0}^{m} (A - ky)\,,$$

therefore $x + ky \in A$ for every $k \in \{0, 1, \dots, m\}$. □

The particular case $m = N = 1$ of Lemma 3.3 was proved by S. Piccard [11]. Piccard's theorem has been generalized by several authors. A quite general result of A. Járai [5] could be applied to our case, but the verification of the validity of the assumptions to our specific settings does not seem essentially shorter then the self contained proof above.

Now we can establish a category version of Theorem 3.1.

THEOREM 3.4. *If a generalized polynomial $f\colon \mathbb{R}^N \to \mathbb{C}$ vanishes on a second category Baire set, then f vanishes everywhere.*

PROOF. Suppose that f is not identically zero. Then f has representation (9) with a not identically zero f_p. Let A denote a second category Baire set such that $f(x) = 0$ for every $x \in A$. Let $m = p$ and choose the neigbourhood U of zero as in Lemma 3.3. Therefore, for every $y \in U$, we have

$$f_p(y) = \frac{\Delta_y^p f_p(x)}{p!} = \frac{\Delta_y^p f(x)}{p!} = \frac{1}{p!} \sum_{k=0}^{p} \binom{p}{k} (-1)^{p-k} f(x + ky) = 0\,.$$

Then f_p vanishes on U and hence, due to the identity $f_p(ru) = r^p f_p(u)$ ($r \in \mathbb{Q}$, $u \in U$), f_p vanishes everywhere on $\mathbb{R}^N$, contrary to our assumption. □

3.2.2. Category constraints in product spaces. In the sequel, for every positive integer N, we shall identify the Cartesian product $\mathbb{R}^N \times \mathbb{R}^N$ with the Euclidean space $\mathbb{R}^{2N}$.

We shall also need the Kuratowski–Ulam theorem [8] (see also, e.g., [7, Theorem 2.1.7] and [10, Theorem 15.1]).

THEOREM 3.5. *Let $T \subseteq \mathbb{R}^{2N}$ be a set of the first category. Then there exists a set $K \subseteq \mathbb{R}^N$ of the first category such that, for every $x \in \mathbb{R}^N \setminus K$, the set T_x is of the first category.*

Now we are ready to formulate our result for the alternative equation.

THEOREM 3.6. *Let $f\colon \mathbb{R}^N \to \mathbb{C}$ be a generalized polynomial fulfilling (4) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ is a second category Baire set. Then $f(x) = 0$ for every $x \in \mathbb{R}^N$.*

PROOF. Let us consider the set

$$P = \big\{ x \in \mathbb{R}^N \mid f(x) = 0 \big\}.$$

Clearly, we have

$$D \subseteq (P \times \mathbb{R}^N) \cup (\mathbb{R}^N \times P).$$

Due to our assumption on D, there exist a non-void open set $G \subseteq \mathbb{R}^{2N}$ and a first category set $T \subseteq \mathbb{R}^{2N}$ such that $D = G \triangle T$, hence we have $G \subseteq D \cup T$. On the other hand, there exist $(x_0, y_0) \in \mathbb{R}^{2N}$ and $r > 0$ such that

$$B_r(x_0) \times B_r(y_0) \subseteq G.$$

Combining these inclusions we obtain

$$B_r(x_0) \times B_r(y_0) \subseteq \big((P \times \mathbb{R}^N) \cup (\mathbb{R}^N \times P)\big) \cup T. \tag{11}$$

We shall also consider the set $K \subseteq \mathbb{R}^N$ with the properties described in Theorem 3.5.

Let

$$A = B_r(x_0) \cap P \quad \text{and} \quad S = B_r(x_0) \setminus P.$$

Now, consider two cases. First let us assume that S is of the second category. Then $S \setminus K \neq \emptyset$, hence there exists $x^* \in S \setminus K$. Then, for every $y \in B_r(y_0)$, $y \in P$ or $y \in T_{x^*}$, hence we have

$$B_r(y_0) \subseteq P \cup T_{x^*}.$$

Due to the choice of x^* the set T_{x^*} is of the first category. Then the set

$$\check{T}_0 = (T_{x^*} \cap B_r(y_0)) \setminus P$$

is of the first category as well, hence $B = B_r(y_0) \cap P$ can be represented in the form

$$B = B_r(y_0) \setminus \check{T}_0 = B_r(y_0) \triangle \check{T}_0 \,.$$

Therefore, B is a second category Baire set and f vanishes on B, so we can apply Theorem 3.4.

If S is of the first category,

$$A = B_r(x_0) \setminus S = B_r(x_0) \triangle S$$

yields that A is a second category Baire set and f vanishes on A, so we can apply Theorem 3.4 as well. □

4. Conditional inequalities for products

In this section we formulate statements concerning the signs of generalized monomials $f\colon \mathbb{R} \to \mathbb{R}$ of even degree that satisfy the inequality (5) for the pairs $(x, y) \in E$, where $E \subseteq \mathbb{R}^2$ has a positive planar Lebesgue measure or it is a second category Baire set.

4.1. Conditional inequalities with measure constraints. Denote by λ^N the N–dimensional Lebesgue measure for fixed $N \geq 1$. Then λ^2 and λ^1 mean the planar Lebesgue measure and the linear Lebesgue measure, respectively.

The discussion of inequality (5) on a set of a positive planar Lebesgue measure is based on the following preliminary results [2, Lemmas 3.5 and 4.1]:

LEMMA 4.1. *Suppose that $P \subset \mathbb{R}$ fulfils the following assumptions*:
(i) *for all $r \in \mathbb{Q}$ and $x \in P$ we have $rx \in P$*;
(ii) *there exists a Lebesgue measurable set A such that $A \subset P$ and $\lambda^1(A) > 0$.*

Then P has full Lebesgue measure, that is, $\mathbb{R} \setminus P$ is Lebesgue measurable and $\lambda^1(\mathbb{R} \setminus P) = 0$.

LEMMA 4.2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree $m \geq 1$ and let H denote a closed subset of $\mathbb{R}$ such that*

$$P = \big\{ x \in \mathbb{R} \mid f(x) \in H \big\}$$

has full Lebesgue measure. Then $f(x) \in H$ for every $x \in \mathbb{R}$.

Applying these results, we can investigate a conditional inequality involving a product.

THEOREM 4.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a generalized monomial of an even degree $m \geq 2$ fulfilling (5) for all $(x, y) \in D$, where $D \subset \mathbb{R}^2$ is a Lebesgue measurable subset with a positive planar Lebesgue measure. Then (5) holds for all $(x, y) \in \mathbb{R}^2$ (i.e., f does not change sign).*

PROOF. Let us consider the sets

$$P_f = \{x \in \mathbb{R} \mid f(x) \geq 0\}, \quad N_f = \{x \in \mathbb{R} \mid f(x) \leq 0\}.$$

We note that both P_f and N_f obviously satisfy assumption (i) in Lemma 4.1. We are going to verify assumption (ii) of the same lemma for one of these sets.

Clearly, we have

$$D \subset (P_f \times P_f) \cup (N_f \times N_f).$$

Let E denote a Lebesgue measurable subset of D such that

$$0 < \lambda^2(E) < +\infty$$

(such a subset E exists since the Lebesgue measure is σ-finite).

According to Fubini's theorem [1, Theorem 3.4.1 and Corollary 3.4.2] E_x is Lebesgue measurable for λ^1-almost every $x \in \mathbb{R}$, the mapping $x \mapsto \lambda^1(E_x)$ is Lebesgue measurable and

$$\lambda^2(E) = \int_{\mathbb{R}} \lambda^1(E_x)\, d\lambda^1(x).$$

Due to our assumption on the set E and the definition of the integral there exists a real number $r > 0$ and a Lebesgue measurable set $B \subset \mathbb{R}$ such that $\lambda^1(B) > 0$ and, for every $x \in B$, $\lambda^1(E_x) \geq r$. Now we have to distinguish three cases.

Case I: If $E_x \subset P_f$ for some $x \in B$, we can verify assumption (ii) of Lemma 4.1 for P_f with $A = E_x$. Clearly, f and P_f satisfy the assumptions of Lemma 4.2 with $H = \{t \in \mathbb{R} \mid t \geq 0\}$. Thus we have $f(x) \geq 0$ for every $x \in \mathbb{R}$.

Case II: If $E_x \subset N_f$ for some $x \in B$, we can verify assumption (ii) of Lemma 4.1 for N_f with $A = E_x$. Clearly, f and N_f satisfy the assumptions of Lemma 4.2 with $H = \{t \in \mathbb{R} \mid t \leq 0\}$. Thus we have $f(x) \leq 0$ for every $x \in \mathbb{R}$.

Case III: For every $x \in B$, we have

$$E_x \setminus P_f \neq \emptyset \quad \text{and} \quad E_x \setminus N_f \neq \emptyset.$$

That is, for each $x \in B$, there exist $y_j \in E_x$ $(j = 1, 2)$ such that $f(y_1) < 0$ and $f(y_2) > 0$. However, we have $(x, y_j) \in E$ and thus $f(x)f(y_j) \geq 0$ for both $j = 1$ and $j = 2$. This yields $f(x) = 0$. Hence $B \subset P_f \cap N_f$. Hence we can verify assumption (ii) of Lemma 4.1 for $P_f \cap N_f$ with $A = B$.

Due to the statement of Lemma 4.1, $Z_f = P_f \cap N_f$ is a set with full linear Lebesgue measure.

Clearly, f and Z_f satisfy the assumptions of Lemma 4.2 with $H = \{0\}$. Thus we have $f(x) = 0$ for every $x \in \mathbb{R}$. □

4.2. Conditional inequalities with category constraints. In order to investigate inequality (5) on a second category Baire set, we have to elaborate analogies of our Lemmas 4.1 and 4.2 with category constraints.

LEMMA 4.4. *Suppose that $P \subset \mathbb{R}$ fulfills the following assumptions*:
(i) *for all $r \in \mathbb{Q}$ and $x \in P$ we have $rx \in P$*;
(ii) *there exists a second category Baire set A such that $A \subset P$.*
Then $\mathbb{R} \setminus P$ is of the first category.

PROOF. Let $A = G \triangle T$, where $G \subseteq \mathbb{R}$ is a non-void open set and $T \subseteq \mathbb{R}$ is of the first category. Hence $G \subseteq A \cup T$. Let I denote a closed subinterval of G with positive length. In fact, several such intervals exist. One can take the center x of I arbitrarily from any open subinterval of G. So let us fix $x \neq 0$. We may consider all subintervals of I with the same center. In addition, we can replace P with $-P$ (and, accordingly, A with $-A$, G with $-G$ and T with $-T$) as well, whence we may assume that $\inf I > 0$ and that $q = \sup I / \inf I$ is rational (since, for any $x \in \mathbb{R}$ and $0 < \delta \leq x$, the set

$$\left\{ \frac{x + r}{x - r} \;\middle|\; 0 < r < \delta \right\}$$

is a proper interval, hence it contains rational numbers). Let

$$A_0 = A \cap I \quad \text{and} \quad T_0 = T \cap I\,.$$

Then we have

$$I = A_0 \cup T_0\,,$$

where $A_0 \subseteq P$ and T_0 is of the first category. Now, for every integer k let

$$I_k = q^k I, \quad A_k = q^k A_0, \quad T_k = q^k T_0,$$
$$\tilde{I}_k = -q^k I, \quad \tilde{A}_k = -q^k A_0 \quad \text{and} \quad \tilde{T}_k = -q^k T_0.$$

Then we have

$$\mathbb{R} \setminus \{0\} = \bigcup_{k \in \mathbb{Z}} I_k \cup \bigcup_{k \in \mathbb{Z}} \tilde{I}_k$$

as well as

$$I_k \subseteq A_k \cup T_k \quad \text{and} \quad \tilde{I}_k \subseteq \tilde{A}_k \cup \tilde{T}_k$$

for every integer k. Moreover, each T_k and $\tilde{T}_k$ is of the first category, hence the set

$$\hat{T} = \{0\} \cup \bigcup_{k\in\mathbb{Z}} T_k \cup \bigcup_{k\in\mathbb{Z}} \tilde{T}_k$$

is of the first category in $\mathbb{R}$ as well. On the other hand, we have $A_k \subseteq P$ and $\tilde{A}_k \subseteq P$ for every integer k, hence we have

$$\mathbb{R} = P \cup \hat{T}$$

and thus $\mathbb{R} \setminus P \subseteq \hat{T}$, which implies that $\mathbb{R} \setminus P$ is of the first category. □

The proof of the following Lemma is quite similar to the proof of [2, Lemma 4.1].

LEMMA 4.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree $m \geq 1$ and let H denote a closed subset of $\mathbb{R}$ such that the set*

$$S = \big\{ x \in \mathbb{R} \mid f(x) \notin H \big\}$$

is of the first category. Then $f(x) \in H$ for every $x \in \mathbb{R}$.

PROOF. Suppose, on the contrary, that $f(x) \in \mathbb{R} \setminus H$ for some $x \in \mathbb{R}$. For each $n \in \mathbb{N}$ let

$$U_n = \Big\{ y \in \mathbb{R} \;\Big|\; f\Big(x + \frac{y}{n}\Big) \in \mathbb{R} \setminus H \Big\}.$$

Then we have $U_n = n \cdot S - nx$ (i.e., U_n is an affine transform of S), hence U_n is of the first category for every positive integer n. On the other hand, the set $\mathbb{R} \setminus H$ is open, hence Lemma 2.1 yields

$$\mathbb{R} = \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} U_n \,,$$

which is a contradiction. □

Now we can establish the category analogue of [2, Theorem 4.3]

THEOREM 4.6. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a generalized monomial of an even degree $m \geq 2$ fulfilling*

$$f(x) \geq 0 \tag{12}$$

for all $x \in A$, where $A \subset \mathbb{R}$ is a second category Baire set. Then (12) holds for every $x \in \mathbb{R}$.

PROOF. Let us consider the set

$$P = \{ x \in \mathbb{R} \mid f(x) \geq 0 \}.$$

Due to the rational homogeneity property (3) and our assumptions in the theorem, the set P satisfies assumptions (i) and (ii) in Lemma 4.4. Hence $\mathbb{R} \setminus P$ is of the first category.

Clearly, f satisfies the assumptions of Lemma 4.5 if H denotes the set of non-negative real numbers. Thus we have $f(x) \geq 0$ for every $x \in \mathbb{R}$. □

The proof of the following corollary of Theorem 4.6 is identical to the proof of [2, Lemma 4.4].

COROLLARY 4.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a generalized monomial of an odd degree $m \geq 1$ fulfilling*

$$xf(x) \geq 0 \tag{13}$$

for all $x \in A$, where $A \subset \mathbb{R}$ is a second category Baire set. Then (13) holds for every $x \in \mathbb{R}$.

Applying Lemmas 4.4 and 4.5, we can investigate the category analogue of Theorem 4.3.

THEOREM 4.8. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a generalized monomial of an even degree $m \geq 2$ fulfilling (5) for all $(x, y) \in D$, where $D \subset \mathbb{R}^2$ is a second category Baire set. Then (5) holds for all $(x, y) \in \mathbb{R}^2$ (i.e., f does not change sign).*

PROOF. Let us consider the sets

$$P_f = \{ x \in \mathbb{R} \mid f(x) \geq 0 \}, \quad N_f = \{ x \in \mathbb{R} \mid f(x) \leq 0 \}.$$

We note that both P_f and N_f obviously satisfy assumption (i) in Lemma 4.4. We are going to verify assumption (ii) of the same lemma for one of these sets.

Clearly, we have

$$D \subset (P_f \times P_f) \cup (N_f \times N_f).$$

Due to our assumption on D, there exist a non-void open set $G \subseteq \mathbb{R}^2$ and a first category set $T \subseteq \mathbb{R}^2$ such that $D = G \triangle T$, hence we have $G \subseteq D \cup T$. On the other hand, there exist $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$ such that

$$B_r(x_0) \times B_r(y_0) \subseteq G.$$

Combining these inclusions we obtain

$$B_r(x_0) \times B_r(y_0) \subseteq ((P_f \times P_f) \cup (N_f \times N_f)) \cup T. \tag{14}$$

We shall also consider the set $K \subseteq \mathbb{R}$ with the properties described in Theorem 3.5 (applied to the case $N = 1$).

Let

$$A = B_r(x_0) \cap P_f \quad \text{and} \quad S = B_r(x_0) \setminus P_f \,.$$

Let us assume that S is of the second category. Then $S \setminus K \neq \emptyset$, hence there exists $x^* \in S \setminus K$. Then $f(x^*) < 0$ and, for every $y \in B_r(y_0)$, inclusion (14) yields $y \in N_f$ or $y \in T_{x^*}$, hence we have

$$B_r(y_0) \subseteq N_f \cup T_{x^*} \,.$$

Due to the choice of x^* the set T_{x^*} is of the first category. Then the set

$$T_0 = (T_{x^*} \cap B_r(y_0)) \setminus N_f$$

is of the first category as well, hence $B = B_r(y_0) \cap N_f$ can be represented in the form

$$B = B_r(y_0) \setminus T_0 = B_r(y_0) \triangle T_0 \,.$$

Therefore, B is a second category Baire set and $B \subseteq N_f$, hence N_f satisfies the assumptions in Lemma 4.4. Thus $\mathbb{R} \setminus N_f$ is of the first category. Then f fulfills the conditions in Lemma 4.5 with $H = \{ t \in \mathbb{R} \mid t \leq 0 \}$, hence $f(x) \leq 0$ for every $x \in \mathbb{R}$.

If S is of the first category,

$$A = B_r(x_0) \setminus S = B_r(x_0) \triangle S$$

yields that A is a second category Baire set and $A \subseteq P_f$, hence P_f satisfies the assumptions in Lemma 4.4. Thus $\mathbb{R} \setminus P_f$ is of the first category. Then f fulfills the conditions in Lemma 4.5 with $H = \{ t \in \mathbb{R} \mid t \geq 0 \}$, hence $f(x) \geq 0$ for every $x \in \mathbb{R}$. □

5. Results for almost polynomial functions

In this final section we extend our results for almost polynomial functions and almost monomial functions.

5.1. An alternative equation for almost polynomials. In order to cover the concept and the description of almost polynomial functions both in the sense of measure and in the sense of category, we recall the concepts of conjugate proper linearly independent ideals ([3, Definitions 2 and 3], [7, Definitions in Section 17.5]).

DEFINITION 5.1. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly independent ideal* if

(i) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;
(ii) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$;
(iii) $\mathbb{R}^N \notin \mathcal{I}$;
(iv) $\alpha \in \mathbb{R}$, $A \in \mathcal{I}$ and $u \in \mathbb{R}^N$ implies $\alpha A + u \in \mathcal{I}$.

DEFINITION 5.2. Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}^N$ and $\mathcal{I}_2$ be a proper linearly independent ideal in $\mathbb{R}^{2N}$. We say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are *conjugate* if, for every set $M \in \mathcal{I}_2$ there exists a set $U \in \mathcal{I}_1$ such that $M_x \in \mathcal{I}_1$ for every $x \in \mathbb{R}^N \setminus U$.

Given a proper linearly independent ideal $\mathcal{I}$ in $\mathbb{R}^N$, we say that a condition is satisfied *$\mathcal{I}$-almost everywhere* in $\mathbb{R}^N$ (written $\mathcal{I}$-(a.e.)) if there exists a set $S \in \mathcal{I}$ such that the condition holds for every $x \in \mathbb{R}^N \setminus S$. Using this terminology, we may say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate if, for every set $M \in \mathcal{I}_2$, the condition $M_x \in \mathcal{I}_1$ holds $\mathcal{I}_1$-almost everywhere in $\mathbb{R}^N$.

The following examples are corollaries of Fubini's theorem [1, Theorem 3.4.1 and Corollary 3.4.2] and Theorem 3.5, respectively.

EXAMPLE 5.3. Let $\mathcal{L}_0^N$ denote the family of Lebesgue measurable subsets A of $\mathbb{R}^N$ fulfilling $\lambda^N(A) = 0$. Then $\mathcal{L}_0^N$ and $\mathcal{L}_0^{2N}$ are conjugate proper linearly independent ideals.

EXAMPLE 5.4. Let $\mathcal{F}^N$ denote the family of first category subsets of $\mathbb{R}^N$. Then $\mathcal{F}^N$ and $\mathcal{F}^{2N}$ are conjugate proper linearly independent ideals.

These examples motivate the following concept ([3, Definition 4], [7, Section 17.7]).

DEFINITION 5.5. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^{2N}$ and let $p \in \mathbb{N}$. We call a function $f\colon \mathbb{R}^N \to \mathbb{R}$ an *$\mathcal{I}$-almost polynomial function of degree at most p* if there exists $S \in \mathcal{I}$ such that

$$\Delta_y^{p+1} f(x) = 0 \tag{15}$$

holds for every $(x, y) \in \mathbb{R}^{2N} \setminus S$.

This concept was introduced by Ger in order to establish an abstract description of almost polynomial functions as follows ([3, Theorem 1], [7, Theorem 17.7.2]).

THEOREM 5.6. *Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}^N$ and $\mathcal{I}_2$ be a proper linearly independent ideal in $\mathbb{R}^{2N}$ such that $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate. If $f\colon \mathbb{R}^N \to \mathbb{R}$ is an $\mathcal{I}_2$-almost polynomial function of degree at most p, then there exists a unique polynomial function $g\colon \mathbb{R}^N \to \mathbb{R}$ such that $f = g$ $\mathcal{I}_1$-almost everywhere in $\mathbb{R}^N$.*

As immediate consequences of Theorem 5.6, Examples 5.3 and 5.4, Theorem 3.2 (typo for $\mathcal{G} = \mathbb{R}^N$) and Theorem 3.6, respectively, we can establish the following corollaries.

COROLLARY 5.7. *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be an $\mathcal{L}_0^{2N}$-almost polynomial function of degree at most $p \geq 1$ fulfilling (4) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ is Lebesgue measurable and $\lambda^{2N}(D) > 0$. Then $f = 0$ $\mathcal{L}_0^N$–almost everywhere in $\mathbb{R}^N$ (i.e., $f(x) = 0$ for λ^N-almost every $x \in \mathbb{R}^N$).*

COROLLARY 5.8. *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be an $\mathcal{F}^{2N}$-almost polynomial function of degree $m \geq 1$ fulfilling (4) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ is a second category Baire set. Then $f = 0$ $\mathcal{F}^N$–almost everywhere in $\mathbb{R}^N$.*

5.2. Inequalities for almost monomials. Since we have considered conditional inequalities for monomial functions in a real variable, we may wish to extend our related results to almost monomial functions defined on $\mathbb{R}$. Combining our definition for almost monomial functions [2] with Definition 5.5, we may introduce the required concept as follows.

DEFINITION 5.9. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^2$ and let $m \in \mathbb{N}$. We call a function $f\colon \mathbb{R} \to \mathbb{R}$ an *$\mathcal{I}$-almost monomial function of degree m* if there exists $S \in \mathcal{I}$ such that

$$\Delta_y^m f(x) = m!\, f(y) \tag{16}$$

holds for every $(x, y) \in \mathbb{R}^2 \setminus S$.

In order to extend [2, Theorem 5.1] to the ideal setting (possibly preserving the ideas in its proof), we need a stronger invariance property with respect to the linear transformations of the members of the ideal. So we consider the following concept.

DEFINITION 5.10. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly invariant ideal* if $\mathcal{I}$ is a proper linearly independent ideal such that $S \in \mathcal{I}$ implies $\phi(S) \in \mathcal{I}$ for every linear bijection $\phi : \mathbb{R}^N \to \mathbb{R}^N$.

In other words, we could say that a non-void family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is a proper linearly invariant ideal if $\mathcal{I} \neq \{\emptyset\}$, $\mathcal{I}$ fulfills properties (i)–(iii) in Definition 5.1, and (instead of (iv)) it satisfies

(iv*) $S \in \mathcal{I}$ implies $\phi(S) \in \mathcal{I}$ for every affine bijection $\phi\colon \mathbb{R}^N \to \mathbb{R}^N$.

Using the notation introduced in Examples 5.3 and 5.4, we can claim that sets having Lebesgue measure zero as well as sets of the first category form proper linearly invariant ideals. The first example follows immediately from the identity for the Lebesgue measure of linear transforms of measurable sets [1, Corollary 3.6.4].

EXAMPLE 5.11. $\mathcal{L}_0^N$ is a proper linearly invariant ideal.

The second example, which is also well known, can be established as follows.

EXAMPLE 5.12. $\mathcal{F}^N$ is a proper linearly invariant ideal.

In view of these examples, it is reasonable to formulate the following generalization of [2, Theorem 5.1].

THEOREM 5.13. *Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}$ and $\mathcal{I}_2$ be a proper linearly invariant ideal in $\mathbb{R}^2$ such that $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an $\mathcal{I}_2$-almost monomial function of degree $m \geq 1$. Then there exists a unique monomial function $g\colon \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ $\mathcal{I}_1$-almost everywhere.*

PROOF. Due to our assumptions, there exist $M \in \mathcal{I}_2$ such that f satisfies (16) for every $(x, y) \in \mathbb{R}^2 \setminus M$. Let

$$E = \left\{ (x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{R}^2 \setminus M \text{ and } (x + y, y) \in \mathbb{R}^2 \setminus M \right\}.$$

Clearly, $(x, y) \in \mathbb{R}^2 \setminus E$ means $(x, y) \in M$ or $(x + y, y) \in M$. In the latter case there exists $(u, y) \in M$ such that $x = u - y$. Let

$$\phi(u, y) = (u - y, y) \quad ((u, y) \in \mathbb{R}^2).$$

According to the previous argument and our assumptions on $\mathcal{I}_2$, we have

$$\mathbb{R}^2 \setminus E = M \cup \phi(M) \in \mathcal{I}_2\,.$$

For every $(x, y) \in E$ we have

$$\begin{aligned} \Delta_y^{m+1} f(x) &= \Delta_y \Delta_y^m f(x) \\ &= \Delta_y^m f(x + y) - \Delta_y^m f(x) = m!\, f(y) - m!\, f(y) = 0, \end{aligned} \tag{17}$$

hence f is an $\mathcal{I}_2$-almost polynomial function of degree at most m. According to Theorem 5.6, there exists a unique polynomial function $g\colon \mathbb{R} \to \mathbb{R}$ of degree at most m such that $f = g$ $\mathcal{I}_1$-almost everywhere.

By definition, g can be represented in the form

$$g(x) = \sum_{k=0}^{m} g_k(x)$$

for every $x \in \mathbb{R}$, where $g_k\colon \mathbb{R} \to \mathbb{R}$ is a monomial function of degree k for every $k \in \{0, 1, \ldots, m\}$ (in particular, g_0 is a constant function). Equations (8) and (7) imply, for every $(x, y) \in \mathbb{R}^2$,

$$\Delta_y^m g_m(x) = m!\, g_m(y) \quad \text{and} \quad \Delta_y^m g_k(x) = 0$$

for every $k \in \{0, 1, \dots, m-1\}$, respectively. Hence we have

$$\Delta_y^m g(x) = \Delta_y^m \left(\sum_{k=0}^{m} g_k(x) \right) = \sum_{k=0}^{m} \Delta_y^m g_k(x) = m! g_m(y)$$

for every $(x, y) \in \mathbb{R}^2$.

On the other hand, there exists $B \in \mathcal{I}_1$ such that $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus B$.

Applying the linear bijection $\psi(x, y) = (y, x)$ on $\mathbb{R}^2$, we have $\psi(M) \in \mathcal{I}_2$ and thus there exists $A \in \mathcal{I}_1$ such that

$$M_y^* = \psi(M)_y \in \mathcal{I}_1$$

holds for every $y \in \mathbb{R} \setminus A$.

Now let us consider an arbitrary $y \in \mathbb{R} \setminus A$. Let

$$C[y] = M_y^* \cup \bigcup_{k=0}^{m} (B - ky).$$

Then $C[y] \in \mathcal{I}_1$. Hence we may choose $x \in \mathbb{R} \setminus C[y]$. Clearly, $(x, y) \in \mathbb{R}^2 \setminus M$ and $x + ky \in \mathbb{R} \setminus B$ for every $k \in \{0, 1, \dots, m\}$. Thus, applying also [7, Corollary 15.1.2] we have

$$m! f(y) = \Delta_y^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x + ky)$$

$$= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g(x + ky) = \Delta_y^m g(x) = m! g_m(y).$$

We have obtained $f(y) = g_m(y)$, where y is an arbitrary element of $\mathbb{R} \setminus A$ and $g_m \colon \mathbb{R} \to \mathbb{R}$ is a monomial function of order m. □

Now we can establish the analogies of Theorems 4.3 and 4.8 for $\mathcal{L}_0^2$-almost monomial functions and $\mathcal{F}^2$-almost monomial functions, respectively.

Corollary 5.14. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be an $\mathcal{L}_0^2$-almost monomial function of an even degree $m \geq 2$ fulfilling (5) for all $(x, y) \in D$, where $D \subset \mathbb{R}^2$ is a Lebesgue measurable subset with a positive planar Lebesgue measure. Then $f(x) \geq 0$ holds for $\mathcal{L}_0^1$-almost every $x \in \mathbb{R}$ or $f(x) \leq 0$ holds for $\mathcal{L}_0^1$-almost every $x \in \mathbb{R}$.*

Proof. From Theorem 5.13 we conclude that there exists a unique generalized monomial $g \colon \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ $\mathcal{L}_0^1$-almost ev-

erywhere. Namely, there exists a Lebesgue measurable set $A \subseteq \mathbb{R}$ fulfilling $\lambda^1(A) = 0$ such that $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus A$. Clearly, the set

$$S_0 = (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$$

is Lebesgue measurable and fulfils $\lambda^2(S_0) = 0$, hence the set $D_0 = D \setminus S_0$ is also Lebesgue measurable with

$$\lambda^2(D_0) = \lambda^2(D) > 0\,.$$

Moreover, for every $(x, y) \in D_0$ we have

$$g(x)g(y) = f(x)f(y) \geq 0\,.$$

Applying Theorem 4.3 we get that g does not change its sign, i.e., $g \geq 0$ or $g \leq 0$ holds identically. Therefore, $f \geq 0$ holds $\mathcal{L}_0^1$-almost everywhere or $f \leq 0$ holds $\mathcal{L}_0^1$-almost everywhere. □

Corollary 5.15. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an $\mathcal{F}^2$-almost monomial function of an even degree $m \geq 2$ fulfilling (5) for all $(x, y) \in D$, where $D \subset \mathbb{R}^2$ is a second category Baire set. Then $f(x) \geq 0$ holds for $\mathcal{F}^1$-almost every $x \in \mathbb{R}$ or $f(x) \leq 0$ holds for $\mathcal{F}^1$-almost every $x \in \mathbb{R}$.*

Proof. From Theorem 5.13 we conclude that there exists a unique generalized monomial $g\colon \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ holds $\mathcal{F}^1$-almost everywhere. Namely, there exists a first category set $A \subseteq \mathbb{R}$ such that $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus A$. We observe (cf. [7, Theorem 2.1.8]) that the set

$$S_0 = (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$$

fulfills $S_0 \in \mathcal{F}^2$, hence the set $D_0 = D \setminus S_0$ is a second category Baire set in $\mathbb{R}^2$ as well. Moreover, for every $(x, y) \in D_0$ we have

$$g(x)g(y) = f(x)f(y) \geq 0\,.$$

Applying Theorem 4.8 we get that g does not change its sign. Therefore, $f \geq 0$ holds $\mathcal{F}^1$-almost everywhere or $f \leq 0$ holds $\mathcal{F}^1$-almost everywhere. □

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References

[1] V. I. Bogachev, *Measure Theory*, Springer-Verlag (Berlin–Heidelberg–New York, 2007).

[2] Z. Boros, R. Menzer, An alternative equation for generalized monomials involving measure, *Publ. Math. Debrecen*, **104** (2024), 171–183.

[3] R. Ger, On almost polynomial functions, *Colloq. Math.*, **24** (1971), 95–101.
[4] P. R. Halmos, *Measure Theory*, Van Nostrand (New York, 1950, 1st ed.); Springer-Verlag (New York–Heidelberg–Berlin, 1974, 2nd ed.).
[5] A. Járai, A generalization of a theorem of Piccard, *Publ. Math. Debrecen*, **52** (1998), 497–506.
[6] Z. Kominek, L. Reich and J. Schwaiger, On additive functions fulfilling some additional condition, *Sitzungsber. Abt. II*, **207** (1998), 35–42.
[7] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe (Warszawa–Kraków–Katowice, 1985, 1st ed.), Birkhäuser (Basel–Boston–Berlin, 2009, 2nd ed.).
[8] K. Kuratowski and S. Ulam, Quelques propriétés topologiques du produit combinatoire, *Fund. Math.*, **19** (1932), 247–251.
[9] Gy. Maksa, A remark on symmetric biadditive functions having nonnegative diagonalization, *Glas. Mat. Ser. III*, **15/35** (1980), 279–282.
[10] J. C. Oxtoby, *Measure and Category*, Springer-Verlag (New York–Heidelberg–Berlin, 1971).
[11] S. Piccard, *Sur les ensembles de distances des ensembles de points d'un espace euclidien*, Mém. Univ. Neuchâtel, vol. 13, Université de Neuchâtel, Secrétariat de l'Université (Neuchâtel, 1939).
[12] H. Steinhaus, Sur les distances dans les points des ensembles de mesure positive, *Fund. Math.*, **1** (1920), 93–104.
[13] L. Székelyhidi, *Convolution Type Functional Equation on Topological Abelian Groups*, World Scientific (Singapore, 1991).
[14] L. Székelyhidi, Regularity properties of polynomials on groups, *Acta Math. Hungar.*, **45** (1985), 15–19.

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