Spectral synthesis on discrete Abelian groups

M. Laczkovich* and L. Székelyhidi†

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Abstract

We prove that spectral synthesis holds on a discrete Abelian group G if and only if the torsion free rank of G is finite.

1 Introduction.

Let G be a locally compact Abelian group and let $C(G)$ denote the linear space of all complex valued continuous functions defined on G equipped with the topology of uniform convergence on compact sets. By a *variety* on G we mean a translation invariant closed subspace of $C(G)$. A nonzero continuous function $m \in C(G)$ is called an *exponential* if m is multiplicative; that is, if $m(x+y) = m(x) \cdot m(y)$ holds for every $x, y \in G$. A function is a *polynomial* if it belongs to the algebra generated by the continuous additive functions. An *exponential monomial* is the product of a polynomial and an exponential. If a variety is spanned by exponential monomials, then we say that *spectral (or harmonic) synthesis holds for* this variety. If spectral synthesis holds for every variety on G, then we say that *spectral (or harmonic) synthesis holds on G*.

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It was proved by M. Lefranc in [7] that for every finite n, spectral synthesis holds on the group $\mathbb{Z}^n$ equipped with the discrete topology. R. J. Elliot claimed in [2] that spectral synthesis holds on every discrete Abelian group. As it turned out, Elliot's proof was defective. In fact, by a recent result of the second author of this note [11], the statement is false. Let $r_0(G)$ denote the torsion free rank of G; that is, the cardinality of a maximal independent system of elements of infinite order. In [11] it is shown that spectral synthesis fails on every Abelian group G with $r_0(G) \geq \omega$ and a problem is formulated: is it true that if spectral synthesis fails to hold on a discrete Abelian group, then its torsion free rank is at least ω ? In this paper we answer this question in the affirmative, and thus obtain the following result.

Theorem 1. *Spectral synthesis holds on a discrete Abelian group G if and only if $r_0(G)$ is finite.*

This result also verifies the conjecture formulated by the second author in [12]: spectral synthesis holds on a discrete Abelian group if and only if any complex bi-additive function on the group is a bilinear function of complex additive functions.

As an immediate corollary of Theorem 1 we obtain the following.

Theorem 2. *If spectral synthesis holds on the discrete Abelian groups G and H, then it also holds on $G \times H$.* $\square$

It would be interesting to find a direct proof of Theorem 2. In fact, such a proof could simplify the proof of Theorem 1 considerably. As we shall see in Section 3, in order to prove Theorem 1 it is enough to check that spectral synthesis holds on the groups $\mathbb{Q}^n \times T$, where n is a positive integer and T is a torsion group. In possession of Theorem 2 it would be enough to prove that spectral synthesis holds on $\mathbb{Q}$ and on every torsion group. The case of torsion groups is relatively simple, using the compactness of their character groups in the product topology (see [1], and Lemma 15 below). Also, the case of $\mathbb{Q}$ is much easier than that of $\mathbb{Q}^n$ with $n > 1$; this is due to the fact that the structure of ideals of $\mathbb{C}[x]$ is much simpler than those of $\mathbb{C}[x_1, \ldots, x_n]$ for $n > 1$.

It should be noted that the statement of Theorem 2 is false for locally compact Abelian groups. Indeed, by L. Schwartz's celebrated theorem, spectral synthesis holds on $\mathbb{R}$ with the usual topology (See [10], [4] and [5].) On the other hand, as D. I. Gurevič showed, spectral synthesis does not hold on $\mathbb{R}^2$ (see [3]). Therefore, any proof of Theorem 2 must use some special properties of varieties on discrete groups. In the next section we present two such properties. First, we show that on discrete groups there is a complete duality between the varieties and the ideals of measures, unlike on some locally compact groups. Also, on discrete groups, the restriction of a variety to a subgroup is again a variety, while on a locally compact group this is not necessarily the case. This latter observation (Theorem 3) is crucial in our proof of Theorem 1.

2 Preliminaries.

The linear space $C(G)$ endowed with the topology of uniform convergence on compact sets is a locally convex topological vector space. The set $\mathcal{M}_c(G)$ of all signed measures on G having compact support is a ring under the operations of addition and convolution. If G is discrete, then every function is continuous on G, and thus $C(G)$ consists of all complex valued functions defined on G. Also, in a discrete group a set is compact if and only if it is finite, and thus $\mathcal{M}_c(G)$ consists of all measures concentrated on finite sets.

We return to general locally compact Abelian groups. According to the Riesz representation theorem, the continuous linear functionals on $C(G)$ are of the form $f \mapsto \int f\, d\mu$ $(f \in C(G))$, where $\mu \in \mathcal{M}_c(G)$.

Let V be a variety on G, and put

$$V^{\perp} = \left\{ \mu \in \mathcal{M}_c(G) : \int_G f\, d\mu = 0 \text{ for every } f \in V \right\}.$$

It is easy to check that $V^{\perp}$ is an ideal of the ring $\mathcal{M}_c(G)$.

If $I \subset \mathcal{M}_c(G)$ is an ideal then we shall denote by $I^{\perp}$ the set of functions $f \in C(G)$ such that $\int f\, d\mu = 0$ for every $\mu \in I$. Then $I^{\perp}$ is a variety on G. Indeed, it is easy to see that $I^{\perp}$ is a closed linear space. In order to prove translation invariance, let $f \in I^{\perp}$, $y \in G$ and $\mu \in I$ be arbitrary. If δ_y denotes the Dirac measure concentrated at the point y, then we have $\delta_y * \mu \in I$, as I is an ideal. Thus

$$\int_G f(x+y)\, d\mu = \int_G f\, d(\delta_y * \mu) = 0,$$

which proves that $I^\perp$ is translation-invariant.

We shall also need the fact that $V^{\perp\perp} = V$ for every variety V. It is clear that $V \subset V^{\perp\perp}$ for every variety V. To prove the other inclusion, suppose that $f \in V^{\perp\perp}$; that is, $\int_G f\,d\mu = 0$ for every $\mu \in V^\perp$, but $f \notin V$. Then the local convexity of the space implies the existence of a continuous linear functional L such that $Lf \neq 0$ and $Lg = 0$ for every $g \in V$. Let $\mu \in \mathcal{M}_c(G)$ be such that $\int_G g\,d\mu = Lg$ for every $g \in C(G)$. Then $\mu \in V^\perp$ and $\int_G f\,d\mu \neq 0$, a contradiction.

It is obvious that $I \subset I^{\perp\perp}$ holds for every ideal $I \subset \mathcal{M}_c(G)$. The inclusion can be proper, as the following simple example shows. Consider $G = \mathbb{R}$ with the usual topology, and let I denote the ideal generated by the measures $\mu_n = \delta_0 - \delta_{1/n}$ $(n = 1, 2, \ldots)$. If $f \in I^\perp$, then f is periodic mod $1/n$ for every n, and thus, by continuity, f must be constant. Therefore $\delta_0 - \delta_\alpha \in I^{\perp\perp}$ for every $\alpha \in \mathbb{R}$. However, $\delta_0 - \delta_\alpha \notin I$ if α is irrational. Indeed, for every positive integer N there is a continuous function f such that f is periodic mod $1/n$ for every $n \leq N$ but f is not periodic mod α. This easily implies that $\delta_0 - \delta_\alpha$ does not belong to the ideal generated by μ_n $(n \leq N)$. Now $\delta_0 - \delta_\alpha \in I$ would imply that $\delta_0 - \delta_\alpha$ belongs to an ideal generated by finitely many of the measures μ_n which is not the case.

Now we prove that if the group G is discrete, then $I^{\perp\perp} = I$ holds for every ideal $I \subset \mathcal{M}_c(G)$. We only have to prove $I^{\perp\perp} \subset I$. If $\nu \in I^{\perp\perp}$, then $\int_G f\,d\nu = 0$ for every $f \in I^\perp$. Suppose $\nu \notin I$. Since I is a linear subspace of $\mathcal{M}_c(G)$ and $\nu \notin I$, there is a linear map $L : \mathcal{M}_c(G) \to \mathbb{C}$ such that L vanishes on I and $L(\nu) \neq 0$. Let $f(x) = L(\delta_x)$ for every $x \in G$. Then

$$L(\mu) = \int_G f\,d\mu \tag{1}$$

for every $\mu \in \mathcal{M}_c(G)$. Indeed, (1) is true for $\mu = \delta_x$ for every $x \in G$ by the definition of f. If G is discrete, then every $\mu \in \mathcal{M}_c(G)$ is a finite linear combination of measures concentrated on singletons, and thus (1) holds by the linearity of both sides. Now, if $\mu \in I$, then $\int_G f\,d\mu = L(\mu) = 0$ by the choice of L, and thus $f \in I^\perp$. On the other hand, $\int_G f\,d\nu = L(\nu) \neq 0$, which contradicts $\nu \in I^{\perp\perp}$.

Let G be a locally compact Abelian group, and let V be a variety on G. If H is a closed subgroup of G, then we denote by $V|_H$ the set of functions $\{f|_H : f \in V\}$.

In general $V|_H$ need not be a variety, as the following example shows. Let $G = \mathbb{R}$ with the usual topology, and let α be a fixed irrational number. We define V as the set of all functions $f \in C(\mathbb{R})$ which are periodic mod α. Clearly, V is a variety on $\mathbb{R}$. We show that $V|_{\mathbb{Z}}$ is *not* a variety on $\mathbb{Z}$, as $V|_{\mathbb{Z}}$ is not closed. To prove this, first we show that $V|_{\mathbb{Z}}$ is everywhere dense in $C(\mathbb{Z})$. Indeed, let $g : \mathbb{Z} \to \mathbb{C}$ be an arbitrary function and let U be an arbitrary neighbourhood of g. Then there are a positive integer N and a positive number ε such that U contains every function $h : \mathbb{Z} \to \mathbb{C}$ for which $|h(n) - g(n)| < \varepsilon$ for every $|n| \le N$. Now it is easy to see that there exists a function $f \in V$ such that $f(n) = g(n)$ for every $|n| \le N$. Indeed, the set $S = \{n + k \cdot \alpha : |n| \le N, \ k \in \mathbb{Z}\}$ is periodic mod α, and only has a finite number of elements in the interval $[0, \alpha]$. Let $(x_i)_{i \in \mathbb{Z}}$ be a double infinite sequence such that $S = \{x_i : i \in \mathbb{Z}\}$ and $x_{i-1} < x_i$ for every i. Let $f(n+k\cdot\alpha) = g(n)$ for every $|n| \le N$ and $k \in \mathbb{Z}$. Then f is defined on S and is periodic mod α. Extend f to $\mathbb{R}$ as a linear function in each interval $[x_{i-1}, x_i]$. It is easy to see that f is continuous and is periodic mod α; that is, $f \in V$. Since $f(n) = g(n)$ for every $|n| \le N$, it follows that $f|_{\mathbb{Z}} \in U$. This implies that g is in the closure of $V|_{\mathbb{Z}}$ and, as g was arbitrary, $V|_{\mathbb{Z}}$ is everywhere dense in $C(\mathbb{Z})$. On the other hand, $V|_{\mathbb{Z}} \ne C(\mathbb{Z})$, since each element of $V|_{\mathbb{Z}}$ is bounded. Therefore, $V|_{\mathbb{Z}}$ is not closed and, consequently, is not a variety on $\mathbb{Z}$.

In the next theorem we show that this phenomenon cannot occur in discrete Abelian groups. For every $\mu \in \mathcal{M}_c(G)$ the support of μ will be denoted by $\operatorname{supp} \mu$. Clearly, if μ has finite support then $\operatorname{supp} \mu = \{x : \mu(\{x\}) \ne 0\}$.

Theorem 3. *Let G be a discrete Abelian group, H a subgroup of G, and let V be a variety on G. Then*

(i) *$V|_H$ is a variety on H.*

(ii) *For every function $f : H \to \mathbb{C}$ we have $f \in V|_H$ if and only if $\int_H f\, d\mu = 0$ whenever $\mu \in V^\perp$ and* $\operatorname{supp} \mu \subset H$.

Proof. First we prove (ii). Suppose that $f \in V|_H$, $\mu \in V^\perp$ and $\operatorname{supp} \mu \subset H$. Let $g \in V$ be such that $g|_H = f$, and let $\mu = \sum_{i=1}^n a_i \delta_{x_i}$, where $a_i \in \mathbb{C}$ and $x_i \in H$ for every $i = 1, \dots, n$. Since $\mu \in V^\perp$, we have

$$\int_H f\, d\mu = \sum_{i=1}^n a_i \cdot f(x_i) = \sum_{i=1}^n a_i \cdot g(x_i) = \int_G g\, d\mu = 0,$$

which proves the 'only if' part of the statement.

Next we prove the 'if' part of (ii). Suppose that $f : H \to \mathbb{C}$ satisfies the condition formulated in (ii); we show $f \in V|_H$. If $\mu \in \mathcal{M}_c(G)$ and $\mu = \sum_{i=1}^n a_i \delta_{x_i}$, then we shall denote by $\phi(\mu)$ the sum of those terms $a_i \delta_{x_i}$ for which $x_i \in G \setminus H$. If there is no such term; that is, if $\operatorname{supp} \mu \subset H$, then we define $\phi(\mu) = 0$. Clearly, ϕ is a linear map from $\mathcal{M}_c(G)$ into itself. We put

$$L(\phi(\mu)) = -\int_H f \, d(\mu - \phi(\mu)) \tag{2}$$

for every $\mu \in V^\perp$. Then L is well-defined; that is, if $\mu, \nu \in V^\perp$ and $\phi(\mu) = \phi(\nu)$, then

$$\int_H f \, d(\mu - \phi(\mu)) = \int_H f \, d(\nu - \phi(\nu)). \tag{3}$$

Indeed, $\phi(\mu) = \phi(\nu)$ implies $\phi(\mu - \nu) = 0$; that is, $\operatorname{supp}(\mu - \nu) \subset H$. Since $\mu - \nu \in V^\perp$, it follows from the condition imposed on f that $\int_H f \, d(\mu - \nu) = 0$. Thus $\int_H f \, d\mu = \int_H f \, d\nu$ and, as $\phi(\mu) = \phi(\nu)$, (3) holds.

Therefore, (2) defines a linear map $L : \phi\left(V^\perp\right) \to \mathbb{C}$. Let $\tilde{L}$ be a linear extension of L onto $\mathcal{M}_c(G)$, and define

$$g(x) = \begin{cases} f(x), & \text{if } x \in H, \\ \tilde{L}(\delta_x), & \text{if } x \in G \setminus H. \end{cases}$$

Then g is an extension of f. We show that $g \in V$. If $\mu \in \mathcal{M}_c(G)$ and $\mu = \sum_{i=1}^n a_i \delta_{x_i}$, then

$$\begin{aligned} \int_G g \, d\mu &= \int_{G \setminus H} g \, d\mu + \int_H g \, d\mu \\ &= \sum_{x_i \in G \setminus H} a_i \cdot g(x_i) + \sum_{x_i \in H} a_i \cdot f(x_i) \\ &= \sum_{x_i \in G \setminus H} a_i \cdot \tilde{L}(\delta_{x_i}) + \int_H f \, d(\mu - \phi(\mu)) \\ &= \tilde{L}(\phi(\mu)) + \int_H f \, d(\mu - \phi(\mu)). \end{aligned}$$

If $\mu \in V^\perp$, then (2) gives

$$\int_G g \, d\mu = \tilde{L}(\phi(\mu)) + \int_H f \, d(\mu - \phi(\mu)) = 0.$$

Thus $g \in V^{\perp\perp} = V$ and $f = g|_H$, which completes the proof of (ii).

Now (i) is an immediate consequence of (ii). Indeed, let I denote the set of measures $\mu \in V^\perp$ satisfying $\operatorname{supp} \mu \subset H$, and let $J = \{\mu|_H : \mu \in I\}$. Since $V^\perp$ is an ideal of $\mathcal{M}_c(G)$, it is easy to check that J is an ideal of $\mathcal{M}_c(H)$. By (ii) we have $V|_H = J^\perp$, and thus $V|_H$ is a variety on H. □

We shall use the following notation. If $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$, then $f \otimes g$ will denote the function defined by $f(x) \cdot g(y)$ for every $(x, y) \in X \times Y$.

We shall need the following simple fact about varieties on products, when one of the factors is finite.

Lemma 4. *Let V be a variety on $G \times T$, where G is a locally compact Abelian group and T is a finite and discrete Abelian group. Let $|T| = k$, and let $\gamma_1, \dots, \gamma_k$ be the characters of T. Then for every $g \in V$ there are functions $f_1, \dots, f_k : G \to \mathbb{C}$ such that $f_i \otimes \gamma_i \in V$ for every $i = 1, \dots, k$, and*

$$g(x, t) = \sum_{i=1}^{k} f_i(x)\gamma_i(t) \tag{4}$$

for every $(x, t) \in G \times T$.

Proof. It is well-known that the characters $\gamma_1, \dots, \gamma_k$ are linearly independent. Since the set $\mathbb{C}^T = \{f : T \to \mathbb{C}\}$ is a linear space of dimension k, it follows that every element of $\mathbb{C}^T$ is a linear combination of $\gamma_1, \dots, \gamma_k$. Then, for every $x \in G$ there are numbers $f_1(x), \dots, f_k(x)$ such that (4) holds for every $t \in T$.

If $T = \{t_j\}_{j=1}^k$, then, by the linear independence of $\gamma_1, \dots, \gamma_k$ we find that the determinant $\operatorname{Det} \{\gamma_i(t_j)\}_{i,j=1}^k$ is nonzero. Substituting $t+t_j$ for t in (4) we obtain a system of linear equations, from which the unknowns $f_i(x)\gamma_i(t)$ can be expressed as linear combinations of expressions of the form $g(x, t+t_j)$. As the functions $(x, t) \mapsto g(x, t + t_j)$ belong to V, this proves that $f_i \otimes \gamma_i \in V$ for $i = 1, \dots, k$. □

We note that the lemma above easily implies that if spectral synthesis holds on a locally compact Abelian group G, then the same is true for $G \times T$ for every finite discrete Abelian group T.

3 Reduction.

In this section our aim is to show that if spectral synthesis holds on the groups $\mathbb{Q}^n \times T$, where n is a positive integer and T is torsion, then it also holds on every group satisfying $r_0(G) < \infty$. This is an immediate consequence of the following two lemmas.

Lemma 5. *Every Abelian group with finite torsion free rank can be embedded into a group $\mathbb{Q}^n \times T$, where $n = r_0(G)$ and T is a torsion group.*

Proof. Suppose that G is an Abelian group with $n = r_0(G) < \infty$. It is well known [9, Theorem 10.23] that G can be embedded as a subgroup in a divisible group H. Let H_1 denote the set of those elements $x \in H$ for which $mx \in G$ for a suitable positive integer m. Clearly, $G \subset H_1$. It is easy to check that H_1 is a divisible subgroup of H and $r_0(H_1) = n$. Let T denote the torsion subgroup of H_1. Then T is also divisible, and thus T is a direct summand of H_1 by [9, Corollary 10.10]. Let $H_1 = H_2 \times T$, then H_2 is divisible, torsion free and its torsion free rank is n. Consequently, H_2 is a vector space over $\mathbb{Q}$ of dimension n. Thus H_2 is isomorphic to $\mathbb{Q}^n$, which completes the proof. $\square$

Lemma 6. *If spectral synthesis holds on the discrete Abelian group G, then the same is true for every subgroup of G.*

Proof. Let H be a subgroup of G, and let $V \subset C(H)$ be a variety on H. Let T_x denote the operator of translation by x. Then $W = \{f \in \mathbb{C}^G : (T_x f)|_H \in V \text{ for every } x \in G\}$ is a variety on G. We claim that every function $f \in V$ can be extended to G as an element of W. Indeed, let U be a subset of G containing exactly one element of each coset of H; we may assume that $U \cap H = \{0\}$. If $f \in V$, then the function $g : G \to \mathbb{C}$ defined by $g(u + x) = f(x)$ $(u \in U,\ x \in H)$ belongs to W.

By assumption, the set of all exponential polynomials contained by W is dense in W. Now, it is easy to check that the restrictions of these exponential monomials to H constitute a dense subset of V, which proves that spectral synthesis holds on H. $\square$

4 Modules of polynomials.

Let $\mathcal{D}$ denote the ring of partial differential operators of n variables. Since the partial derivatives of a polynomial are also polynomials, the ring $\mathbb{C}[x_1, \dots, x_n]$ is a $\mathcal{D}$-module. The submodules of $\mathbb{C}[x_1, \dots, x_n]$ are those linear subspaces of $\mathbb{C}[x_1, \dots, x_n]$ which are closed under partial differentiation. These submodules can be described as follows.

Lemma 7. *Let M be a linear subspace of $\mathbb{C}[x_1, \dots, x_n]$. Then the following are equivalent.*

(i) *M is invariant under translations.*

(ii) *There exists a nonzero complex number s such that M is invariant under translations by vectors of $s \cdot \mathbb{Z}^n$.*

(iii) *M is a submodule of $\mathbb{C}[x_1, \dots, x_n]$.*

Proof. (i)$\Longrightarrow$(ii) is obvious. Suppose (ii), and let $p \in M$ be arbitrary. If the total degree of p is d, then Taylor's formula gives

$$\sum_{i=0}^{d} (ks)^i \cdot \frac{1}{i!} \frac{\partial^i}{\partial x_1^i} p(x_1, \dots, x_n) = p(x_1 + ks, x_2, \dots, x_n) \tag{5}$$

for every $k = 0, 1, \dots, d$. Now (5) is a system of linear equations with unknowns $\frac{1}{i!} \frac{\partial^i}{\partial x_1^i} p(x_1, \dots, x_n)$ $(i = 0, \dots, d)$. Since the determinant of this system is non-zero, it follows that the partial derivatives $\frac{\partial^i}{\partial x_1^i} p$ are linear combinations of the functions $p(x_1 + ks, x_2, \dots, x_n)$ $(k = 0, \dots, d)$. By assumption, these functions belong to M for every k, and thus $\frac{\partial^i}{\partial x_1^i} p \in M$ for every i. By the same argument, $\frac{\partial^i}{\partial x_j^i} p \in M$ for every $j = 1, \dots, n$ and $i \geq 0$; that is, M is a submodule of $\mathbb{C}[x_1, \dots, x_n]$. This proves (ii)$\Longrightarrow$(iii).

The implication (iii)$\Longrightarrow$(i) is an immediate consequence of Taylor's formula. $\square$

The following statement is probably known, but we were unable to find a reference. We are grateful to L. Rónyai for communicating the following proof.

Lemma 8. *The module $\mathbb{C}[x_1, \dots, x_n]$ has the minimal condition. In other words, if $M_1 \supset M_2 \supset \dots$ is a descending sequence of submodules of $\mathbb{C}[x_1, \dots, x_n]$, then there is an N such that $M_i = M_N$ for every $i \geq N$.*

Proof. We order the monomials $x_1^{i_1} \cdots x_n^{i_n}$ according to the lexicographic order of the n-tuples $(i_1, \dots, i_n)$. By the leading monomial of a nonzero polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ we mean the largest monomial that appears in p with a nonzero coefficient.

For every submodule $M \subset \mathbb{C}[x_1, \dots, x_n]$ we denote by $S(M)$ the set of leading monomials of the nonzero elements of M. It is easy to see, using the fact that M is closed under partial differentiation, that if m_1, m_2 are monomials, m_2 is a multiple of m_1 and $m_2 \in S(M)$, then $m_1 \in S(M)$.

We show that if $M_1 \supsetneq M_2$, then $S(M_1) \supsetneq S(M_2)$. It is clear that $S(M_1) \supset S(M_2)$. Let $p_0 \in M_1 \setminus M_2$ have a minimal leading monomial m_0 among all polynomials $p \in M_1 \setminus M_2$. Then $m_0 \notin S(M_2)$. Indeed, suppose that m_0 is a leading monomial of $q \in M_2$. Then $p_0 - c \cdot q \in M_1 \setminus M_2$ for every $c \in \mathbb{C}$, and for a suitable c the leading monomial of $p_0 - c \cdot q$ is smaller than m_0, which is impossible. This proves $m_0 \in S(M_1) \setminus S(M_2)$ and $S(M_1) \supsetneq S(M_2)$.

Now suppose that $M_1 \supsetneq M_2 \supsetneq \dots$ is a strictly decreasing sequence of submodules of $\mathbb{C}[x_1, \dots, x_n]$. Then we have $S(M_1) \supsetneq S(M_2) \supsetneq \dots$. Let $m_i \in S(M_i) \setminus S(M_{i+1})$ for every $i \geq 1$. It is not difficult to prove that in every sequence of monomials (m_i) there is an m_i which is a multiple of some m_j with $j < i$ (see [8, the Theorem on p. 147]). In our case, however, this would imply $m_j \in M_i$, which contradicts $m_j \notin M_{j+1} \supset M_i$. $\square$

5 Varieties on $\mathbb{Z}^n$.

Let n be a fixed positive integer. In this section we shall write $\mathcal{M}$ for $\mathcal{M}_c(\mathbb{Z}^n)$. If $\mu \in \mathcal{M}$ and $\mu = \sum a_{i_1 \dots i_n} \cdot \delta_{(i_1 \dots i_n)}$, then we put

$$q_\mu = \sum a_{i_1 \dots i_n} \cdot x_1^{i_1} \cdots x_n^{i_n}.$$

The map $\mu \mapsto q_\mu$ is a ring isomorphism between $\mathcal{M}$ and the ring R_n of functions of the form $x_1^{j_1} \cdots x_n^{j_n} \cdot q$, where $j_1, \dots, j_n \in \mathbb{Z}$ and $q \in \mathbb{C}[x_1, \dots, x_n]$. If $I \subset \mathcal{M}$, then we put $\widehat{I} = \{q_\mu : \mu \in I\}$. Obviously, if I is an ideal in $\mathcal{M}$, then $\widehat{I}$ is an ideal in R_n.

Because of the isomorphism of the rings $\mathcal{M}$ and R_n we could actually identify them. We shall not do so; however, if A is an ideal of R_n, then we shall write

$$A^\perp = \{\mu \in \mathcal{M} : q_\mu \in A\}^\perp.$$

Note that if A, B are ideals of R_n and $A \cap \mathbb{C}[x_1, \dots, x_n] = B \cap \mathbb{C}[x_1, \dots, x_n]$, then $A = B$. Indeed, for every $q \in A$ there is a monomial $p = x_1^{j_1} \cdots x_n^{j_n}$ such that $p \cdot q$ is a polynomial. Then $p \cdot q \in A \cap \mathbb{C}[x_1, \dots, x_n] = B \cap \mathbb{C}[x_1, \dots, x_n] \subset B$, and hence $q = (p \cdot q) \cdot x_1^{-j_1} \cdots x_n^{-j_n} \in B$. Thus $A \subset B$, and the same argument shows $B \subset A$.

Lemma 9. *For every positive integer n, the family of all varieties on $\mathbb{Z}^n$ satisfies the minimal condition. In other words, if $V_1 \supset V_2 \supset \dots$ is a descending sequence of varieties on $\mathbb{Z}^n$, then there is an N such that $V_i = V_N$ for every $i \geq N$.*

Proof. If $V_1 \supset V_2 \supset \dots$ is a descending sequence of varieties on $\mathbb{Z}^n$, then $V_1^\perp \subset V_1^\perp \subset \dots$ is an ascending sequence of ideals in $\mathcal{M}$. Let $I_i = (V_i^\perp)^\wedge \cap \mathbb{C}[x_1, \dots, x_n]$ for every $i = 1, 2, \dots$. Then $I_1 \subset I_2 \subset \dots$ is an ascending sequence of ideals in $\mathbb{C}[x_1, \dots, x_n]$ and thus, as $\mathbb{C}[x_1, \dots, x_n]$ is Noether, there exists an index N such that $I_i = I_N$ for every $i \geq N$. Then, as we remarked above, we have $(V_i)^\perp = (V_N)^\perp$ and $V_i = (V_N)^{\perp\perp} = (V_N)^{\perp\perp} = V_N$ for every $i \geq N$. $\square$

If $c = (c_1, \dots, c_n) \in (\mathbb{C} \setminus \{0\})^n$, then we shall denote by m_c the function

$$m_c(x_1, \dots, x_n) = c_1^{x_1} \cdots c_n^{x_n} \qquad (x_1, \dots, x_n \in \mathbb{Z}). \tag{6}$$

Clearly, m_c is an exponential on $\mathbb{Z}^n$. A simple calculation shows that

$$q_\mu(c_1, \dots, c_n) = \int_{\mathbb{Z}^n} m_c \, d\mu \tag{7}$$

for every $\mu \in \mathcal{M}$. Let V be a variety on $\mathbb{Z}^n$. Then it follows from (7) that $m_c \in V$ if and only if c is a root of the ideal $(V^\perp)^\wedge$.

In [7] Lefranc gives a condition for $p \cdot m \in V$ for every polynomial p and exponential m. The condition is formulated in terms of the partial derivatives of q_μ ($\mu \in V^\perp$). In order to make the formulas as simple as possible, we shall slightly change the notation used by Lefranc. We denote $x^{[0]} = 1$ and

$x^{[i]} = x \cdot (x-1) \cdots (x-i+1)$ for every $i = 1, 2, \ldots$. Every $p \in \mathbb{C}[x_1, \ldots, x_n]$ can be written uniquely in the form

$$p = \sum a_{i_1 \ldots i_n} \cdot x_1^{[i_1]} \cdots x_n^{[i_n]}.$$

Then we define

$$D_p = \sum a_{i_1 \ldots i_n} \cdot \frac{\partial^{i_1 + \ldots + i_n}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} \in \mathcal{D}.$$

Clearly, the map $p \mapsto D_p$ is a linear isomorphism from $\mathbb{C}[x_1, \ldots, x_n]$ onto $\mathcal{D}$. (Note, however, that $p \mapsto D_p$ is a not a ring isomorphism.) Now we prove that

$$\int_{\mathbb{Z}^n} p \, d\mu = D_p(q_\mu)(1, \ldots, 1) \tag{8}$$

for every $p \in \mathbb{C}[x_1, \ldots, x_n]$ and $\mu \in \mathcal{M}$. Since both sides of (8) are linear in both p and μ, it is enough to check the case when $p = x_1^{[i_1]} \cdots x_n^{[i_n]}$ and $\mu = \delta_{(j_1, \ldots, j_n)}$. Then both sides of (8) equals $j_1^{[i_1]} \cdots j_n^{[i_n]}$, which completes the proof. The next lemma is an immediate corollary of (8).

Lemma 10. *Let V be a variety on $\mathbb{Z}^n$, and let $p \in \mathbb{C}[x_1, \ldots, x_n]$. Then $p \in V$ if and only if $D_p(q_\mu)(1, \ldots, 1) = 0$ for every $\mu \in V^\perp$.* $\square$

Lemma 11. *Let V be a variety on $\mathbb{Z}^n$, and let $(V^\perp)^\wedge = A \cap B$, where A and B are ideals of R_n. Suppose that $p \in V$, where $p \in \mathbb{C}[x_1, \ldots, x_n]$. If $(1, \ldots, 1)$ is not a root of B, then $p \in A^\perp$.*

Proof. By the previous lemma, it is enough to show that $D_p(q_\mu)(1, \ldots, 1) = 0$ for every measure μ with $q_\mu \in A$. Let d denote the (total) degree of p.

Since $(1, \ldots, 1)$ is not a root of B, there is a function $t \in B$ such that $t(1, \ldots, 1) = 1$. If $s = 1 - t$, then $1 - s^{d+1} \in B$, as

$$1 - s^{d+1} = (1-s)(1 + s + \ldots + s^d) = t \cdot (1 + s + \ldots + s^d).$$

If $q_\mu \in A$, then we have

$$(1 - s^{d+1}) \cdot q_\mu \in B \cdot A \subset A \cap B = (V^\perp)^\wedge. \tag{9}$$

Let $\sigma \in \mathcal{M}$ be such that $q_\sigma = (1 - s^{d+1})$. Then $q_{\sigma * \mu} = (1 - s^{d+1}) q_\mu$, and thus $\sigma * \mu \in V^\perp$ by (9). Therefore, $D_p(q_{\sigma * \mu})(1, \ldots, 1) = 0$ by Lemma 10. From $q_{\sigma * \mu} = q_\mu - s^{d+1} \cdot q_\mu$ we obtain

$$D_p(q_\mu)(1, \ldots, 1) = D_p(s^{d+1} \cdot q_\mu)(1, \ldots, 1). \tag{10}$$

Since $s(1,\dots,1) = 0$ and the degree of the partial differential operator D_p equals d, it follows that the right hand side of (10) is zero. Thus $D_p(q_\mu)(1,\dots,1) = 0$, which completes the proof. $\square$

Lemma 12. *Let V be a variety on $\mathbb{Z}^n$, and let H be a subgroup of $\mathbb{Z}^n$ of finite index. Then*

(i) $M = \{p \in \mathbb{C}[x_1,\dots,x_n] : p|_H \in V|_H\}$ *is a submodule of* $\mathbb{C}[x_1,\dots,x_n]$.

(ii) *If* $p \in \mathbb{C}[x_1,\dots,x_n]$ *and* $p|_H \in V|_H$, *then* $\int_{\mathbb{Z}^n} p\, d(\nu * \mu) = 0$ *whenever* $\nu \in \mathcal{M},\ \mu \in V^\perp$, *and* $\operatorname{supp}\mu \subset H$.

Proof. Since H is of finite index, there is a positive integer N such that $N \cdot \mathbb{Z}^n \subset H$. It is clear that M is a linear subspace of $\mathbb{C}[x_1,\dots,x_n]$, and that M is invariant under translations by elements of $N \cdot \mathbb{Z}^n \subset H$. By Lemma 7 we obtain (i).

In order to prove (ii) we may assume that $\nu = \delta_{(j_1\cdots j_n)}$. Then

$$\int_{\mathbb{Z}^n} p\, d(\nu * \mu) = \int_{\mathbb{Z}^n} p(x_1 + j_1,\dots,x_n + j_n)\, d\mu(x_1,\dots,x_n). \tag{11}$$

Since, by (i), the polynomial $p(x_1+j_1,\dots,x_n+j_n)$ belongs to $V|_H$, it follows from Theorem 3 that the value of the right hand side of (11) is zero. $\square$

For every ideal $A \subset R_n$ and $\alpha \in \mathbb{C} \setminus \{0\}$ we define

$$A(\alpha) = \{p \in R_n : p(\alpha x_1, x_2,\dots,x_n) \in A\};$$

it is an ideal of R_n. If I is an ideal of $\mathcal{M}$, then we shall write $I(\alpha) = \{\mu : q_\mu \in \widehat{I}(\alpha)\}$; then $I(\alpha)$ is also an ideal of $\mathcal{M}$.

Lemma 13. *Let $H = \{(x_1,\dots,x_n) \in \mathbb{Z}^n : N \mid x_1\}$, where $N > 1$. Let V be a variety on $\mathbb{Z}^n$, and let $p \in C[x_1,\dots,x_n]$. Then $p|_H \in V|_H$ if and only if*

$$p \in \left(I \cap I(\varepsilon) \cap \ldots \cap I(\varepsilon^{N-1})\right)^\perp, \tag{12}$$

where $I = V^\perp$ and $\varepsilon = e^{2\pi i/N}$.

Proof. We put $I \cap I(\varepsilon) \cap \ldots \cap I(\varepsilon^{N-1}) = J$. Suppose $p \in J^\perp$; we prove $p|_H \in V|_H$. By Theorem 3 it is enough to show that $\int_H p\, d\mu = 0$, whenever

$\mu \in I$ and $\operatorname{supp} \mu \subset H$. For such a μ the exponent of x_1 in each term of q_μ is divisible by N. Therefore, $q_\mu(\varepsilon^j x_1, x_2, \dots, x_n) = q_\mu(x_1, \dots, x_n)$ and thus $\mu \in I(\varepsilon^j)$ for every j. Then $\mu \in J$ and hence, by $p \in J^\perp$, we obtain $\int_H p\, d\mu = \int_{\mathbb{Z}^n} p\, d\mu = 0$. This proves the 'if' direction of the statement.

In order to prove the 'only if' direction, suppose $p|_H \in V|_H$, and let $\mu \in J$ be arbitrary. We have to show

$$\int_{\mathbb{Z}^n} p\, d\mu = 0. \tag{13}$$

Let $q = q_\mu$, then $q \in \widehat{I}(\varepsilon^j)$ for every $j = 0, \dots, N-1$. We denote by q_ℓ the sum of those terms of q in which the exponent of x_1 is congruent to ℓ mod N. Then $q = q_0 + \ldots + q_{N-1}$. The condition $q \in \widehat{I}(\varepsilon^j)$ gives $q(\varepsilon^j x_1, \dots, x_n) \in \widehat{I}$; that is,

$$q_0 + \varepsilon^j q_1 + \varepsilon^{2j} q_2 + \ldots + \varepsilon^{(N-1)j} q_{N-1} = t_j \in \widehat{I} \tag{14}$$

for every $j = 0, \dots, N-1$. The system of equations (14) has a nonzero determinant, since the numbers $1, \varepsilon, \dots, \varepsilon^{N-1}$ are different. Thus each q_ℓ is a linear combination of the functions t_j, hence $q_\ell \in \widehat{I}$ for every $\ell = 0, \dots, N-1$. Let $\mu_\ell \in I$ be such that $q_{\mu_\ell} = q_\ell$. Then $\mu = \mu_0 + \ldots + \mu_{N-1}$ and thus, in order to prove (13), it is enough to show that $\int_{\mathbb{Z}^n} p\, d\mu_\ell = 0$ for every ℓ. Since $q_\ell \in \widehat{I}$, we have $x_1^{-\ell} \cdot q_\ell \in \widehat{I}$, and the exponent of x_1 in each term of $x_1^{-\ell} \cdot q_\ell$ is divisible by N. If $\delta = \delta_{(-\ell, 0, \dots, 0)}$, then $\delta * \mu_\ell \in I$, and $\operatorname{supp}(\delta * \mu_\ell) \subset H$. Since $p \in V|_H$, we find, by (ii) of Lemma 12, that

$$\int_{\mathbb{Z}^n} p\, d\mu_\ell = \int_{\mathbb{Z}^n} p\, d\left(\delta_{(\ell, 0, \dots, 0)} * (\delta * \mu_\ell)\right) = 0,$$

which completes the proof. $\square$

Lemma 14. *Let $N > 1$ and $1 \le i \le n$ be integers, and let*

$$H = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : N \mid x_1, \dots, N \mid x_i\}.$$

Let V be a variety on $\mathbb{Z}^n$, and let $p \cdot m \in V|_H$, where $p \in C[x_1, \dots, x_n]$ and $m : H \to \mathbb{C}$ is an exponential. Then for every finite set $F \subset H$ and for every $\eta > 0$ there are polynomials $p_1, \dots, p_k$ and exponentials $m_1, \dots, m_k$ defined on $\mathbb{Z}^n$ such that $m_j|_H = m$ and $p_j \cdot m_j \in V$ for every $j = 1, \dots, k$, and

$$\left| p(x) - \sum_{j=1}^{k} p_j(x) \right| < \eta \tag{15}$$

for every $x \in F$.

Proof. We shall prove the statement by induction on i.

I. Let $i = 1$; then $H = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : N \mid x_1\}$. First we shall assume that $m \equiv 1$ on H. Then $p|_H \in V|_H$. In $\mathbb{C}[x_1, \ldots, x_n]$ every ideal is the intersection of primary ideals, and thus

$$(V^\perp)^\wedge \cap \mathbb{C}[x_1, \ldots, x_n] = Q_1 \cap \ldots \cap Q_s, \tag{16}$$

where $Q_1, \ldots, Q_s$ are primary ideals. We shall prove the lemma by induction on s. If $s = 0$, then $(V^\perp)^\wedge = R_n,\ V^\perp = \mathcal{M},\ V = \{0\},\ p = 0$, and thus we can choose $m_1 \equiv 1$ and $p_1 = 0$.

Suppose that $s > 0$, and the statement is true for $s - 1$. Let $\varepsilon = e^{2\pi i/N}$. First we assume that each ideal Q_j has a root $c_j = (\varepsilon^{t_j}, 1, \ldots, 1)$, where t_j is an integer. It is easy to check that c_j is also a root of $\left(V^\perp\right)^\wedge$, and thus the exponential m_{c_j} belongs to V for every $j = 1, \ldots, s$. Moreover, as Lefranc proved in [7], the set of functions

$$\left\{\sum_{j=1}^{s} p_j \cdot m_{c_j} : p_j \in \mathbb{C}[x_1, \ldots, x_n],\ p_j \cdot m_{c_j} \in V\ (j = 1, \ldots, s)\right\}$$

is everywhere dense in V. Since $m_{c_j}|_H = 1$ for every j, the statement of the lemma is true.

Next suppose that some of the ideals Q_j does not have roots of the form $(\varepsilon^{t_j}, 1, \ldots, 1)$. We may assume that Q_1 is such an ideal.

We put $V^\perp = I$ and $J = I \cap I(\varepsilon) \cap \ldots \cap I(\varepsilon^{N-1})$. Then, by Lemma 13, we have $p \in J^\perp$.

If A is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$, then we shall denote by A^* the ideal of R_n generated by A. Obviously,

$$A^* = \{x_1^{j_1} \cdots x_n^{j_n} \cdot q : j_1, \ldots, j_n \in \mathbb{Z},\ q \in A\}.$$

By (16) we have $\widehat{I} = (V^\perp)^\wedge = Q_1^* \cap \ldots \cap Q_s^*$. Let $K \subset \mathcal{M}$ be the ideal satisfying $\widehat{K} = Q_2^* \cap \ldots \cap Q_s^*$. Then

$$\widehat{J} = \left[Q_1^* \cap Q_1^*(\varepsilon) \cap \ldots \cap Q_1^*(\varepsilon^{N-1})\right] \cap \left[\widehat{K} \cap \widehat{K}(\varepsilon) \cap \ldots \cap \widehat{K}(\varepsilon^{N-1})\right]. \tag{17}$$

Since Q_1^* does not have roots of the form $(\varepsilon^j, 1, \ldots, 1)$, it follows that $(1, \ldots, 1)$ is not a root of any of the ideals $Q_1^*(\varepsilon^j)$ $(j = 0, \ldots, N-1)$. Let $r_j \in Q_1^*(\varepsilon^j)$

be such that $r_j(1,\ldots,1) \neq 0$. Then $r = r_0 \cdots r_{N-1} \in Q_1^* \cap Q_1^*(\varepsilon) \cap \ldots \cap Q_1^*(\varepsilon^{N-1})$ and $r(1,\ldots,1) \neq 0$. Therefore, $(1,\ldots,1)$ is not a root of the ideal $Q_1^* \cap Q_1^*(\varepsilon) \cap \ldots \cap Q_1^*(\varepsilon^{N-1})$.

As we saw above, $p \in J^\perp$. Then (17) and Lemma 11 imply

$$p \in \left[K \cap K(\varepsilon) \cap \ldots \cap K(\varepsilon^{N-1})\right]^\perp.$$

Therefore, by Lemma 13, $p|_H \in W|_H$, where $W = K^\perp$. Since $I \subset K$, we obtain $W = K^\perp \subset I^\perp = V$. Also,

$$\widehat{K} \cap \mathbb{C}[x_1,\ldots,x_n] = (Q_2^* \cap \ldots \cap Q_s^*) \cap \mathbb{C}[x_1,\ldots,x_n] = R_2 \cap \ldots \cap R_s,$$

where $R_i = Q_i^* \cap \mathbb{C}[x_1,\ldots,x_n]$ $(2 \leq i \leq s)$. It is easy to check that $R_2,\ldots,R_s$ are primary ideals. Then, by the induction hypothesis, there are polynomials $p_1,\ldots,p_k$ and exponentials $m_1,\ldots,m_k$ such that $p_j \cdot m_j \in W \subset V$ and $m_j|_H \equiv 1$ for every $j = 1,\ldots,k$, and (15) holds for every $x \in F$. This proves the case when $m \equiv 1$.

Now we consider the case when m is an arbitrary exponential on H. Let $c_1 \in \mathbb{C}$ be such that $c_1^N = m(N,0,\ldots,0)$, and let $c_j = m(0,\ldots,0,\underset{j}{1},0,\ldots,0)$ for every $j = 2,\ldots,n$. It is easy to see that the exponential m_c defined by (6) is an extension of m onto $\mathbb{Z}^n$. Let $V_1 = \{f/m_c : f \in V\}$. Then V_1 is a variety on $\mathbb{Z}^n$, and $p|_H \in V_1|_H$. Let the finite set $F \subset H$ and $\eta > 0$ be given. We put $\eta' = \eta/\max_{x\in F}|m_c(x)|$. As we proved above, there are polynomials $p_1,\ldots,p_k$ and exponentials $m_1,\ldots,m_k$ such that $p_j \cdot m_j \in V_1$ and $m_j|_H \equiv 1$ for every j, and

$$\left| p(x) - \sum_{j=1}^{k} p_j(x) \right| < \eta'$$

for every $x \in F$. Since $p_j \cdot (m_j m_c) \in V$ by $p_j \cdot m_j \in V_1$ for every j, the proof of the case $i = 1$ is complete.

II. Now let $1 \leq i < n$, and suppose that the statement of the lemma is true for i. We prove the statement for $i+1$. Let

$$A = \{(x_1,\ldots,x_n) \in \mathbb{Z}^n : N \mid x_1,\ldots,N \mid x_{i+1}\},$$

and let p, m, F, η be as in the lemma, with A in place of H. Let

$$B = \{(x_1,\ldots,x_n) \in \mathbb{Z}^n : N \mid x_1,\ldots,N \mid x_i\}.$$

Then A is a subgroup of B, and there is an isomorphism ϕ from $\mathbb{Z}^n$ onto B such that

$$\phi^{-1}(A) = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : N \mid x_1\}.$$

(We define $\phi(x_1, \ldots, x_n) = (Nx_2, \ldots, Nx_{i+1}, x_1, x_{i+2}, \ldots, x_n)$ for every $x_1, \ldots, x_n \in \mathbb{Z}$.) Therefore, the statement of the lemma remains true if we replace $\mathbb{Z}^n$ by B and H by A. Since $V|_B$ is a variety on B and $pm \in V|_A$, we obtain the polynomials $p_1, \ldots, p_k$ and the exponentials $m_1, \ldots, m_k$ defined on B such that $m_j|_A = m$ and $p_j \cdot m_j \in V|_B$ for every $j = 1, \ldots, k$, and

$$\left| p(x) - \sum_{j=1}^{k} p_j(x) \right| < \eta/2$$

for every $x \in F$. By the induction hypothesis, the statement of the lemma is true if we replace H by B. Thus, for every $j = 1, \ldots, k$ we obtain the polynomials $p_{j,\ell}$ and the exponentials $m_{j,\ell}$ defined on $\mathbb{Z}^n$ such that $p_{j,\ell} \cdot m_{j,\ell} \in V$ and $m_{j,\ell}|_A = m_j$ for every $\ell = 1, \ldots, s_j$, and

$$\left| p_j(x) - \sum_{\ell=1}^{s_j} p_{j,\ell}(x) \right| < \eta/(2k)$$

for every $x \in F$. It is clear that the polynomials $p_{j,\ell}$ and the exponentials $m_{j,\ell}$ $(j = 1, \ldots, k,\ \ell = 1, \ldots, s_j)$ satisfy the requirements. $\square$

6 Varieties on $\mathbb{Z}^n \times T$, where T is torsion.

Lemma 15. *Spectral synthesis holds on $\mathbb{Z}^n \times T$ for every positive integer n and for every discrete torsion group T.*

Proof. Let V be a variety on $\mathbb{Z}^n \times T$, and let $g \in V$ be arbitrary. We have to show that for every finite set $F \subset \mathbb{Z}^n \times T$ and for every $\varepsilon > 0$ there is an exponential polynomial $\phi \in V$ such that $|\phi(y) - g(y)| < \varepsilon$ for every $y \in F$. Let $|F| = k$, and let $F_1 \subset \mathbb{Z}^n$ and $F_2 \subset T$ be finite sets such that $F \subset F_1 \times F_2$.

We shall denote by $\mathcal{H}$ the family of all finite subgroups $H \subset T$ satisfying $F_2 \subset H$. Let $H \in \mathcal{H}$, and let $\gamma_1, \ldots, \gamma_s$ be the characters of H. By Lemma 4, there are functions $f_1, \ldots, f_s : \mathbb{Z}^n \to \mathbb{C}$ such that $f_i \otimes \gamma_i \in V|_{\mathbb{Z}^n \times H}$ for every $i = 1, \ldots, s$, and

$$g(x,t) = \sum_{i=1}^{s} f_i(x)\gamma_i(t) \tag{18}$$

for every $(x,t) \in F$. Now, the set of functions $\mathbb{C}^F = \{f : F \to \mathbb{C}\}$ is a linear space of dimension k. The functions $\phi_i = (f_i \otimes \gamma_i)|_F$ $(i = 1, \ldots, s)$ generate a linear subspace L of $\mathbb{C}^F$. Thus the dimension of L is at most k, and it is generated by at most k of the functions ϕ_i. We may assume that L is generated by the functions $\phi_1, \ldots, \phi_k$. Since, by (18), $g|_F \in L$, we have $g(y) = \sum_{i=1}^k c_i\phi_i(y)$ for every $y \in F$ with suitable constants c_i. Now we have $c_i\phi_i = (c_i f_i) \otimes \gamma_i \in V|_{\mathbb{Z}^n \times H}$ for every $i = 1, \ldots, k$. We have proved the following statement: for every $H \in \mathcal{H}$ there are characters $\gamma_1, \ldots, \gamma_k \in \widehat{H}$ and there are functions $f_1, \ldots, f_k : \mathbb{Z}^n \to \mathbb{C}$ such that $f_i \otimes \gamma_i \in V|_{\mathbb{Z}^n \times H}$ for every $i = 1, \ldots, k$, and

$$g(x,t) = \sum_{i=1}^{k} f_i(x)\gamma_i(t) \tag{19}$$

for every $(x,t) \in F$.

If $H \in \mathcal{H}$ and $\gamma \in \widehat{H}$, then we shall denote by $V(\gamma)$ the set of functions $f : \mathbb{Z}^n \to \mathbb{C}$ such that $f \otimes \gamma \in V|_{\mathbb{Z}^n \times H}$. It is easy to check that $V(\gamma)$ is a variety on $\mathbb{Z}^n$. Note that if $H \subset H'$, $\gamma \in \widehat{H}$, $\gamma' \in \widehat{H'}$ and γ' is an extension of γ, then $V(\gamma) \supset V(\gamma')$. We shall refer to this property by saying that the variety $V(\gamma)$ is a decreasing function of γ.

For every $H \in \mathcal{H}$ let Γ_H denote the set of k-tuples $(\gamma_1, \ldots, \gamma_k) \in (\widehat{H})^k$ such that, for suitable functions $f_i \in V(\gamma_i)$ $(i = 1, \ldots, k)$, (19) holds for every $(x,t) \in F$. As we proved above, $\Gamma_H \neq \emptyset$ for every $H \in \mathcal{H}$.

Our next aim is to prove that there are characters $\chi_1, \ldots, \chi_k \in \widehat{T}$ with the following property: for every $H \in \mathcal{H}$, we have $(\chi_1|_H, \ldots, \chi_k|_H) \in \Gamma_H$. This will be shown by a compactness argument.

If $\chi \in \widehat{T}$ and $t \in T$ is an element of order $d(t)$, then $\chi(t)$ is a root of unity of order $d(t)$. Consider the product $P = \prod_{t \in T} X_t$, where X_t is the set of roots of unity of order $d(t)$ equipped with the discrete topology for every $t \in T$. Then P is a compact topological space, and $\widehat{T} \subset P$. The product space P^k is also compact.

For every $H \in \mathcal{H}$, the set Γ_H is finite. Therefore, the set C_H of k-tuples $(p_1, \ldots, p_k) \in P^k$ such that $(p_1|_H, \ldots, p_k|_H) \in \Gamma_H$ is a closed, hence compact subset of P^k. We show that if $H_1, \ldots, H_m \in \mathcal{H}$, then $C_{H_1} \cap \ldots \cap C_{H_m} \neq \emptyset$. Indeed, if H denotes the group generated by $H_1 \cup \ldots \cup H_m$, then H is finite (since T is torsion), and thus $H \in \mathcal{H}$. Let $(\gamma_1, \ldots, \gamma_k) \in \Gamma_H$, and let $p_i \in P$ be an arbitrary extension of γ_i to T $(i = 1, \ldots, k)$. It is clear that $(p_1, \ldots, p_k)$ belongs to C_{H_j} for every $j = 1, \ldots, m$.

Therefore, by the compactness of P^k, we have $\bigcap_{H \in \mathcal{H}} C_H \neq \emptyset$. Let $(\chi_1, \ldots, \chi_k)$ be a common element of each C_H $(H \in \mathcal{H})$. Then $\chi_1, \ldots, \chi_k$ are characters of T. Indeed, let $t_1, t_1 \in T$ be arbitrary, and let $H \in \mathcal{H}$ be a finite subgroup containing t_1 and t_2. Since $\chi_i|_H$ is a character of H, we have $\chi_i(t_1 + t_2) = \chi_i(t_1) \cdot \chi_i(t_2)$, proving that χ_i is multiplicative. We also have $\chi_i(t) \in X_t$ for every $t \in T$, and thus χ_i does not vanish. Therefore, χ_i is a character for every $i = 1, \ldots, k$.

For every fixed $i = 1, \ldots, k$, consider the family of varieties $V(\chi_i|_H)$ $(H \in \mathcal{H})$. Since the varieties on $\mathbb{Z}^n$ satisfy the minimal condition, there is a finite group $B_i \in \mathcal{H}$ such that $V(\chi_i|_{B_i})$ is minimal among these varieties. Now, as the variety $V(\gamma)$ is a decreasing function of γ, it follows that, whenever $H \in \mathcal{H}$ and $B_i \subset H$ then $V(\chi_i|_H) = V(\chi_i|_{B_i})$. Let B denote the group generated by $B_1, \ldots, B_k$; then $B \in \mathcal{H}$. Then, for every $H \in \mathcal{H}$ with $B \subset H$ we have $V(\chi_i|_H) = V(\chi_i|_B)$ for every $i = 1, \ldots, k$.

We selected $\chi_1, \ldots, \chi_k$ in such a way that $(\chi_1|_B, \ldots, \chi_k|_B) \in \Gamma_B$ holds; that is, there are functions $f_i \in V(\chi_i|_B)$ such that

$$g(x,t) = \sum_{i=1}^{k} f_i(x)\chi_i(t) \tag{20}$$

for every $(x,t) \in F$. As we proved above, we have $V(\chi_i|_B) = V(\chi_i|_H)$ for every finite group $H \supset B$, and thus $f_i \otimes (\chi_i|_H) \in V|_{\mathbb{Z}^n \times H}$ for every such H. In other words, if $H \in \mathcal{H}$ and $B \subset H$, then there is a function $g_H \in V$ such that $f_i(x)\chi_i(t) = g_H(x,t)$ $((x,t) \in \mathbb{Z}^n \times H)$. In particular, for every finite subset of $\mathbb{Z}^n \times T$, the function $f_i \otimes \chi_i$ equals the restriction of an element of V to the given finite set. Since V is closed, this means that $f_i \otimes \chi_i \in V$.

We have proved the existence of functions f_i and characters $\chi_i \in \widehat{T}$ such that $f_i \otimes \chi_i \in V$ for every $i = 1, \ldots, k$, and (20) holds for every $(x,t) \in F$. Let W_i denote the set of functions $f : \mathbb{Z}^n \to \mathbb{C}$ such that $f \otimes \chi_i \in V$. Then W_i is a variety on $\mathbb{Z}^n$ containing f_i. By Lefranc's theorem [7], spectral synthesis holds on $\mathbb{Z}^n$, and thus we can find, for every given $\varepsilon > 0$, an exponential polynomial $\phi_i \in W_i$ such that $|f_i(x) - \phi_i(x)| < \varepsilon/k$ for every $x \in F_1$. It is clear that $\phi_i \otimes \chi_i$ is an exponential polynomial on $\mathbb{Z}^n \times T$ and we also have $\phi_i \otimes \chi_i \in V$ by $\phi_i \in W_i$. Then $\phi = \sum_{i=1}^k (\phi_i \otimes \chi_i)$ is an exponential polynomial belonging to V such that $|\phi(y) - g(y)| < \varepsilon$ for every $y \in F$. This completes the proof. $\square$

Remark 16. Lemma 15 actually gives the following. *If V is a variety on $\mathbb{Z}^n \times T$, then V is spanned by all functions in V having the form $(pm) \otimes \chi$, where p is a polynomial, m is an exponential on $\mathbb{Z}^n$ and χ is a character of T.*

Lemma 17. *Let n, N be positive integers, and let*

$$H = N \cdot \mathbb{Z}^n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : N \mid x_i \ (i = 1, \ldots, n)\}.$$

Let T be a discrete torsion group, and let V be a variety on $\mathbb{Z}^n \times T$. Suppose that $(p \cdot m) \otimes \chi \in V|_{H \times T}$, where $p \in C[x_1, \ldots, x_n]$, $m : H \to \mathbb{C}$ is an exponential, and $\chi \in \widehat{T}$. Then for every finite set $F \subset H \times T$ with $|F| = k$ and for every $\eta > 0$ there are polynomials $p_1, \ldots, p_k$ and exponentials $m_1, \ldots, m_k$ defined on $\mathbb{Z}^n$ such that $m_j|_H = m$ and $(p_j \cdot m_j) \otimes \chi \in V$ for every $j = 1, \ldots, k$, and

$$\left| p(x) - \sum_{j=1}^{k} p_j(x) \right| < \eta \tag{21}$$

for every $(x, t) \in F$.

Proof. First we assume that T is finite. Let $F_1 \subset H$ be a finite set such that $F \subset F_1 \times T$ and $|F_1| \leq k$. Let $\chi_1, \ldots, \chi_s$ be the characters of T listed in such a way that $\chi = \chi_1$. Since $(p \cdot m) \otimes \chi \in V|_{H \times T}$, there exists a $g \in V$ such that $(p \cdot m) \otimes \chi = g|_{H \times T}$. By Lemma 4, there are functions $f_1, \ldots, f_s : \mathbb{Z}^n \to \mathbb{C}$ such that $f_i \otimes \chi_i \in V$ for every $i = 1, \ldots, s$, and

$$g(x, t) = \sum_{i=1}^{s} f_i(x) \chi_i(t) \tag{22}$$

for every $(x, t) \in \mathbb{Z}^n \times T$. If $x \in H$, then (22) gives

$$p(x) m(x) \chi(t) = \sum_{i=1}^{s} f_i(x) \chi_i(t) \tag{23}$$

for every $t \in T$. Since the characters χ_i are linearly independent and $\chi = \chi_1$, we find that $p(x)m(x)\chi(t) = f_1(x)\chi(t)$ for every $x \in H$. Therefore, we have $pm = f_1|_H$.

Let $V(\chi)$ denote the set of functions $f : \mathbb{Z}^n \to \mathbb{C}$ such that $f \otimes \chi \in V$. Then $V(\chi)$ is a variety on $\mathbb{Z}^n$. Since $f_1 \in V(\chi)$, we have $pm \in V(\chi)|_H$. Thus, we may apply Lemma 14, and obtain the polynomials $p_1, \dots, p_r$ and the exponentials $m_1, \dots, m_r$ defined on $\mathbb{Z}^n$ such that $p_i m_i \in V(\chi)$ and $m_i|_H = m$ for every $i = 1, \dots, r$, and

$$\left| p(x) - \sum_{i=1}^{r} p_i(x) \right| < \eta \tag{24}$$

for every $x \in F_1$. By $p_i m_i \in V(\chi)$ we have $(p_i m_i) \otimes \chi \in V$ for every i. Since $F \subset F_1 \times T$, this proves the statement of the lemma, apart from the fact that r can be larger than $k = |F|$. In order to reduce the number r we may apply the argument used in the previous proof. The set of functions $\mathbb{C}^{F_1} = \{f : F_1 \to \mathbb{C}\}$ is a linear space of dimension $|F_1| \le k$. The functions $\psi_i = (p_i m_i)|_{F_1}$ $(i = 1, \dots, r)$ generate a linear subspace L of $\mathbb{C}^{F_1}$. Thus the dimension of L is at most k, and it is generated by at most k of the functions ψ_i. We may assume that L is generated by the functions $\psi_1, \dots, \psi_k$. Since

$$S = \left(\sum_{i=1}^{r} p_i m_i \right) \Bigg|_{F_1} \in L,$$

we have $S(x) = \sum_{i=1}^{k} c_i \psi_i(x)$ for every $x \in F_1$ with suitable constants c_i. Now we have $c_i \psi_i = (c_i p_i) \cdot m_i \in V$ for every $i = 1, \dots, k$, which completes the proof in the case when T is finite.

Now let T be an arbitrary torsion group. Let $F_1 \subset H$ and $F_2 \subset T$ be finite sets such that $F \subset F_1 \times F_2$. We shall denote by $\mathcal{H}$ the family of all finite subgroups $G \subset T$ satisfying $F_2 \subset G$. If $G \in \mathcal{H}$ and $\gamma \in \widehat{G}$, then we shall denote by $V(\gamma)$ the set of functions $f : \mathbb{Z}^n \to \mathbb{C}$ such that $f \otimes \gamma \in V|_{\mathbb{Z}^n \times G}$. It is easy to check that $V(\gamma)$ is a variety on $\mathbb{Z}^n$. Note that if $G \subset G'$, $\gamma \in \widehat{G}$, $\gamma' \in \widehat{G'}$ and γ' is an extension of γ, then $V(\gamma) \supset V(\gamma')$. We shall refer to this property by saying that the variety $V(\gamma)$ is a decreasing function of γ.

For every $G \in \mathcal{H}$ we define two sets of k-tuples as follows. D_G will denote the set of k-tuples $(p_1 m_1, \dots, p_k m_k)$, where $p_i \in \mathbb{C}[x_1, \dots, x_n]$, m_i is an exponential defined on $\mathbb{Z}^n$, $m_i|_H = m$ and $(p_i m_i) \otimes (\chi|_G) \in V|_{\mathbb{Z}^n \times G}$ for every $i = 1, \ldots, k$, and (21) holds for every $(x, t) \in F$. Also, we shall denote by E_G the set of those k-tuples $(m_1, \ldots, m_k)$ for which $(p_1 m_1, \ldots, p_k m_k) \in D_G$ for some polynomials $p_1, \ldots, p_k$. As we proved above, D_G and E_G are nonempty for every $G \in \mathcal{H}$. It is clear that if $G, G' \in \mathcal{H}$ and $G \subset G'$ then $D_G \supset D_{G'}$ and $E_G \supset E_{G'}$. Therefore, the intersections $D_{G_1} \cap \ldots \cap D_{G_s}$ and $E_{G_1} \cap \ldots \cap E_{G_s}$ are nonempty for any $G_1, \ldots, G_s \in \mathcal{H}$.

We prove that $\bigcap\{D_G : G \in \mathcal{H}\} \neq \emptyset$. Since $H = N \cdot \mathbb{Z}^n$, it follows that m only has**has only** a finite number of extensions to $\mathbb{Z}^n$ as an exponential. Indeed, if m' is an exponential on $\mathbb{Z}^n$ with $m'|_H = m$ then for every $x \in \mathbb{Z}^n$ the equation $m'(x)^N = m'(Nx) = m(Nx)$ implies that the number of possible values of $m'(x)$ is at most N. Since $\mathbb{Z}^n$ is generated by n elements, it follows that the number of possible exponentials defined on $\mathbb{Z}^n$ is at most N^n. Therefore, the set $\bigcup\{E_G : G \in \mathcal{H}\}$ is finite. Then $\bigcap\{E_G : G \in \mathcal{H}\} = E_{G_1} \cap \ldots \cap E_{G_s}$ for some $G_1, \ldots, G_s \in \mathcal{H}$, and thus $\bigcap\{E_G : G \in \mathcal{H}\} \neq \emptyset$.

Let $(m_1, \ldots, m_k) \in \bigcap\{E_G : G \in \mathcal{H}\}$ be fixed. For every $G \in \mathcal{H}$ we shall denote by $M_i(G)$ the set of polynomials $p \in \mathbb{C}[x_1, \ldots, x_n]$ such that $(p \cdot m_i) \otimes (\chi|_G) \in V|_{\mathbb{Z}^n \times G}$. Clearly, $M_i(G)$ is a linear subspace of $\mathbb{C}[x_1, \ldots, x_n]$. We show that $M_i(G)$ is invariant under translations by vectors of $\mathbb{Z}^n$. Indeed, let $p \in M_i(G)$ and $a \in \mathbb{Z}^n$ be arbitrary, and let $f \in V$ be such that

$$f|_{\mathbb{Z}^n \times G} = (p \cdot m_i) \otimes (\chi|_G).$$

Since V is translation invariant, the function $(x, t) \mapsto f(x + a, t)$ belongs to V. Thus

$$p(x + a) \cdot m_i(x)\chi(t) = \frac{1}{m_i(a)} \cdot p(x + a) \cdot m_i(x + a)\chi(t) = \frac{1}{m_i(a)} \cdot f(x + a, t)$$

for every $(x, t) \in \mathbb{Z}^n \times G$, which proves that the polynomial $x \mapsto p(x + a)$ belongs to $M_i(G)$.

Therefore, by Lemma 7, $M_i(G)$ is a submodule of $\mathbb{C}[x_1, \ldots, x_n]$. It is easy to see that $M_i(G)$ is a decreasing function of G.

For every fixed $i = 1, \ldots, k$, consider the family of modules $M_i(G)$ ($G \in \mathcal{H}$). Since the submodules of $\mathbb{C}[x_1, \ldots, x_n]$ satisfy the minimal condition, there is a finite group $B_i \in \mathcal{H}$ such that $M_i(B_i)$ is minimal among these modules. Now, as the module $M_i(G)$ is a decreasing function of G, it follows that, whenever $G \in \mathcal{H}$ and $B_i \subset G$ then $M_i(G) = M_i(B_i)$ ($i = 1, \ldots, k$).

Let B denote the group generated by $B_1, \dots, B_k$; then $B \in \mathcal{H}$, and for every $G \in \mathcal{H}$ with $B \subset G$ we have $M_i(G) = M_i(B)$ for every $i = 1, \dots, k$.

Since $(m_1, \dots, m_k) \in E_B$, there are polynomials $p_1, \dots, p_k$ such that $(p_1 m_1, \dots, p_k m_k) \in D_B$. Then $(p_i m_i) \otimes (\chi|_B) \in V|_{\mathbb{Z}^n \times B}$; that is, $p_i \in M_i(B)$ for every $i = 1, \dots, k$. Now we prove that

$$(p_1 m_1, \dots, p_k m_k) \in D_G \tag{25}$$

for every $G \in \mathcal{H}$. Let $G \in \mathcal{H}$ be arbitrary; we may assume that $B \subset G$. Then $M_i(B) = M_i(G)$, and thus $p_i \in M_i(G)$ $(i = 1, \dots, k)$, which implies (25). Therefore, $(p_i m_i) \otimes (\chi|_G) \in V|_{\mathbb{Z}^n \times G}$ for every $G \in \mathcal{H}$. In other words, if $G \in \mathcal{H}$ and $B \subset G$, then there is a function $g_G \in V$ such that $p_i(x) m_i(x) \chi(t) = g_G(x, t)$ $((x, t) \in \mathbb{Z}^n \times G)$. In particular, for every finite subset of $\mathbb{Z}^n \times T$, the function $p_i(x) m_i(x) \chi(t)$ equals the restriction of an element of V to the given finite set. Since V is closed, this means that $(p_i m_i) \otimes \chi \in V$. This completes the proof of the lemma. $\square$

7 Spectral synthesis on $\mathbb{Q}^n \times T$, where T is torsion.

Let T be a discrete torsion group. In this section we complete the proof of Theorem 1 by showing that spectral synthesis holds on $\mathbb{Q}^n \times T$ for every positive integer n. Let

$$G_i = \left\{ \left(\frac{k_1}{i!}, \dots, \frac{k_n}{i!} \right) : k_1, \dots, k_n \in \mathbb{Z} \right\}$$

for every $i = 1, 2, \dots$. Then $G_1, G_2, \dots$ is an increasing sequence of subgroups of $\mathbb{Q}^n$ such that $\bigcup_{i=1}^{\infty} G_i = \mathbb{Q}^n$. In addition, for every $i < j$ there exists a isomorphism ϕ from G_j onto $\mathbb{Z}^n$ and there is an integer $N > 1$ such that

$$\phi(G_i) = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : N \mid x_1, \dots, N \mid x_n\}.$$

This implies that the statement of Lemma 17 remains valid if we replace $\mathbb{Z}^n$ by G_j and H by G_i.

Now let V be a variety on $\mathbb{Q} \times T$, and let $g \in V$ be arbitrary. We have to show that for every finite set $F \subset \mathbb{Q}^n \times T$ and for every $\varepsilon > 0$ there is a exponential polynomial $\phi \in V$ such that $|\phi(y) - g(y)| < \varepsilon$ for every $y \in F$.

Let $|F| = k$, and let i_0 be such that $F \subset G_{i_0} \times T$. By Theorem 3, $V|_{G_{i_0} \times T}$ is a variety on G_{i_0}. Since G_{i_0} is isomorphic to $\mathbb{Z}^n$, it follows from Lemma 15 that spectral synthesis holds in $V|_{G_{i_0} \times T}$. In other words, we can find a exponential polynomial $\phi_0 \in V|_{G_{i_0} \times T}$ such that $|g(y) - \phi_0(y)| < \varepsilon/2$ for every $y \in F$. Moreover, by Remark 16, we may assume that $\phi_0(x,t) = \sum_{i=1}^{s} (p_i \cdot m_i) \otimes \chi_i$, where $p_i \in \mathbb{C}[x_1, \ldots, x_n]$, m_i is an exponential defined on G_{i_0}, $\chi \in \widehat{T}$, and $(p_i \cdot m_i) \otimes \chi_i \in V|_{G_{i_0} \times T}$ for every $i = 1, \ldots, s$. Clearly, it is enough to show that for every i there is an exponential polynomial $\phi_i \in V$ such that

$$|(p_i \cdot m_i)(x)\chi_i(t) - \phi_i(x,t)| < \varepsilon/(2s)$$

for every $(x,t) \in F$. Summing up: we have to show that whenever

- p is a polynomial,
- m is an exponential on G_{i_0},
- χ is a character of T,
- $(pm) \otimes \chi \in V|_{G_{i_0} \times T}$,
- $F \subset G_{i_0} \times T$ is a finite set, and
- $\varepsilon > 0$,

then there is an exponential polynomial $\psi \in V$ such that $|(p \cdot m)(x)\chi(t) - \psi(x,t)| < \varepsilon$ for every $(x,t) \in F$. In the course of the following proof p, m, χ, F and ε will be fixed. Let k denote the cardinality of F. We put $\eta = \varepsilon / \max_{(x,t) \in F} |m(x)|$.

Let $i > i_0$ be given. As we remarked above, the statement of Lemma 17 remains valid if we replace $\mathbb{Z}^n$ by G_i and H by G_{i_0}. Since $V|_{G_i \times T}$ is a variety and $(pm) \otimes \chi \in V|_{G_{i_0} \times T}$, we obtain the polynomials $p_1, \ldots, p_k$ and the exponentials $m_1, \ldots, m_k$ defined on G_i such that $m_j|_{G_0} = m$ and $(p_j m_j) \otimes \chi \in V|_{G_i \times T}$ for every $j = 1, \ldots, k$, and

$$\left| p(x) - \sum_{j=1}^{k} p_j(x) \right| < \eta \tag{26}$$

for every $(x,t) \in F$. Let Ξ denote the set of $k+1$-tuples $(i, m_1, \ldots, m_k)$ with the following properties: (i) $i \geq i_0$; (ii) $m_1, \ldots, m_k$ are exponentials defined on G_i such that $m_j|_{G_{i_0}} = m$ for every $j = 1, \dots, k$; (iii) there are polynomials $p_1, \dots, p_k$ such that $(p_j m_j) \otimes \chi \in V|_{G_i \times T}$ for every $j = 1, \dots, k$, and (26) holds for every $x \in F$. As we proved, Ξ contains $k+1$-tuples $(i, m_1, \dots, m_k)$ for every $i \geq i_0$.

We define a partial order on Ξ as follows: if $(i, m_1, \dots, m_k)$ and $(i', m'_1, \dots, m'_k)$ are elements of Ξ then we write

$$(i, m_1, \dots, m_k) < (i', m'_1, \dots, m'_k)$$

if $i < i'$ and $m'_j|_{G_i} = m_j$ for every $j = 1, \dots, k$. It is clear that this partial order makes Ξ a tree. Every level of Ξ is finite. Indeed, if i is fixed then there is an integer N such that $N \cdot G_i = G_{i_0}$. This implies that m only has a finite number of extensions to G_i as an exponential, and thus the number of $k+1$-tuples $(i, m_1, \dots, m_k) \in \Xi$ is finite.

By König's lemma [6, 5.7. Lemma, p. 69], Ξ has an infinite chain. That is, there is a sequence of $k+1$-tuples $(i, m_1^i, \dots, m_k^k) \in \Xi$ $(i \geq i_0)$ such that

$$m_j^{i+1}|_{G_i} = m_j^i \tag{27}$$

for every $i \geq i_0$ and $j = 1, \dots, k$.

For every $i \geq i_0$ and $j = 1, \dots, k$ we shall denote by M_j^i the set of polynomials $p \in \mathbb{C}[x_1, \dots, x_n]$ such that $(p \cdot m_j^i) \otimes \chi \in V|_{G_i \times T}$. It is obvious that M_j^i is a linear subspace of $\mathbb{C}[x_1, \dots, x_n]$. It is easy to see that M_j^i is invariant under translations by elements of G_i. Therefore, by Lemma 7, M_j^i is a submodule of $\mathbb{C}[x_1, \dots, x_n]$ for every i and j.

For every i and j we have $M_j^i \supset M_j^{i+1}$. Indeed, if $p \in M_j^{i+1}$, then $(p \cdot m_j^{i+1}) \otimes \chi \in V|_{G_{i+1} \times T}$. By (27), this implies $(p \cdot m_j^i) \otimes \chi \in V|_{G_i \times T}$; that is, $p \in M_j^i$. Now it follows from Lemma 8, that for every j there is an index $i(j)$ such that $M_j^i = M_j^{i(j)}$ for every $i \geq i(j)$. Let $i_1 = \max\{i_0, i(1), \dots, i(k)\}$. Then we have $M_j^i = M_j^{i_1}$ for every $i \geq i_1$.

By the definition of Ξ we can find polynomials $p_j \in M_j^{i_1}$ $(j = 1, \dots, k)$ such that (26) holds for every $x \in F$. If $i > i_1$ then $M_j^i = M_j^{i_1}$ for every j, and hence $(p_j \cdot m_j^i) \otimes \chi \in V|_{G_i \times T}$ for every $i > i_1$ and $j = 1, \dots, k$.

It follows from (27) that for every $j = 1, \dots, k$ there is a function $m_j : \mathbb{Q}^n \to \mathbb{C}$ such that $m_j|_{G_i} = m_j^i$ for every $i \geq i_1$. It is clear that m_j is an exponential on $\mathbb{Q}^n$. By $(p_j m_j^i) \otimes \chi \in V|_{G_i \times T}$ we can find a function $g_j^i \in V$

such that $g_j^i|_{G_i} = (p_j m_j^i) \otimes \chi$. It is easy to see that the sequence $(g_j^i)_{i \geq i_1}$ converges pointwise to the function $(p_j m_j) \otimes \chi$, and thus $(p_j m_j) \otimes \chi \in V$.

Now $\psi = \sum (p_j m_j) \otimes \chi$ is an exponential polynomial belonging to V. Taking into consideration that m_j is an extension of $m_j^{i_1}$ and that $F \subset G_{i_0} \times T \subset G_{i_1} \times T$, it follows from (26) that $|(p \cdot m)(x)\chi(t) - \psi(x,t)| < \varepsilon$ for every $x \in F$. This completes the proof. $\square$

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Department of Analysis
Eötvös Loránd University
Budapest, Pázmány Péter sétány 1/C
1117 Hungary

and

Department of Mathematics
University College London
Gower Street, London, WC1E 6BT
England.

e-mail: `laczk@cs.elte.hu`

Institute of Mathematics
University of Debrecen
Debrecen, P.O. Box 12
4010 Hungary

e-mail: `szekely@math.klte.hu`