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# Extensions of Taylor's theorem and norm estimations of linear functionals

Thesis for the degree
of Doctor of Philosophy (PhD)

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Doctoral Council of Natural Sciences and Information Technology
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Debrecen, 2024

Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences of the University of Debrecen in order to obtain a PhD Degree in Natural Sciences from the University of Debrecen.
I declare that the results published in this thesis are not reported in any other PhD theses.

Debrecen, April 15, 2024.

Ali Hasan Ali
signature of the candidate

Hereby I confirm that Ali Hasan Ali candidate conducted his studies with my supervision within the Mathematical Analysis Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen between 2020 and 2024. The independent studies and research work of the candidate significantly contributed to the results published in this thesis.
I also declare that the results published in the thesis are not reported in any other PhD theses.
I support the acceptance of the dissertation.
Debrecen, April 15, 2024.

Páles Zsolt
signature of the supervisor

# Extensions of Taylor's theorem and norm estimations of linear functionals

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Written by: Ali Hasan Ali

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Dissertation advisor: Dr. Zsolt Páles

**The official opponents of the dissertation:**

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# Contents

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## Notation and Symbols

| | |
|---|---|
| $\mathbb{N}$ | the set of natural numbers |
| $\mathbb{N}_0$ | the set of nonnegative integer |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}_+$ | the set of positive real numbers |
| $\Re$ | the real part of a complex number |
| $I$ | a non-degenerate real interval |
| $\mathbb{C}$ | the set of complex numbers |
| $\boldsymbol{i}$ | the imaginary unit in $\mathbb{C}$ |
| $\mathbb{K}$ | denotes either the field of real or complex numbers |
| $\mathcal{C}_{\mathbb{K}}(I)$ | the space of continuous $\mathbb{K}$-valued functions defined on $I$ |
| $\mathcal{C}^n_{\mathbb{K}}(I)$ | the space of $n$-times continuously differentiable $\mathbb{K}$-valued functions defined on $I$ |
| $D_c$ | the linear differential operator |
| $P_c$ | the characteristic polynomial of $D_c$ |
| $\omega_c$ | the characteristic solution of the equation $D_c(\omega) = 0$ |
| $\tau_n$ | the unique fixed point of the tangent function in the interval $\big((n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi\big)$ |
| $T_{n;a}(f)$ | the $n$th-order Taylor polynomial of the function $f$ at the base point $a$ |
| $T_{a,c}f$ | the generalized Taylor polynomial at the point $a$ with respect to the differential operator $D_c$ |
| $\sigma_n(D)$ | the set $\{(\lambda_1, \dots, \lambda_n) \in D^n \mid \lambda_i \neq \lambda_j \text{ for all } i, j \in \{1, \dots, n\} \text{ with } i \neq j\}$ |
| $\partial_\ell$ | the differentiation with respect to the variable $\lambda_\ell$, where $\ell \in \{1, \dots, k\}$ |
| $E_t(\lambda)$ | $\exp(\lambda t)$ |
| $\delta_{i,j}$ | the Kronecker delta |
| $\zeta_{n,k,\gamma}$ | the unique solution of the IVP $\zeta^{(n+1)} = \gamma\zeta^{(k+1)}$, $\zeta^{(i)}(0) = \delta_{i,n} \quad (i \in \{0, \dots, n\})$ |
| $\rho^+(h)$ | the infimum of the positive roots of the continuous function $h$ in $[0, +\infty]$ |
| $\rho^-(h)$ | the supremum of the negative roots of the continuous function $h$ in $[-\infty, 0]$ |
| $U_k$ | the $k$th degree Chebyshev polynomial of the second kind |

| | |
|---|---|
| $X, Y, Z$ | normed spaces over the field $\mathbb{K}$ |
| $A$, $B$, $C$ | linear maps |
| $\ker(\cdot)$ | the null space of the corresponding operator |
| $\mathcal{B}_X$ | the closed unit ball of the spaces $X$ |
| $\mu$ | a nonzero bounded $\mathbb{C}$-valued Borel measure on $[a, b]$ |
| $\mathcal{A}_\mu$ | the linear functional related to the measure $\mu$ |
| $\mathcal{S}_\mu$ | the spectral function related to the measure $\mu$ |
| $\Lambda_\mu$ | the spectral set (the set of zeros of $\mathcal{S}_\mu$) related to the measure $\mu$ |
| $m(\mathcal{S}_\mu, \lambda)$ | the multiplicity of an element $\lambda \in \Lambda_\mu$ |
| $x \mapsto x^j e^{\lambda_i x}$ | the exponential polynomials where $(i \in \{1, \ldots, k\}$, $j \in \{0, \ldots, m_i - 1\})$ |
| $\chi_S$ | the characteristic function of any subset $S$ of $[a, b]$ |
| $R_T(f)$ | the remainder term of the classical trapezoidal rule |
| $\delta_t$ | the Dirac measure concentrated at $t$ |
| $\nu$ | the normalized Lebesgue measure on $[a, b]$ |

CHAPTER 1

# Introduction and preliminaries

## 1.1. Historical perspective

Taylor's theorem provides an approximation for a function that is differentiable up to a certain order around a specified point, using a polynomial of a corresponding degree known as the Taylor polynomial. This polynomial truncates the Taylor series of the function at the specified order. The first-order Taylor polynomial yields a linear approximation, while the second-order Taylor polynomial is commonly referred to as a quadratic approximation. Various versions of Taylor's theorem exist, offering explicit estimates for the error in the approximation. Named after mathematician Brook Taylor, who introduced a version of it in 1715 [26], Taylor's theorem was foreshadowed by James Gregory in 1671 [14]. It serves as a fundamental concept taught in introductory calculus courses and serves as a cornerstone tool in mathematical analysis. It furnishes straightforward arithmetic formulas for accurately computing values of numerous transcendental functions such as exponential and trigonometric functions. Moreover, it marks the initiation of the investigation of analytic functions and holds significance across diverse mathematical domains, including numerical analysis and mathematical physics. Taylor's theorem also extends to multivariate and vector-valued functions.

When something is approximated, attention should be given to the quality of the approximation. The simplest form of the error, denoted by $E$, is defined as the absolute difference between the actual value and the approximation. If $T_n(x)$ represents the Taylor or Maclaurin approximation of degree $n$ for a function $f(x)$, then the error is expressed as

$$E = |f(x) - T_n(x)|.$$

In fact, there are several ways that can be used to assess the magnitude of the error, such as the Alternating Series error bound [29], and the Lagrange error bound [15]. In both methods, a value $B$ is obtained, ensuring that the function's approximation at $x = x_0$ within the convergence interval is within $B$ units of the exact value. This can be expressed as:

$$(f(x_0) - B) < T_n(x_0) < (f(x_0) + B),$$

or

$$T_n(x_0) \in (f(x_0) - B, f(x_0) + B).$$

## 1.2. Taylor's theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and suppose that $a \in \mathbb{R}$, then a *Taylor series* of the function $f(x)$ around the point $a$ is given by:

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\qquad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j. \end{aligned} \tag{1.1}$$

Specifically, when $a = 0$, the series expansion (1.1) is referred to as the *Maclaurin series*, and it can be expressed as follows:

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}x^j. \end{aligned}$$

To establish a requirement necessary for the existence of a Taylor series for a function, let us begin by defining the $n$th degree Taylor polynomial of $f(x)$ as follows:

$$\begin{aligned} T_{n;a}(f)(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\qquad + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j. \end{aligned} \tag{1.2}$$

The polynomial (1.2) is called the *$n$th-order Taylor polynomial of the function $f$ at the base point $a$.*

It is clear that the $n$th degree Taylor polynomial (1.2) is just the partial sum of the Taylor series (1.1). Therefore, we can define the *remainder* by the difference between the function $f(x)$ (Taylor series (1.1)) and the Taylor polynomial (1.2), and it is given by:

$$\begin{aligned} R_n(x) &= \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j - \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j \\ &= f(x) - T_{n;a}(f)(x). \end{aligned}$$

With this definition, it is worth noting that we can express the function as:

$$f(x) = T_{n;a}(f)(x) + R_n(x).$$

In fact, the remainder term $R_n(x)$ can be represented by various formulas, and here are some of them:

### 1.2.1. Formulas of the remainder term.

(1) The *integral form* of the remainder, which is given by:

$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt.$$

(2) The *Lagrange form* of the remainder, which is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1},$$

where the point $\xi$ is a number between $a$ and $x$.

(3) The *Cauchy form* of the remainder, which is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n)!}(x-\xi)^n(x-a).$$

(4) The *Schlömilch form* of the remainder, which is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^{n+1-p}\frac{(x-a)^p}{p},$$

which is also known as the *Schlömilch-Roché form* [7], and it is the general case of (2) and (3), where selecting $p = n+1$ represents the Lagrange form, while choosing $p = 1$ corresponds to the Cauchy form.

The first and the second remainders in the previous list will be intensively studied in the next chapter.

## 1.3. Numerical integration formulas

Integrals are fundamental in engineering disciplines as they serve to represent quantities derived from infinitesimal data. For instance, when tracking the motion of an object over a specific time interval, integrating the known velocity function allows us to approximate the change in position. This estimation involves summing the velocities at discrete time points, each multiplied by the corresponding time interval, and as we reduce the time intervals between these points, the precision of the position change estimation improves.

Formally, the integral of a function $f : [a, b] \to \mathbb{R}$ signifies the signed area beneath the curve $f(x)$ between the points $a$ and $b$. In other words, if the function $f(x)$ is positive within the interval $[a, b]$, then the area bounded by the $x$-axis and the graph of $f(x)$ over the same interval is given by the integral

$$\int_a^b f(x)\, dx.$$

However, calculating this integral often requires finding the antiderivative of $f(x)$, which can be challenging. For instance, the Gaussian function

$$f(x) = e^{-x^2},$$

commonly used to model normal distributions, lacks a straightforward antiderivative aside from

$$F(x) = \int_{-\infty}^{x} e^{-t^2} dt.$$

Nevertheless, we can still approximate this integral to any desired level of precision through numerical Integration techniques. Among the simplest methods are the Newton–Cotes formulas [19], alternatively known as the Newton–Cotes quadrature rules or simply Newton–Cotes rules, which comprise a set of numerical integration techniques, or quadrature methods, that involve evaluating the integrand at evenly distributed points. They derive their name from Isaac Newton and Roger Cotes. These formulas prove beneficial when the integrand's values at uniformly spaced intervals are known. However, if there is flexibility in selecting the evaluation points, other approaches like Gaussian quadrature [9] or Clenshaw–Curtis quadrature [27] may be better suited for the task.

Next, we present some of the Newton–Cotes formulas and some other numerical integration approaches.

**1.3.1. The rectangle method.** The rectangle method, operating within the framework $f : [a, b] \to \mathbb{R}$, employs the Riemann integral definition to approximate the area under a curve by employing numerous rectangles of infinitesimal width juxtaposed between the function's graph and the x-axis. This approach simplifies by maintaining a constant width for all rectangles. Let $n$ denote the number of intervals with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and a uniform spacing of $h = x_{i+1} - x_i$. The rectangle method offers three implementation options:

$$I_1 = \int_a^b f(x)\,\mathrm{d}x \approx h\sum_{i=1}^{n} f(x_{i-1}),$$
$$I_2 = \int_a^b f(x)\,\mathrm{d}x \approx h\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right),$$
$$I_3 = \int_a^b f(x)\,\mathrm{d}x \approx h\sum_{i=1}^{n} f(x_i).$$

In the context where $x_{i-1}$ and $x_i$ represent the left and right points defining the $i$-th rectangle, $I_1$ assumes the rectangle's height equals $f(x_{i-1})$, $I_2$ assumes it is $f\left(\frac{x_{i-1}+x_i}{2}\right)$, and $I_3$ assumes $f(x_i)$. Specifically, $I_2$ is known as the *midpoint rule*.

1.3.1.1. *Error analysis of the rectangle method.* Taylor's theorem enables the assessment of how the error varies as the step size $h$ diminishes. Initially, let us analyze $I_1$ (the same applies to $I_3$). The discrepancy in the computation of rectangle $i$ between the points $x_{i-1}$ and $x_i$ is approximated. Utilizing Taylor's theorem, for all $x$ within $[x_{i-1}, x_i]$, there exists $\xi$ within $[x_{i-1}, x]$ such that:

$$f(x) = f(x_{i-1}) + f'(\xi)(x - x_{i-1})$$

The error in the integral employing this rectangle can be computed as:

$$\begin{aligned}|E_i| &= \left|\int_{x_{i-1}}^{x_i} f(x) - f(x_{i-1})\,\mathrm{d}x\right| = \left|\int_{x_{i-1}}^{x_i} f'(\xi)(x - x_{i-1})\,\mathrm{d}x\right| \\ &\leq \max_{\xi\in[x_{i-1},x_i]} |f'(\xi)| \frac{(x - x_{i-1})^2}{2}\bigg|_{x_{i-1}}^{x_i} = \max_{\xi\in[x_{i-1},x_i]} |f'(\xi)|\frac{h^2}{2}.\end{aligned}$$

If $n$ represents the number of subdivisions (rectangles), i.e., $nh = b - a$, then:

$$|E| = |nE_i| \leq \max_{\xi\in[a,b]} |f'(\xi)| n\frac{h^2}{2} = \max_{\xi\in[a,b]} |f'(\xi)|(b-a)\frac{h}{2}.$$

In essence, the total error is constrained by a term directly proportional to $h$. As $h$ decreases, the error limitation diminishes correspondingly. Naturally, when $h$ approaches zero, the error does likewise.

$I_2$ (Midpoint rule) offers a swifter convergence rate, or a more precise approximation, as the error is confined by a term directly proportional to $h^2$, as elucidated below. Utilizing Taylor's theorem, for all $x$ within $[x_{i-1}, x_i]$, there exists $\xi$ within $[x_{i-1}, x]$ such that:

$$f(x) = f(x_m) + f'(x_i)(x - x_m) + f''(\xi)\frac{(x - x_m)^2}{2},$$

where $x_m = \frac{x_{i-1}+x_i}{2}$. The error in the integral utilizing this rectangle can be computed as:

$$\begin{aligned}
|E_i| &= \left|\int_{x_{i-1}}^{x_i} f(x) - f(x_m)\,\mathrm{d}x\right| \\
&= \left|f'(x_i)\int_{x_{i-1}}^{x_i}(x - x_m)\,\mathrm{d}x + \int_{x_{i-1}}^{x_i}\frac{f''(\xi)}{2}(x - x_m)^2\,\mathrm{d}x\right| \\
&= \left|f'(x_i)\frac{(x_i - x_m)^2 - (x_{i-1} - x_m)^2}{2} + \int_{x_{i-1}}^{x_i}\frac{f''(\xi)}{2}(x - x_m)^2\,\mathrm{d}x\right| \\
&= \left|0 + \int_{x_{i-1}}^{x_i}\frac{f''(\xi)}{2}(x - x_m)^2\,\mathrm{d}x\right| \\
&\leq \max_{\xi\in[x_{i-1},x_i]}\frac{|f''(\xi)|}{2}\frac{\left(\frac{h}{2}\right)^3 - \left(\frac{-h}{2}\right)^3}{3} \\
&\leq \max_{\xi\in[x_{i-1},x_i]}\frac{|f''(\xi)|}{24}h^3.
\end{aligned}$$

If $n$ represents the number of subdivisions (rectangles), i.e., $nh = b - a$, then:

$$|E| = |nE_i| \leq \max_{\xi\in[a,b]}|f''(\xi)|n\frac{h^3}{24} = \max_{\xi\in[a,b]}|f''(\xi)|(b-a)\frac{h^2}{24}.$$

In other words, the overall error is constrained by a term directly proportional to $h^2$, facilitating a faster convergence than $I_1$.

**1.3.2. Trapezoidal rule.** Consider the function $f : [a, b] \to \mathbb{R}$. By segmenting the interval $[a, b]$ into numerous subintervals, the trapezoidal rule offers an approximation for the area under the curve. It achieves this by linearly interpolating between the function values at the junctions of these subintervals. Consequently, within each subinterval, the area to be assessed takes the shape of a trapezoid. To maintain simplicity, the trapezoids' widths are uniformly chosen. Let $n$ denote the number of intervals, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and $h = x_{i+1} - x_i$ represents the constant spacing. The trapezoidal method can be expressed as:

$$\begin{aligned} I_T &= \int_a^b f(x)\,\mathrm{d}x \\ &\approx \frac{h}{2}\sum_{i=1}^{n}(f(x_{i-1}) + f(x_i)) \\ &= \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)). \end{aligned}$$

1.3.2.1. *Error analysis of the trapezoidal rule.* An extension of Taylor's theorem provides the way of how the error fluctuates as the step size $h$ diminishes. Given a function $f$ and its interpolating polynomial of degree $n$ ($p_n(x)$), the error term between the interpolating polynomial and the function can be expressed as:

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=1}^{n+1}(x - x_i).$$

Here, $\xi$ lies within the domain of the function $f$ and is contingent on the point $x$. Estimating the error in the calculation of a trapezoidal number $i$ between the points $x_{i-1}$ and $x_i$ is based on the aforementioned formula, assuming linear interpolation between $x_{i-1}$ and $x_i$. Consequently:

$$\begin{aligned} |E_i| &= \left|\int_{x_{i-1}}^{x_i} f(x) - p_1(x)\,\mathrm{d}x\right| \\ &= \left|\int_{x_{i-1}}^{x_i} \frac{f''(\xi)}{2}(x - x_{i-1})(x - x_i)\,\mathrm{d}x\right| \\ &\leq \max_{\xi\in[x_{i-1},x_i]} \frac{|f''(\xi)|h^3}{12}. \end{aligned}$$

If $n$ represents the number of subdivisions (number of trapezoids), i.e., $nh = b - a$, then:

$$|E| = |nE_i| \leq \max_{\xi \in [a,b]} \frac{|f''(\xi)|nh^3}{12} = \max_{\xi \in [a,b]} \frac{|f''(\xi)|(b-a)h^2}{12}.$$

**1.3.3. Simpson's 1/3 rule.** Consider the function $f : [a, b] \to \mathbb{R}$. By partitioning the interval $[a, b]$ into multiple subintervals, Simpson's 1/3 rule provides an approximation for the area under the curve within each subinterval. It achieves this by interpolating between the function values at the midpoint and ends of each subinterval. Consequently, within each subinterval, the curve to be integrated forms a parabola. To maintain simplicity, the width of each subinterval is uniformly chosen and equals $2h$. Let $n$ denote the number of intervals, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and the constant spacing is $2h = x_i - x_{i-1}$. Within each interval with endpoints $x_{i-1}$ and $x_i$, Lagrange polynomials can define the interpolating parabola as follows:

$$\begin{aligned} p_2(x) = f(x_{i-1}) \frac{(x - x_{m_i})(x - x_i)}{2h^2} &+ f(x_{m_i}) \frac{(x - x_{i-1})(x - x_i)}{-h^2} \\ &+ f(x_i) \frac{(x - x_{i-1})(x - x_{m_i})}{2h^2}. \end{aligned}$$

where $x_{m_i}$ is the midpoint in interval $i$. Integrating the above formula yields:

$$\int_{x_{i-1}}^{x_i} p_2(x)\,\mathrm{d}x = \frac{h}{3} \left( f(x_{i-1}) + 4f(x_{m_i}) + f(x_i) \right).$$

The Simpson's 1/3 rule can be implemented as follows:

$$\begin{aligned} I_{S1} &= \int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3} \sum_{i=1}^{n} \left( f(x_{i-1}) + 4f(x_{m_i}) + f(x_i) \right) \\ &= \frac{h}{3} (f(x_0) + 4f(x_{m_1}) + 2f(x_1) + 4f(x_{m_2}) + 2f(x_2) + \cdots \\ &\qquad \cdots + 2f(x_{n-1}) + 4f(x_{m_n}) + f(x_n)). \end{aligned}$$

1.3.3.1. *Error analysis of Simpson's 1/3 rule.* An estimate for the upper bound of the error can be obtained similarly to deriving the upper bound of the error in the trapezoidal rule. Consider a function $f$ and its interpolating polynomial of degree $n$ ($p_n(x)$). The error term between the interpolating polynomial and the function is given by:

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=1}^{n+1} (x - x_i). \tag{1.3}$$

Here, $\xi$ lies within the domain of the function $f$. Estimating the error in the calculation of the integral of the parabola connecting the points $x_{i-1}$, $x_m = \frac{x_{i-1}+x_i}{2}$, and $x_i$ is based on the formula (1.3), assuming $2h = x_i - x_{i-1}$. Therefore:

$$
\begin{aligned}
|E_i| &= \left| \int_{x_{i-1}}^{x_i} f(x) - p_2(x)\, \mathrm{d}x \right| \\
&= \left| \int_{x_{i-1}}^{x_i} \frac{f'''(\xi)}{3 \times 2} (x - x_{i-1})(x - x_m)(x - x_i)\, \mathrm{d}x \right| .
\end{aligned}
$$

Consequently, the upper bound for the error can be expressed as:

$$
\begin{aligned}
|E_i| &\leq \max_{\xi \in [x_{i-1}, x_i]} \frac{|f'''(\xi)|}{6} \int_{x_{i-1}}^{x_i} |(x - x_{i-1})(x - x_m)(x - x_i)|\, \mathrm{d}x \\
&= \max_{\xi \in [x_{i-1}, x_i]} \frac{|f''(\xi)| h^4}{12} .
\end{aligned}
$$

If $n$ is the number of subdivisions, where each subdivision has a width of $2h$, i.e., $n(2h) = b - a$, then:

$$
|E| = |nE_i| \leq \max_{\xi \in [a,b]} \frac{|f'''(\xi)| n h^4}{12} = \max_{\xi \in [a,b]} \frac{|f'''(\xi)|(b-a)h^3}{24} .
$$

However, a superior estimate for the upper bound of the error can be demonstrated. This can be achieved by employing Newton interpolating polynomials [12] through the points $x_{i-1}$, $x_m$, $x_i$, and $x_i + h$. The rationale behind adding an additional point will become apparent during the integration process. The error term between the interpolating polynomial and the function is expressed as:

$$
\begin{aligned}
f(x) = b_1 &+ b_2(x - x_{i-1}) + b_3(x - x_{i-1})(x - x_m) \\
&+ b_4(x - x_{i-1})(x - x_m)(x - x_i) \\
&+ \frac{f''''(\xi)}{4!}(x - x_{i-1})(x - x_m)(x - x_i)(x - x_i - h).
\end{aligned}
$$

where $\xi \in [x_{i-1}, x_i + h]$ and is contingent on $x$. The first three terms on the right-hand side represent the interpolating parabola passing through the points $f(x_{i-1})$, $f(x_m)$, and $f(x_i)$. Hence, an evaluation for the error can be conducted as:

$$
\begin{aligned}
|E_i| &= \left| \int_{x_{i-1}}^{x_i} f(x) - b_1 - b_2(x - x_{i-1}) - b_3(x - x_{i-1})(x - x_m)\,\mathrm{d}x \right| \\
&\le \left| \int_{x_{i-1}}^{x_i} b_4(x - x_{i-1})(x - x_m)(x - x_i)\,\mathrm{d}x \right| + \max_{\xi \in [x_{i-1}, x_i + h]} \left| \frac{f''''(\xi)}{4!} \right| \\
&\qquad\qquad \times \int_{x_{i-1}}^{x_i} |(x - x_{i-1})(x - x_m)(x - x_i)(x - x_i - h)|\,\mathrm{d}x \\
&\le 0 + \max_{\xi \in [x_{i-1}, x_i + h]} \left| \frac{f''''(\xi)}{4!} \right| \frac{4h^5}{15} \\
&\le \max_{\xi \in [x_{i-1}, x_i + h]} |f''''(\xi)| \frac{h^5}{90}.
\end{aligned}
$$

The first term on the right-hand side of the inequality equals to zero. This arises because the point $x_m$ represents the average of $x_{i-1}$ and $x_i$, resulting in the integration of the cubic polynomial term yielding zero. This choice of a third-order polynomial, rather than a second-order one, facilitates expressing the error term in terms of $h^5$. If $n$ denotes the number of subdivisions, where each subdivision has a width of $2h$, i.e., $n(2h) = b - a$, then:

$$
|E| = |nE_i| \le \max_{\xi \in [a,b]} \frac{|f''''(\xi)| n h^5}{90} = \max_{\xi \in [a,b]} \frac{|f''''(\xi)|(b-a)h^4}{180}.
$$

**1.3.4. Simpson's 3/8 rule.** Consider the function $f : [a, b] \to \mathbb{R}$. By partitioning the interval $[a, b]$ into numerous subintervals, the Simpson's 3/8 rule approximates the area under the curve within each subinterval by interpolating between the function values at the ends of the subinterval and at two intermediate points. Consequently, within each subinterval, the curve to be integrated forms a cubic shape. To maintain simplicity, the width of each subinterval is uniformly chosen and equals $3h$. Let $n$ represent the number of intervals, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and constant spacing $3h = x_i - x_{i-1}$, with the intermediate points for each interval $i$ denoted as $x_{li} = x_{i-1} + h$ and $x_{ri} = x_{i-1} + 2h$. Within each interval with endpoints $x_{i-1}$ and $x_i$, Lagrange polynomials can define the interpolating cubic polynomial as follows:

$$\begin{aligned}p_3(x) = f(x_{i-1})&\frac{(x - x_{li})(x - x_{ri})(x - x_i)}{-6h^3}\\
+ f(x_{li})&\frac{(x - x_{i-1})(x - x_{ri})(x - x_i)}{2h^3}\\
+ f(x_{ri})&\frac{(x - x_{i-1})(x - x_{li})(x - x_i)}{-2h^3}\\
+ f(x_i)&\frac{(x - x_{i-1})(x - x_{li})(x - x_{ri})}{6h^3}.\end{aligned}$$

Integrating the above formula yields:

$$\int_{x_{i-1}}^{x_i} p_3(x)\,\mathrm{d}x = \frac{3h}{8}\left(f(x_{i-1}) + 3f(x_{li}) + 3f(x_{ri}) + f(x_i)\right).$$

The Simpson's 3/8 rule can be implemented as follows:

$$\begin{aligned}I_{S2} = \int_a^b f(x)\,\mathrm{d}x &\approx \frac{3h}{8}\sum_{i=1}^{n}\left(f(x_{i-1}) + 3f(x_{li}) + 3f(x_{ri}) + f(x_i)\right)\\
&= \frac{3h}{8}(f(x_0) + 3f(x_{l1}) + 3f(x_{r1}) + 2f(x_1) + \cdots + 3f(x_{rn}) + f(x_n)).\end{aligned}$$

1.3.4.1. *Error analysis of Simpson's 3/8 rule.* The estimation for the upper bound of the error can be obtained similarly to the derivation of the upper bound of the error in the trapezoidal rule. Given a function $f$ and its interpolating polynomial of degree $n$ ($p_n(x)$), the error term between the interpolating polynomial and the function is expressed as follows:

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=1}^{n+1}(x - x_i).$$

Here, $\xi$ belongs to the domain of the function $f$. The error in calculating the integral of the cubic function $i$ connecting the points $x_{i-1}$, $x_{li} = x_{i-1} + h$, $x_{ri} = x_{i-1} + 2h$, and $x_i = x_{i-1} + 3h$ will be estimated based on the above formula assuming $3h = x_i - x_{i-1}$. Therefore:

$$\begin{aligned}|E_i| &= \left|\int_{x_{i-1}}^{x_i} f(x) - p_3(x)\,\mathrm{d}x\right|\\
&= \left|\int_{x_{i-1}}^{x_i} \frac{f''''(\xi)}{4\times 3\times 2}(x - x_{i-1})(x - x_{li})(x - x_{ri})(x - x_i)\,\mathrm{d}x\right|.\end{aligned}$$

Hence, the upper bound for the error can be expressed as:

$$\begin{aligned} |E_i| &\leq \max_{\xi \in [x_{i-1}, x_i]} \frac{|f''''(\xi)|}{4 \times 3 \times 2} \int_{x_{i-1}}^{x_i} |(x - x_{i-1})(x - x_{li})(x - x_{ri})(x - x_i)| \, \mathrm{d}x \\ &= \max_{\xi \in [x_{i-1}, x_i]} \frac{|f''''(\xi)| 3h^5}{80}. \end{aligned}$$

If $n$ represents the number of subdivisions, where each subdivision has a width of $3h$, i.e., $n(3h) = b - a$, then:

$$|E| = |nE_i| \leq \max_{\xi \in [a,b]} \frac{|f''''(\xi)| 3nh^5}{80} = \max_{\xi \in [a,b]} \frac{|f''''(\xi)|(b-a)h^4}{80}.$$

CHAPTER 2

# Taylor-type expansions in terms of exponential polynomials

## 2.1. Preliminaries

The aim of this chapter is to derive an extension of the Taylor theorem related to linear differential operators with constant coefficients. For this aim, using divided differences with repeated arguments, the so-called characteristic element from the kernel of the differential operator is described. The extension of the Taylor theorem related to exponential polynomials and its consequences are established with integral remainder terms as well as in the form of mean value type theorems. The results presented in this chapter are attributed to the published findings in [1].

## 2.2. Introduction

There are two basic variants of the classical Taylor Theorem, which have a huge number of applications and extensions in various settings.

Given a function $f : I \to \mathbb{R}$, which is $n$ times differentiable at $a \in I$ (where $I$ is a non-degenerate real interval), the polynomial $T_{n;a}(f)$ defined by

$$T_{n;a}(f)(x) := \sum_{j=0}^{n} f^{(j)}(a) \cdot \frac{(x-a)^j}{j!} \tag{2.1}$$

is called the *$n$th-order Taylor polynomial of the function $f$ at the base point $a$.*

The form with integral remainder term can be formulated as follows.

THEOREM 2.2.1. *Let $I$ be a real interval and let $f : I \to \mathbb{R}$ be $(n+1)$ times continuously differentiable. Then, for all $a, x \in I$,*

$$f(x) = T_{n;a}(f)(x) + \int_a^x f^{(n+1)}(t) \cdot \frac{(x-t)^n}{n!} dt.$$

The variant as an intermediate value theorem is the following assertion.

THEOREM 2.2.2. *Let $I$ be a real interval and let $f : I \to \mathbb{R}$ be $(n+1)$ times differentiable. Then, for all $a, x \in I$, there exists a point $\xi$ between $a$ and $x$*

*such that*

$$f(x) = T_{n;a}(f)(x) + f^{(n+1)}(\xi) \cdot \frac{(x-a)^{n+1}}{(n+1)!}. \tag{2.2}$$

These two theorems are contained in most of the textbooks on basic analysis (see, e.g., [4], [8], [23], [24]). There have been several papers where extensions, generalizations and applications of the above fundamental results can be found, cf. [3], [17], [20], [18], [21], [22], [31].

The main content of these results is that they give a high order approximation of the function $f$ near the point $a \in I$ in terms of the polynomial $T_{n;a}(f)$ defined by (2.1), which is of degree at most $n$ and therefore it is in the kernel of the differential operator $D$ given by

$$D(f) = f^{(n+1)}.$$

It seems to be a natural problem to obtain similar approximations in terms of linear combinations of a given finite set of functions, in particular, in terms of exponential polynomials, which span the kernel of a linear differential operator with constant coefficients. Of course, the remainder term of such an approximation is of interest both in integral form and in terms of a mean value theorem.

The aim of this chapter is to accomplish the above goal and derive a general form of the Taylor theorem related to a linear differential operator with constant coefficients. For this aim, in Section 2.3, we describe the so-called characteristic element from the kernel of the differential operator using divided differences with repeated arguments. The main results, an extension of the Taylor theorem and its consequences with an integral remainder term, are stated in Section 2.4, while mean value type extensions are established in Section 2.5.

## 2.3. Auxiliary results on linear differential equations

Let $\mathbb{K}$ denote either the field of real or complex numbers. The imaginary unit in $\mathbb{C}$ will be denoted by $\boldsymbol{i}$.

Given an interval $I \subseteq \mathbb{R}$, let $\mathcal{C}_{\mathbb{K}}(I)$ stand for the space of continuous $\mathbb{K}$-valued functions defined on $I$. If additionally $n \in \mathbb{N}$, then let $\mathcal{C}^n_{\mathbb{K}}(I)$ denote the space of $n$-times continuously differentiable $\mathbb{K}$-valued functions defined on $I$. For $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, let $n$th-order linear differential operator $D_c \colon \mathcal{C}^n_{\mathbb{K}}(I) \to \mathcal{C}_{\mathbb{K}}(I)$ be defined by the formula

$$D_c(f) := c_n f^{(n)} + \dots + c_1 f' + c_0 f \qquad (f \in \mathcal{C}^n_{\mathbb{K}}(I)). \tag{2.3}$$

Let $\omega_c \in \mathcal{C}^n_{\mathbb{C}}(\mathbb{R})$ denote the unique solution of the initial value problem

$$(2.4) \qquad D_c(\omega_c) = 0, \qquad \omega_c^{(\ell)}(0) = \delta_{\ell,n-1} \quad (\ell \in \{0, \dots, n-1\}).$$

The function $\omega_c$ will be called the *characteristic solution* of the differential equation

$$D_c(\omega) = 0.$$

One can see that if $c \in \mathbb{R}^{n+1}$, then $\omega_c$ is real-valued and hence it belongs to $\mathcal{C}^n_{\mathbb{R}}(\mathbb{R})$. Let $P_c$ denote the characteristic polynomial of $D_c$, which is given by

$$(2.5) \qquad P_c(\lambda) := c_n\lambda^n + \cdots + c_1\lambda + c_0 \qquad (\lambda \in \mathbb{C}).$$

In order to provide a more or less explicit formula for $P_c$, we recall the notion of divided differences and their limiting properties.

If $D \subseteq \mathbb{K}$ and $n \in \mathbb{N}$, then let $\sigma_n(D)$ denote the set

$$\sigma_n(D) := \{(\lambda_1, \dots, \lambda_n) \in D^n \mid \lambda_i \neq \lambda_j \text{ for all } i, j \in \{1, \dots, n\} \text{ with } i \neq j\}.$$

For $f : D \to \mathbb{C}$, and $(\lambda_1, \dots, \lambda_n) \in \sigma_n(D)$, the $(n-1)$*st order divided difference* $f(\lambda_1, \dots, \lambda_n)$ is defined by

$$f(\lambda_1, \dots, \lambda_n) := \sum_{i=1}^{n} \frac{f(\lambda_i)}{\prod_{j \in \{1,\dots,n\}\setminus\{i\}} (\lambda_i - \lambda_j)},$$

see [10] for more details and alternative definitions. To define divided differences with repeated arguments, for $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, let $(\lambda)^m$ denote the $m$-tuple $(\lambda, \dots, \lambda) \in \mathbb{C}^m$ and, for $n, k, m_1, \dots, m_k \in \mathbb{N}$ with $m_1 + \cdots + m_k = n$ and $(\lambda_1, \dots, \lambda_k) \in \sigma_k(D)$, denote

$$f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) := \lim_{\sigma_n(D) \ni (\mu_1,\dots,\mu_n) \to ((\lambda_1)^{m_1},\dots,(\lambda_k)^{m_k})} f(\mu_1, \dots, \mu_n)$$

provided that the limit exists. In the following lemma, we compute divided differences of $f$ with repeated arguments under natural regularity assumptions.

LEMMA 2.3.1. *Let $D \subseteq \mathbb{K}$ be open, let $n, k, m_1, \dots, m_k \in \mathbb{N}$ with $m_1 + \cdots + m_k = n$, let $(\lambda_1, \dots, \lambda_k) \in \sigma_k(D)$ and define the polynomials $P_1, \dots, P_k, P : \mathbb{C} \to \mathbb{C}$ by*

$$(2.6) \qquad \begin{aligned} P_i(\lambda) &:= \prod_{j \in \{1,\dots,k\}\setminus\{i\}} (\lambda - \lambda_j)^{m_j} \qquad (i \in \{1, \dots, k\}), \\ P(\lambda) &:= \prod_{j=1}^{k} (\lambda - \lambda_j)^{m_j}. \end{aligned}$$

*If $f : D \to \mathbb{C}$ is $(m_i - 1)$ times continuously differentiable at $\lambda_i$ for all $i \in \{1, \dots, k\}$, then*

$$f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) = \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}. \tag{2.7}$$

*Furthermore,*

$$\begin{aligned} &f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) \\ &= \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{f^{(\ell)}(\lambda_i)}{\ell!} \Bigg( \sum_{j=0}^{m_i-1-\ell} (-1)^j \\ &\quad \times \frac{j! B_{m_i-1-\ell,j}\big(\frac{1!}{(m_i+1)!} P^{(m_i+1)}(\lambda_i), \dots, \frac{(m_i-\ell-j)!}{(2m_i-\ell-j)!} P^{(2m_i-\ell-j)}(\lambda_i)\big)}{(m_i-1-\ell)! \big(\frac{0!}{m_i!} P^{(m_i)}(\lambda_i)\big)^{j+1}} \Bigg). \end{aligned}$$

PROOF. In what follows, the symbol $\partial_\ell$ will stand for differentiation with respect to the variable $\lambda_\ell$, where $\ell \in \{1, \dots, k\}$.

Using the well-known formula for divided differences with repeated arguments and also the higher-order Leibniz Rule at the very last equality, we get

$$\begin{aligned} &f((\lambda_1)^{m_1}, \dots, (\lambda_k)^{m_k}) \\ &= \left( \prod_{\ell=1}^{k} \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) f(\lambda_1, \dots, \lambda_k) \\ &= \left( \prod_{\ell=1}^{k} \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) \sum_{i=1}^{k} \frac{f(\lambda_i)}{\prod_{j \in \{1,\dots,k\} \setminus \{i\}} (\lambda_i - \lambda_j)} \\ &= \sum_{i=1}^{k} \left( \prod_{\ell=1}^{k} \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) \frac{f(\lambda_i)}{\prod_{j \in \{1,\dots,k\} \setminus \{i\}} (\lambda_i - \lambda_j)} \\ &= \sum_{i=1}^{k} \frac{\partial_i^{m_i-1}}{(m_i-1)!} \left( \prod_{\ell \in \{1,\dots,k\} \setminus \{i\}} \frac{\partial_\ell^{m_\ell-1}}{(m_\ell-1)!} \right) \frac{f(\lambda_i)}{\prod_{j \in \{1,\dots,k\} \setminus \{i\}} (\lambda_i - \lambda_j)} \\ &= \sum_{i=1}^{k} \frac{\partial_i^{m_i-1}}{(m_i-1)!} f(\lambda_i) \left( \prod_{\ell \in \{1,\dots,k\} \setminus \{i\}} \frac{\partial_\ell^{m_\ell-1} (\lambda_i - \lambda_\ell)^{-1}}{(m_\ell-1)!} \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \frac{\partial_i^{m_i-1}}{(m_i-1)!} f(\lambda_i) \Bigg( \prod_{\ell \in \{1,\dots,k\}\setminus\{i\}} (\lambda_i - \lambda_\ell)^{-m_\ell} \Bigg) \\
&= \sum_{i=1}^{k} \frac{(P_i^{-1} \cdot f)^{(m_i-1)}(\lambda_i)}{(m_i-1)!} \\
&= \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}.
\end{aligned}
$$

This proves the first equality of the lemma.

By applying the Faà di Bruno formula (see [8]) for the computation of the $(m_i - 1 - \ell)$th-order derivative of $P_i^{-1} = Q \circ P_i$ (with $Q(u) := u^{-1}$), we have

$$
\begin{aligned}
&(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i) \\
&= \sum_{j=0}^{m_i-1-\ell} \frac{(-1)^j j!}{P_i^{j+1}(\lambda_i)} B_{m_i-1-\ell,j}(P_i'(\lambda_i), \dots, P_i^{(m_i-\ell-j)}(\lambda_i)).
\end{aligned}
$$

For $i \in \{1, \dots, k\}$ and $\lambda \in \mathbb{C}$, we have that $P(\lambda) = (\lambda - \lambda_i)^{m_i} P_i(\lambda)$. Thus, using the higher-order Leibniz Rule again, for $\ell \geq 0$ and $i \in \{1, \dots, k\}$, we obtain

$$
\begin{aligned}
P^{(m_i+\ell)}(\lambda_i) &= \sum_{j=\ell}^{m_i+\ell} \binom{m_i+\ell}{j} \frac{m_i!}{(j-\ell)!} (\lambda_i - \lambda_i)^{j-\ell} P_i^{(j)}(\lambda_i) \\
&= \frac{(m_i+\ell)!}{\ell!} P_i^{(\ell)}(\lambda_i).
\end{aligned}
$$

Applying these equalities, for $i \in \{1, \dots, k\}$, we conclude that

$$
\begin{aligned}
&(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i) \\
&= \sum_{j=0}^{m_i-1-\ell} \Bigg( (-1)^j \\
&\quad \times \frac{j! B_{m_i-1-\ell,j}\big(\frac{1!}{(m_i+1)!} P^{(m_i+1)}(\lambda_i), \dots, \frac{(m_i-\ell-j)!}{(2m_i-\ell-j)!} P^{(2m_i-\ell-j)}(\lambda_i)\big)}{\big(\frac{0!}{m_i!} P^{(m_i)}(\lambda_i)\big)^{j+1}} \Bigg).
\end{aligned}
$$

Substituting this expression into the first equality of the lemma, we get the second asserted formula. □

REMARK 2.3.2. The $i$th term of the first (equivalently, of the second) formula of the lemma becomes very simple in the particular cases when

$1 \leq m_i \leq 3$. Indeed,

$$\sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}$$
$$= \begin{cases} \dfrac{1}{P'(\lambda_i)} f(\lambda_i) & \text{if } m_i = 1, \\ \dfrac{2}{P''(\lambda_i)} f'(\lambda_i) - \dfrac{2P'''(\lambda_i)}{3P''(\lambda_i)^2} f(\lambda_i) & \text{if } m_i = 2, \\ \dfrac{3}{P'''(\lambda_i)} f''(\lambda_i) - \dfrac{3P^{(4)}(\lambda_i)}{2P'''(\lambda_i)^2} f'(\lambda_i) & \\ \quad + \left( \dfrac{3P^{(4)}(\lambda_i)^2}{8P'''(\lambda_i)^3} - \dfrac{3P^{(5)}(\lambda_i)}{10P'''(\lambda_i)^2} \right) f(\lambda_i) & \text{if } m_i = 3. \end{cases}$$

LEMMA 2.3.3. *Let $n \in \mathbb{N}$, $c = (c_0, \ldots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, and let $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be pairwise distinct roots of the characteristic polynomial $P_c$ with multiplicities $m_1, \ldots, m_k \in \mathbb{N}$, respectively. Then*

$$\omega_c(t) = \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{t^\ell \exp(\lambda_i t)}{\ell!},$$

*where $P_i$ is defined by* (2.6).

PROOF. From the theory of higher-order linear ordinary differential equations, it follows that the functions $t^\ell \exp(\lambda_i t)$ (where $i \in \{1, \ldots, k\}$ and $\ell \in \{0, \ldots, m_i - 1\}$) form a fundamental system of solutions to the linear differential equation $D_c(\omega) = 0$. Therefore, any linear combination of them is a solution, which proves that $D_c(\omega_c) = 0$ holds. To complete the proof, we need to show that $\omega_c$ also satisfies the initial value condition $\omega_c^{(j)}(0) = \delta_{j,n-1}$ for all $j \in \{0, \ldots, n-1\}$.

Denote $E_t(\lambda) := \exp(\lambda t)$. Then, for all $\ell \geq 0$, we have that $E_t^{(\ell)}(\lambda) = t^\ell \exp(\lambda t)$. Therefore, using the definition of $\omega_c$ and the previous lemma, we can conclude that

$$\begin{aligned} \omega_c(t) &= \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{E_t^{(\ell)}(\lambda_i)}{\ell!} \\ &= E_t((\lambda_1)^{m_1}, \ldots, (\lambda_k)^{m_k}) \\ &= \lim_{\sigma_n(D) \ni (\mu_1, \ldots, \mu_n) \to ((\lambda_1)^{m_1}, \ldots, (\lambda_k)^{m_k})} E_t(\mu_1, \ldots, \mu_n). \end{aligned}$$

Thus, for $\ell \geq 0$, we obtain

$$\begin{aligned}\omega_c^{(j)}(t) &= \frac{d^j}{dt^j}\left(\lim_{\sigma_n(D)\ni(\mu_1,\dots,\mu_n)\to((\lambda_1)^{m_1},\dots,(\lambda_k)^{m_k})} E_t(\mu_1,\dots,\mu_n)\right)\\ &= \lim_{\sigma_n(D)\ni(\mu_1,\dots,\mu_n)\to((\lambda_1)^{m_1},\dots,(\lambda_k)^{m_k})} \frac{d^j}{dt^j} E_t(\mu_1,\dots,\mu_n)\\ &= \lim_{\sigma_n(D)\ni(\mu_1,\dots,\mu_n)\to((\lambda_1)^{m_1},\dots,(\lambda_k)^{m_k})} \frac{d^j}{dt^j} \sum_{i=1}^n \frac{\exp(\mu_i t)}{\prod_{\ell\in\{1,\dots,n\}\setminus\{i\}}(\mu_i-\mu_\ell)}\\ &= \lim_{\sigma_n(D)\ni(\mu_1,\dots,\mu_n)\to((\lambda_1)^{m_1},\dots,(\lambda_k)^{m_k})} \sum_{i=1}^n \frac{\mu_i^j\exp(\mu_i t)}{\prod_{\ell\in\{1,\dots,n\}\setminus\{i\}}(\mu_i-\mu_\ell)}.\end{aligned}$$

Finally, substituting $t = 0$, we arrive at

$$\omega_c^{(j)}(0) = \lim_{\sigma_n(D)\ni(\mu_1,\dots,\mu_n)\to((\lambda_1)^{m_1},\dots,(\lambda_k)^{m_k})} \sum_{i=1}^n \frac{\mu_i^j}{\prod_{\ell\in\{1,\dots,n\}\setminus\{i\}}(\mu_i-\mu_\ell)}.$$

Observe that the sum in the above expression is equal to the $(n-1)$st-order divided difference of the $j$th monomial function at $(\mu_1,\dots,\mu_n)$. Therefore, this sum and hence its limit are equal to $\delta_{j,n-1}$ if $j \in \{0,\dots,n-1\}$. □

For the formulation of some consequences of our main results, for $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{C}$, we define the function $\zeta_{n,k,\gamma} : \mathbb{R} \to \mathbb{R}$ by

$$\zeta_{n,k,\gamma}(t) := \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}. \tag{2.8}$$

By applying the ratio test, it follows that the series is convergent for all $t \in \mathbb{R}$. For further properties, we have the following statement.

LEMMA 2.3.4. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{C}$. Then, the function $\zeta_{n,k,\gamma}$ is the (unique) solution of the initial value problem*

$$\zeta^{(n+1)} = \gamma\zeta^{(k+1)}, \qquad \zeta^{(i)}(0) = \delta_{i,n} \quad (i \in \{0,\dots,n\}). \tag{2.9}$$

*In addition, if $j \in \{0,\dots,k\}$, then*

$$\zeta_{n,k,\gamma}^{(j)} = \zeta_{n-j,k-j,\gamma}. \tag{2.10}$$

*If $\gamma \neq 0$, then, for all $t \in \mathbb{R}$,*

$$\zeta_{n,k,\gamma^{n-k}}(t) = \gamma^{-n}\zeta_{n,k,1}(\gamma t). \tag{2.11}$$

*Furthermore, for all $t \in \mathbb{R}$,*
(2.12)
$$\zeta_{n,k,0}(t) = \frac{t^n}{n!},$$
$$\zeta_{n,0,1}(t) = -1 + \frac{1}{n}\sum_{j=0}^{n-1} \exp\left(\cos\left(\frac{2\pi j}{n}\right)t\right) \cdot \cos\left(\sin\left(\frac{2\pi j}{n}\right)t\right).$$

PROOF. The unique solvability of (2.9) is a consequence of standard results of the theory of linear differential equations. It is easy to see that $\zeta = \zeta_{n,k,\gamma}$ satisfies the initial value conditions. To see that it fulfills the differential equation in (2.9), we have the following computation:

$$\begin{aligned}\zeta_{n,k,\gamma}^{(n+1)}(t) &= \left(\sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}\right)^{(n+1)} = \sum_{i=1}^{\infty} \frac{\gamma^i t^{i(n-k)-1}}{(i(n-k)-1)!} \\ &= \gamma \sum_{i=1}^{\infty} \frac{\gamma^{i-1} t^{i(n-k)-1}}{(i(n-k)-1)!} = \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{(i+1)(n-k)-1}}{((i+1)(n-k)-1)!} \\ &= \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n-k-1}}{(i(n-k)+n-k-1)!} = \gamma \left(\sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}\right)^{(k+1)} \\ &= \gamma \zeta_{n,k,\gamma}^{(k+1)}(t).\end{aligned}$$

For the proof of (2.10) when $j \in \{0, \ldots, k\}$, observe that

$$\begin{aligned}\zeta_{n,k,\gamma}^{(j)}(t) &= \left(\sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}\right)^{(j)} \\ &= \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n-j}}{(i(n-k)+n-j)!} \\ &= \sum_{i=0}^{\infty} \frac{\gamma^i t^{i((n-j)-(k-j))+n-j}}{(i((n-j)-(k-j))+n-j)!} = \zeta_{n-j,k-j,\gamma}(t).\end{aligned}$$

We now show that (2.11) holds for $\gamma \neq 0$. Indeed,

$$\begin{aligned}\gamma^{-n}\zeta_{n,k,1}(\gamma t) &= \gamma^{-n} \sum_{i=0}^{\infty} \frac{(\gamma t)^{i(n-k)+n}}{(i(n-k)+n)!} \\ &= \sum_{i=0}^{\infty} \frac{(\gamma^{n-k})^i t^{i(n-k)+n}}{(i(n-k)+n)!} = \zeta_{n,k,\gamma^{n-k}}(t).\end{aligned}$$

The formula stated for $\zeta_{n,k,0}$ in (2.12) is obvious. To compute $\zeta_{n,0,1}$, we use the fact that this function is the unique solution $\zeta : \mathbb{R} \to \mathbb{R}$ of the initial value problem

$$\zeta^{(n+1)} = \zeta', \qquad \zeta^{(\ell)}(0) = \delta_{\ell,n} \quad (\ell \in \{0, \dots, n\}).$$

The characteristic polynomial of this linear differential equation is

$$P(\lambda) = \lambda^{n+1} - \lambda = (\lambda^n - 1)\lambda.$$

The roots of this polynomial are the $n$th roots of unity,

$$\lambda_j = \exp(\boldsymbol{i}\frac{2\pi j}{n}) = \cos(\frac{2\pi j}{n}) + \boldsymbol{i}\sin(\frac{2\pi j}{n}) =: \alpha_j + \boldsymbol{i}\beta_j,$$

where $j \in \{0, \dots, n-1\}$ and $\lambda_n = 0$.

Using that $P(\lambda) = \lambda^{n+1} - \lambda$ and that $\lambda_n = 0$ and $\lambda_j^n = 1$ for all $j < n$, Lemma 2.3.3 implies that

$$\begin{aligned}\zeta(t) &= \sum_{j=0}^{n} \frac{\exp(\lambda_j t)}{(n+1)\lambda_j^n - 1} \\ &= -1 + \sum_{j=0}^{n-1} \frac{\exp(\lambda_j t)}{(n+1)\lambda_j^n - 1} \\ &= -1 + \frac{1}{n}\sum_{j=0}^{n-1} \exp(\lambda_j t).\end{aligned}$$

Taking into consideration that $\zeta$ is real valued, it follows that

$$\begin{aligned}\zeta(t) &= -1 + \frac{1}{n}\sum_{j=0}^{n-1} \Re(\exp(\lambda_j t)) \\ &= -1 + \frac{1}{n}\sum_{j=0}^{n-1} \exp\left(\cos\left(\frac{2\pi j}{n}\right)t\right) \cdot \cos\left(\sin\left(\frac{2\pi j}{n}\right)t\right),\end{aligned}$$

which proves the second formula in (2.12). □

We note that the series involved in the right hand side of (2.8) has closed forms, more precisely, it is the linear combinations of the hyperbolic or trigonometric functions and a polynomial of degree at most $n-3$. Indeed, if $\gamma = 1$, then we get

$$\zeta_{n,n-2,1}(t) = \sum_{i=0}^{\infty} \frac{t^{2i+n}}{(2i+n)!} = \begin{cases} \cosh(t) - \sum_{i=0}^{\frac{n-2}{2}} \frac{t^{2i}}{(2i)!} & \text{if } n \text{ is even,} \\ \sinh(t) - \sum_{i=0}^{\frac{n-3}{2}} \frac{t^{2i+1}}{(2i+1)!} & \text{if } n \text{ is odd.} \end{cases}$$

While for $\gamma = -1$, we can obtain

$$\begin{aligned} \zeta_{n,n-2,-1}(t) &= \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+n}}{(2i+n)!} \\ &= \begin{cases} (-1)^{\frac{n-2}{2}} \left( \cos(t) - \sum_{i=0}^{\frac{n-2}{2}} \frac{(-1)^i t^{2i}}{(2i)!} \right) & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-3}{2}} \left( \sin(t) - \sum_{i=0}^{\frac{n-3}{2}} \frac{(-1)^i t^{2i+1}}{(2i+1)!} \right) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

## 2.4. A generalization of the Taylor theorem

Our first main result can be stated as follows.

THEOREM 2.4.1. *Let $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, and assume that $f : I \to \mathbb{K}$ is $(n-1)$ times differentiable at $a \in I$. Define $T_{a,c}f : \mathbb{R} \to \mathbb{K}$ by*

$$(T_{a,c}f)(x) := \sum_{j=0}^{n-1} \left( f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right),$$

*where $\omega_c$ is defined by (2.4). Then, $T_{a,c}f$ belongs to the kernel of $D_c$ and*

$$f^{(\ell)}(a) = (T_{a,c}f)^{(\ell)}(a) \qquad (\ell \in \{0, \dots, n-1\}). \tag{2.13}$$

The function $T_{a,c}f$ is termed the *generalized Taylor polynomial at the point $a$ with respect to the differential operator $D_c$.*

PROOF. The characteristic function $\omega_c$ satisfies the differential equation (2.3), i.e., we have

$$c_n \omega_c^{(n)} + \cdots + c_1 \omega_c' + c_0 \omega_c = 0. \tag{2.14}$$

Differentiating this equality $i$ times (where $i \in \mathbb{N}$), we can see that $\omega_c^{(i)}$ also solves the differential equation (2.3). It is also obvious that the function $\omega_c^{(i)}$

with the translated argument $(x-a)$ is still a solution to (2.3). Therefore, $T_{a,c}f$ is a linear combination of solutions of the differential equation (2.3), which implies that $T_{a,c}f$ belongs to the kernel of $D_c$.

From the definition of the function $T_{a,c}f$, for $\ell \in \{0,\dots,n-1\}$, we have that

$$(T_{a,c}f)^{(\ell)}(a) = \sum_{j=0}^{n-1}\left(f^{(j)}(a)\sum_{i=0}^{n-1-j} c_{i+j+1}\omega_c^{(i+\ell)}(0)\right).$$

In order to prove that (2.13) holds, it is sufficient to verify that

$$\sum_{i=0}^{n-1-j} c_{i+j+1}\omega_c^{(i+\ell)}(0) = \delta_{j,\ell} \qquad (j,\ell \in \{0,\dots,n-1\}). \tag{2.15}$$

Using the initial value conditions in (2.4), we get

$$\begin{aligned}\sum_{i=0}^{n-1-j} c_{i+j+1}\omega_c^{(i+\ell)}(0) &= \sum_{i=n-1-\ell}^{n-1-j} c_{i+j+1}\omega_c^{(i+\ell)}(0)\\ &= \sum_{\alpha=n+j-\ell}^{n} c_\alpha\omega_c^{(\alpha+\ell-j-1)}(0).\end{aligned} \tag{2.16}$$

If $j>\ell$, then $n+j-\ell>n$, thus the summation is over the empty set, and therefore (2.15) is trivially valid.

If $j=\ell$, then

$$\sum_{i=0}^{n-1-j} c_{i+j+1}\omega_c^{(i+\ell)}(0) = \sum_{\alpha=n}^{n} c_\alpha\omega_c^{(\alpha-1)}(0) = c_n\omega_c^{(n-1)}(0) = 1,$$

which proves that (2.15) holds in this case.

Finally, assume that $j<\ell$. Differentiating the equation (2.14) $(\ell-j-1)$ times and then evaluating it at $0$, we get

$$0 = \sum_{\alpha=0}^{n} c_\alpha\omega_c^{(\alpha+\ell-j-1)}(0) = \sum_{\alpha=n+j-\ell}^{n} c_\alpha\omega_c^{(\alpha+\ell-j-1)}(0),$$

which, combined with (2.16), shows that (2.15) is also valid in this case. □

THEOREM 2.4.2. *Let $n\in\mathbb{N}$, $c=(c_0,\dots,c_n)\in\mathbb{K}^{n+1}$ with $c_n=1$. Then, for all $f\in\mathcal{C}^n_{\mathbb{K}}(I)$ and $x,a\in I$, we have*

$$f(x) = (T_{a,c}f)(x) + \int_a^x D_c(f)(t)\cdot\omega_c(x-t)dt. \tag{2.17}$$

PROOF. Let $k \in \{1, \ldots, n\}$ be fixed. First we show that, for all $j \in \{0, \ldots, k\}$,

$$\int_a^x f^{(k)}(t) \cdot \omega_c(x-t)dt = \sum_{i=0}^{j-1} \Big( f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \Big) + \int_a^x f^{(k-j)}(t) \cdot \omega_c^{(j)}(x-t)dt. \tag{2.18}$$

We show this equality by induction on $j$. Clearly, this equality holds if $j = 0$ (because the domain of summation is then empty, and therefore the sum equals 0). Assume that we have proved (2.18) for some $j \in \{0, \ldots, k-1\}$. Then, integrating by parts, we have

$$\int_a^x f^{(k-j)}(t) \cdot \omega_c^{(j)}(x-t)dt = f^{(k-1-j)}(x) \cdot \omega_c^{(j)}(0) - f^{(k-1-j)}(a) \cdot \omega_c^{(j)}(x-a) + \int_a^x f^{(k-1-j)}(t) \cdot \omega_c^{(j+1)}(x-t)dt.$$

Combining this equality with (2.18), we get

$$\int_a^x f^{(k)}(t) \cdot \omega_c(x-t)dt = \sum_{i=0}^{j} \Big( f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \Big) + \int_a^x f^{(k-1-j)}(t) \cdot \omega_c^{(j+1)}(x-t)dt,$$

which is exactly the statement (2.18) for $j+1$ and completes the induction.

Applying (2.18) for $j = k$, we get that

$$\int_a^x f^{(k)}(t) \cdot \omega_c(x-t)dt = \sum_{i=0}^{k-1} \Big( f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \Big) + \int_a^x f(t) \cdot \omega_c^{(k)}(x-t)dt \tag{2.19}$$

is valid for $k \in \{1, \ldots, n\}$ and, trivially, this equality is also valid for $k = 0$. Multiplying (2.19) by $c_k$ and adding up the equalities so obtained side by side, and applying (2.4), we get

$$\begin{aligned}
\int_a^x D_c(f)(t) \cdot \omega_c(x-t)dt &= \sum_{k=0}^{n} c_k \int_a^x f^{(k)}(t) \cdot \omega_c(x-t)dt \\
&= \sum_{k=0}^{n} c_k \left( \sum_{i=0}^{k-1} \left( f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \right) \right) \\
&\quad + \sum_{k=0}^{n} c_k \left( \int_a^x f(t) \cdot \omega_c^{(k)}(x-t)dt \right) \\
&= \sum_{k=1}^{n} c_k \left( \sum_{i=0}^{k-1} \left( f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \right) \right) \\
&\quad + \int_a^x f(t) \cdot D_c(\omega_c)(x-t)dt \\
&= \sum_{k=1}^{n} \sum_{i=0}^{k-1} c_k f^{(k-1-i)}(x) \cdot \omega_c^{(i)}(0) - \sum_{k=1}^{n} \sum_{i=0}^{k-1} c_k f^{(k-1-i)}(a) \cdot \omega_c^{(i)}(x-a) \\
&= c_n f(x) - \sum_{j=0}^{n-1} \left( f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right) \\
&= f(x) - (T_{a,c} f)(x).
\end{aligned}$$

This completes the proof of (2.17). $\square$

The following result is a consequence of Theorem 2.4.2 in which the main part contains the Taylor expansion of order $k$ and the rest is in terms of the function $\zeta_{n,k,\gamma}$.

THEOREM 2.4.3. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$, and $\gamma \in \mathbb{K}$. Then, for all $f \in \mathcal{C}^n_{\mathbb{K}}(I)$ and $x, a \in I$,*

$$\begin{aligned}
f(x) = \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a) \zeta_{n,k,\gamma}^{(n-j)}(x-a) \\
+ \int_a^x (f^{(n)}(t) - \gamma f^{(k)}(t)) \zeta'_{n,k,\gamma}(x-t)dt.
\end{aligned} \tag{2.20}$$

PROOF. Consider the particular case of Theorem 2.4.2 when $c_n = 1$, $c_k = -\gamma$, and $c_i = 0$ for $i \in \{0, \dots, n-1\} \setminus \{k\}$, i.e., when

$$D_c(f) = f^{(n)} - \gamma f^{(k)}.$$

We show that $\omega_c = \zeta'_{n,k,\gamma}$. The leading term of this power series is $\frac{t^n}{n!}$, therefore, $\zeta^{(i+1)}_{n,k,\gamma}(0) = \delta_{i,n-1}$ for all $i \in \{0, \dots, n-1\}$. On the other hand,

$$\begin{aligned} D_c(\zeta'_{n,k,\gamma})(t) &= \zeta^{(n+1)}_{n,k,\gamma}(t) - \gamma \zeta^{(k+1)}_{n,k,\gamma}(t) \\ &= \sum_{i=1}^{\infty} \frac{\gamma^i t^{i(n-k)-1}}{(i(n-k)-1)!} - \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{(i+1)(n-k)-1}}{((i+1)(n-k)-1)!} = 0. \end{aligned}$$

Thus, we have obtained that $\zeta'_{n,k,\gamma}$ is a solution of the initial value problem (2.4).

For $j \in \{0, \dots, k-1\}$, we get

$$\begin{aligned} \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(t) = c_n \zeta^{(n-j)}_{n,k,\gamma}(t) + c_k \zeta^{(k-j)}_{n,k,\gamma}(t) \\ &= \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+j}}{(i(n-k)+j)!} - \gamma \sum_{i=0}^{\infty} \frac{\gamma^i t^{(i+1)(n-k)+j}}{((i+1)(n-k)+j)!} \\ &= \frac{t^j}{j!}. \end{aligned}$$

Similarly, for $j \in \{k, \dots, n-1\}$, we get

$$\sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(t) = c_n \zeta^{(n-j)}_{n,k,\gamma}(t).$$

Putting these formulas together, we can see that (2.17) simplifies to the equality (2.20), which was to be proved. □

The subsequent results will be corollaries of Theorem 2.4.3. First we note that the classical Taylor theorem with an integral remainder term follows from Theorem 2.4.3 by taking $k = 0$ and $\gamma = 0$.

COROLLARY 2.4.4. *For all $f \in \mathcal{C}^2_{\mathbb{K}}(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) &= f(a)\cos(x-a) + f'(a)\sin(x-a) \\ &\quad + \int_a^x (f''(t) + f(t))\sin(x-t)dt. \end{aligned} \tag{2.21}$$

PROOF. Let $n = 2$, $k = 0$ and $\gamma = -1$ in Theorem 2.4.3. Then,

$$\zeta_{2,0,-1}(t) = 1 - \cos(t).$$

Therefore, the equality (2.20) reduces to (2.21). □

COROLLARY 2.4.5. *For all* $f \in \mathcal{C}^2_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned} f(x) &= f(a)\cosh(x-a) + f'(a)\sinh(x-a) \\ &\qquad + \int_a^x (f''(t) - f(t))\sinh(x-t)dt. \end{aligned} \tag{2.22}$$

PROOF. Let $n = 2$, $k = 0$ and $\gamma = 1$ in Theorem 2.4.3. Then,

$$\zeta_{2,0,1}(t) = \cosh(t) - 1.$$

Hence, the equality (2.20) simplifies to (2.22). □

COROLLARY 2.4.6. *For all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned} f(x) = f(a)&\frac{\cosh(x-a)+\cos(x-a)}{2} \\ &+ f'(a)\frac{\sinh(x-a)+\sin(x-a)}{2} \\ &+ f''(a)\frac{\cosh(x-a)-\cos(x-a)}{2} \\ &+ f'''(a)\frac{\sinh(x-a)-\sin(x-a)}{2} \\ &+ \int_a^x (f''''(t)-f(t))\frac{\sinh(x-t)-\sin(x-t)}{2}dt. \end{aligned} \tag{2.23}$$

PROOF. Let $n = 4$, $k = 0$ and $\gamma = 1$ in Theorem 2.4.3. Then, it is easy to see that

$$\zeta_{4,0,1}(t) = \sum_{i=0}^{\infty} \frac{t^{4i+j}}{(4i+4)!} = \frac{\cosh(t)+\cos(t)}{2} - 1.$$

Thus, the equality (2.20) of Theorem 2.4.3 can be rewritten as (2.23). □

COROLLARY 2.4.7. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$
\begin{aligned}
f(x) = f(a) & \frac{\beta^2 \cos(\alpha(x-a)) - \alpha^2 \cos(\beta(x-a))}{\beta^2 - \alpha^2} \\
& + f'(a) \frac{\beta^3 \sin(\alpha(x-a)) - \alpha^3 \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
& + f''(a) \frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2} \\
& + f'''(a) \frac{\beta \sin(\alpha(x-a)) - \alpha \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\
& + \int_a^x (f''''(t) + (\alpha^2 + \beta^2) f''(t) + \alpha^2\beta^2 f(t)) \\
& \qquad \times \frac{\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t))}{\alpha\beta(\beta^2 - \alpha^2)} dt.
\end{aligned}
\tag{2.24}
$$

PROOF. Let $n = 4$ and apply Theorem 2.4.2 in the setting $c_4 = 1$, $c_3 = c_1 = 0$, $c_2 = \alpha^2 + \beta^2$, $c_0 = \alpha^2\beta^2$, that is, when

$$D_c(f) = f'''' + (\alpha^2 + \beta^2) f'' + \alpha^2\beta^2 f.$$

Then the corresponding characteristic polynomial is

$$P_c(\lambda) = \lambda^4 + (\alpha^2 + \beta^2)\lambda^2 + \alpha^2\beta^2$$

whose roots are $\pm\alpha \boldsymbol{i}$ and $\pm\beta \boldsymbol{i}$. Therefore, $\sin(\alpha x)$, $\cos(\alpha x)$, $\sin(\beta x)$, and $\cos(\beta x)$ form a fundamental system of solutions for the differential equation

$$D_c(f) = 0.$$

Then, the characteristic solution $\omega_c$ is given by

$$\omega_c(t) = \frac{\beta \sin(\alpha t) - \alpha \sin(\beta t)}{\alpha\beta(\beta^2 - \alpha^2)}.$$

We can easily obtain

$$
\sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(t) =
\begin{cases}
\dfrac{\beta^2 \cos(\alpha t) - \alpha^2 \cos(\beta t)}{\beta^2 - \alpha^2} & \text{if } j = 0, \\
\dfrac{\beta^3 \sin(\alpha t) - \alpha^3 \sin(\beta t)}{\alpha\beta(\beta^2 - \alpha^2)} & \text{if } j = 1, \\
\dfrac{\cos(\alpha t) - \cos(\beta t)}{\beta^2 - \alpha^2} & \text{if } j = 2, \\
\dfrac{\beta \sin(\alpha t) - \alpha \sin(\beta t)}{\alpha\beta(\beta^2 - \alpha^2)} & \text{if } j = 3.
\end{cases}
$$

Therefore, the equalities (2.17) and (2.24) turn out to be equivalent. □

The limiting case of the above corollary (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY 2.4.8. *Let* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned} f(x) = f(a)&\frac{2\cos(\alpha(x-a)) + \alpha(x-a)\sin(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sin(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha^3} \\ &+ \int_a^x (f''''(t) + 2\alpha^2 f''(t) + \alpha^4 f(t)) \\ &\qquad \times \frac{\sin(\alpha(x-t)) - \alpha(x-t)\cos(\alpha(x-t))}{2\alpha^3} dt. \end{aligned}$$

The proofs of this and of the next two corollaries are completely similar to that of Corollary 2.4.7, and hence they are omitted. The results in terms of hyperbolic functions in the next corollaries are analogous to Corollary 2.4.7 and Corollary 2.4.8 which are in terms of trigonometric functions.

COROLLARY 2.4.9. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned} f(x) = f(a)&\frac{\beta^2\cosh(\alpha(x-a)) - \alpha^2\cosh(\beta(x-a))}{\beta^2 - \alpha^2} \\ &+ f'(a)\frac{\beta^3\sinh(\alpha(x-a)) - \alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ f''(a)\frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2 - \alpha^2} \\ &+ f'''(a)\frac{\alpha\sinh(\beta(x-a)) - \beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ \int_a^x (f''''(t) - (\alpha^2 + \beta^2) f''(t) + \alpha^2\beta^2 f(t)) \\ &\qquad \times \frac{\alpha\sinh(\beta(x-t)) - \beta\sinh(\alpha(x-t))}{\alpha\beta(\beta^2 - \alpha^2)} dt. \end{aligned} \tag{2.25}$$

COROLLARY 2.4.10. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}^4(I)$ and $a, x \in I$, we have*

$$\begin{aligned} f(x) = f(a) &\frac{2\cosh(\alpha(x-a)) - \alpha(x-a)\sinh(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sinh(\alpha(x-a)) - \alpha(x-a)\cosh(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sinh(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\alpha(x-a)\cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ &+ \int_a^x (f''''(t) - 2\alpha^2 f''(t) + \alpha^4 f(t)) \\ &\qquad \times \frac{\alpha(x-t)\cosh(\alpha(x-t)) - \sinh(\alpha(x-t))}{2\alpha^3}\, dt. \end{aligned}$$

## 2.5. A generalization of the Taylor mean value theorem

Before describing the mean value form of Theorem 2.4.2, we recall the extended mean value theorem for integrals.

LEMMA 2.5.1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $g : [a, b] \to \mathbb{R}$ a nonnegative (or nonpositive) integrable function. Then there exists $\xi \in [a, b]$ such that*

$$\int_a^b fg = f(\xi) \int_a^b g.$$

In the rest of this chapter, for any continuous function $h : \mathbb{R} \to \mathbb{R}$, let $\rho^+(h) \in [0, +\infty]$ (resp. $\rho^-(h) \in [-\infty, 0]$) denote the infimum of the positive roots (resp. the supremum of the negative roots) of $h$.

LEMMA 2.5.2. *Let $n \in \mathbb{N}$ and $c = (c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for $k \in \{1, \dots, n-1\}$,*

$$\big[\rho^-(\omega_c^{(k)}), \rho^+(\omega_c^{(k)})\big] \subseteq \big[\rho^-(\omega_c^{(k-1)}), \rho^+(\omega_c^{(k-1)})\big]. \tag{2.26}$$

*Furthermore, $\big[\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})\big]$ is a neighborhood of* $0$.

PROOF. To prove the statement (2.26), let $t \in \big[\rho^-(\omega_c^{(k)}), \rho^+(\omega_c^{(k)})\big] \setminus \{0\}$, where $k \in \{1, \dots, n-1\}$. Assume first that $t > 0$. Then $\omega_c^{(k)}$ does not vanish in the open interval $]0, t[$. Therefore, $\omega_c^{(k-1)}$ is strictly monotone on $[0, t]$. Hence, for all $s \in ]0, t]$, we have $0 = \omega_c^{(k-1)}(0) \neq \omega_c^{(k-1)}(s)$, which shows that $t \in \big[\rho^-(\omega_c^{(k-1)}), \rho^+(\omega_c^{(k-1)})\big]$. This completes the proof of the inclusion in (2.26) for positive elements. For negative elements, the proof is completely analogous.

We show that 0 is an interior point to the interval $\left[\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})\right]$. In view of $\omega_c^{(n-1)}(0) = 1$, it follows that $\omega_c^{(n-1)}$ is positive on $[-r, r]$ for some $r > 0$. Clearly, $[-r, r] \subseteq \left[\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})\right]$. □

THEOREM 2.5.3. *Let $n \in \mathbb{N}$, $c = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for all $f \in \mathcal{C}^n_{\mathbb{R}}(I)$ and $a, x \in I$ with $\rho^-(\omega_c) \leq x - a \leq \rho^+(\omega_c)$, there exists a point $\xi$ between $a$ and $x$ such that*

$$f(x) = (T_{a,c}f)(x) + D_c(f)(\xi) \cdot \int_0^{x-a} \omega_c(t)dt. \tag{2.27}$$

PROOF. In view of Theorem 2.4.2, we have that (2.17) holds. The statement is trivial if $x = a$. Assume first that $a < x$. Then, by our assumption, $x - a \leq \rho^+(\omega_c)$. If $t \in ]a, x[$, then $0 < x - t < x - a$ and hence $\omega_c(x - t)$ has the same sign for all $t \in ]a, x[$. Using this, by Lemma 2.5.1, we conclude that there exists a point $\xi \in [a, x]$ such that

$$\begin{aligned}\int_a^x D_c(f)(t) \cdot \omega_c(x-t)dt &= D_c(f)(\xi) \cdot \int_a^x \omega_c(x-t)dt \\ &= D_c(f)(\xi) \cdot \int_0^{x-a} \omega_c(t)dt.\end{aligned}$$

Using this equality, formula (2.17) implies the assertion.

In the case when $x \leq a$, the proof is analogous. □

THEOREM 2.5.4. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{R}$ and define $\zeta_{n,k,\gamma} : \mathbb{R} \to \mathbb{R}$ by (2.8). Then, for all $f \in \mathcal{C}^n_{\mathbb{R}}(I)$ and $x, a \in I$ with $\rho^-(\zeta'_{n,k,\gamma}) \leq x - a \leq \rho^+(\zeta'_{n,k,\gamma})$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned}f(x) = \sum_{j=0}^{k-1} f^{(j)}(a)\frac{(x-a)^j}{j!} &+ \sum_{j=k}^{n-1} f^{(j)}(a)\zeta_{n,k,\gamma}^{(n-j)}(x-a) \\ &+ (f^{(n)}(\xi) - \gamma f^{(k)}(\xi))\zeta_{n,k,\gamma}(x-a).\end{aligned} \tag{2.28}$$

PROOF. Let the vector $c \in \mathbb{R}^{n+1}$ be given by $c_n = 1$, $c_k := -\gamma$ and $c_j = 0$ otherwise. Then

$$D_c(f) = f^{(n)} - \gamma f^{(k)}$$

and, as we have seen it in the proof of Theorem 2.4.3, the characteristic solution $\omega_c$ of the differential equation $D_c(\omega) = 0$ is equal to $\zeta'_{n,k,\gamma}$.

Assume first that $a < x$. Then, by our assumption, $x - a \leq \rho^+(\zeta'_{n,k,\gamma})$. If $t \in ]a, x[$, then $0 < x - t < x - a$ and hence $\zeta'_{n,k,\gamma}(x - t)$ has the same sign for all $t \in ]a, x[$. Using this, by Lemma 2.5.1, we conclude that there exists a point $\xi \in [a, x]$ such that

$$\begin{aligned}\int_a^x (f^{(n)}(t) - \gamma f^{(k)}(t))\zeta'_{n,k,\gamma}(x-t)dt \\ &= (f^{(n)}(\xi) - \gamma f^{(k)}(\xi)) \int_a^x \zeta'_{n,k,\gamma}(x-t)dt \\ &= (f^{(n)}(\xi) - \gamma f^{(k)}(\xi))\zeta_{n,k,\gamma}(x-a).\end{aligned}$$

By Theorem 2.4.3, we have formula (2.20), combining it with the above equality, we get (2.28). □

The classical Taylor Mean Value Theorem (stated as Theorem 2.2.2 in the introduction) is the particular case of Theorem 2.5.4 when $k = 0$ and $\gamma = 0$. In this setting, we have that

$$\zeta_{n,0,0}(t) = \frac{t^n}{n!}$$

and hence

$$\rho^{\pm}(\zeta'_{n,0,0}) = \pm\infty$$

and (2.28) simplifies to (2.2).

COROLLARY 2.5.5. *For all $f \in \mathcal{C}^2_{\mathbb{R}}(I)$ and $a, x \in I$ with $|a - x| \le \pi$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned}f(x) = f(a)\cos(x-a) + f'(a)\sin(x-a) \\ + (f''(\xi) + f(\xi))(1 - \cos(x-a)).\end{aligned} \tag{2.29}$$

PROOF. Let $n = 2$, $k = 0$ and $\gamma = -1$ in Theorem 2.5.4. Then,

$$\zeta_{2,0,-1}(t) = 1 - \cos(t)$$

and hence

$$\rho^{\pm}(\zeta'_{2,0,-1}) = \pm\pi.$$

Therefore, we can apply the statement of Theorem 2.5.4 and the equality (2.28) reduces to (2.29). □

To see that the condition $|a - x| \le \pi$ of the above corollary cannot be omitted, consider the function $f(x) := x$. Then, for $x = 2\pi$ and $a = 0$, the equality (2.29) simplifies to $2\pi = \xi \cdot 0$, which cannot be valid for any $\xi \in \mathbb{R}$.

COROLLARY 2.5.6. *For all $f \in \mathcal{C}^2_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned}f(x) = f(a)\cosh(x-a) + f'(a)\sinh(x-a) \\ + (f''(\xi) - f(\xi))(\cosh(x-a) - 1).\end{aligned} \tag{2.30}$$

PROOF. Let $n = 2$, $k = 0$ and $\gamma = 1$ in Theorem 2.5.4. Then,

$$\zeta_{2,0,1}(t) = \cosh(t) - 1$$

and hence

$$\rho^{\pm}(\zeta'_{2,0,-1}) = \pm\infty.$$

Therefore, we can apply the statement of Theorem 2.5.4 and the equality (2.28) simplifies to (2.30). □

COROLLARY 2.5.7. *For all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{\cosh(x-a)+\cos(x-a)}{2} \\ &+ f'(a)\frac{\sinh(x-a)+\sin(x-a)}{2} \\ &+ f''(a)\frac{\cosh(x-a)-\cos(x-a)}{2} \\ &+ f'''(a)\frac{\sinh(x-a)-\sin(x-a)}{2} \\ &+ (f''''(t)-f(t))\frac{\cosh(x-a)+\cos(x-a)-2}{2}. \end{aligned} \tag{2.31}$$

PROOF. Let $n = 4$, $k = 0$ and $\gamma = 1$ in Theorem 2.5.4. Then,

$$\zeta_{4,0,1}(t) = \frac{\cosh(t)+\cos(t)-2}{2}$$

and hence

$$\rho^{\pm}(\zeta'_{4,0,1}) = \pm\infty.$$

Thus, by applying the statement of Theorem 2.5.4, the equality (2.28) can be rewritten as (2.31). □

COROLLARY 2.5.8. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$ and let $t_0$ be the smallest positive root of the equation*

$$\beta\sin(\alpha t) = \alpha\sin(\beta t). \tag{2.32}$$

*Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$ with $|x - a| \leq t_0$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{\beta^2\cos(\alpha(x-a)) - \alpha^2\cos(\beta(x-a))}{\beta^2-\alpha^2} \\ &+ f'(a)\frac{\beta^3\sin(\alpha(x-a)) - \alpha^3\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ f''(a)\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2-\alpha^2} \\ &+ f'''(a)\frac{\beta\sin(\alpha(x-a)) - \alpha\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ (f''''(\xi) + (\alpha^2+\beta^2)f''(\xi) + \alpha^2\beta^2 f(\xi)) \\ &\qquad \times \frac{\alpha^2(\cos(\beta(x-a)) - 1) - \beta^2(\cos(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2-\alpha^2)}. \end{aligned} \tag{2.33}$$

PROOF. Let $n = 4$ and apply Theorem 2.5.3 in the setting $c_4 = 1$, $c_3 = c_1 = 0$, $c_2 = \alpha^2 + \beta^2$, $c_0 = \alpha^2\beta^2$, that is, when

$$D_c(f) = f'''' + (\alpha^2 + \beta^2)f'' + \alpha^2\beta^2 f.$$

Then $\sin(\alpha x)$, $\cos(\alpha x)$, $\sin(\beta x)$, and $\cos(\beta x)$ form a fundamental system of solutions for the differential equation

$$D_c(f) = 0$$

and

$$\omega_c(t) = \frac{\beta\sin(\alpha t) - \alpha\sin(\beta t)}{\alpha\beta(\beta^2-\alpha^2)}$$

is a solution to the initial value problem (2.4). Observe that

$$\rho^{\pm}(\omega_c) = \pm t_0.$$

As we have seen it in the proof of Corollary 2.4.7, the equalities in (2.24) hold. On the other hand,

$$\int_0^{x-a} \omega_c(t)dt = \frac{\alpha^2(\cos(\beta(x-a)) - 1) - \beta^2(\cos(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2-\alpha^2)},$$

Therefore, we can apply Theorem 2.5.3 and hence there exists a point $\xi$ between $a$ and $x$ such that the equality (2.27) holds, which reduces to (2.33). □

REMARK 2.5.9. For the applicability of the previous corollary, it is essential to find the zeroes of the equation (2.32). In general, beyond the trivial solution $t = 0$, the other solutions cannot be established algebraically. On the other hand, if $\frac{\alpha}{\beta}$ is rational, say $|\frac{\alpha}{\beta}| = \frac{n}{m}$, where $n, m$ are coprime natural numbers, let $s := \frac{|\alpha|}{n} = \frac{|\beta|}{m} \neq 0$. Then $\alpha = \pm ns$ and $\beta = \pm ms$ and (2.32) is now equivalent to

$$m \sin(nst) = n \sin(mst).$$

In the case when $t = \frac{k}{s}\pi$ for some $k \in \mathbb{N}$, then both sides are equal to zero. If $t$ is not of this form, then $\sin(st) \neq 0$, thus this equation can be rewritten as

$$mU_{n-1}(\cos(st)) = m\frac{\sin(nst)}{\sin(st)} = n\frac{\sin(mst)}{\sin(st)} = nU_{m-1}(\cos(st)),$$

where $U_k$ denotes the $k$th degree Chebyshev polynomial of the second kind. Therefore, the last equation is an algebraic equation for $\cos(st)$. Solving this equation for $\cos(st)$, the smallest positive solution $t_0$ can easily be computed.

The limiting case of Corollary 2.5.8 (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY 2.5.10. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ and let $t_0$ be the smallest positive root of the equation*

$$\sin(\alpha t) = \alpha t \cos(\alpha t). \tag{2.34}$$

*Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$ with $|x - a| \leq t_0$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{2\cos(\alpha(x-a)) + \alpha(x-a)\sin(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sin(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha^3} \\ &+ (f''''(\xi) + 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\ &\qquad \times \frac{2 - 2\cos(\alpha(x-a)) - \alpha(x-a)\sin(\alpha(x-a))}{2\alpha^4}. \end{aligned} \tag{2.35}$$

The proof of this result is completely analogous to that of Corollary 2.5.8, therefore it is omitted.

COROLLARY 2.5.11. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*
(2.36)

$$\begin{aligned} f(x) = f(a) &\frac{\beta^2 \cosh(\alpha(x-a)) - \alpha^2 \cosh(\beta(x-a))}{\beta^2 - \alpha^2} \\ &+ f'(a)\frac{\beta^3 \sinh(\alpha(x-a)) - \alpha^3 \sinh(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ f''(a)\frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2 - \alpha^2} \\ &+ f'''(a)\frac{\alpha \sinh(\beta(x-a)) - \beta \sinh(\alpha(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ (f''''(\xi) - (\alpha^2 + \beta^2) f''(\xi) + \alpha^2\beta^2 f(\xi)) \\ &\qquad \times \frac{\alpha^2(\cosh(\beta(x-a)) - 1) - \beta^2(\cosh(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)}. \end{aligned}$$

PROOF. Let $n = 4$ and apply Theorem 2.5.3 in the setting $c_4 = 1$, $c_3 = c_1 = 0$, $c_2 = -(\alpha^2 + \beta^2)$, $c_0 = \alpha^2\beta^2$, that is, when

$$D_c(f) = f'''' - (\alpha^2 + \beta^2) f'' + \alpha^2\beta^2 f.$$

Then $\sinh(\alpha x)$, $\cosh(\alpha x)$, $\sinh(\beta x)$, and $\cosh(\beta x)$ form a fundamental system of solutions for the differential equation

$$D_c(f) = 0$$

and

$$\omega_c(t) = \frac{\alpha \sinh(\beta t) - \beta \sinh(\alpha t)}{\alpha\beta(\beta^2 - \alpha^2)}$$

is a solution to the initial value problem (2.4).

Now, we prove that

$$\rho^{\pm}(\omega_c) = \pm\infty.$$

First observe that

$$\left(\frac{\sinh(x)}{x}\right)' = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}\right)' = \frac{1}{x}\sum_{n=1}^{\infty} \frac{2n x^{2n}}{(2n+1)!},$$

which shows that the mapping $x \mapsto \sinh(x)/x$ is strictly monotone on $\mathbb{R}_+$. Assume that $|\alpha| < |\beta|$. Then, for all nonzero $t$,

$$\frac{\sinh(\alpha t)}{\alpha t} = \frac{\sinh(|\alpha t|)}{|\alpha t|} < \frac{\sinh(|\beta t|)}{|\beta t|} = \frac{\sinh(\beta t)}{\beta t}.$$

Therefore, any nonzero number $t$ cannot be a root of $\omega_c$, which implies

$$\rho^{\pm}(\omega_c) = \pm\infty.$$

As we have seen it in the proof of Corollary 2.4.9, the equalities in (2.25) hold. On the other hand,

$$\int_0^{x-a} \omega_c(t)dt = \frac{\alpha^2(\cosh(\beta(x-a))-1) - \beta^2(\cosh(\alpha(x-a))-1)}{\alpha^2\beta^2(\beta^2-\alpha^2)}.$$

Therefore, we can apply Theorem 2.5.3 and hence there exists a point $\xi$ between $a$ and $x$ such that the equality (2.27) holds, which reduces to (2.36). □

COROLLARY 2.5.12. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{2\cosh(\alpha(x-a)) - \alpha(x-a)\sinh(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sinh(\alpha(x-a)) - \alpha(x-a)\cosh(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sinh(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\alpha(x-a)\cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ &+ (f''''(\xi) - 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\ &\qquad \times \frac{2 - 2\cosh(\alpha(x-a)) + \alpha(x-a)\sinh(\alpha(x-a))}{2\alpha^4}. \end{aligned}$$

The proof of the above statement is omitted.

CHAPTER 3

# Estimates of linear expressions through factorization

## 3.1. Preliminaries

The aim of this chapter is to establish various factorization results and then to derive estimates for linear functionals through the use of a generalized Taylor theorem. Additionally, several error bounds are established including applications to the trapezoidal rule as well as to a Simpson formula rule. The results outlined in this chapter are based on the research published in [2].

## 3.2. Introduction

Let $X$, $Y$ and $Z$ be normed spaces over the field $\mathbb{K}$, where $\mathbb{K}$ stands either for the field of real numbers $\mathbb{R}$ or for the field of complex numbers $\mathbb{C}$. Assume that $A : X \to Y$ and $B : X \to Z$ are given linear maps. To describe the properties of $A$ and to connect it to those of $B$, it could be useful if $A$ admits a factorization $A = C \circ B$, where $C : Z \to Y$ is also a linear map. (For instance, if $B$ and $C$ are bounded linear operators, then $\|A\| \leq \|B\|\|C\|$.)

An obvious necessary condition for a decomposition $A = C \circ B$ is that $\ker B \subseteq \ker A$ (where $\ker(\cdot)$ denotes the null space of the corresponding operator). On the other hand, this is also sufficient. Indeed, suppose that $\ker B \subseteq \ker A$. We define $C$ on $B(X)$ first. For each element $z \in B(X)$, there exists $x \in X$ such that $z = B(x)$ and then we define $C(z) := A(x)$. This definition is correct, since if $x' \in X$ also satisfies $z = B(x')$, then $x - x' \in \ker B$ and hence, by the assumption $x - x' \in \ker A$, which implies that $A(x') = A(x)$. It is easy to see that the map $C : B(X) \to Y$ defined in this way is linear. If $B(X) = Z$, then it is obvious that $C$ is uniquely determined. If $B(X)$ is a proper subspace, then $C$ can be extended to a linear map $C : Z \to Y$ (using Hamel bases) arbitrarily. However, it is a more important problem to obtain a factorization $A = C \circ B$ in terms of a bounded linear map $C$ provided that $A$ and $B$ are bounded. The next result establishes a sufficient condition for this factorizability.

THEOREM 3.2.1. *Let $X$ and $Z$ be Banach spaces and $Y$ be a normed space over $\mathbb{K}$. Assume that $A : X \to Y$ and $B : X \to Z$ are bounded linear maps* *such that* $\ker B \subseteq \ker A$ *and* $B(X) = Z$. *Then there exists a unique bounded linear map* $C : Z \to Y$ *such that* $A = C \circ B$.

PROOF. In view of the argument above, there exists a uniquely determined linear map $C : Z \to Y$ such that $A = C \circ B$ holds. We only have to prove that $C$ is bounded using that $A$ and $B$ are bounded linear maps.

By Banach's Open Mapping Theorem, $B$ is an open map. Therefore, there exists $r > 0$ such that $\mathcal{B}_Z \subseteq B(r\mathcal{B}_X)$. (Here $\mathcal{B}_X$ and $\mathcal{B}_Z$ denote the closed unit balls of the spaces $X$ and $Z$, respectively.)

Let $z \in Z$ be a nonzero vector. Then $z/\|z\| \in \mathcal{B}_Z$ and hence there exists $u \in r\mathcal{B}_X$ such that $z/\|z\| = B(u)$. Therefore, $z = \|z\|B(u) = B(\|z\|u)$, where $\|u\| \leq r$. This implies that $C(z) = A(\|z\|u)$, thus

$$\|C(z)\| = \|A(\|z\|u)\| \leq (r\|A\|)\|z\|.$$

This inequality is obviously true also for $z = 0$ and yields that $\|C\| \leq r\|A\|$ and proves the boundedness of $C$. □

The main purpose of this chapter is to investigate the problem of factorization in the following setting. Let $I$ denote the compact interval $[a, b]$ and, for $n \geq 0$, let $\mathcal{C}^n_{\mathbb{K}}(I)$ denote the space of $n$ times continuously differentiable $\mathbb{K}$-valued functions (equipped with the norm $\|f\|_{\mathcal{C}^n} := \|f\|_\infty + \cdots + \|f^{(n)}\|_\infty$). (The space $\mathcal{C}^0_{\mathbb{K}}(I)$ will simply be denoted by $\mathcal{C}_{\mathbb{K}}(I)$.) Furthermore, let $\mu$ be a nonzero $\mathbb{K}$-valued and bounded Borel measure on $[a, b]$ throughout this chapter. The main goal is to obtain various estimates for the linear functional $\mathcal{A}_\mu : \mathcal{C}_{\mathbb{K}}(I) \to \mathbb{K}$ defined by

$$\mathcal{A}_\mu(f) := \int_{[a,b]} f(x)d\mu(x).$$

In order to construct $n \in \mathbb{N}$ and a linear map $B : \mathcal{C}^n_{\mathbb{K}}(I) \to \mathcal{C}_{\mathbb{K}}(I)$ such that $\ker B \subseteq \ker \mathcal{A}_\mu$, we search for exponential polynomials in the kernel of $\mathcal{A}_\mu$. For this aim, let us define the function $\mathcal{S}_\mu : \mathbb{C} \to \mathbb{C}$ by

$$\mathcal{S}_\mu(\lambda) := \int_{[a,b]} e^{\lambda x} d\mu(x).$$

The function $\mathcal{S}_\mu$ will be termed the *spectral function related to the measure* $\mu$. (In fact, the Laplace transform of the measure $\mu$ is strongly connected to this function.) Clearly, (due to the boundedness of $\mu$), $\mathcal{S}_\mu$ is an entire function.

The set of zeros of $\mathcal{S}_\mu$ will be denoted by $\Lambda_\mu$ and will be called the *spectral set related to* $\mu$. For $\lambda \in \Lambda_\mu$, let $m \in \mathbb{N}$ be the *multiplicity of the root* $\lambda$, i.e., the largest number such that

$$\mathcal{S}_\mu^{(j)}(\lambda) = \int_{[a,b]} x^j e^{\lambda x} d\mu(x) = 0 \qquad (j \in \{0, \dots, m-1\}).$$

Since $\mu \neq 0$, it follows that, for some $\ell \in \mathbb{N}$, $\mathcal{S}_\mu^{(\ell)}(\lambda) \neq 0$, and hence the multiplicity of an element $\lambda \in \Lambda_\mu$ is well-defined and will be denoted by $m(\mathcal{S}_\mu, \lambda)$. The above equality shows that, for $\lambda \in \Lambda_\mu$, the exponential polynomials

$$x \mapsto x^j e^{\lambda x} \qquad (j \in \{0, \dots, m(\mathcal{S}_\mu, \lambda) - 1\})$$

belong to the kernel of $\mathcal{A}_\mu$.

For fixed elements $\lambda_1, \dots, \lambda_k \in \Lambda_\mu$, we consider the polynomial $P : \mathbb{C} \to \mathbb{C}$ given by

$$P(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k} = c_n \lambda^n + \cdots + c_1 \lambda + c_0, \tag{3.1}$$

where $1 \leq m_i \leq m(\mathcal{S}_\mu, \lambda_i)$ for all $i$ and $n := m_1 + \cdots + m_k$. Clearly, $c_n = 1$. Then we define the linear differential operator $D_c : \mathcal{C}_\mathbb{K}^n(I) \to \mathcal{C}_\mathbb{K}(I)$ by the formula

$$D_c(f) := c_n f^{(n)} + \cdots + c_1 f' + c_0 f \qquad (f \in \mathcal{C}^n(I)). \tag{3.2}$$

It follows from the theory of ordinary differential equations (see [30]) that $D_c$ is a bounded and surjective linear operator and the exponential polynomials

$$x \mapsto x^j e^{\lambda_i x} \qquad (i \in \{1, \dots, k\},\ j \in \{0, \dots, m_i - 1\}) \tag{3.3}$$

form a fundamental system of solutions of the differential equation $D_c(f) = 0$. In other words, the above exponential polynomials span the kernel of $D_c$ and hence $\ker D_c \subseteq \ker \mathcal{A}_\mu$. In view of Theorem 3.2.1, there exists a unique bounded linear map $C : \mathcal{C}_\mathbb{K}(I) \to \mathbb{K}$ such that $\mathcal{A}_\mu|_{\mathcal{C}_\mathbb{K}^n(I)} = C \circ D_c$. In what follows, we explicitly construct $C$ and then we present several applications to obtain sharp upper bounds for the error terms of quadrature rules. For standard references about error bounds for quadrature rules, we refer to the monographs [5] by Atkinson and [13] by Faires and Burden and to the recent papers [6] by Barnett *et al.*, [11] by Cruz-Uribe and Neugebauer, [16] by Masjed-Jamei *et al.*, [25] by Talman and [28] by Ujević.

## 3.3. Estimating linear functionals

Our basic factorization theorem is stated as follows.

THEOREM 3.3.1. *Let $\mu$ be a nonzero bounded $\mathbb{C}$-valued Borel measure on $[a,b]$, let $\lambda_1, \dots, \lambda_k \in \Lambda_\mu$ and $m_1, \dots, m_k \in \mathbb{N}$ with $m_i \leq m(\mathcal{S}_\mu, \lambda_i)$ for $i \in \{1, \dots, k\}$. Define $c = (c_0, \dots, c_n) \in \mathbb{C}^{n+1}$ by (3.1) (where $n := m_1 + \cdots + m_k$) and the differential operator $D_c : \mathcal{C}_\mathbb{C}^n([a,b]) \to \mathcal{C}_\mathbb{C}([a,b])$ by (3.2).*

*Let $\omega_c \in \mathcal{C}^n_{\mathbb{C}}(\mathbb{R})$ be the characteristic solution of $D_c(\omega) = 0$. Finally, define $g : [a, b] \to \mathbb{C}$ by*

$$g(t) := \int_{[t,b]} \omega_c(x - t) d\mu(x). \tag{3.4}$$

*Then, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a, b])$,*

$$\mathcal{A}_\mu(f) := \int_{[a,b]} f(x) d\mu(x) = \int_{[a,b]} D_c(f)(t) \cdot g(t) dt. \tag{3.5}$$

*In other words, $\mathcal{A}_\mu|_{\mathcal{C}^n_{\mathbb{C}}(I)} = C_g \circ D_c$, where $C_g : \mathcal{C}_{\mathbb{C}}(I) \to \mathbb{C}$ is given by*

$$C_g(h) = \int_{[a,b]} h(t) g(t) dt.$$

PROOF. Due to the assumptions $\lambda_1, \ldots, \lambda_k \in \Lambda_\mu$ and $m_i \leq m(\mathcal{S}_\mu, \lambda_i)$ for $i \in \{1, \ldots, k\}$, it follows that the exponential polynomials given in (3.3) are in the kernel of $\mathcal{A}_\mu$. On the other hand, these functions form a fundamental system of solutions for the differential equation $D_c(\omega) = 0$, which implies that $\ker D_c \subseteq \ker \mathcal{A}_\mu$. Since the differential operator has constant coefficients, it follows that, for all $i \geq 0$, the function

$$\mathbb{R} \ni x \mapsto \omega_c^{(i)}(x - a)$$

is also in the kernel of $D_c$ and hence in the kernel of $\mathcal{A}_\mu$. Therefore, for all $i \geq 0$,

$$\int_{[a,b]} \omega_c^{(i)}(x - a) d\mu(x) = 0.$$

In what follows, let $\chi_S$ denote the characteristic function of any subset $S$ of $[a, b]$. Applying Theorem 2.4.2, the above equalities and finally Fubini's Theorem, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a, b])$, we obtain

$$\begin{aligned}
\mathcal{A}_\mu(f) &= \int_{[a,b]} f(x)d\mu(x) \\
&= \int_{[a,b]} \Bigg( \sum_{j=0}^{n-1} \Bigg( f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1}\omega_c^{(i)}(x-a) \Bigg) \\
&\qquad + \int_a^x D_c(f)(t) \cdot \omega_c(x-t)dt \Bigg) d\mu(x) \\
&= \sum_{j=0}^{n-1} f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \int_{[a,b]} \omega_c^{(i)}(x-a)d\mu(x) \\
&\qquad + \int_{[a,b]} \Bigg( \int_a^x D_c(f)(t) \cdot \omega_c(x-t)dt \Bigg) d\mu(x) \\
&= \int_{[a,b]} \Bigg( \int_a^x D_c(f)(t) \cdot \omega_c(x-t)dt \Bigg) d\mu(x) \\
&= \int_{[a,b]} \int_{[a,b]} \chi_{[a,x]}(t) \cdot D_c(f)(t) \cdot \omega_c(x-t)dtd\mu(x) \\
&= \int_{[a,b]} \int_{[a,b]} \chi_{[a,x]}(t) \cdot D_c(f)(t) \cdot \omega_c(x-t)d\mu(x)dt \\
&= \int_{[a,b]} D_c(f)(t) \int_{[t,b]} \omega_c(x-t)d\mu(x)dt \\
&= \int_{[a,b]} D_c(f)(t) \cdot g(t)dt.
\end{aligned}$$

This proves (3.5). ☐

THEOREM 3.3.2. *Let $\mu$ be a nonzero bounded $\mathbb{C}$-valued Borel measure on $[a, b]$, let $0 \le k \le n$ and $\gamma \in \mathbb{C}$. Assume that*

(3.6)

$$\int_{[a,b]} x^i d\mu(x) = 0, \quad (i \in \{0, \dots, k-1\}),$$

$$\int_{[a,b]} \exp\left( \sqrt[n-k]{\gamma} \exp\left(\tfrac{2j\pi}{n-k}\boldsymbol{i}\right) x\right) d\mu(x) = 0, \quad (j \in \{0, \dots, n-k-1\}),$$

*where* $\sqrt[n-k]{\gamma}$ *denotes the root of order* $(n-k)$ *of* $\gamma$ *with the smallest nonnegative argument in the interval* $[0, 2\pi)$. *Define* $g : [a, b] \to \mathbb{C}$ *by*

$$g(t) := \int_{[t,b]} \zeta'_{n,k,\gamma}(x - t) d\mu(x). \tag{3.7}$$

*Then, for all* $f \in \mathcal{C}^n_{\mathbb{C}}([a, b])$,

$$\mathcal{A}_\mu(f) = \int_{[a,b]} (f^{(n)}(t) - \gamma f^{(k)}(t)) \cdot g(t) dt. \tag{3.8}$$

PROOF. Consider the differential operator $D_c : \mathcal{C}^n_{\mathbb{C}}(I) \to \mathcal{C}_{\mathbb{C}}(I)$ given by $D_c(f) := f^{(n)} - \gamma f^{(k)}$. Its characteristic polynomial $P_c$ is given by $P_c(\lambda) = \lambda^n - \gamma\lambda^k = \lambda^k(\lambda^{n-k} - \gamma)$. The roots of this polynomial are $\lambda = 0$ with multiplicity $k$ and the roots of order $(n - k)$ of $\gamma$ (with multiplicities equal to 1). The second group of roots can be written in the form

$$\sqrt[n-k]{\gamma} \exp\left(\tfrac{2j\pi}{n-k}\boldsymbol{i}\right) \qquad (j \in \{0, \ldots, n - k - 1\}).$$

Therefore, a fundamental solution system of the differential equation

$$D_c(f) = 0$$

can be obtained as

$$\begin{aligned}&\left\{x^i \mid i \in \{0, \ldots, k-1\}\right\}\\ &\qquad \cup \left\{ \exp\left( \sqrt[n-k]{\gamma} \exp\left(\tfrac{2j\pi}{n-k}\boldsymbol{i}\right)x\right) \mid j \in \{0, \ldots, n-k-1\}\right\}.\end{aligned}$$

Thus, the condition (3.6) ensures that $\ker(D_c) \subseteq \ker(\mathcal{A}_\mu)$. Therefore, the polynomial

$$x \mapsto \sum_{j=0}^{k-1} f^{(j)}(a)\frac{(x-a)^j}{j!}$$

(whose degree is at most $k - 1$) is in the kernel of $D_c$ and hence, it belongs to $\ker(\mathcal{A}_\mu)$. On the other hand, by equality (2.9) of Lemma 2.3.4, $\zeta_{n,k,\gamma}$ solves the differential equation $D_c(f) = 0$ and hence, the mapping

$$x \mapsto \sum_{j=k}^{n-1} f^{(j)}(a)\zeta^{(n-j)}_{n,k,\gamma}(x - a)$$

belongs to the kernel of $D_c$. Consequently, it is also in $\ker(\mathcal{A}_\mu)$. Combining these inclusions, we obtain that

$$\int_{[a,b]} \left(\sum_{j=0}^{k-1} f^{(j)}(a)\frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a)\zeta^{(n-j)}_{n,k,\gamma}(x-a)\right)d\mu(x) = 0.$$

Now applying Theorem 2.4.3 and following a similar reasoning as in the proof of Theorem 3.3.1, we can conclude that (3.8) holds, where the function $g$ is defined by (3.7). □

Before formulating the next result, we recall the definition of the $p$th norm of a Lebesgue measurable function $f : I \to \mathbb{C}$ for $p \in [1, \infty]$:

$$\|f\|_p := \begin{cases} \left( \int_{[a,b]} |f(t)|^p dt \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \inf\{s \geq 0 \colon |f(t)| \leq s \text{ for a.e. } t \in I\} & \text{if } p = \infty. \end{cases}$$

COROLLARY 3.3.3. *Under the notation and assumptions of Theorem 3.3.1, for all $f \in \mathcal{C}^n_{\mathbb{C}}(I)$ and for all $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$|\mathcal{A}_\mu(f)| \leq \|D_c(f)\|_p \cdot \|g\|_q. \tag{3.9}$$

*If, in addition, $c = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$, $\mu$ is a real-valued measure, $f \in \mathcal{C}^n_{\mathbb{R}}(I)$, and both $g$ and $D_c(f)$ are nonnegative (or nonpositive) on $[a, b]$, then*

$$\mathcal{A}_\mu(f) \geq 0. \tag{3.10}$$

*If $g$ and $D_c(f)$ have opposite signs over $[a, b]$, then this inequality reverses.*

PROOF. Applying the identity (3.5) and Hölder's inequality, we get

$$|\mathcal{A}_\mu(f)| = \left| \int_{[a,b]} D_c(f)(t) \cdot g(t) dt \right| \leq \left\| D_c(f) \right\|_p \cdot \|g\|_q,$$

which proves (3.9). In the real-valued setting, if $D_c(f) \cdot g$ is nonnegative on $[a, b]$, then we can see that (3.10) holds. If $D_c(f) \cdot g$ is nonpositive, then (3.10) is valid with reversed inequality sign. □

COROLLARY 3.3.4. *Let $\mu$ be a nonzero bounded $\mathbb{C}$-valued Borel measure on $[a, b]$, let $0 \leq k \leq n$ and $\gamma \in \mathbb{C}$. Assume that the equalities in (3.6) hold. Define $g : [a, b] \to \mathbb{C}$ by (3.7). Then, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a, b])$ and $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\left| \int_{[a,b]} f(x) d\mu(x) \right| \leq \|f^{(n)} - \gamma f^{(k)}\|_p \cdot \|g\|_q. \tag{3.11}$$

*If, in addition, $\gamma \in \mathbb{R}$, $\mu$ is a real-valued measure and both $g$ and $f^{(n)} - \gamma f^{(k)}$ are nonnegative (or nonpositive) on $[a, b]$, then the inequality (3.10) holds. If $g$ and $f^{(n)} - \gamma f^{(k)}$ have opposite signs over $[a, b]$, then this inequality reverses.*

PROOF. The statement of this corollary is an immediate consequence of Theorem 3.3.2 and Hölder's inequality. □

### 3.4. Application to the trapezoidal rule

The following lemma is very likely well-known, however, for the sake of completeness, we provide its very simple proof.

LEMMA 3.4.1. *For all* $t \in \mathbb{R}_+$

$$\frac{t}{\sinh(t)} < 1 < t\coth(t) \tag{3.12}$$

*and, for all* $t \in (0, \pi)$*,*

$$t\cot(t) < 1 < \frac{t}{\sin(t)}. \tag{3.13}$$

PROOF. To prove (3.12), let $t$ be a positive number. By the Taylor series expansions of $\sinh$ and $\cosh$, we get that

$$\sinh(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} > \frac{t}{1!} = t,$$

and

$$\sinh(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \le \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k)!} = t\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} = t\cosh(t).$$

These two inequalities directly imply the left and the right hand side inequality in (3.12), respectively.

To verify (3.13), let $t \in (0, \pi)$. Applying the Taylor mean value theorem to the $\sin$ function at $0$, and then using that $\sin$ is positive over $(0, \pi)$, we can conclude that there exists $s \in (0, t)$ such that

$$\sin(t) = \sin(0) + \sin'(0)t + \frac{1}{2}\sin''(s)t^2 = t - \frac{1}{2}\sin(s)t^2 < t,$$

which proves the second inequality in (3.13).

If $t \in [\frac{1}{2}\pi, \pi)$, then $\cot(t) \le 0$, and hence the first inequality in (3.13) is obvious. In the case when $t \in (0, \frac{1}{2}\pi)$, we use the Taylor mean value theorem to the $\tan$ function at $0$, which asserts the existence of $s \in (0, t)$ such that

$$\tan(t) = \tan(0) + \tan'(0)t + \frac{1}{2}\tan''(s)t^2 = t + \frac{\sin(s)}{\cos^3(s)}t^2 > t,$$

where we used that the functions $\sin$ and $\cos$ are positive over $(0, \frac{1}{2}\pi)$. This inequality implies the first inequality in (3.13) also in the case when $t \in (0, \frac{1}{2}\pi)$. $\square$

Given a compact interval $[a,b]$, the classical trapezoidal rule asserts that, for a twice differentiable function $f : [a,b] \to \mathbb{R}$,

$$\frac{1}{b-a}\int_a^b f = \frac{f(a)+f(b)}{2} - R_T(f),$$

where the remainder term $R_T(f)$ has various estimates in terms of the norms of the second derivative of $f$ and the length of the interval $[a,b]$. For instance, (see [5, pp. 252–253]),

$$|R_T(f)| \leq \frac{(b-a)^2}{12}\|f''\|_\infty.$$

The aim of this section is to establish various further estimates for $R_T(f)$, which is defined by

$$R_T(f) := \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f.$$

Observe that, with $\mu := \frac{1}{2}(\delta_a + \delta_b) - \nu$ (where $\delta_t$ denotes the Dirac measure concentrated at $t$ and $\nu$ stands for the normalized Lebesgue measure on $[a,b]$), we can obtain that

$$R_T(f) = \mathcal{A}_\mu(f).$$

The corresponding spectral function is given by

$$\mathcal{S}_\mu(\lambda) = \frac{e^{\lambda a}+e^{\lambda b}}{2} - \frac{1}{b-a}\int_a^b e^{\lambda x}dx \qquad (\lambda \in \mathbb{C}). \tag{3.14}$$

LEMMA 3.4.2. *Let $\lambda \in \mathbb{C}$. Then $\lambda$ is a root of the spectral function $\mathcal{S}_\mu$ given by (3.14) if and only if $u := \lambda\frac{b-a}{2}$ is a fixed point of the tangent hyperbolic function. The multiplicity of $\lambda$ equals 1 if $\lambda \neq 0$ and equals 2 if $\lambda = 0$.*

PROOF. We can see that $\mathcal{S}_\mu(\lambda) = 0$ holds trivially if $\lambda = 0$, and then $u = 0$ is indeed a fixed point of the tangent hyperbolic function.

An easy computation yields that, for $\lambda \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\mathcal{S}_\mu(\lambda) &= \frac{e^{\lambda b}+e^{\lambda a}}{2} - \frac{e^{\lambda b}-e^{\lambda a}}{\lambda(b-a)} \\
&= e^{\lambda\frac{a+b}{2}}\left(\frac{e^{\lambda\frac{b-a}{2}}+e^{\lambda\frac{a-b}{2}}}{2} - \frac{e^{\lambda\frac{b-a}{2}}-e^{\lambda\frac{a-b}{2}}}{\lambda(b-a)}\right) \\
&= e^{\lambda\frac{a+b}{2}}\left(\cosh(\lambda\tfrac{b-a}{2}) - \frac{2\sinh(\lambda\frac{b-a}{2})}{\lambda(b-a)}\right).
\end{aligned}$$

Now, assume that $\lambda \in \mathbb{C} \setminus \{0\}$ is a solution of the equation $\mathcal{S}_\mu(\lambda) = 0$. Then it satisfies

$$\cosh(\lambda \tfrac{b-a}{2}) = \frac{2\sinh(\lambda \frac{b-a}{2})}{\lambda(b-a)}.$$

In this case, $\cosh(\lambda \frac{b-a}{2})$ cannot be zero, because then $\sinh(\lambda \frac{b-a}{2})$ would also be zero, which is impossible. Therefore, with the notation $u := \lambda \frac{b-a}{2}$, the equation $\mathcal{S}_\mu(\lambda) = 0$ is equivalent to $u = \tanh(u)$, which shows that $u$ must be a fixed point of the tangent hyperbolic function. To check the multiplicities of the roots of $\mathcal{S}_\mu$, let $u = \lambda \frac{b-a}{2}$ be a fixed point of the tangent hyperbolic function. Then $\cosh(\lambda \frac{b-a}{2}) \neq 0$ and, for $\lambda \neq 0$, we get

$$\begin{aligned}
\mathcal{S}'_\mu(\lambda) &= \tfrac{a+b}{2} e^{\lambda \frac{a+b}{2}} \left( \cosh(\lambda \tfrac{b-a}{2}) - \frac{2\sinh(\lambda \frac{b-a}{2})}{\lambda(b-a)} \right) \\
&\quad + e^{\lambda \frac{a+b}{2}} \left( \cosh(\lambda \tfrac{b-a}{2}) - \frac{2\sinh(\lambda \frac{b-a}{2})}{\lambda(b-a)} \right)' \\
&= e^{\lambda \frac{a+b}{2}} \left( \tfrac{b-a}{2} \sinh(\lambda \tfrac{b-a}{2}) + \frac{2\sinh(\lambda \frac{b-a}{2})}{\lambda^2(b-a)} - \frac{\cosh(\lambda \frac{b-a}{2})}{\lambda} \right) \\
&= e^{\lambda \frac{a+b}{2}} \cosh(\lambda \tfrac{b-a}{2}) \left( \tfrac{b-a}{2} \tanh(\lambda \tfrac{b-a}{2}) + \frac{2\tanh(\lambda \frac{b-a}{2})}{\lambda^2(b-a)} - \frac{1}{\lambda} \right) \\
&= e^{\lambda \frac{a+b}{2}} \cosh(\lambda \tfrac{b-a}{2}) \left( \lambda \left(\tfrac{b-a}{2}\right)^2 + \frac{1}{\lambda} - \frac{1}{\lambda} \right) \\
&= e^{\lambda \frac{a+b}{2}} \cosh(\lambda \tfrac{b-a}{2}) \lambda \left(\tfrac{b-a}{2}\right)^2 \neq 0,
\end{aligned}$$

which proves that the multiplicity of $\lambda \neq 0$ equals $1$.

In the case when $\lambda = 0$, from the defining formula of $\mathcal{S}_\mu$, we get

$$\begin{aligned}
S'_\mu(\lambda) &= \frac{ae^{\lambda a} + be^{\lambda b}}{2} - \frac{1}{b-a} \int_a^b xe^{\lambda x} dx, \\
S'_\mu(0) &= \frac{a+b}{2} - \frac{1}{b-a} \int_a^b x dx = 0,
\end{aligned}$$

and

$$S''_\mu(\lambda) = \frac{a^2 e^{\lambda a} + b^2 e^{\lambda b}}{2} - \frac{1}{b-a} \int_a^b x^2 e^{\lambda x} dx.$$

Thus,

$$S''_\mu(0) = \frac{a^2+b^2}{2} - \frac{1}{b-a}\int_a^b x^2 dx$$
$$= \frac{a^2+b^2}{2} - \frac{a^2+ab+b^2}{3} = \frac{(b-a)^2}{6} \neq 0,$$

which proves that the multiplicity of $\lambda = 0$ equals 2. □

In order to apply our main theorems to the trapezoidal rule, we shall need to describe the fixed points of the tangent hyperbolic function.

LEMMA 3.4.3. *A number $u \in \mathbb{C}$ is a fixed point of the tangent hyperbolic function, i.e.,*

$$\tanh(u) = u \tag{3.15}$$

*holds if and only if $u = v\boldsymbol{i}$, where $v \in \mathbb{R}$ is a fixed point of the tangent function. Furthermore, for all $k \in \mathbb{Z}$, the open interval $\big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ contains exactly one fixed point of the tangent function.*

PROOF. Assume that $u = v\boldsymbol{i}$, where $v \in \mathbb{R}$ and $\tan(v) = v$. Then

$$\tanh(u) = \tanh(v\boldsymbol{i}) = \frac{\sinh(v\boldsymbol{i})}{\cosh(v\boldsymbol{i})} = \frac{\sin(v)}{\cos(v)}\boldsymbol{i} = \tan(v)\boldsymbol{i} = v\boldsymbol{i} = u,$$

which shows that $u$ is a fixed point of the function $\tanh$.

Assume now that (3.15) holds for $u = w + v\boldsymbol{i}$, where $w, v \in \mathbb{R}$. Then

$$\begin{aligned}\tanh(u) &= \tanh(w+v\boldsymbol{i})\\ &= \frac{\tanh(w)+\tanh(v\boldsymbol{i})}{1+\tanh(w)\tanh(v\boldsymbol{i})}\\ &= \frac{\tanh(w)+\tan(v)\boldsymbol{i}}{1+\tanh(w)\tan(v)\boldsymbol{i}}\\ &= \frac{(\tanh(w)+\tan(v)\boldsymbol{i})(1-\tanh(w)\tan(v)\boldsymbol{i})}{1+\tanh(w)^2\tan(v)^2}.\end{aligned}$$

Thus, (3.15) is equivalent to

$$\frac{\tanh(w)(1+\tan(v)^2)}{1+\tanh(w)^2\tan(v)^2} = w, \qquad \frac{\tan(v)(1-\tanh(w)^2)}{1+\tanh(w)^2\tan(v)^2} = v.$$

We show that these two equalities imply that $w = 0$ (which then easily implies that $\tan(v) = v$). To the contrary, assume that $w \neq 0$. Then $w \neq \tanh(w)$, and hence the first equation implies that $\tan(v) \neq 0$ implying that $v \neq 0$. Then

$$\frac{w}{v} = \frac{\tanh(w)(1+\tan(v)^2)}{\tan(v)(1-\tanh(w)^2)} = \frac{\sinh(2w)}{\sin(2v)},$$

equivalently

$$\frac{\sin(2v)}{2v} = \frac{\sinh(2w)}{2w}.$$

On the other hand, by Lemma 3.4.1, for every nonzero real numbers $x$ and $y$, the following inequalities hold

$$\frac{\sin(x)}{x} < 1 < \frac{\sinh(y)}{y}.$$

Therefore, these inequalities with $x = 2v$ and $y = 2w$ yield an obvious contradiction. This contradiction shows that $w = 0$ must hold and hence $\tan(v) = v$.

Let $k \in \mathbb{Z}$. The function $h$ defined by $h(x) := \tan(x) - x$ is differentiable on the open interval $I_k := \big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ with a derivative $h'(x) = \tan^2(x) > 0$ for all $x \in I_k \setminus \{k\pi\}$. This shows that $h$ is strictly increasing and hence it can have at most one zero in $I_k$. On the other hand, we have that $\lim_{x\to(k\pm\frac{1}{2})\pi} h(x) = \pm\infty$, which shows that $h$ changes sign over $I_k$ and therefore it has a unique zero in $I_k$. □

The unique fixed point of the tangent function in the open interval $\big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ will be denoted by $\tau_k$ in the sequel.

THEOREM 3.4.4. *Let $k \in \mathbb{N}$, $0 \le n_1 < \cdots < n_k$ be integers, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_j := \frac{2}{(b-a)}\tau_{n_j}$ for $j \in \{1, \ldots, k\}$. Define $(c_0, c_1, \ldots, c_{2k}) \in \mathbb{R}^{2k+1}$ by the equality*

$$\big(z^2+\lambda_1^2\big)\cdots\big(z^2+\lambda_k^2\big) = c_{2k}z^{2k} + \cdots + c_1 z^1 + c_0 =: P_c(z) \qquad (z \in \mathbb{C}).$$

*Then, for all $f \in \mathcal{C}^{2k}_{\mathbb{K}}([a,b])$,*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f = \int_a^b D_c(f)(t)\cdot g(t)dt,$$

*where*

$$g(t) := \begin{cases} \displaystyle\sum_{j=1}^{k} \frac{\sin(\lambda_j(b-t)/2)\sin(\lambda_j(t-a)/2)}{\lambda_j Q_j(\lambda_j)\sin(\lambda_j(b-a)/2)} & \text{if } n_1 > 0, \\ \displaystyle\sum_{j=2}^{k} \frac{\sin(\lambda_j(b-t)/2)\sin(\lambda_j(t-a)/2)}{\lambda_j Q_j(\lambda_j)\sin(\lambda_j(b-a)/2)} \\ \qquad + \displaystyle\frac{(b-t)(t-a)}{2Q_1(0)(b-a)} & \text{if } n_1 = 0, \end{cases} \tag{3.16}$$

*and $Q_j(z) := \prod_{\ell\in\{1,\ldots,k\}\setminus\{j\}}(\lambda_\ell^2 - z^2)$ for $j \in \{1, \ldots, k\}$.*

PROOF. Due to the evenness of the characteristic polynomial $P_c$, we have that $c_1 = c_3 = \cdots = c_{2k-1} = 0$ and $c_{2k} = 1$. The roots of $P_c$ are $\lambda_j \boldsymbol{i}$ and $(-\lambda_j \boldsymbol{i})$ for $j \in \{1, \ldots, k\}$ with multiplicities equal to $1$ except if $n_1 = 0$, then $\lambda_1 = 0$ and its multiplicity is equal to $2$.

First, consider the case when $0 < n_1$. Then all roots of the characteristic polynomial have multiplicity $1$. To apply Lemma 2.3.3, observe that the polynomial connected to the root $\lambda_j \boldsymbol{i}$ is given as

$$\prod_{\ell \in \{1,\ldots,k\} \setminus \{j\}} (z - \lambda_\ell \boldsymbol{i}) \prod_{\ell \in \{1,\ldots,k\}} (z + \lambda_\ell \boldsymbol{i}) = (z + \lambda_j \boldsymbol{i}) \prod_{\ell \in \{1,\ldots,k\} \setminus \{j\}} (z^2 + \lambda_\ell^2) = (z + \lambda_j \boldsymbol{i}) Q_j(z\boldsymbol{i}).$$

Similarly, the polynomial which is connected to the root $(-\lambda_j \boldsymbol{i})$ is given as $(z - \lambda_j \boldsymbol{i}) Q_j(z\boldsymbol{i})$. Therefore, in view of Lemma 2.3.3, we can obtain the characteristic solution $\omega_c$ of $D_c$ is given by

$$\omega_c(t) = \sum_{j=1}^{k} \left( \frac{\exp(\lambda_j t \boldsymbol{i})}{2\lambda_j \boldsymbol{i} Q_j(\lambda_j)} - \frac{\exp(-\lambda_j t \boldsymbol{i})}{2\lambda_j \boldsymbol{i} Q_j(\lambda_j)} \right) = \sum_{j=1}^{k} \frac{\sin(\lambda_j t)}{\lambda_j Q_j(\lambda_j)}.$$

In the case when $n_1 = 0$. Then the first root $(\lambda_1 = 0)$ of the characteristic polynomial has multiplicity $2$ and the other roots have multiplicity $1$. Applying Lemma 2.3.3 in a similar manner, we can see that the characteristic solution $\omega_c$ of $D_c$ is given by

$$\omega_c(t) = \frac{t}{Q_1(0)} + \sum_{j=2}^{k} \frac{\sin(\lambda_j t)}{\lambda_j Q_j(\lambda_j)}.$$

By Theorem 3.3.1, $g : [a, b] \to \mathbb{C}$ is defined by formula (3.4). Therefore, for $t \in (a, b]$, we have

$$g(t) = \frac{1}{2}\omega_c(b - t) - \frac{1}{b - a} \int_t^b \omega_c(x - t) dx.$$

If $n_1 > 0$, then

$$\begin{aligned} g(t) &= \sum_{j=1}^{k} \frac{1}{\lambda_j Q_j(\lambda_j)} \left( \frac{1}{2} \sin(\lambda_j(b-t)) - \int_t^b \frac{\sin(\lambda_j(x-t))}{b-a} dx \right) \\ &= \sum_{j=1}^{k} \frac{1}{\lambda_j Q_j(\lambda_j)} \left( \frac{1}{2} \sin(\lambda_j(b-t)) - \frac{1-\cos(\lambda_j(b-t))}{\lambda_j(b-a)} \right) \\ &= \sum_{j=1}^{k} \frac{2\cos(\lambda_j(b-t)) - 2 + \lambda_j(b-a)\sin(\lambda_j(b-t))}{2\lambda_j^2 Q_j(\lambda_j)(b-a)}. \end{aligned}$$

Then, for $j \in \{1, \ldots, k\}$, we have that $\lambda_j(b-a)/2 = \tau_{n_j}$, which is a fixed point of the tangent function, therefore,

$$\lambda_j(b-a) = 2\tan(\lambda_j(b-a)/2). \tag{3.17}$$

Hence

$$\begin{aligned} &\frac{2\cos(\lambda_j(b-t)) - 2 + \lambda_j(b-a)\sin(\lambda_j(b-t))}{2\lambda_j^2 Q_j(\lambda_j)(b-a)} \\ &\qquad = \frac{\cos(\lambda_j(b-t)) - 1 + \tan(\lambda_j(b-a)/2)\sin(\lambda_j(b-t))}{\lambda_j^2 Q_j(\lambda_j)(b-a)}. \end{aligned}$$

To simplify the numerator of this fraction, we use the identity,

$$\begin{aligned} \cos(2u) - 1 + \tan(v)\sin(2u) &= -2\sin(u)^2 + 2\tan(v)\sin(u)\cos(u) \\ &= \frac{2\sin(u)}{\cos(v)}(\cos(u)\sin(v) - \sin(u)\cos(v)) \\ &= \frac{2\sin(u)\sin(v-u)}{\cos(v)} \end{aligned}$$

with $u := \lambda_j(b-t)/2$ and $v := \lambda_j(b-a)/2$. Then, it follows that

$$\begin{aligned} &\frac{2\cos(\lambda_j(b-t)) - 2 + \lambda_j(b-a)\sin(\lambda_j(b-t))}{2\lambda_j^2 Q_j(\lambda_j)(b-a)} \\ &\qquad = \frac{\sin(\lambda_j(b-t)/2)\sin(\lambda_j(t-a)/2)}{\lambda_j Q_j(\lambda_j)\sin(\lambda_j(b-a)/2)}, \end{aligned}$$

which proves the equality (3.16) when $n_1 > 0$.

If $n_1 = 0$, then (3.16) can be obtained analogously. □

In the particular case when $k = 1$, the above theorem simplifies to the following result.

COROLLARY 3.4.5. *Let $n \in \mathbb{N} \cup \{0\}$, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_n := \frac{2\tau_n}{b-a}$. Then, for all $f \in \mathcal{C}^2_{\mathbb{K}}([a,b])$,*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f = \int_a^b (f'' + \lambda_n^2 f)(t) \cdot g(t)dt,$$

*where*

$$g(t) := \begin{cases} \dfrac{\sin(\lambda_n(b-t)/2)\sin(\lambda_n(t-a)/2)}{\lambda_n \sin(\lambda_n(b-a)/2)} & \text{if } n > 0, \\ \dfrac{(b-t)(t-a)}{2(b-a)} & \text{if } n = 0. \end{cases} \tag{3.18}$$

Our final statement is a new error estimate for the trapezoidal rule.

THEOREM 3.4.6. *Let $n \in \mathbb{N} \cup \{0\}$, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_n := \frac{2\tau_n}{b-a}$. Then, for all $f \in \mathcal{C}^2_{\mathbb{K}}([a,b])$,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f\right| \le \begin{cases} \frac{1}{12}(b-a)^2 \cdot \|f''\|_\infty & \text{if } n = 0, \\ \frac{(n+1)n\pi}{2\tau_n^3}(b-a)^2 \cdot \|f'' + \lambda_n^2 f\|_\infty & \text{if } n > 0, \\ \frac{1}{8}(b-a) \cdot \|f''\|_1 & \text{if } n = 0, \\ \frac{1+|\cos(\tau_n)|}{4\tau_n|\sin(\tau_n)|}(b-a) \cdot \|f'' + \lambda_n^2 f\|_1 & \text{if } n > 0. \end{cases} \tag{3.19}$$

PROOF. According to Corollary 3.4.5, we have that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f\right| = \left|\int_a^b (f'' + \lambda_n^2 f)(t) \cdot g(t)dt\right| \le \begin{cases} \|g\|_1 \cdot \|f'' + \lambda_n^2 f\|_\infty, \\ \|g\|_\infty \cdot \|f'' + \lambda_n^2 f\|_1, \end{cases}$$

where $g : [a,b] \to \mathbb{R}$ is given by (3.18). Therefore, to verify the first inequality in (3.19), we have to compute $\|g\|_1$.

If $n = 0$, then $\tau_n = 0$, $\lambda_n = 0$ and, with the substitution $t = a + s(b-a)$, we get

$$\|g\|_1 = \int_a^b \frac{(b-t)(t-a)}{2(b-a)}dt = \frac{1}{2}(b-a)^2 \int_0^1 (1-s)sds = \frac{1}{12}(b-a)^2.$$

If $n > 0$, then, with the substitution $t = a + s(b-a)$, we get

$$\begin{aligned}\|g\|_1 &= \int_a^b \left|\frac{\sin(\lambda_n(b-t)/2)\sin(\lambda_n(t-a)/2)}{\lambda_n \sin(\lambda_n(b-a)/2)}\right| dt \\ &= \frac{(b-a)^2}{2\tau_n|\sin(\tau_n)|}\int_0^1 \big|\sin((1-s)\tau_n)\sin(s\tau_n)\big| ds.\end{aligned}$$

For this computation, we need to find the zeros of $s \mapsto \sin((1-s)\tau_n)\sin(s\tau_n)$ in $[0,1]$:

$$s = \frac{k}{\tau_n}\pi, \qquad s = 1 + \frac{k}{\tau_n}\pi \qquad (k \in \mathbb{Z}).$$

The $n$th positive fixed point $\tau_n$ of the tangent function is in $[n\pi, (n+\frac{1}{2})\pi)$. Then $\frac{k}{\tau_n}\pi \in [0,1]$ if $k \in [0, \frac{\tau_n}{\pi}]$, which holds if and only if $0 \le k \le n$. Analogously, $1 + \frac{k}{\tau_n}\pi \in [0,1]$ if $k \in [-\frac{\tau_n}{\pi}, 0]$, which holds if and only if $-n \le k \le 0$. Therefore, the roots of the map $s \mapsto \sin((1-s)v)\sin(sv)$ in $[0,1]$ in increasing order are

$$\begin{aligned}&s_0 := 0 < s_1 := 1 - \frac{n\pi}{\tau_n} < s_2 := \frac{\pi}{\tau_n} \\ &\qquad < s_3 := 1 - \frac{(n-1)\pi}{\tau_n} < \cdots < s_{2n} := \frac{n\pi}{\tau_n} < s_{2n+1} := 1.\end{aligned}$$

We have that

$$\begin{aligned}\int \sin((1-s)\tau_n)\sin(s\tau_n) ds &= \frac{1}{2}\int (\cos((1-2s)\tau_n) - \cos(\tau_n)) ds \\ &= -\frac{1}{4\tau_n}\big(\sin((1-2s)\tau_n) + 2s\tau_n\cos(\tau_n)\big) \\ &=: F(s).\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^1 & \big|\sin((1-s)v)\sin(sv)\big|ds \\
&= \sum_{j=1}^{2n+1} \int_{s_{j-1}}^{s_j} \big|\sin((1-s)v)\sin(sv)\big|ds \\
&= \sum_{j=1}^{2n+1} |F(s_j) - F(s_{j-1})| \\
&= \Big|\sum_{j=1}^{2n+1} (-1)^j (F(s_j) - F(s_{j-1}))\Big| \\
&= \Big|2\sum_{j=0}^{n} F(s_{2j}) - F(s_0) - 2\sum_{j=0}^{n} F(s_{2j+1}) + F(s_{2n+1})\Big|.
\end{aligned}$$

Using that

$$\begin{aligned}
F(s_0) &= F(0) = -\frac{\sin(\tau_n)}{4\tau_n} = -\frac{\cos(\tau_n)}{4}, \\
F(s_{2n+1}) &= F(1) = \frac{\sin(\tau_n)}{4\tau_n} - \frac{\cos(\tau_n)}{2} = -\frac{\cos(\tau_n)}{4},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{n} F(s_{2j}) &= \sum_{j=0}^{n} F\big(\tfrac{j\pi}{\tau_n}\big) = -\frac{1}{4\tau_n}\sum_{j=0}^{n}\Big(\sin(\tau_n - 2j\pi) + 2j\pi\cos(\tau_n)\Big) \\
&= -\frac{n+1}{4\tau_n}\big(\sin(\tau_n) + n\pi\cos(\tau_n)\big), \\
\sum_{j=0}^{n} F(s_{2j+1}) &= \sum_{j=0}^{n} F\big(1 - \tfrac{(n-j)\pi}{\tau_n}\big) = \sum_{j=0}^{n} F\big(1 - \tfrac{j\pi}{\tau_n}\big) \\
&= -\frac{1}{4\tau_n}\sum_{j=0}^{n}\Big(\sin(2j\pi - \tau_n) + 2(\tau_n - j\pi)\cos(\tau_n)\Big) \\
&= -\frac{n+1}{4\tau_n}\big(-\sin(\tau_n) + (2\tau_n - n\pi)\cos(\tau_n)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
2\sum_{j=0}^{n} F(s_{2j}) - F(s_0) - 2\sum_{j=0}^{n} F(s_{2j+1}) + F(s_{2n+1}) \\
&= -\frac{n+1}{\tau_n}\sin(\tau_n) + \frac{n+1}{\tau_n}(\tau_n - n\pi)\cos(\tau_n) \\
&= (n+1)\cos(\tau_n)\Big( -1 + \frac{1}{\tau_n}(\tau_n - n\pi)\Big) \\
&= -(n+1)n\frac{\cos(\tau_n)}{\tau_n}\pi.
\end{aligned}$$

Putting together the pieces, we can conclude that

$$\begin{aligned}
\|g\|_1 &= \frac{(b-a)^2}{2\tau_n|\sin(\tau_n)|}\int_0^1 \big|\sin((1-s)\tau_n)\sin(s\tau_n)\big|ds \\
&= \frac{(b-a)^2}{2\tau_n|\sin(\tau_n)|}(n+1)n\frac{|\cos(\tau_n)|}{\tau_n}\pi = \frac{(n+1)n\pi}{2\tau_n^3}(b-a)^2.
\end{aligned}$$

Thus, the second inequality in (3.19) is proved.

To verify the third and fourth inequalities in (3.19), we need to compute $\|g\|_\infty$.

If $n = 0$, then $\lambda_n = 0$ and

$$\|g\|_\infty = \sup_{t\in[a,b]} \frac{(b-t)(t-a)}{2(b-a)} = \frac{b-a}{8}.$$

If $n > 1$, then $\lambda_n > 0$ and

$$\|g\|_\infty = \sup_{t\in[a,b]} \left|\frac{\sin(\lambda_n(b-t)/2)\sin(\lambda_n(t-a)/2)}{\lambda_n\sin(\lambda_n(b-a)/2)}\right|.$$

The supremum is attained at a point $t$ belonging to the interior of $[a,b]$, therefore,

$$\begin{aligned}
0 &= g'(t) \\
&= \frac{1}{2\sin(\tau_n)}\big( -\cos(\lambda_n(b-t)/2)\sin(\lambda_n(t-a)/2) \\
&\qquad\qquad + \sin(\lambda_n(b-t)/2)\cos(\lambda_n(t-a)/2)\big) \\
&= \frac{\sin(\lambda_n(a+b-2t)/2)}{2\sin(\tau_n)}.
\end{aligned}$$

It follows from here that there exists $k \in \mathbb{Z}$ such that

$$t = \sigma_k := \frac{a+b}{2} - \frac{k\pi}{\lambda_n}.$$

Clearly, $\sigma_k \in (a, b)$ if and only if

$$\left|\frac{k\pi}{\lambda_n}\right| < \frac{b-a}{2},$$

i.e., if and only if

$$|k|\pi < \tau_n < (n + \frac{1}{2})\pi,$$

which is equivalent to the inequality $|k| \leq n$. For the value of $g$ at $\sigma_k$, we have

$$\begin{aligned} g(\sigma_k) &= \frac{\sin(\lambda_n(b-\sigma_k)/2)\sin(\lambda_n(\sigma_k - a)/2)}{\lambda_n \sin(\lambda_n(b-a)/2)} \\ &= \frac{\sin(\frac{1}{2}\tau_n + k\frac{\pi}{2})\sin(\frac{1}{2}\tau_n - k\frac{\pi}{2})}{\lambda_n \sin(\tau_n)} \\ &= \frac{\cos(k\pi) - \cos(\tau_n)}{2\lambda_n \sin(\tau_n)} = \frac{(-1)^k - \cos(\tau_n)}{2\lambda_n \sin(\tau_n)}. \end{aligned}$$

This shows that the sequence $\big(|g(\sigma_k)|\big)_{k=-n}^{n}$ takes only the following two values:

$$\frac{1-\cos(\tau_n)}{2\lambda_n|\sin(\tau_n)|}, \qquad \frac{1+\cos(\tau_n)}{2\lambda_n|\sin(\tau_n)|}$$

and hence

$$\|g\|_\infty = \frac{1+|\cos(\tau_n)|}{2\lambda_n|\sin(\tau_n)|} = \frac{1+|\cos(\tau_n)|}{4\tau_n|\sin(\tau_n)|}(b-a).$$

Thus, the proof of the theorem is completed. □

## 3.5. An extension of the Simpson formula

THEOREM 3.5.1. *Let $a, b \in \mathbb{R}$ with $a < b$, $u = w + \boldsymbol{i}v$, where $w \in \mathbb{R}_+$, $v \in (0, \pi)$ and define $\alpha_u, \beta_u$ by*

$$\begin{aligned} \alpha_u &:= \frac{\overline{u}\sinh(u) - u\sinh(\overline{u})}{2u\overline{u}(\cosh(u) - \cosh(\overline{u}))}, \\ \beta_u &:= \frac{u\cosh(u)\sinh(\overline{u}) - \overline{u}\cosh(\overline{u})\sinh(u)}{u\overline{u}(\cosh(u) - \cosh(\overline{u}))}. \end{aligned}$$

*Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}([a,b])$,

$$\left|\alpha_u f(a) + \beta_u f\Big(\frac{a+b}{2}\Big) + \alpha_u f(b) - \frac{1}{b-a}\int_a^b f\right|$$
$$\leq \begin{cases} \dfrac{(b-a)^3\big(v\sinh(w) - w\sin(v)\big)^2}{32(w^2+v^2)^2 wv\sinh(w)\sin(v)} \\ \qquad\qquad \times\left\| f'''' + \dfrac{8(v^2-w^2)}{(b-a)^2} f'' + \dfrac{16(w^2+v^2)^2}{(b-a)^4} f\right\|_1, \\ \dfrac{(b-a)^4(2\alpha_u+\beta_u-1)}{16(w^2+v^2)^2} \\ \qquad\qquad \times\left\| f'''' + \dfrac{8(v^2-w^2)}{(b-a)^2} f'' + \dfrac{16(w^2+v^2)^2}{(b-a)^4} f\right\|_\infty. \end{cases}$$

PROOF. We have that

$$\begin{aligned}\cosh(u) - \cosh(\overline{u}) &= \cosh(w+\boldsymbol{i}v) - \cosh(w-\boldsymbol{i}v) \\ &= 2\sinh(w)\sinh(\boldsymbol{i}v) = 2\boldsymbol{i}\sinh(w)\sin(v),\end{aligned} \tag{3.20}$$

which is different from zero due to the assumptions $0 < w$ and $0 < v < \pi$ and hence $\alpha_u, \beta_u$ are well-defined. Observe that $\alpha_u, \beta_u$ are invariant with respect to conjugation, therefore, they are real numbers, and one can also see that $\alpha_u$ and $\beta_u$ are the unique solutions of the following system of equations:

$$2\alpha_u\cosh(u) + \beta_u = \frac{\sinh(u)}{u}, \qquad 2\alpha_u\cosh(\overline{u}) + \beta_u = \frac{\sinh(\overline{u})}{\overline{u}}. \tag{3.21}$$

Therefore, each element $z$ of the set $\{u, -u, \overline{u}, -\overline{u}\}$ is a root of the function $\varphi : \mathbb{C} \to \mathbb{C}$ given by

$$\varphi_u(z) := 2\alpha_u\cosh(z) + \beta_u - \frac{\sinh(z)}{z}.$$

Now, we construct the measure $\mu_u$ on the Borel subsets of $[a,b]$ by

$$\mu_u(A) := \alpha_u\delta_a(A) + \beta_u\delta_{\frac{a+b}{2}}(A) + \alpha_u\delta_b(A) - \frac{1}{b-a}\int_A 1 \qquad (A \subseteq [a,b]),$$

where $\delta_t$ denotes the Dirac measure concentrated at $t \in [a,b]$, and we consider the linear function $\mathcal{A}_{\mu_u} : \mathcal{C}_{\mathbb{K}} \to \mathbb{K}$ defined by

$$\mathcal{A}_{\mu_u}(f) := \int_{[a,b]} f d\mu_u = \alpha_u f(a) + \beta_u f\Big(\frac{a+b}{2}\Big) + \alpha_u f(b) - \frac{1}{b-a}\int_a^b f.$$

It is not difficult to see that

$$|\mathcal{A}_{\mu_u}(f)| \leq (2|\alpha_u| + |\beta_u| + 1)\|f\|_\infty,$$

and the constant $2|\alpha_u| + |\beta_u| + 1$ is the sharpest possible one. Furthermore, if $p \in [1, \infty)$ and $2|\alpha_u| + |\beta_u| > 0$, then $\mathcal{A}_{\mu_u}$ is not bounded with respect to the norm $\|\cdot\|_p$.

The spectral function of $\mathcal{A}_{\mu_u}$ is now of the form

$$\begin{aligned}
\mathcal{S}_{\mu_u}(\lambda) &= \alpha_u e^{\lambda a} + \beta_u e^{\lambda\frac{a+b}{2}} + \alpha_u e^{\lambda b} - \frac{1}{b-a}\int_a^b e^{\lambda x}dx \\
&= \alpha_u e^{\lambda a} + \beta_u e^{\lambda\frac{a+b}{2}} + \alpha_u e^{\lambda b} - \frac{e^{\lambda b} - e^{\lambda a}}{\lambda(b-a)} \\
&= e^{\lambda\frac{a+b}{2}}\left(2\alpha_u \cosh\left(\lambda\frac{b-a}{2}\right) + \beta_u - \frac{2}{\lambda(b-a)}\sinh\left(\lambda\frac{b-a}{2}\right)\right).
\end{aligned}$$

Therefore, the equation $\mathcal{S}_{\mu_u}(\lambda) = 0$ holds if and only if $\varphi_u\big(\frac{1}{2}\lambda(b-a)\big) = 0$ is valid. Consequently, each element of the set $\{\lambda_u, -\lambda_u, \overline{\lambda_u}, -\overline{\lambda_u}\}$, where $\lambda_u := \frac{2u}{b-a}$, is a root of the spectral function $\mathcal{S}_{\mu_u}$. The polynomial whose roots are these numbers is equal to

$$P_u(z) = \big(z^2 - \lambda_u^2\big)\big(z^2 - \overline{\lambda_u}^2\big) = z^4 - (\lambda_u^2 + \overline{\lambda_u}^2)z^2 + \lambda_u^2\overline{\lambda_u}^2.$$

The differential operator whose characteristic polynomial is $P_u$ is given by

$$D_u(f) = f'''' - (\lambda_u^2 + \overline{\lambda_u}^2)f'' + \lambda_u^2\overline{\lambda_u}^2 f.$$

The characteristic solution of the differential equation $D_u(f) = 0$ is now of the form

$$\begin{aligned}
\omega_u(t) &= \frac{\sinh(\lambda_u t)}{\lambda_u(\lambda_u^2 - \overline{\lambda_u}^2)} + \frac{\sinh(\overline{\lambda_u} t)}{\overline{\lambda_u}(\overline{\lambda_u}^2 - \lambda_u^2)} \\
&= \sum_{k=1}^{\infty} \frac{\lambda_u^{2k} - \overline{\lambda_u}^{2k}}{\lambda_u^2 - \overline{\lambda_u}^2}\frac{t^{2k+1}}{(2k+1)!}.
\end{aligned} \tag{3.22}$$

Finally, we compute

$$\begin{aligned}
g_u(t) &= \int_{[t,b]} \omega_u(x-t)d\mu_u(x) = \int_{[a,b]} \chi_{[t,b]}(x)\cdot\omega_u(x-t)d\mu_u(x) \\
&= \begin{cases} \beta_u\omega_u\big(\frac{a+b}{2} - t\big) + \alpha_u\omega_u(b-t) \\ \qquad\qquad -\frac{1}{b-a}\int_0^{b-t}\omega_u(x)dx & \text{if } a \le t \le \frac{a+b}{2}, \\ \alpha_u\omega_u(b-t) - \frac{1}{b-a}\int_0^{b-t}\omega_u(x)dx & \text{if } \frac{a+b}{2} < t \le b, \end{cases}
\end{aligned}$$

and, if $a \le t \le \frac{a+b}{2}$, then, according to (3.22),

$$\begin{aligned} g_u(t) = \beta_u & \Bigg( \frac{\sinh(\lambda_u(\frac{a+b}{2} - t))}{\lambda_u(\lambda_u^2 - \overline{\lambda_u}^2)} + \frac{\sinh(\overline{\lambda_u}(\frac{a+b}{2} - t))}{\overline{\lambda_u}(\overline{\lambda_u}^2 - \lambda_u^2)} \Bigg) \\ & + \alpha_u \Bigg( \frac{\sinh(\lambda_u(b-t))}{\lambda_u(\lambda_u^2 - \overline{\lambda_u}^2)} + \frac{\sinh(\overline{\lambda_u}(b-t))}{\overline{\lambda_u}(\overline{\lambda_u}^2 - \lambda_u^2)} \Bigg) \\ & - \frac{1}{b-a} \Bigg( \frac{\cosh(\lambda_u(b-t))}{\lambda_u^2(\lambda_u^2 - \overline{\lambda_u}^2)} + \frac{\cosh(\overline{\lambda_u}(b-t))}{\overline{\lambda_u}^2(\overline{\lambda_u}^2 - \lambda_u^2)} + \frac{1}{\lambda_u^2 \overline{\lambda_u}^2} \Bigg). \end{aligned}$$

If $\frac{a+b}{2} < t \le b$, then

$$\begin{aligned} g_u(t) = \alpha_u & \Bigg( \frac{\sinh(\lambda_u(b-t))}{\lambda_u(\lambda_u^2 - \overline{\lambda_u}^2)} + \frac{\sinh(\overline{\lambda_u}(b-t))}{\overline{\lambda_u}(\overline{\lambda_u}^2 - \lambda_u^2)} \Bigg) \\ & - \frac{1}{b-a} \Bigg( \frac{\cosh(\lambda_u(b-t))}{\lambda_u^2(\lambda_u^2 - \overline{\lambda_u}^2)} + \frac{\cosh(\overline{\lambda_u}(b-t))}{\overline{\lambda_u}^2(\overline{\lambda_u}^2 - \lambda_u^2)} + \frac{1}{\lambda_u^2 \overline{\lambda_u}^2} \Bigg). \end{aligned}$$

In what follows, we shall prove the following properties of $g_u$.

(1) $g_u$ is symmetric with respect to the midpoint of $[a, b]$, i.e., $g_u(a + b - t) = g_u(t)$ for all $t \in [a, b]$.
(2) $g_u$ is continuous and nonnegative on $[a, b]$, and $g_u(a) = g_u(b) = 0$.
(3) $g_u$ is increasing on $\left[a, \frac{a+b}{2}\right]$ and decreasing on $\left[\frac{a+b}{2}, b\right]$, consequently $\|g_u\|_\infty = g_u(\frac{a+b}{2})$.

To prove assertion (1), for $\frac{a+b}{2} \le t \le b$, we show that the equality $g_u(a + b - t) = g_u(t)$ holds, that is,

$$\begin{aligned} \beta_u & \Bigg( \frac{\sinh\left(\lambda_u(t - \frac{a+b}{2})\right)}{\lambda_u} - \frac{\sinh\left(\overline{\lambda_u}(t - \frac{a+b}{2})\right)}{\overline{\lambda_u}} \Bigg) \\ & + \alpha_u \Bigg( \frac{\sinh(\lambda_u(t-a))}{\lambda_u} - \frac{\sinh(\overline{\lambda_u}(t-a))}{\overline{\lambda_u}} \Bigg) \\ & - \alpha_u \Bigg( \frac{\sinh(\lambda_u(b-t))}{\lambda_u} - \frac{\sinh(\overline{\lambda_u}(b-t))}{\overline{\lambda_u}} \Bigg) \\ = & \frac{1}{b-a} \Bigg( \frac{\cosh(\lambda_u(t-a))}{\lambda_u^2} - \frac{\cosh(\overline{\lambda_u}(t-a))}{\overline{\lambda_u}^2} \Bigg) \\ & - \frac{1}{b-a} \Bigg( \frac{\cosh(\lambda_u(b-t))}{\lambda_u^2} - \frac{\cosh(\overline{\lambda_u}(b-t))}{\overline{\lambda_u}^2} \Bigg). \end{aligned}$$

Using the standard identities for the hyperbolic functions, the above equality is equivalent to

$$\beta_u\left(\frac{\sinh\left(\lambda_u\left(t-\frac{a+b}{2}\right)\right)}{\lambda_u}-\frac{\sinh\left(\overline{\lambda_u}\left(t-\frac{a+b}{2}\right)\right)}{\overline{\lambda_u}}\right)$$
$$+2\alpha_u\left(\frac{\sinh\left(\lambda_u\left(t-\frac{a+b}{2}\right)\right)\cosh(u)}{\lambda_u}-\frac{\sinh\left(\overline{\lambda_u}\left(t-\frac{a+b}{2}\right)\right)\cosh(\overline{u})}{\overline{\lambda_u}}\right)$$
$$=\frac{2}{b-a}\left(\frac{\sinh\left(\lambda_u\left(t-\frac{a+b}{2}\right)\right)\sinh(u)}{\lambda_u^2}-\frac{\sinh\left(\overline{\lambda_u}\left(t-\frac{a+b}{2}\right)\right)\sinh(\overline{u})}{\overline{\lambda_u}^2}\right).$$

To see that this is valid, one should multiply the first and the second equalities in (3.21) by $\frac{1}{\lambda_u}\sinh\left(\lambda_u\left(t-\frac{a+b}{2}\right)\right)$ and $\frac{1}{\overline{\lambda_u}}\sinh\left(\overline{\lambda_u}\left(t-\frac{a+b}{2}\right)\right)$ side by side, respectively, and then subtract the two equations so obtained.

The continuity of $g_u$ at every $t\in[a,b]\setminus\left\{\frac{a+b}{2}\right\}$ is obvious, and, by the definition of $g_u$, this function is also continuous from the left at $t=\frac{a+b}{2}$. Due to the symmetry with respect to the midpoint of $[a,b]$, it follows that $g_u$ is also continuous from the right at $t=\frac{a+b}{2}$, and hence it is also continuous at $t=\frac{a+b}{2}$. The endpoint properties $g_u(a)=g_u(b)=0$ are trivial. The nonnegativity follows from the monotonicity properties of $g_u$, which will be established in what follows. Obviously, in view of the symmetry with respect to the midpoint of $[a,b]$, it is sufficient to show that $g_u$ is decreasing on the upper half of the interval $[a,b]$.

With the substitution $t:=s\frac{a+b}{2}+(1-s)b$, where $s\in[0,1]$, we get that

$$\lambda_u(b-t)=\frac{2u}{b-a}\left(b-\left(s\frac{a+b}{2}+(1-s)b\right)\right)=us.$$

Therefore, applying also the identity (3.20), we get

$$g_u\left(s\frac{a+b}{2}+(1-s)b\right)$$
$$=\frac{(b-a)^3}{16u^2\overline{u}^2}\left(\frac{\overline{u}\sinh(u)-u\sinh(\overline{u})}{\cosh(u)-\cosh(\overline{u})}\left(\frac{\overline{u}\sinh(us)}{u^2-\overline{u}^2}+\frac{u\sinh(\overline{u}s)}{\overline{u}^2-u^2}\right)\right.$$
$$\left.-\left(\frac{\overline{u}^2\cosh(us)}{u^2-\overline{u}^2}+\frac{u^2\cosh(\overline{u}s)}{\overline{u}^2-u^2}+1\right)\right)$$
$$=\frac{-(b-a)^3h_u(s)}{16|u|^4(u^2-\overline{u}^2)(\cosh(u)-\cosh(\overline{u}))}$$
$$=\frac{(b-a)^3h_u(s)}{128(w^2+v^2)^2wv\sinh(w)\sin(v)},$$

where

$$\begin{aligned}
h_u(s) := &-(\overline{u}\sinh(u) - u\sinh(\overline{u}))(\overline{u}\sinh(us) - u\sinh(\overline{u}s)) \\
&+ (\cosh(\overline{u}) - \cosh(u))(u^2\cosh(\overline{u}s) - \overline{u}^2\cosh(us) + \overline{u}^2 - u^2) \\
= &\ 4w^2\cosh((1-s)w)\sin(v)\sin(sv) \\
&+ 4v^2\cos((1-s)v)\sinh(sw)\sinh(w) \\
&+ 4wv\sin(v)\cos(sv)\sinh((1-s)w) \\
&+ 4wv\sinh(w)\cosh(sw)\sin((1-s)v) - 8wv\sinh(w)\sin(v).
\end{aligned}$$

We have

$$\begin{aligned}
h_u'(s) &= -\overline{u}u(\overline{u}\sinh(u) - u\sinh(\overline{u}))(\cosh(us) - \cosh(\overline{u}s)) \\
&\quad + \overline{u}u(\cosh(u) - \cosh(\overline{u}))(u\sinh(\overline{u}s) - \overline{u}\sinh(us)) \\
&= 4(w^2+v^2)\Big(v\sinh(w)\sinh(sw)\sin((1-s)v) \\
&\qquad - w\sin(v)\sin(sv)\sinh((1-s)w)\Big).
\end{aligned}$$

The property that $g_u$ is strictly decreasing on the upper half of the interval $[a,b]$, is equivalent to the assertion that $h_u$ is strictly increasing on $[0,1]$. To verify this, we prove the inequality $h_u'(s) > 0$ for $s \in (0,1)$, which can be rewritten as

$$v\sinh(w)\sinh(sw)\sin((1-s)v) > w\sin(v)\sin(sv)\sinh((1-s)w).$$

After using elementary computations, the above inequality turns out to be equivalent to the following one:

$$v(\cot(sv) - \cot(v)) > w(\coth(sw) - \coth(w)).$$

Actually, we are going to show that, for all $w \in \mathbb{R}_+$, $v \in (0,\pi)$ and $s \in (0,1)$,

$$v(\cot(sv) - \cot(v)) > \frac{1-s}{s} > w(\coth(sw) - \coth(w)). \tag{3.23}$$

We first verify the left hand side inequality in (3.23). Using Lemma 3.4.1, for $v \in (0,\pi)$, we have that

$$\begin{aligned}
\left(\cot(v) - \frac{v}{\sin^2(v)}\right)' &= -\frac{2}{\sin^2(v)} + \frac{2v\cos(v)}{\sin^3(v)} \\
&= \frac{2}{\sin^2(v)}(v\cot(v) - 1) < 0,
\end{aligned}$$

which shows that the map $v \mapsto \cot(v) - \frac{v}{\sin^2(v)}$ is strictly decreasing on $(0, \pi)$. Therefore, for all $v \in (0, \pi)$ and $s \in (0, 1)$, we get

$$\cot(sv) - \frac{sv}{\sin^2(sv)} > \cot(v) - \frac{v}{\sin^2(v)}.$$

Consequently, for each fixed $s \in (0, 1)$, the derivative of the map $v \mapsto v(\cot(sv) - \cot(v))$ is strictly positive on $(0, \pi)$ and hence it is strictly increasing on $(0, \pi)$. Therefore, for all $v \in (0, \pi)$,

$$v(\cot(sv) - \cot(v)) > \lim_{v \to 0+} v(\cot(sv) - \cot(v)) = \frac{1-s}{s}.$$

Secondly, we verify the right hand side inequality in (3.23). Using Lemma 3.4.1, for $w \in \mathbb{R}_+$, we have that

$$\begin{aligned} \Big(\coth(w) - \frac{w}{\sinh^2(w)}\Big)' &= -\frac{2}{\sinh^2(w)} + \frac{2w\cosh(w)}{\sinh^3(w)} \\ &= \frac{2}{\sinh^2(w)}(w\coth(w) - 1) > 0, \end{aligned}$$

which shows that the map $w \mapsto \coth(w) - \frac{w}{\sinh^2(w)}$ is strictly increasing on $\mathbb{R}_+$. Therefore, for all $v \in \mathbb{R}_+$ and $s \in (0, 1)$, we get

$$\coth(sw) - \frac{sw}{\sinh^2(sw)} < \coth(w) - \frac{w}{\sinh^2(w)}.$$

Consequently, for each fixed $s \in (0, 1)$, the derivative of the map $w \mapsto w(\coth(sw) - \coth(w))$ is strictly negative on $\mathbb{R}_+$ and hence it is strictly decreasing on $\mathbb{R}_+$. Therefore, for all $w \in \mathbb{R}_+$,

$$w(\coth(sw) - \coth(w)) < \lim_{w \to 0+} w(\coth(sw) - \coth(w)) = \frac{1-s}{s},$$

which shows that the right hand side inequality in (3.23) is also valid. The inequality (3.23) implies that $h_u' > 0$ on $[0, 1]$, whence the strict increasingness of $h_u$ over $[0, 1]$ follows.

Then, with $s = 1$, we obtain that

$$h_u(1) = 4\big(v\sinh(w) - w\sin(v)\big)^2,$$

therefore,

$$\begin{aligned} \|g_u\|_\infty &= g_u\Big(\frac{a+b}{2}\Big) \\ &= \frac{(b-a)^3\big(v\sinh(w) - w\sin(v)\big)^2}{32(w^2+v^2)^2 wv\sinh(w)\sin(v)}. \end{aligned}$$

In order to determine $\|g_u\|_1$, we need the following computation.

$$
\begin{aligned}
&\int_0^1 h_u(s)ds \\
&= \int_0^1 \Big( - (\overline{u}\sinh(u) - u\sinh(\overline{u}))(\overline{u}\sinh(us) - u\sinh(\overline{u}s)) \\
&\qquad + (\cosh(\overline{u}) - \cosh(u))(u^2\cosh(\overline{u}s) - \overline{u}^2\cosh(us) + \overline{u}^2 - u^2)\Big) ds \\
&= \Big[ - (\overline{u}\sinh(u) - u\sinh(\overline{u}))\Big(\frac{\overline{u}}{u}\cosh(us) - \frac{u}{\overline{u}}\cosh(\overline{u}s)\Big) \\
&\qquad + (\cosh(\overline{u}) - \cosh(u))\Big(\frac{u^2}{\overline{u}}\sinh(\overline{u}s) - \frac{\overline{u}^2}{u}\sinh(us) + (\overline{u}^2 - u^2)s\Big)\Big]_{s=0}^{s=1} \\
&= \Big[ - (\overline{u}\sinh(u) - u\sinh(\overline{u}))\Big(\frac{\overline{u}}{u}\cosh(u) - \frac{u}{\overline{u}}\cosh(\overline{u})\Big) \\
&\qquad + (\cosh(\overline{u}) - \cosh(u))\Big(\frac{u^2}{\overline{u}}\sinh(\overline{u}) - \frac{\overline{u}^2}{u}\sinh(u) + (\overline{u}^2 - u^2)\Big)\Big] \\
&\qquad - \Big[ - (\overline{u}\sinh(u) - u\sinh(\overline{u}))\Big(\frac{\overline{u}}{u} - \frac{u}{\overline{u}})\Big)\Big] \\
&= \frac{(u^2 - \bar{u}^2)}{u\bar{u}}\Big(u\sinh(\bar{u}) - \bar{u}\sinh(u) + \bar{u}(\sinh(u) - u)\cosh(\bar{u}) \\
&\qquad\qquad + u\cosh(u)(\bar{u} - \sinh(\bar{u}))\Big) \\
&= \frac{(u^2 - \bar{u}^2)}{u\bar{u}}\Big(\bar{u}\sinh(u)(\cosh(\bar{u}) - 1) + u\sinh(\bar{u})(1 - \cosh(u)) \\
&\qquad\qquad + u\bar{u}(\cosh(u) - \cosh(\bar{u}))\Big) \\
&= \frac{8vw}{v^2 + w^2}\Big((w\sin(v) + v\sinh(w))(\cosh(w) - \cos(v)) \\
&\qquad\qquad - (v^2 + w^2)\sinh(w)\sin(v)\Big).
\end{aligned}
$$

We have

$$
\begin{aligned}
&(v^2 + w^2)(2\alpha_u + \beta_u)\sinh(w)\sin(v) \\
&\qquad = (v\sinh(w) + w\sin(v))(\cosh(w) - \cos(v)),
\end{aligned}
$$

Therefore,

$$
\int_0^1 h_u(s)ds = 8vw(2\alpha_u + \beta_u - 1)\sinh(w)\sin(v)
$$

and hence

$$\begin{aligned}\int_a^b g_u(x)dx &= 2\int_{\frac{a+b}{2}}^b g_u(x)dx\\&= (b-a)\int_0^1 g_u\Big(s\frac{a+b}{2}+(1-s)b\Big)ds\\&= \frac{(b-a)^4}{128(w^2+v^2)^2 wv\sinh(w)\sin(v)}\int_0^1 h_u(s)ds\\&= \frac{(b-a)^4(2\alpha_u+\beta_u-1)}{16(w^2+v^2)^2}.\end{aligned}$$

Using now Corollary 3.3.3 with $p = 1$ and $p = \infty$, the statement of the theorem follows. □

In the following result, we deduce the Simpson formula with two error terms by taking the limit $u \to 0$ in Theorem 3.5.1.

COROLLARY 3.5.2. *Let $a, b \in \mathbb{R}$. Then, for all $f \in \mathcal{C}^4_{\mathbb{K}}([a,b])$,*

$$\left|\frac{1}{6}f(a)+\frac{2}{3}f\Big(\frac{a+b}{2}\Big)+\frac{1}{6}f(b)-\frac{1}{b-a}\int_a^b f\right| \le \begin{cases}\dfrac{(b-a)^3}{1152}\cdot\left\|f''''\right\|_1,\\[2ex] \dfrac{(b-a)^4}{2880}\cdot\left\|f''''\right\|_\infty.\end{cases}$$

PROOF. In order to derive the statement as the limiting case of Theorem 3.5.1 as $u \to 0$, we have to compute the limits in the inequality stated by this theorem. An easy computation shows that

$$\begin{aligned}\alpha_u &= \frac{\overline{u}\sum_{n=0}^\infty \frac{u^{2n+1}}{(2n+1)!} - u\sum_{n=0}^\infty \frac{\overline{u}^{2n+1}}{(2n+1)!}}{2u\overline{u}\big(\sum_{n=0}^\infty \frac{u^{2n}}{(2n)!} - \sum_{n=0}^\infty \frac{\overline{u}^{2n}}{(2n)!}\big)}\\&= \frac{\sum_{n=1}^\infty \frac{u^{2n}-\overline{u}^{2n}}{(2n+1)!}}{2\sum_{n=1}^\infty \frac{u^{2n}-\overline{u}^{2n}}{(2n)!}}\\&= \frac{\frac{1}{6}+\sum_{n=2}^\infty \frac{u^{2(n-1)}+\cdots+\overline{u}^{2(n-1)}}{(2n+1)!}}{2\big(\frac{1}{2}+\sum_{n=2}^\infty \frac{u^{2(n-1)}+\cdots+\overline{u}^{2(n-1)}}{(2n)!}\big)}.\end{aligned} \tag{3.24}$$

Therefore,

$$\lim_{u\to 0}\alpha_u = \frac{1}{6}.$$

Now, using the first equality in (3.21), we get

$$\begin{aligned}\lim_{u\to 0}\beta_u &= \lim_{u\to 0}\left(\frac{\sinh(u)}{u} - 2\alpha_u\cosh(u)\right)\\ &= 1 - \frac{2}{6}\cdot 1 = \frac{2}{3}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\lim_{(w,v)\to(0,0)}\frac{\big(v\sinh(w) - w\sin(v)\big)^2}{(w^2+v^2)^2 wv\sinh(w)\sin(v)}\\ &= \lim_{(w,v)\to(0,0)}\left(\frac{v\sinh(w) - w\sin(v)}{(w^2+v^2)wv}\right)^2\\ &= \lim_{(w,v)\to(0,0)}\left(\frac{v\sum_{n=0}^{\infty}\frac{w^{2n+1}}{(2n+1)!} - w\sum_{n=0}^{\infty}(-1)^n\frac{v^{2n+1}}{(2n+1)!}}{(w^2+v^2)wv}\right)^2\\ &= \lim_{(w,v)\to(0,0)}\left(\frac{\sum_{n=1}^{\infty}\frac{w^{2n}-(-1)^n v^{2n}}{(2n+1)!}}{w^2+v^2}\right)^2\\ &= \lim_{(w,v)\to(0,0)}\left(\frac{1}{6} + \frac{\sum_{n=2}^{\infty}\frac{w^{2n}-(-1)^n v^{2n}}{(2n+1)!}}{w^2+v^2}\right)^2\\ &= \frac{1}{36}.\end{aligned}$$

Using that $\beta_u = \frac{\sinh(u)}{u} - 2\alpha_u\cosh(u) = \frac{\sinh(\overline{u})}{\overline{u}} - 2\alpha_u\cosh(\overline{u})$ and then applying (3.24), we get

$$\begin{aligned}&\lim_{u\to 0}\frac{2\alpha_u + \beta_u - 1}{(w^2+v^2)^2}\\ &= \lim_{u\to 0}\frac{2\alpha_u(2 - \cosh(u) - \cosh(\overline{u})) + \frac{\sinh(u)}{u} + \frac{\sinh(\overline{u})}{\overline{u}} - 2}{2u^2\overline{u}^2}\\ &= \lim_{u\to 0}\frac{\sum_{n=1}^{\infty}\frac{u^{2n}+\overline{u}^{2n}}{(2n+1)!} - 2\alpha_u\sum_{n=1}^{\infty}\frac{u^{2n}+\overline{u}^{2n}}{(2n)!}}{2u^2\overline{u}^2}\\ &= \lim_{u\to 0}\frac{\sum_{n=1}^{2}\frac{u^{2n}+\overline{u}^{2n}}{(2n+1)!} - 2\alpha_u\sum_{n=1}^{2}\frac{u^{2n}+\overline{u}^{2n}}{(2n)!}}{2u^2\overline{u}^2}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{u\to 0} \frac{\frac{u^2+\overline{u}^2}{3!} + \frac{u^4+\overline{u}^4}{5!} - 2\alpha_u\big(\frac{u^2+\overline{u}^2}{2!} + \frac{u^4+\overline{u}^4}{4!}\big)}{2u^2\overline{u}^2} \\
&= \lim_{u\to 0} \frac{20(u^2+\overline{u}^2) + u^4 + \overline{u}^4 - 2\alpha_u\big(60(u^2+\overline{u}^2) + 5(u^4+\overline{u}^4)\big)}{240u^2\overline{u}^2} \\
&= \lim_{u\to 0} \Bigg( \frac{\big(\frac{1}{2} + \sum_{n=2}^{\infty} \frac{u^{2(n-1)}+\dots+\overline{u}^{2(n-1)}}{(2n)!}\big)(20(u^2+\overline{u}^2) + u^4 + \overline{u}^4)}{240u^2\overline{u}^2\big(\frac{1}{2} + \sum_{n=2}^{\infty} \frac{u^{2(n-1)}+\dots+\overline{u}^{2(n-1)}}{(2n)!}\big)} \\
&\qquad - \frac{\big(\frac{1}{6} + \sum_{n=2}^{\infty} \frac{u^{2(n-1)}+\dots+\overline{u}^{2(n-1)}}{(2n+1)!}\big)\big(60(u^2+\overline{u}^2) + 5(u^4+\overline{u}^4)\big)}{240u^2\overline{u}^2\big(\frac{1}{2} + \sum_{n=2}^{\infty} \frac{u^{2(n-1)}+\dots+\overline{u}^{2(n-1)}}{(2n)!}\big)} \Bigg) \\
&= \lim_{u\to 0} \Bigg( \frac{\big(\frac{1}{2} + \frac{u^2+\overline{u}^2}{4!}\big)(20(u^2+\overline{u}^2) + u^4 + \overline{u}^4)}{120u^2\overline{u}^2} \\
&\qquad - \frac{\big(\frac{1}{6} + \frac{u^2+\overline{u}^2}{5!}\big)\big(60(u^2+\overline{u}^2) + 5(u^4+\overline{u}^4)\big)}{120u^2\overline{u}^2} \Bigg) \\
&= \lim_{u\to 0} \Bigg( \frac{(12 + u^2 + \overline{u}^2)(20(u^2+\overline{u}^2) + u^4 + \overline{u}^4)}{24 \cdot 120u^2\overline{u}^2} \\
&\qquad - \frac{20 + u^2 + \overline{u}^2)\big(12(u^2+\overline{u}^2) + u^4 + \overline{u}^4\big)}{24 \cdot 120u^2\overline{u}^2} \Bigg) \\
&= \frac{1}{180}.
\end{aligned}
$$

Using the above equalities, upon taking the limit $u \to 0$ in Theorem 3.5.1, the result follows. □

# Summary

The following section presents a synopsis of the main discoveries obtained from the doctoral dissertation. Our investigation has produced various significant lemmas, propositions, theorems and corollaries that have been comprehensively explained in papers [1, 2].

We start with Taylor's theorem which involves using a polynomial, known as the Taylor polynomial of corresponding degree, to approximate a differentiable function around a specified point up to a certain order. The polynomial truncates the function's Taylor series at that given order and can provide linear or quadratic approximations depending on its degree. Variations of this method exist with explicit error estimates available. Brook Taylor [26] created one version in 1715 following James Gregory foreshadowing it back in 1671 [14]. The concept serves as an essential tool for introductory calculus courses and mathematical analysis while furnishing formulas for numerous transcendental functions like exponential and trigonometric ones with accuracy. Besides initiating analytic function exploration, it has significance across varied domains such as numerical analysis and mathematical physics by extending into multivariate or vector-valued functions too.

Given a function $f : I \to \mathbb{R}$, which is $n$ times differentiable at $a \in I$ (where $I$ is a non-degenerate real interval), the polynomial $T_{n;a}(f)$ defined by

$$T_{n;a}(f)(x) := \sum_{j=0}^{n} f^{(j)}(a) \cdot \frac{(x-a)^j}{j!}$$

is called the *$n$th-order Taylor polynomial of the function $f$ at the base point $a$.*

The form with integral remainder term can be formulated as follows.

THEOREM. *Let $I$ be a real interval and let $f : I \to \mathbb{R}$ be $(n+1)$ times continuously differentiable. Then, for all $a, x \in I$,*

$$f(x) = T_{n;a}(f)(x) + \int_a^x f^{(n+1)}(t) \cdot \frac{(x-t)^n}{n!} dt.$$

The variant as an intermediate value theorem is the following assertion.

THEOREM. *Let $I$ be a real interval and let $f : I \to \mathbb{R}$ be $(n+1)$ times differentiable. Then, for all $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$f(x) = T_{n;a}(f)(x) + f^{(n+1)}(\xi) \cdot \frac{(x-a)^{n+1}}{(n+1)!}. \tag{1}$$

The formula for the remainder term in (1) is called Lagrange's form of the remainder term.

In addition to the previous two remainders in (1) and (2), there are many other types of remainders such as:

- The *Cauchy form* of the remainder, which is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n)!}(x-\xi)^n(x-a). \tag{2}$$

- The *Schlömilch form* of the remainder, which is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^{n+1-p}\frac{(x-a)^p}{p}.$$

  This is also known as the *Schlömilch-Roché form* [7], and it is the general case of (1) and (2), where selecting $p = n+1$ represents the Lagrange form, while choosing $p = 1$ corresponds to the Cauchy form.

For $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, let the $n$th-order linear differential operator $D_c\colon \mathcal{C}^n_{\mathbb{K}}(I) \to \mathcal{C}_{\mathbb{K}}(I)$ be defined by the formula

$$D_c(f) := c_n f^{(n)} + \dots + c_1 f' + c_0 f \qquad (f \in \mathcal{C}^n_{\mathbb{K}}(I)).$$

Let $\omega_c \in \mathcal{C}^n_{\mathbb{C}}(\mathbb{R})$ denote the unique solution of the initial value problem

$$D_c(\omega_c) = 0, \qquad \omega_c^{(\ell)}(0) = \delta_{\ell,n-1} \quad (\ell \in \{0, \dots, n-1\}). \tag{3}$$

The function $\omega_c$ will be called the *characteristic solution* of the differential equation

$$D_c(\omega) = 0.$$

In order to provide a more or less explicit formula for $P_c$, we define the $(n-1)$*st order divided difference* $f(\lambda_1, \dots, \lambda_n)$ by

$$f(\lambda_1, \dots, \lambda_n) := \sum_{i=1}^{n} \frac{f(\lambda_i)}{\prod_{j\in\{1,\dots,n\}\setminus\{i\}}(\lambda_i - \lambda_j)},$$

see [10] for more details and alternative definitions. Moreover, in the following lemma, we compute divided differences of $f$ with repeated arguments under natural regularity assumptions.

LEMMA. *Let $D \subseteq \mathbb{K}$ be open, let $n, k, m_1, \ldots, m_k \in \mathbb{N}$ with $m_1 + \cdots + m_k = n$, let $(\lambda_1, \ldots, \lambda_k) \in \sigma_k(D)$ and define the polynomials $P_1, \ldots, P_k, P : \mathbb{C} \to \mathbb{C}$ by*

$$
\begin{aligned}
P_i(\lambda) &:= \prod_{j \in \{1,\ldots,k\}\setminus\{i\}} (\lambda - \lambda_j)^{m_j} \qquad (i \in \{1, \ldots, k\}), \\
P(\lambda) &:= \prod_{j=1}^{k} (\lambda - \lambda_j)^{m_j}.
\end{aligned} \tag{4}
$$

*If $f : D \to \mathbb{C}$ is $(m_i - 1)$ times continuously differentiable at $\lambda_i$ for all $i \in \{1, \ldots, k\}$, then*

$$f((\lambda_1)^{m_1}, \ldots, (\lambda_k)^{m_k}) = \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i - 1 - \ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!}.$$

*Furthermore,*

$$
\begin{aligned}
&f((\lambda_1)^{m_1}, \ldots, (\lambda_k)^{m_k}) \\
&= \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{f^{(\ell)}(\lambda_i)}{\ell!} \Bigg( \sum_{j=0}^{m_i-1-\ell} (-1)^j \\
&\quad \times \frac{j! B_{m_i-1-\ell,j}\big(\frac{1!}{(m_i+1)!} P^{(m_i+1)}(\lambda_i), \ldots, \frac{(m_i-\ell-j)!}{(2m_i-\ell-j)!} P^{(2m_i-\ell-j)}(\lambda_i)\big)}{(m_i - 1 - \ell)! \big(\frac{0!}{m_i!} P^{(m_i)}(\lambda_i)\big)^{j+1}} \Bigg).
\end{aligned}
$$

In fact, the $i$th term of the first (equivalently, of the second) formula of the lemma becomes very simple in the particular cases when $1 \leq m_i \leq 3$. Indeed,

$$
\begin{aligned}
&\sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i - 1 - \ell)!} \cdot \frac{f^{(\ell)}(\lambda_i)}{\ell!} \\
&= \begin{cases}
\dfrac{1}{P'(\lambda_i)} f(\lambda_i) & \text{if } m_i = 1, \\[2ex]
\dfrac{2}{P''(\lambda_i)} f'(\lambda_i) - \dfrac{2P'''(\lambda_i)}{3P''(\lambda_i)^2} f(\lambda_i) & \text{if } m_i = 2, \\[2ex]
\dfrac{3}{P'''(\lambda_i)} f''(\lambda_i) - \dfrac{3P^{(4)}(\lambda_i)}{2P'''(\lambda_i)^2} f'(\lambda_i) & \\
\qquad + \left( \dfrac{3P^{(4)}(\lambda_i)^2}{8P'''(\lambda_i)^3} - \dfrac{3P^{(5)}(\lambda_i)}{10P'''(\lambda_i)^2} \right) f(\lambda_i) & \text{if } m_i = 3.
\end{cases}
\end{aligned}
$$

LEMMA. *Let $n \in \mathbb{N}$, $c = (c_0, \ldots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, and let $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be pairwise distinct roots of the characteristic polynomial*

*$P_c$ with multiplicities $m_1, \dots, m_k \in \mathbb{N}$, respectively. Then*

$$\omega_c(t) = \sum_{i=1}^{k} \sum_{\ell=0}^{m_i-1} \frac{(P_i^{-1})^{(m_i-1-\ell)}(\lambda_i)}{(m_i-1-\ell)!} \cdot \frac{t^\ell \exp(\lambda_i t)}{\ell!},$$

*where $P_i$ is defined by (4).*

For the formulation of some consequences of our main results, for $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{C}$, we define the function $\zeta_{n,k,\gamma} : \mathbb{R} \to \mathbb{R}$ by

$$\zeta_{n,k,\gamma}(t) := \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}. \tag{5}$$

By applying the ratio test, it follows that the series is convergent for all $t \in \mathbb{R}$. For further properties, we have the following statement.

LEMMA. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{C}$. Then, the function $\zeta_{n,k,\gamma}$ is the (unique) solution of the initial value problem*

$$\zeta^{(n+1)} = \gamma \zeta^{(k+1)}, \qquad \zeta^{(i)}(0) = \delta_{i,n} \quad (i \in \{0, \dots, n\}).$$

*In addition, if $j \in \{0, \dots, k\}$, then*

$$\zeta_{n,k,\gamma}^{(j)} = \zeta_{n-j,k-j,\gamma}.$$

*If $\gamma \neq 0$, then, for all $t \in \mathbb{R}$,*

$$\zeta_{n,k,\gamma^{n-k}}(t) = \gamma^{-n} \zeta_{n,k,1}(\gamma t).$$

*Furthermore, for all $t \in \mathbb{R}$,*

$$\zeta_{n,k,0}(t) = \frac{t^n}{n!},$$

$$\zeta_{n,0,1}(t) = -1 + \frac{1}{n} \sum_{j=0}^{n-1} \exp\left(\cos\left(\frac{2\pi j}{n}\right)t\right) \cdot \cos\left(\sin\left(\frac{2\pi j}{n}\right)t\right).$$

We note that the series involved in the right hand side of (5) has closed forms, more precisely, it is the linear combinations of the hyperbolic or trigonometric functions and a polynomial of degree at most $n-3$. Indeed, if $\gamma = 1$, then we get

$$\zeta_{n,n-2,1}(t) = \sum_{i=0}^{\infty} \frac{t^{2i+n}}{(2i+n)!} = \begin{cases} \cosh(t) - \displaystyle\sum_{i=0}^{\frac{n-2}{2}} \frac{t^{2i}}{(2i)!} & \text{if } n \text{ is even,} \\ \sinh(t) - \displaystyle\sum_{i=0}^{\frac{n-3}{2}} \frac{t^{2i+1}}{(2i+1)!} & \text{if } n \text{ is odd.} \end{cases}$$

While for $\gamma = -1$, we can obtain

$$\zeta_{n,n-2,-1}(t) = \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+n}}{(2i+n)!}$$
$$= \begin{cases} (-1)^{\frac{n-2}{2}} \left( \cos(t) - \sum_{i=0}^{\frac{n-2}{2}} \frac{(-1)^i t^{2i}}{(2i)!} \right) & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-3}{2}} \left( \sin(t) - \sum_{i=0}^{\frac{n-3}{2}} \frac{(-1)^i t^{2i+1}}{(2i+1)!} \right) & \text{if } n \text{ is odd.} \end{cases}$$

Now, our main results can be stated as follows.

THEOREM. *Let $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$, and assume that $f : I \to \mathbb{K}$ is $(n-1)$ times differentiable at $a \in I$. Define $T_{a,c}f : \mathbb{R} \to \mathbb{K}$ by*

$$(T_{a,c}f)(x) := \sum_{j=0}^{n-1} \left( f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right),$$

*where $\omega_c$ is defined by (3). Then, $T_{a,c}f$ belongs to the kernel of $D_c$ and*

$$f^{(\ell)}(a) = (T_{a,c}f)^{(\ell)}(a) \qquad (\ell \in \{0, \dots, n-1\}).$$

The function $T_{a,c}f$ is termed the *generalized Taylor polynomial at the point $a$ with respect to the differential operator $D_c$.*

THEOREM. *Let $n \in \mathbb{N}$, $c = (c_0, \dots, c_n) \in \mathbb{K}^{n+1}$ with $c_n = 1$. Then, for all $f \in \mathcal{C}^n_{\mathbb{K}}(I)$ and $x, a \in I$, we have*

$$f(x) = (T_{a,c}f)(x) + \int_a^x D_c(f)(t) \cdot \omega_c(x-t)dt.$$

The following result is a consequence of the previous theorem in which the main part contains the Taylor expansion of order $k$ and the rest is in terms of the function $\zeta_{n,k,\gamma}$.

THEOREM. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$, and $\gamma \in \mathbb{K}$. Then, for all $f \in \mathcal{C}^n_{\mathbb{K}}(I)$ and $x, a \in I$,*

$$f(x) = \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a) \zeta_{n,k,\gamma}^{(n-j)}(x-a)$$
$$+ \int_a^x (f^{(n)}(t) - \gamma f^{(k)}(t)) \zeta'_{n,k,\gamma}(x-t)dt.$$

The subsequent results will be corollaries of the previous theorem. First we note that the classical Taylor theorem with an integral remainder term follows from the previous theorem by taking $k = 0$ and $\gamma = 0$.

COROLLARY. *For all* $f \in \mathcal{C}^2_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$f(x) = f(a)\cos(x-a) + f'(a)\sin(x-a) + \int_a^x (f''(t) + f(t))\sin(x-t)dt.$$

COROLLARY. *For all* $f \in \mathcal{C}^2_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$f(x) = f(a)\cosh(x-a) + f'(a)\sinh(x-a) + \int_a^x (f''(t) - f(t))\sinh(x-t)dt.$$

COROLLARY. *For all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned} f(x) = f(a)&\frac{\cosh(x-a)+\cos(x-a)}{2} \\ &+ f'(a)\frac{\sinh(x-a)+\sin(x-a)}{2} \\ &+ f''(a)\frac{\cosh(x-a)-\cos(x-a)}{2} \\ &+ f'''(a)\frac{\sinh(x-a)-\sin(x-a)}{2} \\ &+ \int_a^x (f''''(t) - f(t))\frac{\sinh(x-t)-\sin(x-t)}{2}dt. \end{aligned}$$

COROLLARY. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then, for all $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ and $a, x \in I$, we have*

$$
\begin{aligned}
f(x) = {} & f(a)\frac{\beta^2\cos(\alpha(x-a)) - \alpha^2\cos(\beta(x-a))}{\beta^2-\alpha^2} \\
& + f'(a)\frac{\beta^3\sin(\alpha(x-a)) - \alpha^3\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\
& + f''(a)\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2-\alpha^2} \\
& + f'''(a)\frac{\beta\sin(\alpha(x-a)) - \alpha\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\
& + \int_a^x (f''''(t) + (\alpha^2+\beta^2)f''(t) + \alpha^2\beta^2 f(t)) \\
& \qquad \times \frac{\beta\sin(\alpha(x-t)) - \alpha\sin(\beta(x-t))}{\alpha\beta(\beta^2-\alpha^2)}\,dt.
\end{aligned}
$$

The limiting case of the above corollary (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ and $a, x \in I$, we have*

$$
\begin{aligned}
f(x) = {} & f(a)\frac{2\cos(\alpha(x-a)) + \alpha(x-a)\sin(\alpha(x-a))}{2} \\
& + f'(a)\frac{3\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha} \\
& + f''(a)\frac{(x-a)\sin(\alpha(x-a))}{2\alpha} \\
& + f'''(a)\frac{\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha^3} \\
& + \int_a^x (f''''(t) + 2\alpha^2 f''(t) + \alpha^4 f(t)) \\
& \qquad \times \frac{\sin(\alpha(x-t)) - \alpha(x-t)\cos(\alpha(x-t))}{2\alpha^3}\,dt.
\end{aligned}
$$

COROLLARY. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned} f(x) = f(a)&\frac{\beta^2\cosh(\alpha(x-a)) - \alpha^2\cosh(\beta(x-a))}{\beta^2-\alpha^2} \\ &+ f'(a)\frac{\beta^3\sinh(\alpha(x-a)) - \alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ f''(a)\frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2-\alpha^2} \\ &+ f'''(a)\frac{\alpha\sinh(\beta(x-a)) - \beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ \int_a^x (f''''(t) - (\alpha^2+\beta^2)f''(t) + \alpha^2\beta^2 f(t)) \\ &\qquad \times \frac{\alpha\sinh(\beta(x-t)) - \beta\sinh(\alpha(x-t))}{\alpha\beta(\beta^2-\alpha^2)}dt. \end{aligned}$$

COROLLARY. *Let* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$. *Then, for all* $f \in \mathcal{C}^4(I)$ *and* $a, x \in I$, *we have*

$$\begin{aligned} f(x) = f(a)&\frac{2\cosh(\alpha(x-a)) - \alpha(x-a)\sinh(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sinh(\alpha(x-a)) - \alpha(x-a)\cosh(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sinh(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\alpha(x-a)\cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ &+ \int_a^x (f''''(t) - 2\alpha^2 f''(t) + \alpha^4 f(t)) \\ &\qquad \times \frac{\alpha(x-t)\cosh(\alpha(x-t)) - \sinh(\alpha(x-t))}{2\alpha^3}dt. \end{aligned}$$

To formulate the results of Taylor mean value theorem, we recall the extended mean value theorem for integrals.

LEMMA. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function and* $g : [a, b] \to \mathbb{R}$ *a nonnegative (or nonpositive) integrable function. Then there exists* $\xi \in [a, b]$ *such that*

$$\int_a^b fg = f(\xi)\int_a^b g.$$

Also, for any continuous function $h : \mathbb{R} \to \mathbb{R}$, let $\rho^+(h) \in [0, +\infty]$ (resp. $\rho^-(h) \in [-\infty, 0]$) denote the infimum of the positive roots (resp. the supremum of the negative roots) of $h$.

LEMMA. *Let $n \in \mathbb{N}$ and $c = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for $k \in \{1, \ldots, n-1\}$,*

$$\big[\rho^-(\omega_c^{(k)}), \rho^+(\omega_c^{(k)})\big] \subseteq \big[\rho^-(\omega_c^{(k-1)}), \rho^+(\omega_c^{(k-1)})\big].$$

*Furthermore, $\big[\rho^-(\omega_c^{(n-1)}), \rho^+(\omega_c^{(n-1)})\big]$ is a neighborhood of $0$.*

The generalization of the Taylor mean value theorem is given as follows.

THEOREM. *Let $n \in \mathbb{N}$, $c = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ with $c_n = 1$. Then, for all $f \in \mathcal{C}^n_{\mathbb{R}}(I)$ and $a, x \in I$ with $\rho^-(\omega_c) \le x - a \le \rho^+(\omega_c)$, there exists a point $\xi$ between $a$ and $x$ such that*

$$f(x) = (T_{a,c} f)(x) + D_c(f)(\xi) \cdot \int_0^{x-a} \omega_c(t)dt.$$

THEOREM. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k < n$ and $\gamma \in \mathbb{R}$ and define $\zeta_{n,k,\gamma} : \mathbb{R} \to \mathbb{R}$ by (5). Then, for all $f \in \mathcal{C}^n_{\mathbb{R}}(I)$ and $x, a \in I$ with $\rho^-(\zeta'_{n,k,\gamma}) \le x - a \le \rho^+(\zeta'_{n,k,\gamma})$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} &+ \sum_{j=k}^{n-1} f^{(j)}(a) \zeta_{n,k,\gamma}^{(n-j)}(x-a) \\ &+ (f^{(n)}(\xi) - \gamma f^{(k)}(\xi)) \zeta_{n,k,\gamma}(x-a). \end{aligned}$$

The subsequent results will be corollaries of the previous theorem. Moreover, the classical Taylor Mean Value Theorem is the particular case of the previous theorem when $k = 0$ and $\gamma = 0$. In this setting, we have that

$$\zeta_{n,0,0}(t) = \frac{t^n}{n!}$$

and hence

$$\rho^{\pm}(\zeta'_{n,0,0}) = \pm\infty.$$

COROLLARY. *For all $f \in \mathcal{C}^2_{\mathbb{R}}(I)$ and $a, x \in I$ with $|a - x| \le \pi$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)\cos(x-a) &+ f'(a)\sin(x-a) \\ &+ (f''(\xi) + f(\xi))(1 - \cos(x-a)). \end{aligned}$$

COROLLARY. *For all $f \in \mathcal{C}^2_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)\cosh(x-a) + f'(a)\sinh(x-a) \\ + (f''(\xi) - f(\xi))(\cosh(x-a) - 1). \end{aligned}$$

COROLLARY. *For all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{\cosh(x-a) + \cos(x-a)}{2} \\ &+ f'(a)\frac{\sinh(x-a) + \sin(x-a)}{2} \\ &+ f''(a)\frac{\cosh(x-a) - \cos(x-a)}{2} \\ &+ f'''(a)\frac{\sinh(x-a) - \sin(x-a)}{2} \\ &+ (f''''(t) - f(t))\frac{\cosh(x-a) + \cos(x-a) - 2}{2}. \end{aligned}$$

COROLLARY. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$ and let $t_0$ be the smallest positive root of the equation*

$$\beta \sin(\alpha t) = \alpha \sin(\beta t). \tag{6}$$

*Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$ with $|x - a| \leq t_0$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{\beta^2\cos(\alpha(x-a)) - \alpha^2\cos(\beta(x-a))}{\beta^2 - \alpha^2} \\ &+ f'(a)\frac{\beta^3\sin(\alpha(x-a)) - \alpha^3\sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ f''(a)\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2} \\ &+ f'''(a)\frac{\beta\sin(\alpha(x-a)) - \alpha\sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \\ &+ (f''''(\xi) + (\alpha^2 + \beta^2)f''(\xi) + \alpha^2\beta^2 f(\xi)) \\ &\qquad \times \frac{\alpha^2(\cos(\beta(x-a)) - 1) - \beta^2(\cos(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2 - \alpha^2)}. \end{aligned}$$

For the applicability of the previous corollary, it is essential to find the zeroes of the equation (6). In general, beyond the trivial solution $t = 0$, the other solutions cannot be established algebraically. On the other hand, if $\frac{\alpha}{\beta}$ is rational, say $|\frac{\alpha}{\beta}| = \frac{n}{m}$, where $n, m$ are coprime natural numbers, let $s := \frac{|\alpha|}{n} = \frac{|\beta|}{m} \neq 0$. Then $\alpha = \pm ns$ and $\beta = \pm ms$ and (6) is now equivalent to

$$m \sin(nst) = n \sin(mst).$$

In the case when $t = \frac{k}{s}\pi$ for some $k \in \mathbb{N}$, then both sides are equal to zero. If $t$ is not of this form, then $\sin(st) \neq 0$, thus this equation can be rewritten as

$$mU_{n-1}(\cos(st)) = m\frac{\sin(nst)}{\sin(st)} = n\frac{\sin(mst)}{\sin(st)} = nU_{m-1}(\cos(st)),$$

where $U_k$ denotes the $k$th degree Chebyshev polynomial of the second kind. Therefore, the last equation is an algebraic equation for $\cos(st)$. Solving this equation for $\cos(st)$, the smallest positive solution $t_0$ can easily be computed.

The limiting case of the previous corollary (i.e., when $\alpha^2 = \beta^2 \neq 0$) is formulated as follows.

COROLLARY. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ and let $t_0$ be the smallest positive root of the equation*

$$\sin(\alpha t) = \alpha t \cos(\alpha t).$$

*Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$ with $|x - a| \leq t_0$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{2\cos(\alpha(x-a)) + \alpha(x-a)\sin(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sin(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\sin(\alpha(x-a)) - \alpha(x-a)\cos(\alpha(x-a))}{2\alpha^3} \\ &+ (f''''(\xi) + 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\ &\qquad \times \frac{2 - 2\cos(\alpha(x-a)) - \alpha(x-a)\sin(\alpha(x-a))}{2\alpha^4}. \end{aligned}$$

COROLLARY. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{\beta^2\cosh(\alpha(x-a)) - \alpha^2\cosh(\beta(x-a))}{\beta^2-\alpha^2} \\ &+ f'(a)\frac{\beta^3\sinh(\alpha(x-a)) - \alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ f''(a)\frac{\cosh(\beta(x-a)) - \cosh(\alpha(x-a))}{\beta^2-\alpha^2} \\ &+ f'''(a)\frac{\alpha\sinh(\beta(x-a)) - \beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2-\alpha^2)} \\ &+ (f''''(\xi) - (\alpha^2+\beta^2)f''(\xi) + \alpha^2\beta^2 f(\xi)) \\ &\qquad \times \frac{\alpha^2(\cosh(\beta(x-a)) - 1) - \beta^2(\cosh(\alpha(x-a)) - 1)}{\alpha^2\beta^2(\beta^2-\alpha^2)}. \end{aligned}$$

COROLLARY. *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Then, for all $f \in \mathcal{C}^4_{\mathbb{R}}(I)$ and $a, x \in I$, there exists a point $\xi$ between $a$ and $x$ such that*

$$\begin{aligned} f(x) = f(a)&\frac{2\cosh(\alpha(x-a)) - \alpha(x-a)\sinh(\alpha(x-a))}{2} \\ &+ f'(a)\frac{3\sinh(\alpha(x-a)) - \alpha(x-a)\cosh(\alpha(x-a))}{2\alpha} \\ &+ f''(a)\frac{(x-a)\sinh(\alpha(x-a))}{2\alpha} \\ &+ f'''(a)\frac{\alpha(x-a)\cosh(\alpha(x-a)) - \sinh(\alpha(x-a))}{2\alpha^3} \\ &+ (f''''(\xi) - 2\alpha^2 f''(\xi) + \alpha^4 f(\xi)) \\ &\qquad \times \frac{2 - 2\cosh(\alpha(x-a)) + \alpha(x-a)\sinh(\alpha(x-a))}{2\alpha^4}. \end{aligned}$$

Taylor's theorem plays an important role in the field of the numerical analysis, especially when it comes to understanding errors that arise when approximating functions. This includes analyzing errors related to numerical integration methods such as Simpson's rule and the trapezoidal rule. Here, we establish various factorization results and then derive estimates for linear functionals through the use of a generalized Taylor theorem. Moreover, several error bounds are established including applications to the trapezoidal rule as well as to a Simpson formula rule.

We start with the following theorem which establishes a sufficient condition for such a factorization.

THEOREM. *Let $X$ and $Z$ be Banach spaces and $Y$ be a normed space over $\mathbb{K}$. Assume that $A : X \to Y$ and $B : X \to Z$ are bounded linear maps such that $\ker B \subseteq \ker A$ and $B(X) = Z$. Then there exists a unique bounded linear map $C : Z \to Y$ such that $A = C \circ B$.*

The main goal here is to obtain various estimates for the linear functional $\mathcal{A}_\mu : \mathcal{C}_{\mathbb{K}}(I) \to \mathbb{K}$ defined by

$$\mathcal{A}_\mu(f) := \int_{[a,b]} f(x)d\mu(x).$$

In order to construct $n \in \mathbb{N}$ and a linear map $B : \mathcal{C}^n_{\mathbb{K}}(I) \to \mathcal{C}_{\mathbb{K}}(I)$ such that $\ker B \subseteq \ker \mathcal{A}_\mu$, we search for exponential polynomials in the kernel of $\mathcal{A}_\mu$. For this aim, let us define the function $\mathcal{S}_\mu : \mathbb{C} \to \mathbb{C}$ by

$$\mathcal{S}_\mu(\lambda) := \int_{[a,b]} e^{\lambda x} d\mu(x).$$

The function $\mathcal{S}_\mu$ will be termed the *spectral function related to the measure $\mu$.*

For fixed elements $\lambda_1, \dots, \lambda_k \in \Lambda_\mu$, we consider the polynomial $P : \mathbb{C} \to \mathbb{C}$ given by

$$\text{(7)} \qquad P(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k} = c_n \lambda^n + \cdots + c_1 \lambda + c_0,$$

where $1 \leq m_i \leq m(\mathcal{S}_\mu, \lambda_i)$ for all $i$ and $n := m_1 + \cdots + m_k$. Clearly, $c_n = 1$. Then we define the linear differential operator $D_c : \mathcal{C}^n_{\mathbb{K}}(I) \to \mathcal{C}_{\mathbb{K}}(I)$ by the formula

$$\text{(8)} \qquad D_c(f) := c_n f^{(n)} + \cdots + c_1 f' + c_0 f \qquad (f \in \mathcal{C}^n(I)).$$

Our basic factorization results are stated as follows.

THEOREM. *Let $\mu$ be a nonzero bounded $\mathbb{C}$-valued Borel measure on $[a, b]$, let $\lambda_1, \dots, \lambda_k \in \Lambda_\mu$ and $m_1, \dots, m_k \in \mathbb{N}$ with $m_i \leq m(\mathcal{S}_\mu, \lambda_i)$ for $i \in \{1, \dots, k\}$. Define $c = (c_0, \dots, c_n) \in \mathbb{C}^{n+1}$ by (7) (where $n := m_1 + \cdots + m_k$) and the differential operator $D_c : \mathcal{C}^n_{\mathbb{C}}([a, b]) \to \mathcal{C}_{\mathbb{C}}([a, b])$ by (8). Let $\omega_c \in \mathcal{C}^n_{\mathbb{C}}(\mathbb{R})$ be the characteristic solution of $D_c(\omega) = 0$. Finally, define $g : [a, b] \to \mathbb{C}$ by*

$$g(t) := \int_{[t,b]} \omega_c(x - t)d\mu(x).$$

*Then, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a, b])$,*

$$\mathcal{A}_\mu(f) := \int_{[a,b]} f(x)d\mu(x) = \int_{[a,b]} D_c(f)(t) \cdot g(t)dt.$$

*In other words,* $\mathcal{A}_\mu|_{\mathcal{C}^n_{\mathbb{C}}(I)} = C_g \circ D_c$*, where* $C_g : \mathcal{C}_{\mathbb{C}}(I) \to \mathbb{C}$ *is given by*

$$C_g(h) = \int_{[a,b]} h(t)g(t)dt.$$

THEOREM. *Let* $\mu$ *be a nonzero bounded* $\mathbb{C}$*-valued Borel measure on* $[a,b]$*, let* $0 \leq k \leq n$ *and* $\gamma \in \mathbb{C}$*. Assume that*

$$\begin{aligned} \int_{[a,b]} x^i d\mu(x) = 0, \quad & (i \in \{0,\dots,k-1\}), \\ \int_{[a,b]} \exp\big(\sqrt[n-k]{\gamma}\exp\big(\tfrac{2j\pi}{n-k}\boldsymbol{i}\big)x\big)d\mu(x) = 0, \quad & (j \in \{0,\dots,n-k-1\}), \end{aligned} \tag{9}$$

*where* $\sqrt[n-k]{\gamma}$ *denotes the root of order* $(n-k)$ *of* $\gamma$ *with the smallest nonnegative argument in the interval* $[0,2\pi)$*. Define* $g : [a,b] \to \mathbb{C}$ *by*

$$g(t) := \int_{[t,b]} \zeta'_{n,k,\gamma}(x-t)d\mu(x). \tag{10}$$

*Then, for all* $f \in \mathcal{C}^n_{\mathbb{C}}([a,b])$*,*

$$\mathcal{A}_\mu(f) = \int_{[a,b]} (f^{(n)}(t) - \gamma f^{(k)}(t)) \cdot g(t)dt.$$

Before formulating the next result, we recall the definition of the $p$th norm of a Lebesgue measurable function $f : I \to \mathbb{C}$ for $p \in [1,\infty]$:

$$\|f\|_p := \begin{cases} \Big(\int_{[a,b]} |f(t)|^p dt\Big)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \\ \inf\{s \geq 0 \colon |f(t)| \leq s \text{ for a.e. } t \in I\} & \text{if } p = \infty. \end{cases}$$

COROLLARY. *Under the notation and assumptions of the previous theorems, for all* $f \in \mathcal{C}^n_{\mathbb{C}}(I)$ *and for all* $p, q \in [1,\infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*,*

$$|\mathcal{A}_\mu(f)| \leq \|D_c(f)\|_p \cdot \|g\|_q.$$

*If, in addition,* $c = (c_0,\dots,c_n) \in \mathbb{R}^{n+1}$*,* $\mu$ *is a real-valued measure,* $f \in \mathcal{C}^n_{\mathbb{R}}(I)$*, and both* $g$ *and* $D_c(f)$ *are nonnegative (or nonpositive) on* $[a,b]$*, then*

$$\mathcal{A}_\mu(f) \geq 0. \tag{11}$$

*If* $g$ *and* $D_c(f)$ *have opposite signs over* $[a,b]$*, then this inequality reverses.*

COROLLARY. *Let* $\mu$ *be a nonzero bounded* $\mathbb{C}$*-valued Borel measure on* $[a,b]$*, let* $0 \leq k \leq n$ *and* $\gamma \in \mathbb{C}$*. Assume that the equalities in* (9) *hold. Define*

*$g : [a,b] \to \mathbb{C}$ by (10). Then, for all $f \in \mathcal{C}^n_{\mathbb{C}}([a,b])$ and $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\left| \int_{[a,b]} f(x) d\mu(x) \right| \leq \| f^{(n)} - \gamma f^{(k)} \|_p \cdot \| g \|_q .$$

*If, in addition, $\gamma \in \mathbb{R}$, $\mu$ is a real-valued measure and both $g$ and $f^{(n)} - \gamma f^{(k)}$ are nonnegative (or nonpositive) on $[a,b]$, then the inequality (11) holds. If $g$ and $f^{(n)} - \gamma f^{(k)}$ have opposite signs over $[a,b]$, then this inequality reverses.*

To apply the results to the trapezoidal rule, we begin with the following lemma.

LEMMA. *For all $t \in \mathbb{R}_+$*

$$\frac{t}{\sinh(t)} < 1 < t \coth(t)$$

*and, for all $t \in (0, \pi)$,*

$$t \cot(t) < 1 < \frac{t}{\sin(t)} .$$

The aim is to establish various further estimates for $R_T(f)$, which is defined by

$$R_T(f) := \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f .$$

Observe that, with $\mu := \frac{1}{2}(\delta_a + \delta_b) - \nu$ (where $\delta_t$ denotes the Dirac measure concentrated at $t$ and $\nu$ stands for the normalized Lebesgue measure on $[a,b]$), we can obtain that

$$R_T(f) = \mathcal{A}_\mu(f).$$

The corresponding spectral function is given by

$$\mathcal{S}_\mu(\lambda) = \frac{e^{\lambda a} + e^{\lambda b}}{2} - \frac{1}{b-a} \int_a^b e^{\lambda x} dx \qquad (\lambda \in \mathbb{C}). \tag{12}$$

Given a compact interval $[a,b]$, the classical trapezoidal rule asserts that, for a twice differentiable function $f : [a,b] \to \mathbb{R}$,

$$\frac{1}{b-a} \int_a^b f = \frac{f(a) + f(b)}{2} - R_T(f),$$

where the remainder term $R_T(f)$ has various estimates in terms of the norms of the second derivative of $f$ and the length of the interval $[a,b]$. For instance (see [5, pp. 252–253]),

$$|R_T(f)| \leq \frac{(b-a)^2}{12} \| f'' \|_\infty .$$

LEMMA. *Let $\lambda \in \mathbb{C}$. Then $\lambda$ is a root of the spectral function $\mathcal{S}_\mu$ given by (12) if and only if $u := \lambda\frac{b-a}{2}$ is a fixed point of the tangent hyperbolic function. The multiplicity of $\lambda$ equals $1$ if $\lambda \neq 0$ and equals $2$ if $\lambda = 0$.*

In order to apply our main theorems to the trapezoidal rule, we shall need to describe the fixed points of the tangent hyperbolic function.

LEMMA. *A number $u \in \mathbb{C}$ is a fixed point of the tangent hyperbolic function, i.e.,*

$$\tanh(u) = u$$

*holds if and only if $u = v\boldsymbol{i}$, where $v \in \mathbb{R}$ is a fixed point of the tangent function. Furthermore, for all $k \in \mathbb{Z}$, the open interval $\big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ contains exactly one fixed point of the tangent function.*

The unique fixed point of the tangent function in the open interval $\big((k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big)$ will be denoted by $\tau_k$ in the sequel.

THEOREM. *Let $k \in \mathbb{N}$, $0 \leq n_1 < \cdots < n_k$ be integers, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_j := \frac{2}{(b-a)}\tau_{n_j}$ for $j \in \{1, \ldots, k\}$. Define $(c_0, c_1, \ldots, c_{2k}) \in \mathbb{R}^{2k+1}$ by the equality*

$$\big(z^2+\lambda_1^2\big)\cdots\big(z^2+\lambda_k^2\big) = c_{2k}z^{2k} + \cdots + c_1 z^1 + c_0 =: P_c(z) \qquad (z \in \mathbb{C}).$$

*Then, for all $f \in \mathcal{C}^{2k}_{\mathbb{K}}([a,b])$,*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f = \int_a^b D_c(f)(t)\cdot g(t)dt,$$

*where*

$$g(t) := \begin{cases} \displaystyle\sum_{j=1}^{k} \frac{\sin(\lambda_j(b-t)/2)\sin(\lambda_j(t-a)/2)}{\lambda_j Q_j(\lambda_j)\sin(\lambda_j(b-a)/2)} & \text{if } n_1 > 0, \\ \displaystyle\sum_{j=2}^{k} \frac{\sin(\lambda_j(b-t)/2)\sin(\lambda_j(t-a)/2)}{\lambda_j Q_j(\lambda_j)\sin(\lambda_j(b-a)/2)} & \\ \qquad + \dfrac{(b-t)(t-a)}{2Q_1(0)(b-a)} & \text{if } n_1 = 0, \end{cases}$$

*and $Q_j(z) := \prod_{\ell\in\{1,\ldots,k\}\setminus\{j\}}(\lambda_\ell^2 - z^2)$ for $j \in \{1, \ldots, k\}$.*

In the particular case when $k = 1$, the above theorem simplifies to the following result.

COROLLARY. *Let $n \in \mathbb{N} \cup \{0\}$, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_n := \frac{2\tau_n}{b-a}$. Then, for all $f \in \mathcal{C}^2_{\mathbb{K}}([a,b])$,*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f = \int_a^b (f'' + \lambda_n^2 f)(t) \cdot g(t)dt,$$

*where*

$$g(t) := \begin{cases} \dfrac{\sin(\lambda_n(b-t)/2)\sin(\lambda_n(t-a)/2)}{\lambda_n \sin(\lambda_n(b-a)/2)} & \textit{if } n > 0, \\ \dfrac{(b-t)(t-a)}{2(b-a)} & \textit{if } n = 0. \end{cases}.$$

Consequently, the following statement is a new error estimate for the trapezoidal rule.

THEOREM. *Let $n \in \mathbb{N} \cup \{0\}$, let $a, b \in \mathbb{R}$ with $a < b$ and let $\lambda_n := \frac{2\tau_n}{b-a}$. Then, for all $f \in \mathcal{C}^2_{\mathbb{K}}([a,b])$,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f\right| \leq \begin{cases} \frac{1}{12}(b-a)^2 \cdot \|f''\|_\infty & \textit{if } n = 0, \\ \frac{(n+1)n\pi}{2\tau_n^3}(b-a)^2 \cdot \|f'' + \lambda_n^2 f\|_\infty & \textit{if } n > 0, \\ \frac{1}{8}(b-a) \cdot \|f''\|_1 & \textit{if } n = 0, \\ \frac{1+|\cos(\tau_n)|}{4\tau_n|\sin(\tau_n)|}(b-a) \cdot \|f'' + \lambda_n^2 f\|_1 & \textit{if } n > 0. \end{cases}$$

Our final result is an extension of the Simpson formula.

THEOREM. *Let $a, b \in \mathbb{R}$ with $a < b$, $u = w + \boldsymbol{i}v$, where $w \in \mathbb{R}_+$, $v \in (0, \pi)$ and define $\alpha_u, \beta_u$ by*

$$\alpha_u := \frac{\overline{u}\sinh(u) - u\sinh(\overline{u})}{2u\overline{u}(\cosh(u) - \cosh(\overline{u}))},$$
$$\beta_u := \frac{u\cosh(u)\sinh(\overline{u}) - \overline{u}\cosh(\overline{u})\sinh(u)}{u\overline{u}(\cosh(u) - \cosh(\overline{u}))}.$$

*Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}([a,b])$,

$$\left|\alpha_u f(a) + \beta_u f\Big(\frac{a+b}{2}\Big) + \alpha_u f(b) - \frac{1}{b-a}\int_a^b f\right|$$
$$\leq \begin{cases} \dfrac{(b-a)^3\big(v\sinh(w) - w\sin(v)\big)^2}{32(w^2+v^2)^2 wv\sinh(w)\sin(v)} \\ \qquad\qquad \times \left\| f'''' + \dfrac{8(v^2-w^2)}{(b-a)^2} f'' + \dfrac{16(w^2+v^2)^2}{(b-a)^4} f \right\|_1, \\ \dfrac{(b-a)^4(2\alpha_u + \beta_u - 1)}{16(w^2+v^2)^2} \\ \qquad\qquad \times \left\| f'''' + \dfrac{8(v^2-w^2)}{(b-a)^2} f'' + \dfrac{16(w^2+v^2)^2}{(b-a)^4} f \right\|_\infty. \end{cases}$$

Finally, we deduce the Simpson formula with two error terms by taking the limit $u \to 0$ in the previous theorem.

COROLLARY. *Let* $a, b \in \mathbb{R}$. *Then, for all* $f \in \mathcal{C}^4_{\mathbb{K}}([a,b])$,

$$\left|\frac{1}{6}f(a) + \frac{2}{3}f\Big(\frac{a+b}{2}\Big) + \frac{1}{6}f(b) - \frac{1}{b-a}\int_a^b f\right|$$
$$\leq \begin{cases} \dfrac{(b-a)^3}{1152} \cdot \left\| f'''' \right\|_1, \\ \dfrac{(b-a)^4}{2880} \cdot \left\| f'''' \right\|_\infty. \end{cases}$$

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# List of talks

(1) *A new extension of the Taylor theorem with an application of estimating linear functionals*, The First Sharjah International Conference on Mathematical Sciences, University of Sharjah, Sharjah, UAE, November 6–November 8, 2023.
(2) *Estimating linear functionals via factorization: Theory and applications*, The 59th International Symposium on Functional Equations, Hajdúszoboszló, Hungary, June 18–June 25, 2023.
(3) *Estimates of linear expressions through factorization*, Qualification at the End of the Third Year, Institute of Mathematics, University of Debrecen, June 6, 2023.
(4) *Generalizations of the Taylor theorem with factorization results*, 22nd Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Hungary, February 1–February 4, 2023.
(5) *Taylor-type expansions in terms of exponential polynomials*, Complex Exam Seminar, Institute of Mathematics, University of Debrecen, June 16, 2022.
(6) *Taylor-type expansions in terms of exponential polynomials*, Analysis Research Seminar, Institute of Mathematics, University of Debrecen, May 18, 2022.
(7) *A generalization of the Taylor theorem*, Qualification at the End of the First Year, Institute of Mathematics, University of Debrecen, June 18, 2021.