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ORIGINAL ARTICLE

EXISTENCE OF A PERIODIC AND SEASONAL INAR PROCESS

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A spectral criterion involving the model parameters is given for the existence and uniqueness of a periodically correlated and seasonal non-negative integer-valued autoregressive process. The structure of the mean and covariance functions of the periodically stationary distribution of the model is derived using its implicit state-space representation. Two infinite series representations for the process, the moving average, and the immigrant generation, are established. Based on the latter representation, a novel and parallelizable simulation method is proposed to generate the process.

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1. INTRODUCTION

A discrete-time stochastic process $\{Y_t\} := \{Y_t | t = 0, \pm 1, \pm 2, \dots\}$ is called periodically correlated (PC), or periodically stationary in the wide sense, if its mean function $\mu(t) := \mathsf{E}(Y_t)$ and its autocovariance function $R(s,t) := \mathrm{Cov}(Y_s, Y_t)$ exist for all $s, t \in \{0, \pm 1, \pm 2, \dots\}$ and are periodic functions with integer period $S \geq 1$. If $S = 1$, the process is weakly stationary. PC processes occur in many fields, such as medicine, hydrology, climatology, and air pollution, among others. They were introduced by Gladyshev (1961), for recent reviews see, e.g. Gardner *et al.* (2006) and Hurd and Miamee (2007). A natural way to build models for PC processes is to allow the parameters of stationary models to vary periodically with time. For example, the autoregressive moving average (ARMA) model was extended to the periodic ARMA (PARMA) model for PC processes. The existence of PARMA processes was studied by Pagano (1978), Vecchia (1985) and Ula and Smadi (1997). Statistical inference for PARMA models was addressed in the seminal papers of Lund and Basawa (2000) and Basawa and Lund (2001).

A weakly stationary process is called seasonally correlated (SC) if its autocovariance function γ has a seasonal pattern which is defined as $\gamma(h) := R(t+h, t)$ for all $h, t \in \mathbb{Z}$. A widely used parametric model for weakly stationary seasonal time series, as a part of the Box–Jenkins methodology, is the class of seasonal ARMA (SARMA) processes, pure or multiplicative ones see, e.g. section 3.9 in Shumway and Stoffer (2017) and examples therein. Inference in SARMA models was addressed by Chatfield and Prothero (1973), among others.

In practice, some time series exhibit both seasonal and periodic behavior by showing seasonal patterns in their periodic autocovariance structure. Seasonal ARMA processes with periodically varying parameters (SPARMA) were introduced in Basawa *et al.* (2004). Such time series occur in econometrics, see section 4.2

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in Koopman *et al.* (2006) and section 3.3 in Hindrayanto *et al.* (2010), where SPARIMA models were applied for the monthly US unemployment series. Further discussion on the modeling of seasonality and periodicity in real-valued time series can be found in Lund (2012) and Baek *et al.* (2017).

Nowadays, there is a growing interest in count time series $\{Y_t\}$, where Y_t denotes the number of occurrences of individuals, objects or events at time t, see, e.g. Davis *et al.* (2021) and Weiss (2018). The count time series analog of the standard AR model is the integer-valued AR (INAR) model. This thinning-based model appears as a competitive alternative to other approaches, e.g. parameter-driven models like latent Gaussian transformations, hidden Markov and GLARMA models; observation-driven models such as linear and log-linear Poisson autoregression, and copula-based models. The first-order INAR (INAR(1)) model was introduced by McKenzie (1985) and Al-Osh and Alzaid (1987), independently. The pth-order extension (INAR(p)) of this process proposed by Du and Li (1991) has the same correlation structure as an AR model with order p (AR(p)). Until now, relatively few articles have dealt with periodic and seasonal count time series. Monteiro *et al.* (2010) introduced the periodic INAR(1) (PINAR(1)) model and addressed some statistical properties of the parameter estimation together with some finite sample size investigations. Sadoun and Bentarzi (2020) provided efficient estimation methods for the PINAR(1) model. Moriña *et al.* (2011) presented an INAR(2) model with periodic behavior in immigration to analyze the number of hospital emergency service arrivals caused by diseases. A seasonal INAR model of order 1 (SINAR(1)) was introduced in Bourguignon *et al.* (2016) and Li *et al.* (1994) independently. Buteikis and Leipus (2020) generalized the SINAR(1) model in the sense that the seasonal autoregressive parameter may vary by season, and the immigration process may also be intra-seasonally dependent. Liu *et al.* (2020) proposed a generalization of the r-states random environment INAR(1) model to predict a time series of counts with small values and notable fluctuations. Recently, Bentarzi and Aries (2020a, 2020b) introduced the periodic integer-valued ARMA(p, q) model, denoted by PINARMA$_S$(p, q), and established its existence and statistical inference in some particular cases. A general periodic mixed Poisson autoregression was investigated by Aknouche *et al.* (2018).

In Filho *et al.* (2021), a non-negative integer-valued time series model called PINAR($1, 1_S$) model was introduced, see equation (2). To our knowledge, this was the first simple count time series model that simultaneously presents periodic and seasonal serial correlations. The paper also discussed an application for modeling the daily number of people who received antibiotics for respiratory disease treatment from the public health care system in an emergency service. The PINAR($1, 1_S$) model is a generalization of models in Bourguignon *et al.* (2016), Monteiro *et al.* (2010), and Buteikis and Leipus (2020), and it is a special case of the general PINARMA$_S$(p, q) model. Although the PINAR($1, 1_S$) model is only flexible to deal with non-negative temporal covariance, unlike the latent Gaussian transformation or copula-based models, see, e.g. Kong and Lund (2023), it is suitable for a parsimonious description of the complex phenomena of joint periodicity and seasonality. In this article, we prove the existence and uniqueness of the PINAR($1, 1_S$) process under a spectral criterion involving the model parameters, and we present some of its properties, see Theorem 2. Two infinite series representations are also derived: the infinite immigrant generation based expansion (34) which consists of mutually independent components; and the infinite moving average expansion (33) which involves uncorrelated but dependent components. The results of the article show that, up to the second-order moments, the PINAR($1, 1_S$) process cannot be distinguished from a corresponding PAR($1, 1_S$) process, see equation (31) but the dependency structure of the innovation process is much more complex in the integer-valued case. An efficient and parallelizable simulation algorithm is proposed to generate PINAR($1, 1_S$) processes.

The new main tool for examining the PINAR($1, 1_S$) process is its S-dimensional implicit state-space representation with the help of the matricial binomial thinning operator. The standard or explicit state-space representation as a multi-variate INAR (MINAR) model is widely used in modeling integer-valued vector time series. Franke and Subba Rao (1993) introduced a first-order MINAR (MINAR(1)) model for describing INAR(p) processes, which was later generalized by Latour (1997). Some special cases, e.g. when the state vector is two-dimensional or the coefficient matrices are diagonal have been studied in detail by Pedeli and Karlis (2011, 2013), and Darolles *et al.* (2019); see also Santos *et al.* (2021) for an overview. Fokianos (2021) presents a recent survey of multi-variate integer-valued time series models. In the implicit state-space representation proposed in this article, the coefficient matrices are sparse matrices with a special structure, see (6).

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In the following, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote respectively the sets of positive integers, non-negative integers, integers, reals, non-negative reals, and complex numbers. Let $n, m \in \mathbb{N}$. The standard basis in $\mathbb{R}^n$ is denoted by $\{\boldsymbol{e}_i | i = 1, \dots, n\}$. $I_n := \sum_{i=1}^n \boldsymbol{e}_i \boldsymbol{e}_i^\top$ denotes the identity matrix of dimension n. The symbol $O_{n\times m}$ denotes the zero matrix of dimension $n \times m$. All subscripts are omitted when it is clear from the context. For any vector $\boldsymbol{v} = (v_1, \dots, v_n)^\top \in \mathbb{R}^n$, the operator diag($\boldsymbol{v}$) maps $\boldsymbol{v}$ onto the diagonal matrix D with diagonal elements $d_{i,i} = v_i$, $i = 1, \dots, n$, i.e. $D = \sum_{i=1}^n v_i \boldsymbol{e}_i \boldsymbol{e}_i^\top$, $|\boldsymbol{v}|_{\text{vec}} := (|v_1|, \dots, |v_n|)^\top \in \mathbb{R}_+^n$, where $|v|$ denotes the modulus of $v \in \mathbb{C}$, and $\|\boldsymbol{v}\| := (\sum_{j=1}^n v_j^2)^{1/2}$ denotes the Euclidean norm of $\boldsymbol{v}$. If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{u} \leq \boldsymbol{v}$ means that $\boldsymbol{v} - \boldsymbol{u} \in \mathbb{R}_+^n$. For any matrix $M \in \mathbb{R}^{n\times m}$, $M^\top \in \mathbb{R}^{m\times n}$ denotes its transpose, $\|M\|$ denotes the matrix norm induced by the vector norm, the operator vec(M) maps M onto an nm-dimensional vector obtained by stacking the columns of M below each other successively, and the symbol $\otimes$ denotes the usual Kronecker product, $M^{\otimes 2} := M \otimes M$. For details on the vec operator and the Kronecker product see Appendices A.11 and A.12 in Lütkepohl (2005). For a random vector $\boldsymbol{X} = (X_1, \dots, X_n)^\top$, $\mathsf{E}(\boldsymbol{X})$ ($\mathsf{E}(\boldsymbol{X}|\mathcal{F})$) and $\operatorname{Var}(\boldsymbol{X})$ ($\operatorname{Var}(\boldsymbol{X}|\mathcal{F})$) denote its (conditional) mean vector and variance matrix (with respect to (w.r.t.) a σ-algebra $\mathcal{F}$) respectively. If $\boldsymbol{Y}$ is another random vector, $\operatorname{Cov}(\boldsymbol{X}, \boldsymbol{Y}) := \mathsf{E}((\boldsymbol{X} - \mathsf{E}(\boldsymbol{X}))(\boldsymbol{Y} - \mathsf{E}(\boldsymbol{Y}))^\top)$ ($\operatorname{Cov}(\boldsymbol{X}, \boldsymbol{Y}|\mathcal{F}) := \mathsf{E}((\boldsymbol{X} - \mathsf{E}(\boldsymbol{X}|\mathcal{F}))(\boldsymbol{Y} - \mathsf{E}(\boldsymbol{Y}|\mathcal{F}))^\top|\mathcal{F})$) denotes the (conditional) covariance matrix between $\boldsymbol{X}$ and $\boldsymbol{Y}$. The symbol $\stackrel{D}{=}$ denotes the equality in distribution. Finally, δ stands for the Kronecker symbol.

The organization of the article is as follows. Section 2 introduces the PINAR$(1, 1_S)$ model and presents its state-space representations. Section 3 is devoted to some lemmas on non-negative matrices and matricial binomial thinning operators. In Section 4, by introducing the concept of shifted matricial binomial thinning operator, the successive approximation of the unique solution is investigated in detail. Section 5 presents the main results of the article, among others, a Gladyshev-type result in Theorem 1, and the existence of a unique PC solution for the PINAR$(1, 1_S)$ model in Corollary 1. Section 6 discusses the basic stochastic structure of the model, and derives Yule–Walker (YW) equations. In Section 7, a simulation method based on the infinite immigrant generation representation is proposed to produce PINAR$(1, 1_S)$ processes, and the theoretical results are corroborated by simulation. A real data application is presented in Section 8. Finally, in Section 9, the proofs of the main results are collected, while the proofs of auxiliary equations, lemmas, and propositions are left to the Appendix S1.

2. A PERIODIC AND SEASONAL INAR MODEL

In Filho *et al.* (2021), a first-order periodic and seasonal integer-valued autoregressive process $\{Y_t\}$, called PINAR$(1, 1_S)$ process, with seasonal period S, for some $S \in \{2, 3, \dots\}$, is introduced which is defined by the periodic stochastic difference equation

$$Y_{kS+s} = \sum_{j=1}^{Y_{kS+s-1}} \xi_{k,s,j} + \sum_{j=1}^{Y_{kS+s-S}} \eta_{k,s,j} + \varepsilon_{kS+s}, \tag{1}$$

$k \in \mathbb{Z}$, $s = 1, \dots, S$. In (1), the notation Y_{kS+s} denotes the series during the sth season of period k. For example, in the case of monthly data and yearly seasonality, $S = 12$, s is the month of the year, and k is the index of the year. For convenience, the non-periodic notation $\{Y_t\}$ and periodic notation $\{Y_{kS+s}\}$ will be used interchangeably. For the sake of simplicity, we suppose that the random variables (r.v.'s) $\{\xi_{k,s,j}\}$ and $\{\eta_{k,s,j}\}$ are independent and Bernoulli distributed with mean $\mathsf{E}(\xi_{k,s,j}) = \alpha_s$ and $\mathsf{E}(\eta_{k,s,j}) = \beta_s$, respectively, for all $s = 1, \dots, S$ and $k \in \mathbb{Z}$, $j \in \mathbb{N}$. However, the results of the article also hold for more general thinning operators considered in Joe (1996) and Latour (1997), which involve other infinitely divisible counting distributions. The parameters $\alpha_s, \beta_s \in [0, 1]$, $s = 1, \dots, S$, are called the autoregressive coefficients of the model, where α_s's are responsible for the first-order temporal dependency and β_s's for the seasonal dependency. The immigration process $\{\varepsilon_t\}$ is a sequence of $\mathbb{N}_0$-valued r.v.'s, where the random vectors $\boldsymbol{\varepsilon}_k := (\varepsilon_{kS+1}, \dots, \varepsilon_{kS+S})^\top$, $k \in \mathbb{Z}$, are i.i.d. with finite mean vector $\boldsymbol{\lambda}$. The immigration r.v.'s may be correlated in a period. (Note that this feature makes the model (1) more general than the original model in Filho *et al.* (2021), see also Buteikis and Leipus (2020)). Moreover, the sets $\{\xi_{k,s,j}\}$, $\{\eta_{k,s,j}\}$ and $\{\boldsymbol{\varepsilon}_k\}$ are mutually independent. The empty sum is set to 0 in (1). Finally, all r.v.'s are defined on a common probability space $(\Omega, \mathcal{A}, \mathsf{P})$.

Model (1) can be reformulated, similarly to the standard INAR process, see definition 2.1.1.1 in Weiss (2018), by using the notation of the binomial thinning operator in the following way

$$Y_{kS+s} = \alpha_{k,s} \circ Y_{kS+s-1} + \beta_{k,s} \circ Y_{kS+s-S} + \epsilon_{kS+s}. \tag{2}$$

In (2), we recall that if Y is a $\mathbb{N}_0$-valued r.v., $\alpha \in [0,1]$ and $\{\xi_j\}_{j\in\mathbb{N}}$ is a sequence of i.i.d.r.v.'s which are Bernoulli distributed with parameter α, then $\alpha \circ Y := \sum_{j=1}^{Y} \xi_j$ denotes the binomial thinning operator, see Steutel and Van Harn (1979). We assume that the sequence $\{\xi_j\}$ is mutually independent of Y. The sequence $\{\xi_j\}$ is called a counting sequence. Observe that the probability of success in the thinning is $\mathsf{P}(\xi_j = 1) = \alpha$. Conditionally on Y, $\alpha \circ Y \overset{\mathcal{D}}{=} \mathrm{Bin}(Y, \alpha)$ where $\mathrm{Bin}(n, \alpha)$ denotes a binomial distribution with parameters $n \in \mathbb{N}$ and $\alpha \in [0,1]$. Thus, $\mathsf{E}(\alpha \circ Y|Y) = \alpha Y$ and $\mathrm{Var}(\alpha \circ Y|Y) = v(\alpha)Y$, where $v(\cdot)$ denotes the variance function of the Bernoulli distribution defined as $v(\alpha) := \alpha(1-\alpha)$, $\alpha \in [0,1]$. Recall that $0 \circ Y = 0$ and $1 \circ Y = Y$, see lemma 1 in Silva and Oliveira (2004). The binomial thinnings $\alpha_{k,s}\circ$ and $\beta_{k,s}\circ$ are based on the counting sequences $\{\xi_{k,s,j}\}_{j\in\mathbb{N}}$ and $\{\eta_{k,s,j}\}_{j\in\mathbb{N}}$ of (1) with mean α_s and β_s respectively, for all $k \in \mathbb{Z}$ and $s = 1, \dots, S$. For more details on thinning-based count time series models see, e.g. Scotto *et al.* (2015) and Silva and Oliveira (2004) in the univariate and Latour (1997) in the multi-variate case respectively.

Model (1) has three random components since the set of the elements of the process in the sth season consist of three disjoint subsets of new elements given as follows: the survival elements from the previous generation with survival probability α_s, the seasonally new-born elements by the elements of the process from one period before with probability of birth β_s, and the new entering elements into the system with intensity λ_s, which corresponds to the immigration term. Moreover, the autoregressive coefficients α_s, β_s and immigration means λ_s, $s = 1, \dots, S$, change periodically according to the period S. In this context, the PINAR$(1, 1_S)$ model (2) accommodates both seasonality and periodicity in the autoregressive coefficients, that is, it can be considered a kind of seasonally correlated cyclostationary model similar to the periodic SARIMA model for standard linear time series.

Following the same lines as Vecchia (1985) for a PAR model, the PINAR$(1, 1_S)$ model can be algebraically rewritten with the help of the matricial binomial thinning operator, see definition 2.1 in Latour (1997). The matricial binomial thinning operator, called also matricial Steuel and van Harn operator, is denoted by $M\circ$, where the matrix $M\circ = (m_{i,j}\circ)$ of dimension $S \times S$ consists of mutually independent binomial thinning operators, and $M \circ \mathbf{Z}$ is a $\mathbb{N}_0^S$-valued r.v. for any $\mathbb{N}_0^S$-valued r.v. $\mathbf{Z} = (Z_1, \dots, Z_S)^\top$ defined by

$$(M \circ \mathbf{Z})_i := \sum_{j=1}^{S} m_{i,j} \circ Z_j, \qquad i = 1, \dots, S. \tag{3}$$

The matrix $M = (m_{i,j})$, $m_{i,j} \in [0,1]$ for all $1 \le i,j \le S$, is called the mean matrix of $M\circ$. It is also supposed that the involved counting sequences are independent of $\mathbf{Z}$. Let the matrix $V = (v_{i,j})$ denote the variance matrix of $M\circ$ defined by $v_{i,j} := v(m_{i,j})$ for all $i,j = 1, \dots, S$. Let $\mathcal{F} \subset \mathcal{A}$ be a σ-algebra. If $M\circ$ is independent of $\mathcal{F}$ and $\mathbf{Z}$ is $\mathcal{F}$-measurable, then the conditional mean vector and the conditional variance matrix of $M \circ \mathbf{Z}$ w.r.t. $\mathcal{F}$ can be derived as

$$\mathsf{E}(M \circ \mathbf{Z}|\mathcal{F}) = M\mathbf{Z}, \qquad \mathrm{Var}(M \circ \mathbf{Z}|\mathcal{F}) = \mathrm{diag}(V\mathbf{Z}). \tag{4}$$

Formula (4) implies that $\mathsf{E}(M \circ \mathbf{Z}) = M\mathsf{E}(\mathbf{Z})$, provided $\mathbf{Z}$ has finite first moment, and $\mathrm{Var}(M \circ \mathbf{Z}) = M\mathrm{Var}(\mathbf{Z})M^\top + \mathrm{diag}(V\mathsf{E}(\mathbf{Z}))$, provided $\mathbf{Z}$ has a finite second moment, see lemma 2.1 in Latour (1997). The composition $M \circ N \circ$ of two (or more) independent matricial binomial thinning operators $M\circ$ and $N\circ$ is defined as $M \circ N \circ \mathbf{Z} := M \circ (N \circ \mathbf{Z})$. Note that, in general, $M \circ (N \circ \mathbf{Z}) \ne (MN) \circ \mathbf{Z}$ even in distribution. The kth power of a matricial binomial thinning operator $M\circ$ is defined recursively by $(M\circ)^0 := I\circ$ and $(M\circ)^k := M \circ (M\circ)^{k-1}$, $k \in \mathbb{N}$, where the k copies of matricial binomial thinning operator $M\circ$ in the kth power are mutually independent. The definition of the kth power of the composition of independent matricial binomial thinning operator is similar, e.g. $(M \circ N\circ)^2 := M \circ N \circ M \circ N\circ$, where the different copies of $M\circ$ and $N\circ$ are mutually independent.

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Let us define the state vectors of the process $\{Y_t\}$ as $\boldsymbol{Y}_k := (Y_{kS+1}, \ldots, Y_{kS+S})^\top$, $k \in \mathbb{Z}$, and consider the $\mathbb{N}_0^S$-valued stochastic processes $\{\boldsymbol{Y}_k\}$ and $\{\boldsymbol{\varepsilon}_k\}$. Then, by (2), one can see that, for all $k \in \mathbb{Z}$, the following stochastic equation holds

$$\boldsymbol{Y}_k = A_k \circ \boldsymbol{Y}_k + B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k, \tag{5}$$

where $A_k \circ = (a_{i,j}^{(k)} \circ)$ and $B_k \circ = (b_{i,j}^{(k)} \circ)$ are defined as $a_{s+1,s}^{(k)} \circ := \alpha_{k,s+1} \circ$ for all $s = 1, \ldots, S-1$, $0\circ$ otherwise, and $b_{s,s}^{(k)} \circ := \beta_{k,s} \circ$ for all $s = 1, \ldots, S$, $b_{1,S}^{(k)} \circ := \alpha_{k,1} \circ$ and $0\circ$ otherwise. Equation (5) is called the implicit state-space representation of a PINAR$(1, 1_S)$ process (2), where $\{A_k \circ, B_k \circ, \boldsymbol{\varepsilon}_k\}$ form a mutually independent set. The $S \times S$ dimensional mean matrices A and B of the matricial binomial thinning operators $A_k \circ$ and $B_k \circ$, $k \in \mathbb{Z}$ respectively, can be expressed as

$$A = \sum_{s=2}^{S} \alpha_s \boldsymbol{e}_s \boldsymbol{e}_{s-1}^\top, \qquad B = \alpha_1 \boldsymbol{e}_1 \boldsymbol{e}_S^\top + \sum_{s=1}^{S} \beta_s \boldsymbol{e}_s \boldsymbol{e}_s^\top, \tag{6}$$

where A and B are non-negative matrices, i.e. their entries are non-negative numbers. Let V_A and V_B denote the variance matrices of $A_k \circ$ and $B_k \circ$ respectively, which can be expressed as

$$V_A := \sum_{s=2}^{S} v(\alpha_s) \boldsymbol{e}_s \boldsymbol{e}_{s-1}^\top, \qquad V_B := v(\alpha_1) \boldsymbol{e}_1 \boldsymbol{e}_S^\top + \sum_{s=1}^{S} v(\beta_s) \boldsymbol{e}_s \boldsymbol{e}_s^\top. \tag{7}$$

In (6) and (7), α_s and β_s, $s = 1, \ldots, S$, are the autoregressive coefficients of the PINAR$(1, 1_S)$ model.

To get an explicit expression for the state vector $\boldsymbol{Y}_k$, introduce the notation

$$(I - A_k)^{\circ -1} := A_k^{(S)} \circ A_k^{(S-1)} \circ \cdots A_k^{(2)} \circ \tag{8}$$

where $A_k^{(s)} \circ := I \circ + (\alpha_{k,s} \circ) \boldsymbol{e}_s \boldsymbol{e}_{s-1}^\top$ is a matricial binomial thinning operator for all $s \in \{2, \ldots, S\}$ and $k \in \mathbb{Z}$. $A_k^{(s)} \circ$, $k \in \mathbb{Z}$, are identically distributed with mean matrix $A^{(s)} := I + \alpha_s \boldsymbol{e}_s \boldsymbol{e}_{s-1}^\top$ for all $s \in \{2, \ldots, S\}$. The variance matrix of $A_k^{(s)} \circ$ is given by $V_s := v(\alpha_s) \boldsymbol{e}_s \boldsymbol{e}_{s-1}^\top$. Note that $A_k^{(2)} \circ, \ldots, A_k^{(S)} \circ$ and $B_k \circ$, $k \in \mathbb{Z}$, are mutually independent matricial binomial thinning operators. By (i) of Lemma 3, if $\boldsymbol{Y}_{k-1}$ is independent of $A_k \circ, B_k \circ$ and $\boldsymbol{\varepsilon}_k$, then (5) can be rearranged to the stochastic recursion

$$\boldsymbol{Y}_k = (I - A_k)^{\circ -1} (B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k), \tag{9}$$

see the explanation in Remark 1. Equation (9) is called the explicit state-space representation of a PINAR$(1, 1_S)$ process (2).

The state vector $\boldsymbol{Y}_k$ in state-space representations (5) and (9) is in the forward form which is usual in the theory of real-valued periodic processes, see, e.g. Franses and Paap (2004). Equations (5) and (9) can be considered as extensions of the multi-variate INAR model defined in Latour (1997). However, we should emphasize that the state process $\{\boldsymbol{Y}_k\}$ cannot be described by a multi-type branching process model unlike in the case of INAR processes. Finally, it is noted that this state-space representation is well known for some particular models, see, e.g. eq. (2) in Monteiro *et al.* (2010) in the case of PINAR$(1)_S$ model and eq. (7) in Buteikis and Leipus (2020) in the case of SINAR$(1)_S$ model.

3. AUXILIARY LEMMAS

We need the following technical lemmas to derive and prove the basic properties of the PINAR$(1, 1_S)$ process. Since A is a strictly lower triangular matrix we obtain that $I - A$ is non-singular and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k = \sum_{k=0}^{S-1} A^k$.

Thus, $(I-A)^{-1}$ is a non-negative matrix. Moreover, this inverse can be expressed as

$$(I-A)^{-1} = A^{(S)}A^{(S-1)} \dots A^{(2)}. \tag{10}$$

Recall that the spectral radius $\rho(M)$ of a square matrix M is the maximum of its eigenvalues in modulus.

Lemma 1. *Let the matrices A and B be defined in (6). Then, the following statements are equivalent*:

(i) $\rho((I-A)^{-1}B) < 1$;
(ii) *the roots* $z \in \mathbb{C}$ *of the polynomial* $Q(z) := \prod_{j=1}^{S}(z-\beta_j) - z^{S-1}\prod_{j=1}^{S}\alpha_j$ *satisfy* $|z| < 1$;
(iii) *the roots* $z \in \mathbb{C}$ *of the polynomial* $P(z) := \prod_{j=1}^{S}(z-\beta_j) - \prod_{j=1}^{S}\alpha_j$ *satisfy* $|z| < 1$;
(iv) $\rho(A+B) < 1$.

The polynomial P is called the characteristic polynomial of the PINAR$(1, 1_S)$ model. Note that both $(I-A)^{-1}B$ and $A+B$ are non-negative matrices similarly to A and B. Thus, by theorem 8.3.1 in Horn and Johnson (2012), their spectral radius is an eigenvalue of these matrices respectively. We introduce the following assumption, see assumption 1 in Filho *et al.* (2021).

Assumption 1. The matrices A and B satisfy $\rho(A+B) < 1$.

Lemma 1 gives some equivalent characterizations of Assumption 1. However, there are several further equivalent characterizations, a few of them are: (v) the roots $z \in \mathbb{C}$ of the determinant equation $\det(zI-(I-A)^{-1}B) = 0$ satisfy $|z| < 1$; (vi) the roots $z \in \mathbb{C}$ of the matricial autoregressive polynomial $\boldsymbol{P}(z) := I - A - zB$ satisfy $|z| > 1$; (vii) the roots $z \in \mathbb{C}$ of the polynomial $\prod_{j=1}^{S}(1-\beta_j z) - z\prod_{j=1}^{S}\alpha_j$ satisfy $|z| > 1$. The determinant equation in (v) is well-known in the field of real-valued PC processes, see, e.g., (4) in Vecchia (1985), (12) in Ula and Smadi (1997) and (3.26) in Franses and Paap (2004). The matricial polynomial $\boldsymbol{P}$ can be interpreted as the characteristic polynomial to the implicit state-space representation (5). Moreover, $\boldsymbol{P}$ is the matricial autoregressive polynomial of the VAR process $\{\boldsymbol{X}_k\}$ which satisfies (32), see (3.12) in Franses and Paap (2004).

We remark that a similar spectral assumption is known for many time series models, where the stability of a matrix that depends on the model parameters is required for the existence and uniqueness of a stationary solution. Examples of these models, among others, include the VAR(1) model (see example 11.3.1 in Brockwell and Davis, 2013), the general threshold ARMA model (see (11.21) in De Gooijer, 2017), the vector bilinear model (see (11.8) in De Gooijer, 2017), the vector GARCH model (see (iii) in theorem 10.5 of Francq and Zakoian, 2019), the stochastic recurrence model (see (3.7) in De Gooijer, 2017), and some multi-variate Markovian autoregressive processes, see Debaly and Truquet (2021).

The next lemma provides a simple sufficient condition that implies Assumption 1.

Lemma 2. *Let the matrices A and B be defined in (6). If* $\alpha_s + \beta_s < 1$ *for all* $s = 1, \dots, S$, *then* $\rho(A+B) < 1$.

Example 1. Consider the case when $\beta_j = 0$ for all $j = 1, \dots, S$. Then the PINAR$(1, 1_S)$ model is reduced to the PINAR$(1)_S$ model introduced in Monteiro *et al.* (2010). The characteristic polynomial of this model is $P(z) = z^S - \prod_{j=1}^{S}\alpha_j$ and a necessary and sufficient condition for Assumption 1 is $\prod_{j=1}^{S}\alpha_j < 1$, which is equivalent to $\alpha_s < 1$ for some $s \in \{1, \dots, S\}$ since $\alpha_j \le 1$ for all $j = 1, \dots, S$. Note that $\prod_{j=1}^{S}\alpha_j$ is the spectral radius of the matrix A defined on p. 1531 in Monteiro *et al.* (2010), see also (5.1.4) in Bentarzi and Aries (2020b).

Example 2. Consider the case $S = 2$, i.e. the PINAR$(1, 1_2)$ model. Then

$$A = \begin{bmatrix} 0 & 0 \\ \alpha_2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \beta_1 & \alpha_1 \\ 0 & \beta_2 \end{bmatrix}, \qquad A+B = \begin{bmatrix} \beta_1 & \alpha_1 \\ \alpha_2 & \beta_2 \end{bmatrix}.$$

The characteristic polynomial is given by $P(z) = (z-\beta_1)(z-\beta_2) - \alpha_1\alpha_2$. By solving the characteristic equation we have that $\beta_1 + \beta_2 - \beta_1\beta_2 + \alpha_1\alpha_2 < 1$ is a necessary and sufficient condition for Assumption 1. Note that this condition is equivalent to the condition $\alpha_1\alpha_2 < (1-\beta_1)(1-\beta_2)$, see also proposition 3 in Darolles *et al.* (2019) for an other bivariate INAR model. The example $\alpha_1 = 0.4$, $\alpha_2 = 0.3$, $\beta_1 = 0.7$ and $\beta_2 = 0.5$ shows that the assumption in Lemma 2 is not necessary for Assumption 1.

Example 3. Consider a PINAR$(1, 1_S)$ model with a sparse first-order autoregressive part, i.e. suppose that there exists at least one $j \in \{1, \dots, S\}$ such that $\alpha_j = 0$. Then the characteristic polynomial is given as $P(z) = \prod_{j=1}^{S}(z-\beta_j)$ whose roots are β_j, $j = 1, \dots, S$. Thus, a necessary and sufficient condition for Assumption 1 is that $\beta_j < 1$ for all $j = 1, \dots, S$.

The next lemma is fundamental in the stochastic analysis of the implicit state-space representation in this article.

Lemma 3. *Let A be defined in (6), $\boldsymbol{Z}$ be a $\mathbb{N}_0^S$-valued r.v., and $\mathcal{F} \subset \mathcal{A}$ be a σ-algebra. Assume that $\boldsymbol{Z}$ and $\mathcal{F}$ are mutually independent of the counting sequences involved into the matricial binomial thinning operator $A\circ$. Then, the following statements hold for the implicit stochastic equation*

$$\boldsymbol{Y} = A \circ \boldsymbol{Y} + \boldsymbol{Z}. \tag{11}$$

(i) The stochastic equation (11) has a unique solution $\boldsymbol{Y}$ whose coordinates satisfy the recursion $Y_1 = Z_1$, $Y_s = \alpha_s \circ Y_{s-1} + Z_s$, $s = 2, \dots, S$. Moreover, the unique solution can be expressed explicitly as

$$\boldsymbol{Y} = A^{(S)} \circ A^{(S-1)} \circ \dots \circ A^{(2)} \circ \boldsymbol{Z}, \tag{12}$$

i.e. $\boldsymbol{Y} = (I-A)^{\circ -1}\boldsymbol{Z}$, where $A^{(s)}\circ$ is defined as $A_k^{(s)}\circ$, $s = 2, \dots, S$, in (8).

(ii) If $\boldsymbol{Z}$ has finite first moment, then $\boldsymbol{Y}$ has finite mean, and

$$\mathsf{E}(\boldsymbol{Y}|\mathcal{F}) = (I-A)^{-1}\mathsf{E}(\boldsymbol{Z}|\mathcal{F}). \tag{13}$$

Particularly, $\mathsf{E}(\boldsymbol{Y}|\boldsymbol{Z}) = (I-A)^{-1}\boldsymbol{Z}$ and $\mathsf{E}(\boldsymbol{Y}) = (I-A)^{-1}\mathsf{E}(\boldsymbol{Z})$.

(iii) If $\boldsymbol{Z}$ has finite second moment, then $\boldsymbol{Y}$ has finite variance matrix and

$$\mathrm{Var}(\boldsymbol{Y}|\mathcal{F}) = (I-A)^{-1}\left(\mathrm{diag}\left(V_A\mathsf{E}(\boldsymbol{Y}|\mathcal{F})\right) + \mathrm{Var}(\boldsymbol{Z}|\mathcal{F})\right)\left((I-A)^{-1}\right)^{\top}. \tag{14}$$

Particularly, $\mathrm{Var}(\boldsymbol{Y}|\boldsymbol{Z}) = (I-A)^{-1}(\mathrm{diag}(V_A\mathsf{E}(\boldsymbol{Y}|\boldsymbol{Z})))\left((I-A)^{-1}\right)^{\top}$ and $\mathrm{Var}(\boldsymbol{Y}) = (I-A)^{-1}(\mathrm{diag}(V_A\mathsf{E}(\boldsymbol{Y})) + \mathrm{Var}(\boldsymbol{Z}))\left((I-A)^{-1}\right)^{\top}$. (Note that V_A is defined in (7).)

Remark 1. Formula (12) can be interpreted as the analog of the formula $\boldsymbol{y} = (I-A)^{-1}\boldsymbol{z} = A^{(S)}A^{(S-1)} \dots A^{(2)}\boldsymbol{z}$ which is the unique solution to the linear vector equation $(I-A)\boldsymbol{y} = \boldsymbol{z}$, see (10). Namely, rewrite (11) formally as $(I-A)\circ\boldsymbol{Y} = \boldsymbol{Z}$. Then Lemma 3 states that a unique solution to this formal linear stochastic equation is given by $\boldsymbol{Y} = (I-A)^{\circ -1}\boldsymbol{Z}$. Moreover, (13) and (14) describe the conditional mean vector and variance matrix of the $\mathbb{N}_0^S$-valued r.v. $(I-A)^{\circ -1}\boldsymbol{Z}$ respectively. We also remark that the analog of (10) for matricial binomial thinning operators is not true, i.e. $(I-A)^{-1}\circ \neq A^{(S)} \circ A^{(S-1)} \circ \dots A^{(2)} \circ = (I-A)^{\circ -1}$ even in distribution. Finally, the conditional expectation $\mathsf{E}(\boldsymbol{Y}|\boldsymbol{Z})$ is the unique solution to the linear stochastic equation in $\boldsymbol{Y}$ given by $(I-A)\boldsymbol{Y} = \boldsymbol{Z}$.

Remark 2. One can see by (13) that the conditional mean $\mathsf{E}(\boldsymbol{Y}|\mathcal{F})$ satisfies the linear equation

$$(I-A)\mathsf{E}(\boldsymbol{Y}|\mathcal{F}) = \mathsf{E}(\boldsymbol{Z}|\mathcal{F}).$$

Since $\mathsf{E}(\mathrm{Var}(\boldsymbol{Y}|\boldsymbol{Z})|\mathcal{F}) = (I-A)^{-1}\mathrm{diag}(V_A\mathsf{E}(\boldsymbol{Y}|\mathcal{F}))((I-A)^{-1})^\top$ we have by (14) that $\mathrm{Var}(\boldsymbol{Y}|\mathcal{F}) - \mathsf{E}(\mathrm{Var}(\boldsymbol{Y}|\boldsymbol{Z})|\mathcal{F})$ is a symmetric positive semi-definite matrix and satisfies the equation

$$(I-A)\,(\mathrm{Var}(\boldsymbol{Y}|\mathcal{F}) - \mathsf{E}(\mathrm{Var}(\boldsymbol{Y}|\boldsymbol{Z})|\mathcal{F}))\,(I-A)^\top = \mathrm{Var}(\boldsymbol{Z}|\mathcal{F})$$

Finally, $\mathrm{Cov}(\boldsymbol{Y},\boldsymbol{Z}|\mathcal{F}) = (I-A)^{-1}\mathrm{Var}(\boldsymbol{Z}|\mathcal{F})$.

To prove the uniqueness of the state process $\{\boldsymbol{Y}_k\}$ we need the following lemma.

Lemma 4. *Let $\boldsymbol{Z}$ and $\boldsymbol{Z}'$ be $\mathbb{N}_0^S$-valued r.v.'s with finite means and let the matricial binomial thinning operator $M\circ$ of dimension $S\times S$ be independent of $\boldsymbol{Z}$ and $\boldsymbol{Z}'$. Then $\mathsf{E}|M\circ\boldsymbol{Z} - M\circ\boldsymbol{Z}'|_{\mathrm{vec}} \le M\mathsf{E}|\boldsymbol{Z}-\boldsymbol{Z}'|_{\mathrm{vec}}$.*

4. SUCCESSIVE APPROXIMATION OF PINAR$(1, 1_S)$ PROCESSES

Here, we provide the technical details of the construction of an approximating sequence to a unique stationary solution $\{\boldsymbol{Y}_k\}$ of state-space equations (5) and (9). We apply the method of successive approximation which is based on recursion (9) following the argument in Latour (1997).

First, we need the concept of a shifted binomial thinning operator. If $\alpha\circ$ is a binomial thinning operator with counting sequence $\{\xi_j\}$ and Y is a $\mathbb{N}_0$-valued r.v. which is mutually independent of the counting sequence, then let the shifted binomial thinning operator $\alpha\circ_{|Y}$ be defined as $\alpha\circ_{|Y}Z := \sum_{j=Y+1}^{Y+Z}\xi_j$ for all $\mathbb{N}_0$-valued r.v. Z which is mutually independent of $\alpha\circ$. Note that Y and Z can be dependent. Clearly, $\alpha\circ(Y+Z) = \alpha\circ Y + \alpha\circ_{|Y}Z$ almost surely and the r.v.'s $\alpha\circ Y$ and $\alpha\circ_{|Y}Z$ are conditionally independent w.r.t. Y. Moreover, $\alpha\circ_{|Y}Z \stackrel{\mathcal{D}}{=} \alpha\circ Z$ and thus $\alpha\circ(Y+Z) \stackrel{\mathcal{D}}{=} \alpha\circ Y + \alpha\circ Z$, see the third property of the binomial thinning operator in the introduction of Scotto *et al.* (2015). This concept can easily be extended to matricial binomial thinning operator by (3). Namely, if $M\circ$ is a matricial binomial thinning operator of dimension $S\times S$ and $\boldsymbol{Y}$ is a $\mathbb{N}_0^S$-valued r.v. which is mutually independent of $M\circ$, then the shifted matricial binomial thinning operator $M\circ_{|\boldsymbol{Y}}$ is defined as $(M\circ_{|\boldsymbol{Y}}\boldsymbol{Z})_i = \sum_{j=1}^S m_{i,j}\circ_{|Y_j}Z_j$ for all $\mathbb{N}_0^S$-valued r.v. $\boldsymbol{Z}$ which is mutually independent of $M\circ$. Clearly, $M\circ(\boldsymbol{Y}+\boldsymbol{Z}) = M\circ\boldsymbol{Y} + M\circ_{|\boldsymbol{Y}}\boldsymbol{Z}$ almost surely and the random vectors $M\circ\boldsymbol{Y}$ and $M\circ_{|\boldsymbol{Y}}\boldsymbol{Z}$ are conditionally independent w.r.t. $\boldsymbol{Y}$ which implies $\mathrm{Cov}(M\circ\boldsymbol{Y}, M\circ_{|\boldsymbol{Y}}\boldsymbol{Z}|\boldsymbol{Y}) = 0$. More generally, if $\mathcal{F}\subset\mathcal{A}$ is a σ-algebra which is mutually independent of $M\circ$ and $\boldsymbol{Y}_1, \boldsymbol{Y}_2$ are $\mathcal{F}$-measurable $\mathbb{N}_0^S$-valued r.v.'s such that $\boldsymbol{Y}_1 \le \boldsymbol{Y}_2$ almost surely, then $M\circ\boldsymbol{Y}_1$ and $M\circ_{|\boldsymbol{Y}_2}\boldsymbol{Z}$ are conditionally independent w.r.t. $\mathcal{F}$ for all $\boldsymbol{Z}$ which is mutually independent of $M\circ$ since the counting r.v.'s involved into $M\circ\boldsymbol{Y}_1$ and $M\circ_{|\boldsymbol{Y}_2}\boldsymbol{Z}$ are mutually independent. Particularly, in this case, $\mathrm{Cov}(M\circ\boldsymbol{Y}_1, M\circ_{|\boldsymbol{Y}_2}\boldsymbol{Z}|\mathcal{F}) = 0$. We have that $M_{|\boldsymbol{Y}}\circ\boldsymbol{Z} \stackrel{\mathcal{D}}{=} M\circ\boldsymbol{Z}$ and $M\circ(\boldsymbol{Y}+\boldsymbol{Z}) \stackrel{\mathcal{D}}{=} M\circ\boldsymbol{Y} + M\circ\boldsymbol{Z}$. The shifting concept can be extended to the composition $M\circ N\circ$ of matricial binomial thinning operators $M\circ$ and $N\circ$ as $(M\circ N\circ)_{|\boldsymbol{Y}} := M\circ_{|N\circ\boldsymbol{Y}}N\circ_{|\boldsymbol{Y}}$ for all $\mathbb{N}_0^S$-valued r.v. $\boldsymbol{Y}$ which is mutually independent of $M\circ$ and $N\circ$. For all $\mathbb{N}_0^S$-valued r.v. $\boldsymbol{Y}$ which is mutually independent of $A\circ$, the shifted composite matricial binomial thinning operator $(I-A)^{\circ-1}_{|\boldsymbol{Y}}$ is defined iteratively in the following way. For all $\mathbb{N}_0^S$-valued r.v. $\boldsymbol{Z}$ which is mutually independent of $A\circ$ define the $\mathbb{N}_0^S$-valued r.v.'s $\boldsymbol{Y}_s, \boldsymbol{Z}_s$, $s = 1, \ldots, S$, as $\boldsymbol{Y}_1 := \boldsymbol{Y}$, $\boldsymbol{Z}_1 := \boldsymbol{Z}$ and $\boldsymbol{Y}_s := A^{(s)}\circ\boldsymbol{Y}_{s-1}$, $\boldsymbol{Z}_s := A^{(s)}\circ_{|\boldsymbol{Y}_{s-1}}\boldsymbol{Z}_{s-1}$, $s = 2, \ldots, S$. Then, $(I-A)^{\circ-1}_{|\boldsymbol{Y}}\boldsymbol{Z} = (A^{(S)}\circ \ldots A^{(2)}\circ)_{|\boldsymbol{Y}}\boldsymbol{Z} := \boldsymbol{Z}_S$.

Define the sequences of $\mathbb{N}_0^S$-valued stochastic processes $\{\boldsymbol{Y}_k^{(n)}\}_k$, $n\in\mathbb{N}_0$, and $\{\boldsymbol{Z}_k^{(n)}\}_k$, $n\in\mathbb{N}$, recursively as $\boldsymbol{Y}_k^{(0)} := 0$, $\boldsymbol{Z}_k^{(1)} := (I-A_k)^{\circ-1}\boldsymbol{\varepsilon}_k$ for all $k\in\mathbb{Z}$, and, for all $n\in\mathbb{N}$ and $k\in\mathbb{Z}$,

$$\boldsymbol{Y}_k^{(n)} := \boldsymbol{Y}_k^{(n-1)} + \boldsymbol{Z}_k^{(n)}, \tag{15}$$

$$\boldsymbol{Z}_k^{(n+1)} := (I-A_k)^{\circ-1}_{|B_k\circ\boldsymbol{Y}_{k-1}^{(n-1)}+\boldsymbol{\varepsilon}_k}\left(B_k\circ_{|\boldsymbol{Y}_{k-1}^{(n-1)}}\boldsymbol{Z}_{k-1}^{(n)}\right). \tag{16}$$

Clearly, $\boldsymbol{Y}_k^{(n)} = \sum_{j=1}^n \boldsymbol{Z}_k^{(j)}$ for all $n\in\mathbb{N}$, and thus the sequence $\{\boldsymbol{Y}_k^{(n)}\}_n$ is monotone non-decreasing for all $k\in\mathbb{Z}$. Through the branching process representation (1) of the PINAR$(1, 1_S)$ process and (16), one can see that $(\boldsymbol{Z}_k^{(n)})_s$ is the number of nth generation offspring of $\boldsymbol{\varepsilon}_{k+1-n}$, i.e. the immigrants in the time interval $[(k+1-n)S+1, (k+1-n)S+S]$, at time $kS+s$. Thus, $(\boldsymbol{Y}_k^{(n)})_s$ is the number of offspring of the immigrants in the time interval $[(k+1-n)S+1, kS+S]$ at time $kS+s$. Moreover, $\{\boldsymbol{Y}_k^{(n)}\}$ satisfies (9) in the following sense

$$\boldsymbol{Y}_k^{(n)} = (I - A_k)^{\circ -1} \left(B_k \circ \boldsymbol{Y}_{k-1}^{(n-1)} + \boldsymbol{\varepsilon}_k \right), \tag{17}$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. This explicit recursion can be written in the implicit form, see (5), as

$$\boldsymbol{Y}_k^{(n)} = A_k \circ \boldsymbol{Y}_k^{(n)} + B_k \circ \boldsymbol{Y}_{k-1}^{(n-1)} + \boldsymbol{\varepsilon}_k, \tag{18}$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Finally, by representation (1) of the PINAR$(1, 1_S)$ process, the auxiliary r.v.'s $\{Y_t^{(n)}\}$ defined as the coordinates of $\boldsymbol{Y}_k^{(n)} = (Y_{kS+1}^{(n)}, \dots, Y_{kS+S}^{(n)})^\top$ satisfies the recursion

$$Y_{kS+s}^{(n)} = \sum_{j=1}^{Y_{kS+s-1}^{(n-\delta_{1s})}} \xi_{k,s,j} + \sum_{j=1}^{Y_{kS+s-S}^{(n-1)}} \eta_{k,s,j} + \varepsilon_{kS+s} \tag{19}$$

for all $k \in \mathbb{Z}$, $s = 1, \dots, S$, and $n \in \mathbb{N}$. We will show in Theorem (2) that, for all $k \in \mathbb{Z}$, the sequence $\{\boldsymbol{Y}_k^{(n)}\}_n$ has an almost sure limit $\boldsymbol{Y}_k$ which is a solution to stochastic equations (5) and (9).

The $\mathbb{N}_0^S$-valued r.v.'s $\{\boldsymbol{Y}_k^{(n)}\}$ and $\{\boldsymbol{Z}_k^{(n)}\}$ defined by (15) and (16) respectively, satisfy the following properties.

First, we note that there exists a probability space $(\Omega, \mathcal{A}, \mathsf{P})$ with mutually independent r.v.'s $\{\xi_{k,s,j}\}$, $\{\eta_{k,s,j}\}$, and $\{\boldsymbol{\varepsilon}_k\}$ such that $\boldsymbol{Y}_k^{(n)}$ and $\boldsymbol{Z}_k^{(n+1)}$ are $\mathcal{G}_k$-measurable for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, where $\mathcal{G}_k \subset \mathcal{A}$ denotes the σ-algebra generated by the r.v.'s $\{\xi_{l,s,j}, \eta_{l,s,j}, \varepsilon_{lS+s} | j \in \mathbb{N}, s = 1, \dots, S, l \le k\}$ for all $k \in \mathbb{Z}$. For all $n \in \mathbb{N}_0$, the process $\{\boldsymbol{Y}_k^{(n)}\}_k$ is strictly stationary, non-anticipative, and ergodic. For all $k \in \mathbb{Z}$, $\{\boldsymbol{Z}_k^{(n)}\}_n$ is a sequence of independent $\mathbb{N}_0^S$-valued r.v.'s.

The first moments of $\boldsymbol{Y}_k^{(n)}$ and $\boldsymbol{Z}_k^{(n+1)}$ exist for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, and the mean vectors of $\boldsymbol{Y}_k^{(n)}$ and $\boldsymbol{Z}_k^{(n+1)}$ do not depend on k. The sequence $\{\boldsymbol{\mu}^{(n)}\}$, where $\boldsymbol{\mu}^{(n)} := \mathsf{E}(\boldsymbol{Y}_k^{(n)})$ for all $k \in \mathbb{Z}$, is monotone non-decreasing and satisfies the recursion

$$\boldsymbol{\mu}^{(n)} = (I - A)^{-1} \left(B\boldsymbol{\mu}^{(n-1)} + \lambda \right), \tag{20}$$

$n \in \mathbb{N}$, with starting value $\boldsymbol{\mu}^{(0)} := 0$. Under Assumption 1, the sequence $\{\boldsymbol{\mu}^{(n)}\}$ is bounded and $\boldsymbol{\mu}^{(n)} \to \boldsymbol{\mu}$ as $n \to \infty$ where $\boldsymbol{\mu}$ is defined in (24). Furthermore, the sequence $\{\boldsymbol{\nu}^{(n)}\}$, where $\boldsymbol{\nu}^{(n)} := \mathsf{E}(\boldsymbol{Z}_k^{(n)})$ for all $k \in \mathbb{Z}$, satisfies the recursion $\boldsymbol{\nu}^{(n+1)} = (I - A)^{-1} B\boldsymbol{\nu}^{(n)}$, $n \in \mathbb{N}$, with initial condition $\boldsymbol{\nu}^{(1)} = (I - A)^{-1}\lambda$.

The second moments of $\boldsymbol{Y}_k^{(n)}$ and $\boldsymbol{Z}_k^{(n+1)}$ exist for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, and the variance matrices of $\boldsymbol{Y}_k^{(n)}$ and $\boldsymbol{Z}_k^{(n+1)}$ do not depend on k. The sequence $\{\Sigma^{(n)}\}$ of symmetric positive semi-definite matrices, where $\Sigma^{(n)} := \mathrm{Var}(\boldsymbol{Y}_k^{(n)})$ for all $k \in \mathbb{Z}$, is monotone non-decreasing, i.e. $\Sigma^{(n)} - \Sigma^{(n-1)}$ is a symmetric positive semi-definite matrix for all $n \in \mathbb{N}$, and satisfies the recursion

$$\Sigma^{(n)} = (I - A)^{-1} \left(B\Sigma^{(n-1)}B^\top + \mathrm{diag}\left(V_A \boldsymbol{\mu}^{(n)} + V_B \boldsymbol{\mu}^{(n-1)} \right) + \Sigma_\varepsilon \right) \left((I - A)^{-1} \right)^\top, \tag{21}$$

$n \in \mathbb{N}$, with starting value $\Sigma^{(0)} := O_{S\times S}$. Under Assumption 1, the sequence $\{\Sigma^{(n)}\}$ is bounded and $\Sigma^{(n)} \to \Sigma$ as $n \to \infty$ where Σ is defined in (25). Furthermore, the sequence $\{\Phi^{(n)}\}$ of symmetric positive semi-definite matrices, where $\Phi^{(n)} := \mathrm{Var}(\boldsymbol{Z}_k^{(n)})$ for all $k \in \mathbb{Z}$, satisfies the recursion $\Phi^{(n+1)} = (I - A)^{-1}(B\Phi^{(n)}B^\top + \mathrm{diag}(V_A \boldsymbol{\nu}^{(n+1)} + V_B \boldsymbol{\nu}^{(n)}))((I - A)^{-1})^\top$, $n \in \mathbb{N}$, with initial condition $\Phi^{(1)} = (I - A)^{-1}(\mathrm{diag}(V_A \boldsymbol{\nu}^{(1)}) + \Sigma_\varepsilon)((I - A)^{-1})^\top$.

The autocovariance matrix function $\Gamma^{(n)}$ of $\{\boldsymbol{Y}_k^{(n)}\}_k$ is given by $\Gamma^{(n)}(h) = ((I - A)^{-1}B)^h \Sigma^{(n-h)}$, $\Gamma^{(n)}(-h) = \Sigma^{(n-h)}(((I - A)^{-1}B)^\top)^h$ if $0 \le h \le n$, and $\Gamma^{(n)}(h) = O_{S\times S}$ if $|h| > n$ for all $n \in \mathbb{N}_0$.

Remark 3. One can see that the recursion (20) can be written as $\boldsymbol{\mu}^{(n)} = (I-A)^{-1}B\boldsymbol{\mu}^{(n-1)} + \boldsymbol{\mu}^{(1)}$, $n \in \mathbb{N}$. Moreover, $\boldsymbol{\nu}^{(1)} = \boldsymbol{\mu}^{(1)}$, $\boldsymbol{\mu}^{(n)} = \sum_{j=1}^{n}\boldsymbol{\nu}^{(j)}$, and $\boldsymbol{\mu} = \sum_{j=1}^{\infty}\boldsymbol{\nu}^{(j)}$. Similarly, the recursion (21) can be written as $\Sigma^{(n)} = (I-A)^{-1}B\Sigma^{(n-1)}B^{\top}((I-A)^{-1})^{\top} + \Lambda^{(n)}$, where $\Lambda^{(n)} := (I-A)^{-1}(\mathrm{diag}(V_A\boldsymbol{\mu}^{(n)} + V_B\boldsymbol{\mu}^{(n-1)}) + \Sigma_{\boldsymbol{\varepsilon}})((I-A)^{-1})^{\top}$, $n \in \mathbb{N}$. Moreover, $\Phi^{(n)} = (I-A)^{-1}B\Phi^{(n-1)}B^{\top}((I-A)^{-1})^{\top} + \Psi^{(n)}$, where $\Psi^{(n)} := (I-A)^{-1}(\mathrm{diag}(V_A\boldsymbol{\nu}^{(n)} + V_B\boldsymbol{\nu}^{(n-1)}) + \delta_{1n}\Sigma_{\boldsymbol{\varepsilon}})((I-A)^{-1})^{\top}$, $n \in \mathbb{N}$, with initial conditions $\Phi^{(0)} := O_{S\times S}$ and $\boldsymbol{\nu}^{(0)} := 0$. One can easily check that $\Lambda^{(n)} = \sum_{j=1}^{n}\Psi^{(j)}$ which implies that $\Sigma^{(n)} = \sum_{j=1}^{n}\Phi^{(j)}$ and $\Sigma = \sum_{j=1}^{\infty}\Phi^{(j)}$.

5. MAIN RESULTS

A stochastic process $\{\boldsymbol{Y}_k\}$ is said to be non-anticipative with respect to the state-space representations (5) or (9) if the sets of r.v.'s $\{\boldsymbol{Y}_l | l < k\}$ and $\{\xi_{l,s,j}, \eta_{l,s,j}, \varepsilon_{lS+s} | j \in \mathbb{N}, s = 1, \dots, S, l \geq k\}$ are mutually independent for all $k \in \mathbb{Z}$. In the sequel, it is assumed that the state process $\{\boldsymbol{Y}_k\}$ is non-anticipative. Suppose that the immigration vector has a finite second moment and let $\Sigma_{\boldsymbol{\varepsilon}}$ denote its variance matrix. We will see that the immigration vector $\boldsymbol{\lambda}$, the coefficient matrices A, B, V_A, V_B, and the variance matrix $\Sigma_{\boldsymbol{\varepsilon}}$ determine the first- and second-order structures of the state process $\{\boldsymbol{Y}_k\}$ and thus those of the PC process $\{Y_t\}$. These intermediate parameters depend on the original parameters $\alpha_s, \beta_s, \lambda_s$, $s = 1, \dots, S$, and $\Sigma_{\boldsymbol{\varepsilon}}$ of the PINAR$(1, 1_S)$ model.

In Proposition 1, linear equations are derived for the mean vector and the variance matrix of a stationary solution to the state-space representation (5) (or (9)). Recall that $M^{\otimes 2} := M \otimes M$ for any matrix M.

Proposition 1. *Assume that a $\mathbb{N}_0^S$-valued non-anticipative stochastic process $\{\boldsymbol{Y}_k\}$ is a solution to (5) with stationary mean vector $\boldsymbol{\mu}$ and variance matrix Σ respectively. Then, $\boldsymbol{\mu}$ and Σ satisfy the following equations*:

$$\boldsymbol{\mu} = (I-A)^{-1}(B\boldsymbol{\mu} + \boldsymbol{\lambda}), \tag{22}$$

$$\Sigma = (I-A)^{-1}\left(B\Sigma B^{\top} + \Sigma_{\boldsymbol{M}}\right)\left((I-A)^{-1}\right)^{\top}, \tag{23}$$

where $\Sigma_{\boldsymbol{M}} := \mathrm{diag}((V_A + V_B)\boldsymbol{\mu}) + \Sigma_{\boldsymbol{\varepsilon}}$. Under Assumption 1, these equations have unique solutions given by

$$\boldsymbol{\mu} = \left(I - (I-A)^{-1}B\right)^{-1}(I-A)^{-1}\boldsymbol{\lambda} = (I-A-B)^{-1}\boldsymbol{\lambda}, \tag{24}$$

$$\begin{aligned}\mathrm{vec}(\Sigma) &= \left(I_{S^2} - \left((I_S-A)^{-1}B\right)^{\otimes 2}\right)^{-1}\left((I_S-A)^{\otimes 2}\right)^{-1}\mathrm{vec}\Sigma_{\boldsymbol{M}} \\ &= \left((I_S-A)^{\otimes 2} - B^{\otimes 2}\right)^{-1}\mathrm{vec}\Sigma_{\boldsymbol{M}}.\end{aligned} \tag{25}$$

Equations (22) and (23) can also be derived by using Lemma 3 from the following stationary equation

$$\boldsymbol{Y} \stackrel{\mathcal{D}}{=} (I-A)^{\circ -1}(B\circ\boldsymbol{Y} + \boldsymbol{\varepsilon}),$$

where the $\mathbb{N}_0^S$-valued r.v. $\boldsymbol{Y}$ is mutually independent of $A\circ$, $B\circ$ and $\boldsymbol{\varepsilon}$.

By lemma 5.6.10 and corollary 5.6.16 in Horn and Johnson (2012), $\rho(A+B) < 1$ implies that $I - A - B$ is a non-singular matrix and $(I-A-B)^{-1} = \sum_{j=0}^{\infty}(A+B)^j$. Thus, under Assumption 1 the stationary mean vector $\boldsymbol{\mu}$ in Proposition 1 can be expressed by an infinite series as

$$\boldsymbol{\mu} = \sum_{j=0}^{\infty}(A+B)^j\boldsymbol{\lambda} = \sum_{j=0}^{\infty}\left((I-A)^{-1}B\right)^j(I-A)^{-1}\boldsymbol{\lambda}. \tag{26}$$

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A similar formula holds for the stationary variance matrix Σ since (25) can be written as

$$\mathrm{vec}\,\Sigma = \sum_{j=0}^{\infty}\left(\left((I-A)^{-1}B\right)^{\otimes 2}\right)^{j}\left((I-A)^{-1}\right)^{\otimes 2}\mathrm{vec}\Sigma_{\boldsymbol{M}}, \tag{27}$$

and (27) is equivalent to

$$\Sigma = \sum_{j=0}^{\infty}\left((I-A)^{-1}B\right)^{j}(I-A)^{-1}\Sigma_{\boldsymbol{M}}\left((I-A)^{-1}\right)^{\top}\left(\left((I-A)^{-1}B\right)^{j}\right)^{\top}. \tag{28}$$

Formulae (24) and (25) (or (26) and (28)) provide explicit expressions for the mean vector and variance matrix of a stationary solution to the explicit (or implicit) state-space representation of a PINAR$(1, 1_S)$ process respectively, as functions of the parameters $\alpha_s, \beta_s, \lambda_s$, $s = 1, \dots, S$, and Σ_{ϵ} of model (1) through the matrices A, B, V_A, V_B and Σ_{ϵ}. However, for numerical calculation, solving linear equations (22) and (23) is more efficient. On the numerical solution of Lyapunov-type equation (23) see, e.g. Kitagawa (1977).

We draw attention to the structural similarity of recursions (9) for $\{\boldsymbol{Y}_k\}$, (32) for $\{\boldsymbol{X}_k\}$, (37) for $\{\boldsymbol{\mu}_k\}$ and (20) for $\{\boldsymbol{\mu}^{(n)}\}$, and Lyapunov-type recursions (40) for $\{\Sigma_k\}$ and (21) for $\{\Sigma^{(n)}\}$ respectively.

Example 4. Let $\alpha_s = 0$ for all $s = 2, \dots, S$. Then $A = O_{S\times S}$ which implies that the implicit and explicit state-space representations coincide and are given by

$$\boldsymbol{Y}_k = B\circ\boldsymbol{Y}_{k-1} + \boldsymbol{\epsilon}_k,$$

$k \in \mathbb{Z}$, which is a multi-variate INAR process, see Latour (1997). By Example 3, a necessary and sufficient condition for the existence of a stationary solution is that $\beta_j < 1$ for all $j = 1, \dots, S$. Moreover, in this case, $V_A = O_{S\times S}$ and thus $\Sigma_{\boldsymbol{M}} = \mathrm{diag}(V_B\boldsymbol{\mu}) + \Sigma_{\epsilon}$, see proposition 4.1 in Latour (1997).

By (28), since $(I-A)^{-1}$ and B are non-negative matrices, if $\Sigma_{\epsilon} \geq 0$ then $\Sigma \geq 0$, i.e. the components of the state-vector $\boldsymbol{Y}_k$ are non-negatively correlated. One can easily see, by (26), that $\lambda > 0$ implies $\boldsymbol{\mu} > 0$. Similarly, by (28), if additionally $\alpha_s, \beta_s \in (0,1)$ for all $s = 1, \dots, S$ then $\Sigma_{\epsilon} > 0$ implies $\Sigma > 0$, i.e. the components of the state-vector $\boldsymbol{Y}_k$ are positively correlated. This property limits the applicability of the PINAR$(1, 1_S)$ model somewhat compared to other models, e.g. latent Gaussian transformation or copula-based models, see Kong and Lund (2023), but in practice many time series show non-negative temporal correlation, see the datasets in this article and Filho *et al.* (2021).

If Assumption 1 holds and $\mathsf{E}(\epsilon_t^k)$ is finite for all $t \in \mathbb{Z}$ where $2 < k \in \mathbb{N}$, then the kth-order moments of the PINAR$(1, 1_S)$ process is also finite. This statement can be proved similarly to the cases $k = 1, 2$.

In Theorem 1, necessary and sufficient conditions are given for a PINAR$(1, 1_S)$ process to be PC.

Theorem 1. *Let* $\{Y_t\}$ *be a* PINAR$(1, 1_S)$ *process with S-dimensional state process* $\{\boldsymbol{Y}_k\}$. *Then, the following statements are equivalent*:

(i) $\{Y_t\}$ *is a PC process with period S*;
(ii) The $\mathbb{N}_0^S$*-valued r.v.* $(Y_1, \dots, Y_S)$ *has mean vector* $\boldsymbol{\mu}$ *and variance matrix* Σ *which satisfy (22) and (23) respectively*;
(iii) $\{\boldsymbol{Y}_k\}$ *is a weakly stationary process with mean vector* $\boldsymbol{\mu}$, *variance matrix* Σ *which satisfy (22), (23) respectively, and covariance matrix function* Γ *defined by (42).*

The PINAR$(1, 1_S)$ process has a PAR representation similarly to the AR representation of the INAR models, see section 4 in Latour (1998). Following the right-hand side of (2), let us define the linear predictor P_{kS+s} based on the observations Y_{kS+s-1} and Y_{kS+s-S} as

$$P_{kS+s} := \alpha_s Y_{kS+s-1} + \beta_s Y_{kS+s-S} + \lambda_s,$$

$k \in \mathbb{Z}$ and $s = 1, \ldots, S$, and introduce the innovation sequence $\{M_t\}$ as $M_t := Y_t - P_t$. Let $\mathcal{F}_t \subset \mathcal{A}$ denote the σ-algebra generated by the r.v.'s $\{Y_s | s \le t\}$ for all $t \in \mathbb{Z}$. Clearly, $\mathcal{F}_{kS}$ coincides with the σ-algebra generated by the $\mathbb{N}_0^S$-valued r.v.'s $\{\boldsymbol{Y}_\ell | \ell < k\}$. Remark that if the coordinates of the immigration vector $\boldsymbol{\varepsilon}_k$ are mutually independent, then $P_t = \mathsf{E}(Y_t|\mathcal{F}_{t-1})$, i.e., $\{M_t\}$ is a martingale difference w.r.t. to the filtration $\{\mathcal{F}_t\}$. However, in general, if the coordinates of the immigration vector $\boldsymbol{\varepsilon}_k$ are dependent, then $\mathsf{E}(\varepsilon_{kS+s}|\mathcal{F}_{kS+s-1}) \neq \lambda_s$ provided $s > 1$, which implies that $\{\mathcal{M}_t\}$ is not a martingale difference anymore w.r.t. $\{\mathcal{F}_t\}$. Clearly, by (2), we have

$$M_{kS+s} = (\alpha_{k,s} \circ Y_{kS+s-1} - \alpha_s Y_{kS+s-1}) + (\beta_{k,s} \circ Y_{kS+s-S} - \beta_s Y_{kS+s-S}) + (\varepsilon_{kS+s} - \lambda_s). \tag{29}$$

Thus, $\{M_t\}$ is a centered process. Let $\sigma_s^2 := \mathrm{Var}(\varepsilon_s)$, $s = 1, \ldots, S$, denote the diagonal elements of $\Sigma_{\boldsymbol{\varepsilon}}$. Since the counting r.v.'s involved into the model and the immigration are mutually independent at the same time, one can see that $\{M_t\}$ has the periodic variance

$$\mathsf{E}(M_{kS+s}^2) = v(\alpha_s)\mu(s-1) + v(\beta_s)\mu(s) + \sigma_s^2, \tag{30}$$

for all $s = 1, \ldots, S$ and $k \in \mathbb{Z}$, where μ denotes the mean function of $\{Y_t\}$. Let us introduce the centered random variables $X_t := Y_t - \mu(t)$, $t \in \mathbb{Z}$. By (2), (39) and (29), the process $\{X_t\}$ satisfies a periodic autoregressive (PAR) model, in fact, a subset PAR($\{1, S\}$) model, defined by

$$X_{kS+s} = \alpha_s X_{kS+s-1} + \beta_s X_{kS+s-S} + M_{kS+s}, \tag{31}$$

$k \in \mathbb{N}$ and $s = 1, \ldots, S$. In (31), α_s, β_s, $s = 1, \ldots, S$, are the autoregressive coefficients, and $\{M_t\}$ is a periodic innovation process with zero mean and periodic variance (30).

The state-space representation of the PAR process $\{X_t\}$ is derived in the following way. Define the $\mathbb{R}^S$-valued r.v.'s $\boldsymbol{X}_k := (X_{kS+1}, \ldots, X_{kS+S})^\top$ and $\boldsymbol{M}_k := (M_{kS+1}, \ldots, M_{kS+S})^\top$, $k \in \mathbb{Z}$, and consider the $\mathbb{R}^S$-valued stochastic processes $\{\boldsymbol{X}_k\}$ and $\{\boldsymbol{M}_k\}$. Then, by (31), we obtain the S-dimensional vector autoregressive (VAR) representation of $\{X_t\}$ as

$$\boldsymbol{X}_k = (I - A)^{-1}(B\boldsymbol{X}_{k-1} + \boldsymbol{M}_k), \tag{32}$$

$k \in \mathbb{N}$, where the matrices A and B are defined in (6). One can see that $\boldsymbol{X}_k = \boldsymbol{Y}_k - \mathsf{E}(\boldsymbol{Y}_k)$ and $\boldsymbol{M}_k = (A_k \circ \boldsymbol{Y}_k - A\boldsymbol{Y}_k) + (B_k \circ \boldsymbol{Y}_{k-1} - B\boldsymbol{Y}_{k-1}) + (\boldsymbol{\varepsilon}_k - \lambda)$ for all $k \in \mathbb{N}$. Thus, the autoregressive representation of the implicit multi-variate INAR model (5) coincides with the vector autoregressive model (32), see also proposition 4.1 in Latour (1997). If $\{\boldsymbol{Y}_k\}$ is a non-anticipative weakly stationary stochastic process, then $\{\boldsymbol{M}_k\}$ is an S-dimensional weak white noise with zero mean vector and variance matrix $\Sigma_{\boldsymbol{M}}$ which explains the notation $\Sigma_{\boldsymbol{M}}$ introduced in Proposition 1.

Recursion (32) implies that the $\mathbb{N}_0^S$-valued state process $\{\boldsymbol{Y}_k\}$ has an infinite moving average representation.

Proposition 2. *Under Assumption 1, the centered process $\{\boldsymbol{X}_k\}$ of $\{\boldsymbol{Y}_k\}$ can be expressed as the almost surely and in mean square convergent infinite series*

$$\boldsymbol{X}_k = \boldsymbol{Y}_k - \boldsymbol{\mu} = \sum_{j=0}^{\infty} \left((I - A)^{-1}B\right)^j \boldsymbol{W}_{k-j}, \tag{33}$$

$k \in \mathbb{Z}$, where the $\mathbb{R}^S$-valued r.v.'s $\{\boldsymbol{W}_k\}$ are defined as $\boldsymbol{W}_k := (I - A)^{-1}\boldsymbol{M}_k$ for all $k \in \mathbb{Z}$.

One can see, by (5) and (35), that $\boldsymbol{W}_k = \boldsymbol{Y}_k - \mathsf{E}(\boldsymbol{Y}_k|\mathcal{F}_{kS})$, thus $\{\boldsymbol{W}_k\}$ is a martingale difference vector sequence w.r.t. the filtration $\{\mathcal{F}_{kS}\}_k$. However, neither $\{\boldsymbol{M}_k\}$ nor $\{\boldsymbol{W}_k\}$ are a strong white noise, i.e. their elements are not independent since they depend on the PC process $\{Y_t\}$, see (29). Moreover, the $\mathbb{R}^S$-valued state process $\{\boldsymbol{X}_k\}$ in the VAR representation (32) satisfies a similar equation to (9) where the usual matrix product replaces the matricial binomial thinnings.

The main theorem on the existence and uniqueness of the S-dimensional stationary state-space representation $\{\boldsymbol{Y}_k\}$ of a PINAR$(1, 1_S)$ process is the following.

Theorem 2. *Under Assumption 1, there exists an almost surely unique non-anticipative and weakly stationary $\mathbb{N}_0^S$-valued process $\{\boldsymbol{Y}_k\}$ which satisfies (5) (or (9)). The process $\{\boldsymbol{Y}_k\}$ can be expressed as the almost sure convergent infinite series*

$$\boldsymbol{Y}_k = \sum_{j=1}^{\infty} \boldsymbol{Z}_k^{(j)}, \tag{34}$$

$k \in \mathbb{Z}$, where $\boldsymbol{Z}_k^{(n)}$ denotes the number of nth generation offspring of immigrants at time k defined by (16). $\{\boldsymbol{Y}_k\}$ is strictly stationary and ergodic, its mean vector $\boldsymbol{\mu}$, and variance matrix Σ satisfy (24) and (25) respectively, and its autocovariance matrix function is given by $\Gamma(h) = ((I - A)^{-1}B)^h \Sigma$ for all $h \in \mathbb{N}_0$. Moreover, $\boldsymbol{Y}_k^{(n)} \to \boldsymbol{Y}_k$ almost surely and in mean square as $n \to \infty$ for all $k \in \mathbb{Z}$, where $\boldsymbol{Y}_k^{(n)}$ denotes the number of offspring of immigrants up to generation n at time k defined in (15).

The formula (34) can be interpreted as the infinite immigrant generation representation of the state process of a PINAR$(1, 1_S)$ process. In this representation, $\{\boldsymbol{Z}_k^{(j)} | j \in \mathbb{N}\}$ are independent random vectors for all $k \in \mathbb{Z}$, see Section 4, and $\boldsymbol{Z}_k^{(j)}$ depends only on the immigration vector process at time $k + 1 - j$. Note that the components in the infinite moving average representation (33), see definition 3.1.3 in Brockwell and Davis (2013), are not independent but uncorrelated. Representation (34) also implies that, under Assumption 1, the stationary distribution of the state-space representations (5) and (9) can be expressed as the following formal infinite series

$$\begin{aligned} Y &\overset{\mathcal{D}}{=} \sum_{j=0}^{\infty} \left((I - A)^{\circ -1} B \circ\right)^j (I - A)^{\circ -1} \boldsymbol{\varepsilon}_{-j} \\ &= \sum_{j=0}^{\infty} \left(A^{(S)} \circ A^{(S-1)} \circ \ \dots \ A^{(2)} \circ B \circ\right)^j A^{(S)} \circ A^{(S-1)} \circ \ \dots \ A^{(2)} \circ \boldsymbol{\varepsilon}_{-j}, \end{aligned}$$

where the different copies of the matricial binomial thinning operators $A^{(2)}\circ, \dots, A^{(S)}\circ$ and $B\circ$ are mutually independent. This infinite series expansion can be seen as the infinite immigrant generation representation of the stationary distribution, see, e.g. (2.2) in Al-Osh and Alzaid (1987), proposition 1 in Bourguignon *et al.* (2016), and the formula on the bottom of p. 215 in Pedeli and Karlis (2013).

Remark 4. The connection between the two infinite series representations (33) and (34) can be discussed as follows. Define, for all $k \in \mathbb{Z}$, the conditionally centered $\mathbb{N}_0^S$-valued r.v.'s $\boldsymbol{V}_k^{(n+1)} := \boldsymbol{Z}_k^{(n+1)} - \mathsf{E}(\boldsymbol{Z}_k^{(n+1)} | \boldsymbol{Z}_{k-1}^{(n)})$, $n \in \mathbb{N}$, and $\boldsymbol{V}_k^{(1)} := \boldsymbol{Z}_k^{(1)} - \mathsf{E}(\boldsymbol{Z}_k^{(1)})$. In the notation of $\boldsymbol{V}_k^{(n)}$, the superscript (n) and subscript k refer to generation and time respectively. $\boldsymbol{V}_k^{(n+1)}$ is the conditional fluctuation of the number of offspring of $\boldsymbol{\varepsilon}_{k-n}$ w.r.t. the size of previous nth generation. Clearly, by (16) and Lemma 3, we have $\mathsf{E}(\boldsymbol{Z}_k^{(n+1)} | \boldsymbol{Z}_{k-1}^{(n)}) = (I - A)^{-1} B \boldsymbol{Z}_{k-1}^{(n)}$. Then, one can easily check that, by (34), $\boldsymbol{W}_k = \sum_{j=1}^{\infty} \boldsymbol{V}_k^{(j)}$ for all $k \in \mathbb{Z}$, and $\boldsymbol{Z}_k^{(n)} - \mathsf{E}(\boldsymbol{Z}_k^{(n)}) = \sum_{j=0}^{n-1} ((I - A)^{-1} B)^j \boldsymbol{V}_{k-j}^{(n-j)}$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thus, the innovation vector $\boldsymbol{W}_k$ in the VAR representation of a PINAR$(1, 1_S)$ process is the sum of conditional fluctuations of offspring over generations at a time k, while the centered number of offsprings $\boldsymbol{Z}_k^{(n)} - \mathsf{E}(\boldsymbol{Z}_k^{(n)})$ at generation n and time k is a finite moving average of the same conditional fluctuation of previous n generations at consecutive time points $k, \dots, k - n + 1$.

By the proof of Theorem 2, one can see that $\boldsymbol{Y}_k$ is measurable w.r.t. $\mathcal{G}_k$. However, the process $\{\boldsymbol{Y}_k\}$ is not causal (or physically realizable) in the sense that $\boldsymbol{Y}_k$ is not measurable w.r.t. $\mathcal{F}_{kS}$, i.e., the σ-algebra generated only by the immigration process up to time kS.

The following corollary follows directly from Theorem 2.

Corollary 1. *Under Assumption 1, there exists an almost surely unique non-anticipative and $\mathbb{N}_0$-valued PC process $\{Y_t\}$ with a period S which satisfies the* PINAR$(1, 1_S)$ *model. The process $\{Y_t\}$ is a periodically strictly stationary process also with a period S.*

6. SECOND-ORDER PROPERTIES

Here, the first- and second-order structures of the state process $\{\boldsymbol{Y}_k\}$ are described. First, the conditional mean vector and variance matrix of $\boldsymbol{Y}_k$ are derived. We apply Lemma 3 by choosing $\boldsymbol{Y} = \boldsymbol{Y}_k$, $\boldsymbol{Z} = B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k$ and $\mathcal{F} = \mathcal{F}_{kS}$. Then, $\boldsymbol{Y}_{k-1}$ is $\mathcal{F}_{kS}$-measurable and $B_k\circ$ and $\boldsymbol{\varepsilon}_k$ are independent of $\mathcal{F}_{kS}$. Thus, by (4), we have $\mathsf{E}(B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k | \mathcal{F}_{kS}) = B\boldsymbol{Y}_{k-1} + \lambda$ and $\mathrm{Var}(B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k | \mathcal{F}_{kS}) = \mathrm{diag}(V_B \boldsymbol{Y}_{k-1}) + \Sigma_\varepsilon$. Using (9) and Lemma 3, we have

$$\mathsf{E}(\boldsymbol{Y}_k | \mathcal{F}_{kS}) = (I - A)^{-1}(B\boldsymbol{Y}_{k-1} + \lambda), \tag{35}$$

$$\mathrm{Var}(\boldsymbol{Y}_k | \mathcal{F}_{kS}) = (I - A)^{-1} \left(\mathrm{diag}\left(V_A (I - A)^{-1}(B\boldsymbol{Y}_{k-1} + \lambda) + V_B \boldsymbol{Y}_{k-1}\right) + \Sigma_\varepsilon\right) \left((I - A)^{-1}\right)^\top, \tag{36}$$

where V_A and V_B are defined in (7). Thus, the conditional mean vector and variance matrix of $\boldsymbol{Y}_k$ depend on the past only through $\boldsymbol{Y}_{k-1}$. Note that the right-hand sides of (35) and (36) are affine functions of $\boldsymbol{Y}_{k-1}$. In fact, by (9), one can see that the state process $\{\boldsymbol{Y}_k\}$ is a Markov chain on the state-space $\mathbb{N}_0^S$ and thus the PINAR$(1, 1_S)$ process $\{Y_t\}$ is an inhomogeneous S-step Markov chain on the state-space of non-negative integers.

Let us introduce the S-dimensional vectors $\boldsymbol{\mu}_k := (\mu(kS + 1), \ldots, \mu(kS + S))^\top$, $k \in \mathbb{Z}$. Then $\boldsymbol{\mu}_k = \mathsf{E}(\boldsymbol{Y}_k)$ and, by (9) and Lemma 3 or taking the expectation of (35), we have the explicit linear recursion

$$\boldsymbol{\mu}_k = (I - A)^{-1}(B\boldsymbol{\mu}_{k-1} + \lambda), \tag{37}$$

for all $k \in \mathbb{N}$. The recursion (37) can be rearranged in the implicit form

$$\boldsymbol{\mu}_k = A\boldsymbol{\mu}_k + B\boldsymbol{\mu}_{k-1} + \lambda, \tag{38}$$

for all $k \in \mathbb{N}$ which is the expectation of (5). One can see that both recursions (37) and (38) are non-negative equations. Formula (38) implies that the mean function μ satisfies the periodic linear recursion

$$\mu(kS + s) = \alpha_s \mu(kS + s - 1) + \beta_s \mu(kS + s - S) + \lambda_s, \tag{39}$$

for all $k \in \mathbb{Z}$ and $s = 1, \ldots, S$. Remark that this recursion can also be derived from (2) by taking expectation.

Let us define the $S \times S$-dimensional autocovariance matrix function $\Gamma : \mathbb{Z}^2 \to \mathbb{R}^{S\times S}$ as $\Gamma(k, \ell) := \mathrm{Cov}(\boldsymbol{Y}_k, \boldsymbol{Y}_\ell)$ for all $k, \ell \in \mathbb{Z}$. Clearly, $(\Gamma(k, \ell))_{i,j} = \mathrm{Cov}(Y_{kS+i}, Y_{\ell S+j}) = R(kS + i, \ell S + j)$ for all $k, \ell \in \mathbb{Z}$ and $i, j = 1, \ldots, S$. Since $\Gamma(k, \ell) = \Gamma(\ell, k)^\top$ for all $k, \ell \in \mathbb{Z}$ it is enough to consider the case $k \geq \ell$. Let $\Sigma_k := \mathrm{Var}(\boldsymbol{Y}_k) = \Gamma(k, k)$ for all $k \in \mathbb{Z}$. Since $\boldsymbol{Y}_{k-1}$, $B_k\circ$ and $\boldsymbol{\varepsilon}_k$ are mutually independent, by (4), we obtain that $\mathsf{E}(B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k) = B\boldsymbol{\mu}_{k-1} + \boldsymbol{\lambda}$ and $\mathrm{Var}(B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k) = B\Sigma_{k-1}B^\top + \mathrm{diag}(V_B \boldsymbol{\mu}_{k-1}) + \Sigma_\varepsilon$. Thus, by (9), (37) and (iii) of Lemma 3, we have the recursion for the variance matrices Σ_k as

$$\Sigma_k = (I - A)^{-1} \left(B\Sigma_{k-1}B^\top + \mathrm{diag}(V_A \boldsymbol{\mu}_k + V_B \boldsymbol{\mu}_{k-1}) + \Sigma_\varepsilon\right) \left((I - A)^{-1}\right)^\top, \tag{40}$$

$k \in \mathbb{Z}$. Equation (40) is a Lyapunov-type analog of (37). By rearranging (40) we have

$$(I - A)\Sigma_k (I - A)^\top = B\Sigma_{k-1}B^\top + \mathrm{diag}(V_A \boldsymbol{\mu}_k + V_B \boldsymbol{\mu}_{k-1}) + \Sigma_\varepsilon.$$

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By using the vec operator, the recursion (40) can be written as the following linear recursion for the vectorized variance matrices $\mathrm{vec}\Sigma_k$, $k \in \mathbb{Z}$,

$$\mathrm{vec}\,\Sigma_k = \left((I-A)^{-1}\right)^{\otimes 2}\left(B^{\otimes 2}\,\mathrm{vec}\,\Sigma_{k-1} + \mathrm{vec}\left(\mathrm{diag}(V_A\boldsymbol{\mu}_k + V_B\boldsymbol{\mu}_{k-1}) + \Sigma_\epsilon\right)\right). \tag{41}$$

The autocovariance matrix function Γ of a non-anticipative state process $\{\boldsymbol{Y}_k\}$ is derived in the following way. By the law of total covariance and (35), we have the recursion $\Gamma(k,\ell) = (I-A)^{-1}B\Gamma(k-1,\ell)$ for all $k > \ell$. This recursion implies that Γ can be expressed as

$$\Gamma(k+h,k) = \left((I-A)^{-1}B\right)^h\Sigma_k, \tag{42}$$

for all $h \in \mathbb{N}_0$ and $k \in \mathbb{Z}$. If $\Sigma_k = \Sigma$ for all $k \in \mathbb{Z}$, then $\Gamma(k,\ell)$ depends only on $k-\ell$. Hence, in this case, we may introduce the function $\Gamma : \mathbb{Z} \to \mathbb{R}^{S\times S}$ as $\Gamma(h) := \left((I-A)^{-1}B\right)^h\Sigma$ and $\Gamma(-h) = \Gamma(h)^\top$ for all $h \in \mathbb{N}_0$. The relationship between the autocovariance function R of $\{Y_t\}$ and the autocovariance matrix function Γ of $\{\boldsymbol{Y}_k\}$ can be expressed as

$$(\Gamma(h))_{i,j} = R(hS+i,j), \tag{43}$$

for all $h \in \mathbb{Z}$ and $i,j = 1,\dots,S$.

Define the scalar-valued functions $\gamma_j : \mathbb{Z} \to \mathbb{R}$, $j \in \mathbb{Z}$, as $\gamma_j(h) := R(j+h,j)$, $h \in \mathbb{Z}$. One can see that if R is a periodic function with period S, then $\gamma_{kS+s} = \gamma_s$ for all $k \in \mathbb{Z}$ and $s = 1,\dots,S$. The functions γ_s, $s = 1,\dots,S$, determine the autocovariance function R. Namely, if $s = kS+i$ and $t = \ell S + j$ where $k,\ell \in \mathbb{Z}$ and $i,j \in \{1,\dots,S\}$, then $R(s,t) = R(kS+i,\ell S+j) = \gamma_j((k-\ell)S+i-j)$. Thus, if a PINAR$(1,1_S)$ process $\{Y_t\}$ is a PC process of period S, then the functions γ_s, $s = 1,\dots,S$, determine its covariance kernel R, and the covariance matrix function Γ of its weakly stationary state process $\{\boldsymbol{Y}_k\}$. The functions γ_s, $s = 1,\dots,S$, are called the periodic autocovariance functions (perACF) of the PC process $\{Y_t\}$. The following symmetry property

$$\gamma_j(hS+s) = \gamma_{j+s}(-hS-s), \tag{44}$$

holds for all $h,j \in \mathbb{Z}$ and $s = 1,\dots,S$. Particularly, in the case of $h = 0$, we have $\gamma_j(i) = \gamma_{i+j}(-i)$ for all $i,j \in \{1,\dots,S\}$. According to (43), the covariance matrix function Γ of the state process $\{\boldsymbol{Y}_k\}$ and the perACF's γ_j, $j = 1,\dots,S$, of the PINAR$(1,1_S)$ process $\{Y_t\}$ are related by $(\Gamma(h))_{i,j} = \gamma_j(hS+i-j)$ for all $h \in \mathbb{Z}$ and $i,j \in \{1,\dots,S\}$. By the symmetry property (44), the covariance matrix Σ can also be expressed as $\Sigma = \Gamma(0) = (\gamma_{i\wedge j}(|i-j|))_{i,j=1}^S$ where $i \wedge j := \min\{i,j\}$.

By (42), we have the YW equations for the weakly stationary process $\{\boldsymbol{Y}_k\}$ as $(I-A)\Gamma(h) = B\Gamma(h-1)$ and $\Gamma(-h)(I-A)^\top = \Gamma(-h+1)B^\top$ for all $h \in \mathbb{N}$. Since this equation can be written in the form $\Gamma(h) = A\Gamma(h)+B\Gamma(h-1)$, $h \in \mathbb{N}$, where A and B are defined by (6), we obtain the following YW equations for the perACF of the PC process $\{Y_t\}$

$$\gamma_j(hS+i-j) = \alpha_i\gamma_j(hS+i-j-1) + \beta_i\gamma_j(hS+i-j-S), \tag{45}$$

which can be rewritten by using the periodic autocovariance function R as

$$R(hS+i,j) = \alpha_i R(hS+i-1,j) + \beta_i R(hS+i-S,j),$$

for all $h \in \mathbb{N}$ and $i,j = 1,\dots,S$. This autocovariance function is identical to that of a PAR process, namely the process $\{X_t\}$ in (31), see also theorem 2 in Pagano (1978). When Σ_ϵ is diagonal, then the YW equations can be written in the form

$$\begin{bmatrix} \gamma_{s-1}(0) & \gamma_s(S-1) \\ \gamma_s(S-1) & \gamma_s(0) \end{bmatrix} = \begin{bmatrix} \alpha_s \\ \beta_s \end{bmatrix} = \begin{bmatrix} \gamma_{s-1}(1) \\ \gamma_s(S) \end{bmatrix}, \tag{46}$$

for all $s = 1, \dots, S$. Thus, the parameters α_s, β_s, $s = 1, \dots, S$, are determined by the perACF's γ_s, $s = 1, \dots, S$, at lags $0, 1, S-1, S$ in this particular case.

By (22) and (23) respectively, the mean vector $\boldsymbol{\lambda}$ and the variance matrix $\Sigma_{\boldsymbol{\varepsilon}}$ of the immigration vector can be expressed in the following form

$$\begin{aligned} \boldsymbol{\lambda} &= (I - A - B)\boldsymbol{\mu} \\ \Sigma_{\boldsymbol{\varepsilon}} &= (I - A)\Sigma(I - A)^\top - B\Sigma B^\top - \mathrm{diag}((V_A + V_B)\boldsymbol{\mu}), \end{aligned} \tag{47}$$

as functions of the parameters α_s, β_s, $s = 1, \dots, S$, the mean vector $\boldsymbol{\mu}$ and the variance matrix Σ of the stationary solution.

Finally, the spectral distribution matrix of the weakly stationary state process $\{\boldsymbol{Y}_k\}$ is given by

$$\begin{aligned} f_{\boldsymbol{Y}}(\lambda) &= \frac{1}{2\pi}\boldsymbol{F}^{-1}(\mathrm{e}^{-i\lambda})(I - A)^{-1}\Sigma_{\boldsymbol{M}}\left((I - A)^{-1}\right)^\top \left(\boldsymbol{F}^{-1}(\mathrm{e}^{i\lambda})\right)^\top \\ &= \frac{1}{2\pi}\boldsymbol{G}^{-1}(\mathrm{e}^{-i\lambda})\Sigma_{\boldsymbol{M}}\left(\boldsymbol{G}^{-1}(\mathrm{e}^{i\lambda})\right)^\top, \end{aligned}$$

$\lambda \in (-\pi, \pi]$, where $\boldsymbol{F}(z) := I - (I - A)^{-1}Bz$ and $\boldsymbol{G}(z) := I - A - Bz$, $z \in \mathbb{C}$, see section 11.8 in Brockwell and Davis (2013) and proposition 4.2 in Latour (1997). Note that $\det \boldsymbol{F}(z) \neq 0$ and $\det \boldsymbol{G}(z) \neq 0$ for all $|z| \leq 1$ under Assumption 1.

7. SIMULATION OF PINAR(1,1S) PROCESSES

Two simulation methods are investigated here for generating a PINAR$(1, 1_S)$ process with sample size $T = nS$. We take $S = 4$, the immigration process $\{\varepsilon_t\}$ is independent Poisson distributed, and the parameter set $\boldsymbol{\vartheta}$ is the same as in Filho *et al.* (2021), i.e. $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.1, 0.42, 0.23, 0.39)$, $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.47, 0.25, 0.36, 0.3)$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4, 3, 2, 1)$. The spectral radius of matrices in Lemma 1 are $\rho((I - A)^{-1}B) = 0.5239$ and $\rho(A + B) = 0.6079$, thus Assumption 1 is satisfied for these parameters, hence, by Corollary 1, a unique PC process exists that satisfies the PINAR$(1, 1_S)$ model. The parameters are estimated by the YW equations where the perACF's are replaced by their natural estimates obtained from the sample estimate of the covariance matrix Γ of the weakly stationary state process $\{\boldsymbol{Y}_k\}$, see p. 407 in Brockwell and Davis (2013). Of course, more efficient parameter estimation methods exist for a PINAR$(1, 1_S)$ model, e.g. the conditional quasi-maximum likelihood method in Filho *et al.* (2021). Here, we use the simple YW estimation method only to compare the two proposed simulation methods based on these estimates. The results of YW estimation for sample size $n = 250, 1000, 4000$ are displayed in Table II. The empirical bias and root mean square error (RMSE) correspond to 1000 replications. All simulations were carried out using the Numpy library of Python language, the program codes are available on request.

The first simulation method (Sim1) is the Markovian simulation which is based on equation (9) and the Markov property of the state process $\{\boldsymbol{Y}_k\}$. In this simulation, at each step, we generate an immigration vector $\boldsymbol{\varepsilon}$ and binomial distributed r.v.'s belonging to matricial binomial thinning operators $A\circ$ and $B\circ$ which are mutually independent. To avoid the problem of initial distribution, we start from the integer part of the theoretical mean vector $\boldsymbol{\mu}$, and, before generating n observations from the process, 200 pre-iterations are performed by recursion (9) to ensure that the simulation starts near the stationary distribution.

The second simulation method (Sim2) is based on the infinite immigrant generation representation (34) of the PINAR process. In this case, by recursion (16), we generate mutually i.i.d. stochastic processes $\{\boldsymbol{Z}^{(j)}_{k+j-1} | j \in \mathbb{N}\}$, $k = 1, \dots, n$. Note that it is not necessary to shift the matricial binomial thinning operators during the simulation, only mutually independent copies of $A\circ$ and $B\circ$ are needed. Each process is generated until its extinction, which occurs exponentially fast, see problem I.5 in Athreya and Ney (1972). Then, by Theorem 2, the diagonal sum $\boldsymbol{Y}^{(k)}_k := \sum_{j=1}^{k} \boldsymbol{Z}^{(j)}_k$ is a good approximation of the state vector $\boldsymbol{Y}_k$ when k is large enough. Here, the upper limit k of the summation can be replaced by the maximum M_n of the extinction times of the processes $\{\boldsymbol{Z}^{(j)}_{k+j-1} | j \in \mathbb{N}\}$, $k = 1, \dots, n$, which is approximately $O(\ln n)$. It is also worthwhile to omit the first M_n elements of the series $\{\boldsymbol{Y}^{(k)}_k | k = 1, \dots, n\}$ to improve the accuracy of the approximation.

A precise probabilistic study of the simulation Sim2 is as follows. Let $Q := \min_j\{j : \boldsymbol{Z}_j = 0\}$ denote the extinction time of a process $\{\boldsymbol{Z}_j\}$ which is a copy of i.i.d. processes $\{\boldsymbol{Z}^{(j)}_{k+j-1}\}$, $k = 1, \dots, n$. Then, by Markov's inequality, there exists $C > 0$ such that, for all $j \in \mathbb{N}$,

$$\mathsf{P}(Q > j) = \mathsf{P}(\boldsymbol{Z}_j \neq 0) = \mathsf{P}(\|\boldsymbol{Z}_j\|_1 \geq 1/2) \leq 2\|\boldsymbol{v}^{(j)}\|_1 \leq C\|((I-A)^{-1}B)^j\|_1,$$

where $\|\boldsymbol{v}\|_1 := \sum_i |v_i|$ for a vector $\boldsymbol{v} = (v_i)$ and $\|M\|_1 := \sum_{i,j} |m_{i,j}|$ for a matrix $M = (m_{i,j})$. Let $\rho := \rho((I-A)^{-1}B) < 1$ and $0 < \epsilon < 1 - \rho$. We deduce from corollary 5.6.13 in Horn and Johnson (2012) that there exists $C > 0$ such that, for all $k \in \mathbb{N}$,

$$\mathsf{P}(Q > j) \leq C(\rho + \epsilon)^j.$$

Thus, the tail probabilities of Q decrease exponentially, see a simulation study in Table I. Let $\{c_n\}$ be a sequence of non-negative numbers such that $n^{-1}c_n \to \infty$ as $n \to \infty$ and define the non-negative sequence $\{a_n\}$ with $a_n := -(\ln c_n + \ln C)/\ln(\rho+\epsilon)$, $n \in \mathbb{N}$. Then, for the maximum M_n of the extinction times of the processes $\{\boldsymbol{Z}^{(j)}_{k+j-1} | j \in \mathbb{N}\}$, $k = 1, \dots, n$, we have

$$\begin{aligned}\mathsf{P}(M_n \leq a_n) &= \mathsf{P}(\boldsymbol{Z}^{(j)}_{a_n} = 0,\ j = 1, \dots, n) = (\mathsf{P}(\boldsymbol{Z}_{a_n} = 0))^n \\ &\geq (1 - C(\rho+\epsilon)^{a_n})^n = \left(1 - c_n^{-1}\right)^n \to 1,\end{aligned}$$

as $n \to \infty$. By choosing $c_n := n^{\varkappa}$ where $\varkappa > 1$, we have that $M_n = O_{\mathsf{P}}(\ln n)$.

Thus, the algorithmic complexity of the simulation algorithms Sim1 and Sim2 is $O(n)$ and $O(n \ln n)$ respectively, where n is the length of the simulation as the number of periods. The running times of the simulations confirm these asymptotics. However, while the first algorithm (Sim1) is not parallelizable, the second algorithm (Sim2) has a parallelized version with n threads, since the processes $\{\boldsymbol{Z}^{(j)}_{k+j-1} | j \in \mathbb{N}\}$, $k = 1, \dots, n$, are mutually independent. The algorithmic complexity of the parallelized version of Sim2 is $O(\ln n)$ which allows a significantly faster execution.

Finally, we compare the statistical efficiency of the two simulation algorithms. In Table II, the bias and the RMSE of YW estimator of the parameters of a PINAR$(1, 1_4)$ model with Poisson immigration are presented. As was expected, in general, the performance of the YW estimator presents estimates quite accurate even for a small sample size. By increasing n, the bias and RMSE of the estimates decrease and show $n^{1/2}$ asymptotic (the ratio of

Table I. Empirical tail probabilities of extinction time Q of $\{\boldsymbol{Z}_j\}$ with sample size $n = 10{,}000$

j	1	2	3	4	5	6	7	8	9
$P(Q > j)$	0.9895	0.8834	0.6643	0.4282	0.2551	0.1419	0.0768	0.0393	0.0213
j	10	11	12	13	14	15	16	17	18
$P(Q > j)$	0.0109	0.005	0.0025	0.0013	0.0008	0.0006	0.0003	0.0002	0.0001

Table II. Results of the simulation to estimate the parameter by YW method

	$n = 250$		$n = 1000$		$n = 4000$	
Parameter	Sim1	Sim2	Sim1	Sim2	Sim1	Sim2
α_1	−0.00664	0.00045	0.00122	−0.00254	0.00009	0.00031
	(0.0758)	(0.0756)	(0.0355)	(0.0356)	(0.0184)	(0.0178)
α_2	0.00508	−0.00168	−0.00279	0.00053	0.00042	0.00017
	(0.0603)	(0.0608)	(0.0286)	(0.0298)	(0.0145)	(0.0147)
α_3	0.00148	0.00207	0.00021	0.00212	0.00015	−0.00014
	(0.0498)	(0.0499)	(0.0233)	(0.0243)	(0.0117)	(0.0118)
α_4	0.00028	0.00206	0.00043	0.001	−0.00022	−0.00008
	(0.0483)	(0.0527)	(0.025)	(0.0244)	(0.0123)	(0.0125)
β_1	−0.01672	−0.01151	−0.00231	−0.00431	−0.00136	−0.00058
	(0.0621)	(0.0575)	(0.0281)	(0.0295)	(0.0147)	(0.0141)
β_2	−0.01121	−0.00948	0.00048	−0.003	0.00005	−0.00017
	(0.0586)	(0.0574)	(0.0281)	(0.0293)	(0.0143)	(0.0137)
β_3	−0.01031	−0.01178	−0.0031	−0.00218	−0.00044	−0.00023
	(0.0593)	(0.059)	(0.0293)	(0.0285)	(0.0142)	(0.0144)
β_4	−0.00908	−0.01162	−0.00179	−0.00326	−0.00083	−0.00034
	(0.0552)	(0.0585)	(0.0273)	(0.0286)	(0.014)	(0.0138)
λ_1	0.17246	0.09378	0.01641	0.05198	0.011	0.00317
	(0.626)	(0.6013)	(0.2805)	(0.295)	(0.149)	(0.1467)
λ_2	0.04446	0.08857	0.01376	0.02345	−0.00616	0.0014
	(0.6374)	(0.6571)	(0.3166)	(0.3192)	(0.1533)	(0.1537)
λ_3	0.05084	0.05645	0.01673	−0.00503	0.00061	0.00235
	(0.5327)	(0.5292)	(0.2596)	(0.2616)	(0.1248)	(0.1257)
λ_4	0.03626	0.0417	−0.00197	0.00942	0.00538	0.00028
	(0.3548)	(0.3816)	(0.1809)	(0.1831)	(0.0915)	(0.0928)

Note: The table contains the bias (RMSE) of each estimate.

the RMSE values of the estimates of the same parameters are around 2 between two consecutive sample sizes). Furthermore, the accuracy of the second simulation algorithm (Sim2) is at least as good as the first one (Sim1). Comparing Tables II and III, we see that the estimates of immigration parameters λ_s's are more precise by taking the diagonal of the estimated immigration variance matrix than the estimated immigration means, see (47). In figures of Appendix S1, we compare the distributions of the YW estimates of autoregressive parameters α_s's and β_s's and immigration parameters λ_s's for the two simulation methods. These figures show the asymptotic normality of the estimates and corroborate the fact that the accuracy of the two simulation algorithms is similar. In Tables IV and V, simulation results are presented for the parameters as mean and covariance matrix of the stationary distribution of a PINAR$(1, 1_4)$ model which also shows satisfactory behavior. Note that the bias values of all elements of the variance matrix are negative confirming that the variance matrix $\Sigma^{(n)}$ of $\boldsymbol{Y}_k^{(n)}$ approximates the variance matrix Σ of the stationary distribution from below.

8. REAL DATA APPLICATION

In this application, the time series of counts (Pickup data) refers to the daily number of parcels picked up from one pickup point (PUP) at a PUP management company. It consists of daily aggregation of all parcels picked up from a PUP stored in the file *Data PUP1.csv*, available at https://github.com/cabani/ForecastingParcels, see Nguyen *et al.* (2023). This real data set corresponds to the period of July 3, 2017, to December 29, 2019, resulting in $T = 910$ daily ($n = 130$ weeks) observations. The series displayed in Figure 1 contains persistence oscillation in the sense that the mean and the variance change periodically. This is evidenced in the plots of the sample ACF

Table III. Results of the simulation to estimate the immigration parameters by the diagonal of the estimated immigration variance matrix.

	$n = 250$		$n = 1000$		$n = 4000$	
Parameter	Sim1	Sim2	Sim1	Sim2	Sim1	Sim2
λ_1	−0.0172	−0.02572	−0.00119	0.01895	−0.00385	−0.00029
	(0.6391)	(0.686)	(0.3293)	(0.3263)	(0.165)	(0.1752)
λ_2	−0.03092	−0.01963	−0.00317	0.01266	−0.01056	−0.00134
	(0.6636)	(0.6532)	(0.3187)	(0.322)	(0.1648)	(0.1578)
λ_3	0.00145	−0.00913	−0.00272	−0.01016	0.00031	−0.00292
	(0.5058)	(0.5086)	(0.2565)	(0.2543)	(0.1251)	(0.1324)
λ_4	−0.01518	0.01671	−0.00238	0.00097	−0.00035	−0.0017
	(0.3353)	(0.3566)	(0.1731)	(0.1782)	(0.0824)	(0.0881)

Note: The table contains the bias (RMSE) of each estimate.

Table IV. Results of the simulation to estimate the means

	$n = 250$		$n = 1000$		$n = 4000$	
Parameter	Sim1	Sim2	Sim1	Sim2	Sim1	Sim2
μ_1	−0.00139	0.00004	0.00393	0.00678	−0.0002	−0.00051
	(0.3133)	(0.3218)	(0.1549)	(0.16)	(0.0801)	(0.0778)
μ_2	−0.01271	−0.00966	−0.00497	0.0065	−0.00295	0.00149
	(0.2847)	(0.3006)	(0.1388)	(0.1428)	(0.0721)	(0.0733)
μ_3	−0.00312	0.000276	−0.00245	0.00266	−0.00236	0.00023
	(0.2512)	(0.2463)	(0.1282)	(0.1209)	(0.0617)	(0.0618)
μ_4	−0.00891	−0.00093	−0.01231	0.00125	−0.00144	−0.00239
	(0.2311)	(0.2252)	(0.1098)	(0.1068)	(0.0538)	(0.0535)

Note: The table contains the bias (RMSE) of each estimate.

Table V. Results of the simulation (Sim2) to estimate the variance matrix with $n = 4000$.

	$\sigma^2_{\cdot,1}$	$\sigma^2_{\cdot,2}$	$\sigma^2_{\cdot,3}$	$\sigma^2_{\cdot,4}$
$\sigma^2_{1,\cdot}$	−0.00185	−0.00093	−0.00121	−0.00463
	(0.263)	(0.2018)	(0.15)	(0.1358)
$\sigma^2_{2,\cdot}$		−0.00198	−0.00188	−0.00257
		(0.2451)	(0.148)	(0.1273)
$\sigma^2_{3,\cdot}$			−0.00501	−0.00413
			(0.1853)	(0.1287)
$\sigma^2_{4,\cdot}$				−0.0063
				(0.1548)

Note: The table contains the bias (RMSE) of each estimate.

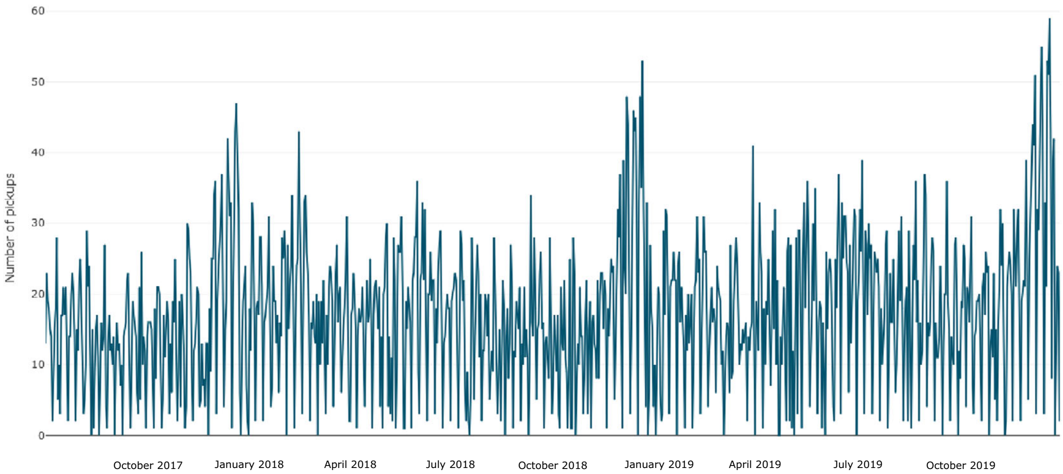

Figure 1. Daily number of parcels picked up from one Pick-Up Point (PUP) at a PUP management company.

and periodogram, as discussed below. The time series is available on request from the authors, together with the R and Python codes analyzing it. The data have been processed with the standard time series libraries of R with perARMA for analyzing periodic time series, and Python libraries such as csv, numpy, pandas, and scipy. The results were verified by comparing the outputs of the two softwares.

Figure 2 shows the sample periodic mean and variance of the series over seasons, the sample ACF, and the periodogram of the series. The periodic mean and variance curves show a similar shape, with a minimum on Sunday (2.377 and 3.989 respectively) and a maximum on Thursday-Friday (23.915 and 94.048 respectively). The analysis of the sample ACF suggests that the series has seasonal autocorrelation of period $S = 7$ which is an expected result since the series consists of daily data. The periodogram provides a high peak at frequency 0.14, which corresponds to the period $1/0.14 = 7$. Tables VI and VII present the sample periodic autocorrelation (perACF) and the sample periodic partial autocorrelation (perPACF) functions. In these tables, the values in bold are the sample correlations that exceed the critical value $1.96/\sqrt{130} = 0.172$. All values of the sample perACF are positive except Sunday, which supports the applicability of the PINAR model. The autoregressive order identification of PAR processes is based on finding the lowest lag for which the sample perPACF cuts off, see McLeod (1994). The characteristics of the Pickup data show complex and contradictory behavior, which is not surprising for real data. For example, the clear weekly seasonality ($S = 7$) seen on the sample ACF and periodogram in Figure 2 is spread over the lags 6, 7 and 8 in the sample perPACF in Table VII. However, if we want to describe the pickup time series with a consistent and parsimonious model, based on this preliminary model identification step, the PINAR$(1, 1_7)$ model may be adequate to capture the basic dynamic of the data.

We fit a PINAR$(1, 1_7)$ model to the data using the YW method which consists in solving the vector equation (46), where the perACF's are replaced by their sample estimates, to find estimates of α_s and β_s and to use (39), where the periodic means are replaced by their sample estimates, to derive estimate of λ_s for $s = 1, \dots, 7$. This estimation method does not require any assumptions about the thinning and immigration distribution. The YW estimates of the parameters of the PINAR$(1, 1_7)$ model are displayed in Table VIII. These parameter values provide deeper insights to understand the dynamics of the time series. They show that the first-order temporal dependence is significant each day except Sunday and it is stronger than the seasonal dependence. The largest first-order autoregressive coefficient is Thursday, showing strong dependence on the previous Wednesday's value, while the largest seasonal autoregressive coefficients are Friday and Saturday, indicating a stronger seasonal dependence between consecutive weekends. In contrast, the impact of immigration is largest at the beginning of the week (Monday and Tuesday). Finally, since the autoregressive parameters are not significant on Sunday, the time series behaves as white noise here, which is corroborated by the corresponding correlation values in Tables VI and VII.

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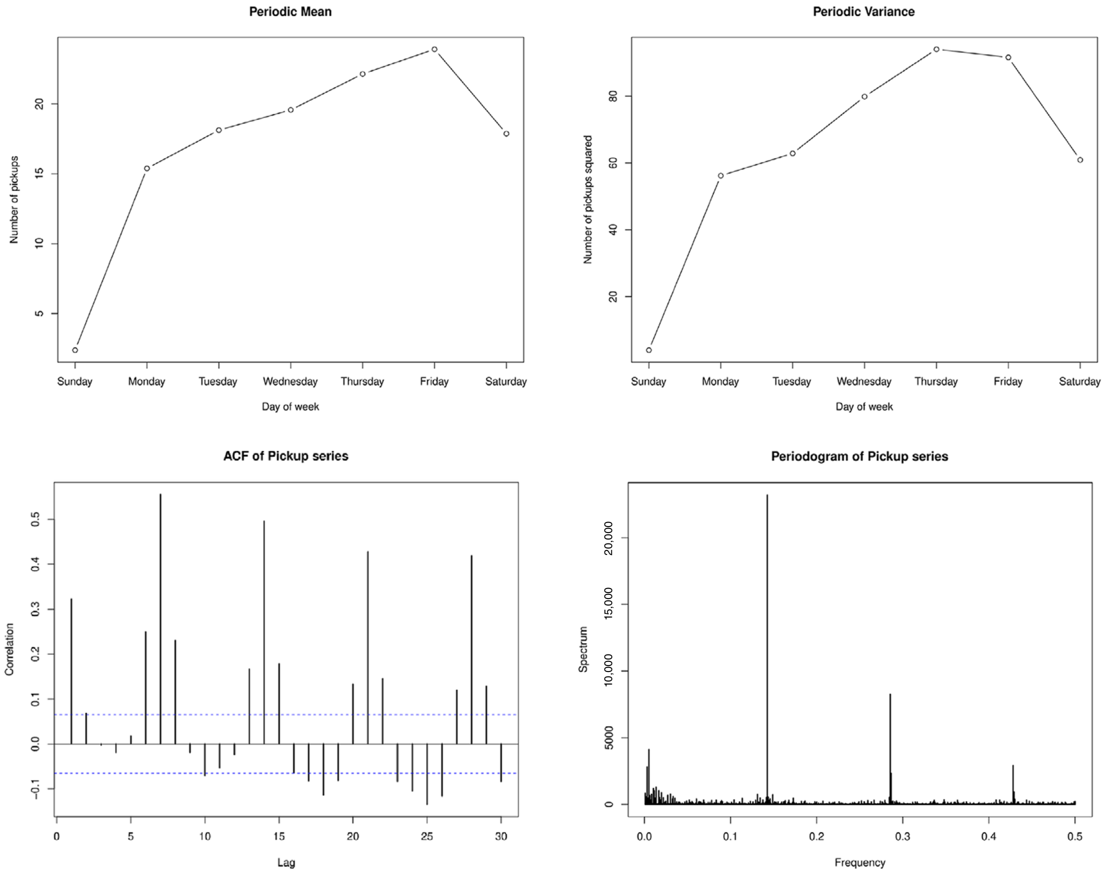

Figure 2. The sample periodic mean and variance over days of the week, the sample ACF, and the periodogram of Pickup data

Table VI. Sample periodic ACF of the pickup data

	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$	$h = 7$	$h = 8$	$h = 9$	$h = 10$
Sunday	0.072	0.008	0.118	−0.021	0.036	0.120	−0.042	0.000	−0.058	−0.012
Monday	**0.261**	**0.215**	**0.370**	**0.287**	**0.321**	0.075	0.169	0.084	0.184	0.186
Tuesday	**0.328**	**0.438**	**0.241**	**0.208**	0.081	**0.281**	0.060	0.168	**0.205**	0.115
Wednesday	**0.548**	**0.479**	**0.373**	**0.215**	**0.342**	0.171	**0.222**	**0.238**	**0.238**	**0.232**
Thursday	**0.486**	**0.450**	**0.196**	**0.278**	**0.196**	**0.222**	**0.308**	**0.406**	**0.245**	0.096
Friday	**0.521**	0.149	**0.381**	**0.351**	**0.314**	**0.398**	**0.368**	**0.363**	0.097	**0.337**
Saturday	**0.244**	**0.332**	**0.238**	**0.341**	**0.312**	**0.443**	**0.406**	**0.260**	**0.234**	0.135

9. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. (i) ⇒ (ii) Suppose that $\{Y_t\}$ is a PC process of period S. Then, we have

$$\mu(j) = \mu(kS + j), \qquad R(i, j) = R(kS + i, kS + j),$$

for all $i, j \in \{1, \dots, S\}$ and $k \in \mathbb{N}$. This implies that $\boldsymbol{\mu}_0 = \boldsymbol{\mu}_k = \boldsymbol{\mu}$ and $\Sigma_0 = \Sigma_k = \Sigma$ for all $k \in \mathbb{N}$. Thus, the random vector $(Y_1, \dots, Y_S)^\top$ has mean $\boldsymbol{\mu}$ and covariance matrix Σ which satisfy (22) and (23) by (37) and (40) respectively.

Table VII. Sample periodic PACF of the pickup data

	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$
Sunday	0.072	−0.011	**0.174**	**0.237**	0.009	−0.037	−0.068	−0.045	−0.088	−0.021
Monday	**0.261**	0.142	0.114	**0.240**	0.091	0.015	0.033	0.081	0.067	−0.022
Tuesday	**0.328**	**0.326**	**0.260**	−0.116	0.085	0.142	0.118	−0.045	0.027	0.005
Wednesday	**0.548**	**0.290**	0.002	0.134	−0.004	**0.297**	0.065	**0.204**	−0.086	−0.156
Thursday	**0.486**	**0.264**	0.064	−0.008	0.152	0.117	0.152	0.063	−0.104	0.017
Friday	**0.521**	0.027	0.096	0.109	−0.021	−0.025	−0.081	**0.202**	−0.080	0.010
Saturday	**0.244**	**0.327**	**0.258**	0.074	0.164	0.157	0.003	−0.052	0.020	0.158

Table VIII. Application of PINAR$(1, 1_7)$ model to the pickup data

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
α	0.065	0.224	0.280	0.337	0.547	0.398	0.346
β	−0.072	0.165	−0.014	0.171	0.196	0.207	0.218
λ	1.393	12.321	14.072	10.122	7.092	10.137	5.698

Note: The parameters are estimated by the YW method.

(ii) ⇒ *(iii)* Suppose that the random vector $\boldsymbol{Y}_0 = (Y_1, \ldots, Y_S)^\top$ has mean $\boldsymbol{\mu}$ and variance matrix Σ which satisfy (22) and (23) respectively. Then, by recursion (37), we have that $\boldsymbol{\mu}_k = \boldsymbol{\mu}$ for all $k \in \mathbb{Z}$. (Note that B is an invertible matrix.) Similarly, by recursion (40), we have that $\Sigma_k = \Sigma$ for all $k \in \mathbb{Z}$. Moreover, by (42), $\Gamma(k, \ell)$ depends only on $k - \ell$. Thus, $\{\boldsymbol{Y}_k\}$ is a weakly stationary process.

(iii) ⇒ *(i)* Suppose that $\{\boldsymbol{Y}_k\}$ is a weakly stationary process with mean $\boldsymbol{\mu}$, variance matrix Σ and covariance matrix function Γ. For all $s, t \in \mathbb{Z}$ there exists $k, \ell \in \mathbb{Z}$ and $i, j \in \{1, \ldots, S\}$ such that $s = kS + i$ and $t = \ell S + j$. Thus,

$$\mu(s+S) = \mu((k+1)S + i) = (\boldsymbol{\mu}_{k+1})_i = \mu_i = (\boldsymbol{\mu}_k)_i = \mu(kS+i) = \mu(s),$$

and

$$\begin{aligned} R(s,t) = R(kS+i, \ell S + j) &= (\Gamma(k,\ell))_{i,j} = (\Gamma(k+1, \ell+1))_{i,j} \\ &= R((k+1)S + i, (\ell+1)S + j) = R(s+S, t+S). \end{aligned}$$

Hence $\{Y_t\}$ is a PC process of period S. ■

Proof of Theorem 2. Almost sure convergence of $\{\boldsymbol{Y}_k^{(n)}\}_n$. Consider the infinite sum (34) of $\mathbb{N}_0^S$-valued r.v.'s, which can take infinite value if it is necessary. By the monotone convergence theorem for Lebesgue integral, we obtain

$$\mathsf{E}(\boldsymbol{Y}_k) = \mathsf{E}\left(\lim_{n\to\infty} \boldsymbol{Y}_k^{(n)}\right) = \lim_{n\to\infty} \mathsf{E}\left(\boldsymbol{Y}_k^{(n)}\right) = \lim_{n\to\infty} \boldsymbol{\mu}^{(n)} = \boldsymbol{\mu},$$

see Section 4, which implies that $\boldsymbol{Y}_k$ is finite almost surely and is almost sure limit of the sequence of r.v.'s $\{\boldsymbol{Y}_k^{(n)}\}_n$ for all $k \in \mathbb{Z}$.

Mean square convergence of $\{\boldsymbol{Y}_k^{(n)}\}_n$. Since $\mathsf{E}\|\boldsymbol{Z}\|^2 = \|\mathsf{E}(\boldsymbol{Z})\|^2 + \mathrm{tr}\mathrm{Var}(\boldsymbol{Z})$ for any random vector $\boldsymbol{Z}$, we have

$$\mathsf{E}\|\boldsymbol{Y}_k^{(n)} - \boldsymbol{Y}_k\|^2 = \mathsf{E}\left\|\sum_{j=n+1}^{\infty} \boldsymbol{Z}_k^{(j)}\right\|^2 = \left\|\sum_{j=n+1}^{\infty} \boldsymbol{\nu}^{(j)}\right\| + \mathrm{tr}\sum_{j=n+1}^{\infty} \Phi^{(j)}. \tag{48}$$

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Then, the assertion follows, since, by Remark 3, $\sum_{j=n+1}^{\infty}\boldsymbol{\nu}^{(j)} = \boldsymbol{\mu} - \boldsymbol{\mu}^{(n)}$ and $\sum_{j=n+1}^{\infty}\Phi^{(j)} = \Sigma - \Sigma^{(n)}$, where the right-hand sides tend 0 as $n \to \infty$.

Non-negative integer-valued property of $\{\boldsymbol{Y}_k\}$. The sequence $\{\boldsymbol{Y}_k^{(n)}\}_n$ is a non-decreasing sequence of $\mathbb{N}_0^S$-valued r.v.'s and has an almost-sure finite limit $\boldsymbol{Y}_k$ for all $k \in \mathbb{Z}$. Consequently, for each $k \in \mathbb{Z}$, there exists $E_k \in \mathcal{A}$ such that $\mathsf{P}(E_k) = 1$ and $\boldsymbol{Y}_k^{(n)}(\omega) \nearrow \boldsymbol{Y}_k(\omega)$ as $n \to \infty$ for all $\omega \in E_k$. Thus, for each $\omega \in E_k$, there exists $N_k(\omega) \in \mathbb{N}$ such that $\boldsymbol{Y}_k(\omega) = \boldsymbol{Y}_k^{(n)}(\omega)$ for all $n > N_k(\omega)$, and hence $\boldsymbol{Y}_k(\omega)$ is a $\mathbb{N}_0^S$-valued r.v. for all $\omega \in E_k$.

Strict and second-order stationarity of $\{\boldsymbol{Y}_k\}$. Clearly, for all $k_1 < \cdots < k_m$, $m \in \mathbb{N}$, $(\boldsymbol{Y}_{k_1}^{(n)}, \ldots, \boldsymbol{Y}_{k_m}^{(n)})$ converges to $(\boldsymbol{Y}_{k_1}, \ldots, \boldsymbol{Y}_{k_m})$ almost surely. Thus, $\{\boldsymbol{Y}_k\}$ is a strictly stationary process because it is the almost sure limit of a sequence of strictly stationary processes. Moreover, the monotone convergence theorem implies that $\{\boldsymbol{Y}_k\}$ has mean vector $\boldsymbol{\mu}$ and variance matrix Σ. Finally, $\Gamma_n(h) \to \Gamma(h)$ for all $h \in \mathbb{Z}$ implies the formula for the autocovariance matrix function.

$\{\boldsymbol{Y}_k\}$ *is a solution to equation* (9). Since $\boldsymbol{Y}_k = \boldsymbol{Y}_k^{(n)} + \sum_{j=n+1}^{\infty}\boldsymbol{Z}_k^{(j)}$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have by (17) that

$$(I - A_k)^{\circ -1}(B_k \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k) = \boldsymbol{Y}_k^{(n)} + \sum_{j=n}^{\infty}\left((I - A_k)^{\circ -1}B_k\circ\right)_{|\boldsymbol{Y}_{k-1}^{(j-1)}}\boldsymbol{Z}_{k-1}^{(j)},$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. The right-hand side of this equation tends to $\boldsymbol{Y}_k$ almost surely as $n \to \infty$ because the $\mathbb{N}_0^S$-valued r.v.'s in the infinite sum are non-negative and

$$\begin{aligned}\mathsf{E}\left(\sum_{j=n}^{\infty}\left((I - A_k)^{\circ -1}B_k\circ\right)_{|\boldsymbol{Y}_{k-1}^{(j-1)}}\boldsymbol{Z}_{k-1}^{(j)}\right) &= (I - A)^{-1}B\sum_{j=n}^{\infty}\mathsf{E}\left(\boldsymbol{Z}_{k-1}^{(j)}\right)\\ &= (I - A)^{-1}B\left(\boldsymbol{\mu} - \boldsymbol{\mu}^{(n-1)}\right) \to 0,\end{aligned}$$

as $n \to \infty$. Thus, $\{\boldsymbol{Y}_k\}$ is a solution to equation (9).

Ergodicity and non-anticipativity of $\{\boldsymbol{Y}_k\}$. It is proved in Section 4 that, for each $n \in \mathbb{N}$, $\boldsymbol{Y}_k^{(n)}(T\omega) = \boldsymbol{Y}_{k+1}^{(n)}(\omega)$ for all $\omega \in \Omega$ and $k \in \mathbb{Z}$ where T is an ergodic measure preserving transformation of $(\Omega, \mathcal{A}, \mathsf{P})$. By almost sure convergence of $\{\boldsymbol{Y}_k^{(n)}\}_n$ for all $k \in \mathbb{Z}$, there exists $A \in \mathcal{A}$ with $\mathsf{P}(A) = 1$ such that $\boldsymbol{Y}_k(T\omega) = \boldsymbol{Y}_{k+1}(\omega)$ for all $\omega \in A$ and $k \in \mathbb{Z}$. Thus, $\{\boldsymbol{Y}_k\}$ is ergodic. By definition of $\boldsymbol{Y}_k$, this r.v. depends on the r.v.'s $\{\boldsymbol{Z}_k^{(n)}\}_n$, i.e., $\boldsymbol{Y}_k$ is $\mathcal{G}_k$-measurable for all $k \in \mathbb{Z}$, which implies the non-anticipativity of $\{\boldsymbol{Y}_k\}$.

Uniqueness of $\{\boldsymbol{Y}_k\}$. Suppose that $\{\boldsymbol{Y}'_k\}$ is another non-anticipative weakly stationary solution to (9). Define recursively the $\mathbb{N}_0^S$-valued r.v.'s $\boldsymbol{Z}_{k,1} := B_k\circ\boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k$, $\boldsymbol{Z}'_{k,1} := B_k\circ\boldsymbol{Y}'_{k-1} + \boldsymbol{\varepsilon}_k$, $\boldsymbol{Z}_{k,s} := A_k^{(s)}\circ\boldsymbol{Z}_{k,s-1}$, $\boldsymbol{Z}'_{k,s} := A_k^{(s)}\circ\boldsymbol{Z}'_{k,s-1}$, $s = 2, \ldots, S$. Then, $\boldsymbol{Y}_k = \boldsymbol{Z}_{k,S}$ and $\boldsymbol{Y}'_k = \boldsymbol{Z}'_{k,S}$ and, by Lemma 4 and formula (10), we have

$$\begin{aligned}\mathsf{E}|\boldsymbol{Y}_k - \boldsymbol{Y}'_k|_{\text{vec}} &= \mathsf{E}|A_k^{(S)}\circ\boldsymbol{Z}_{k,S-1} - A_k^{(S)}\circ\boldsymbol{Z}'_{k,S-1}|_{\text{vec}} \le A^{(S)}\mathsf{E}|\boldsymbol{Z}_{k,S-1} - \boldsymbol{Z}'_{k,S-1}|_{\text{vec}} \le \cdots \le\\ &\le A^{(S)}A^{(S-1)} \ldots A^{(2)}\mathsf{E}|\boldsymbol{Z}_{k,1} - \boldsymbol{Z}'_{k,1}|_{\text{vec}} = (I - A)^{-1}\mathsf{E}|B_k\circ\boldsymbol{Y}_{k-1} - B_k\circ\boldsymbol{Y}'_{k-1}|_{\text{vec}}\\ &\le (I - A)^{-1}B\mathsf{E}|\boldsymbol{Y}_{k-1} - \boldsymbol{Y}'_{k-1}|_{\text{vec}}\end{aligned}$$

for all $k \in \mathbb{Z}$ since $A_k\circ$ and $B_k\circ$ are mutually independent of $\boldsymbol{Y}_{k-1}$ and $\boldsymbol{Y}'_{k-1}$ by non-anticipativity. Thus, by Proposition 1, we obtain that

$$\mathsf{E}|\boldsymbol{Y}_k - \boldsymbol{Y}'_k|_{\text{vec}} \le \left((I - A)^{-1}B\right)^n\mathsf{E}|\boldsymbol{Y}_{k-n} - \boldsymbol{Y}'_{k-n}|_{\text{vec}} \le 2\left((I - A)^{-1}B\right)^n\boldsymbol{\mu} \to \boldsymbol{0},$$

as $n \to \infty$, see theorem 5.6.12 in Horn and Johnson (2012), and hence $\boldsymbol{Y}_k = \boldsymbol{Y}'_k$ almost surely for all $k \in \mathbb{Z}$. ■

Proof of Corollary 1. By Theorem 2, let $\{\boldsymbol{Y}_k\}$ be the almost surely unique non-anticipative and weakly stationary solution to (5). Define the process of $\mathbb{N}_0$-valued r.v.'s $\{Y_t\}$ as $Y_{kS+s} := (\boldsymbol{Y}_k)_s$ for all $k \in \mathbb{Z}$, $s = 1, \ldots, S$. One can easily see that, by considering the coordinates of the stochastic vector equation (5), we obtain equation (1) of the PINAR$(1, 1_S)$ model. Since $\{\boldsymbol{Y}_k\}$ is second-order stationary, $\{Y_t\}$ is PC with period S. Moreover, the strict stationarity of $\{\boldsymbol{Y}_k\}$ implies the periodically strict stationarity of $\{Y_t\}$, i.e. $(Y_{t_1}, \dots, Y_{t_n}) \stackrel{\mathcal{D}}{=} (Y_{t_1+S}, \dots, Y_{t_n+S})$ for all $t_1 < \cdots < t_n$, $n \in \mathbb{N}$. Finally, the uniqueness of $\{Y_t\}$ follows from the uniqueness in Theorem 2. ■

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author on reasonable request.

SUPPORTING INFORMATION

Additional Supporting Information may be found online in the supporting information tab for this article.

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