Design of a Modified Linear Quadratic Regulator for Vibration Control of Suspension Systems

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Abstract-**This paper is concerned with the construction of a prototype active vehicle suspension system for a one-wheel car model by using a modified Linear Quadric Regulator (LQR). The experimental system is approximately described by non-linear system with two degrees of freedom subject to excitation from a road profile. The active control at the suspension location is designed by using feedback constant gain controller structure. The experimental results show that the active suspension system with LQR more improves the control performance than standard PID controller. On the other hand, the results improved that the modified LQR has superior performance for controlling suspension systems in real time.**

Keywords: Suspension system, Linear Quadratic Regulator, Vibration control

I. INTRODUCTION

Recently, the problem of active suspensions for car models has been much interested in the design and development because the active suspensions relatively improve ride comfort of passengers in high-speed ground transportation.

The investigations of active suspension systems for car models have been drawn much interest in recent years. The active suspension system is more effective to ride comfort of passengers, but its construction generally provides higher production cost then semi-active and passive suspension systems [1]. The construction of the active suspension system is mainly based on linear quadric control theory where a car model is assumed to be a linear or approximate linear system. However, as the car model is practically denoted as a complicated system including non-negligible non-linearity and uncertainty, the derivation of the active control becomes relatively complicated. Recently, various kinds of active suspension systems have been derived for such complicated systems using the concepts of fuzzy reasoning [2], neural network [3] and sliding mode theory [4]. The obtained active suspension systems provide more effective performance in the vibration isolation of the car body, but need more complicated structure in the suspension system than the linear active suspension system derived on the basis of LQR theory.

A quarter car model has been controlled using sliding mode control concept [5]. In their investigation, the sliding mode control denoted as the active control is relatively simpler in the structure of the controller than those based on fuzzy reasoning and neural network, and it guarantees the system stability. The quarter car model to be considered here is approximately described as a non-linear two degrees-of-freedom (DOF) system subject to excitation from a road profile. The time variation of the road profile is assumed to be unknown, and it is estimated by using the minimum order observer on the basis of a linear system transformed from the exact non-linear system.

To achieve a compromise between several performance requirements for uncertain active suspension systems, very recently a robust multi-objective controller was designed for quarter-car model whose system matrices are subject to parameter uncertainties characterized by a given polytope [6]. The main objective there is to use a robust state-feedback controller to achieve multiple performance objectives for different controlled output signals. It is worth mentioning that the solutions are given in the quadric framework.

A semi-active control of vehicle suspension system with magnetorheological MR damper has been presented in [7]. In their paper, MR damper working in flow mode was designed. Performance testing was done for this damper with INSTRON machine.

II. PROTOTYPE VEHICLE SUSPENSION SYSTEM

The vehicle suspension systems basically consist of wishbones, the spring, and shock absorber to transmit and also filter all forces between body and road. The task of the spring is to carry the body-mass and to isolate the body from road disturbances and thus contributes to drive comfort. The damper contributes to both driving safety and comfort. Its task is the damping of body and vehicle oscillations, where the avoidance of wheel oscillations directly refers to drive safety, as a non-bouncing wheel is the condition for transferring road-contact forces.
Considering the vertical dynamics and taking into account the vehicle's symmetry, a suspension can in a first step be reduced to the so-called quarter-car model as shown in Figure 1(a). A quarter-car model consists of one-fourth of the body mass, suspension components and

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one wheel. The model has been used extensively in the literature and captures many essential characteristics of a real suspension system. Besides, the schematic representation of the system is shown in Figure 1(b). The governing equations of motion for sprung and unsprung masses of the quarter-car model are given by

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = F(t). \tag{1}$$

Similarly for m2:

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0. \tag{2}$$

These may be expressed in a state space realization as:

$$\begin{aligned} \dot{x} &= Ax + BF(t) \\ Y &= Cx \end{aligned} \tag{3}$$

where:

$$X = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2)/m_1 & -c_1/m_1 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -(k_2+k_3)/m_2 & -c_2/m_2 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix} \tag{4}$$

and $C_i = 1$ ($i = 1,2,3,4$) when X_i is an output and equals 0 otherwise.

By Laplace transform of Equations (1, 2) and assuming zero valued initial conditions we may solve for the transfer functions:

$$\frac{x_1(s)}{F(s)} = \frac{m_2s^2 + c_2s + k_2 + k_3}{D(s)} \tag{5}$$

$$\frac{x_2(s)}{F(s)} = \frac{k_2}{D(s)} \tag{6}$$

where:

$$\begin{aligned} D(s) = {} & m_1m_2s^4 + (m_1c_2 + m_2c_1)s^3 + (m_1(k_2+k_3) + m_2(k_1+k_2) + c_1c_2)s^2 \\ & + (c_1(k_2+k_3) + c_2(k_1+k_{23}))s + k_1k_2 + k_1k_3 + k_2k_3 \end{aligned}$$

which may also be expressed in the form:

$$\frac{x_1(s)}{F(s)} = \frac{K_1\left(s^2 + 2\zeta_z\omega_z s + \omega_z^2\right)}{\left(s^2 + 2\zeta_{p1}\omega_{p1}s + \omega_{p1}^2\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{8}$$

$$\frac{x_2(s)}{F(s)} = \frac{K_2}{\left(s^2 + 2\zeta_{p1}\omega_{p1}s + \omega_{p1}^2\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{9}$$

where the ω_i's and ζ_i's are the natural frequencies and damping ratios respectively, and the gains K_1 & K_2, are nominally equal to $1/m_1$ and k_1/m_1m_2 (but often may be measured more directly).

For the case $k_1 = k_3 = 0$ ($\omega_1 = 0$), a damped rigid body motion exists and Equations (8, 9) become:

$$\frac{x_1(s)}{F(s)} = \frac{K_1\left(s^2 + 2\zeta_z\omega_z s + \omega_z^2\right)}{s\left(s + c^*\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{10}$$

$$\frac{x_2(s)}{F(s)} = \frac{K_2}{s\left(s + c^*\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{11}$$

II. Modified Linear Quadric Regulator

Linear Quadratic Regulator (LQR) design technique is well known in modern optimal control theory and has been widely used in many applications. It has very nice robustness property, if the process is of single-input and single-output, then the control system has at least the phase margin of 60° and gain margin of infinity. This attractive property appeals to the practicing engineers. Thus, the LQR theory has received considerable attention since 1950s. In the context of optimal PID tuning, typical performance indices are the integral of squared error and time weighted error.

The main idea of the recursive approximating sequence approach is introduced. Consider a general nonlinear system of Equation (12) with a quadratic cost function of Equation (13):

$$\dot{x} = A(x)\,x + B(x)\,u, \quad x(0) = x_0 \tag{12}$$

$$J = \int_{t_0}^{t_f} \left(x^T Q\, x + u^T R\, u\right) dt \tag{13}$$

where $x \in R^n$, $u \in R^m$, $A(x) \in R^{nxn}$ and $B(x) \in R^{nxm}$, Q is n x n positive semi-definite state weighting matrix; and R is m x m positive define control weighting matrix.

By introducing the approximating sequence of Equations (12)-(13), the quadratic cost can be rewritten as Equation (13)

$$\begin{aligned} \dot{x}^{(i)} &= A\left(x^{(i-1)}(t)\right)x^{(i)} + B\left(x^{(i-1)}(t)\right)u^{(i)}, \\ \dot{x}^{(i)}(t_0) &= x_0 \end{aligned} \tag{14}$$

$$\dot{x}^{(1)} = A(x_0)\, x^{(1)} + B(x_0)\, u^{(1)}\,, \qquad (15)$$
$$x^{(1)}(t_0) = x_0$$

$$J^{(i)} = \int_{t_0}^{t_f} \left(x^{(i)^T} Q x^{(i)} + u^{(i)^T} R u^{(i)} \right) dt$$

It can be seen that the first approximation of the nonlinear system of Equation (12) with Equation (15) is a linear time-invariant system. The successive approximating systems are linear and time- varying as in Equation (14). Moreover matrices of A and B in the *ith* approximation are functions of state $x^{(i-1)}(t)$ obtained in the $(i-1)^{th}$ approximation.

Suppose that successive approximating systems of Equations (8) and (9) are controllable. Using Equations (14)-(15), the optimal modified LQR can be written as

$$u^{(i)}(t) = -R^{-1} B^T(x^{(i-1)}(t))\, P^{(i)}(t)\, x^{(i)}(t) \qquad (16)$$

where $P^{(i)}(t)$ is the solution of the following algebraic Riccati equation

$$A^T(x^{(i-1)}(t))\, P^{(i)}(t) + P^{(i)}(t)\, A(x^{(i-1)}(t)) + Q$$
$$- P^{(i)}(t) B(x^{(i-1)}(t)) R^{-1} B^T(x^{(i-1)}(t)) P^{(i)}(t) = 0$$

The *ith* closed-loop dynamic system can be written as

$$\dot{x}^{(i)}(t) = \left[A(x^{(i-1)}(t)) - B(x^{(i-1)}(t)) R^{-1} B^T \times (x^{(i-1)}(t)) P^{(i)}(t) \right] x^{(i)}(t) \qquad (17)$$

Equation (16) can simply be written as Equation (17),

$$\dot{x}^{(i)}(t) = \hat{A}(x^{(i-1)}(t))\, x^{(i)}(t) \qquad (18)$$

where

$$\hat{A}(x^{(i-1)}(t)) = \left[A(x^{(i-1)}(t)) - B(x^{(i-1)}(t)) R^{-1} B^T \times (x^{(i-1)}(t)) P^{(i)}(t) \right]$$

The convergence of the closed loop approximating sequence of Equation (18) can now be proved. Define $\Phi^{(i-1)}(t,t_0)$ as the transition matrix of $\hat{A}(x^{(i-1)}(t))$, then $\Phi^{(i-1)}(t,t_0)$ meets the following inequality.

$$\left\| \Phi^{(i-1)}(t,t_0) \right\| \le exp\left[\int_{t_0}^{t} \mu(\hat{A}(x^{(i-1)}(\tau)))\, d\tau \right] \qquad (19)$$

where $\mu(\hat{A}(x^{(i-1)}(\tau)))$ is the logarithmic norm the closed loop matrix of $\hat{A}(x^{(i-1)}(t))$.
The following lemma is introduced to prove the convergence.

Lemma 1. Suppose that $\mu(\hat{A}(x)) \le \mu_0$ for all x, and that

$$\left\| \hat{A}(x) - \hat{A}(y) \right\| \le \alpha \left\| x - y \right\| \;,\; \forall x, y \in R^n \qquad (20)$$

$$\left\| \Phi^{(i-1)}(t,t_0) - \Phi^{(i-2)}(t,t_0) \right\| \le \alpha\, e^{\mu_0(t-t_0)} (t-t_0) \sup_{s\in(t,t_0)} \left\| x^{(i-1)}(s) - x^{(i-2)}(s) \right\| \qquad (21)$$

The convergence of the closed-loop approximating sequence of Equation (18) can be begun to prove.

The solution of Equation (18) is

$$x^{(i)}(t) = \Phi^{(i-1)}(t,t_0) x_0 + \int_{t_0}^{t} \Phi^{(i-1)}(t,s) x^{(i)}(s)\, ds \qquad (22)$$

Hence

$$x^{(i)}(t) - x^{(i-1)}(t) = \left[\Phi^{(i-1)}(t,t_0) - \Phi^{(i-2)}(t,t_0) \right] x_0 + \int_{t_0}^{t} \Phi^{(i-1)}(t,s) \left[x^{(i)}(s) - x^{(i-1)}(s) \right] ds$$
$$+ \int_{t_0}^{t} \left[\Phi^{(i-1)}(t,s) - \Phi^{(i-2)}(t,s) \right] x^{(i-1)}(s)\, ds \qquad (23)$$

From *lemma 1* we have

$$\left\| x^{(i)}(t) - x^{(i-1)}(t) \right\| \le \alpha\, e^{\mu_0(t-t_0)} (t-t_0) \sup_{s\in(t_0,t)} \left\| x^{(i-1)}(s) - x^{(i-2)}(s) \right\| \left\| x_0 \right\|$$
$$+ \int_{t_0}^{t} \alpha e^{\mu_0(t-s)} (t-s) \sup_{s\in(t_0,t)} \left\| x^{(i-1)}(s) - x^{(i-2)}(s) \right\| e^{\mu_0(s)} \left\| x_0 \right\| ds$$
$$- \int_{t_0}^{t} e^{\mu_0(t-s)} \left\| x^{(i)}(s) - x^{(i-1)}(s) \right\| ds \qquad (24)$$

Now define

$$\xi^{(i)}(t) = \sup_{s\in(t_0,t)} \left\| x^{(i)}(s) - x^{(i-1)}(s) \right\|, \quad (25)$$

then

$$\left[1+\frac{1}{\mu_0}-\frac{1}{\mu_0}e^{\mu_0(t-t_0)}\right]\xi^{(i)}(t) \le \|x_0\| \left[\alpha\, e^{\mu_0(t-t_0)}(t-t_0)+\frac{1}{2}(t-t_0)^2\right]\xi^{(i-1)}(t)$$

(26)

Suppose that $\left[1+\frac{1}{\mu_0}-\frac{1}{\mu_0}e^{\mu_0(t-t_0)}\right] \neq 0$. Thus

$$\xi^{(i)}(t) \le \lambda(t,\mu_0,\alpha)\,\xi^{(i-1)}(t) \quad (27)$$

where

$$\lambda(t,\mu_0,\alpha) = \frac{\left[\alpha\, e^{\mu_0(t-t_0)}(t-t_0)+\frac{1}{2}(t-t_0)^2\right]}{\left[1+\frac{1}{\mu_0}-\frac{1}{\mu_0}e^{\mu_0(t-t_0)}\right]}$$

Soif

$$|\lambda(t,\mu_0,\alpha)| \langle 1 \;\; for \;\; t \in [t_0 \quad t_f] \;\; then \;\; x^{(i)}(t) \to x(t) \quad (28)$$

Hence over a finite time interval, the state $x^{(i)}(t)$ of the approximating sequence of Equation (12) converges to the state $x(t)$ of nonlinear system. Meanwhile the convergence of the approximating sequence is related to the modified optimal LQR gain matrix. So if the successive approximating system Equations (12)-(13) is controllable, for selected state weighting matrix Q and control weighting matrix R, it is possible to satisfy $|\lambda(t,\mu_0,\alpha)| \langle 1$ by choosing proper control gain to guarantee the convergence of the approximating sequence.

III. Experimental Results

In this experiment a proposed LQR was implemented on a prototype suspension system using state feedback on Windows Operating system. The prototype suspension plant used was a 2 degrees of freedom configuration. Presented in this section are the results of experimental research that have been conducted to evaluate the performance of the proposed LQR system of this research. Figure 2(a) shows the amplitude of Encoder 2 under modified LQR structure. It is seen that under LQR the amplitude in the frequency range between body resonance and wheel is acceptable. Travel response of the suspension (Encoder 1) is shown in Figure 2(b). It can be seen that the proposed LQR provides improved performances compared these of the LQR system.

The optional disturbance drive is useful in studying the important system properties of regulation performance and disturbance rejection at any of the system outputs (inertia disks). It may also be used to apply viscous friction at any of these locations.

The important concept of frequency response phase behavior is easily demonstrated with the experimental system. By exciting the system a particular frequency and plotting the Commanded Position and encoder 1 position data, the phase is found from the expression:

$$\phi = -360^{o}\frac{t_{w0}}{t_{wi}}$$

Shown in Figure 3(a) is the sweep output at Encoder 2 with sweep frequency of 0.1 to 10 Hz and count 600 amplitude. Here, the collocated magnitude is relatively high including a rise following the zero (due to derivative lead) out to roughly 4 Hz. The non-collocated response is flat-out to the other loop design bandwidth of 4 Hz (see also Figure 3(b) for Encoder 1).

Shown in Figures 4(a) and 4(b) are the Encoder 2 and Encoder 1 responses to a 3000 count, 3500 count/sec step trajectory with viscous disturbances. The irregular shape at encoder 1 as it leads and then makes dynamic corrections to minimize the error at Encoder 2. The Encoder 2 tracking here closely follows the reference input. This is consistent with the relatively flat high bandwidth frequency response.

On the other hand, Figure 5 shows the experimental results of the suspension system for step signal using standard PID controller. As can be seen figures, the results of the PID controller case are poorer than LQR approach.

III. Conclusion

In this paper the use of a modified Linear Quadric Regulator as a controller of the suspension system has been investigated. Simple PID controller was also employed to control of the suspension system for comparison. The modified LQR has been designed to meet the performance requirements taking into account the intrinsic physical limitations of the controller. It is clear from the experimental works that controllers designed based on LQR, perform well for system with viscous disturbance. Therefore, the control algorithm has been shown to have good robustness properties with viscous disturbances on the suspension system. Finally, the proposed LQR control scheme has also been shown to perform better then the conventional PID control scheme.

In future, an artificial neural network controller should be designed for controlling suspension systems in real time applications.

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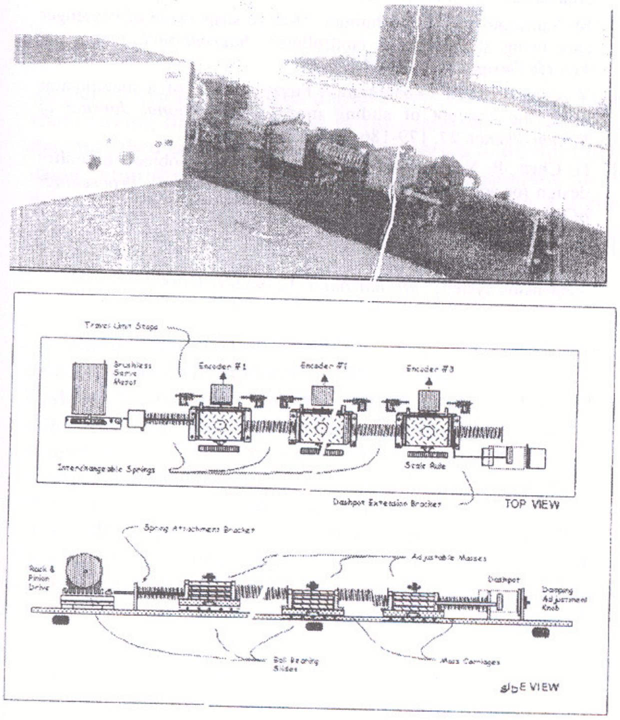

Figure 1 (a). Representation of experimental system

(b). Schematic view of the suspension

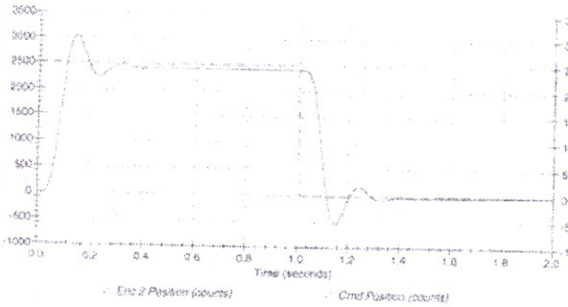

Figure 2(a). Step Response at Encoder 2

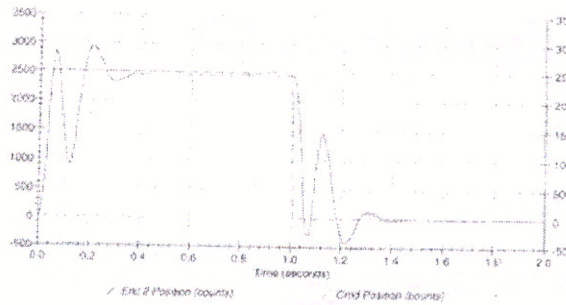

Figure 2(b). Step Response at Encoder 1

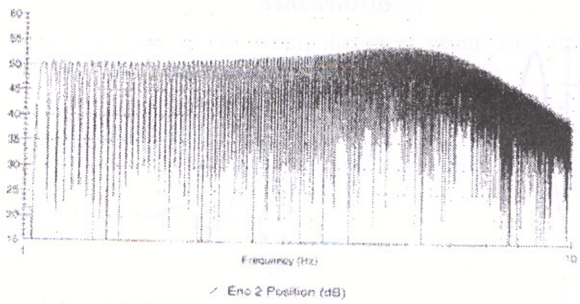

Figure 3(a). Sine Sweep Response at Encoder 2

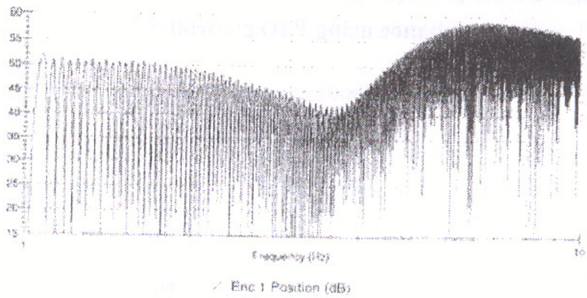

Figure 3(b). Sine Sweep Response at Encoder 1

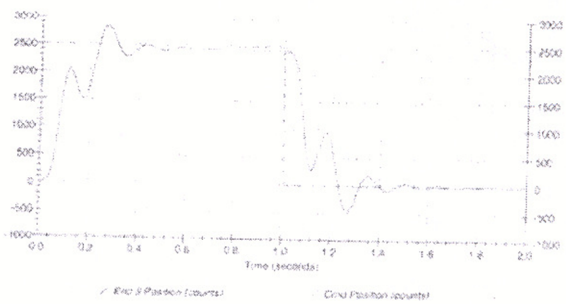

Figure 4(a). Step Response at Encoder 2 with viscous disturbance

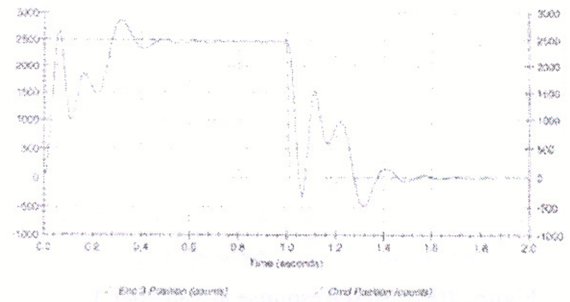

Figure 4(b). Step Response at Encoder 1 with viscous disturbance

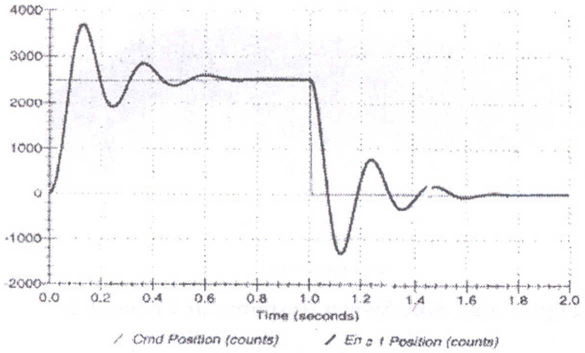

Figure 5. Step Response at Encoder 1 with viscous disturbance using PID controller

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