Investigations into non-classical logic

(Axiomatizability of spatio-temporal theories and complexity of interval-valued computations)

Doktori (Ph. D.) értekezés

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Tanúsítom, hogy Vályi Sándor doktorjelölt 2007-ben a fent megnevezett Doktori Iskola *Tudáskezelés elmélete és alkalmazásai* programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javasolom.

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Ezúton köszönetemet fejezem ki mindazoknak, akik segítettek az értekezés és az abban foglalt eredmények létrehozásában, megírásában. Ezen emberek nevei nem is férnének el egy lapra (együtt). Talán nem csak azért hosszú ez a lista, mert az eredmények elérése igen sokáig tartott. Akiket ki szeretnék emelni, sorrendiséget nem feltétlenül figyelembevevén:

– Albert Grigorjevics Dragalin
– Andréka Hajnal, Németi István
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Tartalomjegyzék

1. Introduction, motivation

In this dissertation we will investigate two recently flourishing areas of non-classical logic, namely spatio-temporal logic and interval-valued logic. In both cases our aim is to explore areas that are interesting for formalization of analog (in the sense of non-digital) computation processes.

Spatio-temporal logic is a kind of temporal logic, where the so-called chronological accessability relation plays the role of time flow. (We will denote this relation by $\blacktriangleleft$.) This logic is relevant –among other topics– for the formalization of time-dynamical properties of spatially distributed mobil systems. To deal with finite-state systems, it is enough to consider propositional spatio-temporal logic, but for more complex systems, it would be interesting to find first-order spatio-temporal theories having nice metamathematical properties. Here *nice* is to be understood – for us – at least as recursive enumerability.

To formalize non-digital processes in spacetime, the first time flow that could be relevant is $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$. Unfortunately, we will establish that over any of these flows, no reasonable first-order temporal logic is recursively enumerable (axiomatizable, to be more concise). We cannot be surprised at this, since it is known, that the corresponding first-order temporal theories over $(\mathbb{R}, <)$ are not axiomatizable. What gives more hope, is that first-order temporal theories over $(\mathbb{Q}, <)$ are known to be axiomatizable. This fact raises the question, whether first-order spatio-temporal theories over $(\mathbb{Q}^n, \blacktriangleleft)$ $(n \geqslant 2)$ are axiomatizable. In this dissertation we will establish that this holds exactly for $n = 2$.

In addition, we will determine which fragments of monadic second-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ $(n \geqslant 2)$ and $(\mathbb{R}^n, \blacktriangleleft)$ $(n \geqslant 2)$ are axiomatizable. Fragments of monadic second-order theories are important at the decidability problems of propositional spatio-temporal theories.

In a nutshell, in the first half of the thesis we determine some important limits of axiomatizability in first-order spatio-temporal theories.

Temporal logic can be naturally interpreted as non-classical logic. Our second topic, interval-valued logic (IVL) falls into the same

category, in the following sense. In both area, the possible extensions of formulæ form an infinite Boolean algebra. IVL has also a temporal logical interpretation, more closely, an interval temporal logic interpretation, which we will demonstrate in Section 13.

In the second part of the dissertation, we start with arguing why one should introduce IVL-based computations, based on Benedek Nagy's work. After this, we will give a possible formalization of interval-valued computations. We also prove an essential result relating the interval-valued computational complexity hierarchy to the classical Turing-hierarchy. Namely, we will prove that the class of languages decidable by a restricted polynomial interval-valued computation coincides with $PSPACE$.

This dissertation is organized as follows. In the first section we give a short summary of our motivations to do these pieces of work. In Section 2 we give a more detailed introduction and a summary of preliminary results – both in temporal/spatio-temporal logic and interval-valued logic. Next (Section 3) we formulate some new results on spatio-temporal logic and the definitions needed for this. In Sections 4 – 9 we give the proofs of the results announced in the previous section. After this, we formulate our new results on interval-valued logic and computing and the definitions needed for this (Section 10). In the following Section (no. 11) we also prove these results. In the section after this we try to connect the area of interval-valued logics with temporal logics by giving a temporal logical interpretation of our results on interval-valued computing. In the final section we give possible extensions of this work and suggestions possibly leading to further development of the topics of this dissertation.

2. Survey (background and new developments)

2.1. Temporal logic

The well-known „possible world" semantics of modal logic is dynamical, opposite to the static interpretation of the classical logic ([GG01]). One of the successful modal logical branches is temporal logic. It involves built-in sentential operators to describe temporal changes of semantic values of expressions. For example, *„. . . will be true in the future"* or *„. . . will be true before . . . will turn to be true"* are usual temporal operators.

In a nutshell, the main feature of temporal logic is that it draws qualitative reference to the temporal changes of truth values into the logical part of syntax and semantics. ([GHR94]). In an other paradigm, it would be possible to say explicitly, for example, that „*there is a time point* t *that . . . will be true at* t", instead of *„. . . will be true in the future"*. This paradigm would involve quantitative time references to truth values in different time points. (See a related translation in Definition 14. In the theory of specification of temporal properties of computing devices, the qualitative paradigm is more accepted ([BSz95], [WZ00]). Also temporal phenomena of natural languages are more manageable in our framework ([P57], [PN01]).

While formalizing non-parallel, single-thread processes in digital computers, obviously the well-known discretely ordered linear sets, mainly $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$, come forward to serve as an appropriate time flow. The natural time flow for continuous processes is $(\mathbb{R}, <)$ or $(\mathbb{Q}, <)$.

Branching, non-linear time temporal logic is widely used to model parallel and non-deterministic phenomena, such as future tenses of natural languages or random choice in computation sequences. Its propositional version is exploited successfully in designing reliable finite-state computing devices (see e.g. [CBES85], [CES86], [NASA]). Its full first-order version can express properties of arbitrary computing paradigms ([MP81], [BK00]). Automatic decision and/or proof-searching algorithms support automatic specification and verification of such systems (see [ANS91], [AGMNS95], [MP92] or in the subseries *Computer Aided Verification* of Lecture Notes in Computer Science).

Temporal logics nowadays are widely used in the theory of specification and verification of computational systems, they provide tools for formulating and proving dynamic properties of computational devices, either software or hardware (c.f. [ANS79], [ANS91], [ANS95], [K87]). It is also possible to write specifications in a temporal logic language directly and to allow an automated process to plan and construct an appropriate computing device (e.g. [AM87], [G87]). Temporal logics built on a first-order signature have significantly greater expressive power than logics based on a propositional one. However, the price of this power is non-axiomatizability, at least in the case of most of these theories. Here and in what follows we understand axiomatizability of a theory as its recursively enumerability.

The first-order temporal logics over classical structures as time flow (like $(\mathbb{N}, <)$, $(\mathbb{Z}, <)$, $(\mathbb{R}, <)$) are usually not axiomatizable. This is a well-known fact, which was observed first by D. Scott (see [G84]). M. Reynolds [R96] axiomatized first-order temporal theory over $(\mathbb{Q}, <)$ with the temporal operators $Until$ and $Since$ and proved its completeness in quite a novel way. In Theorem 11 we present a short argument for the axiomatizability of first-order temporal theories with arbitrary temporal connectives over $(\mathbb{Q}, <)$. $Until$ and $Since$ cannot express arbitrary temporal connectives over this time flow ([K68]), hence this result strengthens the result of [R96]. A general and simple reason of axiomatizability of a first-order temporal theory is the recursive axiomatizability and ω-categoricity of the first-order theory of the underlying time flow structure. However, due to its generality, this method does not provide an explicit axiom system in terms of axioms and deduction rules. This method is our first – quite modest – contribution to the development of *linear time* first-order temporal logic. It constitutes a part of [V07b] and was presented in [V00].

What can be more relevant is that our proof method for non-axiomatizability of some first-order spatio-temporal theories over $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ can be modificated to prove that a first-order temporal theory with a very basic first-order signature (only one monadic predicate without equality) over $(\mathbb{R}, <)$ is not axiomatizable. We do not know any proofs for this in the literature. The presented non-axiomatizability proofs utilize a three-argument signature or are not

valid for $(\mathbb{R}, <)$ ([GHR94], [HWZ00], [Me92]). This proof is given in Section 9, Subsection 9.2 (Theorem 13).

2.2. Spatio-temporal logic

What can temporal logic offer to designers of mobile distributed computing systems? Apart from having dynamics in time, these systems have dynamics in space, too. To cover this area, an analogue of temporal logic has been developed, which is usually called spatio-temporal logic. The need for appropriate knowledge representation systems has generated a big boom of investigations into this direction in the past ten years. One way to follow this is to combine a spatial language with a temporal language in such a way that in the hybrid language there are separate modalities for time and space ([RS85], [BC99], [WZ00], [WZ03], [BC02], [GN02], [GKKWZ05]). This idea originated from research on multi-dimensional modal logics ([S73], [S78], [MV97], [RZ01]). In this formalization there are separate modalities for space and time. In advance, we assert that our non-axiomatizability results on $(\mathbb{Q}^n, \blacktriangleleft)$ and $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 2)$ are not consequences of non-axiomatizability results on multi-modal logics over $(\mathbb{R}, <)$ or on products of modal logics over that time flow.

There is another long-standing tradition to deal with time and space, namely to speak jointly about spacetime and use its geometrical relations and objects to express various properties of the dynamics of processes in spacetime. Assuming that these processes have no synchronized time one come to consider hyperbolic geometry of Minkowski spacetime, as in the works of F. Mattern ([Ma92], [CM96]). He proposed investigating so-called causal connectability relations of spacetime from the viewpoint of specification and verification of distributed computing. In the present introduction we will distinguish five relations related to causality: $(x \blacktriangleleft y)$ for pure material causal connectability usually called chronological accessability in the literature, while $(x \triangleleft y)$ for optical accessability, $(x \ll y)$ for the disjunction of the previous two, $(x =\!\ll y)$ for $(x \ll y \vee x = y)$ and finally $(x =\!\blacktriangleleft y)$ stands for $x \blacktriangleleft y \vee x = y$. Exact definitions will be given when our theorems are developed.

A theory of the causal relation $\ll$ of spacetime was axiomatized as early as in 1914 by A. Robb [R14] and later on similar results were

obtained among others by B. Mundy and J. P. Ax ([M86a], [M86b], [A78]). R. Goldblatt elaborated the first-order theory of some spacetime relations – including causal relations – in [G87] and [G89]. V. Pambuccian reinterpreted the Alexandrov–Zeeman theorem concerning causality-preserving mappings from the definitional viewpoint of these relations ([P05]).

A relevant new approach of logic to causality and relativity is to axiomatize the whole relativity theory, including facts about observers, co-ordinates or even co-ordinate transformations, and not only the the geometrical core. This approach may be called the analytic version of formalized relativity theory ([AMN04]). Moreover, since this formalization does not utilize second-order or set-theoretical notions, for instance the set of real numbers, only their first-order approximations, we can call it the non-standard analytic version of formalized relativity theory in the sense of non-standard analysis.

This approach is extremely useful when we formalize real physical statements and experiments, as in [AMN98], [MNS05]. Further, in the conceptual analysis of relativity theory, it allows varying not only the axioms but even the basic vocabulary [M02]. Finally, an extra advantage can be the uniform handling of different theories of different space dimensions.

This complex theory just described can serve as a metatheory of the above mentioned first-order and propositional temporal logics and as a motivation for investigating higher-order theories concerning causal structures over other co-ordinatizing fields besides that of the real numbers.

As one can prove non-axiomatizability of a temporal logic over time flow $(\mathbb{R}, <)$ (this is the situation, see above) it does not seem much harder to refute axiomatizability over $(\mathbb{R}^n, \blacktriangleleft)$. Nevertheless, it is not a trivial consequence of non-axiomatizability results on first-order temporal theories over $(\mathbb{R}, <)$, especially for our monadic base first-order signature, as in our Theorem 9.

We can draw the lesson from this, that we cannot hope to find any nice first-order temporal logic over the real spacetime. Only a restricted fragment of first-order temporal logic, the so-called monodic fragment is decidable. ([HWZ00]). J. van Benthem drew attention to

the spacetime flow $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 1)$ in [B83]. At least for $n = 2$, he proved that its first-order theory is ω-categorical (countably categorical), finitely axiomatizable and consequently, complete and decidable. Further, that $(\mathbb{Q}^2, \blacktriangleleft)$ is an elementary substructure of $(\mathbb{R}^2, \blacktriangleleft)$ so their first-order theories coincide. Anyway, the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$ is decidable through semantic interpretation into the first-order theory of $(\mathbb{R}, +, *, <)$, which is known to be decidable by a well-known result of A. Tarski. To the best of our knowledge, for $n > 2$, the decidability of the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ has neither been proved nor disproved. The previous method does not work, since for $n > 2$, $(\mathbb{Q}^n, \blacktriangleleft)$ is not an elementary substructure of $(\mathbb{R}^n, \blacktriangleleft)$.

Any causal accessability relation of spacetime can be considered to be a generalization of time flows in temporal logic, when it serves as an alternativity relation of a Kripke frame for propositional modal logic as it was done first by V. Shehtman and R. Goldblatt, independently. In [S83] and [G80], modal logics of $(\mathbb{R}^n, =\!\!\ll)$ was proved to be decidable. The more than 20-year-long open problems of decidability and axiomatization of modal logics of the frames $(\mathbb{R}^n, \blacktriangleleft)$ were solved by Shapirovsky and Shehtman ([SS03]. Modal logics of other spacetime relations on $\mathbb{R}^n$ and more abstract spaces were analysed in [BG02] and [R97]. Modal logics of the frames $(\mathbb{Z}, \ll)$ and $(\mathbb{Z}, =\!\!\ll)$ were investigated in [P98].

Now, if we are interested in the propositional modal logic of the frame $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ then we face difficulties at this point. $(\mathbb{Q}^n, \blacktriangleleft)$ is no more isomorphic to $(\mathbb{Q}^n, L_n)$ where $(x_1, \ldots, x_n)L_n(y_1, \ldots, y_n) :\Leftrightarrow (x_1 < y_1 \wedge \ldots \wedge x_n < y_n)$. So this modal logic cannot be regarded as a product of modal logics of the frame $(\mathbb{Q}, <)$.

We have some methods of proving decidability of modal and temporal logics. The most popular one is to prove that the monadic second-order theory of the time flow structure is decidable ([R68], [GHR94]). Unfortunately, Theorem 2 shows that even the $\forall\exists$-fragment of monadic second-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is not recursively enumerable (therefore not decidable), for all $n > 1$. We measure here the quantifier complexity of subset quantifications only. What is more, in Theorem 6 and 7 we prove that the $\forall$-fragment of this theory is recursively enumerable if and only if $n = 2$. It implies that proposi-

tional temporal theories over $(\mathbb{Q}^2, \blacktriangleleft)$ are decidable but one cannot give a complete axiomatization of a propositional temporal logic over $(\mathbb{Q}^n, \blacktriangleleft)$ $((n > 2)$ with an expressively complete temporal operator set. However it does not exclude axiomatizations with some specific temporal operator set. It remains the subject of further investigations.

Thus, $(\mathbb{Q}^2, \blacktriangleleft)$ shows an interesting example when the ($\forall\exists$-fragment of) the monadic second-order theory of a structure with ω-categorical and finitely axiomatizable first-order theory is not recursively enumerable.

Theorem 7 is valid, with a little simplification in the proof, for the monadic second-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$, too. This is stated in Theorem 3. One can conclude a similar but weaker result also from S. Shelah's paper [S75] that states the (full) monadic second-order theory of $(\mathbb{R}, <)$ to be not recursively enumerable. This conclusion can be drawn by Lemma 15.5.3 (p. 567) of [GHR94]. However, our theorem strengthens this result by establishing non-axiomatizability even for the $\forall$-fragment of the monadic second-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$.

Theorem 2 can be proven according to the non-axiomatizability proofs in second-order logics (not restricted to be monadic), except for the absence of binary relations. To cope with this, we introduce some spacetime geometric objects to make possible pairing and other constructions assisting the representation of binary relations on non-negative integers. The proof of Theorem 7 is more difficult. We have developed new definitions for some spacetime geometrical relations in the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ – the most remarkable being for spacelike betweenness – and made a first-order formula which substitutes the second-order condition of the proof of Theorem 2. These definitions can be found in Section 4.

We note that Theorem 1 and 2 are not superfluous, although, if $n > 2$, they are partial cases of Theorem 7. Nevertheless, they were separately stated and proved, because they are valid also for the case $(\mathbb{Q}^2, \blacktriangleleft)$. If such a situation occurs, then it seems reasonable to construct the proof of Theorem 7 as a modification of the proof for the mentioned two theorems.

The results on monadic second-order theories appear in my paper [V07a] which is accepted for publication in Journal of Philosophical Logic.

After surveying our results on monadic second-order theories, we turn to first-order spatio-temporal theories. In [V07c] and [V07b] (Theorem 8–11) we obtain axiomatizability results on first-order spatio-temporal theories of $(\mathbb{Q}^n, \blacktriangleleft)$ and $(\mathbb{R}^n, \blacktriangleleft)$. Based on similar spacetime geometric considerations, we establish that all first-order spatio-temporal theories are axiomatizable over $(\mathbb{Q}^2, \blacktriangleleft)$ but not over $(\mathbb{F}^n, \blacktriangleleft)$ if $\mathbb{F} = \mathbb{Q}$ and $n > 2$ or $\mathbb{F} = \mathbb{R}$ and $n \geqslant 2$. As an extra technical contribution, we develop our non-axiomatizability results for a very simple first-order signature, namely, we allow only one unary predicate symbol, without the equality.

We will continue the spatio-temporal part of this dissertation with demonstrating the usefulness of our axiomatizable spatio-temporal logic (provided by Theorem 10) by showing the expressive power of this logic. We formalize a relevant property of distributed computing systems of mobile agents in it. This is a part of the paper [V07b].

Additionally, we also prove two new results. First, if $n > 2$ then the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is not ω-categorical. Second, the first-order temporal theory of time flow $(\mathbb{R}, <)$ is not recursively enumerable even we have only a very basic first-order signature: one monadic predicate symbol without the equality.

2.3. Interval-valued computations

In the second logical half of the dissertation (Section 10 – 12) we give two interesting results on a newly arisen non-classical computational theory. In the following, we give an introduction to this topic.

Based on the theory of the universal Turing machines the principle of classical computers were developed by John von Neumann. From theoretical point of view, these computers can compute everything that is Turing computable. There are some significant features of these wide-spread computers, such as sequential run and usage of binary data representation.

Although Neumann-type computers work sequentially, they work with cells (fixed length sequences of bits) and some operations, such

as the Boolean operations, are performed parallel on the bits of the cells (inner parallelism). The 'resolution level' of the CPU is measured by the numbers of bits in a cell it uses. (It also can be related to the size of the alphabet of Turing machines, it is the information unit of the sequential process.) In the last decades the number of bits of cells of computers has permanently increased. Considering the increasing bit size of cells, increasing Boolean algebras are needed to formalize the behavior of computations. Many-valued Boolean algebras with increasing number of elements can describe this situation. Instead of taking different Boolean algebras for different number of bits in cells, one can employ an infinite Boolean algebra for a uniform representation and a limit of all the cases.

The most important idealization in the Turing model of traditional computers is that the memory (built up by cells of a given number of bits) can be linearly extended in an unrestricted way. This corresponds to the straightforward model of everyday practice when one can use as much memory as one needs to solve a given problem.

In conference paper [N05b], Benedek Nagy proposed a simple discrete time / continuous space computational model, the so-called interval-valued computing. It involves another type of idealization – the density of the memory can be raised unlimitedly instead of its length. This new paradigm keeps some of the features of the traditional Neumann-Turing type computations. It works on 1-dimensional continuous data, namely, on specific subsets of the interval $[0, 1)$, more specifically, on finite unions of [)-type subintervals. This system is similar to the optical computing in [WN05] in some features.

In a nutshell, an interval-valued computation starts with $\left[0, \frac{1}{2}\right)$ and continues with a finite sequence of operator applications. It works sequentially in a deterministic manner. The allowed operations are motivated by the classical operations on finite bit sequences in traditional computers: Boolean operations and shift operations. There is only an extra operator, the product. The role of the introduced product is to connect interval-values on different 'resolution levels'. Essentially, it shrinks interval-values. So, in interval-valued computing systems, an important restriction is eliminated, i.e. there is no limit on the number of bits of a cell in the system; we have to sup-

pose only that we always have a finite number of bits. Of course, in the case of a given computation an upper bound (the bit height of the computation sequence) always exists, and it gives the maximum number of bits the system needs for that computation process. Hence our model still fits into the framework of the Church-Turing paradigm, but it faces different limitations than the classical Turing model. Although the computation in this model is sequential, the inner parallelism is extended. One can consider the system without restriction on the size of the information coded in an information unit (interval-value). It allows to increase the size of the alphabet unlimitedly in a computation.

As our results will show, interval-valued computations are suitable for dealing with polynomial space problems. First, interval-values and interval-valued computations are explained based on conference paper [N05b]. In that paper, the problem SAT was solved by a linear interval-valued computation and the question was posed, whether there are $PSPACE$-complete problems decidable by linear interval-valued computations. In conference paper [NV06] we answer this question in the affirmative, namely, we prove it for $QSAT$, the problem whether a quantified propositional formula is true or false. By observing the method of this proof, in our paper [NV07] we also determined a natural syntactic class of interval-valued computations that the resulting class of decided problems coincides with $PSPACE$.

In Section 13, we construct a connection of interval-valued computations to interval temporal logic.

3. Definitions and new results in spatio-temporal logic

We start with the definitions needed to precisely formulate our theorems. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{Q}$.

Definition 1. Let $n > 1$. We recall the definition for the function *Minkowskian distance* $\mu : \mathbb{F}^n \to \mathbb{F}$. It is defined by $\mu((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = (x_1 - y_1)^2 - (x_2 - y_2)^2 - \ldots (x_n - y_n)^2$. †

Definition 2. For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $(x \blacktriangleleft y)$ for $\mu(x, y) > 0 \wedge x_1 < y_1$. †

In special relativity theory, this relation is also known as *chronological accessability* because it holds iff there is a possibility for an event occuring in y to take a material (below-lightspeed) effect from an event in x. In this case we say also that y is inside of the upper lightcone of x.

3.1. Monadic second-order theories

First we present our results concerning second-order theories. In the present dissertation we use the standard version of the monadic second-order theory of a structure $(T, \prec)$. We give its definition below.

Definition 3. In *monadic second-order language*, we have variables $(x, y, \ldots)$ for individuals as well as variables for subsets $(X, Y, \ldots)$. There is a binary predicate symbol $(x \in X)$ connecting an individual and a subset variable and a binary predicate symbol $(x < y)$ connecting two individual variables.
The formal inductive definition of the monadic second-order terms and formulae can be given as follows. The two type of variables are given. No other terms are present in the language. The atomic formulae are given above by the predicate symbols, too. The set of monadic second-order formulae is defined as the smallest set of strings including the atomic formulae and satisfying the following inductive rules:

– if A and B are monadic second-order formulae then $(A \wedge B)$, $\neg A$ are monadic second-order formulae, too,
– if A is a monadic second-order formula and v is a variable (either individual or subset variable) then $\forall vA$ and $\exists vA$ are monadic second-order formulae, too. †

Definition 4. We obtain *standard model* $\mathcal{M}(T, \prec)$ of this language built on $(T, \prec)$ when the domain is T, the interpretation of $<$ is $\prec$, further, the variables of the second sort range over *all* subsets of T and the interpretation of $\in$ is the standard inclusion. †

Definition 5. Variable valuations can be defined as expected. A *variable valuation* into $\mathcal{M}(T, \prec)$ is no other than a finite partial mapping from the variables to $T \cup \mathcal{P}T$ satisfying that the value for an individual variable is always an element of T while the value of an set variable is always a subset of T.
Further, $\Theta@(x|s)$ denotes a valuation Σ such that $dom\ \Sigma = dom\Theta \cup \{x\}$, $\Sigma(x) = s$ but for each variable $v \in dom\ \Sigma \backslash dom\ \Theta$, $\Sigma(v) = \Theta(v)$ holds. †

For example, we write $(x, y, X||\ o_1, o_2, O_1)$ for a valuation Θ which satisfies $\Theta(x) = o_1 \in T$, $\Theta(y) = o_2 \in T$, $\Theta(X) = O_1 \subseteq T$ and $dom\ \Theta = \{x, y, X\}$. This implies that () denotes the empty evaluation.

Definition 6. The *satisfaction relation* in $\mathcal{M}(T, \prec)$ between variable valuations and monadic second-order formulae is the standard one, too, it is denoted by $\mathcal{M}(T, \prec) \models A\Theta$, where A is a formula and Θ is a variable valuation. We write in this definition only $\mathcal{M}$ instead of $\mathcal{M}(T, \prec)$.

For $A = (x < y)$, $\mathcal{M} \models A\Theta$ if and only if $\Theta(x) \prec \Theta(y)$,

For $A = (x \in X)$, $\mathcal{M} \models A\Theta$ if and only if $\Theta(x) \subseteq \Theta(X)$,

For $A = (B \wedge C)$, $\mathcal{M} \models A\Theta$ if and only if $\mathcal{M} \models B\Theta$ and $\mathcal{M} \models C\Theta$,

For $A = \neg B$, $\mathcal{M} \models A\Theta$ if and only if $\mathcal{M} \models B\Theta$ does not hold,

For $A = \forall xB$, $\mathcal{M} \models A\Theta$ if and only if $\mathcal{M} \models B\Theta@(x|t)$ holds for every $t \in T$,

For $A = \forall XB$, $\mathcal{M} \models A\Theta$ if and only if
$\mathcal{M} \models B(\Theta@(X|S))$ holds for every $S \subseteq T$.

Definition 7. The *monadic second-order theory* of $(T, \prec)$ is the set of closed monadic second-order formulae true in $\mathcal{M}(T, \prec)$. $MSOTH$ $(T, \prec)$ will abbreviate this set. We define its $\forall\exists$-fragment as the set of the formulae in this theory of form $\forall V_1 \ldots \forall V_n \exists W_1 \ldots \exists W_m \; B$, where $n \geqslant 0$, $m \geqslant 0$, $V_1, \ldots V_n$ and $W_1, \ldots, W_m$ are subset variables and B itself is free from subset quantifiers, that is, one measure only the complexity of subset quantifications. If $m = 0$ we obtain the definition of the $\forall$-fragment. For the sake of conciseness, we say a set of formulae *axiomatizable* iff it is recursively enumerable. †

We work in this dissertation only with $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ and $MSOTH$ $(\mathbb{R}^n, \blacktriangleleft)$.

Our first result can be formulated as

Theorem 1. [V00], [V07a]
For any $n > 1$, $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ is not axiomatizable. †

By a deeper complexity analysis of the previous proof we can also show

Theorem 2. [V00], [V07a]
For any $n > 1$, not even the $\forall\exists$-fragment of $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ is axiomatizable. †

By adopting our proof to $\mathbb{R}^n$ and carrying out the needed simplification, we get

Theorem 3. [V07a]
For any $n > 1$, not even the $\forall$-fragment of $MSOTH(\mathbb{R}^n, \blacktriangleleft)$ is axiomatizable. †

J. van Benthem established the following theorem.

Theorem 4. [B83]
The first-order theory of $(\mathbb{Q}^2, \blacktriangleleft)$ is both ω-categorical and recursively enumerable. †

We made a useful note with the aim to utilize the previous theorem in [V07a].

Theorem 5. *For any countable time flow $(T, \prec)$, if its first-order theory is ω-categorical and recursively enumerable then the $\forall$-fragment of $MSOTH(T, \prec)$ is also axiomatizable.* †

From the two previous items we conclude

Theorem 6. [V07a]
The $\forall$-fragment of $MSOTH(\mathbb{Q}^2, \blacktriangleleft)$ is axiomatizable. †

By a more sophisticated argument than the provided one for the $\forall\exists$-fragment, we can also prove

Theorem 7. [V07a]
For $n > 2$, not even the $\forall$-fragment of $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ is axiomatizable. †

3.2. First-order spatio-temporal theories

We continue with our results on first-order spatio-temporal theories. We assume the reader to be familiar with the basic semantic and syntactic notions of first-order logic.

Definition 8. A *temporal operator* is a triple $(\odot, k, \tau)$ where $\odot$ is a symbol, k is a positive integer and τ is a first-order formula in the signature $\mathcal{S}_k$ having a denumerably infinite set $\{t_0, t_1, \ldots\}$ of variables, a binary predicate symbol $\prec$, a finite set $\{P_1, P_2, \ldots P_k\}$ of unary predicate symbols and nothing else. Further requirement on τ is to contain exactly one parameter (namely, t_0). †

$\odot$ is the visual form of the operator, k is its arity – the operator is a connective which connects k formulae. The role of τ is to provide a sentential semantics to the operator – it describes the truth in time point t_0. We will name the operators just by their first component, to avoid unnecessary complication of notations. Two examples of temporal operators are presented:

$(G, 1, \forall t_1(t_0 \prec t_1 \rightarrow P_1(t_1))$ and
$(U, 2, \exists t_1\{t_0 \prec t_1 \wedge P_2(t_1) \wedge \forall t_2[t_0 \prec t_2 \wedge t_2 \prec t_1 \rightarrow P_1(t_2)]\}$).

The reading of the first is *P_1 holds always in the future*, while of the second is *from now on, P_1 holds until a timepoint where P_2 will come true.*

Definition 9. A *temporal language* TL_L^{Op}, based on a first-order signature L and a finite set Op of temporal operators, is the smallest set of strings (on an appropriate alphabet) satisfying the following requirements:

– any atomic formula of L is an atomic temporal formula of TL_L^{Op},
– $(A \wedge B)$ and $\neg A$ are formulæ of TL_L^{Op} , if A and B belong to that set,
– $\forall x A$ is formula of TL_L^{Op} if A is a formula of TL_L^{Op} and x is a variable of L,
– $\odot(A_1, \ldots A_k)$ is a formula of TL_L^{Op} if $(\odot, k, \tau) \in Op$ for some table τ and $A_1, \ldots A_k$ are formulæ of TL_L^{Op}.

The set of terms of TL_L^{Op} coincides with the set of terms of pure L. †

We assume usual syntactic notions – as subformula, free and quantified variable, term substitution etc., modified in the adequate way – to be understood. We accept usual abbreviations of first-order logic, as $(A \vee B)$, $(A \to B)$, $\exists x A$ etc., and use their well-known semantic properties without any extra remark.

Definition 10. A *time flow* is by definition a non-empty partially ordered set $(T, \ll)$. †

$(T, \ll)$ is the intended time notion. The essence of the semantics for temporal logic is to have time-dynamical interpretations. There are many variations on what part of signature is interpreted dynamically – all of them may find an own application area. The most simple case is if the interpretation of all the terms including interpreting domain and variable valuations are time-independent, only the predicate interpretations vary on time. While we investigate axiomatizability questions of theories of temporal logic, the chosen kind of temporal interpretation is indifferent– our following results proving or refuting axiomatizability are insensitive to this variations. So we employ the most simple formalization of first-order temporal semantics – the chosen one is of [GHR94] because it is a basic monograph of temporal logic.

Definition 11. Let L be a first-order signature and let Op be a finite set of temporal operators. A *temporal interpretation* $\mathcal{I}$ for TL_L^{Op} on the time flow $(T, \ll)$ consists of a triple $(D_\mathcal{I}, \mathcal{I}^f, \mathcal{I}^p)$ where $D_\mathcal{I}$ is a non-empty set (the time-independent domain of $\mathcal{I}$), $\mathcal{I}^f$ is a usual first-order interpretation for the terms of L while $\mathcal{I}^p$ is a function mapping a usual first-order interpretation $\mathcal{I}_t^p$ of the predicate symbols of L to each $t \in T$, where the domain of $\mathcal{I}_t^p$ is always $D_\mathcal{I}$. †

Definition 12. The definition for a *valuation* Θ of the variables of TL_L^{Op} in interpretation $\mathcal{I}$ is the same as the definition of the first-order valuation of variables of L in a first-order interpretation. It is a finite partial fuction mapping from the variables of L to $D_{\mathcal{I}}$. We denote the valuation $\{(x_1, v_1), \ldots, (x_m, v_m)\}$, as usual, by $\binom{x_1 \ldots x_m}{v_1 \ldots v_m}$. This implies, that () denotes the empty valuation. Further, $\Theta@\binom{x}{d}$ stands for a valuation Π whose domain is $dom\ \Theta \cup \{x\}$, Π and Θ agree on $dom\ \Theta \setminus \{x\}$ but $\Pi(x) = d$. †

We remind that the domain and the interpretation of the terms is constant in time.

Definition 13. The value $|t\Theta|_{\mathcal{I}}$ of the term t in the interpretation $\mathcal{I}$ after the variable valuation Θ can be defined just as in first-order case. †

The temporal satisfaction relates more objects than its classical counterpart. It involves, besides an interpretation, a variable valuation and a formula, also a time flow and an evaluation time point.

Definition 14. Let L be a first-order signature and let Op be a finite set of temporal operators. For any time flow $(T, \ll)$, any temporal interpretation $\mathcal{I}$ for TL_L^{Op}, any variable valuation Θ on $\mathcal{I}$, any time point $t(\in T)$ and any temporal formula A of the temporal language just mentioned, the *satisfaction relation* $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ is defined as follows:

- if A is an atomic formula then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff $\mathcal{I}_t^p \models A\Theta$, ($\models$ denotes the classical first-order satisfaction relation)
- if $A = (B \wedge C)$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff $(T, \ll), \mathcal{I}, \Theta, t \Vdash B$ and $(T, \ll), \mathcal{I}, \Theta, t \Vdash C$,
- if $A = \neg B$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff $(T, \ll), \mathcal{I}, \Theta, t \Vdash B$ does not hold,
- if $A = \forall x B$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff for all $d \in D_{\mathcal{I}}$, $(T, \ll), \mathcal{I}, \Theta@\frac{x}{d},\ t \Vdash B$,
- if $A = \odot(B_1, \ldots B_n)$ for a temporal operator $(\odot, n, \tau) \in Op$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff $\mathcal{B} \models \tau(t_0/t)$ where $\mathcal{B}$ is an interpretation for signature $\mathcal{S}_n$ (c.f. Def. 2.1) whose domain is T, further, $\prec^{\mathcal{B}} = \ll$ and the interpretation of P_i in $\mathcal{B}$ can be given as the subset of T consisting of time points where

B_i holds, that is, $(P_i)^{\mathcal{B}} = \{s \in T : (T, \ll), \mathcal{I}, \Theta, s \Vdash B_i\}$ for any integer i between 1 and n. †

Consequently, if $A = Until(B, C)$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff there exists an $s \in T$ such that $t \ll s$, $(T, \ll), \mathcal{I}, \Theta, s \Vdash B$ and for all $r \in T$ such that $t \ll r \ll s$, $(T, \ll), \mathcal{I}, \Theta, r \Vdash C$.
Further, if $A = GB$ then $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ iff for all $s \in T$ such that $t \ll s$, $(T, \ll), \mathcal{I}, \Theta, s \Vdash B$.

Definition 15. The *Op-temporal theory* $\mathtt{Th}_L^{Op}(T, \ll)$ of time flow $(T, \ll)$ on signature L is the set of such closed TL_L^{Op}-formulæ A, that for any temporal interpretation $\mathcal{I}$, any $t \in T$ and any variable valuation Θ, $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ holds. †

Definition 16. To be concise, we say a set S of temporal formulæ *axiomatizable* iff it is recursively enumerable. †

Definition 17. $GN := \{G, N\}$, where G is given after Definition 8 and the second operator is $(N, 1, \forall t_1(\neg\forall t_2(t_2 \ll t_0 \leftrightarrow t_2 \ll t_1) \rightarrow P_1(t_1)))$. †

N will have a special reading in our spacetime flow which is to specify later. The time has come to fix which first-order signature we prove our theorems for. We choose it as minimal as possible.

Definition 18. Signature L includes no equality symbol just one unary predicate symbol, namely, r. We postulate also that the set of variables of L includes $\{\alpha, \gamma, \delta, \varepsilon\}$ and $\{x, y, z, u, v, w\}$. †

Theorem 8. [V01], [V07c]
Let $n > 2$. $\mathtt{Th}_L^{GN}(\mathbb{Q}^n, \blacktriangleleft)$ is not axiomatizable. †

This result may be interesting in contrast with the following theorems.

Theorem 9. [V07c]
Let $n \geqslant 2$. $\mathtt{Th}_L^{GN}(\mathbb{R}^n, \blacktriangleleft)$ is not axiomatizable. †

Theorem 10. [V07b]
For any first-order signature L and arbitrary finite set of temporal operators Op, $\mathtt{Th}_L^{Op}(\mathbb{Q}^2, \blacktriangleleft)$ is axiomatizable. †

The proof of Theorem 10 can be based on the following theorem and J. van Benthem's Theorem 4.

Theorem 11. [V07b]
If the first-order theory of a countable time flow $(T, \prec)$ is ω-categorical and recursively enumerable, then for any first-order signature L and arbitrary finite set of temporal operators Op, $\mathtt{Th}_L^{Op}(T, \prec)$ is axiomatizable. †

Consequently, the first-order theories of $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ are either ω-categorical or not recursively enumerable. It is hard to imagine it to be ω-categorical without being recursively enumerable. In Subsection 9.1 we find the following theorem.

Theorem 12. The first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is not ω-categorical if $n > 2$.

In Section 9 also other additional results on first-order temporal logic and spacetime are proved. We do not know any proof of non-axiomatizability of a first-order temporal logic over the reals, with such a pure signature and operator set. For this reason we give a proof for the following theorem.

Theorem 13. $\mathtt{Th}_L^G(\mathbb{R}, <)$ is not axiomatizable.

4. Preparing results on definability in the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$

We start with showing that some relations on spacetime points are definable within the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$. The first two definitions, for $=$ and $\triangleleft$, work well for every $n \geqslant 2$. In the appendix of [G87], definitions are given for these relations starting from $(\mathbb{R}^n, <<)$, where $<<$ is the disjunction of $\blacktriangleleft$ and $\triangleleft$. All, but one of these definitions for $(\mathbb{Q}^n, \blacktriangleleft)$ follow [G87] with the difference that they are tailored to the starting point $\blacktriangleleft$ instead of $<<$. For these definitions one should check whether these defining formulae work in the case of $\mathbb{Q}^n$, too. For spacelike collinearity we have to develope a new idea. In the book cited above, space-like collinearity is defined in terms of $\triangleleft$ in the following way. Three distinct, space-like connectible points x, y, z are collinear iff $\neg\exists u\ (x \triangleleft u \wedge y \triangleleft u \wedge z \triangleleft u)$. This is not valid for the $\mathbb{Q}^n$, this method works only in the case of quadratic ordered fields, as R. Goldblatt himself stated. We developed a characterization of linear spacelike betweenness which is also valid for $(\mathbb{Q}^n, \blacktriangleleft)$, if $n > 2$. Further, we note that σ is used to denote more than one relation – the number of arguments decides which meaning is to be understood. In this subsection $\mathcal{M}_n$ denotes $\mathcal{M}(\mathbb{Q}^n, \blacktriangleleft)$. Further, throughout in this dissertation we write $\not{R}$ for the negation of R.

Definition 19.

(i) $(x = y) \rightleftharpoons \forall z(z \blacktriangleleft x \leftrightarrow z \blacktriangleleft y)$,
(ii) $(x \triangleleft y) \rightleftharpoons \forall z(y \blacktriangleleft z \rightarrow x \blacktriangleleft z) \wedge \neg x \blacktriangleleft y \wedge \neg x = y$,
(iii) $\sigma(x, y) \rightleftharpoons x \not\blacktriangleleft y \ \wedge\ x \not\triangleleft y \ \wedge\ y \not\blacktriangleleft x \ \wedge\ y \not\triangleleft x \ \wedge\ y \neq x$,
(iv) $\underline{\beta_\sigma}(x, z, y) \rightleftharpoons$
$\quad \overline{\sigma(x, y)} \wedge \forall u(x \blacktriangleleft u \wedge y \blacktriangleleft u \rightarrow z \blacktriangleleft u) \wedge$
$\quad \forall u(u \blacktriangleleft x \wedge u \blacktriangleleft y \rightarrow u \blacktriangleleft z)$,
(v) $\beta_\sigma(x, z, y) \rightleftharpoons \underline{\beta_\sigma}(x, z, y) \wedge x \neq z \wedge z \neq y$,
(vi) $\sigma(x, y, z) \rightleftharpoons \overline{\beta_\sigma}(x, y, z) \vee \beta_\sigma(x, z, y) \vee \beta_\sigma(y, x, z)$,
(vii) $cpl_\sigma(x, , y, z, w) \rightleftharpoons$
$\quad [\sigma(x, y, z) \wedge \sigma(x, w) \wedge \sigma(y, w) \wedge \sigma(z, w)] \vee$
$\quad [\neg\sigma(x, y, z) \wedge \sigma(x, y) \wedge \sigma(x, z) \wedge \sigma(y, z) \wedge$
$\qquad \{\sigma(x, y, w) \vee \sigma(x, z, w) \vee$
$\qquad \exists y' z' (\sigma(x, y, y') \wedge \sigma(x, z, z') \wedge \sigma(y', z', w))\}]$,
(viii) $\beta_\lambda(x, y, z) \rightleftharpoons x \triangleleft y \wedge y \triangleleft z \wedge x \triangleleft z$,

(ix) $(x, y||_\sigma z, w) \rightleftharpoons cpl_\sigma(x, y, z, w) \wedge \neg\exists u(\sigma(x, y, u) \wedge \sigma(z, w, u))$,
(x) $H(x, y, z) \rightleftharpoons \exists u\ v[(x, y||_\sigma z, v) \wedge (x, v||_\sigma u, z) \wedge \beta_\sigma(u, y, v) \wedge \beta_\lambda(x, y, z)$. †

Somewhat loosely, but for the sake of easier readability, we write below –for example– $\mathcal{M}_n \models p \triangleleft q$ or simply $p \triangleleft q$ instead of $\mathcal{M}_n \models (x \triangleleft y)(x, y|p, q)$.

Statement 1. The following items hold.

For each $n \geqslant 2$ and for each $p, q, r, s \in \mathbb{Q}^n$:

- (i) $\mathcal{M}_n \models p = q$ if and only if p and q coincide,
- (ii) $\mathcal{M}_n \models p \triangleleft q$ if and only if $\mu(p, q) = 0$ and $p_1 < q_1$, that is, q is on the boundary of the upper lightcone of p (directed optical accessability).

For each $n > 2$ and for each $p, q, r, s \in \mathbb{Q}^n$:

- (iii) $\mathcal{M}_n \models \sigma(p, q)$ if and only if the line joining p to q is spacelike,
- (iv) $\mathcal{M}_n \models \underline{\beta_\sigma}(p, q, r)$ if and only if p, q, r lie on a common spacelike straight line and q is between the other two points,
- (v) $\mathcal{M}_n \models \beta_\sigma(p, q, r)$ if and only if p, q, r lie on a common spacelike straight line and q is between the other two points and p,q and r are pairwise distinct,
- (vi) $\mathcal{M}_n \models \sigma(p, q, r)$ if and only if p, q and r lie on a common spacelike straight line, i.e., they are spacelikely collinear,
- (vii) $\mathcal{M}_n \models cpl_\sigma(p, q, r, s)$ if and only if p, q, r and s are spacelikely coplanar,
- (viii)$\mathcal{M}_n \models \beta_\lambda(p, q, r)$ if and only if p, q and r lightlikely collinear and q is between the other two,
- (ix) $\mathcal{M}_n \models (p, q||_\sigma r, s)$ if and only if p is spacelikely connected with q, r is also spacelikely connected with s and the straight line joining p and q is parallel to the straight line joining r to s,
- (x) If $\mathcal{M}_n \models H(p, q, r)$ then q is lightlikely between p and r, furthermore, the Euclidean distance between p and q is equal to the Euclidean distance between q and r. †

Proof. First, we give a short outline. Only the definition for $\underline{\beta_\sigma}$ [(iv)] differs from the way of defining the corresponding relations in $(\mathbb{R}^n, \blacktriangleleft)$. (However it does not imply that the other items do not need

a correctness proof in the case of $(\mathbb{Q}^n, \blacktriangleleft)$.) If a definition is a Boolean combination of expressions involving only already defined notions [(iii), (v), (vi), (viii)], then the argumentation remains the same as for $(\mathbb{R}^n, \blacktriangleleft)$. Some definitions involve quantifiers on spacetime points, namely (i),(ii), (iv), (vii), (ix) and (x). However, in (vii) and (ix) only intersections of straight lines are described by the existential quantifiers and if two rational straight lines have an intersection in $\mathbb{R}^n$, then this intersection is a rational point. In (x), q is the intersection of two diagonals of a paralellogram; one of this diagonals is joining p and r – this guarantees the equality of the Euclidean distances between p and q, q and r, resp. Nevertheless, we do not state that all such p, q and r satisfy $\mathcal{M}_n \models H(p,q,r)$. Finally, (iv) can be verified by a rather lengthy but elementary calculation.

To give a chance to check the above outlined proof, we attach the detailed proofs of the more complex items.

4.1. Detailed proofs

(i)
$\Leftarrow$ follows from the properties of the real equality relation. If $p \in \mathbb{Q}^n$ then obviously $\forall z(z \blacktriangleleft p \leftrightarrow z \blacktriangleleft p)$ holds.

$\Rightarrow$. Conversely, we prove that if $p \neq q$ then $(\mathbb{Q}^n, \blacktriangleleft) \models \neg\forall z(z \blacktriangleleft p \leftrightarrow z \blacktriangleleft q)$. If $p = q$ does not hold then there are five other cases.

- if $p \blacktriangleleft q$ then $p \blacktriangleleft q$ but $q \not\blacktriangleleft p$, similarly,
- if $q \blacktriangleleft p$ then $q \blacktriangleleft p$ but $p \not\blacktriangleleft q$,
- if $p \triangleleft q$ then $\delta := q - 1 - p_1 > 0$ and with $r := (p_1 - \frac{\delta}{2}, p - 2, \ldots, p_n)$ we have $r \blacktriangleleft p$ but $r \not\blacktriangleleft q$, similarly,
- if $q \triangleleft p$ then $\delta := p - 1 - q_1 > 0$ and with $r := (q_1 - \frac{\delta}{2}, q_2, \ldots, p_n)$ we have $r \blacktriangleleft q$ but $r \not\blacktriangleleft p$, and finally,
- if $\sigma(p,q)$ then
  - if $p_1 < q_1$ then $(p_1 - \frac{p1-q_1}{2}, p_2, \ldots, p_n) \blacktriangleleft p$ but $(p_1 - \frac{p1-q_1}{2}, p_2, \ldots, p_n) \not\blacktriangleleft q$,
  - symmetrically, if $q_1 < p_1$ then $(q_1 - \frac{q1-p_1}{2}, q_2, \ldots, q_n) \blacktriangleleft q$ but $(q_1 - \frac{q1-p_1}{2}, q_2, \ldots, q_n) \not\blacktriangleleft p$,
  - and otherwise, if $p_1 = q_1$, we can go on as follows. We know $(M :=)\mu(p,q) < 0$. Let $\varepsilon := \frac{1}{2} \cdot \sqrt{|M|}$ (consequently, $\varepsilon^2 < |M|$). Obviously, $(p_1 - \varepsilon, p_2, \ldots, p_n) \blacktriangleleft p$ but

$(p_1-\varepsilon, p_2, \ldots, p_n) \not\blacktriangleleft$ because
$\mu(q,(p_1-\varepsilon,p_2,p_n)) = ((q_1-p_1)+\varepsilon)^2-(q_2-p_2)^2-\ldots-(q_n-p_n)^2$
$= M - 2(q_1-p_1)\varepsilon + \varepsilon^2 = M + \varepsilon^2 < 0.$

(ii)

$\Leftarrow$ is clear because of the reverse Minkowski inequality.

$\Rightarrow$. By contraposition, we prove that if $p \not\blacktriangleleft q$ then
$\exists z(q \blacktriangleleft z \wedge p \not\blacktriangleleft z) \vee p \blacktriangleleft q \vee p = q$. There are no more than four cases when $p \not\blacktriangleleft q$ does not hold: $\mu(p,q) > 0$, $\mu(p,q) < 0$, $p_1 > q_1$ and $p_1 = q_1$.

- if $\mu(p,q) > 0$ then
  - if $p_1 < q_1$ then $p \blacktriangleleft q$,
  - $p_1 = q_1$ is impossible without $p = q$,
  - if $p_1 > q_1$ then $q \blacktriangleleft p$ so by valuation $\binom{z}{p}$, $\exists z(q \blacktriangleleft z \wedge p \not\blacktriangleleft z)$ holds;
- if $\mu(p,q) < 0$ then by the method of proof of (i) of this Statement we can choose an r satisfying $q \blacktriangleleft r \wedge p \not\blacktriangleleft r$;
- if $p_1 > q_1$ and $\mu(p,q) = 0$ then $q \triangleleft p$ and $\exists z(q \blacktriangleleft z \wedge p \not\blacktriangleleft z)$ holds by valuation $\binom{z}{(q_1+\frac{p_1-q_1}{2}, p_2, \ldots, p_n)}$;
- if $p_1 = q_1$ and $\mu(p,q) = 0$ then $p = q$.

(iv)

For each $n > 2$ and for each $p, q, r \in \mathbb{Q}^n$: $(\mathbb{Q}^n, \blacktriangleleft) \models \underline{\beta_\sigma}(p,q,r)$ if and only if p, q, r lie on a common spacelike straight line and q is between the other two points, that is, $\forall n > 2\ \forall p, q, r \in \mathbb{Q}^n$:

$[\sigma(p,r) \wedge (\forall s \in \mathbb{Q}^n)(s \blacktriangleleft p \wedge s \blacktriangleleft r \rightarrow s \blacktriangleleft q) \wedge$
$(\forall s \in \mathbb{Q}^n)(p \blacktriangleleft s \wedge r \blacktriangleleft s \rightarrow q \blacktriangleleft s) \Leftrightarrow$
$\exists \lambda \in \mathbb{Q} \cap [0,1]\ \forall i \in \{1, \ldots, n\} : q_i = \lambda \cdot p_i + (1-\lambda) \cdot r_i]$.

We prove this equivalence by a 4-length chain of equivalent conditions, for some arbitrarily fixed $n > 2$, p, q, $r \in \mathbb{Q}^n$.

$\sigma(p,r) \wedge (\forall s \in \mathbb{Q}^n)(s \blacktriangleleft p \wedge s \blacktriangleleft r \rightarrow s \blacktriangleleft q) \wedge$
$(\forall s \in \mathbb{Q}^n)(p \blacktriangleleft s \wedge r \blacktriangleleft s \rightarrow q \blacktriangleleft s) \ \Leftrightarrow^a$
$\sigma(p,r) \wedge (\forall s \in \mathbb{R}^n)(s \blacktriangleleft p \wedge s \blacktriangleleft r \rightarrow s \blacktriangleleft q) \wedge$
$(\forall s \in \mathbb{R}^n)(p \blacktriangleleft s \wedge r \blacktriangleleft s \rightarrow q \blacktriangleleft s) \ \Leftrightarrow^b$
$\exists \lambda \in [0,1]\ \forall i \in \{1, \ldots, n\} : q_i = \lambda \cdot p_i + (1-\lambda) \cdot r_i \ \Leftrightarrow^c$
$\exists \lambda \in \mathbb{Q} \cap [0,1]\ \forall i \in \{1, \ldots, n\} : q_i = \lambda \cdot p_i + (1-\lambda) \cdot r_i$.

The first equivalence (a) follows from the facts that first, μ and the mapping $p \mapsto p_1$ are continuous functions on $\mathbb{Q}^n$ and if $\mu(p,q) = 0$ then any open ball with center p includes a point $r_1 \in \mathbb{Q}^n$ satisfying $\mu(r_1,q) > 0$ and another point $r_2 \in \mathbb{Q}^n$ satisfying $\mu(r_2,q) < 0$. The third equivalence (c) is trivial because p, q and r are elements of $\mathbb{Q}^n$. It is enough to prove the second equivalence (b) for $p = (0,\ldots,0)$ and $r = (0,1,0,\ldots,0)$, because in $\mathbb{R}^n$, for every p and r satisfying $\sigma(p,r)$ there exists an affine transformation $\mathbb{R}^n \to \mathbb{R}^n$ preserving both $\blacktriangleleft$ and betweenness and taking $(0,\ldots,0)$ to p and $(0,1,0,\ldots,0)$ to q. This transformation can be composed from the following transformations: a uniform expansion, a distance-preserving transformation which leaves the time co-ordinate fixed, a Lorentz-transformation in specific configuration (changing only the first two co-ordinates) and finally, a translation.

So, what is left to prove, is the following. Let $q \in \mathbb{Q}^n$. Then
$(\forall s \in \mathbb{R}^n)(s \blacktriangleleft (0,\ldots 0) \wedge s \blacktriangleleft (0,1,0,\ldots,0) \to s \blacktriangleleft q)$
$\quad \wedge(\forall s \in \mathbb{R}^n)((0,\ldots 0) \blacktriangleleft s \wedge (0,1,0,\ldots,0) \blacktriangleleft s \to q \blacktriangleleft s)$
$\Leftrightarrow$
$\exists \lambda \in [0,1] : q = (0,\lambda,0,\ldots,0)$.

In algebraic form, it amounts to prove that for any $(q_1,\ldots,q_n) \in \mathbb{Q}^n$:
$[(\forall s \in \mathbb{R}^n)(s_1^2 > s_2^2 + \ldots + s_n^2 \wedge s_1 < 0 \wedge s_1^2 > (s_2-1)^2 + \ldots + s_n^2 \wedge s_1 < 0 \to (s_1-q_1)^2 > (s_2-q_2)^2 + \ldots + (s_n-q_n)^2 \wedge s_1 < q_1)$
$\wedge(\forall s \in \mathbb{R}^n)(s_1^2 > s_2^2 + \ldots + s_n^2 \wedge s_1 > 0 \wedge s_1^2 > (s_2-1)^2 + \ldots + s_n^2 \wedge s_1 > 0 \to (s_1-q_1)^2 > (s_2-q_2)^2 + \ldots + (s_n-q_n)^2 \wedge s_1 > q_1)]$
$\Leftrightarrow$
$q_1 = 0 \wedge q_3 = 0 \wedge \ldots \wedge q_n = 0 \wedge 0 \leqslant q_2 \leqslant 1$.

We have checked this equivalence by a rather lengthy calculation.

Case $\Rightarrow$. First we justify that $q_1 = 0$. For this, first we assume that $q_1 < 0$ and we will deduce a contradiction.

It is enough to prove that if $q_1 \neq 0$ then
$\exists s_1,\ldots,s_n[s_1 < 0 \wedge s_1^2 - (s_2^2 + \ldots + s_n^2) > 0 \wedge$
$\quad s_1^2 - (s_2^2 + \ldots + s_n^2) > 1 - 2s_2 \wedge$
$\quad s_1^2 - (s_2^2 + \ldots + s_n^2) \leqslant 2(s_1q_1 - (s_2q_2 + \ldots + s_nq_n) + q_2^2 + \ldots + q_n^2 - q_1^2]$.

If $q_1 < 0$ then by restriction $s_2 = \frac{1}{2}$ and $s_1 = -\sqrt{\frac{5}{4} + s_3^2 + \ldots + s_n^2}$ it is enough to show that $(\star)\exists s_3, \ldots, s_n[1 \leqslant -2q_1\sqrt{\frac{5}{4} + s_3^2 + \ldots + s_n^2} - q_2 - 2(s_3q_3 + \ldots + s_nq_n) + q_2^2 + \ldots + q_n^2 - q_1^2]$.

Let R denote the relation $<$ if $q_3 > 0$ and $\geqslant$ otherwise. Then, by restricting $s_4, \ldots, s_n$ to 0, it is enough to provide an $s_3 R 0$ that also satisfies $1 \leqslant -2q_1\sqrt{\frac{5}{4} + s_3^2} - q_2 - 2s_3q_3 + q_2^2 + \ldots + q_n^2 - q_1^2$. But it is extremely simple by selecting an s_3 with a big enough absolute value since $-2q_1$ is positive.

In a similar way one can conclude the contradictory
$\exists s_1, \ldots, s_n[s_1 > 0 \wedge s_1^2 - (s_2^2 + \ldots + s_n^2) > 0 \wedge$
$s_1^2 - (s_2^2 + \ldots + s_n^2) > 1 - 2s_2 \wedge$
$s_1^2 - (s_2^2 + \ldots + s_n^2) \leqslant 2(s_1q_1 - (s_2q_2 + \ldots + s_nq_n) + q_2^2 + \ldots + q_n^2 - q_1^2]$
from the assumption $q_1 > 0$. In such a way we have established that $q_1 = 0$.

After this, we prove that $q_3 = \ldots = q_n = 0$. If $q_3 \neq 0$, for a contradiction, by the same specializing it is enough to prove $(\star)$. Then it is easy to see by restricting $s_4, \ldots, s_n$ to 0 and choosing a big enough or small enough s_3 (it depends on the signum of q_3). The same method works for $i \in \{4, \ldots, n\}$ for proving $q_i = 0$.

Finally, we have to show $0 \leqslant q_2 \leqslant 1$. In algebraic form, we have to validate that

$[(\forall s \in \mathbb{R}^n)(s_1^2 > s_2^2 + \ldots + s_n^2 \wedge s_1 < 0 \wedge s_1^2 > (s_2 - 1)^2 + \ldots + s_n^2 \rightarrow$
$s_1^2 > (s_2 - q_2)^2 + s_3^2 + \ldots + s_n^2)$
$\wedge(\forall s \in \mathbb{R}^n)(s_1^2 > s_2^2 + \ldots + s_n^2 \wedge s_1 > 0 \wedge s_1^2 > (s_2 - 1)^2 + \ldots + s_n^2 \rightarrow$
$s_1^2 > (s_2 - q_2)^2 + s_3^2 + \ldots + s_n^2)]$
$\Rightarrow$
$q_2 \geqslant 0 \wedge q_2 \leqslant 1.$

First, assume $q_2 < 0$. For a contradiction, it is enough to provide $s_1, \ldots, s_n$ such that $s_1^2 - (s_2^2 + \ldots + s_n^2) > 0 \wedge s_1^2 - (s_2^2 + \ldots + s_n^2) > 1 - 2s_2 \wedge s_1^2 - (s_2^2 + \ldots + s_n^2) \leqslant -2s_2q_2 + q_2^2$ holds. By fixing $s_3 = \ldots = s_n = 0$, $s_1 = \sqrt{1 + s_2^2}$ and $s_2 > 0$, it is enough to give an $s_2 > 0$ satisfying $1 - q_2^2 \leqslant (-2q_2)s_2$ which is trivial by taking a big enough s_2.

Second, assume $q_2 > 1$. As we did above, we can fix $s_3, \ldots, s_n$ as 0. For contradiction, we have to provide such s_1 and s_2 that $s_1^2 - s_2^2 > 0 \wedge s_1^2 - s_2^2 > 1 - 2s_2$ but $s_1^2 - s_2^2 \leqslant q_2^2 - 2s_2q_2$ holds. By fixing $s_2 < 0$ and $s_1 = \sqrt{(s_2 - 1)^2 - (q_2 - 1)s_2}$ (consequently, $s_1^2 - s_2^2$ is fixed to $1 - s_2(q_2 + 1)$), we get the needed inequalities in the following form.

$0 < 1 - s_2(q_2 + 1) = s_1^2 - s_2^2$,
$2 < q_2 + 1 \Rightarrow -2s_2 < -(q_2 + 1)s_2 \Rightarrow 1 - 2s_2 < s_1^2 - s_2^2$ and
$s_2 \leqslant q_2 + 1 \Rightarrow s_2(q_2 - 1) \leqslant q_2^2 - 1 \Rightarrow 1 - s_2(q_2 + 1) \leqslant q_2^2 - 2s_2q_2 \Rightarrow s_1^2 - s_2^2 \leqslant q_2^2 - 2s_2q_2$.

Case $\Leftarrow$. Let λ be an arbitrary element of $[0, 1]$. We have to prove that
$(\forall s \in \mathbb{R}^n)(s \blacktriangleleft (0, \ldots, 0) \wedge s \blacktriangleleft (0, 1, 0, \ldots, 0) \rightarrow s \blacktriangleleft (0, \lambda, 0, \ldots, 0))$
and
$(\forall s \in \mathbb{R}^n)((0, \ldots, 0) \blacktriangleleft s \wedge (0, 1, 0, \ldots, 0) \blacktriangleleft s \rightarrow (0, \lambda, 0, \ldots, 0) \blacktriangleleft s)$.

The proof of the second conjunct is constructed in an analogous way to the first, so we omit it. Let us consider an element $(s_1, \ldots, s_n)$ of $\mathbb{R}^n$ such satisfies the above two conditions. In algebraic form, these conditions appear as $s_1 < 0 \wedge s_1^2 - (s_2^2 + \ldots + s_n^2) > 0 \wedge s_1^2 - ((s_2 - 1)^2 + s_3^2 + \ldots + s_n^2) > 0$. The only thing that is left to show that $s_1^2 - ((s_2 - \lambda)^2 + s_3^2 + \ldots + s_n^2) > 0$. By cases, if $s_2 \leqslant \lambda$ then $s_2 \leqslant \frac{\lambda+1}{2}$, $(s_2 - \lambda)^2 \leqslant (s_2 - 1)^2 < s_1^2 - (s_3^2 + \ldots + s_n^2)$ and $s_1^2 - ((s_2 - \lambda)^2 + s_3^2 + \ldots + s_n^2) > 0$ else if $s_2 > \lambda$ then $\lambda(\lambda - 2s_2) < 0$, $(s_2 - \lambda)^2 < s_2^2$ and again we get $s_1^2 - ((s_2 - \lambda)^2 + s_3^2 + \ldots + s_n^2) > 0$.

(vii), (ix)

They can be established by the same reasoning as did it Professor Goldblatt for $(\mathbb{R}^n, \blacktriangleleft)$ in [G87]. This is because the existential quantifiers concern intersection of linear subsets of $\mathbb{Q}^n$ and from this point of view there is no difference between $\mathbb{Q}^n$ and $\mathbb{R}^n$.

(x)

In $(\mathbb{R}^n, \blacktriangleleft)$ H expresses the halfpoint property adequatly, it describes that x, y and z constitute a diagonal of a parallelogram. It remains true also for $(\mathbb{Q}^n, \blacktriangleleft)$ that if $H(p, q, r)$ then q is the halfpoint of sector pr. Only validity of the reverse direction vanishes.

The other items ((iii), (v), (vi) and (viii)) are Boolean combinations of previously defined notions and the reasoning can be borrowed from the respective reasonings valid also for $(\mathbb{R}^n, \blacktriangleleft)$ ([G87]). $\surd$

5. Proofs of theorems on monadic second-order theories

Definition 20. We define $\cup(x, y; z)$ as $(x \lhd z \wedge y \lhd z)$. †

We do not continue providing the concrete formulae expressing our intuitive description below. We leave this to the reader. Further, we note that the existence and uniqueness of this upper bound z for x and y is not guaranteed, except for the case $n = 2$ (one space and one time dimension). This is a consequence of the fact that the intersection of light-cones of two rational points (elements of $\mathbb{Q}^n$) is allowed not to include rational points, if $n > 2$. (At the same time, this fact prevents to exist any elementary equivalence between $(\mathbb{Q}^n, \blacktriangleleft)$ and $(\mathbb{R}^n, \blacktriangleleft)$ if $n > 2$). In this dissertation we do not use the latter fact so we did not emphasize it as a separate statement.)

Definition 21. Let ν be the conjunction of the monadic second-order formulae ν_1, ν_2, ν_3 defined below (Definitions 22,23,25). We postulate that besides the variables given in the definition of monadic second-order language (Definition 3) we also have set variables N_1, N_2, N_{12} , N and two point variables n_1, n_2. †

Definition 22. ν_1 is the monadic second-order formula expressing the following.

For each $i \in \{1, 2\}$:

- (i) N_i is a discrete linear ordering with respect to $\lhd$ with the minimum n_i but without any upper bound,
- (ii) Every lower Dedekind-cut Y (with respect to $\lhd$) of the light-like line M through N_i has the following property: there exist two consecutive points x, y in N_i such that $x \in Y \wedge y \notin Y$ holds. †

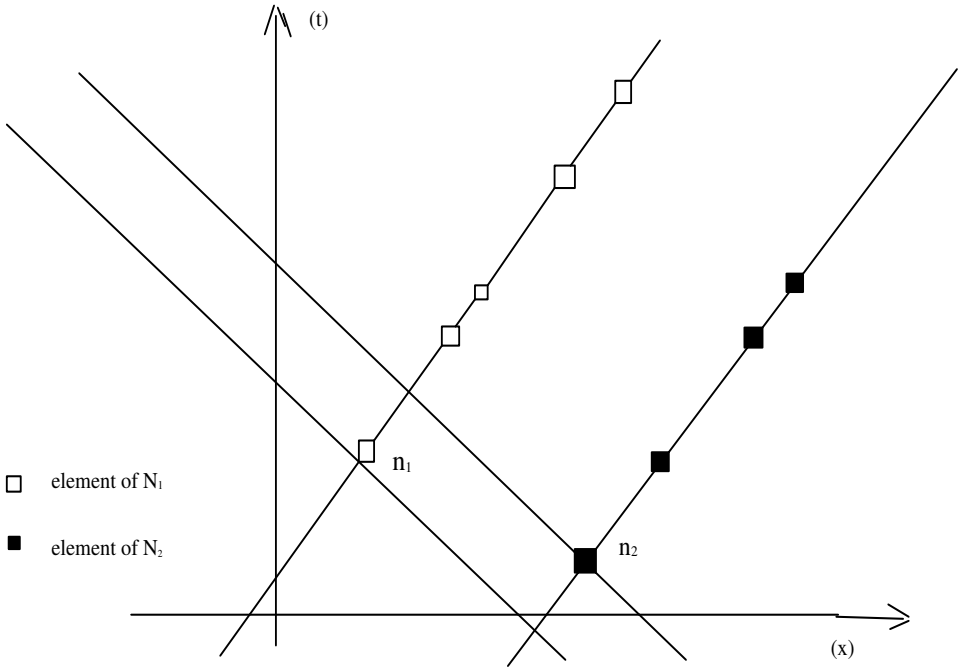
In [V07a] and also in this dissertation, ν_1 (ii) forms the only exception where we provided at least an outline of the second-order formula expressing it. Formalization of the other parts of ν can be executed in the framework of first-order logic (no other subset quantification is necessary).

$\forall M[N_i \subseteq M \wedge \forall x(x \in M \leftrightarrow (\forall y \in N_i)(y \lhd x \vee x \lhd y \vee x = y)) \rightarrow$
$\forall Y\{Y \subset M \wedge Y \neq \emptyset \wedge (\forall x \in M)(\forall y \in Y)(x \lhd y \rightarrow x \in Y) \rightarrow$

$\exists xy(x \lhd y \wedge x \in N_i \cap Y \wedge y \notin Y \wedge y \in N_i \wedge \forall u(x \lhd u \lhd y \rightarrow u \notin N_i)\}]$.

†

For the sake of the easier interpretability but not of any kind of visual reasoning we attach a picture of an outline of a possible –quite general– model of ν_1. Of course, only a finite starting cut of N_1 and N_2 fitted to the picture. At this stage the possibility is not excluded when N_1 and N_2 are in coincidence or at least lay on a common light line.



1. ábra. a model for ν_1

Statement 2. ν_1 makes $(N_i, \lhd)$ isomorphic to $(\mathbb{N}, <)$. More rigorously, for $i \in \{1, 2\}$ and $n > 1$, if $\mathcal{M}_n \models \nu_1 \Pi$ for a monadic variable valuation Π, then the structure $(\Pi(N_i), \lhd)$ is isomorphic to $(\mathbb{N}, <)$. This holds since $\nu_1(ii)$ prevents $\Pi(N_i)$ from having any accumulation point even in $\mathbb{R}^n$. Moreover, there exists a unique enumeration validating this, denoted by $k_i^{\Pi} : \mathbb{N} \rightarrow \mathbb{Q}^n$. This enumeration can be

defined by the following. Let $k_1^{\Pi}(0)$ be the minimal element of $\Pi(N_1)$, that is, $\Pi(n_1)$ and let $k_1^{\Pi}(m+1)$ be the $\lhd$-successor of $k_1^{\Pi}(m)$ in N_1 ($k \geqslant 1$). In that case for all natural numbers m, j: $k_i^{\Pi}(m) \lhd k_i^{\Pi}(j)$ iff $m < j$. †

Proof. Assume the hypothesis of the statement. The only question is how ν_1 (ii) prevents $\Pi(N_i)$ from having an accumulation point in $\mathbb{R}^n$. For a contradiction, assume that $p \in \mathbb{R}^n$ is such an accumulation point. We can see that p must be on the light-like straight line l containing all the points of N_i. If we consider valuation $\Pi' := \Pi@(M, Y|l, \{r \in l : r \lhd p\})$ then

$\mathcal{M}_n \models (N_i \subset M) \wedge \forall x(x \in M \leftrightarrow (\forall y \in N_i)(y \lhd x \vee x \lhd y \vee x = y))\ \Pi'$ (1) and

$\mathcal{M}_n \models (y \subset M \wedge Y \neq \emptyset \wedge (\forall x \in M)(\forall y \in Y)(x \lhd yx \in Y))\ \Pi'$ (2) but

$\mathcal{M}_n \models \neg\exists xy(x \lhd y \wedge x \in N_i \cup Y \wedge y \notin Y \wedge y \in N_i \wedge$
$\forall u(x \lhd u \lhd y \rightarrow u \notin N_i))\ \Pi'$, so

$\mathcal{M}_n \not\models \nu_1(\text{ii})\Pi'$.

(1) is trivial, (2) is obvious if one consider that $\Pi'(M)$ is a lightlike line and if (3) wouldn't be true then p couldn't be an accumulation point of $\Pi'(N_i)$.

The properties of k_1^{Π} are obvious by its definition. The same is true for k_2^{Π}.$\surd$

Definition 23. ν_2 is the conjunction of (i)-(iv), where

- (i) $(\forall x \in N_1)(\forall y \in N_2)$: x and y are both $\lhd$- and $\blacktriangleleft$-incomparable $\wedge\ x \neq y$,
- (ii) $(\forall x \in N_1)(\forall y \in N_2)(\exists! z \in N_{12}) \cup (x, y; z)$,
- (iii) $(\forall z \in N_{12})(\exists! x \in N_1)(\exists! y \in N_2) \cup (x, y; z)$,
- (iv) $(\forall x_1, x_2 \in N_1)(\forall y_1, y_2 \in N_2)(\forall z_1, z_2 \in N_{12})$:
  $\cup(x_1, y_1; z_1) \wedge \cup(x_2, y_2; z_2) \rightarrow$
  $[(x_1 \lhd x_2 \wedge y_1 \lhd y_2 \rightarrow z_1 \blacktriangleleft z_2) \wedge$
  $(x_1 \lhd x_2 \wedge y_1 = y_2 \rightarrow z_1 \lhd z_2) \wedge$
  $(x_1 = x_2 \wedge y_1 \lhd y_2 \rightarrow z_1 \lhd z_2)]$. †

The use of $\exists!$ is legitimate since we have already defined $=$.

Statement 3. For $n > 1$ and for any monadic variable valuation Π, if $\mathcal{M}_n \models (\nu_1 \wedge \nu_2)\Pi$ then $\Pi(N_{12})$ can be bi-enumerated in a unique way as $\{n^{\Pi}_{t,s} | (t,s) \in \mathbb{N}^2\}$ that the following properties hold for any $m, j, t, s \in \mathbb{N}$:
$\cup(k^{\Pi}_1(m), k^{\Pi}_2(j); n^{\Pi}_{t,s})$ iff $m = t \wedge j = s$, further,
$n^{\Pi}_{m,j} = n^{\Pi}_{t,s}$ iff $m = t \wedge j = s$,
$n^{\Pi}_{m,j} \lhd n^{\Pi}_{t,s}$ iff $(m = t \wedge j < s) \vee (m < t \wedge j = s)$, finally,
$n^{\Pi}_{m,j} \blacktriangleleft n^{\Pi}_{t,s}$ iff $m < t \wedge j < s$. †

Definition 24. A virtual partial term $\bar{\cup}(x, y)$ can be introduce for the unique $z \in \mathbb{Q}^n$ for which $z \in N_{12} \wedge \cup(x, y; z)$ holds. Partiality means that this notion can be only used only for those x and y satisfy $x \in N_1 \wedge y \in N_2$. Use of this virtual term is justified by $\nu_2(ii)$. †

Statement 4. If a valuation Π makes $\nu_1 \wedge \nu_2$ true then the semantic value of $\bar{\cup}$ is such a partial function that its domain is $\Pi(N_1) \times \Pi(N_2)$ and $(\forall x \in N_1)(\forall y \in N_2)(\forall z \in N_{12}) : \cup(x, y; z) \leftrightarrow z = \bar{\cup}(x, y)$ is true in Π. †

The virtual terms above have been introduced only for the sake of convenience. Evidently, such virtual terms can be simulated in first-order logic.

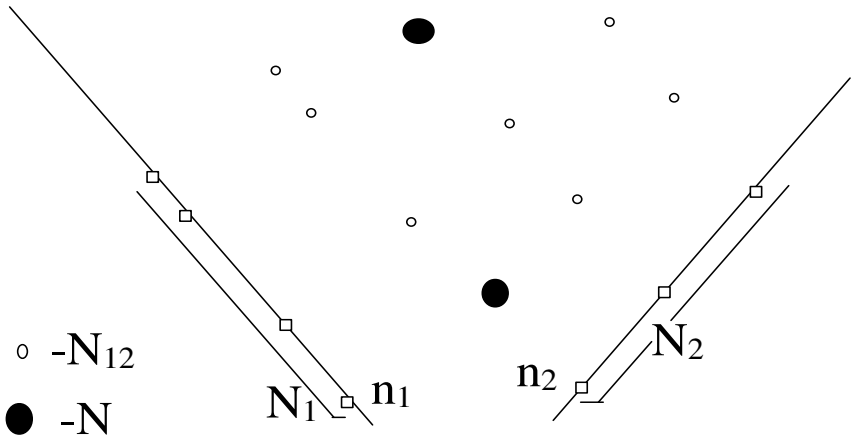
Definition 25. ν_3 expresses the following properties:

- (i) $\bar{\cup}(n_1, n_2) \in N$,
- (ii) $N \subset N_{12}$,
- (iii) $(\forall x \in N_1)(\exists! y \in N_2)\ \bar{\cup}(x, y) \in N$,
- (iv) $(\forall y \in N_2)(\exists! x \in N_1)\ \bar{\cup}(x, y) \in N$.
- (v) $(\forall x_1, x_2 \in N_1)(\forall y_1, y_2 \in N_2)$
  $[\{\bar{\cup}(x_1, y_1) \in N \wedge \neg\exists z(x_1 \lhd z \lhd x_2 \wedge z \in N_1) \wedge \neg\exists z(y_1 \lhd z \lhd y_2 \wedge z \in N_2)\} \rightarrow$
  $\bar{\cup}(x_2, y_2) \in N]$. †

Definition 26. Another virtual partial term $p(x)$ can be introduced for the unique y for which $y \in N_2 \wedge \bar{\cup}(x, y) \in N$ holds when $x \in N_1$. By $\nu_3(iii)$, it is legitimate to use $p(x)$ when $\nu \wedge x \in N_1$ holds in a given valuation. †

Statement 5. If ν holds for a monadic variable valuation Π, then the semantic value of p in Π is a bijection from $\Pi(N_1)$ onto $\Pi(N_2)$ that satisfies $\bar{\cup}(x, p(x))$ whenever $x \in N_1$. Moreover, for all $t, s \in \mathbb{N}$, $p(k_1^{\Pi}(t)) = k_2^{\Pi}(t)$ and $n_{t,s}^{\Pi} = \bar{\cup}(k_1^{\Pi}(t), p(k_1^{\Pi}(s))$ hold. †

In Figure 2 We present (a finite part of) a model of ν.



2. ábra. A model for ν

Statement 6. For $n > 1$, ν is satisfiable in the standard model $\mathcal{M}_n$. †

Proof. This can be justified by $\mathcal{M}_n \models \nu\Pi_0$ for the valuation Π_0 given below.

For any non-negative integers t and s we define the following objects:

$k_{1t} := (t+1, 0, ..., 0, -t-1)$, $k_{2t} := (t+1, 0, ..., 0, t+1)$,

$n_{ts} := (t+s+2, 0, ..., 0, s-t)$,

and then we define Π_0 as follows:

$\Pi_0(N_1) := \{k_{1t} | t \in \mathbb{N}\}$, $\Pi_0(N_2) := \{k_{2t} | t \in \mathbb{N}\}$,

$\Pi_0(N_{12}) := \{n_{ts} | t, s \in \mathbb{N}\}$, $\Pi_0(N) := \{n_{tt} | t \in \mathbb{N}\}$,

$\Pi_0(n_1) := k_{10}$ and $\Pi_0(n_2) := k_{20}$.

One should check the following conditions for all non-negative integers t, r, s, q:

$k_{1t} \triangleleft k_{1(t+r+1)}$, $k_{2t} \triangleleft k_{2(t+r+1)}$, $k_{1t} \not\triangleleft k_{2s}$, $k_{2s} \not\triangleleft k_{1t}$, $k_{1t} \blacktriangleleft k_{2s}$, $k_{2s} \blacktriangleleft k_{1t}$, $k_{1t} \neq k_{2s}$

and $(\cup(k_{1t}, k_{2s}; n_{rq})$ iff $t = r$ and $s = q)$. $\surd$

Let us recall that the *dyadic second-order language* of a structure $(T, <)$ contains variables $(x, y, \ldots)$ for elements, variables $(X, Y, \ldots)$ for binary relations on T, a binary predicate symbol $(x < y)$ connecting two individual variables and a ternary predicate symbol $((x, y) \in X)$ connecting two individual variables and a relation variable.

We describe its *standard model* $\mathcal{N}$ over $(\mathbb{N}, <)$. The domain of $\mathcal{N}$ is $\mathbb{N}$ and the interpretation of $<$ is the ordering of $\mathbb{N}$, further, relation variables and symbol $\in$ are used in the standard way – *all* the binary relations on $\mathbb{N}$ are considered.

The *satisfaction relation* of $\mathcal{N}$ between dyadic variable valuations Θ and dyadic formulae A is defined in the expected way.

The *dyadic second-order theory* of $(\mathbb{N}, <)$ is the set of the true closed formulae of $\mathcal{N}$. †

Statement 7. Even the $\forall$-fragment of the dyadic second-order theory of $(\mathbb{N}, <)$ is not recursively enumerable, where only the complexity of relation quantifications is measured. †

This is a well-known fact. I try to recall of Professor Albert Dragalin's proof from 1996 in a lecture note about „Automata and complexity of theories". By the way, it is the first, natural way of proof one should try.

Proof. We express the relations of addition and multiplication by some dyadic second-order formulæ. We prepare for this by some definitions.

$x = y$, $x \leqslant y$ etc. can be defined in the well-known first-order way.
$null(x) \leftrightharpoons \forall z(x \leqslant z)$, this just expresses that $x = 0$.
$s(x, y) \leftrightharpoons x < y \wedge \neg\exists z(x < z \wedge z < y)$ says that y is the successor of x.

Now we are ready to provide the promised formulæ.
$+^r(m, n, k) \leftrightharpoons$
$\exists u, A :$
$null(u) \wedge$
$\forall x \exists! y(x, y) \in A \wedge$
$(u, n) \in A \wedge$
$\forall x, y, x', y'((x, y) \in A \wedge s(x, x') \wedge s(y, y') \rightarrow (x', y') \in A) \wedge$
$(m, k) \in A.$
$\cdot^r(m, n, k) \leftrightharpoons$
$\exists u, M :$
$null(u) \wedge\ (u, u) \in M$
$\forall x \exists! y(x, y) \in M \wedge$
$\forall x, y, x', z((x, y) \in M \wedge s(x, x') \wedge +^r(y, m, z) \rightarrow (x', z) \in M) \wedge$
$(n, k) \in M.$

It can be validated by induction that $+^r$ and $\cdot^r$ are the graphs of addition and multiplication, respectively. After this, a direct translation of true first-order arithmetics (the first-order theory of structure $(\mathbb{N}, +, \cdot)$) is possible so our dyadic second-order theory is far from being recursively enumerable. $\surd$

We will represent below the standard dyadic second-order model $\mathcal{N}$ over $(\mathbb{N}, <)$ in the standard monadic second-order model $\mathcal{M}_n$ $(n > 1)$ by a translation m of the dyadic formulae and dyadic variable valuations.

Definition 27. $(x < y)^m, (A \wedge B)^m, (\neg A)^m$ are just $(x \lhd y)$, $(A^m \wedge B^m)$ and $\neg A^m$, respectively; let $((x, y) \in X)^m$ be the monadic formula expressing $\bar{\cup}(x, p(y)) \in X$; let $(\forall x A)^m$ be the monadic formula $(\forall x \in N_1)A^m$; let $(\forall X A)^m$ be the monadic formula $(\forall X \subseteq N_{12})A^m$. †

Definition 28. We postulate that the set of variables of the monadic language consists of the disjoint union of the set of the variables of

the dyadic language with $\{N_1, N_2, N_{12}, N, n_1, n_2\}$ (the set of parameters of ν).
We associate a monadic variable valuation $\Sigma = (\Theta + \Pi)^m$ with every pair of a dyadic variable valuation Θ and a monadic variable valuation Π satisfying ν in the following way:

- (i) for any parameter v of ν, $\Sigma(v) = \Pi(v)$,
- (ii) for any individual variable x of the dyadic language, $\Sigma(x) = k_1^{\Pi}(\Theta(x))$,
- (iii) for any relation variable X of the dyadic language, $\Sigma(X) = \{n_{t,s}^{\Pi} | (t,s) \in \Theta(X)\}$,

  where k_1^{Π} is the enumeration of $\Pi(N_1)$ defined in Statement 2 and n^{Π} is the bi-enumeration of $\Pi(N_{12})$ as given in Statement 3. †

Lemma 1 (Main lemma). If $n > 1$, then for all dyadic formulae A, the following holds:
if Θ is dyadic and Π is a monadic variable valuation under which $\mathcal{M}_n \models \nu\Pi$ holds, then $\mathcal{N} \models A\Theta$ iff $\mathcal{M}_n \models A^m(\Theta + \Pi)^m$. †
Proof. We apply a structural induction on the complexity of A. We write only $\mathcal{M}$ instead of $\mathcal{M}_n$ in this proof.

- * for an atomic formula $A = x < y$,
  $\mathcal{N} \models (x < y)\Theta \Leftrightarrow \Theta(x) < \Theta(y) \Leftrightarrow^{(a)} k_1^{\Pi}(\Theta(x)) \lhd k_1^{\Pi}(\Theta(y)) \Leftrightarrow$
  $\mathcal{M} \models (x < y)^m(\Theta + \Pi)^m$.
  The equivalence (a) follows from Statement 2.
- * for an atomic formula $A = (x, y) \in X$,
  $\mathcal{N} \models A\Theta \Leftrightarrow (\Theta(x), \Theta(y)) \in \Theta(X) \Leftrightarrow n^{\Pi}_{\Theta(x),\Theta(y)} \in \{n_{t,s}^{\Pi} | (t,s) \in \Theta(X)\} \Leftrightarrow^{(b)}$
  $\mathcal{M} \models (\bar{\cup}(x, p(y)) \in X)(x, y, X || k_1^{\Pi}(\Theta(x)), k_1^{\Pi}(\Theta(y)), \{n_{t,s}^{\Pi} | (t,s) \in \Theta(X)\} \Leftrightarrow$
  $\mathcal{M} \models A^m(\Theta + \Pi)^m$.
  (b) can be concluded from Statement 5.
- * for $A = (B \wedge C)$ and $A = \neg B$,

  the proof can be completed in a straightforward way by the induction hypothesis.

  For example, $\mathcal{M} \models (\neg A)^m(\Theta + \Pi)^m \Leftrightarrow \mathcal{M} \not\models A^m(\Theta + \Pi)^m \Leftrightarrow$
  $\mathcal{N} \not\models A\Theta \Leftrightarrow \mathcal{N} \models (\neg A)\Theta$.

– * for $A = \forall xB$,
$\mathcal{M} \models (\forall xB)^m(\Theta + \Pi)^m \Leftrightarrow$
$\mathcal{M} \models \forall x(x \in N_1 \to B^m)(\Theta + \Pi)^m \Leftrightarrow$
for all $s \in \mathbb{Q}^n$, $\mathcal{M} \models x \in N_1 \to B^m(\Theta + \Pi)^m@(x|s) \Leftrightarrow$
for all $s \in \Pi(N_1)$, $\mathcal{M} \models B^m(\Theta + \Pi)^m@(x|s) \Leftrightarrow^{(c)}$
for all $i \in \mathbb{N}$, $\mathcal{M} \models B^m(\Theta + \Pi)^m@(x|k_1^{\Pi}(\Theta(i)) \Leftrightarrow^{(d)}$
for all $i \in \mathbb{N}$, $\mathcal{M} \models B^m(\Theta@(x|i) + \Pi)^m \Leftrightarrow$
for all $i \in \mathbb{N}$, $\mathcal{N} \models B\Theta@(x|i) \Leftrightarrow$
$\mathcal{N} \models (\forall xB)\Theta$,

where for any valuation Σ, variable x and value D, $\Sigma@(x|D)$ denotes the valuation Γ whose domain is $\mathrm{dom}(\Sigma) \cup \{x\}$, further, it agrees with Σ on variables different from x, but $\Gamma(x) = D$. Equivalence (c) holds because of Statement 2, while Definition 11 explains (d).

– *for $A = \forall XB$,
$\mathcal{M} \models (\forall XB)^m(\Theta + \Pi)^m \Leftrightarrow$
for all $R \subseteq \mathbb{Q}^n$, $\mathcal{M} \models (X \subseteq N_{12} \to B^m)(\Theta + \Pi)^m@(X|R) \Leftrightarrow$
for all $R \subseteq \Pi(N_{12})$, $\mathcal{M} \models B^m(\Theta + \Pi)^m@(X|R) \Leftrightarrow^{(e)}$
for all $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$, $\mathcal{M} \models B^m(\Theta + \Pi)^m@(X|\{n_{a,b}^{\Pi}|(a,b) \in \mathcal{R}\})$
$\Leftrightarrow^{(f)}$
for all $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$, $\mathcal{M} \models B^m(\Theta@(X|\mathcal{R}) + \Pi)^m \Leftrightarrow$
for all $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$, $\mathcal{N} \models B\Theta@(X|\mathcal{R}) \Leftrightarrow$
$\mathcal{N} \models (\forall XB)\Theta$.

Equivalence (e) holds because of Statement 2, while Definition 11 explains (f). $\surd$

Lemma 2. For all closed dyadic formulae A and $n > 1$,
A is in the dyadic second-order theory of $(\mathbb{N}, <)$ iff
$\forall N_1, N_2, N_{12}, N, n_1, n_2(\nu \to A^m) \in MSOTH(\mathbb{Q}^n, \blacktriangleleft)$. †

Proof. The following conditions are equivalent:
(i) $\mathcal{N} \models A()$
(ii) for any monadic valuation Π satisfying ν: $\mathcal{M}_n \models A^m(()+\Pi)^m$
(iii) for any monadic valuation Σ, $\mathcal{M}_n \models \nu\Sigma \Rightarrow \mathcal{M}_n \models A^m\Sigma$
(iv) $\forall N_1, N_2, N_{12}, N, n_1, n_2(\nu \to A^m) \in MSOTH(\mathbb{Q}^n, \blacktriangleleft)$.

We can prove (i)$\Leftrightarrow$ (ii) from the previous lemma, (iii) $\Leftrightarrow$ (iv) by standard logic.

Finally, (ii) $\Leftrightarrow$ (iii) follows from the fact that $((\,) + \Pi)^m$ and Π agree on parameters of ν. $\surd$

Proof. [Proof of Theorem 1] The translation m of dyadic formulae is recursive. Assume the monadic second-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is recursively enumerable. Give another recursive enumeration of the monadic formula set

$\{\forall N_1, N_2, N_{12}, N, n_1, n_2(\nu \to A^m) :$

A is a dyadic second-order formula$\}$.

The intersection of these two recursively enumerable formula sets is again recursively enumerable but this intersection is the m-translation of the dyadic second-order theory of $(\mathbb{N}, <)$, by Lemma 2. This contradicts Statement 7. $\surd$

Proof. [Proof of Theorem 2] In ν, subset quantifiers occur only in $\nu_1(ii)$. $\nu_1(ii)$ can be converted to an equivalent form of $\forall M \forall Y A$ where A does not contain subset quantifications. So ν is equivalent to a suitable formula of the $\forall$-fragment of the monadic language.

The m-translations of universal dyadic formulae are universal monadic formulae. Consequently, any formula of the form

$\forall N_1, N_2, N_{12}, N, n_1, n_2(\nu \to A^m)$ fits in the $\forall\exists$-fragment of the monadic language, considering the one-sided quantifier movement logical laws. So, if we intersect $MSOTH(\mathbb{Q}^n, \lhd)$ with

$\{\forall N_1, N_2, N_{12}, N, n_1, n_2(\nu \to A^m) :$

A is a **universal** *dyadic second-order formula*$\}$,

we get again a non-axiomatizable part of the $\forall\exists$-fragment, which is an intersection of this fragment with a recursive set nevertheless. $\surd$

Proof. [Proof of Theorem 3] The proof goes through in the case of $\mathbb{R}^n$ $(n > 1)$ with the simplification that in $\nu_2(ii)$ one can avoid the use of any second-order formula. We can substitute it by $\forall x \exists y z(y \blacktriangleleft x \wedge x \blacktriangleleft z \wedge \forall w(z \blacktriangleleft w \wedge w \blacktriangleleft y \wedge x \neq w \to w \notin N_i))$. In this way, ν still guarantees the needed properties but does not contain any subset quantification. Thus, the intersection in the proof of Theorem 2 is indeed a part of the $\forall$-fragment of the monadic second-order language. $\surd$

Proof. [Proof of Theorem 5] If the first-order theory of a countable structure $(T, \prec)$ is ω-categorical and recursively enumerable then the $\forall$-fragment of its monadic second-order theory is recursively enumerable, as well.

The proof of this is rather routine. For the proof of a similar statement, see [V07b], also Theorem 11 in this dissertation. The idea is to take the first-order theory as an axiom set in a signature for $(T, \prec)$ extended by a finite number of unary predicate symbols. $\surd$

Proof. [Proof of Theorem 6] This can be concluded from the previous Theorem taking also into account van Benthem's Theorem 4 from [B83]. $\surd$

Proof. [Proof of Theorem 7] In this proof, let us fix n as an integer greater than 2. First we notice that once we have a (partial) equidistance formula on lightlikely connectible spacetime points (this is the situation since Statement 1), in $\nu_1(ii)$ we can avoid the use of any second-order conditions and still we can ensure the validity of Statement 2, more specifically, that $(\Pi(N_i), \lhd)$ is isomorphic to $(\mathbb{N}, <)$. We simply require in $\nu_1(ii)$ that for each three $\lhd$-consecutive members x, y, z of N_i: $H(x, y, z)$ holds.

All the remaining parts of the proof of Theorem 2 can be retained, except for the proof of consistency of ν (Statement 6). The original proof of this part would be supplemented with checking whether the new $\nu_1(ii)$ is satisfied by the original Π_0. This amounts to finding a parallelogram for each nonnegative integer t which satisfies the following conditions: the two end-points of its first diagonal are $(t, 0, \ldots, 0, t)$ and $(t + 2, 0, \ldots, 0, t + 2)$, the end-points of the other diagonal are also rational points, moreover the sides of the parallelogram are spacelike. $(t + 3, 0, \ldots, t + 2)$ and $(t - 1, 0, \ldots, 0, t)$ can be chosen for the other end-points of the other parallelogram. Now the parallelograms are provided for any three $\lhd$-consecutive points of $\Pi_0(N_2)$. For $\Pi_0(N_1)$ an analogous argument can be constructed. $\surd$

6. Proofs for theorems on first-order temporal and spatio-temporal theories

6.1. Ideas of non-axiomatizability proofs in first-order temporal logic

A significant part of the non-axiomatizability proof of Theorem 8 –which is the most relevant result in the first-order temporal part of this dissertation– can be constructed following the proof for non-axiomatizability of $\mathtt{Th}_S^{\{F,P\}}(\mathbb{R},<)$, for a ternary signature S and for the standard temporal operators F („*sometimes in the future, ... will hold*") and P („*sometimes in the past, ... held*"). Proofs for this appear in [GHR94] (theorem 4.6.1) or later in [HWZ00].

We outline here a usual non-axiomatizability proof, say one for time flow $(\mathbb{R},<)$. It usually starts with copying the time structure of the time flow to a representation by some rigid (time-independent) predicates on the domain of the temporal interpretation. Secondly, by some other rigid predicates, a subset of this copy is separated which is isomorphic to $(\mathbb{N},<)$ with respect to the interpretation of a binary predicate symbol. Then the operations of succession, addition and multiplication are realized on this subset by some predicates and postulates on them . Finally, the true arithmetics (the first-order theory of $(\mathbb{N},\ +,\ \cdot)$) is translated into the temporal theory in a recursive way.

The second step of the original proof is quite transparent. Concerning a unary and a binary predicate symbol, a formula of the temporal language can be written whose satisfaction guarantees that the interpretation of these predicate symbols constitutes a discrete linear ordering with a minimal but without any maximal point and requires that this ordering has no accumulation point in (the isomorphic copy of) $(\mathbb{R},<)$. This condition can be formulated by a first-order temporal formula and imply isomorphism to $(\mathbb{N},<)$.

To prove Theorem 13, the proof just mentioned is not directly applicable, for the reason that the mentioned theorem concerns monadic signature (one unary predicate symbol without equality). We have some difficulties with the representation of three-argument relations.

Furthermore, in proving Theorem 8, we face another difficulty, too. For one thing, saying that „a discrete linear ordering has no accumu-

lation point among the rational points" does not imply isomorphism to $(\mathbb{N}, <)$. This difficulty is so serious that it prevents $\mathtt{Th}_L^{Op}(\mathbb{Q}^2, \blacktriangleleft)$ from being non-axiomatizable. However, this fact can be regarded as pleasure for those who want to discover axiomatizable first-order spatio-temporal theories, see the section on use of an axiomatizable first-order spatio-temporal theory (Section 8).

In [G89], R. Goldblatt has given a definitional development of Minkowski geometry on $\mathbb{R}^4$, starting from the causal relation *after* ($\ll$). The author noticed that his construction is valid for Minkowski geometry on $\mathbb{F}^n$, for arbitrary finite $n > 2$ and arbitrary quadratic ordered field $\mathbb{F}$, without changes. However not all of these definitions are valid for $\mathbb{Q}^n$. Particularly, spacelike collinearity cannot be defined in terms of $\lhd$ by the method of [G87], presented next. (Here $\lhd$ is the so-called directed optical accessability relation, which is definable within $(\mathbb{Q}^n, \blacktriangleleft)$.) Three distinct, space-like connectible points x, y, z are collinear iff $\neg\exists u\ (x \lhd u \wedge y \lhd u \wedge z \lhd u)$. This characterization is no more valid for the rationals. In Statement 1 (iv,v) an alternative definition has been given which describes the relation *spacelike betweenness* in $(\mathbb{Q}^n, \blacktriangleleft)$, if $n > 2$. By means of this, we construct a formula and prove that its satisfaction by the temporal interpretation will imply isomorphism to $(\mathbb{N}, <)$.

The first difficulty mentioned above concerns that the base first-order signature is limited to one unary predicate symbol. A representation of predicates of more than one argument can be employed to work out this problem. This solution will throw the proof to a more technical level, but this is the strongest result we can prove. The method of the existing proofs of non-axiomatizability of *monadic* first-order temporal logic (see [HC68] –it proves only non-decidability of some first-order modal logic–, [HWZ00] and [Me92]) cannot be followed directly but it is unnecessary to deny their motivation to our work.

6.2. Representation of the spacetime structure by temporal interpretation

We begin with the proof of Theorem 8.

Definition 29. We also introduce some defined temporal operators. If A is an arbitrary temporal formula then $\Box A$ stands for

$A \vee NA$ and $\Diamond A$ for $\neg\Box\neg A$. Further we write $\boxdot A$ and $\diamondsuit A$ instead of NA and $\neg N\neg A$, respectively, only for the sake of the unity of our notation. It turns to be clear, that $\Box A$ expresses that A is true at all the points of spacetime and $\Diamond A$ is its existential counterpart, if we realize that in spatio-temporal setting, NA holds in a spacetime point q if and only if A in all spacetime points maybe except for q itself. As usual, F denotes the existential counterpart of G, we write them also in visual form $\diamondsuit\!\!\rightarrow$ and $\boxdot\!\!\rightarrow$.

We fix some formulæ of temporal language TL_L^{GN}, as follows.

Definition 30. $Id := \Diamond(r(x) \wedge \boxdot\!\!\rightarrow \neg r(x))$, $\nu_1 := \Box\exists x(r(x) \wedge \boxdot\!\!\rightarrow \neg r(x))$. †

Id tells about x that it falls in the extension of r in exactly one time point. ν_1 postulates that in every time point there has to be such an object that satisfies $r(x) \wedge \boxdot \neg r(x)$ in that time.

Let us fix a temporal interpretation $\mathcal{I}$ for TL_L^{GN} on the flow $(\mathbb{Q}^n, \blacktriangleleft)$ and a rational point $e \in \mathbb{Q}^n$ satisfying $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, (), e \Vdash \nu_1$. It is clear that then the same holds for all $e' \in \mathbb{Q}^n$. We will see later that such an interpretation exists, it follows from the stronger result of Lemma 4. Through this and the next two subsections (6.4), these $\mathcal{I}$ and e remain fixed.

Definition 31. We define the relation $\varphi_0 \subseteq D_{\mathcal{I}} \times \mathbb{Q}^n$ by the condition $(d, q) \in \varphi_0 \Leftrightarrow (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q \Vdash r(x)$. The set $\{d \in D_{\mathcal{I}} : (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, e \Vdash Id\}$ will be denoted by ID_1. Further, φ_1 stands for the restriction of φ_0 to $ID_1 \times \mathbb{Q}^n$. †

Statement 8. With the notations of the previous definition, φ_1 is a surjective function taking ID_1 onto $\mathbb{Q}^n$. †

This follows from the way we have fixed $\mathcal{I}$. In details, this sounds as follows.

Proof. First, that for an arbitrary $\in ID_1$ uniquely exists a $q \in \mathbb{Q}^n$ that $\varphi_1(d, q)$ holds. $d \in ID_1$ means that $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, e \mid\!\vdash \Diamond(r(x) \wedge \boxdot \neg r(x))$ ergo there exists $q \in \mathbb{Q}^n$ that $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{}, q \mid\!\vdash r(x) \wedge \boxdot \neg r(x)$ and so

$(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q \mid\!\vdash r(x)$,
$\varphi_1(d, q)$,

$\forall q' \in \mathbb{Q}^n \setminus \{q\}$:
$(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q' \mid\!\vdash \neg r(x)$ and so
$\neg\varphi(d, q')$ hold, respectively.

Second, that arbitrary $q \in \mathbb{Q}^n$ occurs as an φ_1-image of an ID_1-element. Let $q \in \mathbb{Q}^n$. Then we have the followings holding, each of them follow from its preceding in an obvious way.

$(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, (), e \mid\!\vdash \Box\exists(r(x) \wedge \boxdot\neg r(x))$,
$(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, (), q \mid\!\vdash \exists(r(x) \wedge \boxdot\neg r(x))$,
there exists a $d \in D_{\mathcal{I}}$ satisfying $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q \mid\!\vdash (r(x) \wedge \boxdot \neg r(x))$, and consequently, $\varphi_0(d, q)$,
there exists a $d \in D_{\mathcal{I}}$ satisfying $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, e \mid\!\vdash \Diamond(r(x) \wedge \boxdot \neg r(x))$, so $d \in ID_1$ and consequently, $\varphi_0(d, q)$. $\surd$

Definition 32. The binary relation ρ on ID_1 is defined by the condition $(d_1, d_2) \in \rho \Leftrightarrow \varphi_1(d_1) = \varphi_1(d_2)$. †

By standard algebraic arguments, the following is true.

Statement 9. ρ is an equivalence on ID_1. †

Definition 33. With the notations of the previous items, the set of the equivalence classes of ρ is denoted by ID. The function on ID corresponding to φ_1 is denoted by φ. More exactly, if the ρ-equivalence class of d is denoted by $[d]_\rho$, then $\varphi([d]_\rho)$ is just $\varphi_1(d)$. †

All the following statements of this subsection can be verified by standard algebraic arguments.

Statement 10. φ is a bijection from ID onto $\mathbb{Q}^n$. †

Definition 34. Let Ord denote the formula $\Diamond(r(x) \wedge \diamondsuit\!\!\rightarrow r(y)) \wedge Id \wedge Id\binom{x}{y}$, where $A\binom{x}{y}$ is the result of substituting y for free occurrences of x in formula A. If $d_1, d_2 \in ID_1$ then we write $Ord_1(d_1, d_2)$ for $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y}{d_1 d_2}, e \Vdash Ord$. †

Statement 11. The relation Ord_1 on ID_1 is compatible with ρ. That is, $Ord_1(d_1, d_2) \Leftrightarrow Ord_1(d_3, d_4)$ whenever $\{(d_1, d_2), (d_3, d_4)\} \subseteq \rho$. †

Definition 35. Let $Ord^{\mathcal{I}}$ denote the inherited relation of ρ-equivalence classes, that is, $Ord^{\mathcal{I}}$ is a binary relation on ID and for any $d_1, d_2 \in ID_1$, $Ord_1(d_1, d_2)$ if and only if $Ord^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho)$. †

Statement 12. φ is an isomorphism from $(ID, Ord^{\mathcal{I}})$ onto $(\mathbb{Q}^n, \blacktriangleleft)$. †
Proof. For this, we have to see that if $d_1, d_2 \in ID_1$, then $Ord_1([d_1]_\rho, [d_2]_\rho) \Leftrightarrow \varphi([d_1]_\rho) \blacktriangleleft \varphi([d_2]_\rho)$. For it, it is enough that $Ord_1(d_1, d_2) \Leftrightarrow \varphi(d_1) \blacktriangleleft \varphi(d_2)$. The left side of the last equivalence is equivalent to $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y}{d_1 d_2}, e \Vdash \Diamond(r(x) \wedge \Diamond\!\!\!\!-\!\!\!\!\rightarrow r(y)) \wedge Id \wedge Id\binom{x}{y}$ and so, to that there exist $q_1, q_2 \in \mathbb{Q}^n$ such that $q_1 \blacktriangleleft q_2$, $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d_1}, q_1 \mid\!\!\!- r(x)$ and $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{y}{d_2}, q_2 \mid\!\!\!- r(x)$. The last equivalent form can be seen equivalent to the original right side in an easy way. $\surd$

Definition 36. Writing Ord into the place of $\blacktriangleleft$ in formulæ $\beta_\sigma(x, y, z)$, $x \lhd y$, $(x = y)$ and $\sigma(x, y)$ of Definition 19 and relativizing to Id, we obtain temporal formulæ $Betw(x, y, z)$, $Opt(x, y)$, $Eq(x, y)$ and $Sim(x, y)$ of TL_L^{GN}, respectively. For example, $Eq(x, y)$ is $\forall z(Id_z^x \rightarrow (Ord(z, x) \leftrightarrow Ord(z, y)))$. The corresponding relations on ID are also denoted by $Betw^{\mathcal{I}}$, $Opt^{\mathcal{I}}$,$Eq^{\mathcal{I}}$ and $Sim^{\mathcal{I}}$, respectively. For example, for each $d_1, d_2 \in ID_1$, $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y}{d_1 d_2}, e \Vdash Opt(x, y)$ is also denoted by $Opt^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho)$. This is reasonable, because all these relations are ρ-compatible on ID_1, since they are defined from $\blacktriangleleft$. †

Statement 13. φ is an isomorphism from $(ID, Ord^{\mathcal{I}}, Betw^{\mathcal{I}}, Opt^{\mathcal{I}}, Eq^{\mathcal{I}}, Sim^{\mathcal{I}})$ onto $(\mathbb{Q}^n, \blacktriangleleft, \beta_\sigma, \lhd, =, \sigma)$. †
Proof. For example, we have to justify that for any $d_1, d_2 \in ID_1$, $Eq^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho) \Leftrightarrow \varphi_1(d_1) = \varphi_1(d_2)$. These family of conditions can be checked based on Lemma 1. $\surd$

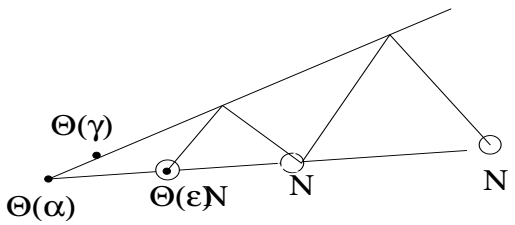
6.3. Isomorphism to the ordering of $\mathbb{N}$

Definition 37. Formulæ $\Diamond(r(\delta) \wedge r(x))$, $Betw(\alpha, x, y)$ (in TL_L^{GN}) will be abbreviated as $N(x)$ and $O(x, y)$, respectively. Further, we write $O(x, y) \wedge \neg\exists z(N(z) \wedge O(x, z) \wedge O(z, y))$ also in form $S(x, y)$. †

Parameter δ is used only for separating a subset of ID without adding a new predicate symbol into L. This is natural enough. Dealing with predicates with more than one arguments requires a more sophisticated representation, as the next subsection will show.

Definition 38. Let ν_2 be defined as conjunction of ν_1 and the following formulæ :
(1) $Sim(\alpha,\varepsilon) \wedge Sim(\alpha,\gamma) \wedge Id\binom{x}{\alpha} \wedge Id\binom{x}{\varepsilon} \wedge Id\binom{x}{\gamma}$,
(2) $\forall x(N(x) \rightarrow Id \wedge (Betw(\alpha,\varepsilon,x) \vee Eq(x,\varepsilon)))$,
(3) $\forall x(N(x) \rightarrow \exists! y(N(y) \wedge S(x,y))$,
where $\exists!$ is to understand with respect to the defined Eq,
(4) $\forall xy(N(x) \wedge N(y) \wedge S(x,y) \rightarrow$
$\exists z(N(z) \wedge Betw(\alpha,\gamma,z) \wedge Opt(x,z) \wedge Opt(y,z))$,
(5) $\forall xyzw(N(x) \wedge N(y) \wedge N(z) \wedge N(w) \wedge S(x,y) \wedge Opt(x,z) \wedge Opt(y,z) \wedge$
$\neg Opt(x,w) \wedge Opt(y,w) \wedge Betw(\alpha,\gamma,z) \wedge Betw(\alpha,\gamma,w)$
$\rightarrow Betw(\alpha,z,w))$. †

Figure 3 shows the φ-image (projection into $\mathbb{Q}^n$) of a model of ν_2.



3. ábra. A model for ν_2

Definition 39. $N^{\mathcal{I}} := \{[d]_\rho : (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q \Vdash N(x)\}$,
$O^{\mathcal{I}} := \{([d_1]_\rho, [d_2]_\rho) : (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y}{d_1 d_2}, q \Vdash O(x,y)\}$, and $S^{\mathcal{I}}$ can be defined in an analogous way. †

Let Θ denote a fixed valuation satisfying $dom\Theta \supseteq \{\alpha, \delta, \varepsilon, \gamma\}$, till the end of the next subsection.

Lemma 3. If $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_2$ then $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}})$ is isomorphic to
$(\mathbb{N}, <, succ^r)$, where $succ^r = \{(n, n+1) : n \in \mathbb{N}\}$. †

Proof. The way of defining ν_2 results in that $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}})$ is a discrete linear ordering whose minimal element is $\Theta(\varepsilon)$ but without any maximal element. We only have to take into account also the properties of the betweenness and that the existing isomorphism φ allows us to use the mentioned spacetime geometrical notions also for the elements of ID. By the way, if the admitting of the above statement would be an overloading task, then we simply attach to ν_2 the extra condition that O is a linear ordering on the elements satisfying N.

Thus, the only interesting point is to show that $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}})$ is isomorphic to
$(\mathbb{N}, <, succ^r)$. For this, it is enough to prove that $N^{\mathcal{I}}$ is exhausted by the set $\{\Theta(\varepsilon), s_{\mathcal{I}}(\Theta(\varepsilon)), s^2_{\mathcal{I}}(\Theta(\varepsilon)), \ldots\}$, where $s_{\mathcal{I}}$ is the function denotation of relation $S^{\mathcal{I}}$ which is actually a function by $\nu_2(3)$ and by the fact that $Eq^{\mathcal{I}}$ coincides with the real equality on ID. In this proof, we write simply s instead of $s_{\mathcal{I}}$.

The fulfilment of the above exhaust can be verified by means of the statement, that for all non-negative integer m, $\delta(s^{m+2}\Theta(\varepsilon), s^{m+1}\Theta(\varepsilon))$ $> \delta(s^{m+1}\Theta(\varepsilon), s^m\Theta(\varepsilon))$ holds, where δ is the Euclidean distance and s^0 is, as usual, the identity function. (Please remind that usage of spacetime geometrical notions on the elements of ID is reasonable through the isomorphism φ. For example, for $d_1, d_2 \in ID_1$, $\delta([d_1]_\rho, [d_1]_\rho)$ is just the Euclidean distance between $\varphi([d_1]_\rho)$ and $\varphi([d_1]_\rho)$.) This inequality can be shown by properties of spacelike betweenness, parallelity of $\lhd$-linear straight lines and similar triangles, as follows.

We interject a remark, namely, that this condition cannot be attached directly to ν_2, as a first-order condition in terms of $\blacktriangleleft$, because we are not able to define equidistance in the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ – it is an important difference to $(\mathbb{R}^n, \blacktriangleleft)$ which makes this proof more complicated.

Let us fix an integer $m \geqslant 0$ and $a_0 := s^m(\Theta(\varepsilon))$, $a_1 := s^{m+1}(\Theta(\varepsilon))$, $a_2 := s^{m+2}(\Theta(\varepsilon))$. Then $Betw^{\mathcal{I}}(\Theta(\alpha), a_0, a_1)$ and $Betw^{\mathcal{I}}(\Theta(\alpha), a_1, a_2)$

follow from $\nu_2(2)$ and from the definition for S. This implies $\delta(\Theta(\alpha), a_0) < \delta(\Theta(\alpha), a_1) < \delta(\Theta(\alpha), a_2)$ because of the properties of betweenness. By $\nu 2(4)$–(5), there exist $b, c \in ID$ that the following properties hold:
$Opt^{\mathcal{I}}(a_0, b)$, $Opt^{\mathcal{I}}(a_1, b)$, $Opt^{\mathcal{I}}(a_1, c)$, $Opt^{\mathcal{I}}(a_2, c)$,$\neg Opt^{\mathcal{I}}(a_0, c)$, $\neg Opt^{\mathcal{I}}(a_2, b)$, $Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\gamma), b)$, $Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\gamma), c)$ and $Betw^{\mathcal{I}}(\Theta(\alpha), b, c)$.

One can conclude from them on $\delta(\Theta(\alpha), b) < \delta(\Theta(\alpha), c)$, furthermore, on that the lightlike line a_1b is parallel to the lightlike line a_2b and the lightlike line a_0b is parallel to the lightlike line a_1c since b and c both have to be in the plane $\Theta(\alpha)\Theta(\gamma)\Theta(\beta)$.

Using the properties of similar triangles $\Theta(\alpha)a_0b$ and $\Theta(\alpha)a_1c$, $\Theta(\alpha)a_1b$ and $\Theta(\alpha)a_2c$, respectively, and applying the inequality on the geometrical and arithmetic mean, we can derive now the desired inequality in the form (1) $\delta(\Theta(\alpha), a_2) - \delta(\Theta(\alpha), a_1) > \delta(\Theta(\alpha), a_1) - \delta(\Theta(\alpha), a_0)$ regarding that the mentioned elements of this inequality are collinear. (1) can be derived as follows.

$\frac{\delta(\Theta(\alpha),a_2)}{\delta(\Theta(\alpha),a_1)} > \frac{\delta(\Theta(\alpha),a_1)}{\delta(\Theta(\alpha),a_0)}$,

$\delta(\Theta(\alpha), a_2) \cdot \delta(\Theta(\alpha), a_0) > \delta(\Theta(\alpha), a_1) \cdot \delta(\Theta(\alpha), a_1)$,

$\sqrt{\delta(\Theta(\alpha), a_2) \cdot \delta(\Theta(\alpha), a_0)} > \delta(\Theta(\alpha), a_1)$,

$\frac{\delta(\Theta(\alpha),a_2)+\delta(\Theta(\alpha),a_0)}{2} > \delta(\Theta(\alpha), a_1)$ and this implies (1) directly.

Finally, the set $\{\Theta(\varepsilon), s(\Theta(\varepsilon)), s^2(\Theta(\varepsilon)), \ldots\}$ exhausts $N^{\mathcal{I}}$ because any $r \in N^{\mathcal{I}} \setminus \{\Theta(\varepsilon)\}$ satisfies also $Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\varepsilon), r)$ by $\nu_2(2)$, and $\Theta(\varepsilon), s(\Theta(\varepsilon)), s^2(\Theta(\varepsilon)), \ldots$ form a growing distance ω-sequence on the half-line $\{d \in ID : Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\varepsilon), d) \vee Eq^{\mathcal{I}}(d, \Theta(\varepsilon))\}$, without any accumulation point. Since there is no element of $N^{\mathcal{I}}$ between two consecutive points of the sequence or outside of the half-line mentioned, the above exhaust and consequently, Lemma 3 is proved. $\surd$

6.4. Representation of predicates of more than one argument

We introduce the following abbreviations in TL_L^{GN}. They allow to represent predicates with more than argument.

Definition 40. $A(x,y,z) :=$
$\{Eq(x,\varepsilon) \wedge Eq(y,z)\} \vee \{Eq(y,\varepsilon) \wedge Eq(x,z)\} \vee$
$\{Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge$
$\quad \exists uv\, [Id\binom{x}{u} \wedge Opt(u,x) \wedge Sim(u,z) \wedge \Box(r(v) \leftrightarrow r(x) \vee r(z) \vee r(u))]\} \vee$
$\{\neg Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge \neg Eq(y,\varepsilon) \wedge$
$\quad \exists uv\, [Id\binom{x}{u} \wedge Opt(u,x) \wedge Ord(u,y) \wedge Sim(u,z) \wedge$
$\qquad \Box(r(v) \leftrightarrow r(x) \vee r(y) \vee r(z) \vee r(u))]\},$

$M(x,y,z) :=$
$\{Eq(x,\varepsilon) \wedge Eq(z,\varepsilon)\} \vee \{Eq(y,\varepsilon) \wedge Eq(z,\varepsilon)\} \vee$
$\{S(\varepsilon,x) \wedge Eq(y,z)\} \vee \{S(\varepsilon,y) \wedge Eq(x,z)\} \vee$
$\{Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge \neg S(\varepsilon,x) \wedge$
$\quad \exists uv\, [Id\binom{x}{u} \wedge Opt(x,u) \wedge Sim(z,u) \wedge \Box(r(v) \leftrightarrow r(x) \vee r(z) \vee r(u))]\} \vee$
$\{\neg Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge \neg Eq(y,\varepsilon) \wedge \neg S(\varepsilon,x) \wedge \neg S(\varepsilon,y) \wedge$
$\quad \exists uv\, [Id\binom{x}{u} \wedge Opt(x,u) \wedge Ord(y,u) \wedge Sim(z,u) \wedge$
$\qquad \Box(r(v) \leftrightarrow r(x) \vee r(y) \vee r(z) \vee r(u))]\}.$ †

Once we have represented two predicates of three arguments by the means of our sole unary predicate symbol r (it was the harder to provide than what follows), we can endow these formulæ to represent addition and multiplication, in the expected way, by postulating the following ν_3 on them.

Definition 41. Let ν_3 is the conjuntion of ν_2 and the following conditions (the usual primitive recursive definitions addition and multiplication – in our representation):
(1) $\forall xy(N(x) \wedge N(y) \rightarrow \exists!^{Eq} z(N(z) \wedge A(x,y,z)) \wedge \exists!^{Eq} w(N(w) \wedge M(x,y,w)))$,
where $\exists!^{Eq}$ is to understand regarding Eq as equality,
(2) $\forall x(N(x) \rightarrow A(\varepsilon,x,x))$,
(3) $\forall xyzvw(N(x) \wedge N(y) \wedge N(z) \wedge N(v) \wedge N(w) \wedge$
$\quad S(x,y) \wedge A(x,z,v) \wedge S(v,w) \rightarrow A(y,z,w))$,
(4) $\forall x(N(x) \rightarrow M(\varepsilon,x,\varepsilon))$,
(5) $\forall xyzvw(N(x) \wedge N(y) \wedge N(z) \wedge N(v) \wedge N(w) \wedge$
$\quad S(x,y) \wedge M(x,z,v) \wedge A(v,z,w) \rightarrow M(y,z,w))$. †

Definition 42. Let $A^{\mathcal{I}}$ and $M^{\mathcal{I}}$ denote the meaning of A and M, resp., on ID. So, for $d_1, d_2, d_3 \in ID_1$, we write also
$A^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho, [d_3]_\rho)$ for $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y\ z}{d_1 d_2 d_3}, e \Vdash A(x,y,z)$, and similar applies to $M^{\mathcal{I}}$.

Statement 14. If $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, e \Vdash \nu_3$ then $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}}, A^{\mathcal{I}}, M^{\mathcal{I}})$ is isomorphic to $(\mathbb{N}, <, succ^r, +^r, *^r)$, where
$succ^r = \{(n, n+1) : n \in \mathbb{N}\}$, $+^r = \{(k,l,m) \subseteq \mathbb{N}^3 : k + l = m\}$ and $*^r$ is $\{(k,l,m) \subseteq \mathbb{N}^3 : k \cdot l = m\}$.

More specifically, the isomorphism can be given by $\psi : \mathbb{N} \to ID$, where $\psi(k) = s^k_{\mathcal{I}}(\Theta(\varepsilon))$. This follows from the following and similar other facts: for any $k, l, m \in \mathbb{N}$, the conditions $k + l = m$ and $A^{\mathcal{I}}(s^k_{\mathcal{I}}(\Theta(\varepsilon)), s^l_{\mathcal{I}}(\Theta(\varepsilon)), s^m_{\mathcal{I}}(\Theta(\varepsilon))$ are equivalent.

This statement can be verified by Lemma 3 and induction on k and l, taking into account that on $(\mathbb{N}, <, succ^r)$ only the functions of addition and multiplication satisfy their defining primitive recursive equations. We omit the routine details. $\surd$

6.5. Translation of true arithmetics into our theory

In this section, the proof is finished by the usual way of non-axiomatizability proofs of first-order temporal theories. The only difference is that consistency is not straightforward because of the rather complex way of representing the three-argument predicate symbols.

Lemma 4. There exists a temporal interpretation for TL^{GN}_L on the time flow $(\mathbb{Q}^n, \blacktriangleleft)$, a rational point $q \in \mathbb{Q}^n$ and a valuation Θ in $\mathcal{I}$ satisfying $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_3$. †

Proof. We supply only the asked $\mathcal{I}$, q and Θ and leave checking satisfaction of ν_3 to the reader. q will be specified as $(0, \ldots, 0)$. Let $D_{\mathcal{I}}$ be the set

$\mathbb{Q}^n \cup \{D\} \cup$

$\{A_{k,l,m} : k + l = m, k \neq 0, l \neq 0, k \neq l\} \cup \{A^{=}_{k,m} : k + k = m, k \neq 0\} \cup$

$\{M_{k,l,m} : k \cdot l = m, k > 1, l > 1, k \neq l\} \cup \{M^{=}_{k,m} : k \cdot k = m, k > 1\}$,

where D and the other objects are just formal symbols.

The interpretation $\mathcal{I}^p_t$ is defined via its value on predicate symbol r. We write shortly r_t for $\mathcal{I}^p_t(r)$. r_t can be defined via the definition for the truth values $r_t(d)$, for arbitrary $d \in D_{\mathcal{I}}$.
if $d = q \in \mathbb{Q}^n$ then $r_t(q) = (t = q)$,
if $d = D$ then $r_t(D) = (\exists m \geqslant 1)\ t = (0, \ldots, 0, 2^m)$,

if $d = A_{k,l,m}$ for $k, l, m \in \mathbb{N}$ satisfying $k + l = m, k \neq 0, l \neq 0, k \neq l$ then

$r_t(A_{k,l,m}) = t \in \{(0, \ldots, 0, k), (0, \ldots, 0, l), (0, \ldots, 0, m),$
$\big(-\big|\frac{k}{2} - \frac{l}{2}\big| + \frac{1}{4}, 0, \ldots, 0, \frac{k}{2} + \frac{l}{2} + \frac{1}{4}\big)\}$,

if $d = A^{=}_{k,m}$ for $k, m \in \mathbb{N}$ satisfying $2 \cdot k = m, k \neq 0$, then

$r_t(A^{=}_{k,m}) = t \in \{(0, \ldots, 0, k), (0, \ldots, 0, m), \big(-\frac{1}{4}, 0, \ldots, 0, k + \frac{1}{4}\big)\}$,

if $d = M_{k,l,m}$ for $k, l, m \in \mathbb{N}$ satisfying $k \cdot l = m, k \geqslant 2, l \geqslant 2, k \neq l$ then

$r_t(M_{k,l,m}) = t \in \{(0, \ldots, 0, k), (0, \ldots, 0, l), (0, \ldots, 0, m),$
$\big(\big|\frac{l}{2} - \frac{k}{2}\big| + \frac{1}{4}, 0, \ldots, 0, \frac{l}{2} + \frac{k}{2} + \frac{1}{4}\big)\}$,

if $d = M^{=}_{k,m}$ for $k, m \in \mathbb{N}$ satisfying $2 \cdot k = m, k \neq 0$, then

$r_t(M^{=}_{k,m}) = t \in \{(0, \ldots, 0, k), (0, \ldots, 0, m), \big(\frac{1}{4}, 0, \ldots, 0, k + \frac{1}{4}\big)\}$.

The valuation Θ can be determined by setting $\Theta(\alpha)$, $\Theta(\varepsilon)$, $\Theta(\gamma)$, $\Theta(\delta)$ to $(0, \ldots, 0)$, $(0, \ldots, 0, 1)$, $(\frac{1}{4}, 0, \ldots, 0, \frac{3}{4})$ and $\Theta(\delta) = D$, respectively.

Definition 43. For any first-order formula A in the signature of $(\mathbb{N}, <, succ^r, +^r, *^r)$ (somewhat loosely, we do not differentiate the predicate symbol from the corresponding interpreting relation), we give a translation A^t into TL^{GN}_L, by structural induction, as follows. We assume that the variables of the arithmetical language are exactly those of L minus $\{\alpha, \gamma, \delta, \varepsilon\}$.

$(x < y)^t = O(x, y)$ $(= Betw(\alpha, x, y))$,
$(succ^r(x, y))^t = S(x, y)$, where S is defined in 37,
$(+^r(x, y, z))^t = A(x, y, z)$, where A is defined in 6.4,
$(*^r(x, y, z))^t = M(x, y, z)$, where M is defined in 6.4,
$(A \wedge B)^t = (A^t \wedge B^t)$, $(\neg A)^t = \neg A^t$ and
$(\forall x A)^t = \forall x (N(x) \rightarrow A^t)$. †

Definition 44. Assume that $\mathcal{I}$ is a temporal interpretation for TL^{GN}_L on the time flow $(\mathbb{Q}^n, \blacktriangleleft)$ which also satisfies ν_3, and Θ is a valuation in $\mathcal{I}$. We associate a valuation $\Sigma \oplus \Theta$ of the variables of that temporal language in $\mathcal{I}$, to every valuation Σ of the variables of the arithmetical language into $\mathbb{N}$.

Values for $\alpha, \gamma, \varepsilon, \delta$ come from Θ, that is, for example, $(\Sigma \oplus \Theta)(\alpha) = \Theta(\alpha)$, while the other variables gets value $(\Sigma \oplus \Theta)(x) = s_{\mathcal{I}}^{\Sigma(x)}(\Theta(\varepsilon))$, where $s_{\mathcal{I}}$ is described in the proof of Lemma 3. †

Lemma 5. Let us assume that $\mathcal{I}$ is a temporal interpretation for TL_L^{GN} on the time flow $(\mathbb{Q}^n, \blacktriangleleft)$, $q \in \mathbb{Q}^n$, Θ is a valuation in $\mathcal{I}$ such that $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_3$, further, that A is a first-order formula in the language of $(\mathbb{N}, <, succ^r, +^r, *^r)$ and Σ is a valuation of the variables of the last language into $\mathbb{N}$. Then we have

$(\mathbb{N}, <, succ^r, +^r, *^r) \models A\Sigma$ if and only if $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Sigma \oplus \Theta, q \Vdash A^t$. †

Proof. By induction on the complexity of the arithmetical formula A. For atomic formulæ, this follows from Statement 14. For $\wedge$-formulae this is a trivial consequence of the induction hypothesis, while, for $\forall x B$, it is enough to consider that ID is exhausted by $\{s_{\mathcal{I}}^k(\Theta(\varepsilon)) : k \geqslant 0\}$ (Statement 3). $\surd$

Lemma 6. If A is a closed first-order formula in the language of $(\mathbb{N}, <, succ^r, +^r, *^r)$ then we have $A \in Th\ (\mathbb{N}, <, succ^r, +^r, *^r)$ if and only if $\forall\alpha\delta\gamma\varepsilon(\nu_3 \rightarrow A^t) \in \mathtt{Th}_L^{GN}(\mathbb{Q}^n, \blacktriangleleft)$, where $Th\ K$ denotes the first-order theory of structure K. †

Proof. We can prove this by specializing the previous lemma to $\Sigma = ()$, remembering that there exist $\mathcal{I}$, Θ, $q \in \mathbb{Q}^n$ such that $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \mid\!\vdash \nu_3$, and observing that the left side of equivalence in the previous lemma is independent of $\mathcal{I}$, Θ and q. $\surd$

Proof. [Proof of Theorem 8] If $\mathtt{Th}_L^{GN}(\mathbb{Q}^n, \blacktriangleleft)$ was recursively enumerable then $\mathtt{Th}_L^{GN}(\mathbb{Q}^n, \blacktriangleleft) \cap$
$\{\forall\alpha\delta\gamma\varepsilon\ (\nu_3 \rightarrow A^t) | A$ *is an arithmetical formula*$\}$ would be recursively enumerable, too. This is impossible by the previous lemma. $\surd$

Proof. [Proof of Theorem 9] The proof of Theorem 8 goes through also for $(\mathbb{R}^n, \blacktriangleleft)$, even with the following simplification. In [G87] an equidistance formula was presented in the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$. In Lemma 3, we could simply require that for all three neighbor elements a, b, c of $N^{\mathcal{I}}$ satisfies that the distance between a and b is equal to the distance of b and c. This way covers cases when $n > 2$. Otherwise, if $n = 2$, another way has to be chosen. The above mentioned equidistance formula cannot work well for $(\mathbb{R}^2, \blacktriangleleft)$. See the next remark. But $(\mathbb{R}^2, \blacktriangleleft)$ is isomorphic to $(\mathbb{R}^2, L_2)$ where L_n is defined on $\mathbb{R}^n$ is defined as
$\{((p_1, \ldots, p_n), (q_1, \ldots, q_n)) : p_1 < q_1 \wedge \ldots \wedge p_n < q_n\}$. These makes possible to transfer the proof of non-axiomatizability of linear tempo-

ral logic over $(\mathbb{R}, <)$ by an affordable amount of effort, see Theorem 13.

We give only the points of difference related to the Proof of Theorem 8.

Only the first three items of Statement 1 is valid. The rest of Subsection 6.2 can be taken word-by-word with the difference that in Definition 36 where we are not able to define $Betw$ only the remaining temporal formulæ. So in Statement 13, we have isomorphism between $(ID, Ord^{\mathcal{I}}, Opt^{\mathcal{I}}, Eq^{\mathcal{I}}, Sim^{\mathcal{I}})$ and $(\mathbb{R}^2, \blacktriangleleft, \lhd, =, \sigma)$. In Definition 37, $N(x)$ can remain defined as $\Diamond(r(\delta) \wedge r(x))$ and we can keep the definition of $S(x, y)$ but $O(x, y)$ abbreviates now simply $Opt(x, y)$. In Definition 38, it is enough to set ν_2 expressing that $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}})$ is a discrete linear ordering with a minimum but without any maximum and there is no accumulation point for it. It will result Lemma 3 because we are now over the reals. The representation of three-argument relations (Definition 6.4, Lemma 4) also has to be tailored to the possibilities. For example, $A(x, y, z)$ could be taken as
$Id \wedge Id^x_y \wedge Id^x_z \wedge \exists u, v(Id^x_v \wedge Ord(x, v) \wedge Opt(v, y) \wedge Ord(v, z) \wedge \Box(r(u) \leftrightarrow r(x) \vee r(y) \vee r(z) \vee r(v)))$. Similar idea works for M. The rest of the reasoning can be taken word-by-word. $\surd$

Remark 1. Actually, we can prove that equidistance and even betweenness is not definable in $(\mathbb{R}^2, \blacktriangleleft)$. If it would be definable then this must be work for $(\mathbb{Q}^2, \blacktriangleleft)$, as well, being the latter an elementary substructure of the first structure. Then we could achieve the non-axiomatizability proof of Theorem 8 even for $n = 2$, which is impossible. Moreover, we cannot define betweenness in $(\mathbb{R}^2, \blacktriangleleft)$ otherwise the other definitions of Statement 1 would work well for $(\mathbb{Q}^2, \blacktriangleleft)$ to get a defining formula for equidistance.

7. Proofs of theorems on axiomatizability of first-order temporal theories

We construct these proofs regarding the temporal operator set of *Until* and *Since*. *Until* is given after Definiton 8 while *Since* is its time mirror. Its semantics can be read back also from Definition 46. The proof can be transformed into the proof for another operator sets without much more effort.

Definition 45. For a first-order language L, let $M(L)$ denote a two-sorted language which is derived by adding a new sort (so-called time sort, while the original one is called data sort) to L, also a new binary predicate symbol $\prec$ on the new sort and finally, an extra time argument to each predicate symbol of L. The terms are left unchanged. †

Below the well-known first-order translation of temporal formulæ is given, which are used to establish our axiomatizability results on first-order temporal theories. Having defined this translation, the strong results of classical first-order logic apply.

Definition 46. For any temporal formula A of $TL^{US}(L)$, a translation formula A^* of $M(L)$ is defined in the following way:

for an atomic formula $A = P(x_1, ..., x_n)$ let A^* be $P(x_1, ..., x_n, t_1)$,

$(\neg A)^* := \neg A^*, \quad (A \wedge B)^* := A^* \wedge B^*,$

$(\forall x A)^* := \forall x A^*,$

$Until(A, B)^* := \exists t_k(t_1 \prec t_k \wedge A^*(t_1||t_k) \wedge$
$\forall t_{k+1}(t_1 \prec t_{k+1} \prec t_k \rightarrow B^*(t_1||t_{k+1})),$
where k is the smallest index such that no free or bounded variable
of $A^* \wedge B^*$ occurs in $\{t_k, t_{k+1}, t_{k+2}, ...\}$,

$Since(A, B)^* := \exists t_k(t_k \prec t_1 \wedge A^*(t_1||t_k) \wedge$
$\forall t_{k+1}(t_k \prec t_{k+1} \prec t_1 \rightarrow B^*(t_1||t_{k+1})),$
where k is the smallest index such that no free or bounded variable
of $A^* \wedge B^*$ occurs in $\{t_k, t_{k+1}, t_{k+2}, ...\}$.

$A(x||t)$ denotes the result of substituting the term t for the variable x in the formula A. †

The translation above is just a formalized version of the semantics of first-order temporal formulae. Now, the time references have been built explicitly into the language. Next we specify the translation of any temporal interpretation into an interpretation of $M(L)$.

Definition 47. For every time flow $(T, <)$ and interpretation $\mathcal{I}$ of $TL^{US}(L)$, an interpretation $\mathcal{I}^*$ of $M(L)$ is defined as follows. The interpretation of the terms of the original sort are inherited from $\mathcal{I}$. The time domain is T and the interpretation of $\prec$ is $<$, while for any predicate symbol $P(x_1, ..., x_n)$ of L, for any $d_1, ..., d_n \in D_{\mathcal{I}}$ and $t \in T$, we set $(d_1, ..., d_n, t) \in \mathcal{I}^*(P)$ iff $(d_1, ..., d_n) \in \mathcal{I}_t^p(P)$. †
The following lemma is also standard.

Lemma 7 (translation lemma). If $(T, <)$ is a frame, $\mathcal{I}$ is an interpretation of $TL^{US}(L)$, Θ is a valuation and $t \in T$ then $(T, <), \mathcal{I}, \Theta, t \models\!\!\!\vdash A$ iff $\mathcal{I}^* \models A^*\Theta^{@}(t_1/t)$. †
Proof. The proof is an induction on the complexity of A due to the corresponding definition of the semantics of first-order temporal logic and the appropriate translation into many-sorted first-order formulae. $\surd$
Now we prove the fundamental lemma which provides a *short* way to get the result of [R96] mentioned above and a slightly stronger result, too. Moreover, by the help of this lemma, another interesting theorem will be proved about the axiomatizability of first-order temporal theories on space-time flows. It coincides with Theorem 11. Practically, we prove this now for the given special operator set.

Lemma 8 (fundamental lemma). If the first-order theory of a countable time flow $(T, <)$ is ω-categorical and recursively enumerable, then $\mathbf{Th}_L^{US}(T, <)$ is axiomatizable.
Proof. Let Q denote the first-order theory of $(T, <)$ (Q is a subset of the set of the closed L-formulae). $\models_L$ and $\models_{M(L)}$ denote the consequence relations in L and $M(L)$, respectively.

We will show that for any closed $TL^{US}(L)$-temporal formula A
$A \in \mathbf{Th}_L^{US}(T, <)$ iff $Q \models_{M(L)} \forall t_1 A^*$.

This result yields the proof of the lemma, since being a consequence of a recursively enumerable set of formulae is a recursively enumerable relation.

Assume first that $A \notin \mathbf{Th}_L^{US}(T,<)$. We have then a temporal interpretation $\mathcal{I}$ over time flow $(T,<)$ such that there exists $t \in T$: $(T,<),\mathcal{I},(),t \mid\!\vdash \neg A$. On the basis of translation lemma, $\mathcal{I}^* \models_{M(L)} \neg A^*(t_1/t)$ and so $\mathcal{I}^* \models_{M(L)} \neg\forall t_1 A^*$. However, the time fragment of $\mathcal{I}^*$ is $(T,<)$, thus $\mathcal{I}^*$ is also a model for Q, so $Q \not\models_{M(L)} \forall t_1 A^*$.

On the other hand, assume second that $Q \not\models_{M(L)} \forall t_1 A^*$. Due to a Löwenheim—Skolem type theorem there exists a model $\mathcal{K}$ of $Q \cup \{\neg\forall t_1 A^*\}$ (in $M(L)$) whose time fragment $(\mathcal{K}^{time}, \prec_{\mathcal{K}})$ is of cardinality ω and by the ω-categoricity of Q this fragment is isomorphic to $(T,<)$. Let us fix such an isomorphism $f : (T,<) \to (\mathcal{K}^{time}, <_{\mathcal{K}})$. We will build an interpretation M (of $M(L)$) which is isomorphic to $\mathcal{K}$ and whose time fragment is $(T,<)$.

The first step in the construction is to require that the data domain of $\mathcal{M}$ and the interpretation of the terms involving only the data domain remain the same as in $\mathcal{K}$. The second step is to set $(d_1,...,d_k,t) \in \mathcal{M}(P)$ iff $(d_1,...,d_k,f(t)) \in \mathcal{K}(P)$, for any predicate symbol $P(x_1,...,x_k)$ of L, for any $t \in T$ and $d_1,...,d_k \in \mathcal{K}^{data}$.

It can be checked that an isomorphism is given between $\mathcal{M}$ and $\mathcal{K}$ by the disjoint union of f on $\mathcal{M}^{time}$ and the identity on $\mathcal{M}^{data}$. So $\mathcal{M}$ is a model in $M(L)$ such that its time domain is $(T,<)$ and $\mathcal{M} \models_{M(L)} \neg\forall t_1 A^*$.

If we provide a temporal interpretation $\mathcal{I}$ based on the frame $(T,<)$ such that $\mathcal{I}^* = \mathcal{M}$ then the proof of the fundamental lemma can be finished, due to the translation lemma. We accomplish this task by the following settings. $D_{\mathcal{I}} := \mathcal{K}^{data}$, the interpretation of the terms can be copied without any changes from $\mathcal{M}$ (and so from $\mathcal{K}$) and for the predicate interpretation of $\mathcal{I}^*$, for any predicate symbol $P(x_1,...,x_k)$, for any $t \in T$ and for any $d_1,...,d_k \in D_{\mathcal{I}}$, we set

$(d_1,...,d_k) \in \mathcal{I}_t^p(P)$ if and only if $(d_1,...,d_k,t) \in \mathcal{M}(P)$. $\surd$

One can repeat the proof of the fundamental lemma for arbitrary operator sets. In this way we can prove Theorem 11 and slightly strengthen the following theorem of Mark Reynolds in [R96] saying that the first-order temporal logic $\mathbf{Th}_L^{US}(\mathbb{Q},<)$ with arbitrary signature L and operator set $\{Until,\ Since\}$ is axiomatizable. †

Proof. [A simple one, for the above theorem of Reynolds and for slightly more] It is well-known that the first-order theory of $(\mathbb{Q},<)$

is both ω-categorical and recursive. (Usually this is established by a back-and-forth argumentation.) Thus, $\mathbf{Th}^U S_L(\mathbb{Q}, <)$ is recursively enumerable due to our fundamental lemma. $\surd$

Proof. [Theorem 10] In the same way we can obtain a proof for Theorem 10 considering van Benthem's theorem 4. $\surd$

8. Use of the axiomatizable spatio-temporal theory

Why is it important to establish axiomatizability results concerning temporal theories? As it was mentioned in the introduction, temporal logic can be employed in specifying time dynamics of computing devices. After specifying such a device by a temporal logic description, its partial or total correctness can be formalized by a temporal logic formula. To have correctness, it is sufficient to know the validity of this formula. An axiomatic proof system can help in the automated searching of the proof of the validity. If axiomatizability is established then one can start to develop a practically useful algorithm for proof searching.

To demonstrate the expressive power of our axiomatizable first-order spatio-temporal logic, we formalize in this language a relevant property of distributed systems of mobile agents. In the forthcoming example, we consider a spatially distributed system of mobile agents $A_1, A_2, \ldots, A_n$ (in this case, mobility is of a rather restricted kind, only 1 dimension the agents have to move along) which can send/receive messages to/from each other. For the sake of keeping simplicity we formalize a finite-state computing system that needs only a propositional temporal logic to formalize. First-order version is needed to describe agents with full computability power say of Turing machines. We do not plan to do this in this dissertation.

Let us recall Definition 1.4 of [Ma92]. There an n-fold distributed computation over a finite event set E as an n-tuple $(E_1, \ldots, E_n)$ with a relation $\Gamma \subset S \times R$ of corresponding send events S and receive events R was defined, such that each E_i is linearly ordered by a relation $\prec_i$ and the following three conditions hold:

1. The event sets $E_1, \ldots, E_n$ are pairwise disjoint,
2. Γ is left-unique and right-unique,
3. The smallest transitive relation $\prec$ which fulfils $(\forall ab \in E)\ (a \prec_i b \vee (a, b) \in \Gamma \rightarrow a \prec b)$ is an irreflexive partial order.

It would be more intuitive to call this object simply the communication trace of a distributed computation. We will do this. Mathematically, the communication trace can be described by an object

$(n, \{(E_i, L_i) \mid 0 \leqslant i < n\}, \Gamma)$, where $n \in \mathbb{N} \setminus \{0\}$ (the number of the processes), (E_i, L_i) is a finite linearly ordered chain for each $i < n$ (the distinguished events of A_i), and Γ is a binary relation on $E = \bigcup_{i<n} E_i$ satisfying conditions 1–3 and $(\forall e, f \in E_i) : (e, f) \notin \Gamma$, for each $i < n$.

We say such an object a realizable mobile distributed computation (on 1 space dimension) if it is possible to arrange a set of n mobile systems and its events in $(\mathbb{Q}^2, \blacktriangleleft)$,with the possibility of realizing the specified communications – that is, the specified causal connectabilities hold. In our example, the speed of the messages does not reach that of the light. More exactly, there are continuous timelike curves $A_1, \ldots, A_n$ in $\mathbb{Q}^2$ and points of the appropriate number on them in the appropriate order such that the corresponding causal connectabilities also hold. We do not give this condition in an even more formal manner, the translation into spatio-temporal language explains it.

We fix L as a propositional language, the temporal operators G and N are given in Definition 1. We are now in a position to understand the meaning of operator N. NA holds in a spacetime point q if and only if A holds in all the spacetime points $r \neq q$. $\neg N \neg A$ is abbreviated as OA. It holds in q if and only if A holds in an other spacetime point. Similarly, FA stands for $\neg G \neg A$.

For every object $C = (n, \{(E_i, L_i) \mid 0 \leqslant i < n\}, \Gamma)$, one can construct a propositional spatio-temporal formula ϕ_C of $T^{\{G,N\}}(L)$ such that C is realizable on 1 space dimension if and only if $\neg\phi_C \notin \mathbf{Th}_L^{\{G,N\}}(\mathbb{Q}^2, \blacktriangleleft$). Since the event set is finite, in this case we do not have to employ the full power of first-order spatio-temporal logic. Assume that E_i is enumerated by $e_{i,1}, \ldots, e_{i,|E_i|}$ in $\prec$-order where the definition of $\prec$ is the same as in Def. 1.4 of [Ma92] cited above.

The propositional variables of ϕ_C are $\{p_{ij} : i < n, j < |E_i|\}$ and ϕ_C itself is

$$\bigwedge_{i<n} \bigwedge_{j<|E_i|} O\left(p_{ij} \wedge \neg O p_{ij} \ \wedge \bigwedge_{k<n, k\neq i} \bigwedge_{j<|E_k|} \neg p_{kj}\right) \wedge$$

$$\bigwedge_{(e_{i,k} \prec\ e_{l,j})} N(p_{ik} \rightarrow F p_{lj}).$$

9. Additional results on spacetime theories

In this section we give some additional results on several spacetime-related theories, including first-order and first-order temporal ones.

9.1. Non-ω-categoricity

The first question is the following. From Theorem 11 and 8 we know that the first-order theory $\mathtt{Th}(\mathbb{Q}^n, \blacktriangleleft)$ is either not ω-categorical or not recursively enumerable if $n > 2$. It is hardly plausible that it is not recursively enumerable but ω-categorical but is not sure until it is proven. So we prove in this subsection that it is not ω-categorial indeed. For this subsection, let us fix an integer $n > 2$. We use a standard logical tool for this purpose, the theorem of characterization of ω-categoricity can be used from the book [CK]:

The first-order theory T of a countable ω-categorical structure has a finite number of types in $x_1, \ldots, x_m$, for every $m > 0$, where a type of T in $x_1, \ldots, x_m$ means a maximal consistent set of formulæ written with free variables only from the given set.

So, we provide an ω-sequence of different consistent formula sets with free variables from $\{x_1, x_2, x_3\}$.

By Statement 1, we have a partial equidistance first-order formula $H(x, y, z)$ written in the language of $(\mathbb{Q}^n, \blacktriangleleft)$ which expresses that x, y and z are lightlikely collinear, and y is the halfpoint of sector xz.

Definition 48. By recursion, we define a sequence $H_k(x, y, z)$ ($k \geqslant 1$) of first-order formulæ in this signature. Let $H_1(x, y, z)$ be just $H(x, y, z)$. For $k \geqslant 1$, let $H_{k+1}(x, y, z)$ be $\exists w(H(x, y, w) \wedge H_k(x, w, z))$. †

Statement 15. For any $p, q, r \in \mathbb{Q}^n$ and $k \geqslant 1$, if $H_k(p, q, r)$ then p, q, r are lightlikely collinear, q is between p and r and $2^k \cdot \delta(p, q) = \delta(p, r)$.†

Proof. For $k = 1$ it is just Statement 1 (x). Assume that k satisfies the inductive hypothesis and $H_{k+1}(p, q, r)$ holds, that is,
$\exists w(H(p, q, w) \wedge H_k(p, w, r))$. Let us choose an $s \in \mathbb{Q}^n$ satisfying $H(p, q, s) \wedge H_k(p, s, r)$. Then p, q, s and r are lightlikely collinear in

the given order and $2 \cdot \delta(p,q) = \delta(p,s)$ and $2^k \cdot \delta(p,s) = \delta(p,r)$ so $2^{k+1} \cdot \delta(p,q) = \delta(p,r)$. $\surd$
Proof. [Theorem 3.2] The formula sequence $\langle H_k(x,y,z) : k \geqslant 1\rangle$ means an infinite set of different 3-types of the given theory. $\surd$

We also observe the following.

Theorem 14. In the first-order theory $(\mathbb{Q}^2, \blacktriangleleft)$ there are no formulæ to express the properties of halfpoint, parallelity or even betweenness.
Proof. Probably, it would be more elegant to give $(\mathbb{Q}^n, \blacktriangleleft)$-automorphisms not preserving these relations but considering the preceding proof, we can observe that if we had such formulæ then we could construct similar proofs for non-ω-categoricity of the first-order theory of $(\mathbb{Q}^2, \blacktriangleleft)$ which contradicts to Theorem 4. For example, if we had a formula $B(x,y,z)$ expressing spacelike betweenness in $(\mathbb{Q}^2, \blacktriangleleft)$ then the items (v)-(x) of Definition 19 would work well also in the case of $n = 2$ so we could define the partial halfpoint relation in (x).

9.2. Monadic first-order theory over $(\mathbb{R}, <)$ is not axiomatizable

In this subsection we prove Theorem 13. We produce the proof as a variant of the proof of Theorem 8. We do not feel necessary to change the sentences in the proof only at the places of relevant differences and do not write down the proofs which are only slight modifications of their correspondent originals.

Definition 49. We introduce some defined temporal operators. If A is an arbitrary temporal formula then $\boxplus\!\!\rightarrow A$ stands for GA, only for the sake of easier visual reference. As usual, F denotes the existential counterpart of G, we write it also in visual form $\diamondplus\!\!\rightarrow$.

We fix some formulæ of temporal language TL_L^G, as follows.

Definition 50. $Id := \diamondplus\!\!\rightarrow (r(x) \wedge \boxplus\!\!\rightarrow \neg r(x))$, $\nu_1 := \boxplus\!\!\rightarrow \exists x(r(x) \wedge \boxplus\!\!\rightarrow \neg r(x))$. †

Let us fix a temporal interpretation $\mathcal{I}$ for TL_L^G on the flow $(\mathbb{R}, <)$ and a rational point $e \in \mathbb{R}$ satisfying $(\mathbb{R}, <), \mathcal{I}, (), e \Vdash \nu_1$. It is clear that then the same holds for all $e' \geqslant e (\in \mathbb{R})$. We will see later that

such an interpretation exists, it follows from the stronger result of Lemma 10. Through this and the next two subsections (till 9.4), these $\mathcal{I}$ and e remain fixed.

Definition 51. We define the relation $\varphi_0 \subseteq D_{\mathcal{I}} \times \mathbb{R}$ by the condition $(d,q) \in \varphi_0 \Leftrightarrow (\mathbb{R},<),\mathcal{I},\binom{x}{d},q \Vdash r(x) \wedge \boxplus\!\!\rightarrow \neg r(x)$. The set $\{d \in D_{\mathcal{I}} : (\mathbb{R},<),\mathcal{I},\binom{x}{d},e \Vdash Id\}$ will be denoted by ID_1. Further, φ_1 stands for the restriction of φ_0 to $ID_1 \times \mathbb{R}$. Let $\mathbb{R}_e$ denote $\{x \in \mathbb{R} : x > e\}$.†

Statement 16. With the notations of the previous definition, φ_1 is a surjective function taking ID_1 onto $\mathbb{R}_e$. †

Definition 52. The binary relation ρ on ID_1 is defined by the condition $(d_1,d_2) \in \rho \Leftrightarrow \varphi_1(d_1) = \varphi_1(d_2)$. †

Statement 17. ρ is an equivalence on ID_1. †

Definition 53. With the notations of the previous items, the set of the equivalence classes of ρ is denoted by ID. The function on ID corresponding to φ_1 is denoted by φ. More exactly, if the ρ-equivalence class of d is denoted by $[d]_\rho$, then $\varphi([d]_\rho)$ is just $\varphi_1(d)$. †

Statement 18. φ is a bijection from ID onto $\mathbb{R}_e$. †

Definition 54. Let Ord denote the formula $\diamondsuit\!\!\rightarrow (r(x) \wedge \diamondsuit\!\!\rightarrow r(y)) \wedge Id \wedge Id\binom{x}{y}$, where $A\binom{x}{y}$ is the result of substituting y for free occurrences of x in formula A. If $d_1,d_2 \in ID_1$ then we write $Ord_1(d_1,d_2)$ for $(\mathbb{R}_e,<),\mathcal{I},\binom{x\ y}{d_1 d_2},e \Vdash Ord$. †

Statement 19. The relation Ord_1 on ID_1 is compatible with ρ. That is, $Ord_1(d_1,d_2) \Leftrightarrow Ord_1(d_3,d_4)$ whenever $\{(d_1,d_2),(d_3,d_4)\} \subseteq \rho$. †

Definition 55. Let $Ord^{\mathcal{I}}$ denote the inherited relation of ρ-equivalence classes, that is, $Ord^{\mathcal{I}}$ is a binary relation on ID and for any $d_1,d_2 \in ID_1$, $Ord_1(d_1,d_2)$ if and only if $Ord^{\mathcal{I}}([d_1]_\rho,[d_2]_\rho)$. †

Statement 20. φ is an isomorphism from $(ID,Ord^{\mathcal{I}})$ onto $(\mathbb{R}_e,<)$. †

9.3. Isomorphism to the ordering of $\mathbb{N}$

Definition 56. The formula $\diamondsuit\!\!\!\!-\!\!\!\rightarrow (r(\delta) \wedge r(x) \wedge \boxdot\!\!\!\rightarrow \neg r(x))$ (in TL_L^G) will be abbreviated as $N(x)$. Further, we write $Ord(x,y) \wedge N(y) \wedge \neg\exists z(N(z) \wedge Ord(x,z) \wedge Ord(z,y))$ also in form $S(x,y)$. †

δ is used only as a parameter for separating a subset of ID without adding a new predicate symbol into L. This is natural enough. Dealing with predicates with more than one arguments requires a more sophisticated representation, as the next subsection will show. We use more than one parameter in the scope of a temporal operator, that is, we transgress the bounds the syntactic restrictions of monodic first-order temporal logic. This is the price of using only one predicate symbol without equality.

Definition 57. Let ν_2 be defined as conjunction of ν_1 and the following formulæ:

1 $\exists x(N(x) \wedge \forall y(Ord(y,x) \rightarrow \neg N(y)))$,
2 $\forall x \exists y S(x,y)$,
3 $\forall y(N(y) \wedge \exists x(Ord(x,y) \wedge N(x)) \rightarrow \exists x(Ord(x,y) \wedge S(x,y)))$.†

In the present case we do not need a complex geometrical representation. These simple requirements are enough to guarantee the following analogy of Lemma 3. We remind the reader that despite the first look, ν_2 is a temporal formula of TL_L^G, it involves only r as only predicate symbol.

Definition 58. $N^{\mathcal{I}} := \{[d]_\rho : (\mathbb{R},<), \mathcal{I}, \binom{x}{d}, q \Vdash N(x)\}$
and $S^{\mathcal{I}}$ can be defined in an analogous way. †

Let Θ denote a fixed valuation satisfying $dom\Theta \supseteq \{\delta\}$, till the end of the next subsection.

Lemma 9. If $(\mathbb{R},<), \mathcal{I}, \Theta, q \Vdash \nu_2$ then $(N^{\mathcal{I}}, Ord^{\mathcal{I}}, S^{\mathcal{I}})$ is isomorphic to $(\mathbb{N}, <, succ^r)$, where $succ^r = \{(n, n+1) : n \in \mathbb{N}\}$. †
Proof. By the requirements given in Definition 57, $(N^{\mathcal{I}}, Ord^{\mathcal{I}}, S^{\mathcal{I}})$ is a discrete linear ordering with a least element but without any maximum point. Its ϕ-image is in $(\mathbb{R},<)$ has the same properties by the isomorphism and, by Definiton 57[3], it has no accumulation point, consequently, it is isomorphic to $(\mathbb{N}, <, succ^r)$. If we denote the latter isomorphism by ϕ_2 then $\psi := \phi \circ \phi_2$ is an isomorphism from $(N^{\mathcal{I}}, Ord^{\mathcal{I}}, S^{\mathcal{I}})$ to $(\mathbb{N}, <, succ^r)$.

9.4. Representation of predicates of more than one argument

We introduce the following abbreviations in TL_L^G. They allow to represent predicates with more than argument.

Definition 59. $x \prec y := \diamondarrow (r(x) \wedge \diamondarrow r(y))$,
$x \approx y := \neg(x \prec y) \wedge \neg(y \prec x)$. †

Statement 21. If $d_1, d_2 \in ID_1$ then
$(\mathbb{R}, <), \mathcal{I}, , \binom{x\ y}{d_1 d_2}, e \Vdash x \prec y \Leftrightarrow Ord^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho)$ and
$(\mathbb{R}, <), \mathcal{I}, , \binom{x\ y}{d_1 d_2}, e \Vdash x \approx y \Leftrightarrow [d_1]_\rho = [d_2]_\rho)$.

The following is the main new idea of the present proof related to existing non-axiomatizability proofs in first-order temporal logic.

Definition 60. Let $null(x)$ denote the formula $N(x) \wedge \neg\exists u(N(u) \wedge u \prec x)$ while $one(x)$ denotes $\exists y(null(y) \wedge S(y, x))$.†

Definition 61.

$A(x, y, z) :=$
$(null(x) \wedge y \approx z) \vee$
$(null(y) \wedge x \approx z) \vee$
$[\neg(x \approx y) \wedge$
$\exists u \boxdot\!\!\rightarrow (r(u) \leftrightarrow$
$(r(x) \wedge \boxdot\!\!\rightarrow \neg r(x)) \vee$
$(r(y) \wedge \boxdot\!\!\rightarrow \neg r(y)) \vee$
$(r(z) \wedge \boxdot\!\!\rightarrow \neg r(z)))]$
$\vee$
$[x \approx y \wedge$
$\exists u \boxdot\!\!\rightarrow (r(u) \leftrightarrow$
$(r(x) \wedge \boxdot\!\!\rightarrow \neg r(x)) \vee$
$(r(z) \wedge \boxdot\!\!\rightarrow \neg r(z)))]$,

$M(x, y, z) :=$
$(null(x) \wedge null(z)) \vee$
$(null(y) \wedge null(z)) \vee$
$(one(x) \wedge y \approx z) \vee$
$(one(y) \wedge x \approx z) \vee$
$[\neg(x \approx y) \wedge$
$\exists uv(Id\binom{x}{v} \wedge z \prec v \wedge$

$$
\begin{array}{l}
\quad \boxminus\!\!\rightarrow (r(u) \leftrightarrow \\
\qquad\qquad (r(x) \wedge \boxminus\!\!\rightarrow \neg r(x)) \vee \\
\qquad\qquad (r(y) \wedge \boxminus\!\!\rightarrow \neg r(y)) \vee \\
\qquad\qquad (r(z) \wedge \boxminus\!\!\rightarrow \neg r(z)) \vee \\
\qquad\qquad (r(v) \wedge \boxminus\!\!\rightarrow \neg r(v))))] \\
\vee \\
[x \approx y \wedge \\
\exists uvw(Id\binom{x}{v} \wedge Id\binom{x}{w} \wedge z \prec v \prec w \wedge \\
\quad \boxminus\!\!\rightarrow (r(u) \leftrightarrow \\
\qquad\qquad (r(x) \wedge \boxminus\!\!\rightarrow \neg r(x)) \vee \\
\qquad\qquad (r(y) \wedge \boxminus\!\!\rightarrow \neg r(y)) \vee \\
\qquad\qquad (r(z) \wedge \boxminus\!\!\rightarrow \neg r(z)) \vee \\
\qquad\qquad (r(v) \wedge \boxminus\!\!\rightarrow \neg r(v)) \\
\qquad\qquad (r(w) \wedge \boxminus\!\!\rightarrow \neg r(w))))]\dagger.
\end{array}
$$

Once we have represented two predicates of three arguments by the means of our sole unary predicate symbol r (it was the harder to provide than what follows), we can endow these formulæ to represent addition and multiplication, in the expected way, by postulating the following ν_3 on them.

Definition 62. Let ν_3 is the conjuntion of ν_2 and the following conditions (the usual primitive recursive definitions addition and multiplication – in our representation):

(1) $\forall xy(N(x) \wedge N(y) \rightarrow \exists!^{\approx} z(N(z) \wedge A(x,y,z)) \wedge \exists!^{\approx} w(N(w) \wedge M(x,y,w)))$,

where $\exists!^{\approx}$ is to understand regarding $\approx$ as equality,

(2) $\forall xyzvw(N(x) \wedge N(y) \wedge N(z) \wedge N(v) \wedge N(w) \wedge$
$S(x,y) \wedge A(x,z,v) \wedge S(v,w) \rightarrow A(y,z,w))$,

(3) $\forall xyzvw(N(x) \wedge N(y) \wedge N(z) \wedge N(v) \wedge N(w) \wedge$
$S(x,y) \wedge M(x,z,v) \wedge A(v,z,w) \rightarrow M(y,z,w))$. $\dagger$

Definition 63. Let $A^{\mathcal{I}}$ and $M^{\mathcal{I}}$ denote the meaning of A and M, resp., on ID. So, for $d_1, d_2, d_3 \in ID_1$, we write also $A^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho, [d_3]_\rho)$ for $(\mathbb{R}, <), \mathcal{I}, \binom{x\ y\ z}{d_1 d_2 d_3}, e \Vdash A(x,y,z)$, and similar applies to $M^{\mathcal{I}}$.

Statement 22. If $(\mathbb{R}_e, <), \mathcal{I}, \Theta, e \Vdash \nu_3$ then $(N^{\mathcal{I}}, Ord^{\mathcal{I}}, S^{\mathcal{I}}, A^{\mathcal{I}}, M^{\mathcal{I}})$ is isomorphic to $(\mathbb{N}, <, succ^r, +^r, *^r)$, where

$succ^r = \{(n, n+1) : n \in \mathbb{N}\}$, $+^r = \{(k,l,m) \subseteq \mathbb{N}^3 : k + l = m\}$ and $*^r$ is $\{(k,l,m) \subseteq \mathbb{N}^3 : k \cdot l = m\}$. †

The isomorphism is given by ψ defined in the proof of Lemma 9.

9.5. Translation of true arithmetics into the first-order temporal theory

In this section, the proof is finished by the usual way of non-axiomatizability proofs of first-order temporal theories. The only difference is that consistency is not straightforward because of the rather complex way of representing the three-argument predicate symbols.

Lemma 10. There exist a temporal interpretation for TL_L^G on the time flow $(\mathbb{R}, <)$, a point $q \in \mathbb{R}$ and a valuation Θ in $\mathcal{I}$ such that $(\mathbb{R}, <), \mathcal{I}, \Theta, q \Vdash \nu_3$. †

Proof. We supply only the asked $\mathcal{I}$, q and Θ and leave checking satisfaction of ν_3 to the reader. q can be specified as 0. Let $D_\mathcal{I}$ be the set

$$\mathbb{R}_+ \cup \{D\} \cup$$
$$\{A_{k,l,m} : k + l = m, k \neq 0, l \neq 0, k \neq l\} \cup \{A^{=}_{k,m} : k + k = m, k \neq 0\} \cup$$
$$\{M_{k,l,m} : k \cdot l = m, k > 1, l > 1, k \neq l\} \cup \{M^{=}_{k,m} : k \cdot k = m, k > 1\},$$

where D and the other objects are just formal symbols.

The interpretation $\mathcal{I}_t^p$ is defined via its value on predicate symbol r. We write shortly r_t for $\mathcal{I}_t^p(r)$. r_t can be defined via the definition for the truth values $r_t(d)$, for arbitrary $d \in D_\mathcal{I}$. We utilize a unique enumeration $[[[k,m,n]]]$ of triples of natural numbers satisfying $[[[k,m,n]]] > max(k,m,n)$ and another unique enumeration $[[k,m]]$ of pairs of natural numbers satisfying $[[k,m]] > max(k,m)$.
If $d = q \in \mathbb{R}_e$ then $r_t(q) = (t = q)$,
if $d = D$ then $r_t(D) = (t \in \mathbb{N})$,
if $d = A_{k,l,m}$ for $k,l,m \in \mathbb{N}$ satisfying $k + l = m, k \neq 0, l \neq 0, k \neq l$ then $r_t(A_{k,l,m}) = (t = k \vee t = l \vee t = m)$,
if $d = A^{=}_{k,m}$ for $k,m \in \mathbb{N}$ satisfying $2 \cdot k = m, k \neq 0$, then
$r_t(A^{=}_{k,m}) = (t = k \vee t = m)$,
if $d = M_{k,l,m}$ for $k,l,m \in \mathbb{N}$ satisfying $k \cdot l = m, k \geqslant 2, l \neq 2, k \neq l$ then $r_t(M_{k,l,m}) = (t = k \vee t = l \vee t = m \vee t = [[[k,l,m]]])$,
if $d = M^{=}_{k,m}$ for $k,m \in \mathbb{N}$ satisfying $2 \cdot k = m, k \neq 0$, then

$r_t(M^{=}_{k,m}) = (t = k \vee t = m \vee t = [[k,m]] \vee t = [[k,m]] + 1)$.

The valuation Θ can be determined by setting $\Theta(\delta) = D$.

Definition 64. For any first-order formula A in the signature of $(\mathbb{N}, <, succ^r, +^r, *^r)$ (somewhat loosely, we does not differ the predicate symbol from the corresponding interpreting relation), we give a translation A^t into TL^G_L, by structural induction, as follows. We assume that the variables of the arithmetical language are exactly that of L minus $\{\delta\}$.

$(x < y)^t = Ord(x,y)$,
$(succ^r(x,y))^t = S(x,y)$, where S is defined in 56,
$(+^r(x,y,z))^t = A(x,y,z)$, where A is defined in 9.4,
$(*^r(x,y,z))^t = M(x,y,z)$, where M is defined in 9.4,
$(A \wedge B)^t = (A^t \wedge B^t)$, $(\neg A)^t = \neg A^t$ and
$(\forall x A)^t = \forall x(N(x) \to A^t)$. †

Definition 65. Assume that $\mathcal{I}$ is a temporal interpretation for TL^G_L on the time flow $(\mathbb{R}_e, <)$ which also satisfies ν_3, and Θ is a valuation in $\mathcal{I}$. We associate a valuation $\Sigma \oplus \Theta$ of the variables of that temporal language in $\mathcal{I}$, to every valuation Σ of the variables of the arithmetical language into $\mathbb{N}$.

Value for δ comes from Θ, that is, $(\Sigma \oplus \Theta)(\alpha) = \Theta(\alpha)$, while the other variables gets value by $(\Sigma \oplus \Theta)(x) = \underline{\psi}^{-1}(\Sigma(x))$ where $\underline{\psi}$ is the isomorphism ψ from $(ID, Ord^{\mathcal{I}}, S^{\mathcal{I}}, A^{\mathcal{I}}, M^{\mathcal{I}})$ to $(\mathbb{N}, <, s^r, \underline{+^r}, \cdot^r)$ given in Lemma 9 prefixed by a class-selection function from ID onto ID_1.†

Statement 23. Let us assume that $\mathcal{I}$ is a temporal interpretation for TL^G_L on the time flow $(\mathbb{R}, <)$, $q \in \mathbb{R}$, Θ is a valuation in $\mathcal{I}$ such that $(\mathbb{R}, <), \mathcal{I}, \Theta, q \Vdash \nu_3$, further, that A is a first-order formula in the language of $(\mathbb{N}, <, succ^r, +^r, *^r)$ and Σ is a valuation of the variables of the last language into $\mathbb{N}$. Then we have

$(\mathbb{N}, <, succ^r, +^r, *^r) \models A\Sigma$ if and only if $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Sigma \oplus \Theta, q \Vdash A^t$. †

Statement 24. If A is a closed first-order formula in the language of $(\mathbb{N}, <, succ^r, +^r, *^r)$ then we have $A \in Th\ (\mathbb{N}, <, succ^r, +^r, *^r)$ if and only if $\forall\delta(\nu_3 \to A^t) \in \mathtt{Th}^G_L(\mathbb{R}, <)$, where $Th\ K$ denotes the first-order theory of structure K. †

Proof. [Proof of Theorem 13] If $\mathtt{Th}_L^G(\mathbb{R}, <)$ would be recursively enumerable then $\mathtt{Th}_L^G(\mathbb{R}, <) \cap$
$\{\forall\delta(\nu_3 \rightarrow A^t) | A$ *is an arithmetical formula*$\}$ would be recursively enumerable, too. This is impossible by the previous lemma. $\surd$

10. Definitions and results on interval-valued computations

10.1. Former results in interval-valued computation

As we mentioned in the introduction, in conference paper [N05b], Benedek Nagy was proposed an interval-valued computing system, furthermore, a solution of the most typical NP-complete problem (SAT) was presented. The idea of investigating the many-valued logic of the interval-values arised in [N05a] and before this, in a thesis of Benedek Nagy.

10.2. Definitions and our results on interval-valued computation

In this section we formalize the interval-valued computing system of [N05b] following the definitions of our joint paper [NV06].

First we define what an interval-value means. Then we present the allowed operations which can be used to build and evaluate computation sequences. Finally, we give the notions concerning decidability and computational complexity.

10.3. Interval-values

We note in advance that we do not distinguish interval-values (specific functions from [0,1) into $\{0,1\}$) from their subset representations (subsets of [0,1)) and we use always the more convenient notation.

Definition 66. The set $\mathbb{V}$ of *interval-values* coincides with the set of finite unions of [)-type subintervals of $[0,1)$. †

Definition 67. The set $\mathbb{V}_0$ of *specific interval-values* coincides with
$\left\{ \bigcup_{i=1}^{k} \left[\frac{l_i}{2^m}, \frac{1+l_i}{2^m}\right) \ : m \in \mathbb{N}, k \leqslant 2^m, 0 \leqslant l_1 < \ldots < l_k < 2^m \right\}$. †

We note that the set of finite unions includes the empty set ($k = 0$), that is, $\emptyset$ is also an allowed interval-value. Essentially, the notion of interval-value coincides with the notion of generalized interval

([BCFO98], [L91]). In interval temporal logic ([A83]), these intervals represent occurring, non-contiguous events. The main difference between the proposed interval-valued computational model and the existing interval logic approach is that the latter deals with problems about interval-values while the proposed system computes classical decision problems with the help of computations on such interval-values. For example, the proposed fractalian product is an operation that cannot be expressed by usual interval logic relations. However, generalized interval logic can provide tools for reasoning about interval-valued computations.

10.4. Operators on interval-values

Similarly to traditional computers working on bytes, of course, we allow bitwise Boolean operations. If we consider interval-values as subsets of [0,1) then the corresponding operations coincide with the set-theoretical operations of complementation ($\overline{A}$), union ($A \cup B$) and intersection ($A \cap B$). $\mathbb{V}$ forms an infinite Boolean set algebra with these operations. $\mathbb{V}_0$ is an infinite subalgebra of the last algebra.

Before we add some other operators, we introduce a function assisting the formulation of the following definition. Intuitively, it provides the length of the left-most component (included maximal subinterval) of an interval-value A.

Definition 68. We define the function $Flength : \mathbb{V} \to \mathbb{R}$ as follows. If there exist $a, b \in [0, 1]$ satisfying $[a, b) \subseteq A$, $[0, a) \cap A = \emptyset$ and $[a, b') \not\subseteq A$ for all $b' \in (b, 1]$, then $Flength(A) = b - a$, otherwise $Flength(A) = 0$. †

$Flength$ helps us to define the binary shift operators on $\mathbb{V}$. The *left-shift* operator will shift the first interval-value to the left by the first-length of the second operand and remove the part which is shifted out of the interval $[0, 1)$. As opposed to this, the *right-shift* operator is defined in a circular way, i.e. the parts shifted above 1 will appear at the lower end of $[0, 1)$. In this definition we write interval-values in their original, „characteristic function" notation.

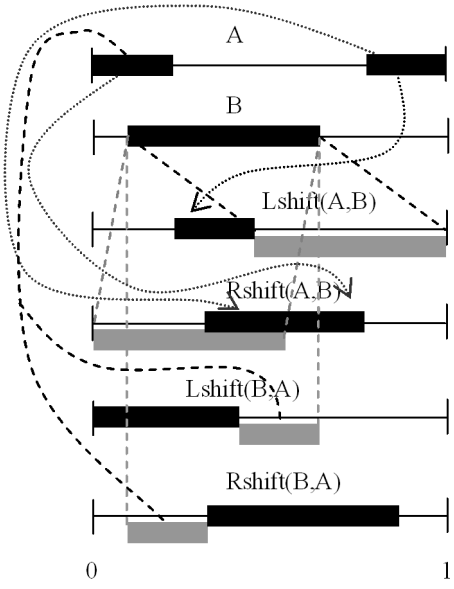
Definition 69. The binary operators $Lshift$ and $Rshift$ on $\mathbb{V}$ are defined in the following way. If $x \in [0, 1]$ and $A, B \in \mathbb{V}$ then

$$Lshift(A,B)(x) = \begin{cases} A(x + Flength(B)), & \text{if } 0 \leqslant x + Flength(B) \leqslant 1, \\ 0 \text{ in other cases.} \end{cases}$$

$$Rshift(A,B)(x) = \begin{cases} A(frac(x - Flength(B))), & \text{if } x < 1, \\ 0 \text{ if } x = 1. \end{cases}$$

Here the function *frac* gives the fractional part of a real number, i.e., $frac(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer which is not greater than x. †

In Figure 4 some examples can be seen for both operations $Rshift$ and $Lshift$. The second (ancillary-) operands are shown in grey to assist understanding, but they are not the real parts of the resulted interval-values. Now we explain the so-called *fractalian product* on intervals.



4. ábra. Examples of shift operators with interval-values

Definition 70. Let A and B be general interval-values and $x \in [0,1)$. Then the fractalian product $A * B$ includes x if and only if

$B(x) = 1$ and $A\left(\frac{x-x_B}{x^B-x_B}\right) = 1$, where x_B denotes the lower end-point of the B-component including x and x^B denotes the upper end-point of this component, that is, $[x_B, x^B)$ is the maximal subinterval of B containing x. †

We can give this operation in a more descriptive manner. If A contains k interval components with ends $a_{i,1}, a_{i,2}$ $(1 \leqslant i \leqslant k)$ and B contains l components with ends $b_{i,1}, b_{i,2}$ $(1 \leqslant i \leqslant l)$, then we determine the value of $C = A * B$ as follows: we set the number of components of C to be $k \cdot l$. For this process we can use double indices for the components of C. The starting- and end points of the ij-th component are $a_{i1} + b_{j1}(a_{i2} - a_{i1})$ and $a_{i1} + b_{j2}(a_{i2} - a_{i1})$, respectively.

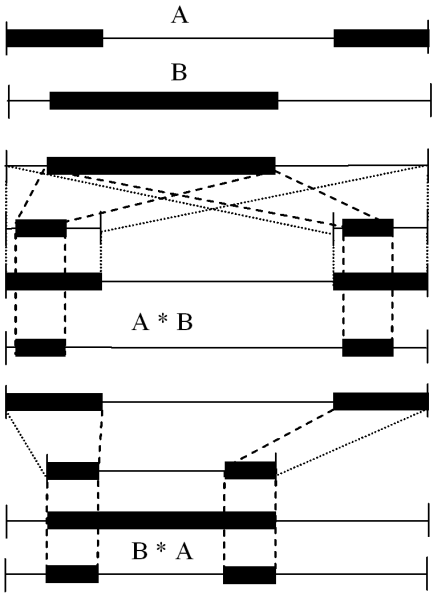
The idea and the role of this operation is similar to that of unlimited shrinking of 2-dimensional images in [WN05]. It will be used to connect interval-values of different resolution. We note, that for the results of the present paper, it would be enough to introduce a restricted version of product operation, taking products by only with $\left[0, \frac{1}{2}\right)$ as an operand. For future extensions of this research, we keep the binary product in the definition.

As we can observe in Figure 5, as well, the fractalian product of two interval-values is the result of shrinking the first operand to each component of the second one.

10.5. Syntax and semantics of computation sequences

In this subsection, we formalize the interval-valued computations of [N05b]. This formalization is of Boolean network style, since equality or similar tests do not seem to be easily implementable for interval-values, just like in the case of optical computing (no tests for equalities on images). As usual, the length of a sequence S is denoted by $|S|$ and its i-th element by S_i. If $j \leqslant |S|$ then the j-length prefix of S is denoted by $S_{\rightarrow j}$.

Definition 71. An *interval-valued computation sequence* is a nonempty finite sequence S satisfying $S_1 = FIRSTHALF$ and further,



5. ábra. Examples for product of interval-values

for any $i \in \{2, \ldots, |S|\}$, S_i is (op, l, m) for some $op \in \{AND, OR, LSHIFT, RSHIFT, PRODUCT\}$ or S_i is (NOT, l) where $\{l, m\} \subseteq \{1, \ldots, i-1\}$. The *bit height* of a computation is the number of the applied $PRODUCT$ operators in it. †

The semantics of interval-valued computation sequences is defined by induction on the length of the sequences. The *interval-value* of such a sequence S is denoted by $\|S\|$ and defined by induction on the length of the computation sequence, as follows.

Definition 72. First, we fix $\|(FIRSTHALF)\|$ as $\left[0, \frac{1}{2}\right)$. Second, if S is an interval-valued computation sequence and $|S|$ is its length, then

$$\|S\| = \begin{cases} \|S_{\to j}\| \cap \|S_{\to k}\|, \text{ if } S_{|S|} = (AND, j, k), \\ \|S_{\to j}\| \cup \|S_{\to k}\|, \text{ if } S_{|S|} = (OR, j, k) \\ \|S_{\to j}\| * \|S_{\to k}\|, \text{ if } S_{|S|} = (PRODUCT, j, k) \\ Rshift(\|S_{\to j}\|, \|S_{\to k}\|), \text{ if } S_{|S|} = (RSHIFT, j, k) \\ Lshift(\|S_{\to j}\|, \|S_{\to k}\|), \text{ if } S_{|S|} = (LSHIFT, j, k) \\ \overline{\|S_{\to j}\|}, \text{ if } S_{|S|} = (NOT, j). \dagger \end{cases}$$

One can notice, that in this formulation of interval-valued computations, only *specific* interval-values (cf. Definition 67) appear as values of computation sequences. However, this observation only strengthens our main result (Theorem 15) and makes it more likely to find implementations.

10.6. Decidability

In this subsection, we give the definitions concerning interval-valued computability and complexity.

Definition 73. Let Σ be a finite alphabet and let $L \subseteq \Sigma^*$ be a language. We say that L is *decidable by an interval-valued computation* if there is an algorithm A that for each input word $w \in \Sigma^*$ constructs an appropriate computation sequence $A(w)$ such that $w \in L$ if and only if $\|A(w)\|$ is nonempty. Furthermore, we consider $\overline{L}$ also decidable in this case. †

This last remark makes it possible to test emptiness and, by applying set-theoretical operators, also to test whether $\|A(w)\| = [0, 1)$. The following statement is straightforward, since, algorithm A can be abitrary, on the one hand, and by the obvious fact, that if a language is decidable by an interval-valued computation then one can calculate and track the sequence of the limiting points (rationals in this model) of all the components of the actual interval-values, on the other.

Statement 25. The class of languages decidable by an interval-valued computation coincides with the class of recursive languages. †

This fact shows that we have to narrow down the notion of acceptable interval-valued computations. In [N05b], SAT was solved by a linear interval-valued computation in the following meaning.

Definition 74. We say that a language $L \subseteq \Sigma^*$ is *decidable by a linear interval-valued computation* if and only if there is a positive constant c and a logarithmic space algorithm A with the following properties. For each input word $w \in \Sigma^*$, A constructs an appropriate interval-valued computation sequence $A(w)$ such that $|A(w)|$ is not greater than $c \cdot (|w|)$ and $w \in L$ if and only if $\|A(w)\|$ is nonempty. Again, deciding $\overline{L}$ instead of L itself is allowed. †

In this operator network style formulation of interval-valued computations, the size of the network is constrained. The question was raised in [N05b] whether there exists a $PSPACE$-complete language decidable by a linear interval-valued computation. We will answer this question in the next section. To solve all the problems in $PSPACE$ by interval-valued computations, it is useful to introduce the following notions.

Definition 75. We say that a language $L \subseteq \Sigma^*$ is *decidable by a restricted polynomial interval-valued computation* if and only if there is a polynomial P and a logarithmic space algorithm A with the following properties. For each input word $w \in \Sigma^*$, A constructs an appropriate interval-valued computation sequence $A(w)$ containing product operators only of the form $(PRODUCT, 1, n)$ such that $|A(w)|$ is not greater than $P(|w|)$, further, $w \in L$ if and only if $\|A(w)\|$ is nonempty. Again, deciding $\overline{L}$ instead of L itself is allowed. If we omit the condition on the $PRODUCT$ operators, we obtain the notion of *polynomial interval-valued computations.* †

Having this restriction on products, one can take products of an interval-value only by the starting interval-value $\left[0, \frac{1}{2}\right)$. As the main result of the paper we will show that this restriction leads to a class of interval-valued computations that decide exactly the languages of $PSPACE$.

Our motivation to define linear interval-valued computations in this way was to make explicit in what sense [N05b] stated that a linear computation exists to decide SAT.

Under this restriction on products, one can multiply an interval-value only by the starting interval-value $\left[0, \frac{1}{2}\right)$. In this paper, we set this extra condition on products compared to the respective definition of [NV06] but one can check that this restriction does not break down neither the result nor the proof of [NV06] (its linear interval-valued computation to decide $QSAT$ can be performed in the restricted linear way, too). Our main results are the following.

Theorem 15 ([NV06]). There is a $PSPACE$-complete language which can be decided by a linear inter-valued computation.

Theorem 16 ([NV07]). The class of languages decidable by a restricted polynomial interval-valued computation coincides with $PSPACE$.

11. A linear interval-valued computation to decide a $PSPACE$-complete problem

11.1. The language of true quantified propositional formulae (QSAT)

We recall now the definition of (a suitable variant of) the language $QSAT$ of *true quantified propositional formulae*. It is a subset of satisfiable propositional formulae, say, built from the propositional variables $\{x_1, x_2, \ldots\}$, by the logical operators $\neg, \wedge, \vee$. We *do not explicitly put the quantifier prefix to the propositional formulae*, only the definition of the language is given this way. Variables with odd indices are meant to quantify universally while those with even indices to quantify existentially. It can be shown by renaming of variables and using fictive quantifiers that this variant is equally $PSPACE$-complete as the original $QSAT$ ([P94]). Before we define $QSAT$, we have to make some preparations.

Definition 76. A *valuation* is a function with range $\{0, 1\}$ on the domain $\{x_1, \ldots, x_n\}$ for some positive integer n. If $t_1, \ldots, t_n$ are truth values then we write $(t_1, \ldots, t_n)$ for the valuation v that $v(x_1) = t_1, \ldots, v(x_n) = t_n$ and $dom(v) = \{x_1, \ldots, x_n\}$. For a quantifier-free formula ϕ, $[[\phi v]]$ denotes the *truth value* of ϕ by the valuation v. For any positive integer i, *the quantifier* Q_i is $\forall$ if i is odd otherwise it is $\exists$. †

Definition 77. For any propositional formula ϕ, ϕ belongs to $QSAT$ if and only if there exists a positive integer n such that the propositional variables in ϕ are exactly $x_1, \ldots, x_n$ and $(\forall t_1 \in \{0, 1\})(\exists t_2 \in \{0, 1\}) \ldots (Q_n t_n \in \{0, 1\}) : [[\phi(t_1, \ldots, t_n)]] = 1$ holds. †

Example 1. $\phi = (((x_1 \equiv x_2) \wedge \neg x_4)) \vee (x_3 \wedge ((\neg x_1 \wedge x_2 \wedge \neg x_4) \vee (x_1 \wedge \neg x_2 \wedge x_4)))$ is in $QSAT$, since $(\forall t_1 \in \{0, 1\})(\exists t_2 \in \{0, 1\})(\forall t_3 \in \{0, 1\})(\exists t_4 \in \{0, 1\}) : [[\phi(t_1, t_2, t_3, t_4)]] = 1$ holds. (Here $\equiv$ is the usual abbreviation of the logical connective 'equivalence'.) The index

of a propositional variable determines if it is universally or existentially quantified.

11.2. A linear interval-valued computation to decide QSAT

The following result implies Theorem 15.

Theorem 17. QSAT is decidable by a linear interval-valued computation. †

Proof. We give an algorithm to construct the computation sequence $K_1, \ldots, K_{11n+m-1}$ for any input formula ϕ that contains exactly the variables $x_1, \ldots, x_n$ and the number of its subformulae is m. The length of this list is less than $13 \cdot |\phi|$, where $|\phi|$ is the length of ϕ. The algorithm provides the above computation sequence in such a way that its interval-value will be empty if and only if $\phi \in QSAT$.

Let K_1 be $FIRSTHALF$. For all positive integers $k \leqslant n$, we define $K_{3k-1} = (PRODUCT, 1, 3k-2)$, $K_{3k} = (RSHIFT, 3k-1, 3k-2)$ and $K_{3k+1} = (OR, 3k, 3k-1)$.

By induction on k one can establish the following statement.

Lemma 11. For all positive integer k, if $k \leqslant n$ then

$$\|K_{\rightarrow 3k-2}\| = \bigcup_{l=0}^{2^{k-1}-1} \left[\frac{2l}{2^k}, \frac{2l+1}{2^k}\right).$$

†

The n independent truth values of $x_1, \ldots, x_n$ will be represented by the interval-values $\|K_{\rightarrow 1}\|, \|K_{\rightarrow 4}\|, \ldots, \|K_{\rightarrow 3n-2}\|$. In the first four lines of Figure 6 one can observe $\|K_{\rightarrow 1}\|$, $\|K_{\rightarrow 4}\|$, $\|K_{\rightarrow 7}\|$ and $\|K_{\rightarrow 11}\|$.

Now we establish some further properties of $\|K_{\rightarrow 1}\|$, $\|K_{\rightarrow 4}\|, \ldots,$ $\|K_{\rightarrow 3n-2}\|$.

Lemma 12. For every $r \in [0, 1)$ and positive integer $j \leqslant n$ hold the following conditions.

(1) if $r \in \|K_{\rightarrow 3j-2}\|$ then for all $i < j$, $r + \frac{1}{2^j} \in \|K_{\rightarrow 3i-2}\|$ if and only if $r \in \|K_{\rightarrow 3i-2}\|$,

(2) if $r \notin \|K_{\to 3j-2}\|$ then for all $i < j$, $r - \frac{1}{2^j} \in \|K_{\to 3i-2}\|$ if and only if $r \in \|K_{\to 3i-2}\|$,
(3) $r + \frac{1}{2^j} \in \|K_{\to 3j-2}\|$ if and only if $r \notin \|K_{\to 3j-2}\|$.

†

Let $\phi_1, \ldots, \phi_m$ be an enumeration of all the subformulae of ϕ such that any formula is preceded by its subformulae (consequently, $\phi_m = \phi$). The algorithm gives the next part of the computation sequence $(K_{3n-2+1}, \ldots, K_{3n-2+m})$ in the following way. For each $i \in \{1, \ldots, m\}$,

$$K_{3n-2+i} = \begin{cases} (AND, 3n-2+j, 3n-2+k) \text{ if } \phi_i = \phi_j \wedge \phi_k, \\ (OR, 3n-2+j, 3n-2+k) \text{ if } \phi_i = \phi_j \vee \phi_k, \\ (NOT, 3n-2+j) \text{ if } \phi_i = \neg\phi_j, \\ (AND, 3j-2, 3j-2) \text{ if } \phi_i = x_j. \end{cases}$$

By induction on j the following statement can be verified.

Lemma 13. For each $j \in \{1, \ldots, m\}$, $\|K_{\to 3n-2+j}\| = \{r \in [0,1) : [[\phi_j(r \in \|K_{\to 1}\|, r \in \|K_{\to 4}\|, \ldots, r \in \|K_{\to 3n-2}\|)]] = 1\}$ holds. †

So far, we have obtained a linear size computation sequence to decide the satisfiability of $\phi(= \phi_m)$ by the validity of the following equivalence: ϕ is satisfiable if and only if $\|K_{\to 3n-2+m}\|$ is nonempty. This can be concluded from the fact, that for each n-tuple $(t_1, \ldots, t_n)$ of truth values there is an $r \in [0,1)$ such that $(\forall i \in \{1, \ldots, n\})$: $t_i = r \in \|K_{\to 3i-2}\|$.
The computation sequence continues with
$K_{3n-2+m+1}, \ldots, K_{3n-2+m+8n}$ so that for each integer $j < n$, the following holds.

$\|K_{\to 3n-2+m+8(j+1)}\| =$
$((Lshift(\|K_{\to 3n-2+m+8j}\|, \|K_{\to 3(n-j)-2}\|) \cap \|K_{\to 3(n-j)-2}\|)$
$\quad \cup \|K_{\to 3n-2+m+8j}\|)$
$\cup((Rshift(\|K_{\to 3n-2+m+8j}\|, \|K_{\to 3(n-j)-2}\|) \cap \overline{\|K_{\to 3(n-j)-2}\|})$
$\quad \cup \|K_{\to 3n-2+m+8j}\|)$, if $n-j$ is even,
and $(Lshift(\|K_{\to 3n-2+m+8j}\|, \|K_{\to 3(n-j)-2}\|) \cap \|K_{\to 3(n-j)-2}\|$
$\quad \cap \|K_{\to 3n-2+m+8j}\|)$
$\cup(Rshift(\|K_{\to 3n-2+m+8j}\|,$
$\quad \|K_{\to 3(n-j)-2}\|) \cap \overline{\|K_{\to 3(n-j)-2}\|} \cap \|K_{\to 3n-2+m+8j}\|)$,
in the other case.

In this definition, we do not specify all the intermediate expressions between $K_{3n-2+m+8j}$ and $K_{3n-2+m+8(j+1)}$, they are the subexpressions of $K_{3n-2+m+8(j+1)}$ needed to express $K_{3n-2+m+8(j+1)}$ from $K_{3n-2+m+8j}$ and $K_{3(n-j)-2}$.

To make the next lemma more readable, we assume without any further mention, that variables $t_1, t_2, \ldots, t_n$ range over the truth values. We recall that the quantifier sequence $Q_1, Q_2, Q_3, \ldots$ is defined as $\forall, \exists, \forall, \ldots$, respectively.

Lemma 14. For each $j \in \{0, \ldots, n\}$ and for all $r \in [0,1)$:
$r \in \|K_{\to 3n-2+m+8j}\|$ if and only if
$Q_{n-j+1}t_{n-j+1} \ldots Q_n t_n$
$\big[\big[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3(n-j)-2}\|, t_{n-j+1}, \ldots, t_n)\big]\big] = 1$. †
Proof. The proof goes by induction on j from 0 up to n. For $j = 0$, Lemma 13 implies the needed equivalence, which is $r \in \|K_{\to 3n-2+m}\|$ if and only if $[[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3\cdot n-2}\|)]] = 1$.

Assume $j < n$. Let the induction hypothesis be the following. For any $r \in [0,1)$, $r \in \|K_{\to 3n-2+m+8j}\|$ if and only if
$Q_{n-j+1}t_{n-j+1} \ldots Q_n t_n$
$\big[\big[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3(n-j)-2}\|, t_{n-j+1}, \ldots, t_n)\big]\big] = 1.$

We have to show that $r \in \|K_{\to 3n-2+m+8(j+1)}\|$ if and only if
$Q_{n-j}t_{n-j} \ldots Q_n t_n$
$\big[\big[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3(n-(j+1))-2}\|, t_{n-j}, \ldots, t_n)\big]\big] = 1,$
for arbitrary $r \in [0,1)$.

As a proof, we write a sequence of equivalent conditions starting with
$r \in \|K_{\to 3n-2+m+8(j+1)}\|$ and closing with the right side of the equivalence. We prove the case when $n-j$ is even and Q_{n-j} is $\exists$, the proof, when $n-j$ is odd, can be constructed analogously.

(i) $r \in \|K_{\to 3n-2+m+8(j+1)}\|$;
(ii) $r \in \|K_{\to 3n-2+m+8j}\|$ or
$\big(r \in Lshift(\|K_{\to 3n-2+m+8j}\|, \|K_{\to 3(n-j)-2}\|) \wedge r \in \|K_{\to 3(n-j)-2}\|\big)$
or
$\big(r \in Rshift(\|K_{\to 3n-2+m+8j}\|, \|K_{\to 3(n-j)-2})\| \wedge r \in \overline{\|K_{\to 3(n-j)-2}\|}\big)$;
(iii) $\forall t_{n-j+1} \ldots Q_n t_n$
$\big[\big[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3(n-j)-2}\|, t_{n-j+1}, \ldots, t_n)\big]\big] = 1$

or
$(r \in \|K_{\to 3(n-j)-2}\| \wedge \forall t_{n-j+1} \ldots Q_n t_n$
$\big[\big[\phi(r + \frac{1}{2^{n-j}} \in \|K_{\to 3\cdot 1-2}\|, \ldots, r + \frac{1}{2^{n-j}} \in \|K_{\to 3(n-j)-2}\|, t_{n-j+1}, \ldots, t_n)\big]\big]$
$= 1)$
or
$(r \notin \|K_{\to 3(n-j)-2}\| \wedge \forall t_{n-j+1} \ldots Q_n t_n$
$\big[\big[\phi(r - \frac{1}{2^{n-j}} \in \|K_{\to 3\cdot 1-2}\|, \ldots, r - \frac{1}{2^{n-j}} \in \|K_{\to 3(n-j)-2}\|, t_{n-j+1}, \ldots, t_n)\big]\big]$
$= 1)$;

(iv) $\forall t_{n-j+1} \ldots Q_n t_n$
$\big[\big[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3(n-j)-2}\|, t_{n-j+1}, \ldots, t_n)\big]\big] = 1$ or
$\forall t_{n-j+1} \ldots Q_n t_n$
$\Big[\Big[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3(n-(j+1))-2}\|, r \notin \|K_{\to 3(n-j)-2}\|,$
$t_{n-j+1}, \ldots, t_n)\Big]\Big] = 1$;

(v) $\exists t_{n-j} \forall t_{n-j+1} \ldots Q_n t_n$
$\big[\big[\phi(r \in \|K_{\to 3\cdot 1-2}\|, \ldots, r \in \|K_{\to 3(n-(j+1))-2}\|, t_{n-j}, \ldots, t_n)\big]\big] = 1.$

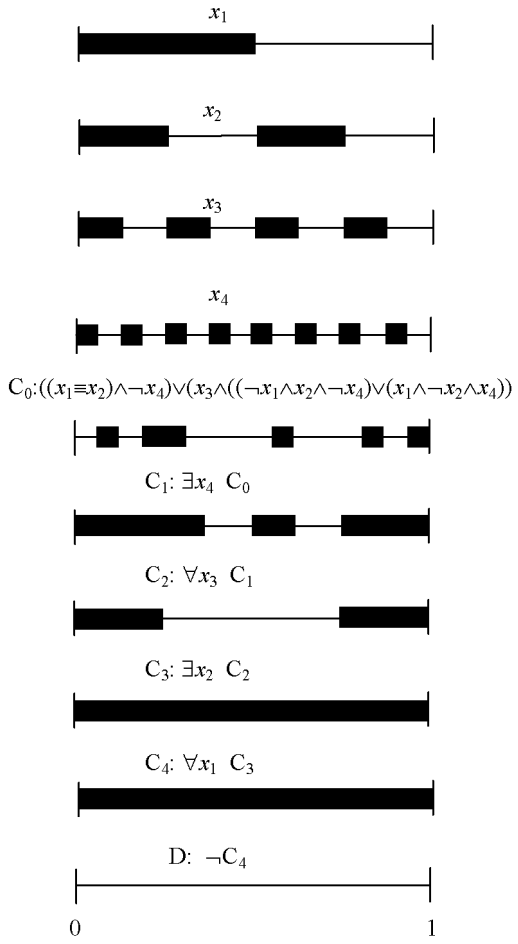
The equivalence of (i) and (ii) is due to the definition of $K_{3n-2+m+8(j+1)}$. The equivalence of (ii) and (iii) follows from the following three properties: $Flength(\|K_{\to 3(n-j)-2}\|) = \frac{1}{2^{n-j}}$ (cf. Lemma 11); for every $r \in [0,1)$ and interval-values A, B: $r \in Lshift(A,B)$ if and only if $r + Flength(B) \in A$ and an analogous fact concerning $Rshift$. The equivalence of (iii) and (iv) can be shown by the propositions (1)–(3) of Lemma 12. Finally, the equivalence of (iv) and (v) can be shown by considering that only two possible truth values exist. The proof of the lemma is finished. Now we are ready to finish the proof of Theorem 17.

Proof[of Theorem 17] Letting $j = n$, the above lemma ensures that $r \in \|K_{\to 3n-2+m+8n}\|$ if and only if
$Q_1 t_1 \ldots Q_n t_n \ : [[\phi(t_1, \ldots, t_n)]] = 1$ holds for any $r \in [0,1)$. Since the right side of the last equivalence is independent from r, we can state that $Q_1 t_1 \ldots Q_n t_n \ : [[\phi(t_1, \ldots, t_n)]] = 1$ if and only if $\|K_{\to 3n-2+m+8n}\|$ is equal to $[0,1)$. Finally, by setting $K_{3n-2+m+8n+1}$ to $(NOT, 3n-2+m+8n)$ the algorithm constructs a computation sequence whose interval-value is empty if and only if $\phi \in QSAT$. $\surd$

So we have proved one direction of Theorem 16, namely, that every language in $PSPACE$ is decidable by a restricted polynomial interval-valued computation.

This can be proved in a way similar to that of proving the transitivity of log-space reducibility. One should only observe one more thing: the given interval-valued computation for $QSAT$ also satisfies the restriction on the applications of product.

Figure 6 gives an example of the computation on a formula. $(((x_1 \equiv x_2) \wedge \neg x_4)) \vee (x_3 \wedge ((\neg x_1 \wedge x_2 \wedge \neg x_4) \vee (x_1 \wedge \neg x_2 \wedge x_4)))$ is shown to be in $QSAT$.



6. ábra. $[\forall x_1 \exists x_2 \forall x_3 \exists x_4](((x_1 \equiv x_2) \wedge \neg x_4)) \vee (x_3 \wedge ((\neg x_1 \wedge x_2 \wedge \neg x_4) \vee (x_1 \wedge \neg x_2 \wedge x_4))) \in QSAT$ holds.

12. Interval-valued computations that characterize $PSPACE$

The second achievement of the present part of the dissertation is the following. It accomplishes the proof of Theorem 16.

Theorem 18. The class of languages decidable by a restricted polynomial interval-valued computation is included in $PSPACE$. †

For this (the reverse) direction of the class equation to prove, we will construct a quadratic space algorithm which decides whether the value of an input interval-valued computation sequence is equal to the full [0,1). First we give a recursive algorithm to decide this problem. This guarantees only that the problem is solvable in exponential time. We also show how the execution of this recursive program can be equipped by a back-track like control in such a way that the needed memory is limited by a quadratic function of the length of the input computation sequence.

Lemma 15. For any interval-valued computation sequence S of bit height m, $x, y \in \mathbb{R}$ and nonnegative integer l such that $l < 2^{m+1}$, if $\{x, y\} \subseteq \left[\frac{l}{2^{m+1}}, \frac{l+1}{2^{m+1}}\right)$ then $x \in \|S\|$ if and only if $y \in \|S\|$. In other words, $\left[\frac{l}{2^{m+1}}, \frac{l+1}{2^{m+1}}\right) \subseteq \|S\|$ or $\left[\frac{l}{2^{m+1}}, \frac{l+1}{2^{m+1}}\right) \cap \|S\| = \emptyset$. †

Proof. It can be formulated for an induction on $|S|$. If $|S|$=1 then $\|S\| = \left[0, \frac{1}{2}\right)$ and the concerning number is 0. If the last applied operator is a Boolean one or one of the shifts then the inductive hypothesis yields the needed statement in a straightforward way. One has only to check the case when the last applied operator is STAR (product). If this is the case and we use the induction hypothesis for shorter computation sequences, then we have to compute the product of interval-value $\left[0, \frac{1}{2}\right)$ by $\bigcup_{i=1}^{k} \left[\frac{l_i}{2^m}, \frac{l_i+1}{2^m}\right)$, where m is equals to the number of products in S, $k < 2^m$ and $0 \leqslant l_1 < l_2 < \ldots < l_k < 2^m$. This product is $\bigcup_{i=1}^{k} \left[\frac{2l_i}{2^{m+1}}, \frac{2l_i+1}{2^{m+1}}\right)$, so is of the required form. $\surd$

Below we introduce a notation naming some subintervals of $[0, 1)$ that occur as values of computational sequences.

Definition 78. We define a subinterval $i(w)$ for an arbitrary word $w \in \{0, 1\}^*$ in the following way. Let us denote the length of w by

$m = |w|$ and the k-th element of this sequence by w_k. If $v = \sum_{k=1}^{m} w_k 2^k$ then $i(w)$ is $\left[\frac{v}{2^m}, \frac{v+1}{2^m}\right)$. Under these circumstances, we call $i(w)$ an *m-elementary subinterval*. We denote the set of m-elementary subintervals by $\mathbb{E}_m$, that is, $\mathbb{E}_m = \left\{\left[\frac{l-1}{2^m}, \frac{l}{2^m}\right) \ : \ l \in \{1, \ldots, 2^m\}\right\}$. Furthermore, let $\mathbb{E}$ be $\bigcup_{m \in \mathbb{N}} \mathbb{E}_m$. †

Remark 2. $i(\lambda) = [0, 1)$ holds, also $\bigcup_{w \in \{0,1\}^m} i(w) = [0, 1)$, if $m \geqslant 0$. Moreover, i is a bijection from $\{0, 1\}^m$ onto $\mathbb{E}_m$.

Now we can continue describing the algorithm, let us denote it by $\mathcal{B}$. It takes a computation sequence S as input and decide whether $\|S\| = [0, 1)$. Hence one of the non-basic data types of this algorithm is the type of the computation sequences, or, specified in the narrowest sense, all the nonempty prefixes of S. The set of these prefixes is denoted by Seq. Clearly, its elements can be identified with positive integers not greater than $|S|$. The other type of data structure is given by the set of elementary intervals $\mathbb{E}$, which we represent by i as words in $\{0, 1\}^*$. Let m denote the bit height of the input computation sequence. All the words while $\mathcal{B}$ is running on this sequence correspond to m-elementary subintervals, that is, elements of $\mathbb{E}_m$.

The algorithm uses both recursively and non-recursively definable functions.

Definition 79. The functions of $\mathcal{B}$ computable in a recursive way are the following:

$\sqsubset\colon \mathbb{E} \times Seq \to \{TRUE, FALSE\}$,
$\lhd\colon \mathbb{E} \times Seq \to \mathbb{E} \times (\mathbb{E} \cup \{\lambda\})$,
$\prec\colon \mathbb{E} \times Seq \to \mathbb{E} \times (\mathbb{E} \cup \{\lambda\})$.

The meaning of $(w \sqsubset S)$ is $i(w) \subset \|S\|$, $(w \lhd S)$ returns the starting and ending m-elementary subintervals of the maximal connected component of $\|S\|$ containing $i(w)$ where m is the bit height of S if such a component exists, (w, λ) otherwise. Finally, $(w \prec S)$ returns the starting and ending m-elementary subinterval of the maximal connected component of $[0, 1) \setminus \|S\|$ containing $i(w)$ if such a component exists, (w, λ) otherwise. †

Definition 80. The directly, non-recursively definable (partial) functions of $\mathcal{B}$ are the following. (m is the bit height of the input computational sequence.)

$bitheight : Seq \to \mathbb{N}$,
$< : \mathbb{E}_m \times \mathbb{E}_m \to \{TRUE, FALSE\}$,
$min, max : \mathbb{E}_m \times \mathbb{E}_m \to \mathbb{E}_m$,
$rotate\_left, rotate\_right : \mathbb{E}_m \times (\mathbb{E}_m \times \mathbb{E}_m) \to \mathbb{E}_m$,
$pred, succ : \mathbb{E}_m \to \mathbb{E}_m$,
$center, upper\_center : \mathbb{E}_m \times \mathbb{E}_m \to \mathbb{E}_m$.

The meaning of *bitheight* is straightforward. $w_1 < w_2$ holds if and only if w_1 is strictly left to w_2. *min* and *max* works with respect to the just defined liner ordering $<$. $rotate\_left(w, w_1, w_2)$ returns the result of the shifting of w towards left by the length of the subinterval starting point w_1 and ending point w_2. If overflow occurs then the result or a part of it appears right to w. If (w_1, w_2) is empty then no shifting occurs. *rotate_right* is interpreted analogously. $pred(w)$ determines the left neighbour of w among the m-elementary intervals, $pred(0^{|w|}) = \lambda$, $pred(\lambda)$ is undefined. *succ* is the mirror of *pred* moving right, $succ(1^m)$ is undefined. $center(w_1, w_2)$ is the central $|w|$-elementary subinterval between w_1 and w_2 if it is unambiguous, that is, there is an odd number of $|w_1|$-elementary subintervals strictly between w_1 and w_2. $upper\_center(w_1, w_2)$ returns the bigger of the two central $|w|$-elementary subintervals between w_1 and w_2. †

If m is the bit height of S, then by Lemma 15, instead of deciding $\|S\| = [0, 1)$, it is enough to decide in polynomial space that $i(w) \subset \|S\|$ for every $w \in \{0, 1\}^m$. It is clear that for this purpose it is enough to decide $i(w) \subset \|S\|$ one by one, for each $w \in \{0, 1\}^m$, in a uniformly sized quadratic space. So $\mathcal{B}$ has to answer $(w \sqsubset S)$, for each $w \in \{0, 1\}^m$.

We give the recursive algorithm in a self-explaining pseudo-code in which w, $w_1, \ldots, w_9$ denote (i-codes of) m-elementary subintervals while K, K_1, K_2 denote prefixes of S. For the sake of easier readability we write $op(K)$ for $(K_{|K|})_1$ and $arg_{j-1}(K)$ for the $(K_{|K|})_j$-length prefix of K if $j \in \{2, 3\}$. Let $op(K)$ be $FIRSTHALF$ if $K = (FIRSTHALF)$. We omit conditions on some of the cases, since there can be constructed analogously to the cases given. Further, we exclude the case $\lhd PRODUCT$ due to lack of space. To

compensate for that, we include the case of $\prec PRODUCT$ which is no less complex.

We establish the following recursive conditions on $\sqsubset$, $\lhd$ and $\prec$.

$(w \sqsubset K) =$
$(K_1, K_2) := (arg_1(K), arg_2(K));$
case $op(K)$
$=FIRSTHALF \rightarrow$ return $(first\_character\_of(w) = 0);$
$=NOT \rightarrow$ return the negation of $w \sqsubset K_1$;
$=AND \rightarrow$ return the conjunction of $w \sqsubset K_1$ and $w \sqsubset K_2$;
$=LSHIFT \rightarrow$
$(w_1, w_2) := 0^{|w|} \prec K_2,$
if $w_2 = 1^{|w|}$ then return $(w \sqsubset K_1),$
% not a real shift, $\|K_2\| = \emptyset$
if $w_2 = \lambda$ then $w_2 := 0^{|w|}$ else $w_2 := succ(w_2),$
% now w_2 is the first m-elementary subinterval
% included in $\|K_2\|$
$(w_3, w_4) := w_2 \lhd K_2,$
% the first component of K_2 starts with w_3
% and ends with w_4
$w_5 := rotate\_right(w, w_3, w_4),$
if $w < w_5$ then return $rotate\_right(w) \sqsubset K_1$ else $FALSE.$
% $RSHIFT$ is slightly different since
% it is cyclic
$=PRODUCT \rightarrow$
$(w_1, w_2) := w \lhd K_2,$
if (w_1, w_2) is empty then return $FALSE,$
% by Statement 15, the number of
% $|w|$-elementary subintervals is even
return $(w < upper\_center(w_1, w_2)).$
% remember K_1=$FIRSTHALF$

$(w \lhd K) =$
$(K_1, K_2) := (arg_1(K), arg_2(K));$
case $op(K)$
$=FIRSTHALF \rightarrow$
if $first\_character\_of(w) = 1$ then
return $(0^{|w|}, \lambda)$

else return $(0^{|w|}, 01^{|w|-1})$;
$=NOT \rightarrow$ return $w \prec K_1$;
$=OR \rightarrow$
$(w_1, w_2) := w \lhd K_1$,
$(w_3, w_4) := w \lhd K_2$,
if $w_2 = \lambda$ then return (w_3, w_4)
else if $w_4 = \lambda$ then return (w_1, w_2)
else return $(min(w_1, w_3), max(w_2, w_4))$;
$=LSHIFT \rightarrow$
$(w_1, w_2) := 0^{|w|} \prec K_2$,
if $w_2 = 1^{|w|}$ then return $(w \sqsubset K_1)$,
% not a real shift, $\|K_2\| = \emptyset$
if $w_2 = \lambda$ then $w_2 := 0^{|w|}$ else $w_2 := succ(w_2)$,
$(w_3, w_4) := w_2 \lhd K_2$,
$w_5 := rotate\_right(w, (w_3, w_4))$,
if $w_5 < w$ then return (w, λ),
% w is shifted out from $[0, 1)$ by $LShift(K_1, K_2)$
$(w_6, w_7) := w_5 \lhd K_2$,
if $w_7 = \lambda$ then return (w, λ),
$w_8 := rotate\_left(w_6, (w_3, w_4))$,
if $w_6 < w_8$ then $w_8 := 0^{|w|}$,
$(w_9, w_1 0) := (w_8, rotate\_left(w_7, (w_3, w_4)))$,
return $(w_9, w_1 0)$.
% The idea is to move our interval right, find
% out $\lhd K_1$ and transform it back to the left

$(w \prec K) =$
$K_1 := arg_1(K)$,
case $op(K) = STAR \rightarrow$
$(K_1, K_2) := (arg_1(K), arg_2(K))$,
$(w_1, w_2) := w \lhd K_2$,
if (w_1, w_2) is empty then
$(w_3, w_4) := w \prec K_2$,
if $w_3 = 0^{|w|}$ then return $(0^{|w|}, w_4)$,
else $(w_5, w_6) := w \lhd pred(w_3)$,
return $(upper\_center(w_5, w_6), w_4)$,
else % the case when $w \lhd K_2$ is nonempty
$w_3 := upper\_center(w_1, w_2)$,

if $w < w_3$ then return (w, λ),
    % $w \sqsubset FIRSTHALF * K_2$
else
  if $w_2 = 1^{|w|}$ then return (w_3, w_2),
    % (w_1, w_2) `is the last component`
  $(w_8, w_9) := succ(w_2) \prec K_2$,
    % (w_8, w_9) `is the next component of` $\neg K_2$
  return (w_3, w_9).

The given set of recursive conditions describes a terminating recursive algorithm. This can be shown by observing that each recursive call operates on a shorter computation sequence and that the cases of $FIRSTHALF$ are directly given. The correctness of the conditions can be proved by examining the various cases.

Unfortunately, the existence of a recursive algorithm deciding a problem guarantees only its solvability in exponential time. Hence we have to proceed further. We equip this recursive algorithm with a back-track type control. The memory use of the resulting equipped algorithm is quadratic in the input interval-valued computation sequence S. The expression $c \cdot |S| \cdot bitheight(S) \leqslant c \cdot |S|^2$ describes a sufficient space limit. First we realize that the non-recursive functions are all computable in linear space. To carry out these computations the same memory can always be recycled.

For the organization of back-track type control, the algorithm stores the following data additionally to the input computational sequence.

- ◇ An integer $j \leqslant log|S|$ stores which prefix of S is actually under processing;
- ◇ for each prefix of S, the index of its caller prefix is stored;
- ◇ for each prefix of S, the the actual task is stored by a word of length
  $bitheight(S)$ and an element of $\{\sqsubset, \lhd, \prec\}$;
- ◇ for each prefix of S, the whole cumulative information that is needed to answer the actual task is stored.

This amount of data fits into the mentioned quadratic space since no description of the gathered information per prefix (*local description of the process of the stored task*) exceeds the size $10 * bitheight(S)$.

This can be proved by examining the various cases. We give these local descriptions only in one case when the actual task is $(w, \lhd)$ on a prefix K whose last operation is $LSHIFT$. It is clear, that the full description fits into the given space limit. We use the concept *anti-component* containing w in the following sense: the component of the complement set containing w.

The cumulative information about the stored task can be:

1 There is no information about $w \lhd K$ yet.
2 The anti-component around $0^{|w|}$ is already known and it is (w_1, w_2).
3 In addition to 2, it is known that the anti-component is the whole [0,1).
4 In addition to 2, it has turned out that the anti-component of $0^{|w|}$ is empty.
5 In addition to 2, the anti-component of $0^{|w|}$ is neither empty nor the whole [0,1).
6 In addition to 2 the values (w_3, w_4) are known (they determine the first component of $arg_2(K)$.
. and so on ...
. at last, the the answer is known, it is stored in (w_9, w_{10}).

The notion of local descriptions can be described in a more formal manner. We introduce a relation called *local comparison* between the local descriptions of the states of the computation at the same stored task, based on their amount of gathered information. The local comparison is a partial ordering with two (in case of a task of type $w \sqsubset K$) or one (in the other two cases) maximal element(s). The maximal element(s) belong(s) to the finished, answered task. A *global description* for a state of a computation of $\mathcal{B}$ for S is a sequence $(L_1, \ldots, L_n)$ where each L_i is a local description belonging to $S \rightarrow j$ if $i \in \{1, \ldots, n\}$ or L_i is $\emptyset$ satisfying that $L_i = \emptyset$ and $j < i$ always imply $L_j = \emptyset$.

The execution of the algorithm (equipped by the back-track type control) is as follows. While it works on the task of the actual prefix, there are two possible types of steps. If the answer to the actual task is already known, then the actual task is terminated, the answer returns to the caller prefix. Another possible step is to gather further

information to answer the actual task. This is done by calling another task belonging to a prefix of a less index. Practically, it means that we take a step in the execution in the part of the algorithm answering the actual task. This organization guarantees that at most one task has to be stored per prefix. Every task has to be executed as many times as it is called.

One can observe that if the control goes back to the caller prefix then the global amount of gathered information strictly grows, in the following sense. If G_1 and G_2 are two global descriptions then $G_1 < G_2$ if and only if there exists a positive index $j < |S|$ such that G_1 and G_2 agree on $S_{j+1}, \ldots, S_{|S|}$, G_1 and G_2 have the same actual tasks at S_j but G_2 has more information about it. Intuitively speaking, G_2 is closer to answering the original question than G_1. We can ascertain that if the actual task finishes then the caller's information will increase. So, in this sense, the global amount of gathered information always – at each return to the caller – strictly increases. At the same time, it has an upper bound, since we know the answer to the original question $w \sqsubset S$. Earlier we have established that the algorithm always halts. Moreover, it terminates with the answer to the original question.

The previous arguments complete the proof of Theorem 18. $\surd$

The proof of the last theorem completes also the proof of Theorem 16, by consideration that the length of the restricted interval-valued computation sequence constructed to the original input word w is polynomial in $|w|$. $\surd$

On the spur of the given algorithm, further results can be concluded – we give them in the section about conclusions.

13. A connection to interval temporal logic

If we think of [0,1) as a time flow then we may investigate its temporal logic. It trivially coincides with the temporal logic of $(\mathbb{R}^{+0}, <)$ where $\mathbb{R}^{+0}$ is the set of nonnegative reals. It may be an interesting question, what happens if we add our operators to the temporal logic over [0,1) as binary modal operators. For an interesting class of such modal-temporal formulae we can provide a decidability theorem.

Definition 81. Let us call the members of the following set of formulae interval-valued modal-temporal formula. It is the minimal set of strings satisfying the following:

- $a, b, \ldots$ are (atomic) formulae,
- $FirstHalf$ is a formula,
- if φ, θ are formulae, then $(\varphi \wedge \theta)$, $(\varphi \vee \theta)$ and $\neg\varphi$ are formulae, too,
- if φ, θ are formulae, $\Box\!\!\rightarrow \varphi$ and $\leftarrow\!\!\Box\varphi$ are formulae, too,
- if φ, θ are formulae then $R(\varphi, \theta)$, $L(\varphi, \theta)$ and $P(\varphi, \theta)$ are formulae, too.

†

Definition 82. An interval-valuation v is no other than a function assigning to each member of $\{a, b, \ldots\}$ an interval-value. Then for any interval-valued modal-temporal formula $\|\varphi\|_v$ is an interval-value of the interval-valued modal-temporal formula φ. The definition of this notion is the expected one. We just write three clauses of this definition.

- $\|FirstHalf\|_v = \left[0, \frac{1}{2}\right)$,
- $\|\Box\!\!\rightarrow \varphi\|_v$ is $\{t \in [0,1) \ : \ (t,1) \subseteq \|\varphi\|_v\}$,
- $\|P(\varphi, \theta)\|_v = \|\varphi\|_v * \|\theta\|_v$.

†

A modal-temporal formula is said to be modal-temporal logical law if with every valuation v its interval-value is [0,1).

Problem 1. How to axiomatize this kind of modal-temporal logic? Is it decidable? If yes, what is its complexity?

We have a partial answer to this question.

Claim. The problem if a modal-temporal formula built up only from $FirstHalf$ but without other propositional variables is decidable by exponential time. If the usage of the product operator is restricted to always taking product with $FirstHalf$ then the arising problem is solvable in polynomial space, moreover, is $PSPACE$-complete.

This answer can be extracted by Theorem 18. To any formula one can find an appropriate interval-valued computation whose result have to be [0,1) if and only if the formula is a law.

14. Ideas and suggestions for further work

14.1. Interval-valued computing

Using the algorithm of Theorem 18, one can decide the problem whether the values of two input restricted interval-valued computation sequences are equal. Inclusions are also decidable. The space complexity of these methods quadratic in the length of the computation sequences. If we do not restrict the product operators to the product by $FIRSTHALF$ then the number of elementary subintervals can depend double exponentially on the length of the sequences. Hence we can only establish that the equality or inclusion problem of unrestricted computational sequences is in $SPACE(c^{n^2}) \subseteq EXPSPACE$. It does not mean that the computational power of unrestricted interval-valued computations is small. It could be established rather by deciding whether two algorithm (denoted by A in the definition of decidability of languages) producing computation sequences for the original input words accepts the same words but this task seems to be hopeless.

From the algebraic/logical point of view, our result (Theorem 16) implies that the equations and inequalities of closed terms of the Boolean Algebra $\mathbb{V}$ equipped with the shift operators and the unary operation of product with $FIRSTHALF$ (can be taken also the other side product by $FIRSTHALF$) are in quadratic space decidable. If we release the condition on product and raise it to the status of a binary operation then we can get in this case exponential space decidability.

Recently we do not know problems in $EXPSPACE \setminus PSPACE$ (say $EXP- SPACE$-complete problems) that are decidable by an unrestricted but polynomial interval-valued computation. We would like to see one. It may need a more sophisticated data representation. The following question is also reasonable: what other operators should be added to the system to get a solution of the previous problem.

Another model should be worked out and analyzed where we let the interval-valued mechanism work more, in the following sense. A digital-to-(interval-value) converter translates the input into some

interval-values, and the interval-values are processed by a usual imperative control. It is closer to the intuitive motivation given in the introduction. One may adjust the selection of allowed interval-values, operators and tests. The interrelation to interval temporal logic and fuzzy logic should be more deeply developed. The various theories(equations with parameters, quasi-equations, first-order theory) of the underlying Boolean algebra remain a subject of future research. Finally, one may ask what advantages we can obtain by allowing more space dimensional objects to store information.

Let us extend the structure $(\mathbb{V}, \wedge, \neg)$ by the new operators. Is its first-order theory decidable? Or at least, the set of its valid equations is decidable? A similar question is also possible for $\mathbb{V}_0$.

14.2. Spatio-temporal logic

In the area of temporal and spatio-temporal logics, the following questions are to be mentioned. What is the situation in the case of other temporal operators? The non-axiomatizability proofs utilize strongly the irreflexivity of the time flow. What if we choose the reflexive ones? An especially important question is whether there are axiomatizable first-order temporal theories over $(\mathbb{R}^n, \blacktriangleleft)$ and $(\mathbb{Q}^n, \blacktriangleleft)$ if $n > 2$. The first-order theory of the latter structure is decidable? (We conjecture the answer is not.) Does the first one allow quantifier elimination? It is decidable by algebraic translation into the first-order theory of $(\mathbb{R}, +, \cdot)$.

15. Summary

On these pages I will give a summary on my investigations into two recently flourishing areas of non-classical logic, namely spatio-temporal logic and interval-valued logic. In both cases my aim is to explore topics that are interesting for formalization of analog (in the sense of non-digital) computation processes.

Results in temporal logic

The first-order temporal logics over classical structures as time flow (like $(\mathbb{N}, <)$, $(\mathbb{Z}, <)$, $(\mathbb{R}, <)$) are usually not axiomatizable. This is a well-known fact, which was observed first by D. Scott. M. Reynolds [R96] axiomatized first-order temporal theory over $(\mathbb{Q}, <)$ with the temporal operators *Until* and *Since* and proved its completeness in quite a novel way. In Theorem 11 we present a short argument for the axiomatizability of first-order temporal theories with arbitrary temporal connectives over $(\mathbb{Q}, <)$. *Until* and *Since* cannot express arbitrary temporal connectives over this time flow ([K68]), hence this result strengthens the result of [R96]. A general and simple reason of axiomatizability of a first-order temporal theory is the recursive axiomatizability and ω-categoricity of the first-order theory of the underlying time flow structure. However, due to its generality, this method does not provide an explicit axiom system in terms of axioms and deduction rules. This method is our first – quite modest – contribution to the development of *linear time* first-order temporal logic. It constitutes a part of [V07b] and was presented in [V00].

What can be more relevant is that our proof method for non-axiomatizability of some first-order spatio-temporal theories over $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ can be modificated to prove that a first-order temporal theory with a very basic first-order signature (only one monadic predicate without equality) over $(\mathbb{R}, <)$ is not axiomatizable (Theorem 13). We do not know any proofs for this in the literature. The presented non-axiomatizability proofs utilize a three-argument signature or are not valid for $(\mathbb{R}, <)$ ([GHR94], [HWZ00], [Me92]).

Results in spatio-temporal logic

What can temporal logic offer to designers of mobile distributed computing systems? Apart from having dynamics in time, these systems have dynamics in space, too. To cover this area, an analogue of temporal logic has been developed, which is usually called spatio-temporal logic. One way to follow this is to combine a spatial language with a temporal language in such a way that in the hybrid language there are separate modalities for time and space. This idea originated from research on multi-dimensional modal logics. In this formalization there are separate modalities for space and time.

There is another long-standing tradition to deal with time and space, namely to speak jointly about spacetime and use its geometrical relations and objects to express various properties of the dynamics of processes in spacetime. Assuming that these processes have no synchronized time one come to consider hyperbolic geometry of Minkowski spacetime, as in the works of F. Mattern ([Ma92], [CM96]). He proposed investigating so-called causal accessability relations of spacetime from the viewpoint of specification and verification of distributed computing. In the present introduction we will distinguish five relations related to causality: $(x \blacktriangleleft y)$ for pure material causal connectability usually called chronological accessability, while $(x \triangleleft y)$ for optical accessability, $(x \ll y)$ for the disjuntion of the previous two, $(x =\!\ll y)$ for $(x \ll y \vee x = y)$ (causal accessability in the literature) and finally $(x =\!\blacktriangleleft y)$ stands for $x \blacktriangleleft y \vee x = y$.

Any causal accessability relation of spacetime can be considered to be a generalization of time flows in temporal logic, when it serves as an alternativity relation of a Kripke frame for propositional modal logic as it was done first by V. Shehtman and R. Goldblatt, independently. In [S83] and [G80], modal logic of $(\mathbb{R}^n, =\!\ll)$ was proved to be decidable. The more than 20-year-long open problems of decidability and axiomatization of modal logics of the frame $(\mathbb{R}^n, \blacktriangleleft)$ was solved by Shapirovsky and Shehtman ([SS03].

Now, if we are interested in the propositional modal logic of the frame $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ then we face difficulties at this point. $(\mathbb{Q}^n, \blacktriangleleft)$ is no more isomorphic to $(\mathbb{Q}^n, L_n)$ where $(x_1, \ldots, x_n) L_n (y_1, \ldots, y_n) :\Leftrightarrow$

$(x_1 < y_1 \wedge \ldots \wedge x_n < y_n)$. So this modal logic cannot be regarded as a product of modal logics of the frame $(\mathbb{Q}, <)$.

We have some methods of proving the decidability of temporal logics. The most popular one is to prove that the monadic second-order theory of the time flow structure is decidable. Unfortunately, in Theorem 2 show that even the $\forall\exists$-fragment of monadic second-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is not recursively enumerable (therefore not decidable), for all $n > 1$. We measure here the quantifier complexity of subset quantifications only. What is more, in Theorem 6 and 7 we prove that the $\forall$-fragment of this theory is recursively enumerable if and only if $n = 2$. It implies that propositional temporal theories over $(\mathbb{Q}^2, \blacktriangleleft)$ are decidable but one cannot give a complete axiomatization of a propositional temporal logic over $(\mathbb{Q}^n, \blacktriangleleft)$ $((n > 2)$ with an expressively complete temporal operator set. However it does not exclude axiomatizations with some specific temporal operator set. It remains the subject of further investigations.

Thus, $(\mathbb{Q}^2, \blacktriangleleft)$ shows an interesting example when the ($\forall\exists$-fragment of) the monadic second-order theory of a structure with ω-categorical and finitely axiomatizable first-order theory is not recursively enumerable.

Theorem 7 is valid, with a little simplification in the proof, for the monadic second-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$, too. This is stated in Theorem 3. One can conclude a similar but weaker result also from S. Shelah's paper [S75] that states the (full) monadic second-order theory of $(\mathbb{R}, <)$ to be not recursively enumerable. This conclusion can be drawn by Lemma 15.5.3 (p. 567) of [GHR94]. However, our theorem strengthens this result by establishing non-axiomatizability even for the $\forall$-fragment of the monadic second-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$.

Theorem 2 can be proven according to the non-axiomatizability proofs in second-order logics (not restricted to be monadic), except for the absence of binary relations. To cope with this, we introduce some spacetime geometric objects to make possible pairing and other constructions assisting the representation of binary relations on non-negative integers. The proof of Theorem 7 is more difficult. We have developed new definitions for some spacetime geometrical relations

in the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ – the most remarkable being for spacelike betweenness – and made a first-order formula which substitutes the second-order condition of the proof of Theorem 2.

We note that Theorem 1 and 2 are not superfluous, although, if $n > 2$, they are partial cases of Theorem 7. Nevertheless, they were separately stated and proved, because they are valid also for the case $(\mathbb{Q}^2, \blacktriangleleft)$. If such a situation occurs, then it seems reasonable to construct the proof of Theorem 7 as a modification of the proof for the mentioned two theorems.

The results on monadic second-order theories appear in my paper [V07a] which is accepted for publication in Journal of Philosophical Logic.

After surveying the new results on monadic second-order theories, we turn to first-order spatio-temporal theories. In [V07c] and [V07b] (Theorem 8–11) we obtain axiomatizability results on first-order spatio-temporal theories of $(\mathbb{Q}^n, \blacktriangleleft)$ and $(\mathbb{R}^n, \blacktriangleleft)$. Based on similar spacetime geometric considerations, we establish that all first-order spatio-temporal theories are axiomatizable over $(\mathbb{Q}^2, \blacktriangleleft)$ but not over $(\mathbb{F}^n, \blacktriangleleft)$ if $\mathbb{F} = \mathbb{Q}$ and $n > 2$ or $\mathbb{F} = \mathbb{R}$ and $n \geqslant 2$. As an extra technical contribution, we develop our non-axiomatizability results for a very simple first-order signature, namely, we allow only one unary predicate symbol, without the equality.

The above results implies that the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is either not recursively enumerable or not ω-categorical. It is hardly plausible that the latter holds but it has to be proved. We do this after the non-axiomatizability proofs (Theorem 12).

We close the spatio-temporal part of the theses with demonstrating the usefulness of our axiomatizable spatio-temporal logic (provided by Theorem 10) by showing the expressive power of this logic. We formalize a relevant property of distributed computing systems of mobile agents in it. This is a part of the paper [V07b].

Theorems on monadic second-order theories

For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $(x \blacktriangleleft y)$ for $\mu(x, y) > 0 \wedge x_1 < y_1$ where μ denotes the Minkowskian distance.

In this summary, we have no space to give the formal definitions concerning monadic second-order formulae and interpetations. For the details the reader has to consult the dissertation itself. We just remind the reader that the monadic second-order theory of a structure $\mathcal{T}$ is denoted by $MSOTH(\mathcal{T})$ which consists of monadic second-order formulae satisfied in every monadic second-order interpretation over $\mathcal{T}$. We repeat that we call a theory *axiomatizable* if and only if it is recursively enumerable.

Our first result can be formulated as

Theorem 1. [V00], [V07a]
For any $n > 1$, $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ is not axiomatizable. †

By a deeper complexity analysis of the previous proof we can also show

Theorem 2. [V00], [V07a]
For any $n > 1$, not even the $\forall\exists$-fragment of $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ is axiomatizable. †

By adopting our proof to $\mathbb{R}^n$ and carrying out the needed simplification, we get

Theorem 3. [V07a]
For any $n > 1$, not even the $\forall$-fragment of $MSOTH(\mathbb{R}^n, \blacktriangleleft)$ is axiomatizable. †

J. van Benthem established the following theorem.

Theorem 4. [B83]
The first-order theory of $(\mathbb{Q}^2, \blacktriangleleft)$ is both ω-categorical and recursively enumerable. †

We made a useful note with the aim to utilize the previous theorem in [V07a].

Theorem 5. *For any countable time flow $(T, \prec)$, if its first-order theory is ω-categorical and recursively enumerable then the $\forall$-fragment of*
$MSOTH(T, \prec)$ is also axiomatizable. †

From the two previous items we conclude

Theorem 6. [V07a]
The $\forall$-fragment of $MSOTH(\mathbb{Q}^2, \blacktriangleleft)$ is axiomatizable. †

By a more sophisticated argument than the provided one for the $\forall\exists$-fragment, we can also prove

Theorem 7. [V07a]
For $n > 2$, not even the $\forall$-fragment of $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ is axiomatizable. †

Theorems on first-order spatio-temporal theories

Let $\mathtt{Th}_L^{Op}(\mathcal{T})$ denote the first-order spatio-temporal theory of time flow $\mathcal{T}$ based on a signature L and a temporal operator set Op. The underlying notions can be found in the dissertation in detail. To be concise, we say a set S of temporal formulæ *axiomatizable* iff it is recursively enumerable.

Let G denote the universal modality concerning future – in an intuitive reading, it expresses that *... will hold permanently after now* and let N denote an operator whose intuitive reading is *(... holds in every spacetime point maybe except for now).*

Signature L includes no equality symbol just one unary predicate symbol, namely, r. There is no weaker first-order signature. If we have non-axiomatizability for this signature then there is not much hope for the axiomatizability of the time flow in question.

Theorem 8. [V01], [V07c]
Let $n > 2$. $\mathtt{Th}_L^{GN}(\mathbb{Q}^n, \blacktriangleleft)$ is not axiomatizable. †

This result may be interesting in contrast with the following theorems.

Theorem 9. [V07c]
Let $n \geqslant 2$. $\mathtt{Th}_L^{GN}(\mathbb{R}^n, \blacktriangleleft)$ is not axiomatizable. †

Theorem 10. [V07b]
For any first-order signature L and arbitrary finite set of temporal operators Op, $\mathtt{Th}_L^{Op}(\mathbb{Q}^2, \blacktriangleleft)$ is axiomatizable. †

The proof of the last theorem is based on the following theorem and J. van Benthem's Theorem 4. We recall that ω-categoricity of a

structure means that, up to isomorphism, its first-order theory has only one model.

Theorem 11. [V07b]
If the first-order theory of a countable time flow $(T, \prec)$ is ω-categorical and recursively enumerable, then for any first-order signature L and arbitrary finite set of temporal operators Op, $\mathtt{Th}_L^{Op}(T, \prec)$ is axiomatizable. †

From the previous theorem, the first-order theories of $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ are either not ω-categorical or not recursively enumerable. It is hard to imagine it to be ω-categorical without being recursively enumerable. But it has to be proved.

Theorem 12. The first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is not ω-categorical if $n > 2$.

We do not know any proofs of non-axiomatizability of a first-order temporal logic over the reals with a monadic signature. For this reason we give

Theorem 13. $\mathtt{Th}_L^G(\mathbb{R}, <)$ is not axiomatizable.

Results concerning interval-valued computations

In the second half of the dissertation we give two interesting results on a newly arisen non-classical computational theory. In the conference paper [N05b], Benedek Nagy proposed a simple discrete time / continuous space computational model, the so-called interval-valued computing. It involves another type of idealization – the density of the memory can be raised unlimitedly instead of its length. This new paradigm keeps some of the features of the traditional Neumann-Turing type computations. It works on 1-dimensional continuous data, namely, on specific subsets of the interval $[0, 1)$, more specifically, on finite unions of [)-type subintervals. This system is similar to the optical computing in [WN05] in some features.

In a nutshell, interval-valued computations start with $\left[0, \frac{1}{2}\right)$ and continue with a finite sequence of operator applications. They work sequentially in a deterministic manner. The allowed operations are

motivated by the classical operations on finite bit sequences in traditional computers: Boolean operations and shift operations. There is only an extra operator, the product. The role of the introduced product is to connect interval-values on different 'resolution levels'. Essentially, it shrinks interval-values. So, in interval-valued computing systems, an important restriction is eliminated, i.e. there is no limit on the number of bits of a cell in the system; we have to suppose only that we always have a finite number of bits. Of course, in the case of a given computation an upper bound (the bit height of the computation sequence) always exists, and it gives the maximum number of bits the system needs for that computation process. Hence our model still fits into the framework of the Church-Turing paradigm, but it faces different limitations than the classical Turing model. Although the computation in this model is sequential, the inner parallelism is extended. One can consider the system without restriction on the size of the information coded in an information unit (interval-value). It allows to increase the size of the alphabet unlimitedly in a computation.

A language is said to *decidable by interval-valued computations* iff there exists an algorithm that for every word produces an interval-valued computation sequence such that this sequence ends with the empty interval-value if and only if the word is in the language. Polynomial/linear decidability constrains the length of the produced computation and the memory size of the algorithm producing it, too.

As our results will show, interval-valued computations are suitable for dealing with polynomial space problems. First, interval-values and interval-valued computations are explained based on conference paper [N05b]. In that paper, the problem SAT was solved by a linear interval-valued computation and the question was posed, whether there are $PSPACE$-complete problems decidable by linear interval-valued computations. In conference paper [NV06] we answer this question in the affirmative, namely, we prove it for $QSAT$, the problem whether a quantified propositional formula is true or false.

Theorem 14 ([NV06]). There is a $PSPACE$-complete language which can be decided by a linear inter-valued computation.

By observing the needed syntactical power in the above mentioned computations, in our paper [NV07] (*accepted for publication in Theoretical Computer Science*) we also have determined a natural syntactic class of interval-valued computations that the resulting class of decided problems coincides with $PSPACE$. We have given a concrete algorithm for this purpose.

Theorem 15 ([NV07]). The class of languages decidable by a restricted polynomial interval-valued computation coincides with $PSPACE$.

Finally, in the last section we construct a connection of interval-valued computations to interval temporal logic. More specifically, we interpret the developed interval-valued computation as a deciding algorithm for a specific class of interval temporal formulae.

16. Összefoglaló (Summary in hungarian language)

Ezeken az oldalakon összefoglalom a kutatásaimat a következő két, egyaránt élénken kutatott nem-klasszikus logikai területen: spatio-temporális és általánosított intervallum-értékű logika. Mindkét esetben a munkám hajtóereje az volt, hogy a nem-digitális, ún. analóg számítások tudományához járuljak hozzá.

Eredmények a temporális logikában

Az olyan klasszikus időfolyamok feletti elsőrendű temporális elméletek, mint $(\mathbb{N}, <)$, $(\mathbb{Z}, <)$, $(\mathbb{R}, <)$, általában nem axiomatizálhatók. Ez közismert tény, amelyet elsőként D. Scott figyelt meg. M. Reynolds [R96] a $(\mathbb{Q}, <)$ időfolyam feletti elsőrendű temporális logikára adott axiomatizációt, az $\{Until, Since\}$ temporális operátorkészlettel. A bizonyítás eléggé terjedelmes és összetett. A 11. tételben egy egyszerű érvelést mutatunk be, amely ugyanezen időfolyam felett tetszőleges operátorkészlettel axiomatizálhatóságot biztosít. Az *Until* és a *Since* operátorok nem adnak funkcionálisan teljes operátorkészletet, így az eredményünk Reynolds eredményének kis erősítése, bár azon az áron, hogy olyan explicit axiómarendszert nem ad, amiben szokásos axiómák és levezetési szabályok szerepelnének. Az axiomatizálhatóság egyszerű oka esetünkben az időfolyam elsőrendű elméletének rekurzív felsorolhatósága és ω-kategoricitása. Ez a megállapítás a dolgozatunk első – szerény – hozzájárulása a lineáris idejű temporális logika fejlődéséhez. Ez a rész a [V07b] papír részét képezi és a [V00] konferencián mutattam be.

Ami ennél talán jelentősebb, az az, hogy a nemlineáris, spatio-temporális elméletek nemaxiomatizálhatósági eredményeinek bizonyítási módszerét alkalmazva, ki tudjuk mutatni (13. tétel), hogy a valós időfolyam feletti elsőrendű temporális logika már egy nagyon kicsi szignatúrával sem axiomatizálható, nevezetesen, egy olyannal, amely csak egyetlen monadikus predikátumjelet tartalmaz, de egyenlőségjelet nem. Az irodalomban nem ismert ennek az eredménynek a bizonyítása. Az ismert nem-axiomatizálhatósági eredmények 3-argumentumú szignatúrát alkalmaznak vagy módszerük nem alkalmazható az $(\mathbb{R}, <)$ időfolyam esetében. ([GHR94], [HWZ00], [Me92]).

Eredmények a spatio-temporális logikában

Mit kínál a temporális logika a mobil elosztott rendszerek tervezőinek? Az időbeli dinamikán kívül, ezek térbeli dinamikát is mutatnak. Hogy ezzel meg lehessen birkózni, a temporális logika egy variánsát fejlesztették ki, amelyet szokás spatio-temporális logikának nevezni. Az egyik irányzat, a multi-modális logika szerint külön tér- és külön időbeli modalitásokat vezetünk be.

A másik lehetőség a térrel és idővel együtt foglalkozni, nevezetesen, a téridőt véve, annak téridő-geometriai relációit használni téridőbeli folyamatok dinamikájának kifejezésére. Azt feltételezve, hogy a folyamatok nem rendelkeznek szinkronizált idővel, oda jutunk, hogy a Minkowski téridő hiperbolikus geometriáját vesszük vizsgálatunk eszközéül, mint F. Mattern munkáiban. ([Ma92], [CM96]). Ő kezdeményezte a téridő ok-okozat szerinti összeköthetőségi reláció bevonását az elosztott számítások specifikációjának és verifikációjának vizsgálatába. A disszertációban öt ilyen típusú relációt fogunk megemlíteni. $(x \blacktriangleleft y)$ jelöli a tiszta anyagi ok-okozati összeköthetőséget (anyagi=fénysebesség alatti), $(x \triangleleft y)$ jelöli az optikai összeköthetőséget, $(x \ll y)$ jelöli az előző kettő diszjunkcióját, az ok-okozati összeköthetőséget. $(x =\!\ll y)$ jelöli $(x \ll y \vee x = y)$-t és végül $(x =\!\blacktriangleleft y)$ $x \blacktriangleleft y \vee x = y$-t.

Bármely ok-okozati téridőbeli összeköthetőségi reláció szerepelhet temporális logika időfolyamaként, amikor is alternativitási relációként szerepel a propozicionális modális logika Kripke-féle modelljében. Ez a felfogás V. Shehtman és R. Goldblatt cikkeiben szerepel elsőként. A [S83] and [G80] cikkekben, a $(\mathbb{R}^n, =\!\ll)$ feletti propozicionális modális elmélet eldönthetőnek bizonyíttattik. A $(\mathbb{R}^n, \blacktriangleleft)$ feletti propozicionális elmélet eldönthetőségének több mint 20 évig nyitott problémáját Shapirovsky and Shehtman oldotta meg ([SS03]).

Hogyha a $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ folyam propozicionális logikája iránt érdeklődünk, néhány nehézséggel szembesülünk. Például $(\mathbb{Q}^n, \blacktriangleleft)$ nem izomorf $(\mathbb{Q}^n, L_n)$-lel, ahol $(x_1, \ldots, x_n) L_n (y_1, \ldots, y_n) :\Leftrightarrow (x_1 < y_1 \wedge \ldots \wedge x_n < y_n)$. Ezért is állítottuk fentebb, hogy a $(\mathbb{Q}^n, \blacktriangleleft)$ feletti temporális logika nem modellezhető a $(\mathbb{Q}, <)$ feletti modális logikák szorzatával.

A propozicionális elméletek eldönthetőségének bizonyítása legkönnyebben azon az úton haladhat, hogy bebizonyítjuk, hogy az illető időfolyam, mint struktúra monadikus másodrendű elmélete, vagy legalább annak $\forall$-fragmentuma eldönthető. Balszerencsére, a 2. tételünk azt mutatja, hogy a $(\mathbb{Q}^n, \blacktriangleleft)$ struktúra monadikus másodrendű elméletének még a $\forall\exists$-fragmentuma sem eldönthető (ill. rekurzívan nem felsorolható), bármely $n > 1$-re, sőt, ha $n > 2$, akkor még a $\forall$-fragmentum sem axiomatizálható, csak $n = 2$ esetben. Ezt a 6. és 7. tételben bizonyítottuk. Ezeket az eredményeket a [V07a] cikkemben publikáltam. Ezek következményül adják, hogy $(\mathbb{Q}^2, \blacktriangleleft)$ feletti propozicionális temporális elméletek eldönthetők, de
$(\mathbb{Q}^n, \blacktriangleleft)$ felett nem adható olyan funkcionálisan teljes temporális operátorkészlet, amely elmélete eldönthető. De ez nem zárja ki, hogy egyes operátorkészletek elmélete eldönthető legyen. Ez további kutatások tárgya lehet.

Így $(\mathbb{Q}^2, \blacktriangleleft)$ egy olyan példát szolgáltat, amikor egy struktúra monadikus másodrendű elméletének $\forall\exists$-fragmentuma nem rekurzívan felsorolható, holott a struktúra elsőrendű elmélete ω-kategorikus és végesen axiomatizálható.

A 7. tétel a $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$ időfolyamra is bizonyítható, olyan egyszerűsítéssel a bizonyításában, amivel az működik az $n = 2$ esetben is, azaz valósak esetén már a $(\mathbb{R}^2, \blacktriangleleft)$ monadikus másodrendű elméletének $\forall$-fragmentuma sem aximatizálható. Ezt állítja a 3. tétel. Hasonló, bár kevésbé erős eredményt lehet levonni S. Shelah [S75] cikkéből, miszerint $(\mathbb{R}, <)$ (teljes) monadikus másodrendű elmélete rekurzívan nem felsorolható. Ezt a következtetést a [GHR94] monográfia 15.5.3 tételével lehet levonni.

A 2. tétel bizonyítása hasonlóan konstruálható a (nem feltételenül monadikus) másodrendű logikai nem-axiomatizálhatósági bizonyításokhoz, kivéve a bináris relációk használatát. Ezt pótlandó, néhány téridő-geometriai objektumot és relációt definiálunk a másodrendű elméletünkben, ami párosítást és más objektumokat enged szimulálni, amik a bináris relációk reprezentálását segítik elő. A 7. tétel bizonyítása bonyolultabb. Új definíciókat fejlesztettem ki a $(\mathbb{Q}^n, \blacktriangleleft)$ struktúra elsőrendű elméletében – a leginkább figyelemreméltót a térszerű lineáris közrefogás (spacelike betweenness) reláció számára –

és egy olyan elsőrendű formulát, amelyik pótolja a 2. tétel másodrendű feltételét.

Megjegyezzük, hogy a 1. és 2. számú tételek nem felelegesek, bár, ha $n > 2$, részesetei a 7. tételnek. Az előbbiek érvényesek $(\mathbb{Q}^2, \blacktriangleleft)$-re is. Ha ez a szituáció, akkor értelmesnek láttam a 7. tétel bizonyítását mint az előző kettő bizonyításának finomítását megépíteni.

A másodrendű logikáról szóló eredményeim a [V07a] közleményben találhatók meg, amelyet a Journal of Philosophical Logic c. folyóiratban közlésre elfogadtak.

A monadikus másodrendű logikáról szóló áttekintés után figyelmünket az elsőrendű spatio-temporális elméletek irányába fordítjuk. A [V07c] és [V07b] írásokban (8–11. tételek) $(\mathbb{Q}^n, \blacktriangleleft)$ és $(\mathbb{R}^n, \blacktriangleleft)$ struktúrák, mint időfolyamok feletti elsőrendű temporális elméletekre vonatkozó axiomatizálhatósági kérdéseket válaszoltunk meg. Megállapítjuk, hogy $(\mathbb{Q}^2, \blacktriangleleft)$ felett minden elsőrendű temporális elmélet axiomatizálható, de $(\mathbb{F}^n, \blacktriangleleft)$ felett nem, ha $\mathbb{F} = \mathbb{Q}$ és $n > 2$ vagy $\mathbb{F} = \mathbb{R}$ és $n \geqslant 2$. Extra technikai hozzájárulásként, egy szélsőségesen egyszerű elsőrendű szignatúra felett bizonyítunk, amely csak egyetlen egyváltozós predikátumjelet tartalmaz, még egyenlőségjelet sem.

A fenti eredmények azt imlikálják, hogy $(\mathbb{Q}^n, \blacktriangleleft)$ elsőrendű elmélete nem rekurzívan felsorolható vagy nem ω-kategorikus. Kevéssé plauzibilis, hogy bár ω-kategorikus de nem rekurzívan felsorolható, mindenesetre bizonyítás nélkül nem tudhatjuk. A 12. tételben bebizonyítjuk, hogy tényleg nem ω-kategorikus. A disszertáció spatio-temporális részét az axiomatizálható spatio-temporális elmélet (10. tétel) egy gyakorlati felhasználásának demonstrálásával fejezzük be.

Tételek monadikus másodrendű elméletekről

Legyen $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Ekkor $(x \blacktriangleleft y)$-t úgy definiáljuk, mint $\mu(x, y) > 0 \wedge x_1 < y_1$ fennállását, ahol μ jelöli a Minkowski-féle távolságot. Amint már jeleztük, ez a reláció az irányított anyagi ok-okozati összeköthetőséget fejezi ki, más szóval azt, hogy y az x felső fénykúpjában helyezkedik el, azaz, lehetséges az, hogy x a fénynél lassabb jelet küldjön y-nak.

A formális definíciókat hely híján nem idézem a disszertációból. Csak azt ismétlem meg, hogy egy $\mathcal{T}$ struktúra monadikus másodrendű

elméletét $MSOTH(\mathcal{T})$-val jelöljük és akkor hívjuk axiomatizálhatónak, ha rekurzívan felsorolható.

Az első eredményünk:

TÉTEL 1. [V00], [V07a] Egyetlen $n > 1$-re sem axiomatizálható $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$.†

A bizonyítás alaposabb elemzésével megmutatjuk, hogy

TÉTEL 2. [V00], [V07a] Egyetlen $n > 1$-ra sem axiomatizálható az $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ elméletnek már a $\forall\exists$-fragmentuma sem.†

A bizonyítás $\mathbb{R}^n$-re való egyszerűsítésével kapható:

TÉTEL 3. [V07a] Egyetlen $n > 1$-re sem axiomatizálható már a $\forall$-fragmentuma sem az $MSOTH(\mathbb{R}^n, \blacktriangleleft)$ elméletnek.†

J. van Benthem bizonyította:

TÉTEL 4. [B83] $(\mathbb{Q}^2, \blacktriangleleft)$ elsőrendű elmélete ω-kategorikus és rekurzívan felsorolható.†

[V07a]-ben tettünk egy hasznos megfigyelést:

TÉTEL 5. Ha egy megszámlálható $(T, \prec)$ időfolyam elsőrendű elmélete ω-kategorikus és rekurzívan megszámlálható, akkor másodrendű monadikus elméletének $\forall$-fragmentuma axiomatizálható.†

Az előző kettőből jön:

TÉTEL 6. [V07a] $MSOTH(\mathbb{Q}^2, \blacktriangleleft)$ $\forall$-fragmentuma axiomatizálható.†

Egy bonyolultabb konstrukcióval bizonyítható, hogy

TÉTEL 7. [V07a] Ha $n > 2$, akkor a $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$ elmélet $\forall$-fragmentuma sem axiomatizálható.†

Tételek elsőrendű temporális és spatio-temporális elméletekről

$\mathtt{Th}_L^{Op}(\mathcal{T})$ jelölje a $\mathcal{T}$ időfolyam elsőrendű temporális elméletét az L szignatúrával és az Op temporális operátorkészlettel. Az ezt megelőző definíciók a disszertációban találhatók. Axiomatizálhatóság alatt ismételten rekurzív felsorolhatóságot értünk.

GA fogja jelölni a jövőre vonatkozó univerzális modalitást, ami azt fejezi ki, hogy A *a jövőben végig igaz*, míg NA intuitív jelentése az, hogy A *minden téridőpontban teljesül, kivéve esetleg a mostanit.*

A tételeinkben az az L szignatúra szerepel, amiben egyetlen 1-argumentumú predikátumjel van és még egyenlőségjel sincs. Ennél kisebb kifejező erejű elsőrendű szignatúra nincs.

TÉTEL 8. [V01], [V07c] Legyen $n > 2$. Ekkor $\mathtt{Th}_L^{GN}(\mathbb{Q}^n, \blacktriangleleft)$ nem axiomatizálható.†

Ez érdekes lehet a következő tételekkel kontrasztban.

TÉTEL 9. [V07c] Legyen $n \geqslant 2$. Ekkor $\mathtt{Th}_L^{GN}(\mathbb{R}^n, \blacktriangleleft)$ nem axiomatizálható.†

TÉTEL 10. [V00], [V07b] $\mathtt{Th}_S^{Op}(\mathbb{Q}^2, \blacktriangleleft)$ axiomatizálható, tetszőleges Op operátorkészlettel és tetszőleges S szignatúrával.†

Ennek a tételnek a bizonyítása J. van Benthem tételén (4) és a következő megfigyelésen alapszik. Emlékeztetünk, hogy egy struktúrát akkor hívunk ω-kategorikusnak, ha izomorfizmus erejéig csupán egyetlen megszámlálható modellje van.

TÉTEL 11. [V07b] Ha egy megszámlálható $(T, \prec)$ időfolyam elsőrendű elmélete ω-kategorikus és rekurzívan megszámlálható, akkor bármely Op temporális operátorkészlet és bármely S szignatúra esetén $\mathtt{Th}_L^{Op}(T, \prec)$ axiomatizálható.†

Az előző tételből az következik, hogy ha $n > 2$, akkor $(\mathbb{Q}^n, \blacktriangleleft)$ elsőrendű elmélete vagy nem rekurzívan felsorolható, vagy nem ω-kategorikus, vagy egyik sem.

TÉTEL 12. $(\mathbb{Q}^n, \blacktriangleleft)$ elsőrendű elmélete nem ω-kategorikus, ha $n > 2$.†

Nem ismeretes számunkra olyan nem-axiomatizálhatósági bizonyítás a valósak feletti elsőrendű temporális logika esetén, amelyik monadikus szignatúrát használ, egyenlőségjel nélkül. Ebből az okból kifolyólag bebizonyítjuk még a következőt:

TÉTEL 13. $\mathtt{Th}_L^G(\mathbb{R}, <)$ nem axiomatizálható.†

Tételek az általánosított intervallum-értékű számításokról

A disszertáció második részében két érdekes eredményünket ismertetjük egy újonnan keletkező nem-klasszikus számítási elméletetről. Az [N05b] konferenciaközleményben Nagy Benedek egy új, diszkrét idejű/folytonos tárú számítási modellt javasolt, az ún. általánosított intervallum-értékű számítást. Ez a hagyományos Neumann-Turing-féle modellhez képest másféle idealizációt vezetett be: nem az a helyzet ez esetben, hogy a memóriacellák mindegyikének mérete univerzálisan korlátozott mértékű információt hordozhat, hanem, a memóriacellák információtartalma emelhető bármely határ fölé. A cellák tartalma 1-dimenziós folytonos adat, konkrétan, a [0,1) intervallum bizonyos részhalmazai a lehetséges értékek, még közelebbről, [)-típusú intervallumok véges uniói. Ez a rendszer rokon a [WN05] cikkben bevezetett optikai kiszámítási modellel.

Dióhéjban, az általánosított intervallum-értékű számítások a $\left[0, \frac{1}{2}\right)$ intervallumértékből kiindulva kezdenek el dolgozni és munkájuk bizonyos operátorok véges számú alkalmazásából áll. Ezen operátorok alkalmazása szekvenciális és determinisztikus. Az operátorok a szokásos számítógépek véges bitsorozatain elvégzett szokásos műveletek általánosításai az intervallum-értékekre, a Boole-műveletek mellett a balra és jobbra eltolás műveletei jelennek meg. Egyetlen extra operátor ehhez képest az ún. fraktálszorzat, amelynek szerepe, hogy különböző rezolúciós szinten lévő intervallum-értékeket kössön össze. Alapvetően, rokon az optikai számítások ([WN05]) zoom műveletével, kicsinyíti az intervallum-értékeket, bár a definíció ennél kissé összetettebb.

Ebben a számítási rendszerben a Turing-paradigma egy lényeges megszorítását elengedtük: nincs limit az adott számításban megjelenő cellák információtartalmának méretére. Mivel intervallumok véges uniójáról van szó, ezért az ábrázolt információ mindig véges. Természetesen, egy adott számításban mindig létezik egy felső korlát (a számítási sorozat bitsúlya) Így a modellünk a Church-Turing tézis kereteit nem feszíti szét, de más limitációk hatálya alá esik, mint a klasszikus Turing-modell. Bár a számítási folyamat szekvenciális, a belső párhuzamosság szélesebb.

Egy nyelvet akkor mondunk *általánosított intervallum-számítással eldönthetőnek*, ha van olyan (klasszikus) algoritmus, amely minden bemenő szóhoz produkál egy olyan általánosított intervallum-értékű számítási sorozatot, amely akkor és csakis akkor eredményezi az üres intervallum-értéket, ha a szó benne van a vizsgált nyelvben. Ez a definíció triviálisan a rekurzív nyelveket írja le. Az viszont érdekes, ha polinomiális ill. lineáris megszorítást teszünk a produkált intervallum számítási sorozatra, valamint az azt kiszámító algoritmus tárigényére logaritmikus megszorítást teszünk. Az ilyen nyelveket *polinomiális/lineáris általánosított intervallum-számítással eldönthetőnek* nevezzük.

Amint az eredményeink mutatni fogják, az intervallum-értékű számítások a polinomiális tárral megoldható problémák megoldására hivatottak. A dolgozatban először az általánosított intervallum-értékeket és a számításokat magyarázzuk meg. Ezen fogalmak a [N05b] cikk fogalmainak formalizálásával és elemzésével alakultak ki. Abban a SAT probléma került megoldásra egy lineáris intervallum-értékű számítással, és az a kérdés került nyilvánosságra, hogy van-e olyan $PSPACE$-teljes probléma, ami szintén megoldható lineáris intervallum-értékű számítással. Nagy Benedekkel közös konferenciacikkünkben [NV06] ezt a kérdést jóváhagyólag válaszoltuk meg: a kvantifikált propozicionális formulák igazságának problémáját sikerült ilyen számítással megoldani, amely probléma $PSPACE$-teljes.

TÉTEL 14. [NV06] Van olyan $PSPACE$-teljes nyelv, amelyik eldönthető lineáris általánosított intervallum-értékű számítással.†

Megfigyelve a fenti számítások szintaktikus tulajdonságait, a Theoretical Computer Science folyóiratban megjelent cikkünkben [NV07]

intervallum-értékű számítások olyan szintaktikus osztályát határoztuk meg, amely által polinomiális intervallum-értékű számításokkal eldönthető nyelvek osztálya egybeesik $PSPACE$-szel. Ezen megszorítás a következő: a fraktálszorzat egyik oldalán mindig a [0,1/2) kezdő intervallum-értéknek kell állnia. Ezen állítás céljából egy konkrét polinomiális tárú rekurzív algoritmust adtunk arra, hogy eldöntse, adott intervallum-értékű számítási sorozat elfogadó-e.

TÉTEL 15. [NV07] A megszorított értelemben vett polinomiális intervallum-számítások éppen a $PSPACE$-beli nyelveket képesek eldönteni. †

Az utolsó tartalmi fejezetben egy lehetséges kapcsolatot építünk ki az intervallum-értékű számítások és az intervallum temporális logika között. Közelebbről, az imént említett konkrét algoritmust úgy értelmezzük, mint eldöntő algoritmust az intervallum temporális logika formuláinak bizonyos szintaktikus részosztálya számára.

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