Short Thesis for The Degree of Doctor of Philosophy (PhD)

Exponential polynomials and polynomial equations

written by
Mehak Iqbal

and supervised by
Eszter Novák–Gselmann, PhD, dr. habil

University of Debrecen
Doctoral School of Mathematical and Computational Sciences
Debrecen, 2025

This thesis outlines the main achievements of the doctoral dissertation. Our research has led to the development of various lemmas, propositions, theorems, examples, and corollaries, all of which are derived from the works of Gselmann–Iqbal [**11, 13, 12**].

Introduction

Additive functions play an important role in the field of functional equations and commutative algebra. It is an important and challenging question how special morphisms can be characterized among additive mappings in general. The investigation of additive mappings that satisfy a polynomial equation from one ring to another was started by G. Ancochea in [**2**], who specifically studied additive functions that preserve squares. Subsequently, these findings were further developed among others by Kaplansky [**18**] and Jacobson-Rickart [**17**]. The results of [**2**] were generalized and extended in several ways, see for instance [**17**], [**18**], [**24**]. In [**16**] I.N. Herstein showed that if φ is a Jordan homomorphism of a ring R *onto* a prime ring R' of characteristic different from 2 and 3, then either φ is a homomorphism or an anti-homomorphism.

Besides homomorphisms, derivations also play a key role in the theory of rings and fields. Concerning this notion, we will follow [**19**, Chapter 14]. It is well-known that in case of additive functions, Hamel bases play an important role. As [**19**, Theorem 14.2.1] shows in case of derivations, algebraic bases are fundamental. Similar to homomorphisms, characterization theorems related to derivations also have extensive literature, see e.g. the monographs [**19, 23**].

According to a classical result in connection to derivations, if $\mathbb{F}$ is a subfield of the field $\mathbb{K}$ with characteristic zero, $P \in \mathbb{F}[x]$ is a polynomial and the additive function $a\colon \mathbb{F} \to \mathbb{K}$ fulfills

$$a(P(x)) = P'(x) \cdot a(x) \qquad (x \in \mathbb{F}),$$

then a is a derivation.

Note that this problem can be viewed as a special case of a more general problem. Indeed, let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield, let further $P \in \mathbb{F}[x]$ and $Q \in \mathbb{K}[x_1, x_2]$ be given polynomials and $a\colon \mathbb{F} \to \mathbb{K}$ be an additive function such that

$$a(P(x)) = Q(x, a(x)) \qquad (x \in \mathbb{F}).$$

The question arises: Does the above identity imply that this additive function a has some 'special form'? For certain polynomials P and Q, in the case of classical results, the unknown additive function a is a homomorphism, a derivation, or a linear combination of these. Naturally, the question arises as to whether this is the case for all polynomials P

and Q. Similar questions can be raised about generalized monomial functions instead of additive functions: assume that n and k are positive integers, $P_{i,j} \in \mathbb{F}[x]$ and $P \in \mathbb{K}[z, x_{1,1}, \ldots, x_{n,k}]$ are given polynomials for $i = 1, \ldots, n; j = 1, \ldots, k$. Suppose further that $f_1, \ldots, f_n \colon \mathbb{F} \to \mathbb{K}$ are generalized monomials (of possibly different degree) such that

(•)

$$P(x, f_1(P_{1,1}(x)), \ldots, f_1(P_{1,k}(x)), \ldots, f_n(P_{n,1}(x)), \ldots, f_n(P_{n,k}(x))) = 0$$

holds for all $x \in \mathbb{F}$? Is it true that the monomial functions here necessarily have a 'special' form? If so, is there a method to determine these special forms?

In the case where the unknown generalized monomial functions are additive, some results can be found e.g. in [**6, 7, 8, 9, 14, 15**]. The papers [**10, 11**] contain related results, but there the unknown functions were assumed to be quadratic. Our papers [**11**] and [**12**] are continuations of these.

Preliminaries. We first need to familiarize ourselves with some basic concepts, such as multi-additive functions, generalized monomial functions, and generalized polynomial functions. In this section the most important notations and statements are summarized, based on the monograph Székelyhidi [**21**].

DEFINITION 1. *Let G, S be commutative semigroups, $n \in \mathbb{N}$ and let $A \colon G^n \to S$ be a function. We say that A is* n-additive *if it is a homomorphism of G into S in each variable. If $n = 1$ or $n = 2$ then the function A is simply termed to be* additive *or* bi-additive*, respectively.*

The *diagonalization* or *trace* of an n-additive function $A \colon G^n \to S$ is defined as

$$A^*(x) = A(x, \ldots, x) \qquad (x \in G).$$

One of the most important theoretical results concerning multiadditive functions is the so-called *Polarization formula*, that briefly expresses that every n-additive symmetric function is *uniquely* determined by its diagonalization under some conditions on the domain as well as on the range. Suppose that G is a commutative semigroup and S is a commutative group. The action of the *difference operator* Δ on a function $f \colon G \to S$ is defined by the formula

$$\Delta_y f(x) = f(x + y) - f(x) \qquad (x, y \in G).$$

Note that the addition in the argument of the function is the operation of the semigroup G and the subtraction means the inverse of the operation of the group S.

Theorem 1 (Polarization formula). *Suppose that G is a commutative semigroup, S is a commutative group, $n \in \mathbb{N}$. If $A\colon G^n \to S$ is a symmetric, n-additive function, then for all $x, y_1, \dots, y_m \in G$ we have*

$$\Delta_{y_1,\dots,y_m} A^*(x) = \begin{cases} 0 & \text{if} \quad m > n \\ n!A(y_1, \dots, y_m) & \text{if} \quad m = n. \end{cases}$$

Definition 2. *Let G and S be commutative semigroups, a function $p\colon G \to S$ is called a* generalized polynomial *from G to S if it has a representation as the sum of diagonalizations of symmetric multi-additive functions from G to S. In other words, a function $p\colon G \to S$ is a generalized polynomial if and only if, it has a representation*

$$p = \sum_{k=0}^{n} A_k^*,$$

where n is a nonnegative integer and $A_k\colon G^k \to S$ is a symmetric, k-additive function for each $k = 0, 1, \dots, n$. In this case we also say that p is a generalized polynomial of degree at most n.

Let n be a nonnegative integer, functions $p_n\colon G \to S$ of the form

$$p_n = A_n^*,$$

where $A_n\colon G^n \to S$ is symmetric and n-additive function are the so-called generalized monomials of degree n.

Remark 1. *Obviously, generalized monomials of degree 0 are constant functions and generalized monomials of degree 1 are additive functions.*

Furthermore, generalized monomials of degree 2 will be termed as quadratic functions.

Definition 3. *Let G be a commutative semigroup. We say that the nonzero function $m\colon G \to \mathbb{C}$ is an* exponential *if*

$$m(x + y) = m(x)m(y)$$

holds for all x, y in G.

Remark 2. *Recall that on a commutative semigroup, the identically 1 function is always an exponential.*

Definition 4. *Let G be a commutative semigroup, n be a positive integer and $m\colon G \to \mathbb{C}$ be an exponential. The function $f\colon G \to \mathbb{C}$ is called a* generalized exponential monomial of degree at most n *corresponding to the exponential m, if there exists a generalized polynomial $p\colon G \to \mathbb{C}$ such that*

$$f(x) = p(x)m(x) \qquad (x \in G)\,.$$

Finite sums of generalized exponential monomials are called *generalized exponential polynomials*

Definition 5. *Polynomials are elements of the algebra generated by additive functions over G. Namely, if n is a positive integer, $P\colon \mathbb{C}^n \to \mathbb{C}$ is a (classical) complex polynomial in n variables and $a_k\colon G \to \mathbb{C}$ ($k = 1, \ldots, n$) are additive functions, then the function*

$$x \longmapsto P(a_1(x), \ldots, a_n(x))$$

is a polynomial and, also conversely, every polynomial can be represented in such a form.

Remark 3. *For the sake of easier distinction, at some places polynomials will be called normal polynomials.*

Definition 6. *Let G be commutative semigroup, n be a positive integer and $m\colon G \to \mathbb{C}$ be an exponential. The function $f\colon G \to \mathbb{C}$ is called a* exponential monomial of degree at most n *corresponding to the exponential m, if there exists a polynomial $p\colon G \to \mathbb{C}$ such that*

$$f(x) = p(x)m(x) \qquad (x \in G)\,.$$

Finite sums of exponential monomials are called *exponential polynomials.*

The notion of derivations can be extended in several ways. We will employ the concept of higher order derivations according to Reich [**20**] and Unger–Reich [**22**]. For further results on characterization theorems on higher order derivations consult e.g. [**6, 7, 9**] and [**14**].

Definition 7. *Let $\mathbb{F} \subset \mathbb{C}$ be a field. The identically zero map is the only* derivation of order zero. *For each $n \in \mathbb{N}$, an additive mapping $f\colon \mathbb{F} \to \mathbb{C}$ is termed to be a* derivation of order n, *if there exists $B\colon \mathbb{F} \times \mathbb{F} \to \mathbb{C}$ such that B is a bi-derivation of order $n-1$ (that is, B is a derivation of order $n-1$ in each variable) and*

$$f(xy) - xf(y) - f(x)y = B(x, y) \qquad (x, y \in \mathbb{F})\,.$$

The set of derivations of order n of the field $\mathbb{F}$ will be denoted by $\mathscr{D}_n(\mathbb{F})$.

Definition 8. *Let G be an Abelian group, r a positive integer, and for each multi-index α in $\mathbb{N}^r$ let $f_\alpha\colon G \to \mathbb{C}$ be a continuous function. We say that $(f_\alpha)_{\alpha \in \mathbb{N}^r}$ is a* generalized moment sequence of rank r, *if*

$$f_\alpha(x + y) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha-\beta}(y) \tag{1}$$

holds whenever x, y are in G. The function f_0, where 0 is the zero element in $\mathbb{N}^r$, is called the generating function of the sequence.

Monomial functions, normal polynomials and polynomial equations. Let $\mathbb{F} \subset \mathbb{C}$ be a field, n be a positive integer and $P \in \mathbb{F}[x]$ be a polynomial. In [**11**], firstly we study generalized monomials $f : \mathbb{F} \to \mathbb{C}$ of degree n under the condition that the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a (normal) polynomial.

This problem is not an end in itself. We undertake this investigation because it allows the original problem to be reduced, in many cases, to a much simpler one.

Accordingly, at first we show that instead of polynomials P in the original problem, we always may restrict ourselves to (classical) monomials. In fact, we can always restrict ourselves to homogeneous normal polynomials. More precisely, to carry out the aforementioned reduction, we prove the following two things.

Let $k, n \in \mathbb{N}$, $k \geq 2$, $\mathbb{F} \subset \mathbb{C}$ be a field, $P \in \mathbb{F}[x]$ be a (classical) polynomial of degree k with leading coefficient 1 and $f : \mathbb{F} \to \mathbb{C}$ be a generalized monomial of degree n. If the mapping

$$\mathbb{F} \ni x \longmapsto f(P(x))$$

is a normal polynomial, then the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polynomial as well.

Secondly we show that if the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polynomial, then there exists a *homogeneous* complex polynomial $\tilde{P}$ and there are linearly independent, complex valued additive functions $a_1, \ldots, a_m$ on $\mathbb{F}$ such that

$$f(x^k) = \tilde{P}(a_1(x), \ldots, a_m(x)) \qquad (x \in \mathbb{F}),$$

in other words, we have

$$f(x^k) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = kn}} \lambda_\alpha a^\alpha(x) = \sum_{\substack{\alpha_1, \ldots, \alpha_m \geq 0 \\ \alpha_1 + \cdots + \alpha_m = kn}} \lambda_{\alpha_1, \ldots, \alpha_m} a_1^{\alpha_1}(x) \cdots a_m^{\alpha_m}(x)$$

for each $x \in \mathbb{F}$.

At first glance, the assumption that the mapping

$$\mathbb{F} \ni x \longmapsto f(x^k)$$

is a normal polyomial, seems a bit artificial. Nevertheless, the following examples show that this is indeed the case in connection to characterization theorems of homomorphisms and derivations.

(1) Let k be a positive integer, $\varphi_1, \ldots, \varphi_k \colon \mathbb{F} \to \mathbb{C}$ be linearly independent homomorphisms and $\lambda_{i,j} \in \mathbb{C}$ for all $i, j = 1, \ldots, k$. Then the mapping $f \colon \mathbb{F} \to \mathbb{C}$ defined by

$$f(x) = \sum_{i,j=1}^{k} \lambda_{i,j}\varphi_i(x)\varphi_j(x) \qquad (x \in \mathbb{F})$$

is a quadratic function. Further if $n \in \mathbb{N}$, then we also have

$$f(x^n) = \sum_{i,j=1}^{k} \lambda_{i,j}\varphi_i(x)^n\varphi_j(x)^n \qquad (x \in \mathbb{F}).$$

In other words, we have

$$f(x^n) = P(\varphi_1(x), \ldots, \varphi_k(x)) \qquad (x \in \mathbb{F}),$$

where the k-variable complex, homogeneous polynomial P is defined by

$$P(x_1, \ldots, x_k) = \sum_{i,j=1}^{k} \lambda_{i,j}x_i^n x_j^n \qquad (x_1, \ldots, x_k \in \mathbb{C}).$$

(2) Suppose now that $\mathbb{F}$ is a subfield of $\mathbb{C}$. Let k be a positive integer, $d_1, \ldots, d_k \colon \mathbb{F} \to \mathbb{C}$ be linearly independent derivations and $\lambda_{i,j} \in \mathbb{C}$ for all $i, j = 1, \ldots, k$. Then the mapping $f \colon \mathbb{F} \to \mathbb{C}$ defined by

$$f(x) = \sum_{i,j=1}^{k} \lambda_{i,j}d_i(x)d_j(x) \qquad (x \in \mathbb{F})$$

is a quadratic function. Further if $n \in \mathbb{N}$, then we also have

$$f(x^n) = \sum_{i,j=1}^{k} \lambda_{i,j}n^2x^{2n-2}d_i(x)d_j(x) \qquad (x \in \mathbb{F}).$$

In other words,

$$f(x^n) = P(x, d_1(x), \ldots, d_k(x)) \qquad (x \in \mathbb{F}),$$

where the $(k+1)$-variable complex polynomial P is defined by

$$P(x_1, \ldots, x_k, z) = \sum_{i,j=1}^{n} \lambda_{i,j}n^2z^{2n-2}x_ix_j \qquad (x_1, \ldots, x_k, z \in \mathbb{C}).$$

Regarding 'polynomial equations' for generalized monomials, we note that in the literature, there are several results for additive functions $a \colon \mathbb{F} \to \mathbb{C}$ that also satisfy a polynomial equation. Based on the results presented above, the following statement can be deduced.

Let $\mathbb{F} \subset \mathbb{C}$ be a field, $k \in \mathbb{N}$, $k \geq 2$ and $P \in \mathbb{Q}[x]$ be a (classical) polynomial of degree k.

(1) If the additive function $a\colon \mathbb{F} \to \mathbb{C}$ fulfills
$$a(P(x)) = P(a(x)) \qquad (x \in \mathbb{F}),$$
then there exists a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that $a(x) = a(1)\varphi(x)$ for all $x \in \mathbb{F}$. Further, we also have $a(1) \in \{0, 1\}$.

(2) If the additive function $a\colon \mathbb{F} \to \mathbb{C}$ fulfills
$$a(P(x)) = P'(x)a(x) \qquad (x \in \mathbb{F}),$$
then a is a derivation.

We emphasize that the results of the previous statement are classical ones. Nevertheless, we would like to indicate that from one hand the problem we consider in this thesis has some prior results both in algebra and the theory of functional equations. Further, with the help of the statements presented here, the proofs can be significantly simplified (at least for mappings $a\colon \mathbb{F} \to \mathbb{C}$).

In the papers [**6, 7, 9, 14, 15**] further results can be found concerning additive functions that also fulfill certain polynomial equations.

We also note that related problems have already been considered by Z. Boros and E. Garda–Mátyás in [**4, 3**] by Z. Boros and R. Menzer in [**5**] and also by M. Amou in [**1**]. In these papers the authors consider real monomial functions, which satisfy certain conditional equations on a specified planar curve. In [**10**], the polynomial equation $f(P(x)) = Q(f(x))$ for the monomial function $f\colon \mathbb{F} \to \mathbb{C}$ was considered.

A result for additive functions. The simplest special case of the problem we are interested in is when the generalized monomial $f\colon \mathbb{F} \to \mathbb{C}$ is of degree 1, i.e., when f is an additive function. In this regard, we have the following statement.

Let k be a positive integer, $\mathbb{F} \subset \mathbb{C}$ be a field and $a\colon \mathbb{F} \to \mathbb{C}$ be an additive function. If the mapping
$$\mathbb{F} \ni x \longmapsto a(x^k)$$
is a (normal) polynomial, then a is a generalized higher order derivation.

Results for quadratic functions. As a continuation, the question arises what happens when the generalized monomial $f\colon \mathbb{F} \to \mathbb{C}$ is of degree 2, i.e., when f is a quadratic function. In this case, we have the following results.

Let $n \in \mathbb{N}$, $n \geq 2$ and $\mathbb{F} \subset \mathbb{C}$ be a field. Assume that $f\colon \mathbb{F} \to \mathbb{C}$ is a quadratic function, while $a\colon \mathbb{F} \to \mathbb{C}$ is additive and we have

(2) $$f(x^n) = a(x)^{2n} \qquad (x \in \mathbb{F}).$$

Then there exists a complex constant $\alpha \in \mathbb{C}$ and a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{C}$ such that

$$a(x) = \alpha\varphi(x) \qquad \text{and} \qquad f(x) = \alpha^{2n}\varphi(x)^2 \qquad (x \in \mathbb{F}).$$

And also conversely, if we define the functions a and f through the above formula then they satisfy equation (2) for all $x \in \mathbb{F}$.

The second result states the following.

Let $\mathbb{F} \subset \mathbb{C}$ be a field, $f\colon \mathbb{F} \to \mathbb{C}$ be a quadratic function, while $a_1, a_2\colon \mathbb{F} \to \mathbb{C}$ be additive functions. Then equation

$$f(x^2) = a_1(x)^2 a_2(x)^2$$

holds for all $x \in \mathbb{F}$ if and only if there exist homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{C}$ such that

$$f(x) = f(1)\cdot\varphi_1(x)\varphi_2(x) \qquad \text{and} \qquad a_i(x) = a_i(1)\varphi_i(x)\,(x \in \mathbb{F}, i = 1, 2).$$

Qudaratic functions as the solutions of polynomial equations. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield of $\mathbb{K}$. The main objective of [**12**] is to determine all those quadratic functions $q\colon \mathbb{F} \to \mathbb{K}$ that satisfy a Levi-Civita equation on the multiplicative structure, i.e., that can be written as

$$q(xy) = \sum_{i=1}^{k} g_i(x)h_i(y) \qquad \left(x, y \in \mathbb{F}^{\times}\right)$$

with some positive integer k and with some appropriate functions g_i, h_i, $i = 1, \ldots, k$. For this, those quadratic functions q that satisfy the equations

$$q(xy) = q(x)q(y) \qquad \left(x, y \in \mathbb{F}^{\times}\right)$$

and

$$q(xy) = x^2 q(y) + q(x)y^2 \qquad \left(x, y \in \mathbb{F}^{\times}\right),$$

respectively, must first be determined.

Let us endow $\mathbb{F}^{\times}$ and also $\mathbb{K}$ with the discrete topology and let $\mathscr{C}(\mathbb{F}^{\times}, \mathbb{K})$ denote the linear space of all those functions $f\colon \mathbb{F}^{\times} \to \mathbb{K}$ that are continuous. Then the set

$$\mathscr{V} = \{f|_{\mathbb{F}^{\times}} \mid f\colon \mathbb{F} \to \mathbb{K} \text{ is quadratic}\}$$

is a closed, translation-invariant linear subspace of $\mathscr{C}(\mathbb{F}^{\times}, \mathbb{K})$. So $\mathscr{V} \subset \mathscr{C}(\mathbb{F}^{\times}, \mathbb{K})$ is a *variety*. Note that the translate of a function $f \in \mathscr{C}(\mathbb{F}^{\times}, \mathbb{K})$ by an element $y \in \mathbb{F}^{\times}$ is defined as

$$(\tau_y f) = f(x \cdot y) \qquad \left(x \in \mathbb{F}^{\times}\right).$$

We also have some illustrative examples

(1) Let $\varphi_1, \varphi_2 \colon \mathbb{F} \to \mathbb{K}$ be homomorphisms and let us consider the function $q\colon \mathbb{F} \to \mathbb{K}$ defined by

$$q(x) = \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F})\,.$$

Then q is the trace of the symmetric and bi-additive mapping

$$B(x,y) = \frac{1}{2}\left[\varphi_1(x)\varphi_2(y) + \varphi_1(y)\varphi_2(x)\right] \qquad (x, y \in \mathbb{F})\,.$$

So q is a quadratic function. Further, we have

$$\begin{aligned} q(xy) &= \varphi_1(xy)\varphi_2(xy) \\ &= (\varphi_1(x)\varphi_1(y)) \cdot (\varphi_2(x)\varphi_2(y)) \\ &= (\varphi_1(x)\varphi_2(x)) \cdot (\varphi_1(y)\varphi_2(y)) = q(x)\cdot q(y) \qquad (x, y \in \mathbb{F})\,. \end{aligned}$$

(2) Let $d\colon \mathbb{F} \to \mathbb{K}$ be a derivation and define the quadratic mapping $q\colon \mathbb{F} \to \mathbb{K}$ by

$$q(x) = d(x^2) \qquad (x \in \mathbb{F})\,.$$

An easy computation shows that in this case, q determines uniquely a symmetric and bi-additive mapping, namely

$$B(x,y) = d(xy) \qquad (x, y \in \mathbb{F})\,.$$

Moreover, we have

$$\begin{aligned} \frac{q(xy)}{x^2y^2} &= \frac{1}{x^2y^2} d((xy)^2) = \frac{1}{x^2y^2}\cdot 2xyd(xy) = \frac{2}{xy}\left[xd(y) + d(x)y\right] \\ &= 2\frac{d(x)}{x} + 2\frac{d(y)}{y} = \frac{d(x^2)}{x^2} + \frac{d(y^2)}{y^2} \\ &= \frac{q(x)}{x^2} + \frac{q(y)}{y^2} \end{aligned}$$

for all $x, y \in \mathbb{F}^{\times}$.

With the above notations, the first example shows that the mapping q defined with the aid of the homomorphisms φ_1 and φ_2 by the formula

$$q(x) = \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F})\,,$$

is an exponential in the variety $\mathscr{V}$.

Similarly, the second example shows that the mapping q defined with the help of the derivation $d\colon \mathbb{F} \to \mathbb{K}$ by

$$q(x) = d(x^2) \qquad (x \in \mathbb{F})$$

is a moment function of degree 1 corresponding to the exponential $q_0(x) = x^2$ $(x \in \mathbb{F})$.

In [**12**], we are mainly interested in converse direction.

Quadratic functions that are multiplicative and additive on the multiplicative group $\mathbb{F}^{\times}$. First we have a result for the quadratic functions that are multiplicative on $\mathbb{F}^{\times}$. The theorem states the following.

(1) Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function. If the mapping q is multiplicative on $\mathbb{F}^{\times}$, that is,

$$q(xy) = q(x)q(y)$$

holds for all $x, y \in \mathbb{F}^{\times}$, then there exist homomorphisms $\varphi_1, \varphi_2\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = c \cdot \varphi_1(x)\varphi_2(x) \qquad (x \in \mathbb{F}),$$

where $c \in \{0, 1\}$.

The second theorem for the quadratic functions that are additive on $\mathbb{F}^{\times}$ states following.

(2) Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let further $q\colon \mathbb{F} \to \mathbb{K}$ be a quadratic function. If the mapping $\dfrac{q}{\pi_2}$ is additive on $\mathbb{F}^{\times}$, that is,

$$q(xy) = q(x)y^2 + x^2 q(y)$$

holds for all $x, y \in \mathbb{F}^{\times}$ and $\pi_2(x) = x^2$ $(x \in \mathbb{F})$, then there exists a second order derivation $d\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = 4xd(x) - d(x^2) \qquad (x \in \mathbb{F}).$$

Note that the quadratic mapping that appears in Example 2 is covered in Theorem 2. Indeed, if $d\colon \mathbb{F} \to \mathbb{K}$ is a derivation, then $d \in \mathscr{D}_1(\mathbb{F}, \mathbb{K}) \subset \mathscr{D}_2(\mathbb{F}, \mathbb{K})$. So d is a derivation of order two, too. Thus

$$q(x) = 4xd(x) - d(x^2) = 2d(x^2) - d(x^2) = d(x^2) \qquad (x \in \mathbb{F}),$$

showing that mappings appearing in Example 2 can indeed be written as mappings appearing in Theorem 2.

However, Theorem 2 shows that the equation

$$q(xy) = x^2 q(y) + y^2 q(x) = 0 \qquad (x, y \in \mathbb{F})$$

has other quadratic solutions than those found in Example 2. Indeed, if $d_1, d_2\colon \mathbb{F} \to \mathbb{K}$ are non-identically zero derivations, then $d_1 \circ d_2 \in \mathscr{D}_2(\mathbb{F}, \mathbb{K}) \setminus \mathscr{D}_1(\mathbb{F}, \mathbb{K})$. Then the mapping $q\colon \mathbb{F} \to \mathbb{K}$ defined by

$$q(x) = 4xd_1 \circ d_2(x) - d_1 \circ d_2(x^2) \qquad (x \in \mathbb{F})$$

is quadratic. Further, we have

$$\begin{aligned}q(x^2) &= 4x^2 d_1\circ d_2(x^2) - d_1\circ d_2(x^4)\\ &= 4x^2\left(2x d_1\circ d_2(x) + 2d_1(x)d_2(x)\right)\\ &- \big(4x^3 d_1\circ d_2(x) + 12x^2 d_1(x)d_2(x)\big)\\ &= 4x^3 d_1\circ d_2(x) - 4x^2 d_1(x)d_2(x)\end{aligned}$$

and

$$\begin{aligned}2x^2 q(x) &= 2x^2\big(4x d_1\circ d_2(x) - d_1\circ d_2(x^2)\big)\\ &= 8x^3 d_1\circ d_2(x) - 2x^2\left(2x d_1\circ d_2(x) + 2d_1(x)d_2(x)\right)\\ &= 4x^3 d_1\circ d_2(x) - 4x^2 d_1(x)d_2(x)\end{aligned}$$

for all $x\in\mathbb{F}$. Thus

$$q(x^2) = 2x^2 q(x) \qquad (x\in\mathbb{F})\,.$$

Note, however, that due to the methods described in the proof of Theorem 2, this equation is equivalent to

$$q(xy) = x^2 q(y) + y^2 q(x) \qquad (x,y\in\mathbb{F})\,.$$

In view of Theorem 1, if $\varphi\colon\mathbb{F}\to\mathbb{K}$ is a homomorphism then the mapping q defined on $\mathbb{F}$ by

$$q(x) = \varphi(x)^2 \qquad (x\in\mathbb{F})$$

is quadratic and is also an exponential on the multiplicative group $\mathbb{F}^\times$ Thus, using the previous theorem, we obtain the following statement.

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F}\subset\mathbb{K}$ be a subfield. Let further $q\colon\mathbb{F}\to\mathbb{K}$ be a quadratic function, while $\varphi\colon\mathbb{F}\to\mathbb{K}$ be a homomorphism such that

$$q(xy) = \varphi(x)^2 q(y) + q(x)\varphi(y)^2 \qquad (x,y\in\mathbb{F})\,.$$

Then there exists a second-order derivation $d\in\mathscr{D}_2(\mathbb{F},\mathbb{K})$ such that

$$q(x) = \varphi\left(4xd(x) - d(x^2)\right) \qquad (x\in\mathbb{F})\,.$$

And vice versa, that is, if d is a second-order derivation and φ is a homomorphism, then the function q given by the above formula is a quadratic function that also satisfies the above equation.

We also obtain some extension of these results, in which instead of the identity mapping, homomorphisms appear.

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F}\subset\mathbb{K}$ be a subfield. Let further $q\colon\mathbb{F}\to\mathbb{K}$ be a quadratic function, while $\varphi_1,\varphi_2\colon\mathbb{F}\to\mathbb{K}$ be homomorphisms such that

$$q(xy) = \varphi_1(x)\varphi_2(x)q(y) + \varphi_1(y)\varphi_2(y)q(x) \qquad (x,y\in\mathbb{F})\,.$$

Then there exists an additive function $a\colon \mathbb{F} \to \mathbb{K}$ such that

$$q(x) = 2(\varphi_1(x) + \varphi_2(x))a(x) - a(x^2) \qquad (x \in \mathbb{F}),$$

where the additive function also fulfills

$$(\spadesuit) \qquad \begin{aligned} &2\,a\,(x\,y\,z) - (\varphi_1(x) + \varphi_2(x))\;a\,(y\,z) - (\varphi_1(y) + \varphi_2(y))\;a\,(x\,z) \\ &\quad - (\varphi_1(z) + \varphi_2(z))\;a\,(x\,y) + (\varphi_1(x)\,\varphi_2(y) + \varphi_2(x)\,\varphi_1(y))\;a\,(z) \\ &\quad + (\varphi_1(x)\,\varphi_2(z) + \varphi_2(x)\,\varphi_1(z))\;a\,(y) \\ &\quad + (\varphi_1(y)\,\varphi_2(z) + \varphi_2(y)\,\varphi_1(z))\;a\,(x) = 0 \end{aligned}$$

for all $x, y, z \in \mathbb{F}$. And also conversely, if a is an additive function fulfilling ($\spadesuit$) with the homomorphisms φ_1, φ_2, then the function q given by the above formula is a quadratic function that also satisfies the above equation.

If $\varphi_1 = \varphi_2 = \mathrm{id}$ in equation ($\spadesuit$), then we have

$$a(x^3) - 6xa(x^2) + 3x^2a(x) = 0 \qquad (x \in \mathbb{F})$$

and $a(1) = 0$ for the additive function $a\colon \mathbb{F} \to \mathbb{K}$. In view of [**14**, Corollary 2] we deduce that $a \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$.

Similarly, if $\varphi\colon \mathbb{F} \to \mathbb{K}$ is a non-identically zero homomorphism and we take $\varphi_1 = \varphi_2 = \varphi$, then equation ($\spadesuit$) reduces to

$$a(x^3) - 6\varphi(x)a(x^2) + 3\varphi(x)^2a(x) = 0 \qquad (x \in \mathbb{F}).$$

This means that the mapping $d\colon \mathbb{F} \to \mathbb{K}$ defined by

$$d(x) = \varphi^{-1}(a(x)) \qquad (x \in \mathbb{F})$$

satisfies

$$d(x^3) - 6xd(x^2) + 3x^2d(x) = 0 \qquad (x \in \mathbb{F}).$$

So $a = \varphi \circ d$ with an appropriate $d \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$.

These two special cases allow us to conclude that in the general case (see identity ($\spadesuit$)), the function a can be represented with the help of a second-order derivation and homomorphisms.

Quadratic functions as solutions of polynomial equations. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield. Let r be a positive integer and suppose that for all multi-index $\alpha \in \mathbb{N}^r$ the mapping $q_\alpha\colon \mathbb{F} \to \mathbb{K}$ is quadratic such that

$$q_0(x) = x^2 \qquad (x \in \mathbb{F}).$$

Assume further that we have

$$q_\alpha(xy) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} q_\beta(x) q_{\alpha-\beta}(y) \qquad \left(x, y \in \mathbb{F}^\times\right)$$

for all multi-index $\alpha \in \mathbb{N}^r$. Then there exists a sequence of $\mathbb{K}$-valued 'additive functions' $a = (a_\alpha)_{\alpha\in\mathbb{N}^r}$ on $\mathbb{F}^\times$ such that

$$q_\alpha(x) = B_\alpha(a(x))x^2$$

holds for all $x \in \mathbb{F}$ and for each multi-index $\alpha \in \mathbb{N}^r$. Here the fact that $a_\alpha\colon \mathbb{F} \to \mathbb{K}$ is an 'additive function' means that there exists $d_\alpha \in \mathscr{D}_2(\mathbb{F}, \mathbb{K})$ such that

$$a_\alpha(x) = 4\frac{d_\alpha(x)}{x} - \frac{d_\alpha(x^2)}{x^2} \qquad \left(x \in \mathbb{F}^\times\right).$$

A functional equation for monomials. In [**13**] our study, inspired by [**4**], seeks to build upon the work of the authors who investigated real quadratic functions f that satisfy the equation

$$f(x^2) = K \cdot x^2 f(x) \qquad (x \in \mathbb{R}).$$

Their research laid the foundation for our further exploration of functional equations involving monomial functions. Assuming that the functions involved are monomials, we started the systematic examination of polynomial equation in the papers [**11, 12**]. Our research work delves into a natural extension of the aforementioned equation, as we set out to uncover the properties of monomial functions $f\colon \mathbb{F} \to \mathbb{K}$ of degree n that satisfy the equation

$$f(x^2) = \kappa \cdot x^n f(x) \qquad (x \in \mathbb{F}).$$

As we will see, similarly to additive and quadratic functions, monomial functions in the equation in some cases can be represented with the aid of homomorphisms and higher-order derivations.

Let $\mathbb{F}, \mathbb{K}$ be fields with characteristic zero such that $\mathbb{F} \subset \mathbb{K}$, and n be a positive integer. Define

$$\mathscr{M}_n(\mathbb{F}, \mathbb{K}) = \{f\colon \mathbb{F} \to \mathbb{K} \mid f \text{ is a monomial of degree } n\}.$$

At first instance, the above problem may seem quite specific, and naturally the question arises whether we should consider the more general equation

$$f(x^2) = \kappa \cdot x^l f(x) \qquad (x \in \mathbb{F})$$

with a fixed positive integer l. In this regard, we prove firstly the following statement.

Let n, l be positive integers, κ be a nonzero element of the field $\mathbb{K}$ and $f \in \mathscr{M}_n(\mathbb{F}, \mathbb{K})$ such that

$$f(x^2) = \kappa \cdot x^l f(x) \qquad (x \in \mathbb{F})$$

holds. If $n \neq l$, then the function f is identically zero.

This shows that the above equation can have nonidentically zero solutions only when $n = l$. So it is enough to restrict ourselves to the case $n = l$. In this direction, we have the following result.

Let n be a given positive integer, $\kappa \in \mathbb{K}$ and $f \in \mathscr{M}_n(\mathbb{F}, \mathbb{K})$ such that

(3) $$f(x^2) = \kappa \cdot x^n f(x)$$

holds for all $x \in \mathbb{F}$. Then

(i) if $\kappa \notin \left\{2^k \,|\, k = 0, 1, \dots, n\right\}$, then f is identically zero,

(ii) if $\kappa = 1$, then

$$f(x) = f(1) \cdot x^n \qquad (x \in \mathbb{F}),$$

(iii) if $\kappa = 2$, then there exists an $a \in \mathscr{D}_{2n-1}(\mathbb{F}, \mathbb{K})$ such that

$$f(x) = \sum_{j=1}^{n} \lambda_{n,j} x^{n-j} a(x^j) \qquad (x \in \mathbb{F})$$

holds with appropriate constants $\lambda_1, \dots, \lambda_n \in \mathbb{K}$,

(iv) if $\kappa = 2^n$, then there exists a symmetric and n-additive mapping $A_n \colon \mathbb{F}^n \to \mathbb{K}$ such that

$$\sum_{\sigma \in \mathscr{S}_{n+1}} \left\{A_n(x_{\sigma(1)} \cdot x_{\sigma(2)}, x_{\sigma(3)}, ..., x_{\sigma(n+1)}) - x_{\sigma(1)} \cdot A_n(x_{\sigma(2)}, \dots, x_{\sigma(n+1)}) - x_{\sigma(2)} \cdot A_n(x_{\sigma(1)}, \dots, x_{\sigma(n+1)})\right\} = 0 \qquad (x_1, \dots, x_{n+1} \in \mathbb{F})$$

and

$$f(x) = A_n(x, \dots, x) \qquad (x \in \mathbb{F})$$

holds.

We also deal with the special case $n = 3$ separately. Compared to the above result, for $n = 3$ we can prove two more things. On the one hand, we show that in the case $\kappa = 2$, the order of the higher-order derivation appearing in the representation of the function f is at most 3, not at most 5. On the other hand, we show that in the case $\kappa = 4$ the function f is identically zero. In this regard, we have the following outcome:

Let $\kappa \in \mathbb{K}$ be arbitrarily fixed and $f \in \mathscr{M}_3(\mathbb{F}, \mathbb{K})$ be a monomial for which

(4) $$f(x^2) = \kappa \cdot x^3 f(x)$$

holds for all $x \in \mathbb{F}$. Then the following cases are possible

(i) $\kappa = 1$ and then

$$f(x) = f(1) \cdot x^3 \qquad (x \in \mathbb{F}).$$

(ii) $\kappa = 2$ and then there exists a $d \in \mathscr{D}_3(\mathbb{F}, \mathbb{K})$ such that

$$f(x) = d\left(x^3\right) - \frac{9\,x\,d\,(x^2)}{2} + 9\,x^2\,d\,(x) \qquad (x \in \mathbb{F})\,.$$

(iii) $\kappa = 8$ and then there exists a symmetric and 3-additive mapping D_3 for which

$$\sum_{\sigma\in\mathscr{S}_4}\Big[D_3(x_{\sigma(1)}x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) - x_{\sigma(1)}D_3(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) \\ -x_{\sigma(2)}D_3(x_{\sigma(1)}, x_{\sigma(3)}, x_{\sigma(4)})\Big] = 0 \\ (x_1, x_2, x_3, x_4 \in \mathbb{F})$$

such that

$$f(x) = D_3(x, x, x) \qquad (x \in \mathbb{F})\,.$$

(iv) $\kappa \in \mathbb{K} \setminus \{1, 2, 8\}$ and f is identically zero.

Bibliography

[1] Masaaki Amou. Quadratic functions satisfying an additional equation. *Acta Math. Hungar.*, 162:40–51, 2020.

[2] Germán Ancochea. Le théorème de von Staudt en géométrie projective quaternionienne. *J. Reine Angew. Math.*, 184:193–198, 1942.

[3] Zoltán Boros and Edit Garda-Mátyás. Conditional equations for monomial functions. *Publ. Math. Debr.*, 100(3-4):263–276, 2022.

[4] Zoltán Boros and Edit Garda-Mátyás. Conditional equations for quadratic functions. *Acta Math. Hungar.*, 154:389–401, 2018.

[5] Zoltán Boros and Rayene Menzer. An alternative equation for generalized monomials. *Aequationes Math.*, 97(1):113–120, 2023.

[6] Bruce Ebanks. Characterizing ring derivations of all orders via functional equations: results and open problems. *Aequationes Math.*, 89(3):685–718, 2015.

[7] Bruce Ebanks. Polynomially linked additive functions. *Aequationes Math.*, 91(2):317–330, 2017.

[8] Bruce Ebanks. Polynomially linked additive functions—II. *Aequationes Math.*, 92(3):581–597, 2018.

[9] Bruce Ebanks, Thomas Riedel, and Prasanna K. Sahoo. On the order of a derivation. *Aequationes Math.*, 90(2):335–340, 2016.

[10] Eszter Gselmann. Polynomial identities satisfied by generalized polynomials. *Publ. Math. Debr.*, 101(3-4):421–450, 2022.

[11] Eszter Gselmann and Mehak Iqbal. Monomial functions, normal polynomials and polynomial equations. *Aequationes Math.*, 97(5-6):1059–1082, 2023.

[12] Eszter Gselmann and Mehak Iqbal. Quadratic functions as solutions of polynomial equations, 2024.

[13] Eszter Gselmann and Mehak Iqbal. A functional equation for monomial functions. *J.Appl.Anal.*, 2025.

[14] Eszter Gselmann, Gergely Kiss, and Csaba Vincze. On functional equations characterizing derivations: methods and examples. *Results Math.*, 73(2):Art. 74, 27, 2018.

[15] Eszter Gselmann, Gergely Kiss, and Csaba Vincze. Characterization of field homomorphisms through Pexiderized functional equations. *Journal of Difference Equations and Applications*, 25(12):1645–1679, 2019.

[16] Israel Nathan Herstein. Jordan homomorphisms. *Trans. Amer. Math. Soc.*, 81:331–341, 1956.

[17] Nathan Jacobson and Charles E. Rickart. Jordan homomorphisms of rings. *Trans. Amer. Math. Soc.*, 69:479–502, 1950.

[18] Irving Kaplansky. Semi-automorphisms of rings. *Duke Math. J.*, 14:521–525, 1947.

[19] Marek Kuczma. *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality.* Basel: Birkhäuser, 2^{nd} edition, 2009.

[20] Ludwig Reich. Derivationen zweiter Ordnung als Lösungen von Funktionalgleichungen—ein überblick. In *Gyula Maurer zum 70. Geburtstag*, volume 337 of *Grazer Math. Ber.*, pages 45–65. Karl-Franzens-Univ. Graz, Graz, 1998.

[21] László Székelyhidi. *Convolution Type Functional Equations on Topological Abelian Groups.* World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[22] Josef Unger and Ludwig Reich. *Derivationen höherer Ordnung als Lösungen von Funktionalgleichungen*, volume 336 of *Grazer Mathematische Berichte [Graz Mathematical Reports].* Karl-Franzens-Universität Graz, Graz, 1998.

[23] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. 1.* Springer-Verlag, New York-Heidelberg-Berlin, 1975. With the cooperation of I. S. Cohen, Corrected reprinting of the 1958 edition, Graduate Texts in Mathematics, No. 28.

[24] Wiesław Żelazko. A characterization of multiplicative linear functionals in complex Banach algebras. *Studia Math.*, 30:83–85, 1968.

List of talks

[1] A functional equation for monomial functions, *The* 24^{th} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Debrecen, Hungary, February 6-9, 2025.*

[2] Monomial functions, normal polynomials and polynomial equations, *Dr. Varecza Árpád Memorial Conference, Nyíregyháza, Hungary, November 14–15, 2024*

[3] A functional equation for monomial functions, *The* 60^{th} *International Symposium on Functional Equations and Inequalities, Kościelisko, Poland, June 9-15, 2024*

[4] Quadratic functions as solutions of polynomial equations, *The* 23^{rd} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, January 31 – February 3, 2024.*

[5] Generalised polynomials and polynomial equations, *Islamic university of Science and technology, Awantipora, Kashmir, India, August 16, 2023.*

[6] Monomial functions, normal polynomials and polynomial equations, *The* 59^{th} *International Symposium on Functional Equations and Inequalities, Hajdúszoboszló, Debrecen, Hungary, June 18-25, 2023.*

[7] Monomial functions, normal polynomials and polynomial equations, *The* 22^{nd} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Debrecen, Hungary, February 1-4 2023.*

[8] On the existence of monotone sequences in generalised ordered sets, *The* 21^{st} *Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, February 2-5, 2022.*

UNIVERSITY of DEBRECEN

UNIVERSITY AND NATIONAL LIBRARY
UNIVERSITY OF DEBRECEN
H-4002 Egyetem tér 1, Debrecen
Phone: +3652/410-443, email: publikaciok@lib.unideb.hu

Registry number: DEENK/362/2025.PL
Subject: PhD Publication List

Candidate: Mehak Iqbal
Doctoral School: Doctoral School of Mathematical and Computational Sciences
MTMT ID: 10100156

List of publications related to the dissertation

Foreign language scientific articles in international journals (2)

1. Gselmann, E., **Iqbal, M.**: A functional equation for monomial functions.
   J. Appl. Anal. [Epub ahead of print], 1-10, 2025. ISSN: 1425-6908.
   DOI: http://dx.doi.org/10.1515/jaa-2025-0013
   IF: 0.6 (2023)

2. Gselmann, E., **Iqbal, M.**: Monomial functions, normal polynomials and polynomial equations.
   Aequ. Math. 97, 1059-1082, 2023. ISSN: 0001-9054.
   DOI: http://dx.doi.org/10.1007/s00010-023-00972-z
   IF: 0.9

List of other publications

Foreign language scientific articles in international journals (4)

3. **Iqbal, M.**, Száz, Á.: An instructive treatment of the Brezis-Browder ordering and maximality principles.
   Contrib. Math. 8, 61-74, 2023. EISSN: 2709-3646.
   DOI: http://dx.doi.org/10.47443/cm.2023.021

4. **Iqbal, M.**, Száz, Á.: On the existence of monotone subsequences in generalized ordered sets.
   Math. stud. 92 (3-4), 65-77, 2023. ISSN: 0025-5742.

5. Boros, Z., **Iqbal, M.**, Száz, Á.: A relational improvement of a true particular case of Fierro's maximality theorem.
   Filomat. 36, 7091-7101, 2022. ISSN: 0354-5180.
   DOI: https://doi.org/10.2298/FIL2220091B
   IF: 0.8

DEBRECENI EGYETEM * Egyetemi és Nemzeti Könyvtár *

6. Ali, A. H., Meften, G. A., Bazighifan, O., **Iqbal, M.**, Elaskar, S., Awrejcewicz, J.: A Study of Continuous Dependence and Symmetric Properties of Double Diffusive Convection: Forchheimer Model.
   Symmetry. 14 (4), 1-18, 2022. EISSN: 2073-8994.
   DOI: http://dx.doi.org/10.3390/sym14040682
   IF: 2.7

<u>Foreign language abstracts</u> (6)

7. **Iqbal, M.**: A functional equation for monomial functions.
   Ann. Math. Silesianae. 39, [1], 2025. ISSN: 0860-2107.
8. **Iqbal, M.**: A functional equation for monomials.
   Aequ. Math. 98 (6), 1694, 2024. ISSN: 0001-9054.
9. **Iqbal, M.**: Quadratic functions as solutions of polynomial equations.
   Ann. Math. Silesianae. 38 (2), 386, 2024. ISSN: 0860-2107.
10. **Iqbal, M.**: Monomial functions, normal polynomials and polynomial equations.
    Aequ. Math. 97 (5-6), 1265-1266, 2023. ISSN: 0001-9054.
11. **Iqbal, M.**: Monomial functions, normal polynomials and polynomial equations.
    Ann. Math. Silesianae. 37 (2), 322, 2023. ISSN: 0860-2107.
12. **Iqbal, M.**: On the existence of monotone sequences in generalized ordered sets.
    Ann. Math. Silesianae. 36 (1), 96-97, 2022. ISSN: 0860-2107.

Total IF of journals (all publications): 5
Total IF of journals (publications related to the dissertation): 1,5

The Candidate's publication data submitted to the Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

03 June, 2025

DEBRECENI EGYETEM * Egyetemi és Nemzeti Könyvtár *